Existence of Avalanching Flows

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Abstract Avalanches, landslides and debris flows are devastingly powerful natural phenomena that are far too little understood. These granular matters are mixtures of solid particles and of an interstitial fluid and are easily modelled on the microscopic level by the laws of classical mechanics. On mesoscopic and macroscopic levels the different scales of the influence of the particles, the fluid and their interaction lead to various models of avalanching flows. In this survey we consider several models of granular materials characterized by height and in case also by momentum, discuss the existence of similarity solutions, existence of arbitrary solutions and particle segregation. The main part concerns the Savage-Hutter equations for dense flow avalanches.

1 Introduction

The number of catastrophes induced by snow avalanches, landslides and debris flows has been increasing during the last decades. The reasons are a possible change of climate with heavy rainfalls, but also the activities of human beings in endangered mountainous regions. Therefore the determination of runout zones and of endangered regions by analytical and numerical methods for the different types of "avalanches" is of the utmost importance. A related physical, but less desastrous behavior can be observed in the motion of sand dunes, in the pouring of grains leading to free surfaces of stock piles and in hopper flows.

The main feature of the phenomena of *granular materials* is the mixture of solid particles with water or air leading to a behavior different from that of solids, fluids or gases. On the other hand, the main differences between the various kinds of granular flows are due to the small or large fluid-solid interaction, the size and shape of the grains and due to the predominance either of the solid particles whilst the influence of the interstitial fluid can be neglected or of the fluid carrying the small particles.

On the scale of individual grains the behavior of granular material is described by the laws of classical mechanics. But due to the huge variety of particle sizes, shapes and densities, the abrasion of particles, the interaction with the fluid, with different layers or with the bed leading to an exchange of particles, it is very difficult to model granular flow on mesoscopic or macroscopic scales. Further typical features are the dilatancy and particle segregation. In every day life it is observed that after stirring or shaking grains of different size but not necessarily with different specific weight the bigger grains tend to move upwards (*inverse grading*) and to the nose of an avalanche.

As an example consider the different scales of a typical snow avalanche with weight 10^6 kg consisting of ice particles with radii less than 1 mm. In the lower part of the avalanche sliding on a fluidized layer the solid particles dominate whereas the interstitial air can be neglected. Above this *dense flow avalanche* there may be a *powder snow avalanche* in which turbulent air carries the ice particles. In between there is a thin layer called *resuspension layer* or *saltation layer* feeding the powder snow avalanche, see [**?**, **?**].

		Dense Flow Avalanche	Powder Snow Avalanche
Volume fraction of particles		$0.3 - 0.5$	$1.10^{-3} - 1.10^{-4}$
Density	$[\frac{\mathrm{kg}}{m^3}]$	$200 - 300$	$0.1 - 10$
Length	$[m]$	$10 - 1000$	$100 - 2000$
Height	$[m]$	$0.5 - 20$	$10 - 500$
Speed	$[\frac{m}{s}]$	$0 - 70$	$10 - 200$

Table 1: Characteristic parameter ranges in dense flow and powder snow avalanches [**?**, **?**]

Although powder snow avalanches have a much lower density than dense flow avalanches, their length, height, velocity and consequently their runout zone is much larger.

This review article is organized as follows. In Section 2 we discuss particle segregation and a simple model based on diffusion and convection [**?**]. Further we consider stationary solutions of the BCRE model for (dry) sand piles. Section 3 is devoted to the Savage-Hutter model [**?**] on dense flow avalanches, its similarity solutions and the mathematical analysis of weak entropy solutions within the theory of systems of conservation laws with source terms.

2 On Models of Cohesionless Granular Materials

2.1 Particle Segregation

It is well-known that in granular flow the large particles tend to move upward and to the nose of an avalanche while the small particles lie at the bottom and at the rear end of an avalanche. This *inverse grading* can be explained by the percolation effect or the so-called *random fluctuating sieve mechanism* [**?**]: the probability for a small particle to find a hole in the granular material to fall into is larger than for large particles. But since this gravity-induced hole-filling mechanism would lead to a net mass flux downwards, Savage and Lun [**?**] also propose a *squeezing expulsion mechanism*; by this mechanism the forces exerted by the particles to each other lead to a squeezing of particles up- or downward.

A further discussion of possible reasons for the usual grading and for inverse grading can be found in [**?**] and in references therein. In addition to percolation effects, to geometrical reorganization and to segregation driven by convection, inertia or entropy the authors propose a so-called *condensation of hard spheres* as the driving force.

A simple mathematical model for segregation in a mixture of n species has been proposed by J. Braun [**?**]. Let $u_i = u_i(x,t)$ denote the concentration of the i–th species, $1 \le i \le n$, in a one-dimensional container $\Omega = (0,L)$ of height $L > 0$. Then the change $\frac{\partial u_i}{\partial t}$ of the concentration u_i is balanced by the negative of the flux $J_i = J_i(u)$ which is the sum of a convectional part $J_i^c = f_i(u)$ and of a diffusional part $J_i^d = -d(u)\,\frac{\partial u_i}{\partial x}$ with $d(u) > 0$. Thus we get the system of reaction – diffusion equations

$$u_t - \big(d(u)u_x - f(u)\big)_x = 0\,, \quad u = (u_1, \dots, u_n)\,. \tag{1}$$

The convective part $f(u)_x$ is related to the random fluctuating sieve mechanism whilst the diffusive term $\big(-d(u)u\big)_x$ accounts for the random effects of collisions and could lead to the squeezing expulsion mechanism. In order to guarantee that

$$\sum_{i=1}^{n} u_i \equiv 1 \quad \text{and } u_i \geq 0 \tag{2}$$

the structural conditions

$$\sum_{i=1}^{n} f_i(u) \equiv 0 \quad \text{and } u_i = 0 \Rightarrow f_i(u) = 0 \tag{3}$$

are imposed. We note that (2) is a consequence of (3) due to the maximum principle for parabolic equations. Besides an initial value

$$u(\cdot,0) = u_0(\cdot) \quad \text{with } \sum_{i=1}^{n} u_{i0} \equiv 1$$

the flux condition $J\big(u(x,\cdot)\big) = 0$ for $x = 0$ and $x = L$ is used to impose the (non-linear) boundary condition

$$d(u)u_x - f(u) = 0 \quad \text{at } x = 0,\ x = L\,.$$

Then the effect of segregation is reflected by the long-time behavior of solutions of (1). For $n = 2$ species with concentrations $u := u_1$ and $u_2 = 1 - u_1$ and convectional part $f(u) := f_1(u, 1-u)$ where $f_2 = -f_1$ by (3), the system (1) simplifies to one non-linear parabolic equation

$$\begin{aligned} u_t - \big(d(u)u_x - f(u)\big)_x &= 0 \\ d(u)u_x - f(u) &= 0 \quad \text{in } x = 0,\ x = L \\ u(\cdot,0) &= u_0(\cdot)\,. \end{aligned} \tag{4}$$

Theorem 1 [?] *Assume that d and f are twice continuously differentiable.*

(1) *For every prescribed mean concentration*

$$\overline{u} = \frac{1}{L}\int_0^L u(x)dx \in [0,1]$$

the stationary problem

$$d(u)u_x = f(u)\,, \tag{5}$$

cf. (4)*, has exactly one solution $u(x)$ with mean value $\overline{u}$.*

(2) *For every initial value $u_0 \in C^0([0,L])$ with mean value $\overline{u} \in [0,1]$ problem* (4) *has a unique global solution u on $[0,L] \times (0,\infty)$ converging to the stationary solution u of* (5) *with mean value $\overline{u}$ for $\to \infty$.*

Sketch of Proof (i) A solution u of (5) is defined by the ordinary differential equation

$$\frac{du}{dx} = g(u) := \frac{f(u)}{d(u)} \tag{6}$$

where $f(0) = f(1) = 0$, cf. (3), yields $g(0) = g(1) = 0$. Due to the unique solvability of (6) every solution $u(x)$ of (6) with initial value $u_0 \in [0,1]$ will exist for all $x \in [0,L]$ and satisfy $u(x) \in [0,1]$. By the same argument two solutions u_1 and u_2 with $u_1(0) < u_2(0)$ will satisfy $u_1(x) < u_2(x)$ for all $x \in [0,L]$. Since the solution $u = u(\cdot, u_0)$ is a continuous and even a monotonically increasing function of its initial value $u_0 = u(0)$ and since $u_0 = 0$ or $u_0 = 1$ yield $u \equiv 0$ or $u \equiv 1$ respectively, we conclude that the map

$$\overline{u} : [0,1] \to [0,1]\,, \quad u_0 \mapsto \overline{u(\cdot, u_0)}\,,$$

is a homeomorphism.

(ii) Given an initial value $u_0(x)$ with $u_0(x) \in [0,1]$ the solution u of (4) exists for all $t > 0$. Then $v = d(u)u_x - f(u)$ satisfies the parabolic equation

$$v_t = a(t,x)v_{xx} + b(t,x)v_x$$

with bounded functions $a = d(u)$, $b = d'(u)u_x - f'(u)$ and vanishing boundary values in $x = 0$, $x = L$. By classical theorems v and $v_x = u_t$ converge to zero for $t \to \infty$. In particular u converges to a stationary solution u_∞ of (4), i.e., u_∞ solves (6). Furthermore (4) easily implies that $\overline{u(\cdot,t)}$ is constant; hence u_∞ is the unique solution of (6) satisfying $\overline{u}_\infty = \overline{u}_0$. □

The proof of Theorem 1(1) is based on topological arguments. Therefore degree theoretical arguments are used in the case of more than two species leading to the existence of at least one stationary solution. Thus uniqueness of a final segregation of particles cannot be guaranteed for more than two species in general. Note that in this model empty space is evenly distributed in the vessel and that compressibility or dilatancy effects are ignored.

2.2 Stationary and Self-Similar Solutions

Consider a granular material such as dry sand poured at a rate $s = s(x,t) \geq 0$ and piling up to form heaps. First the material builds up without further motion, but eventually starts to roll down when the pile has a critical slope $k = \tan\alpha > 0$. The pile consists of two main parts, the standing layer of height $h = h(x,t)$ (and of constant density) and a thin rolling layer of relative height $r = r(x,t)$. In the BCRE model established by Bouchaud et al. [**?**, **?**] and modified by de Gennes [**?**] by omitting diffusion terms the exchange of grains from the rolling to the standing layer is described by the exchange term

$$\Gamma(t,r) = \gamma r \big(1 - \frac{|\nabla h|^2}{k^2}\big)\,, \quad \gamma > 0\,; \tag{7}$$

thus it is proportional to the thickness $r \geq 0$ of the rolling layer and vanishes iff $r \equiv 0$ or the slope of the bulk equals the critical slope k. Since the grains in the bulk are motionless except for the exchange $-\Gamma$ to the rolling layer, h satisfies the equation

$$h_t = \Gamma(h,r)\,.$$

However, for the rolling layer, there are two source terms s and $-\Gamma$, such that the continuity equation for r reads

$$r_t + \mathrm{div}(vr) = s - \Gamma(h, r)$$

where v is the horizontal projection of the velocity vector of rolling grains. Assuming that particles are rolling in the steepest descent direction $-\nabla h$, the term v is modelled by

$$v = -\mu \nabla h\,, \quad \mu > 0\,.$$

Summarizing we get the system of partial differential equations

$$\begin{aligned} h_t &= \gamma r \big(1 - \frac{|\nabla h|^2}{k^2}\big) \\ r_t - \mathrm{div}(\mu r \nabla h) &= s - \gamma r \big(1 - \frac{|\nabla h|^2}{k^2}\big) \end{aligned} \tag{8}$$

for $(x, t) \in \Omega \times (0, \infty)$ together with the initial conditions $r(x, 0) = 0$ and $h(x, 0) = h_0(x)$, where $h_0(x)$ describes the bottom on which the granular material is poured. If the domain $\Omega \subset \mathbb{R}^1$ or $\Omega \subset \mathbb{R}^2$ is not the whole space and surplus material drops down at $\partial\Omega$ [**?**, **?**, **?**], we prescribe

$$h(x, t) = 0 \quad \text{for } x \in \partial\Omega\,. \tag{9}$$

Since $h \geq 0$ close to $\partial\Omega$, the scalar product of ∇h with the exterior normal vector ν on $\partial\Omega$ is nonpositive. For x close to $\partial\Omega$ by $(8)_2$ $r_t = \mu \nabla r \cdot \nabla h + \ldots$ indicating that $r(x, t)$ behaves like an outgoing wave near $\partial\Omega$. Thus no boundary value for r may be prescribed.

In the *silo problem* with walls of infinite height at $\partial\Omega$ such that no material can leave the silo [**?**, **?**, **?**], (8) yields the equation

$$\frac{d}{dt} \int_\Omega (h + r) dx = \mu \int_{\partial\Omega} r \, \frac{\partial h}{\partial \nu} \, do + \int_\Omega s \, dx$$

for the balance of the total mass $\int_\Omega (h + r) dx$. Hence $\int_{\partial\Omega} r \, \frac{\partial h}{\partial \nu} \, do = 0$; since r may be arbitrary on $\partial\Omega$, see the discussion above, we get the Neumann boundary condition

$$\frac{\partial h}{\partial \nu}(x, t) = 0 \quad \text{on } \partial\Omega\,. \tag{10}$$

The system (8) is also closely related to an earlier model of L. Prigozhin [**?**, **?**] using variational inequalities. In [**?**] the authors introduce three length scales:

$L_r = \frac{\bar{s}}{\gamma}$	denotes a typical thickness of the rolling layer given a characteristic (mean) source intensity $\bar{s}$
$L_p = \frac{\mu}{\gamma}$	denotes the mean path of a rolling grain before being trapped in the standing layer
L	denotes the pile size.

Then rescaling variables by

$$x' = \frac{x}{L}\,,\; h' = \frac{h}{L}\,,\; r' = \frac{r}{L_r}\,,\; s' = \frac{s}{\overline{s}}\,,\; t' = \frac{t\overline{s}}{L}\,,$$

and omitting primes ($'$) for the new dimensionless variables and functions, h and $m = \frac{L_p}{L} r$ solve the system

$$\begin{aligned} h_t &= \frac{1}{(L_p/L)} m\big(1 - \frac{|\nabla h|^2}{k^2}\big) \\ \big(\frac{L_r}{L}\big) r_t - \mathrm{div}(m\nabla h) &= s - \frac{1}{(L_p/L)} m\,\big(1 - \frac{|\nabla h|^2}{k^2}\big)\,. \end{aligned}$$

Assuming $L_r << L_p$ and $\frac{L_p}{L} \to 0$ the second equation implies that

$$\frac{1}{(L_p/L)} m\big(1 - \frac{|\nabla h|^2}{k^2}\big) \approx s + \mathrm{div}(m\nabla h) \tag{11}$$

and consequently that

$$h_t - \mathrm{div}(m\nabla h) \approx s\,.$$

Actually, if $|\nabla h(x,t)| < k$, the term m has to vanish when $L_p/L \to 0$, see (11). Summarizing, in the limit we get the equation

$$h_t - \mathrm{div}(m\nabla h) = s$$

with the restrictions $m \geq 0$, $|\nabla h| \leq k$ and

$$|\nabla h(x,t)| < k \Rightarrow m = 0\,.$$

Under suitable assumptions and with additional diffusive terms for h and r, this formal analysis is rigorously proved in [**?**] for a related discretized system with respect to time $t \geq 0$. Furthermore the equation for h and its Lagrange multiplier m is equivalent to a variational problem in the convex set $K = \{\varphi \in H^1(\Omega) : |\nabla\varphi| \leq k \text{ a.e.}\}$:

$$\begin{cases} \text{find } h(x,t) \text{ such that } h(\cdot,t) \in K \text{ for a.a. } t > 0, \\ (h_t - s, \varphi - h)_{L^2(\Omega)} \geq 0 \quad \forall\varphi \in K \end{cases}$$

together with an initial condition $h(\cdot,0) = h_0$, see [**?**].

Note that the original BCRE equations included diffusion terms such as $\varepsilon\Delta r$ in $(8)_2$ leading to a parabolic rather than to a hyperbolic equation for r. However diffusion may lead to grains rolling upwards instead of downwards. The advantage of the system (8) is the fact that the exchange between the standing and the rolling layer is easily modelled by the exchange term Γ in (7). The other terms in (8) are just based on the conservation of masses. On the other hand inertia, momenta, longitudinal and lateral pressures as well as density changes are neglected. These effects are incorporated in the Savage-Hutter models for wet snow avalanches, see Section 3 below, leading to a highly nonlinear system of conservation laws.

In [?, ?, ?] another constitutive law for the exchange term Γ is used:

$$\Gamma(h,r) = \gamma r\big(1 - \frac{|\nabla h|}{k}\big)$$

leading to the avalanche model

$$\begin{aligned} h_t &= \gamma r\big(1 - \frac{|\nabla h|}{k}\big) \\ r_t - \operatorname{div}(\mu r \nabla h) &= s - \gamma r\big(1 - \frac{|\nabla h|}{k}\big) \end{aligned} \tag{12}$$

together with initial conditions and the boundary conditions (9) or (10) for h. To our knowledge there is no rigorous proof of existence and uniqueness of solutions to (12) up to now. Even the stationary case with $s = 0$ or $s \neq 0$ poses several open problems. One main property and difficulty of the stationary case with $s = 0$, i.e. for the system

$$r\big(1 - \frac{|\nabla h|}{k}\big) = 0, \quad \operatorname{div}(r\nabla h) = 0 \tag{13}$$

is the *non-uniqueness* of solutions: Every pair of functions h, r satisfying

$$h \geq 0, \quad |\nabla h| \leq k, \quad r = 0$$

(and even with $|\nabla h| \geq k$ leading to unstable situations) is a solution of (13).

Even in one dimension the boundary value problem

$$|\nabla h| = k \text{ a.e. in } \Omega, \quad h = 0 \text{ on } \partial\Omega\,,$$

the so-called *eikonal equation* known from geometrical optics, has uncountably many solutions, namely all piecewise linear functions on an interval $\Omega \subset \mathbb{R}^1$ with slope $\pm k$ a.e. However, uniqueness may be obtained in the setting of viscosity solutions of fully nonlinear equations, see [?, ?], or when looking for the *maximum volume solution.*

Theorem 2 *Let $\Omega \subset \mathbb{R}^1$ be a bounded open interval or let $\Omega \subset \mathbb{R}^2$ be an open bounded domain. Let the function $\psi \in C^{0,1}(\overline{\Omega})$ describe the bottom topography (bed) and let $\phi : \partial\Omega \to [0,\infty]$ with $\phi \not\equiv \infty$, $\psi \leq \phi$ on $\partial\Omega$, describe the rim (wall) of the container. Then there exists a unique maximum volume solution $h \in C^{0,1}(\overline{\Omega})$ such that*

$$\begin{aligned} &\psi(x) \leq h(x) \text{ in } \overline{\Omega}, \quad h(x) \leq \phi(x) \text{ on } \partial\Omega \\ &\psi(x) < h(x) \text{ for } x \in \Omega \;\Rightarrow\; |h|_{C^{0,1}_x} \leq k \\ &\int_\Omega (h - \psi)dx = \max\,. \end{aligned} \tag{14}$$

Here for $x \in \Omega$ the condition $|h|_{C^{0,1}_x} \leq k$ means that there exists an open ball with center x in Ω such that $|h(y) - h(y')| \leq k|y - y'|$ for all $y, y' \in B$. Note that $h(x) = \psi(x)$ iff no granular material lies on the bed at $x \in \overline{\Omega}$. The term $\int_\Omega (h - \psi)dx$ measures the total mass poured onto the bed.

Proof [?] Let

$$\begin{aligned} M \;=\; &\{h \in C^0(\overline{\Omega}) : h \geq \psi \text{ on } \overline{\Omega}, \quad h \leq \phi \text{ on } \partial\Omega\,, \\ &\quad h(x) > \psi(x) \text{ for } x \in \Omega \Rightarrow |h|_{C^{0,1}_x} \leq k\}\,. \end{aligned}$$

Since Ω is a bounded domain with Lipschitz boundary and since $\psi \in C^{0,1}(\overline{\Omega})$ there exists a constant $K = K(k, \psi)$ such that

$$|h(y) - h(y')| \leq K|y - y'| \quad \text{for all } y, y' \in \overline{\Omega} \tag{15}$$

and for all $h \in M$. The assumption $\phi \not\equiv \infty$, i.e., there exists $\xi \in \partial\Omega$ with $\phi(\xi) < \infty$, implies that M is a set of uniformly bounded functions. Furthermore M is closed in $C^0(\overline{\Omega})$. Thus M is bounded in $C^{0,1}(\overline{\Omega})$ and by Arzelà-Ascoli's Theorem even compact in $C^0(\overline{\Omega})$. Since $V(h) = \int_\Omega (h - \psi)dx$ is a continuous functional on $C^0(\overline{\Omega})$ we get the existence of $h \in M$ maximizing the volume $V(\cdot)$. Given $h' \in M$ with $V(h') = V(h)$, but different from h, the continuity of h, h' on $\overline{\Omega}$ will lead to the function $\max(h, h') \in M$ with $V(\max(h, h')) > V(h)$ contradicting the maximality of V at h. □

There exists a remarkable analogy [?, ?, ?] between (14) and the Dirichlet problem for the Laplacian, i.e.,

$$\Delta u = 0 \text{ on } \Omega, \quad u = g \text{ on } \partial\Omega\,. \tag{16}$$

Under suitable assumptions on $\partial\Omega$ and on g Perron's method characterizes the unique solution u of (16) by subharmonic functions:

$$u(x) = \sup\{v \in C^2(\Omega) \cap C^0(\overline{\Omega}) : \Delta v \geq 0 \text{ in } \Omega,\ v \leq g \text{ on } \partial\Omega\}\,.$$

Calling a function $h \in C^{0,1}(\overline{\Omega})$ satisfying $(14)_{1,2}$ *subeikonal* we get the following result.

Proposition 3 *The solution h of problem* (14) *given by Theorem 2 can be characterized for every $x \in \overline{\Omega}$ by*

$$h(x) = \sup\{g(x) : g \in M\}\,,$$

i.e., $h(x)$ is the supremum and even the maximum of $g(x)$ among all subeikonal functions in (14).

Proof [?] To show that $\tilde{h}(x) := \sup\{g(x) : g \in M\}$ is Lipschitz continuous fix $y, y' \in \overline{\Omega}$. Then there are sequences $(h_j), (h'_j) \subset M$ such that $h_j(y) \to \tilde{h}(y)$, $h'_j(y') \to \tilde{h}(y')$. Replacing h_j and h'_j by $\max(h_j, h'_j) \in M$ we may assume that $h'_j = h_j$. Then the estimate

$$|h_j(y) - h_j(y')| \leq K(k, \psi)|y - y'| \quad \text{for all } j \in \mathbb{N},$$

see (15), yields the desired estimate for $\tilde{h}$ when $j \to \infty$. In particular $\tilde{h}$ is continuous.

To prove $(14)_2$ let $\tilde{h}(x) > \psi(x)$ for some $x \in \Omega$. Having the "maximum" Lipschitz constant $K(k, \psi)$ in mind we find an open ball B with center x in Ω such that $\tilde{h} > \psi$ on B and that even every $g \in M$ with $g(y) > \frac{1}{2}\big(\tilde{h}(y) + \psi(y)\big)$ for some $y \in B$ satisfies $g > \psi$ on B. Given arbitrary $y, y' \in B$ there exists a sequence $(h_j) = (h'_j) \subset M$ such that $h_j(y) \to \tilde{h}(y)$, $h_j(y') \to \tilde{h}(y')$. Since the "global" Lipschitz constant of h_j on B is easily seen to be bounded by k for every $j \in \mathbb{N}$, the same holds for $\tilde{h}$ proving that $|\tilde{h}|_{C^{0,1}_x} \leq k$.

Consequently $\tilde{h} \in M$, $\tilde{h}(x) \geq h(x)$ for all $x \in \overline{\Omega}$ and $V(\tilde{h}) \leq V(h)$. If $\tilde{h}(x) > h(x)$, then $\max(\tilde{h}, h) \in \tilde{M}$ would lead to a contradiction to the maximality of $V(h)$. Thus $\tilde{h} \equiv h$. □

The solution h of (14) may also be characterized by transport paths. If for simplicity $\psi \equiv 0$, then for $x \in \overline{\Omega}$

$$h(x) = \inf_\chi\{\phi\big(\chi(1)\big) + k\ell(\chi)\}$$

where χ runs through the set of all continuous piecewise linear paths in $\overline{\Omega}$ connecting x with any point $\chi(1) \in \partial\Omega$; here $\ell(\chi)$ denotes the length of χ [?]. In the most elementary case $\psi \equiv 0$ and $\phi \equiv 0$ (no wall), we easily get the solution

$$h(x) = k\,\mathrm{dist}(x, \partial\Omega)\,.$$

Note that h will have points or lines in Ω where it is not differentiable; for a discussion of these singular sets for concrete examples and for several general classes of domains, see [?].

Besides the maximum volume solution in Theorem 2 we consider the time-independent standing/rolling layer of thickness h and r, resp. when granular material is constantly poured onto a flat table $\psi \equiv 0$ with source intensity $s(x)$. For a point source located in y, i.e., formally $s(x) = \delta_y(x)$, we get a cone with vertex y and with slope k, i.e., $h(x)$ equals

$$\Gamma(x,y) = \begin{cases} k\big(\mathrm{dist}(y,\partial\Omega) - |x-y|\big) \quad , & |x-y| < \mathrm{dist}\,(y,\partial\Omega) \\ 0, & \text{otherwise}\,. \end{cases}$$

Then, for more general source distributions, we take the maximum (not the sum or integral) of $\Gamma(x,y)$ on $\mathrm{supp}\, s$, i.e.,

$$h(x) = \max_{y \in \mathrm{supp}\, s} \Gamma(x,y) = \max_{y\in\Omega} \Gamma(x,y) \cdot \chi_{\mathrm{supp}\, s}(y)\,. \tag{17}$$

This formula is similar to the solution $u(x) = \int_\Omega G(x,y) f(y) dy$ of Poisson's problem $-\Delta u = f$ on Ω, $u = 0$ on $\partial\Omega$ using Green's function $G(x,y)$. In the one-dimensional case $\Omega = (0,\ell)$, $\psi \equiv 0$, $\phi \equiv 0$, problem (12) has a unique stationary solution (h,r). Based on (17) $h(x)$ and also $r(x)$ can be written down explicity; in 2D this problem is not completely solved, see [?].

Finally we consider the silo problem (11), i.e.

$$\begin{aligned} h_t &= \gamma r\big(1 - \frac{|\nabla h|}{k}\big) && \text{in} \quad \Omega \times (0,\infty) \\ r_t - \mathrm{div}(\mu r \nabla h) &= s - \gamma r\big(1 - \frac{|\nabla h|}{k}\big) && \text{in} \quad \Omega \times (0,\infty) \\ \frac{\partial h}{\partial \nu} &= 0 && \text{on} \quad \partial\Omega \times (0,\infty)\,. \end{aligned} \tag{18}$$

This instationary hyperbolic system is not yet solved rigorously. In the one-dimensional case exact solutions have been described in [?] by parametrizing h, r and also x, t in a new coordinate system (μ_1, μ_2). If $s \equiv 0$ and if $\mathrm{div}(\mu r \nabla h)$ is replaced by $c r_x$, the authors find a $4{\times}4$-system of PDEs in which each equation contains only partial derivatives with respect to μ_1 or to μ_2. This system can be solved "explicitly" and yields solutions in the form $h(\mu_1,\mu_2)$, $r(\mu_1,\mu_2)$, $x(\mu_1,\mu_2)$ and $t(\mu_1,\mu_2)$. From these formulae several profiles (h,r) and shock lines can be analyzed.

The analysis gets much easier in the quasi-stationary case where $s \geq 0$ is independent of t with a mean source intensity

$$\overline{s} = \frac{1}{|\Omega|} \int_\Omega s(x) dx > 0\,. \tag{19}$$

In this case, for large t, we expect a *similarity solution*

$$h(x,t) = h_0(x) + \overline{s}t\,, \quad r(x,t) = r(x)\,.$$

Then (18) simplifies to the stationary system

$$\begin{aligned} \overline{s} &= \gamma r\big(1 - \frac{|\nabla h|}{k}\big) && \text{in } \Omega \\ -\mathrm{div}(\mu r \nabla h) &= s - \gamma r\big(1 - \frac{|\nabla h|}{k} && \text{in } \Omega \\ \frac{\partial h}{\partial \nu} &= 0 && \text{on } \partial\Omega\,. \end{aligned} \tag{20}$$

Since $\overline{s} > 0$, we conclude from $(20)_1$ that $|\nabla h| < k$ a.e. A simple calculation leads to the highly nonlinear Neumann problem

$$-\mathrm{div}\big(\frac{\nabla h}{1 - |\nabla h|/k}\big) = \frac{s(x) - \overline{s}}{\mu\overline{s}/\gamma} \text{ in } \Omega, \quad \frac{\partial h}{\partial \nu} = 0 \text{ on } \partial\Omega\,. \tag{21}$$

Proposition 4 *In space dimension one* (20) *has a unique similarity solution (up to additive constants in h). For $\Omega = (0,\ell)$ define*

$$U(x) = \int_0^\ell \big(\frac{x}{\ell} - \chi_{(0,x)}(y)\big) f(y)\,.$$

Then the rolling layer is given by

$$r(x) = \frac{1}{k\mu}\big(\frac{\mu}{\gamma k}\,\overline{s} + |U(x)|\big)\,,$$

and the slope h_x of the standing layer is given by

$$h_x(x) = k\,\frac{U(x)}{\frac{\mu}{\gamma k}\,\overline{s} + |U(x)|}\,.$$

The proof is given in [**?**]. Since $\overline{s} > 0$, actually $|h_x| < \alpha$ in $(0,\ell)$. In the two-dimensional case an explicit solution can be found for the disc $\Omega = B_R(0)$ and a point source $s(x) = \delta_0(x)$, see [**?**]. The general problem in 2D is not yet completely solved.

3 Existence Results for the Savage-Hutter Avalanche Model

3.1 Modelling

The Savage-Hutter equations model the flow of a dense snow avalanche with small aspect ratio on an inclined plane or on a rough bed by considering the avalanche as a cohesionless granular material in which the interstitial air plays a negligible role. In contrast with the BCRE model of Section 2 this model accounts for an exchange of momentum and goes far beyond simple particle models [**?**, **?**]; on the other hand it ignores abrasion and exchange of particles between the avalanche and the bed.

In a plane curvilinear coordinate system let x denote the coordinate along the rough incline and let z denote the perpendicular coordinate. Looking for the velocity u of the avalanche and the height h of the free surface the main assumptions of the Savage-Hutter model [**?**] are as follows:

- The granular material obeys a Mohr-Coulomb-type plastic yield criterion expressed by a constant angle of internal friction ϕ, i.e., given the stress tensor T and the exterior normal vector n on an internal surface the shear traction $S = n\cdot T - n(n\cdot T\cdot n)$ and the normal stress $N = n\cdot T\cdot n$ are related to each other by the formula $|S| = N\tan\phi$. Since shear traction depends on the direction of the velocity vector u,

  $$S = -\frac{u}{|u|} N \tan\phi\,,$$

  giving rise to a jump discontinuity.

- At the base there exists a very thin fluidized layer (about 10 grain diameters) obeying a Coulomb dry friction law with a bed friction angle $\delta < \phi$, i.e., $S = -\frac{u}{|u|} N\tan\delta$.

- The longitudinal stress component T_{xx} is related to the perpendicular component T_{zz} by

  $$T_{xx} = K_{\mathrm{act/pass}}\, T_{zz}$$

  where

  $$\left.\begin{array}{l} K_{\mathrm{act}} \\ K_{\mathrm{pass}} \end{array}\right\} = \frac{2(1 \mp \sqrt{(1-\cos^2\phi/\cos^2\delta)})}{\cos^2\phi} - 1 \qquad \begin{array}{l}\text{iff } \partial u/\partial x > 0 \\ \text{iff } \partial u/\partial x < 0\end{array} \tag{22}$$

  is the active and passive earth pressure coefficient, resp. Note that $0 < K_{\mathrm{act}} < K_{\mathrm{pass}}$, where K_{act} applies iff the flow is locally expanding.

- As a major assumption the velocity profile is blunt (except for the fluidized layer): for every $x \in \mathbb{R}$, $t > 0$

  $$\int_0^{h(x,t)} u(x,z,t)dz = h(x,t)u(x,t),\ \int_0^h u^2 dz = hu^2 \text{ etc.}$$

  Thus all macroscopic quantities are considered to be y–independent.

- Given a characteristic height H and length L of the avalanche assume that the aspect ratio $\varepsilon = H/L$ is small compared to 1, i.e., $\varepsilon \ll 1$. If the bed is curved with a characteristic radius of curvature R, assume that $\frac{L}{R} = O(\varepsilon^{1/2})$. Finally assume that $\tan\delta = O(\varepsilon^{1/2})$.

Typical values of δ, ϕ and K for glass, quartz, marmor or plastic grains are as follows, see [**?**]:

$$20^o < \delta < 40^o,\ 30^o < \phi < 46^o,\ 5^o < \phi - \delta < 20^o$$

where the bed friction angle also depends on the roughness of the bed. Thus typical earth pressure coefficients are

$$K_{\mathrm{act}} \in (0.7, 0.9),\ \ K_{\mathrm{pass}} \in (2.8, 4.6)\,.$$

Ignoring all terms of order higher than ε the Savage-Hutter equations for a thin *two-dimensional* avalanche of height h, velocity $u = (u_1, u_2)$ and momentum hu on a two-dimensional basal profile $z = b(x, y)$ with main down slope direction $\binom{1}{0}$ take the form [**?**, **?**, **?**]

$$\begin{aligned} \partial_t h + \operatorname{div}(hu) &= 0 \\ \partial_t(hu) + \operatorname{div}\big(hu \otimes u + \frac{1}{2}\varepsilon h^2 K(\cos\xi)\big) &= hs(u, x) \end{aligned} \tag{23}$$

with the source term

$$s = \sin\xi \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \frac{u}{|u|} h \tan\delta \cos\xi - \varepsilon h K(\cos\xi)\nabla b\,. \tag{24}$$

Here $\xi = \xi(x)$ is the local inclination angle along the direction $\binom{1}{0}$ whereas the influence of the curvature has been omitted. Furthermore K denotes the diagonal $2{\times}2$-matrix of earth pressure coefficients such that

$$\operatorname{div}\big(\frac{1}{2}\varepsilon h^2(\cos\xi)K\big) = \varepsilon h \cos\xi \big(K_{x,\mathrm{act/pass}} \frac{\partial h}{\partial x}, K^{y,\mathrm{act/pass}}_{x,\mathrm{act/pass}} \frac{\partial h}{\partial y}\big)^T + \dots$$

with $K_{x,\mathrm{act/pass}}$ as in (22) and $K^{y,\mathrm{act/pass}}_{x,\mathrm{act/pass}}$ depending on the signs of $\frac{\partial u_1}{\partial x}$ and of $\frac{\partial u_2}{\partial y}$. This term together with the term $\varepsilon h K(\cos\xi)\nabla b$ represents the variation of the normal pressure in x- and y-directions, whereas the first and second term of (24) are due to gravity normalized to 1 and to friction of the avalanche with the bed, respectively. To be more precise in the two-dimensional case, $\varepsilon = H/L$ has to be replaced by a diagonal 2×2 matrix with entries $\varepsilon_x = H/L_x$ and $\varepsilon_y/\varepsilon_{xy} = (H/L_y)/(L_y/L_x)$ for characteristic lengths L_x and L_y.

System (23) is written in the form of a system of conservation laws for (h, hu) with a source on the right-hand side depending on h and u. Looking at the leading terms and ignoring the term containing K, (23) is similar to the shallow water equations and to the Euler equations of gas dynamics. However, besides the fact that there exists no satisfying mathematical theory for systems of conservation laws in more than one space dimension, the jump discontinuity $\frac{u}{|u|}$ and of course the piecewise constant function K depending on signs of ∇u pose new analytical and numerical difficulties. Thus, in every analytical approach – even when looking for similarity solutions, see §3.2 – K is assumed to be constant.

Of course solutions of (23) may evolve shocks even when the data are smooth. Shocks will mainly occur in the run-out zone when a part of the material has already been deposited. Furthermore shocks can be observed in beautiful experiments on granular matter in rotating drums, see [**?**].

Proposition 5 *Let* $(h, hu) \in \mathbb{R}^3$ *be a weak solution of* (23) *in a domain* $\Omega \subset \mathbb{R}^2 \times (0, \infty)$, *i.e., for all* $\varphi \in C_0^\infty(\Omega)^3$

$$\iint_\Omega \Big\{ \begin{pmatrix} h \\ hu \end{pmatrix} \cdot \varphi_t + \begin{pmatrix} hu \\ hu \otimes u + \frac{1}{2}\varepsilon h^2(\cos\xi)K \end{pmatrix} \cdot \nabla\varphi \Big\}\, dxdt$$
$$= -\iint_\Omega \begin{pmatrix} 0 \\ hs \end{pmatrix} \cdot \varphi\, dxdt\,.$$

Assume that Ω is separated by a smooth, regular surface Γ into two parts Ω_ℓ and Ω_r such that

$$\begin{pmatrix} h \\ hu \end{pmatrix}\Big|_{\Omega_r} \in C^1(\overline{\Omega_\ell})^3, \quad \begin{pmatrix} h \\ hu \end{pmatrix}\Big|_{\Omega_r} \in C^1(\overline{\Omega_r})^3 .$$

Let $\nu = (\nu_t, \nu_x)$ denote the unit normal vector on Γ directed into Ω_ℓ. Then (h, hu) satisfies the Rankine-Hugoniot jump condition

$$\Big[\begin{pmatrix} h \\ hu \end{pmatrix}\Big]\nu_t + \nu_x \cdot \Big[\begin{pmatrix} hu \\ hu \otimes u + \frac{1}{2}\varepsilon h^2 (\cos\xi) K \end{pmatrix}\Big] = 0 ,$$

where as usual $[\cdot]$ denotes the difference of the limits of (h, hu) on Γ taken from Ω_ℓ and from Ω_r.

Coming back to a one-dimensional avalanche on a basal profile $z = b(x)$, $x \in \mathbb{R}$, let a line of discontinuity Γ be given in parameterized form $\big(\gamma(t), t\big)$. Then $\gamma'(t)$ is the speed of propagation of the discontinuity, and the Rankine-Hugoniot condition takes the simple form

$$\Big[\begin{pmatrix} h \\ hu \end{pmatrix}\Big]\, \gamma'(t) = \Big[\begin{pmatrix} hu \\ hu^2 + \frac{1}{2}\varepsilon h^2 (\cos\xi) K_{\mathrm{act/pass}} \end{pmatrix}\Big] . \tag{25}$$

A more recent generalization of the Savage-Hutter model considers compressible avalanches of density ρ satisfying a constitutive equation $\rho = \rho(h, u)$, see [**?**, **?**]. Since there is no physical evidence for a (monotonically decreasing) dependence on $|u|$, up to now the constitutive equation

$$\rho(h) = h^\alpha, \quad \alpha > 0 ,$$

has been investigated; see [**?**] for the mathematically easier case $\alpha = -\frac{1}{2}$. In the one-dimensional case we get the system

$$\begin{aligned} \partial_t(\rho h) + \partial_x(\rho h u) &= 0 \\ \partial_t(\rho h u) + \partial_x(\rho h u^2 + \frac{1}{2}\beta(x)\rho h^2) &= \rho h s(u, x) , \end{aligned} \tag{26}$$

where

$$\begin{aligned} \beta(x) &= \varepsilon K_{\mathrm{act/pass}} \cos\xi(x) , \\ s &= \sin\xi - \varepsilon \cos\xi \, b_x - \frac{u}{|u|} \tan\delta \cos\xi . \end{aligned}$$

Assuming an overall constant $K = K_{\mathrm{act/pass}}$ it is convenient to introduce new functions to get rid of the x-dependence in the term $\frac{1}{2}\beta(x)\rho h^2$ and to refind the standard form of conservation laws. Let

$$(u_1, u_2) = \left((\frac{\beta}{2\kappa}\Big)^{1+\alpha} h^{1+\alpha}, \Big(\frac{\beta}{2\kappa}\Big)^{1+\alpha} h^{1+\alpha}\, u \right) , \tag{27}$$

where $\kappa = \big(4(1+\alpha)(2+\alpha)\big)^{-1}$, and

$$F(u_1, u_2) = \begin{pmatrix} u_2 \\ \frac{u_2^2}{u_1} + \kappa u_1^{\frac{2+\alpha}{1+\alpha}} \end{pmatrix}, \; S_0 = \frac{(1+\alpha)\beta'}{\beta} F + \begin{pmatrix} 0 \\ u_1 s(\frac{u_2}{u_1}, x) \end{pmatrix} . \tag{28}$$

Then (26) takes on the form

$$\partial_t u + \partial_x F(u) = S_0(u,x), \quad u = (u_1, u_2)\,. \tag{29}$$

Note that in (28) also the first component of the source term S_0 is different from zero. But the Rankine-Hugoniot condition for a shock line Γ with speed of propagation $\gamma'(t)$ has the simple form

$$[u_1]\gamma' = [u_2],\ [u_2]\gamma' = [F_2(u_1,u_2)]$$

yielding the compatibility condition

$$[u_2]^2 = [F_2(u_1,u_2)]\cdot[u_1] \quad \text{on } \Gamma\,.$$

Thus a discontinuity of u_1 or of h w.r.t. x, say $h(x-,t) > 0 = h(x+,t)$, is not admissable.

Up to now problems arising from the jump discontinuity in the source terms s and S_0 have been ignored. In Section 3.3 we propose to introduce set-valued maps to deal with this discontinuity, see Definition 7 and Remark 8 below.

3.2 Self-Similar Solutions

Consider the Savage-Hutter model for a one-dimensional incompressible avalanche on a plane moving downwards everywhere, i.e. the system

$$\begin{aligned} \partial_t h + \partial_x(hu) &= 0 \\ \partial_t u + u\partial_x u &= \sin\xi - \tan\delta\cos\xi - \beta\, h_x\,, \end{aligned} \tag{30}$$

when $\operatorname{sgn} u = +1$ is constant. Also $K_{\text{act/pass}}$ is assumed to be constant yielding a constant $\beta = \varepsilon K \cos\xi$. In order to discuss the existence of self-similar solutions we subtract the motion of the center of mass. To this end, define

$$u_0(t) = t(\sin\xi - \tan\delta\cos\xi), \quad \tilde{u} = u - u_0(t)$$

and the moving variable

$$\xi = x - \int_0^t u_0(s)ds\,.$$

Let $g(t)$ denote a typical length of the avalanche at time t, e.g. half the spread of an avalanche with compact support. Now use new coordinates

$$y = \frac{\xi}{g(t)}, \quad \tau = t$$

in (30) and the notation $(\cdot)'$ and $(\cdot)_y$ for derivatives w.r.t. τ and y, resp., to find the system

$$\begin{aligned} \partial_\tau h - y\,\frac{g'}{g}\,\partial_y h + \frac{1}{g}\,\partial_y(h\tilde{u}) &= 0 \\ \partial_\tau \tilde{u} - y\frac{g'}{g}\,\partial_y\tilde{u} + \frac{1}{g}(\tilde{u}\partial_y\tilde{u} + \beta\partial_y h) &= 0 \end{aligned} \tag{31}$$

for $(h,\tilde{u})$. Then a solution of the form

$$h(y,\tau)=\ell(\tau)H(y), \quad \tilde{u}(y,\tau)=k(\tau)U(y) \tag{32}$$

is called a *self similar solution* of (31). Due to the conservation of mass

$$M\equiv\int_{\mathbb{R}}h(\xi,t)d\xi=\int_{\mathbb{R}}h(y,\tau)g(\tau)dy=\ell(\tau)g(\tau)\int_{\mathbb{R}}H(y)dy$$

we get that

$$\ell=\frac{1}{g},$$

at least when the total mass is finite. It will be seen below that this assumption is not satisfied in general. Inserting $\ell=1/g$ in (31) yields the system

$$\begin{aligned} H+yH_y-\frac{k}{g'}(HU)_y &= 0 \\ U-\frac{g'k}{gk'}yU_y+\frac{k^2}{gk'}UU_y+\frac{\beta}{g^2k'}H_y &= 0\,. \end{aligned} \tag{33}$$

¿From $(33)_1$ we see that

$$0=(\frac{k}{g'})'(HU)_y\,.$$

Thus either

$$\frac{k}{g'}\equiv \text{const} \quad \text{or} \quad HU\equiv\text{const}\,.$$

Since $k(\tau)$ denotes an overall increase or decrease of the velocity $\tilde{u}$, the change of the characteristic length $g'(\tau)$ has to be proportional to $k(\tau)$. Hence k/g' has to be independent of τ. Actually, if $k/g'\not\equiv$ const, $(33)_1$ would imply that $(HU)_y\equiv 0$ and that $H+yH_y\equiv 0$. These equations yield the general solution $H(y)=\frac{c_0}{y}$, $U=c_1y$. Then $(33)_2$ can be interpreted as a vanishing linear combination of the functions y and $\frac{1}{y^2}$ with τ-depending coefficients. Now we may conclude that $c_0=0$ and consequently $H\equiv 0$ yielding the trivial solution $h\equiv 0$.

In the following assume w.l.o.g. that

$$k\equiv g'\,,$$

since a constant k/g' different from 1 can be subsumed by the functions H or U, see (32). Then (33) can be written in the simple form

$$\begin{aligned} \big((U-y)H\big)_y &= 0 \\ U+\frac{g'^2}{gg''}(U-y)U_y+\frac{\beta}{g^2g''}H_y &= 0\,. \end{aligned} \tag{34}$$

Case 1: $U\equiv y$ In this case the velocity $\tilde{u}(y,\tau)$ is linear in y for every time τ. From $(34)_2$ we get the equation $g^2g''=-\beta H_y/y$. Consequently both sides are constants leading to the identities

$$g^2g''=\frac{G_0}{2} \quad \text{and} \quad H(y)=H_0-\frac{G_0}{4\beta}y^2 \tag{35}$$

with constants H_0, G_0. Let us ignore the elementary case $G_0 = 0$ where $H(y)$ is constant and $g(\tau)$ is linear. If $G_0 \neq 0$, by $(35)_1$ $g'g'' = G_0 g'/(2g^2)$ and consequently

$$g'^2 = -G_0(\frac{1}{g} + \frac{1}{g_0}) \quad \text{with } g_0 \in \overline{\mathbb{R}}\backslash\{0\}\,.$$

If $g_0 = \infty$, then $g \geq 0$ implies that $G_0 < 0$ and $g'g^{1/2} = \pm\sqrt{|G_0|}$. Hence

$$g(\tau) = \big(g(0)^{3/2} + \frac{3}{2}\sqrt{|G_0|}\,\tau\big)^{2/3}, \; H(y) = H_0 + \frac{|G_0|}{4\beta}\,y^2\,; \tag{36}$$

the case $g'g^{1/2} = -\sqrt{|G_0|}$ leads to an unphysical compression of the avalanche or – in other words – to a time reversal in (36). The solution (36), considered only for $|y| < y_0$, defines an avalanche with the shape of an M, called an M–*wave* in [?]. As $\tau \to \infty$,

$$h(y,\tau) \sim c\tau^{-2/3} H(y)\,.$$

If $g_0 \in \mathbb{R}^* = \mathbb{R}\backslash\{0\}$, then $g'(\tau) = \pm\sqrt{|G_0|}\,\frac{\sqrt{|g_0+g|}}{\sqrt{|g_0 g|}}$. First we consider the case when $g_0 > 0$ and $g' > 0$. Then

$$\begin{aligned}\sqrt{\frac{|G_0|}{g_0}}\tau &= \int_{g(0)}^{g(\tau)} \frac{\sqrt{g}\,dg}{\sqrt{g_0+g}} = 2\int_{\sqrt{g(0)}}^{\sqrt{g(\tau)}} \frac{h^2\,dh}{\sqrt{g_0+h^2}} \\ &= \sqrt{g(\tau)}\,\sqrt{g_0+g(\tau)} - g_0\ln\big(\sqrt{g(\tau)} + \sqrt{g_0+g(\tau)}\big) - C_0\,.\end{aligned}$$

For $\tau \to \infty$ we deduce the linear behavior

$$g(\tau) \sim \sqrt{\frac{|G_0|}{g_0}}\,\tau\,, \quad \tau \to \infty\,.$$

Since $g > 0$ and $g_0 > 0$ necessarily imply that $G_0 < 0$, again $H(y) = m + \frac{|G_0|}{4\beta}\,y^2$ defines an M–wave on $|y| \leq y_0$. But compared to the M-wave above we now get an M-wave with

$$h(y,\tau) \sim \tau^{-1} H(y) \quad \text{as } \tau \to \infty\,.$$

When $g_0 > 0$ but $g' < 0$, then the differential equation for $g(\tau)$ immediately implies that $g(\tau) \to 0$ and consequently that $\ell(\tau) \to \infty$ in finite time. Thus this case is unphysical.

Next consider the case when $g_0 < 0$, but $g + g_0 \geq 0$ and $g' \geq 0$. Then $G_0 > 0$ and

$$\begin{aligned}\sqrt{\frac{G_0}{|g_0|}}\tau &= 2\int_{\sqrt{g(0)}}^{\sqrt{g(\tau)}} \frac{h^2\,dh}{\sqrt{h^2-|g_0|}} \\ &= \sqrt{g(\tau)}\,\sqrt{g(\tau)-|g_0|} + |g_0|\,\ln\big(\sqrt{g(\tau)} + \sqrt{g(\tau)-|g_0|}\big) - C_0\,.\end{aligned}$$

For $\tau \to \infty$ we deduce the asymptotic behavior $g(\tau) \sim \tau$. The shape of the avalanche is described by $H(y) = m - \frac{G_0}{4\beta}\,y^2$ for $|y| \leq \sqrt{4m\beta/G_0}$ forming a *parabolic cap*, see [?] for the special case $\tau_0 = 0$, $g_0 = -1$, $g(0) = 1$ such that $g'(0) = 0$ (avalanche is starting at rest).

In the case $g_0 < 0$ and $g + g_0 < 0$, but $g' > 0$, the function $g(\tau)$ is strictly increasing until $g(\tau) \to |g_0|$ where $g'(\tau) \to 0$. For $g(\tau)$ close to $|g_0|$, but less than $|g_0|$, the differential equation for $g(\tau)$ is related to the equation $g'(\tau) = 2\sqrt{|g_0| - g(\tau)}$ showing that $g(\tau)$

actually approaches $|g_0|$ like the parabola $g(\tau) = |g_0| - (\tau_1 - \tau)^2$ as $\tau \to \tau_1-$. Then $k(\tau_1) = g'(\tau_1) = 0$ and $\tilde{u}(\tau_1, y) = 0$ yielding an avalanche at rest! From standard theory it is known that the solution $g(\tau)$ is not uniquely determined for $\tau > \tau_1$. When the avalanche restarts to move and $g(\tau)$ becomes larger than $|g_0|$ for some $\tau > \tau_1$, then we refer to the previous case.

The case $g_0 < 0$, but $g + g_0 > 0$ and $g' < 0$ also leads to an avalanche at rest in finite time. Finally, if $g_0 < 0$, $g + g_0 < 0$ and $g' < 0$, then $g(\tau)$ converges to 0 in finite time. Thus $\ell(\tau) \to \infty$ in finite time leading to an infinite velocity and an unphysical solution.

Case 2: $U \not\equiv y$. In this case $(34)_1$ yields a constant $m \neq 0$, a characteristic momentum, such that

$$H = \frac{m}{U - y}\,; \tag{37}$$

the case $m = 0$ is trivial. Inserting this identity into $(34)_2$ we get that

$$U + \frac{g'^2}{gg''}(U - y)U_y - \frac{\beta m}{g^2 g''} \cdot \frac{U_y - 1}{(U - y)^2} = 0\,. \tag{38}$$

Dividing by U, multiplying with $g^2 g''$ and differentiating w.r.t y we are led to the equation

$$g'^2 g\Big(\frac{(U - y)U_y}{U}\Big)_y - \beta m\Big(\frac{U_y - 1}{(U - y)^2 U}\Big)_y = 0\,.$$

In order to conclude that $g'^2 g$ is constant we have to exclude the possibility that both terms depending on y vanish. If these terms vanish, we would get *two* ordinary differential equations for $U(y)$ leading after elementary calculations to a contradiction. Thus we get a constant $c \neq 0$ (since $g > 0$) such that

$$g'\sqrt{g} = c \quad \text{and} \quad g(\tau) \sim \Big(\frac{3}{2}c\tau\Big)^{2/3} \quad \text{as } \tau \to \infty\,,$$

cf. Case 1 with $g_0 = \infty$. Hence $g'' g^2 = -c^2/2$, and (38) yields the differential equation

$$U - 2(U - y)U_y + 2a^3 \frac{U_y - 1}{(U - y)^2} = 0\,, \quad a^3 = \frac{\beta m}{c^2} \neq 0\,, \tag{39}$$

and, defining

$$V(y) = U(y) - y\,, \tag{40}$$

the equation

$$\frac{dV}{dy} = \frac{1}{2}\,\frac{V^2(V - y)}{a^3 - V^3}\,. \tag{41}$$

Note that the lines $V \equiv 0$ and $V \equiv a$ as well as points y_0 where $V(y_0) = y_0$ are important in the discussion of local and global properties of solutions of the differential equation (41). Concerning the size of $V_0 = V(0)$ and of a we have to distinguish between several cases.

Since $H(0) = \frac{m}{U(0)} = \frac{m}{V(0)}$ and $\operatorname{sgn} a = \operatorname{sgn} m$, the cases $V_0 < 0 < a$ and $a < 0 < V_0$ lead to unphysical solutions with $H(0) < 0$ and will not be discussed.

Case 2.1: $0 < a < V_0$ *and* $\exists\, y_0 > a : V(y_0) = y_0$ (this case will occur for $V_0 \gg a$).

In this case $V'(0) < 0$ and even $V'(y) < 0$ for all $y \in (-\infty, y_0)$. However $V'(y_0) = 0$ and $V''(y_0) = \frac{y_0^2}{2(y_0^3 - a^3)} > 0$ implying that V has a local minimum at y_0. Since even $V' > 0$ on (y_0, ∞) and $V' < 0$ on $(-\infty, y_0)$, V has a global minimum at y_0. To determine the shape of the corresponding avalanche it is crucial to discuss the asymptotic behavior of V for $y \to \pm\infty$. By (41) we may exclude that $V(y)$ is bounded for $y \to \infty$. If there exists $y_1 > y_0$ such that $V(y_1) = \frac{1}{2} y_1$, the inequality $V'(y_1) = \frac{1}{2} y_1^3/(y_1^3 - 8a_3) > \frac{1}{2}$ leads to a contradiction. Consequently $\frac{y}{2} < V(y) < y$ for $y > 0$. Defining $w(y) = V(y)/y$ we see that $w(y) \in (\frac{1}{2}, 1)$ and that

$$w' = -\frac{w^2}{2y} \, \frac{(2w-1)(w+1) - 2a^3/(wy^3)}{w^3 - a^3/y^3} \,. \tag{42}$$

If there exists $0 < \delta < \frac{1}{2}$ such that $w(y) \geq \frac{1}{2} + \delta$ for large $y > 0$, then $w'(y) \approx -\frac{1}{y}$ for these y leading to a logarithmic decay $-\log y$. Then finally $w(y)$ will cross the line $w = \frac{1}{2} + \delta$ with a negative slope. Thus $w(y) < \frac{1}{2} + \delta$ for all large y, even $w(y) \to \frac{1}{2}$ and $V(y) \sim \frac{y}{2}$ for $y \to \infty$. For $y \to -\infty$ (41) implies that $V(y)$ is unbounded. If there exists $y_2 < -a$ such that $V(y_2) = -y_2$, then $V'(y_2) = -y_2^3/(a^3 + y_2^3) < -1$ leads to a contradiction. Thus $w(y) = \frac{V(y)}{y} < -1$ for all negative y. The possibility that $w(y) \leq w_1 < -1$ for all large $y < 0$ can be excluded since under this assumption $w' \leq \frac{\delta}{y}$ for some $\delta > 0$. Hence $w(y) \to -1$, $V(y) = -y\big(1 + o(1)\big)$ and $U(y) = V(y) + y = o(|y|)$ for $y \to -\infty$. To prove that even $U(y) = O(|y|^{-1/2})$ for $y \to -\infty$ we introduce the auxiliary function $\varphi(y) = |y|^{1/2} U(y)$. By (40), (41)

$$\varphi'(y) = \frac{|y|^{1/2}}{2} \, \frac{w(y)^2 (w(y)+1)^2 + O(|y|^{-3})}{w(y)^3 \big(1 + O(|y|^{-3})\big)}$$

for $y \to -\infty$. Since w is bounded for large negative y, and $1 + w = -\frac{U}{|y|} = -\frac{\varphi(y)}{|y|^{3/2}}$, we get that

$$\varphi' = \frac{c(y)}{|y|^{5/2}} - \frac{1}{2|y|^{5/2}|w(y)|} \, \varphi^2 \,, \tag{43}$$

where $|c(y)|$ is bounded. Assuming that φ is not bounded for $y \to -\infty$ there exists $y_1 < 0$ such that $\varphi'(y_1)$ is negative. We may even assume that φ is strictly decreasing for $y < y_1$. Thus there are constants $c_1, c_2 > 0$ such that

$$-\frac{c_1}{|y|^{5/2}} \, \varphi^2 \leq \varphi' \leq -\frac{c_2}{|y|^{5/2}} \, \varphi^2 \,.$$

However, this differential inequality can be satisfied only for bounded functions. Now the boundedness φ implies that $U(y) = O(|y|^{-1/2})$ for $y \to -\infty$. Summarizing the previous results we get for the avalanche characterized by $U = V + y$ and $H = \frac{m}{U - y}$ where $m > 0$, that

$$\begin{aligned} 0 < U(y) &\sim \frac{c}{|y|^{1/2}} \quad \text{for } y \to -\infty, \quad & U(y) &\sim \frac{3}{2} y \quad \text{for } y \to \infty \\ H(y) &\sim \frac{m}{|y|} \quad \text{for } y \to -\infty, \quad & H(y) &\sim \frac{2m}{y} \quad \text{for } y \to \infty \,. \end{aligned}$$

The rate of decay of H for $y \to \pm\infty$ shows that the mass of the avalanche is infinite.

Case 2.2: $0 < a < V_0$ *and* $V(y)$ will not cross or touch the main diagonal (this case will occur for $V_0 > a$ close to a).

For $y > 0$ the solution $V(y)$ is strictly decreasing. Since it will not cross the main diagonal and cannot cross the line $y \equiv a$, it will approach a in finite time $y_1 < a$ with a slope approaching $-\infty$. The behavior of $V(y)$ for $y \to -\infty$ is the same as in the previous case. Thus the avalanche has the properties

$$\begin{array}{llll} 0 < U(y) < \frac{c}{|y|^{1/2}} & \text{for } y \to -\infty, & U(y) \downarrow y_1 + a & \text{for } y \to y_1- \\ H(y) \sim \dfrac{m}{|y|} & \text{for } y \to -\infty, & H(y) \uparrow \dfrac{m}{a} & \text{for } y \to y_1- \end{array}$$

where U and H have infinite slope at $y = y_1$ in which the solution breaks down.

Case 2.3: $0 < V_0 < a$ *and* $\exists\, y_0 > 0 : V(y_0) = y_0$ (this case will occur for small V_0).

Then $V'(0) > 0$, $y_0 < a$, $V'(y_0) = 0$ and $V''(y_0) = -y_0^2/\big(2(a^3 - y_0^3)\big) < 0$. Thus V has a local maximum at y_0 and even $V'(y) < 0$ for all $y > y_0$. For $y \to \infty$ the behavior of $V(y)$ is modelled by the differential equation $V' = -cyV^2$ with a constant $c > 0$ leading to the asymptotic behavior $V(y) = O(y^{-2})$. Analogously $V(y)$ will tend to 0 as $y \to -\infty$ and $V(y) = O(y^{-2})$. Thus the velocity $U(y)$ has the properties

$$U(y) = y + O(y^{-2}) \quad \text{for } y \to \pm\infty\,.$$

However the height $H(y) = m/V(y)$ where $m > 0$ diverges as y^2 for $y \to \pm\infty$. Hence the avalanche is unphysical in this case; the "explicit" solutions U, H may be used only locally.

Case 2.4: $0 < V_0 < a$ *and* $V(y)$ will not cross or touch the main diagonal (this case will occur for $V_0 < a$ close to a).

Then V is strictly increasing for $y > 0$ until it will reach the level $y = a$ in finite time $y_1 < a$ with slope $+\infty$. For $y \to -\infty$ the behavior is the same as in Case 2.3. Consequently

$$\begin{array}{lll} U(y) = y + O(y^{-2}) & \text{for } y \to -\infty, & U(y) \uparrow y_1 + a \text{ as } y \to y_1- \\ H(y) = O(y^2) & \text{for } y \to -\infty, & H(y) \downarrow \frac{m}{a} > 0 \text{ as } y \to y_1- \ . \end{array}$$

This solution is unphysical since it breaks down in finite time and since H is unbounded.

There are four further cases when $a < 0$ and $V_0 < 0$. However, since $W(y) = -V(-y)$ satisfies (41) with a replaced by $-a$, it suffices to refer to Case 2.1–2.4. The corresponding avalanche is described by $U(y) \widehat{=} -U(-y)$ and $H(y) \widehat{=} H(-y)$.

Case 2.5: $V_0 < a < 0$ *and* $V(y)$ crosses the main diagonal. Looking at Case 2.1 we get an avalanche with the properties ($m < 0$)

$$\begin{array}{llll} U(y) \sim \dfrac{3}{2}y & \text{for } y \to -\infty, & -\dfrac{c}{|y|^{1/2}} < U(y) < 0 & \text{for } y \to +\infty \\ H(y) \sim \dfrac{2m}{y} & \text{for } y \to -\infty, & H(y) \sim \dfrac{|m|}{y} & \text{for } y \to \infty\,. \end{array}$$

Due to the large negative velocities for $y \to -\infty$ the avalanche seems to move upwards everywhere. However, the real self-similar avalanche has the height

$$h(x,t) = \frac{1}{g(t)} H\Big(\frac{x - \alpha t^2/2}{g(t)}\Big)$$

where the acceleration $\alpha = \sin\xi - \tan\delta\cos\xi$ is assumed to be a positive constant ($\xi > \delta$) and $g(t) \sim \big(\frac{3}{2}ct\big)^{2/3}$ for $t \to \infty$. Moreover, its velocity including the motion of the center of mass is given by

$$u(x,t) = \alpha t + g'(t) U\big(\frac{x - \alpha t^2/2}{g(t)}\big)\,.$$

Since $g'(t) = \frac{c}{\sqrt{g(t)}}$ and $g(t)^{3/2} \sim \frac{3}{2}ct$ for $t \to \infty$, the asymptotic behavior of $u(x,t)$ for $t \to +\infty$, i.e. $y \to -\infty$, is given by

$$u(x,t) \sim \alpha t + \frac{3}{2}\,\frac{c}{\sqrt{g(t)}}\,\frac{x - \alpha t^2/2}{g(t)} \sim \frac{\alpha}{2}t + \frac{x}{t}\,. \tag{44}$$

Hence the physical velocity is positive for large t and approaches half the velocity of the corresponding center of mass. This behavior is reflected by the height of the avalanche:

$$h(x,t) \sim \frac{m}{|x - \alpha t^2/2|}\begin{cases} 2 & \text{for } x \to -\infty \\ 1 & \text{for } x \to +\infty\,. \end{cases}$$

Case 2.6: *$a < V_0 < 0$ and $V(y)$ crosses the main diagonal.* Referring to Case 2.3 we find a solution such that $V(y) = O(y^{-2})$ for $y \to \pm\infty$ and consequently that $U(y) \sim y$, $H(y) = O(y^2)$ for $y \to \pm\infty$.

We drop the two cases analogous to Case 2.3 and 2.4 when a and V_0 are negative since these solutions fail to exist for large negative y. Finally we mention that even when a is positive there are further solutions existing on an y-semiaxis. E.g., if $a > 0$, consider $y_0 > a$ and $V_0 := V(y_0) > a$ or $< a$, but close to a. Then there exists a solution $V(y)$ for $y > y_0$ evolving an infinite slope for $y < y_0$ where V converges to a; for $y \to +\infty$ $V(y)$ will diverge as $\frac{y}{2}$ or converge to 0 as y^{-2}, see Case 2.1 or 2.3 for this asymptotic behavior, resp.

Note that all self-similar solutions have been found under the assumption that $\operatorname{sgn} u(x,t) = 1$, i.e., $|\tilde{u}(x,t)| < u_0(t)$. Now the speed of the center of mass of the self-similar avalanche is larger than $\tilde{u}$ iff

$$|g'(t)U(y)| < \alpha t \tag{45}$$

for all admissable y. In several cases, see e.g. the M-wave on a compact interval $|y| \le y_0$, (45) is satisfied for large t since $g'(t)$ is bounded, cf. [**?**]. For the parabolic cap which has a compact support w.r.t. y and where $g'(t) \approx t$ a largeness condition for α, i.e. for the constant slope ξ, has to be assumed [**?**]. In the important Case 2.1 and Case 2.5 condition (45) may be violated locally, but not globally, see (44).

Recall the overall assumption that the earth pressure coefficient $K_{\text{act/pass}}$ was constant ignoring the fact that the avalanche may be compressed or stretched locally. In [**?**, **?**] the authors carefully analyse a parabolic cap solution when even the bed friction angle δ varies either with y or with the center of mass velocity or with both of them. Although in these cases the height h defines a strict parabolic cap and the velocity u is linear, the equation for the spread g is much more complicated since $g''g^2$ is no longer constant.

Finally we mention that also for a two-dimensional avalanche on an inclined plane there exist self-similar parabolic cap solutions starting with a circular support evolving like an ellipse for increasing t, see [**?**, **?**]. Let $g_1(t), g_2(t)$ denote the length of the semiaxes of the supporting ellipse in the longitudinal and traverse direction resp., such that the height of the avalanche equals $h(y_1, y_2, t) = (1 - y_1^2 - y_2^2)/(g_1(t)g_2(t)$. Then g_1, g_2 satisfy a second order highly nonlinear system of ordinary differential equations with coefficients depending on the aspect ratios $\varepsilon_1 = [H/L_{x_1}]$ and $\varepsilon_2 = [h/L_{x_2}]$. Numerical results [**?**, **?**] show a crucial dependence of the spreads $g_1(t), g_2(t)$ on $\varepsilon_1, \varepsilon_2, \frac{\varepsilon_1}{\varepsilon_2}$ and on the angles ξ and δ.

Note added in proof. When preparing this manuscript we learned that also V. Chugunov, J.M.N.T. Gray and K. Hutter [**?**] found almost the same set of self-similar solutions. However they use abstract Lie group theory to find invariance proerpties of (30), then they discuss several cases in more details.

3.3 Existence Results

In this section we present the mathematical analysis of the 2×2-conservation law (23) when the density $\rho \equiv \rho_0 > 0$ is constant and of (26) for a density function $\rho(h) = h^\alpha$. For a bed with varying slope $\xi(x)$ we consider the modified height or mass distribution $u_1 = \left(\frac{\beta}{2\kappa}\right)^{1+\alpha}\rho h$ and momentum $u_2 = \left(\frac{\beta}{2\kappa}\right)^{1+\alpha}\rho h u$, cf. (27) – (29). As indicated in Section 3.1 system (23), (26) or (29) will evolve shocks and will allow multiple, even unphysical solutions. Therefore we are looking for suitable (physical) solutions satisfying a sufficiently large set of entropy conditions.

Definition 6 Let $\eta = \eta(u_1, u_2)$, $q = q(u_1, u_2)$ be scalar C^2-functions satisfying

$$\nabla_u \eta(u) \cdot \nabla_u F(u) = \nabla_u q(u) \quad \text{for all } u \in \mathbb{R}_+^* \times \mathbb{R}\,.$$

If η is convex and $\eta(0, \cdot) = 0$, then (η, q) is called a *convex weak entropy-flux pair* (for the flux $F : \mathbb{R}^2 \to \mathbb{R}^2$).

Since the source terms s in (26) and S_0 in (29) have jump discontinuities, it is reasonable – also in view of the striking non-uniqueness of solutions of (13) for sand piles – to use the notion of set-valued maps. Looking at $s = \sin\xi - \varepsilon b_x \cos\xi - \frac{u}{|u|}\tan\delta\cos\xi$ we introduce the set-valued sign function

$$\operatorname{sig} u = \begin{cases} [-1, 1] & \text{for } u = 0 \\ \frac{u}{|u|} & \text{for } u \neq 0 \end{cases}$$

and

$$\tilde{s}(u, x) = \sin\xi - \varepsilon b_x \cos\xi - \operatorname{sig}(u)\tan\delta\cos\xi\,.$$

Then $\tilde{S}$ is defined by

$$\tilde{S}(u_1, u_2, x) = \frac{(1+\alpha)\beta'}{\beta} F(u_1, u_2) + \begin{pmatrix} 0 \\ u_1 \tilde{s}\left(\frac{u_2}{u_1}, x\right) \end{pmatrix}\,.$$

Finally the system (29) is replaced by the differential inclusion

$$\partial_t u + \partial_x F(u) \in \tilde{S}(u,x) \tag{46}$$

which is made more precise in the following definition.

Definition 7 Given an initial value $u^0 = (u_1^0, u_2^0) \in L^\infty(\mathbb{R})^2$ with $u_1^0 \geq 0$ and $\frac{u_2^0}{u_1^0} \in L^\infty(\mathbb{R})$ we call a function $u = (u_1, u_2) \in L^\infty\big((0,T) \times \mathbb{R};\ \mathbb{R}_+ \times \mathbb{R}\big)$ a *weak entropy solution* of (46) iff u has the following properties:

(1) there exists $S \in L^\infty_{\text{loc}}\big(\mathbb{R} \times [0,T)\big)^2$ such that

$$S(x,t) \in \tilde{S}\big(u(x,t),x\big) \quad \text{for a.a. } (x,t) \in \mathbb{R} \times [0,T)\,.$$

(2) u is a *weak solution* of (46), i.e., for all $\psi \in C_0^1\big(\mathbb{R} \times [0,T)\big)^2$

$$\int_{\mathbb{R}} \int_0^T (u \cdot \partial_t \psi + F(u) \cdot \partial_x \psi + S \cdot \psi) dt dx = \int_R u^0 \cdot \psi(x,0) dx$$

(3) u satisfies the *entropy inequality*

$$\int_{\mathbb{R}} \int_0^T \big(\eta(u)\partial_t \phi + q(u)\partial_x \phi + \nabla_u \eta(u) \cdot S\phi\big) dt dx \geq \int_R \eta(u^0)\phi(\cdot,0)$$

for every test function $0 \leq \phi \in C_0^1\big(\mathbb{R} \times [0,T)\big)$ and every convex-weak entropy flux pair (η, q) for which $\nabla_u \eta(u)$ is locally bounded on $\mathbb{R} \times [0,T)$.

Remark 8 The non-classical part (1) in Definition 7 states the selection of an L^∞_{loc}-function $S(x,t)$ coinciding with $S_0(u_1, u_2, x)$ from (28) when the physical velocity $\frac{u_2}{u_1}$ does not vanish. In contrast to the usual definition of convex entropy flux pairs the degeneracy of (26) when $u_1 \to 0$ ($h \to 0$) requires to add the condition that $\eta(0,\cdot) = 0$ (weak entropy).

Theorem 9 [**?**] *Let $u^0 = (u_1^0, u_2^0) \in L^\infty(\mathbb{R})^2$ denote an initial value such that $u_1^0, \frac{u_2^0}{u_1^0} \in L^\infty(\mathbb{R})$ and*

$$u_1^0 \geq 0\,,\ u_1^0(x) \to 0\,,\ u_2^0(x) \to 0 \ \text{as } |x| \to \infty$$

and let $\beta \in W^{1,\infty}(\mathbb{R})$ be given such that $\beta(x) \geq \beta_0 > 0$. Then there exists a local weak entropy solution $u = (u_1, u_2)$ of (46). If β is constant, there exists a global weak entropy solution.

Sketch of Proof In a first step we consider the viscous approximation

$$u_t + \partial_x F(u) = S_\varepsilon(u,x) + \varepsilon \partial_x^2 u, \quad \varepsilon > 0\,, \tag{47}$$

where S_ε is defined by $\tilde{S}$ via smoothing the jump discontinuity of $\tilde{s}(u,x)$. To prove the existence of classical solutions u^ε and some a priori estimates it is convenient to consider smooth initial values u_ε^0. E.g., we assume that $u_\varepsilon^0 \in C^2(\mathbb{R})^2$, $u_{1\varepsilon}^0 \geq u_1^0 + \varepsilon$. This smoothing and lifting up in addition to the viscous approximation is not contained in the proof in [**?**], but can easily be included [**?**]. Standard parabolic theory yields a unique classical solution u^ε in some interval $[0, T_\varepsilon)$. To prove that T_ε can be chosen independently of $\varepsilon > 0$ we apply the theory of invariant regions.

Lemma 10 *Let $u : \mathbb{R}\times[0,T) \to \mathbb{R}_+^* \times \mathbb{R}$ be a classical solution of the system (47). Further let $R_i : \mathbb{R}^2 \to \mathbb{R}$ and $M_i : [0,T) \to \mathbb{R}$, $i = 1,2$, be smooth functions defining the regions*

$$\Sigma(t) = \bigcap_{i=1}^{2} \{u \in \mathbb{R}^2 : R_i(u) \le M_i(t)\}, \quad t \in [0,T) .$$

Assume the following properties:

1. $u(\cdot,0) \in \Sigma(0)$
2. $\nabla_u R_i(a)$ *is a left eigenvector of* $\nabla_u F(a)$ *for all* $a \in \partial\Sigma_i(t)$, $t \in [0,T)$
3. $\Sigma(t)$ *is convex for all* $t \in [0,T)$
4. $\sup_{x\in\mathbb{R}} \sup_{a\in\partial\Sigma_i(t)} \nabla_u R_i(a) \cdot S_\varepsilon(a,x) \le M_i'(t)$

where $\partial\Sigma_i(t) = \partial\Sigma(t) \cap \{a \in \mathbb{R}^2 : R_i(a) = M_i(t)\}$. Then $\Sigma(t)$ is a one-parameter family of invariant regions, i.e., $u(x,t) \in \Sigma(t)$ for all $t \in [0,T)$.

This lemma will be applied with the Riemann invariants $R_\pm(t) = \pm\frac{u_2}{u_1} + u_1^{1/(2(1+\alpha))}$ and a function $M(t) = M_\pm(t)$ satisfying $M(0) = \max_{+,-} \|R_\pm(u^0)\|_\infty$ and the differential inequality $M'(t) \ge C\big(M(t)^2 + \|\sin\xi\|_\infty\big)$ where $C = C(\|\beta'\|_\infty, \beta_0, \alpha)$. As a conclusion we get a $T_0 > 0$ such that

$$\|u_1^\varepsilon\|_{L^\infty(\mathbb{R}\times(0,T))} + \Big\|\frac{u_2^\varepsilon}{u_1^\varepsilon}\Big\|_{L^\infty(\mathbb{R}\times(0,T))} \le C(T)$$

for all $0 < T < T_0$ independent of $\varepsilon > 0$; if the slope ξ is constant, then $T_0 = \infty$ and $C(T)$ is linear in T.

Besides L^2-estimates of u_1^ε and u_2^ε on $\mathbb{R} \times (0,T_0)$ with bounds depending on ε and the crucial non-negativity of u_1^ε, see [**?**], it is important to have sufficiently many local L^2-estimates of ∇u at hand.

Lemma 11 *Let (η,q) be a convex entropy-flux pair such that $\nabla\eta$ is bounded and $\eta \in C^2$ on $(0,\infty)\times\mathbb{R}$. Further let $u = (u_1^\varepsilon, u_2^\varepsilon)$ be a strong solution of (47) such that u^ε, $F(u^\varepsilon)$ and $G_\varepsilon(u^\varepsilon,x)$ are bounded independendly of $\varepsilon \in (0,1)$. Then for every bounded set $\Omega \subset \mathbb{R}\times\mathbb{R}_+$ there exists a constant $C(\Omega) > 0$ such that*

$$\varepsilon \iint_\Omega \partial_x u^\varepsilon \cdot \nabla_u^2 \eta(u^\varepsilon) \cdot \partial_x u^\varepsilon \, dxdt \le C(\Omega) \quad \forall\, \varepsilon \in (0,1) .$$

If η is even strongly convex, then a similar estimate holds for $\varepsilon|\partial_x u^\varepsilon|^2$.

Proof Given a solution u^ε of (47) and an entropy-flux pair (η,q) the functions $\eta(u^\varepsilon)$, $q(u^\varepsilon)$ will satisfy the equation

$$\partial_t \eta(u^\varepsilon) + \partial_x q(u^\varepsilon) = \varepsilon\big(\partial_x^2 \eta(u^\varepsilon) - \partial_x u^\varepsilon \cdot \nabla_u^2 \eta(u^\varepsilon) \cdot \partial_x u^\varepsilon\big) + \nabla_u \eta(u^\varepsilon) \cdot G_\varepsilon . \tag{48}$$

Testing with $0 \le \varphi \in C_0^\infty\big(\mathbb{R} \times [0,\infty)\big)$ such that $\varphi|_\Omega = 1$ will yield the a priori estimate. If η is strictly convex, the estimate $r \cdot \nabla_u^2 \eta(v) \cdot r \ge \delta|r|^2$ with some $\delta > 0$ will prove the second assertion. □

The second main step deals with the limit $\varepsilon \to 0$. Since only very few a priori estimates on u^ε are available and since (46) contains several nonlinear terms, we need to refer to Young measure solutions as limits of $\big(u_1^\varepsilon, \frac{u_2^\varepsilon}{u_1^\varepsilon}\big)$.

Lemma 12 [?] *Assume that* $(z_k) \subset L^\infty(\Omega)^2$ *is a sequence on* $\Omega \subset [0,T)\times\mathbb{R}$ *with* $z_k(\Omega) \subset K$ *where* $K \subset \mathbb{R}^2$ *is compact and convex. Then there exists a subsequence* $(z_{k_j}) \subset (z_k)$ *and a family of Borel probability measures* $(\mu_{(x,t)})_{(x,t)\in\Omega}$ *on* K *such that for every* $H \in C^0(K)^2$

$$\lim_{j\to\infty} \iint\limits_{\Omega} H\big(z_{k_j}(x,t)\big)\varphi(x,t)\,dxdt = \iint\limits_{\Omega} \overline{H}(x,t)\varphi(x,t)\,dxdt \quad \forall\,\varphi \in L^1(\Omega)^2 \tag{49}$$

where

$$\overline{H}(x,t) = \int_K H(\cdot)\,d\mu_{(x,t)} \quad \textit{for a.a. } (x,t)\in\Omega\,.$$

The proof of this famous lemma is based on the Theorem of Banach-Alaoglu applied to the spaces $L^1(\Omega; C^0(K))$ and $L^\infty(\Omega;\mathcal{M}(K))$. When applied to a sequence (z_k) of approximate solutions of a (partial) differential equation this lemma yields a measure-valued solution $\int_K y\,d\mu_{(x,t)}(y)$ in the limit. By these means the notion of strong or weak solutions is generalized to a great extent: there are no longer function values a.e. in Ω, but only *probabilities* of them. E.g., the sequence of Rademacher functions $r_j(x) = \operatorname{sgn}\sin(2^j\pi x)$ on $(0,1)$ attains the values ± 1 with probability $\frac{1}{2}$ for every $j\in\mathbb{N}$. Its limit for $j\to\infty$ in the usual weak sense of $(L^\infty)^*$ is 0, but in the sense of Lemma 12 we get the probability measure $\mu_x \equiv \mu = \frac{1}{2}(\delta_{-1}+\delta_{+1})$ for all $x\in(0,1)$. Then $\overline{H}(x) \equiv \frac{1}{2}\big(H(-1)+H(+1)\big)$ is the limit of $H\big(r_j(x)\big)$ for all $H\in C^0([-1,1])$.

Lemma 12 will be applied to $z^k = \big(u_1^\varepsilon,\, \frac{u_2^\varepsilon}{u_1^\varepsilon}\big)$ where $\varepsilon = \frac{1}{k}$ and to

$$H_1(z) = \begin{pmatrix} z_1 \\ z_1\, z_2 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad H_2(z) = F(z_1, z_1 z_2)\,.$$

Then for every test function $\varphi \in C_0^1\big(\mathbb{R}\times[0,T)\big)^2$

$$\int_{\mathbb{R}}\int_0^T \big(H_1(z^\varepsilon)\partial_t\varphi + H_2(z^\varepsilon)\partial_x\varphi + H_3^\varepsilon(z^\varepsilon,x)\varphi\big)\,dt\,dx = \varepsilon\int_{\mathbb{R}}\int_0^T \partial_x u^\varepsilon\cdot\partial_x\varphi\,dtdx + \int_{\mathbb{R}} u^0\varphi(x,0)\,dx$$

where $H_3^\varepsilon(z,x) = S_\varepsilon(z_1, z_1 z_2, x)$ is discontinuous in z. By Lemma 12 there exist Young measures $\mu_{(x,t)}$ for a.a. $(x,t)\in\Omega$ such that

$$\int_{\mathbb{R}}\int_0^T (\overline{H}_1\partial_t\varphi + \overline{H}_2\partial_x\varphi + \overline{H}_3\varphi)\,dtdx = \int_{\mathbb{R}} u^0\varphi(x,0)\,dx$$

where $\overline{H}_j(x,t) = \int_K H_j(y)\,d\mu_{(x,t)}(y)$, $j = 1,2$. However, $\overline{H}_3 \in L^\infty(\mathbb{R}\times\mathbb{R}_+)^2$ cannot be characterized by using the measures $\mu_{(x,t)}$. To deal with functions with jump discontinuities as $\operatorname{sgn} u_1$ or in two dimensions with discontinuities of the type $\frac{u}{|u|}$ a further decomposition of the measures $\mu_{(x,t)}$ is needed, see [?].

To show for the Young measures that $\mu_{(x,t)} = \delta_{z(x,t)}$ for a.a. (x,t) we need the Div-Curl Lemma and sufficiently many entropy-flux pairs.

Lemma 13 [**?**] *Let $\Omega \subset \mathbb{R}\times\mathbb{R}_+$ be bounded, (u^k) a sequence of functions on Ω and (η^1, q^1), (η^2, q^2) weak entropy-flux pairs. Assume that $\eta^i(u^k)$, $q^i(u^k)$ converge weakly in $L^2(\Omega)$ to $\eta^i(u)$, $q^i(u)$ and that $\partial_t\eta^i(u^k) + \partial_x q^i(u^k)$ is relatively compact in $W^{-1,2}(\Omega)$ for $i = 1, 2$. Then, as $k \to \infty$,*

$$\iint\limits_{\Omega} \big(\eta^1(u^k)q^2(u^k) - \eta^2(u^k)q^1(u^k)\big)\varphi\, dxdt \to \iint\limits_{\Omega} (\eta^1 q^2 - \eta^2 q^1)\varphi\, dxdt$$

for every $\varphi \in C_0^\infty(\Omega)$. Furthermore

$$\int_K (\eta^1 q^2 - \eta^2 q_1) d\mu_{(x,t)} = \int_K \eta^1 d\mu_{(x,t)} \cdot \int_K q^2 d\mu_{(x,t)} - \int_K \eta^2 d\mu_{(x,t)} \cdot \int_K q^1 d\mu_{(x,t)}\,.$$

Given an entropy-flux pair (η, q) Definition 6 implies that η satisfies a second order wave equation w.r.t. u_1, u_2 with (u_1, u_2)-depending coefficients. By [**?**] every weak entropy η, i.e., $\eta(0,\cdot) = 0$, can be written in the form

$$\begin{aligned} \eta(u_1,u_2) &= \int_{\mathbb{R}} f(\xi)\,\big(u_1^{\frac{1}{2(1+\alpha)}} - (\xi - \frac{u_2}{u_1})^2\big)_+^{2+4\alpha}\, d\xi \\ &= u_1 \int_{-1}^{1} f\big(\frac{u_2}{u_1} + \xi\, u_1^{\frac{1}{2(1+\alpha)}}\big)(1-\xi^2)^{2+4\alpha} d\xi\,. \end{aligned} \tag{50}$$

In particular, for $f(s) = \frac{1}{2}s^2$, the entropy η coincides up to multiplicative constant with the mechanical energy $\eta_E = \frac{1}{2}\frac{u_2^2}{u_1} + \kappa' u_1^{\frac{2+\alpha}{1+\alpha}}$. It is easily seen that for $f \in C_0^\infty(\mathbb{R})$ the sequence $\big(\nabla_u \eta(u^k)\big)$ is bounded on Ω and that

$$|r\cdot\nabla_u^2\eta(u^k)\cdot r| \le C_f r\cdot\nabla_u^2\eta(u^k)\cdot r \quad \forall\, r \in \mathbb{R}^2$$

independent of $k \in \mathbb{N}$. A further analysis based on previous a priori estimates shows that the right-hand side of (48) is precompact in $W^{-1,2}(\Omega)^2 + W^{-1,p}(\Omega)^2$ $(p < 2)$ and that the left-hand side of (48) is uniformly bounded in $W^{-1,\infty}(\Omega)^2$. Then by Murat's Lemma $\big(\partial_t\eta(u^k) + \partial_x q(u^k)\big)_{k\in\mathbb{N}}$ is precompact in $W^{-1,2}(\Omega)$ for every weak entropy-flux pair (η, q) generated by $f \in C_0^\infty(\mathbb{R})$, see (**??**). Hence Lemma **??** may be applied.

This is the starting point to show in a lengthy technical proof [**?**, **?**] that the Young measures $\mu_{(x,t)}$ are δ-measures. To be more precise, using the (z_1, z_2)-functions,

$$\mu_{(x,t)} = \begin{cases} \delta_{(z_1,z_2)} & \text{if } z_1 > 0 \\ \delta_{z_1} \times \nu_{(x,t)} & \text{if } z_1 = 0 \end{cases}$$

where $\operatorname{supp}\nu_{(x,t)} \subset [\liminf_{k\to\infty} z_1^k(x,t),\ \limsup_{k\to\infty} z_1^k(x,t)] \subset\subset \mathbb{R}$. For (u_1, u_2) we conclude that also in the case when $u_1(x,t) = z_1(x,t) = 0$ the measure $\mu_{(x,t)}$ is concentrated in one single point $u(x,t)$. Note that the analysis from [**?**, **?**] simplifies to elementary algebraic considerations when $\alpha = -\frac{1}{2}$, see [**?**]. However, in this case, the constitutive relation $\rho = h^{-1/2}$ seems to be unphysical.

In the final step Lemma 12 implies that $u^k \rightharpoonup u$ weakly in $L^\infty(\Omega)^2$ and using $H(y) = |y|^p$, $1 < p < \infty$, that $\iint\limits_{\Omega} |u^k|^p\, dxdt \to \iint\limits_{\Omega} |u|^p dxdt$ for $k \to \infty$. Hence $u^k \to u$ in $L^p(\Omega)^2$ by the Theorem of Radon-Riesz. By similar arguments u can be shown to be an entropy solution of (29). Looking more carefully at $\tilde{S}$ with its jump discontinuity $\operatorname{sig} u$ we may even select a function $S \in L^\infty_{\rm loc}(\mathbb{R} \times [0,T))^2$ such that u is a solution of (46) in the sense of Definition 7. □

A detailed analysis of the time interval in which a weak entropy solution exists may be performed by the theory of invariant regions, see Lemma 10. In the case when $\rho \equiv \rho_0 > 0$ the following results depending on the behavior of the slope $\xi(x)$ have been obtained [**?**, **?**]:

Theorem 14 *Let a curved base with variable slope angle $\xi(x)$ be given such that for $\beta(x) = \varepsilon K \cos \xi(x)$*

$$\frac{\beta'}{\beta} \in L^\infty(\mathbb{R}) .$$

Consider an incompressible avalanche with initial values h_0, u_0 such that

$$\begin{aligned} 0 \le \rho_0 = \beta h_0 \in L^\infty(\mathbb{R}), \quad & m_0 = \beta h_0 u_0 \in L^\infty(\mathbb{R}) \\ \rho_0(x) \to 0, \quad m_0(x) \to 0 \quad & \textit{for } |x| \to \infty . \end{aligned}$$

Finally let $s(x,u) = \sin \xi(x) - \operatorname{sgn}(u) \tan \delta(x) \cos \xi(x)$, where the bed friction angle $\delta(x)$ may depend on $x \in \mathbb{R}$, let $P = \frac{5}{8} \|\beta'/\beta\|_\infty$, $Q = \|s\|_\infty$ and let $E_0 = \max\left(\|2\rho_0\|_\infty, \|\frac{1}{2}\, u_0^2\|_\infty\right)$ measure the initial energy of the avalanche.

1. *If $\beta' = 0$ (constant slope), then the Savage-Hutter equations admit a global weak entropy solution. Furthermore ρ grows at most quadratically as $t \to \infty$ and $u = \frac{m}{\rho}$ grows at most linearly as $t \to \infty$. If in addition $|\xi| \le |\delta|$ for all $x \in \mathbb{R}$, i.e., the slope angle is bounded by the bed friction angle, then ρ and u are uniformly bounded independent of time with a bound depending only on E_0.*

2. *Assume that*

   $$\beta' \le 0,\ u_0 > 0,\ \frac{1}{2} u_0^2 \ge 2\rho_0 \ \textit{and}\ \xi \ge \delta ,$$

   i.e., the slope angle $\xi(x)$ is constant or even steepening, the initial velocity is positive and sufficiently large and the slope angle is greater than or equal to the bed friction angle. Then there exists a global weak entropy solution. Furthermore ρ grows at most quadratically and $u = m/\rho$ grows at most linearly as $t \to \infty$. Finally $u \ge 0$ (down slope) and $\frac{1}{2}u^2 \ge 2\rho$.

3. *If the slope is arbitrarily curved ($\beta' \ne 0$), then there exists a weak entropy solution on a time interval $(0, T_{\max})$. Here it is sufficient to take*

   $$T_{\max} = \frac{1}{PQ}\Big(\frac{\pi}{2} - \arctan \sqrt{2E_0 P/Q}\Big) .$$

In [**?**, **?**] the term $\operatorname{sgn}(u)$ in $s(x,u)$ has been smoothed. However, the same results hold in the set-valued formulation, cf. (46), Definition 7, for an incompressible avalanche [**?**].

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