# Solutions of the Navier-Stokes Initial Value Problem in Weighted $L^q$-Spaces

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**Abstract**

The problem of strong solvability of the nonstationary Navier-Stokes equations is considered in weighted $L^q$-spaces $L^q_\omega(\Omega)$, where the domain $\Omega \subset \mathbb{R}^n$ is equal to the half space $\mathbb{R}^n_+$ or to a bounded domain with boundary of class $C^{1,1}$ and the weight $\omega$ belongs to the Muckenhoupt class $A_q$. We give general conditions on the weight function ensuring the existence of a unique strong solution at least locally in time. In particular, these conditions admit weight functions $\omega \in A_q$, which become singular at the boundary or, in the case $\Omega = \mathbb{R}^n_+$, grow for $|x| \to \infty$.

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## 1 Introduction

In this paper we investigate the existence of strong solutions in weighted $L^q$-spaces of the instationary Navier-Stokes equations

$$u_t + u \cdot \nabla u - \nu \Delta u + \nabla p = f \quad \text{in } (0,T) \times \Omega \tag{1a}$$

$$\operatorname{div} u = 0 \quad \text{in } (0,T) \times \Omega \tag{1b}$$

$$u = 0 \quad \text{on } (0,T) \times \partial\Omega \tag{1c}$$

$$u(0) = u_0 \quad \text{in } \Omega. \tag{1d}$$

Here $\Omega \subset \mathbb{R}^n$ is a bounded domain with boundary of class $C^{1,1}$ or the half space $\mathbb{R}^n_+$; $u$ denotes the unknown velocity field, $p$ is the unknown pressure, while the external force $f$ and the initial velocity $u_0$ are given and $\nu > 0$ is the constant viscosity.

There are numerous references concerning the problem of strong solvability of the Navier-Stokes system in $L^q$-spaces; we mention only [8], [14] for $q = 2$ and [12], [22] for general $L^q$-spaces and refer to [12] for a more complete survey.

Our aim is to embed the results in $L^q$-spaces without weights [12], [22] into the weighted context for a large class of weight functions. For this purpose we exploit results from [6], [7] on the Stokes operator in weighted $L^q$-spaces, $1 < q < \infty$, for general weight functions of Muckenhoupt class $A_q$ (see Definition 2.1).

To define the Stokes operator in weighted $L^q$-spaces let $1 < q < \infty$, $\omega \in A_q$,

$$L^q_\omega(\Omega) := \{u \in L^1_{loc}(\overline{\Omega}) \ : \ \|u\|^q_{q,\omega} = \int_\Omega |u|^q \omega \, dx < \infty\}$$

and use the existence of the Helmholtz decomposition, see [5],

$$L^q_\omega(\Omega)^n = L^q_{\omega,\sigma}(\Omega) \oplus G^q_\omega(\Omega)$$

where $L^q_{\omega,\sigma}(\Omega)$ is the closure of $C^\infty_{0,\sigma}(\Omega) = \{u \in C^\infty_0(\Omega)^n \ : \ \operatorname{div} u = 0\}$ in $L^q_\omega(\Omega)^n$ and $G^q_\omega(\Omega)$ are the gradient fields in $L^q_\omega(\Omega)^n$. Using the bounded Helmholtz projection $P_{q,\omega} : L^q_\omega(\Omega)^n \to L^q_{\omega,\sigma}(\Omega)$ we define the Stokes operator $\mathcal{A}_{q,\omega}$ in $L^q_{\omega,\sigma}(\Omega)$ by

$$\begin{aligned} D(\mathcal{A}_{q,\omega}) &= W^{2,q}_\omega(\Omega)^n \cap L^q_{\omega,\sigma}(\Omega) \cap \{u \in W^{1,q}_\omega(\Omega)^n \ : \ u\big|_{\partial\Omega} = 0\} \\ \mathcal{A}_{q,\omega} &= -\nu \, P_{q,\omega}\Delta \qquad \text{on } D(\mathcal{A}_{q,\omega}), \end{aligned}$$

where $W^{1,q}_\omega(\Omega)$ and $W^{2,q}_\omega(\Omega)$ denote the Sobolev spaces over $L^q_\omega(\Omega)$ of order 1 and 2 respectively. It was shown in [6], [7] that the Stokes operator $-\mathcal{A}_{q,\omega}$ generates a bounded analytic semigroup in $L^q_{\omega,\sigma}(\Omega)$. Applying the Helmholtz projection $P$ to the Navier-Stokes equations (1) yields the initial value problem

$$u_t + \mathcal{A}_{q,\omega} u = Pf - P(u \cdot \nabla u), \quad u(0) = u_0 \tag{2}$$

in $L^q_{\omega,\sigma}(\Omega)$ or more generally in the spaces $D^{\alpha,q}_\omega(\Omega)$ defined as

$$D^{\alpha,q}_\omega(\Omega) := D((I + \mathcal{A}_{q,\omega})^\alpha) \quad \text{for } \alpha > 0$$

equipped with the norm $\|u\|_{D^{\alpha,q}_\omega(\Omega)} := \|(I + \mathcal{A}_{q,\omega})^\alpha u\|_{q,\omega}$, where $D((I + \mathcal{A}_{q,\omega})^\alpha)$ is the domain of the fractional power $(I + \mathcal{A}_{q,\omega})^\alpha$ in $L^q_{\omega,\sigma}(\Omega)$. For $\alpha < 0$ we define $D^{\alpha,q}_\omega(\Omega)$ to be the dual space of $D^{-\alpha,q'}_{\omega'}(\Omega)$, where $q' = \frac{q}{q-1}$ and $\omega' = \omega^{-1/(q-1)}$. For $\alpha = 0$ let $D^{0,q}_\omega(\Omega) := L^q_{\omega,\sigma}(\Omega)$.

Using that the Stokes operator $-\mathcal{A}_{q,\omega}$ generates a bounded analytic semigroup $\{e^{-t\mathcal{A}_{q,\omega}}\}_{t \geq 0}$ we can reformulate (2) in integral form

$$u(t) = e^{-t\mathcal{A}_{q,\omega}} u_0 + \int_0^t e^{-(t-s)\mathcal{A}_{q,\omega}} \{Pf(s) - P(u \cdot \nabla u)(s)\} \, ds \tag{3}$$

for all $t \in (0, T)$.

Then our main result on solvability of (3) reads as follows:

**Theorem 1.1** *Let* $1 < q < \infty$, $\omega \in A_q$ *and let* $\Omega$ *be a bounded* $C^{1,1}$*-domain or the half space* $\mathbb{R}^n_+$. *Moreover let* $\frac{n}{2q} - \frac{1}{2} < \alpha < 1$ *and assume that there exists an* $\bar{\alpha} \in (-\frac{1}{2}, \alpha)$, *such that*

$$|Q|^{(1+2\bar{\alpha})\frac{q}{n}} \leq C \, \omega(Q) \tag{4}$$

*for some* $C \in \mathbb{R}_+$ *and for all cubes* $Q \subset U$*, where* $U = \mathbb{R}^n_+$ *if* $\Omega = \mathbb{R}^n_+$ *and* $U \subset \mathbb{R}^n$ *is a neighbourhood of* $\overline{\Omega}$ *if* $\Omega$ *is a bounded* $C^{1,1}$*-domain. Set* $A := \mathcal{A}_{q,\omega}$ *and* $Fu := -P(u \cdot \nabla u)$*.*

*Finally let* $u_0 \in D^{\alpha,q}_\omega(\Omega)$ *and for some* $\delta \in [0, \frac{1}{2})$ *with* $-\alpha < \delta < 1 - \alpha$ *and some* $T > 0$ *let*

$$f \in C((0,T], D^{-\delta,q}_\omega(\Omega)) \quad \text{and} \quad \|f(t)\|_{D^{-\delta,q}_\omega} = o(t^{\alpha+\delta-1}) \text{ for } t \to 0.$$

*Then there is a* $T^* > 0$ *and a unique curve* $u : [0,T^*] \longrightarrow D^{\alpha,q}_\omega(\Omega)$ *with the properties*

*a)* $u \in C([0,T^*], D^{\alpha,q}_\omega(\Omega))$ *and* $u(0) = u_0$*.*

*b)* $u \in C([0,T^*], D^{\mu,q}_\omega(\Omega))$ *for* $\alpha < \mu < 1 - \delta$ *and* $\lim_{t\to 0} t^{\mu-\alpha}\|u(t)\|_{D^{\mu,q}_\omega} = 0$*.*

*c)* $u(t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A}\{Fu(s) + f(s)\}\, ds$ *for all* $t \in [0,T^*]$*.*

*The mapping* $u : [0,T^*] \to D^{\alpha,q}_\omega(\Omega)$ *is uniquely determined if it satisfies a), b) for some* $\mu \geq \frac{\alpha-\delta+1}{2}$ *and c).*

Comparing this result with results in the case without weights we see that in [22] and [12] the same lower bound $n/2q - 1/2$ for the choice of $\alpha$ was found. In [12] it was possible to include even the limit case $\alpha = n/2q - 1/2$ because of the result in [13]. Thus except of this limit case we have the same restrictions on the choice of $q$ and $\alpha$ but we can additionally admit a weight function satisfying the condition (4). Roughly spoken, (4) means that the weight function is not allowed to become too small. In particular, we can choose weight functions $\omega \in A_q$ which are bounded from below by positive constants but become singular at the boundary $\partial\Omega$ or grow for $|x| \to \infty$ in the case $\Omega = \mathbb{R}^n_+$. Hence we obtain more precise informations about the behaviour of the solution near $\partial\Omega$ and for $|x| \to \infty$. Some simple examples of weight functions obeying (4) are given in Section 7.

Finally, Theorem 1.1 implies the unique local solvability of the Navier-Stokes equations in the evolutionary formulation (2) in weighted $L^q$-spaces under additional regularity assumptions on $f$ (Theorem 7.1 in Section 7 below). Moreover we get even global existence results under smallness assumptions on the norms of $u_0$ and $f$ in the case of a bounded domain $\Omega$ (Corollary 7.1).

This paper is organized as follows: In Section 2 we introduce the class of Muckenhoupt weights and fix some notation.

Section 3 discusses with imbedding properties of weighted Bessel potential spaces $H^{\alpha,q}_\omega(\Omega)$ for $\alpha \in \mathbb{R}$ exploiting results on the boundedness of fractional integral operators in weighted $L^q$-spaces [18], extension theorems in weighted Sobolev spaces [1] and the complex interpolation method. For these weighted imbedding theorems more restrictive assumptions than $\omega \in A_q$ are needed. These assumptions finally lead to the condition (4) in Theorem 1.1.

After summarizing results on the Stokes operator $\mathcal{A}_{q,\omega}$ from [6], [7] and discussing properties of the spaces $D^{\alpha,q}_\omega(\Omega)$ in Section 4, we compare the spaces

$D^{\alpha,q}_{\omega}(\Omega)$ with the weighted Bessel potential spaces $H^{\alpha,q}_{\omega}(\Omega)$ : Since up to now there is no proof of boundedness of the purely imaginary powers of the Stokes operator $\mathcal{A}_{q,\omega}$ for general Muckenhoupt weights $\omega \in A_q$, we can only use the imbeddings

$$D^{\overline{\alpha},q}_{\omega}(\Omega) \hookrightarrow H^{2\alpha,q}_{\omega}(\Omega) \hookrightarrow D^{\underline{\alpha},q}_{\omega}(\Omega)$$

for $\underline{\alpha} < \alpha < \overline{\alpha}$.

Combining the results of Section 3 and Section 4 we derive weighted estimates of the nonlinearity $P(u \cdot \nabla u)$ in Section 5.

In Section 6 an abstract existence theorem for integral equations of the form (3) is proved with the help of Banach's fixed point theorem generalizing a result of [22].

In Section 7 we apply the abstract results of Section 6 and the estimates of the nonlinear term in Section 5 to obtain Theorem 1.1. Then by standard arguments Theorem 1.1 yields results on unique local and global solvability of the initial value problem (2) in weighted spaces stated as Theorem 7.1 and Corollary 7.1 in Section 7 below.

## 2 Preliminaries

By a cube $Q$ we mean a subset of $\mathbb{R}^n$ of the form $\Pi_{j=1}^n I_j$, where $I_1, \ldots, I_n \subset \mathbb{R}$ are bounded intervals of the same length. Thus cubes have always sides parallel to the axes.

**Definition 2.1** *Let $1 < q < \infty$. A function $0 \le \omega \in L^1_{loc}(\mathbb{R}^n)$ is called an $A_q$-weight if*

$$A_q(\omega) := \sup_Q \left( \frac{1}{|Q|} \int_Q \omega \, dx \right) \left( \frac{1}{|Q|} \int_Q \omega^{-\frac{1}{q-1}} dx \right)^{q-1} < \infty, \tag{5}$$

*where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ and $|Q|$ assigns the Lebesgue measure of $Q$. $A_q(\omega)$ is called the $A_q$-constant of $\omega$.*

*We use the abbreviation $\omega(A)$ for $\int_A \omega(x)\,dx$ and set $\omega' := \omega^{-\frac{1}{q-1}}$ if $q \in (1,\infty)$ is fixed.*

Note that if $1 < q < \infty$ and $q' = q/(q-1)$, then $(L^q_\omega(\Omega))' \cong L^{q'}_{\omega'}(\Omega)$ with respect to the usual dual product $(f,g) = \int_\Omega f\,g\,dx$. Note that $\omega' \in A_{q'}$.

Simple examples of $A_q$-weights are radially symmetric weights of the form $\omega(x) = |x - x_0|^\alpha$ for $-n < \alpha < n(q-1)$ or more generally distance functions of the form $\omega(x) = \mathrm{d}ist\,(x, M)^\alpha$ for a $k$-dimensional compact Lipschitzian manifold $M$ and $-(n-k) < \alpha < (n-k)(q-1)$. For further examples we refer to [2].

For Muckenhoupt weights there is a weighted version of the Hörmander-Michlin multiplier theorem (see [9], Chapter IV, Theorem 3.9 or [2], Theorem 3.3).

**Theorem 2.1 (Hörmander-Michlin multiplier theorem with weights)** *Let $m \in C^n(\mathbb{R}^n \backslash \{0\})$ have the property that*

$$\exists\, M \in \mathbb{R} : \qquad |D^\alpha m(\xi)| \le M\, |\xi|^{-|\alpha|} \quad \forall\, \xi \in \mathbb{R}^n \setminus \{0\},\ |\alpha| = 0, 1, \dots, n.$$

*Then for all $1 < q < \infty$ and $\omega \in A_q$ the multiplier operator $\widehat{Tf} = m\widehat{f}$ defined for Schwartz functions $f \in \mathcal{S} = \mathcal{S}(\mathbb{R}^n)$ can be extended uniquely to a bounded linear operator from $L^q_\omega(\mathbb{R}^n)$ to $L^q_\omega(\mathbb{R}^n)$ witha norm depending only on $n, q, M$ and $A_q(\omega)$.*

For a domain $\Omega \subset \mathbb{R}^n$, $1 < q < \infty$, $\omega \in A_q$ and $k \in \mathbb{N}$ we define the weighted Sobolev space

$$\begin{aligned} W^{k,q}_\omega(\Omega) &:= \{u \in L^q_\omega(\Omega) \ :\ D^\alpha u \in L^q_\omega(\Omega),\ |\alpha| \le k\} \\ \|u\|_{k,q,\omega} &:= \Big( \sum_{|\alpha| \le k} \|D^\alpha u\|^q_{q,\omega} \Big)^{\frac{1}{q}}. \end{aligned}$$

If $\Omega$ is the half space $\mathbb{R}^n_+$ or a bounded Lipschitz domain, the trace of functions in $W^{1,q}_\omega(\Omega)$ on the boundary is well defined (see [6], [7]). By $W^{1,q}_{0,\omega}(\Omega)$ we denote the subspace of functions from $W^{1,q}_\omega(\Omega)$ with zero trace.

Next we prove an extension theorem following from results of Chua [1]. For the definition of an $(\varepsilon, \infty)$-domain see [1] or [6], Definition 3.1. It is easy to check that every bounded Lipschitz domain and the half space $\mathbb{R}^n_+$ are $(\varepsilon, \infty)$-domains.

**Theorem 2.2** *Let $1 < q < \infty$ and $\omega \in A_q$. Let $\Omega \subset \mathbb{R}^n$ be an $(\varepsilon, \infty)$-domain and $m \in \mathbb{N}$. Then there exists a linear, bounded extension operator $E_m : W^{m,q}_\omega(\Omega) \to W^{m,q}_\omega(\mathbb{R}^n)$ and a constant $C > 0$ such that for $k = 0, 1, \dots, m$*

$$\|E_m u\|_{k,q,\omega,\mathbb{R}^n} \le C\, \|u\|_{k,q,\omega,\Omega} \tag{6}$$

*for all $u \in W^{k,q}_\omega(\Omega)$.*

**Proof:** For unbounded $(\varepsilon, \infty)$-domains the result follows from Theorem 1.5 in [1].

For a bounded $(\varepsilon, \infty)$-domain Theorem 1.4 in [1] yields the existence of a bounded neighbourhood $U$ of $\overline{\Omega}$ and the existence of an extension operator $E : W^{m,q}_\omega(\Omega) \to W^{m,q}_\omega(U)$ such that

$$\|\nabla^k E u\|_{q,\omega,U} \le C\, \|\nabla^k u\|_{q,\omega,\Omega}$$

for $k = 0, 1, \dots, m$. Let $\varphi \in C_0^\infty(U)$ with $\varphi \equiv 1$ on $\overline{\Omega}$. Then $E_m := \varphi E$ is the desired extension operator. □

# 3 Weighted Bessel potential spaces

On $\mathcal{S}'(\mathbb{R}^n)$ we define for $s \in \mathbb{R}$ the operator

$$\Lambda^s f := \mathcal{F}^{-1}(1 + |\xi|^2)^{\frac{s}{2}} \mathcal{F} f \qquad \forall f \in \mathcal{S}',$$

where $\mathcal{F}$ is the Fourier transformation on $\mathcal{S}'$. Then for $1 < q < \infty$ and $\omega \in A_q$ the weighted Bessel potential space is defined by

$$H^{s,q}_\omega(\mathbb{R}^n) := \{f \in \mathcal{S}' \ : \ \|f\|_{H^{s,q}_\omega} = \|\Lambda^s f\|_{q,\omega} < \infty\}\,. \tag{7}$$

Note that $H^{s,q}_\omega(\mathbb{R}^n)$ is a reflexive Banach space. The weighted Multiplier Theorem 2.1 yields the imbedding $H^{s_1,q}_\omega(\mathbb{R}^n) \hookrightarrow H^{s_2,q}_\omega(\mathbb{R}^n)$ for $s_1 \geq s_2$.

In the sequel the complex interpolation method (see e.g. [17] or [21]) will be used. Given two compatible Banach spaces $X$ and $Y$ and $\theta \in (0,1)$ the respective complex interpolation space is denoted by $[X,Y]_\theta$. Further let $[X,Y]_\theta = X$ for $\theta = 0$ and $[X,Y]_\theta = Y$ for $\theta = 1$.

**Lemma 3.1** *Let $1 < q < \infty$, $\omega \in A_q$ and $k \in \mathbb{N}$.*

*i) $H^{k,q}_\omega(\mathbb{R}^n) = W^{k,q}_\omega(\mathbb{R}^n)$ with equivalent norms.*

*ii) Let $0 < s < k$. Then $[L^q_\omega(\mathbb{R}^n), H^{k,q}_\omega(\mathbb{R}^n)]_\theta = H^{s,q}_\omega(\mathbb{R}^n)$ for $\theta = \frac{s}{k}$.*

**Proof:** i) The assertion follows from Theorem 2.1 (cf. Propostion 6.1 in [20], Chapter 13). ii) The proof is based on the boundedness of the purely imaginary powers $\Lambda^{iy}$ in $L^q_\omega(\mathbb{R}^n)$, which is an easy consequence of the weighted Multiplier Theorem 2.1. We refer to the proof of Proposition 6.2 in [20], Chapter 13. $\square$

## 3.1 A weighted embedding lemma

For $0 < \alpha < n$ define the fractional integral operator

$$I_\alpha g(x) = \int_{\mathbb{R}^n} \frac{g(y)}{|x-y|^{n-\alpha}}\, dy. \tag{8}$$

In the sequel $Q$ assigns a cube in $\mathbb{R}^n$ with sides parallel to the axes. A weight function $\omega$ has the *reverse doubling property* (RD), if

$$(RD) \qquad \exists\, \epsilon, \delta \in (0,1) \ : \quad \omega(\delta Q) \leq \epsilon\, \omega(Q) \quad \forall\, Q \subset \mathbb{R}^n\,.$$

From [18] (Theorem 1 (B)) we cite the following theorem about boundedness of the operator $I_\alpha$ in weighted $L^p$-spaces.

**Theorem 3.1** *Let $0 < \alpha < n$ and $1 < p < q < \infty$. Let $v, \omega \geq 0$ be measurable functions on $\mathbb{R}^n$ such that both $\omega$ and $v^{-\frac{1}{p-1}}$ have the property (RD). If*

$$(A^\alpha_{p,q}) \qquad |Q|^{\frac{\alpha}{n}-1}\Big(\int_Q \omega\Big)^{\frac{1}{q}}\Big(\int_Q v^{-\frac{1}{p-1}}\Big)^{\frac{1}{p'}} \leq C \quad \forall Q \subset \mathbb{R}^n,$$

*then*

$$\|I_\alpha f\|_{q,\omega} \leq C\, \|f\|_{p,v} \tag{9}$$

*for all $f \in L^p_v(\mathbb{R}^n)$.*

**Remark:** Every Muckenhoupt weight $\omega \in A_\infty := \bigcup_{1<q<\infty} A_q$ has the property (RD) (see e.g. [9], Chapter IV, Lemma 2.4).

The operator $I_\alpha$ can be written as a multiplier operator in the form

$$\widehat{I_\alpha f} = c\,|\xi|^{-\alpha}\widehat{f} \qquad \forall f \in \mathcal{S} \tag{10}$$

(see [19], Chapter V, Lemma 2 (b)), where $c > 0$ depends only on $n, \alpha$. Since $I_\alpha$ neither maps $\mathcal{S}$ into itself nor can be well defined on $\mathcal{S}'$ we consider the space

$$\mathcal{M} := \{f \in \mathcal{S} \,:\, \widehat{f} \equiv 0 \text{ in a neighbourhood of } 0\}, \tag{11}$$

which is obviously mapped by $I_\alpha$ into itself. For $f \in \mathcal{M}$ the composition $I_\alpha\, \Lambda^\alpha f = c\,\mathcal{F}^{-1}|\xi|^{-\alpha}(1+|\xi|^2)^{\frac{\alpha}{2}}\mathcal{F} \in \mathcal{M}$ is well-defined and with

$$J_\alpha g := c^{-1}\,\mathcal{F}^{-1}|\xi|^{\alpha}(1+|\xi|^2)^{-\frac{\alpha}{2}}\mathcal{F}g \quad \forall g \in \mathcal{M}$$

we have $J_\alpha\, I_\alpha\, \Lambda^\alpha f = f$ for all $f \in \mathcal{M}$.

We show the density of $\mathcal{M}$ in weighted Bessel potential spaces $H^{s,q}_\omega(\mathbb{R}^n)$.

**Lemma 3.2** *Let $1 < q < \infty$, $\omega \in A_q$ and let $s \in \mathbb{R}$. Then $\mathcal{M}$ is dense in $H^{s,q}_\omega(\mathbb{R}^n)$.*

**Proof:** Choose a cut-off function $\eta \in C_0^\infty(\mathbb{R}^n)$ with $\eta \equiv 1$ in a neighbourhood of 0. Set $\eta_\varepsilon(\xi) = \eta(\frac{\xi}{\varepsilon})$ for $\varepsilon > 0$ and consider the multiplier operator

$$\widehat{T_\varepsilon f}(\xi) = \eta_\varepsilon(\xi)\widehat{f}(\xi) \tag{12}$$

for $f \in \mathcal{S}'$. We claim that

$$\forall f \in \mathcal{S}: \quad T_\varepsilon f \to 0 \quad \text{in } L^q_\omega(\mathbb{R}^n) \tag{13}$$

for $\varepsilon \to 0$. Since for $k = 1, \ldots, n$

$$|\xi|^k|\nabla^k\eta_\varepsilon(\xi)| \le \left|\frac{\xi}{\varepsilon}\right|^k\left|(\nabla^k\eta)\left(\frac{\xi}{\varepsilon}\right)\right| \le C_\eta$$

with $C_\eta$ independent of $\varepsilon > 0$, Theorem 2.1 implies the uniform boundedness of $\{T_\varepsilon \,:\, \varepsilon > 0\}$ in $\mathcal{L}(L^q_\omega(\mathbb{R}^n)$. Since $\widehat{f} \in \mathcal{S}$ we can use Theorem 2.1 again to obtain

$$\|T_\varepsilon f\|_{q,\omega} \le C_{\omega,f}\,\|\mathcal{F}^{-1}\eta_\varepsilon\|_{q,\omega} = \varepsilon^n C_{\omega,f}\,\|(\mathcal{F}^{-1}\eta)(\varepsilon\,\cdot)\|_{q,\omega}. \tag{14}$$

Since $\omega \in A_q$ there is a $\delta > 0$ such that $\omega \in A_{q-\delta}$ (see [9], Chapter IV, Theorem 2.6). For $\bar{\delta} := q^{-1}n\delta > 0$ we get because of $\mathcal{F}^{-1}\eta \in \mathcal{S}$ the estimate

$$|(\mathcal{F}^{-1}\eta)(\varepsilon x)| \le c'\,(1+|\varepsilon x|)^{\bar{\delta}-n} \le c\,\varepsilon^{\bar{\delta}-n}(1+|x|)^{\bar{\delta}-n}.$$

Applying this estimate to the right hand side of (14) yields

$$\varepsilon^n\|(\mathcal{F}^{-1}\eta)(\varepsilon\,\cdot)\|_{q,\omega} \le c\,\varepsilon^n\varepsilon^{\bar{\delta}-n}\Big(\int_{\mathbb{R}^n}\frac{\omega(x)}{(1+|x|)^{n(q-\delta)}}\,dx\Big)^{\frac{1}{q}} \to 0$$

for $\varepsilon \to 0$, since the integral is finite for $\omega \in A_{q-\delta}$ (see [2], Lemma 2.2 iii)). This shows (13).

Obviously $(I - T_\varepsilon)f \in \mathcal{M}$ for $f \in \mathcal{S}$. By (13) we obtain

$$\|f - (I - T_\varepsilon)f\|_{H^{s,q}_\omega} = \|T_\varepsilon f\|_{H^{s,q}_\omega} = \|T_\varepsilon \Lambda^s f\|_{q,\omega} \to 0$$

for $\varepsilon \to 0$ and $f \in \mathcal{S}$.

It remains to show the density of $\mathcal{S}$ in $H^{s,q}_\omega(\mathbb{R}^n)$. But this fact follows from the density of $\mathcal{S}$ in $L^q_\omega(\mathbb{R}^n)$ (see [6]) and the fact that $\Lambda^{-s} : L^q_\omega(\mathbb{R}^n) \to H^{s,q}_\omega(\mathbb{R}^n)$ is an isomorphism mapping $\mathcal{S}$ onto $\mathcal{S}$. □

Now it is easy to prove a weighted Sobolev imbedding theorem:

**Theorem 3.2** *Let* $1 < q < \infty$, $\omega \in A_q$, $0 < \beta < n$ *and* $1 < q < s < \infty$. *Let*

$$|Q|^{\frac{\alpha}{n}-1}\Big(\int_Q \omega\Big)^{\frac{1}{s}}\Big(\int_Q \omega^{-\frac{1}{q-1}}\Big)^{\frac{1}{q'}} \le C \tag{15}$$

*for some* $0 < \alpha \le \beta$ *and for all cubes* $Q \subset \mathbb{R}^n$. *Then it holds the imbedding*

$$H^{\beta,q}_\omega(\mathbb{R}^n) \hookrightarrow L^s_\omega(\mathbb{R}^n). \tag{16}$$

**Proof:** Since $A_q \subset A_s$ the operator $J_\alpha = c^{-1}\mathcal{F}^{-1}|\xi|^\alpha(1+|\xi|^2)^{-\frac{\alpha}{2}}\mathcal{F}$ extends by the multiplier theorem 2.1 to a linear bounded operator from $L^s_\omega(\mathbb{R}^n)$ to $L^s_\omega(\mathbb{R}^n)$ . Hence for $f \in \mathcal{M}$ it follows by Theorem 3.1 that

$$\|f\|_{s,\omega} = \|J_\alpha I_\alpha \Lambda^\alpha f\|_{s,\omega} \le C\,\|I_\alpha\,\Lambda^\alpha f\|_{s,\omega} \le C\,\|\Lambda^\alpha f\|_{q,\omega} = C\,\|f\|_{H^{\alpha,q}_\omega}.$$

Since by Lemma 3.2 the space $\mathcal{M}$ is dense in $H^{\alpha,q}_\omega(\mathbb{R}^n)$, we get the embedding $H^{\alpha,q}_\omega(\mathbb{R}^n) \hookrightarrow L^s_\omega(\mathbb{R}^n)$. Since $\beta \ge \alpha$ the embedding (21) is proved. □

**Remark:** The assumption (15) is satisfied if $\omega \in A_q$ and

$$|Q|^{\frac{\alpha}{n}}\,\omega(Q)^{\frac{1}{s}-\frac{1}{q}} \le C \tag{17}$$

for all cubes $Q \subset \mathbb{R}^n$ with a constant $C > 0$ independent of $Q$. In fact,

$$|Q|^{\frac{\alpha}{n}-1}\,\Big(\int_Q \omega\Big)^{\frac{1}{s}}\Big(\int_Q \omega^{-\frac{1}{q-1}}\Big)^{\frac{1}{q'}} = |Q|^{\frac{\alpha}{n}}\,\omega(Q)^{\frac{1}{s}-\frac{1}{q}}\,|Q|^{-1}\omega(Q)^{\frac{1}{q}}\omega^{-\frac{1}{q-1}}(Q)^{\frac{1}{q'}} \le C\,A_q(\omega)^{\frac{1}{q}}.$$

Since $1 \le |Q|^{-1}\omega(Q)^{\frac{1}{q}}\omega^{-\frac{1}{q-1}}(Q)^{\frac{1}{q'}}$ the condition (17) is also necessary for (15). In the sequel we will work with the condition (17) instead of (15).

## 3.2 Bessel potential spaces on domains

**Definition 3.1** *A domain* $\Omega \subset \mathbb{R}^n$ *is called an extension domain if for all* $m \in \mathbb{N}$ *there exists an extension operator* $E_m$ *such that for all* $1 < q < \infty$, $\omega \in A_q$ *and* $k = 0, 1, \ldots, m$

$$E_m : W^{k,q}_\omega(\Omega) \to W^{k,q}_\omega(\mathbb{R}^n)$$

*is linear and bounded.*

Theorem 2.2 states that every $(\varepsilon, \infty)$-domain - and thus the half space $\mathbb{R}^n_+$ and every bounded Lipschitz domain - is an extension domain.

In the sequel let $1 < q < \infty$, $\omega \in A_q$ and $\Omega \subset \mathbb{R}^n$ be an extension domain. For $\alpha \in \mathbb{R}$ we define

$$H^{\alpha,q}_\omega(\Omega) := \left\{ g\big|_\Omega \ : \ g \in H^{\alpha,q}_\omega(\mathbb{R}^n) \right\}$$

with norm $\|u\|_{H^{\alpha,q}_\omega(\Omega)} = \inf\{\|g\|_{H^{\alpha,q}_\omega} \ : \ g \in H^{\alpha,q}_\omega(\mathbb{R}^n),\ g\big|_\Omega = u\}$. Lemma 3.1 and the extension property of the domain imply that $H^{k,q}_\omega(\Omega) = W^{k,q}_\omega(\Omega)$ for $k \in \mathbb{N}$. We have the following interpolation property:

**Theorem 3.3** *Let $1 < q < \infty$, $\omega \in A_q$ and let $\Omega \subset \mathbb{R}^n$ be an extension damain. Then for $k \in \mathbb{N}$ und $0 \le \alpha \le k$*

$$H^{\alpha,q}_\omega(\Omega) = [L^q_\omega(\Omega), W^{k,q}_\omega(\Omega)]_{\frac{\alpha}{k}}.$$

**Proof:** Since $\Omega$ is an extension domain, there is a linear bounded extension operator

$$E : \ W^{k,q}_\omega(\Omega) \to W^{k,q}_\omega(\mathbb{R}^n) \quad \text{and} \quad E : \ L^q_\omega(\Omega) \to L^q_\omega(\mathbb{R}^n).$$

By complex interpolation and Lemma 3.1 it follows that

$$E : [L^q_\omega(\Omega), W^{k,q}_\omega(\Omega)]_{\frac{\alpha}{k}} \to [L^q_\omega(\mathbb{R}^n), W^{k,q}_\omega(\mathbb{R}^n)]_{\frac{\alpha}{k}} = H^{\alpha,q}_\omega(\mathbb{R}^n)$$

is linear and bounded. This implies the embedding

$$[L^q_\omega(\Omega), W^{k,q}_\omega(\Omega)]_{\frac{\alpha}{k}} \hookrightarrow H^{\alpha,q}_\omega(\Omega). \tag{18}$$

Replacing in the arguments above the extension operator $E$ by the respective restriction operator, we get the embedding (18) in the other direction. □

The reiteration property of complex interpolation (see [21] or [17]) yields:

**Corollary 3.1** *Under the assumptions of the preceding theorem on $q, \omega, \Omega$ the interpolation property*

$$H^{\gamma,q}_\omega(\Omega) = [H^{\alpha,q}_\omega(\Omega), H^{\beta,q}_\omega(\Omega)]_\theta$$

*holds with $\gamma = (1-\theta)\alpha + \theta\beta$ for all $0 \le \alpha \le \beta$ and $0 \le \theta \le 1$.*

Theorem 3.3 implies the existence of a linear bounded extension operator $E : H^{\beta,q}_\omega(\Omega) \to H^{\beta,q}_\omega(\mathbb{R}^n)$ for all $\beta \ge 0$. Therefore Theorem 3.2 yields weighted Sobolev embeddings for extension domains $\Omega$.

**Theorem 3.4** *Let $1 < q < \infty$, $\omega \in A_q$, $0 < \beta < n$ and $1 < q < s < \infty$ and let $\Omega \subset \mathbb{R}^n$ be an extension domain. If there exists an $0 < \alpha \le \beta$ such that*

$$|Q|^{\frac{\alpha}{n}}\, \omega(Q)^{\frac{1}{s}-\frac{1}{q}} \le C \tag{19}$$

*for all cubes $Q \subset \mathbb{R}^n$, then it holds the imdedding*

$$H^{\beta,q}_\omega(\Omega) \hookrightarrow L^s_\omega(\Omega). \tag{20}$$

As we already mentioned the half space $\mathbb{R}^n_+$ and every bounded Lipschitz domains are $(\varepsilon, \infty)$-domains and therefore by Theorem 2.2 extension domains. Furthermore it can be shown that for these domains $\Omega$ it is sufficient to verify the condition (19) on the weight function $\omega$ only for cubes $Q$ contained in a neighbourhood of $\Omega$, if $\Omega$ is bounded, or contained in $\mathbb{R}^n_+$, for $\Omega = \mathbb{R}^n_+$. Since the proof of this fact is elementary but rather lengthy it will be omitted and we refer to [4], proof of Satz 8.8, for details. This yields the following result:

**Corollary 3.2** *Let $1 < q < \infty$, $\omega \in A_q$, $0 < \beta < n$ and $1 < q < s < \infty$.*

*i) Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and let $U$ be a neighbourhood of $\overline{\Omega}$. If there exists an $\alpha \in (0, \beta]$ such that (19) is satisfied for all cubes $Q \subset U$, then it holds the imbedding*

$$H^{\beta,q}_\omega(\Omega) \hookrightarrow L^s_\omega(\Omega). \tag{21}$$

*ii) Let $\Omega = \mathbb{R}^n_+$. Then the imbedding (21) holds if (19) is satisfied for some $\alpha \in (0, \beta]$ and for all cubes $Q \subset \mathbb{R}^n_+$.*

# 4 The Stokes operator

Let $1 < q < \infty$, $\omega \in A_q$ and let $\Omega = \mathbb{R}^n, \mathbb{R}^n_+$ or a bounded $C^{1,1}$-domain. Recall the definition of the Stokes operator $\mathcal{A}_{q,\omega}$ from the introduction.

The results from [6], [7] yield the following properties of the Stokes operator:

**Theorem 4.1** *i) The Stokes operator $\mathcal{A}_{q,\omega} : D(\mathcal{A}_{q,\omega}) \subset L^q_{\omega,\sigma}(\Omega) \longrightarrow L^q_{\omega,\sigma}(\Omega)$ is densely defined and closed.*

*ii) For every $f \in L^q_{\omega,\sigma}(\Omega)$ and $\lambda \in \Sigma_\varepsilon = \{\lambda \in \mathbb{C} \setminus \{0\} : |arg\, \lambda| < \pi - \varepsilon\}$, $0 < \varepsilon < \frac{\pi}{2}$, the resolvent problem*

$$\lambda u + \mathcal{A}_{q,\omega} u = f \tag{22}$$

*has a unique solution $u \in D(\mathcal{A}_{q,\omega})$. There is a constant $C_\varepsilon$ such that*

$$|\lambda| \|u\|_{q,\omega} + \|\mathcal{A}_{q,\omega} u\|_{q,\omega} \leq C_\varepsilon \|f\|_{q,\omega}. \tag{23}$$

*For all $1 < p < \infty$, $v \in A_p$ and $f \in L^q_{\omega,\sigma}(\Omega) \cap L^p_{v,\sigma}(\Omega)$ it holds*

$$(\lambda + \mathcal{A}_{q,\omega})^{-1} f = (\lambda + \mathcal{A}_{p,v})^{-1} f. \tag{24}$$

*For a bounded $C^{1,1}$-domain $\Omega$ the Stokes operator $\mathcal{A}_{q,\omega}$ is boundedly invertible. For $\lambda \in \Sigma_\varepsilon \cup \{0\}$ the Stokes resolvent problem (22) has a unique solution satisfying the estimate*

$$|\lambda| \|u\|_{q,\omega} + \|u\|_{W^{2,q}_\omega(\Omega)} \leq C_\varepsilon \|f\|_{q,\omega}. \tag{25}$$

*iii) For* $\Omega = \mathbb{R}^n$ *or* $\mathbb{R}^n_+$

$$\|\nabla^2 u\|_{q,\omega} \le C\,\|\mathcal{A}_{q,\omega} u\|_{q,\omega}, \tag{26}$$
$$\|u\|_{W^{2,q}_\omega(\Omega)} \le C\,\|(I+\mathcal{A}_{q,\omega})u\|_{q,\omega} \tag{27}$$

*for all* $u \in D(\mathcal{A}_{q,\omega})$.

*iv) For* $\Omega = \mathbb{R}^n$, $\mathbb{R}^n_+$ *and a bounded* $C^{1,1}$*-domain the Stokes operator* $-\mathcal{A}_{q,\omega}$ *generates a bounded analytic semigroup in* $L^q_{\omega,\sigma}(\Omega)$.

For $\Omega = \mathbb{R}^n$ or $\mathbb{R}^n_+$ it holds $0 \notin \rho(\mathcal{A}_{q,\omega})$, but $0 \in \rho(I+\mathcal{A}_{q,\omega})$ and $-(\mathcal{A}_{q,\omega}+I)$ also generates a bounded analytic semigroup $\{e^{-t\,(I+\mathcal{A}_{q,\omega})}\}$. Hence the fractional powers $(I+\mathcal{A}_{q,\omega})^\alpha$, $\alpha \in \mathbb{R}$, can be defined in the usual way (see e.g. [16]).

**Definition 4.1** *Let* $\Omega = \mathbb{R}^n, \mathbb{R}^n_+$ *or a bounded* $C^{1,1}$*-domain and* $\alpha > 0$. *Then define*

$$D^{\alpha,q}_\omega(\Omega) := D((I+\mathcal{A}_{q,\omega})^\alpha)$$

*equipped with the norm* $\|u\|_{D^{\alpha,q}_\omega(\Omega)} := \|(I+\mathcal{A}_{q,\omega})^\alpha u\|_{q,\omega}$. *Let* $D^{-\alpha,q}_\omega(\Omega) := [D^{\alpha,q'}_{\omega'}(\Omega)]'$ *and* $D^{0,q}_\omega(\Omega) := L^q_{\omega,\sigma}(\Omega)$.

Since $L^q_{\omega,\sigma}(\Omega)$ - as a closed subspace of $L^q_\omega(\Omega)^n$ - is a reflexive Banach space of $L^q_\omega(\Omega)^n$ and since for $\alpha \ge 0$

$$(I+\mathcal{A}_{q,\omega})^{-\alpha} : L^q_{\omega,\sigma}(\Omega) \longrightarrow D^{\alpha,q}_\omega(\Omega) \tag{28}$$

is an isometric isomorphism $D^{\alpha,q}_\omega(\Omega)$ is a reflexive Banach space.

By [16] Theorem 2.7 the intersection $\bigcap_{n=1}^\infty D^{n,q}_\omega(\Omega)$ is dense in $L^q_{\omega,\sigma}(\Omega)$. Then the isometric isomorphism (28) implies that $D^{\beta,q}_\omega(\Omega)$ is dense in $D^{\alpha,q}_\omega(\Omega)$ for $\beta > \alpha \ge 0$. In the sequel we will show (see Lemma 4.1) that $L^q_{\omega,\sigma}(\Omega)$ is dense in $D^{\alpha,q}_\omega(\Omega)$ for $\alpha < 0$; this implies the density of $D^{\beta,q}_\omega(\Omega)$ in $D^{\alpha,q}_\omega(\Omega)$ for arbitrary $\beta > \alpha$.

Since for a bounded $C^{1,1}$-domain $\Omega \subset \mathbb{R}^n$ the Stokes operator $\mathcal{A}_{q,\omega}$ is invertible, the spaces $D^{\alpha,q}_\omega(\Omega)$ and $D(\mathcal{A}^\alpha_{q,\omega})$ coincide for $0 \le \alpha \le 1$ in this case with equivalent norms $\|\cdot\|_{D^{\alpha,q}_\omega(\Omega)}$ and $\|\mathcal{A}^\alpha_{q,\omega}\cdot\|_{q,\omega}$ (see e.g. [15]).

**Lemma 4.1** *Let* $\Omega = \mathbb{R}^n, \mathbb{R}^n_+$ *or a bounded* $C^{1,1}-$*domain and* $\alpha > 0$. *Then* $D^{-\alpha,q}_\omega(\Omega)$ *is isomorphic to the completion of the space* $L^q_{\omega,\sigma}(\Omega)$ *with respect to the norm* $\|(I+\mathcal{A}_{q,\omega})^{-\alpha}\cdot\|_{q,\omega}$.

*Let* $0 \le \alpha \le 1$ *and let* $\Omega$ *a bounded* $C^{1,1}$*-domain. Then the norms* $\|\mathcal{A}^{-\alpha}_{q,\omega}\cdot\|_{q,\omega}$ *and* $\|(I+\mathcal{A}_{q,\omega})^{-\alpha}\cdot\|_{q,\omega}$ *are equivalent on* $L^q_{\omega,\sigma}(\Omega)$.

**Proof:** From Theorem 4.1 ii) and the fact that $(L^q_\omega(\Omega))' = L^{q'}_{\omega'}(\Omega)$ it follows by standard arguments that $(\mathcal{A}_{q,\omega})' = \mathcal{A}_{q',\omega'}$ and $[(\mathcal{A}_{q,\omega}+\lambda)^{-1}]' = (\mathcal{A}_{q',\omega'}+\lambda)^{-1}$ for $\lambda \in \mathbb{C}\setminus\mathbb{R}_-$. This implies $[(I+\mathcal{A}_{q',\omega'})^{-\alpha}]' = (I+\mathcal{A}_{q,\omega})^{-\alpha}$ by the definition of fractional powers (see [16]).

The last assertion follows from the remark above that the spaces $D^{\alpha,q}_\omega(\Omega)$ and $D(\mathcal{A}^\alpha_{q,\omega})$ coincide with equivalent norms $\|\cdot\|_{D^{\alpha,q}_\omega(\Omega)} \sim \|\mathcal{A}^\alpha_{q,\omega}\cdot\|_{q,\omega}$ for $0 \le \alpha \le 1$ if $\Omega$ is a bounded $C^{1,1}$-domain. $\square$

Using the relation $e^{-t\,\mathcal{A}_{q,\omega}} = e^t\,e^{-t\,(I+\mathcal{A}_{q,\omega})}$, $t \ge 0$, and the well known relations between the semigroup $e^{-t\,(I+\mathcal{A}_{q,\omega})}$ and the fractional powers $(I+\mathcal{A}_{q,\omega})^\alpha$ (see e.g. [16]) we obtain the following results:

**Lemma 4.2** *Let $\Omega = \mathbb{R}^n$, $\mathbb{R}^n_+$ or a bounded $C^{1,1}$-domain.*

*i) Let $\alpha > 0$. Then the Stokes semigroup $\{e^{-t\mathcal{A}_{q,\omega}}, t > 0\}$ extends uniquely to a bounded $C_0$-semigroup on $D^{-\alpha,q}_\omega(\Omega)$ - again denoted by $\{e^{-t\mathcal{A}_{q,\omega}}, t > 0\}$.*

*ii) For all $-\infty < \alpha < \beta < \infty$ and $T > 0$ there is a constant $C_T$ such that*

$$\|e^{-t\mathcal{A}_{q,\omega}}u\|_{D^{\beta,q}_\omega(\Omega)} \le C_T\,t^{\alpha-\beta}\|u\|_{D^{\alpha,q}_\omega(\Omega)} \quad \forall u \in D^{\alpha,q}_\omega(\Omega)$$

*for all $t \in (0,T]$. If $\Omega$ is a bounded $C^{1,1}$-domain, $C_T$ is independent of $T > 0$.*

*iii) For $\beta > \alpha$ and $u \in D^{\alpha,q}_\omega(\Omega)$*

$$\lim_{t\to 0} t^{\beta-\alpha}\|e^{-t\mathcal{A}_{q,\omega}}u\|_{D^{\beta,q}_\omega(\Omega)} = 0.$$

**Proof:** i) For $u \in L^q_{\omega,\sigma}(\Omega)$ we get by Lemma 4.1

$$\begin{aligned}&\|e^{-t\mathcal{A}_{q,\omega}}u\|_{D^{-\alpha,q}_\omega(\Omega)} = \|(I+\mathcal{A}_{q,\omega})^{-\alpha}e^{-t\mathcal{A}_{q,\omega}}u\|_{q,\omega}\\ &= \|e^{-t\mathcal{A}_{q,\omega}}(I+\mathcal{A}_{q,\omega})^{-\alpha}u\|_{q,\omega} \le C\|(I+\mathcal{A}_{q,\omega})^{-\alpha}u\|_{q,\omega} = C\|u\|_{D^{-\alpha,q}_\omega(\Omega)}.\end{aligned}$$

Then the density of $L^q_{\omega,\sigma}(\Omega)$ in $D^{-\alpha,q}_\omega(\Omega)$ yields i).

ii) For $u \in D^{\beta,q}_\omega(\Omega) \cap L^q_{\omega,\sigma}(\Omega)$ Lemma 4.1, Theorem 6.13 in [16] and the relation $e^{-t\,\mathcal{A}_{q,\omega}} = e^t\,e^{-t\,(I+\mathcal{A}_{q,\omega})}$ imply

$$\begin{aligned}&\|e^{-t\mathcal{A}_{q,\omega}}u\|_{D^{\beta,q}_\omega(\Omega)} = \|(I+\mathcal{A}_{q,\omega})^{\beta}e^{-t\mathcal{A}_{q,\omega}}u\|_{q,\omega}\\ &= \|(I+\mathcal{A}_{q,\omega})^{\beta-\alpha}e^{-t\mathcal{A}_{q,\omega}}(I+\mathcal{A}_{q,\omega})^{\alpha}u\|_{q,\omega} \le C_T\,t^{\alpha-\beta}\|u\|_{D^{\alpha,q}_\omega(\Omega)}\end{aligned}$$

for all $t \in (0,T]$. Since for a bounded $C^{1,1}$-domain $\Omega$ in the above calculation $(I+\mathcal{A}_{q,\omega})^{\beta-\alpha}$ can be replaced by $\mathcal{A}^{\beta-\alpha}_{q,\omega}$, the constant $C_T$ is independent of $T > 0$ in this case. The density of $D^{\beta,q}_\omega(\Omega) \cap L^q_{\omega,\sigma}(\Omega)$ in $D^{\alpha,q}_\omega(\Omega)$ completes the proof.

iii) Let $\varepsilon > 0$ be given. Since $D^{\beta,q}_\omega(\Omega) \cap L^q_{\omega,\sigma}(\Omega)$ is dense in $D^{\alpha,q}_\omega(\Omega)$ there is a $v \in D^{\beta,q}_\omega(\Omega) \cap L^q_{\omega,\sigma}(\Omega)$ such that $\|u-v\|_{D^{\alpha,q}_\omega} < \varepsilon$. From part i) and ii) of this lemma it follows that

$$\begin{aligned}t^{\beta-\alpha}\|e^{-t\mathcal{A}_{q,\omega}}u\|_{D^{\beta,q}_\omega} &\le t^{\beta-\alpha}\|e^{-t\mathcal{A}_{q,\omega}}v\|_{D^{\beta,q}_\omega} + t^{\beta-\alpha}\|e^{-t\mathcal{A}_{q,\omega}}(u-v)\|_{D^{\beta,q}_\omega}\\ &\le C_T(\,t^{\beta-\alpha}\|v\|_{D^{\beta,q}_\omega} + \varepsilon),\end{aligned}$$

for $0 < t \le T$. Letting $t \to 0$ and noting that $\varepsilon > 0$ was arbitrarily given yields the assertion. $\square$

**Lemma 4.3** *Let $0 \le \alpha \le 1$ and let $\Omega = \mathbb{R}^n$, $\mathbb{R}^n_+$ be a bounded $C^{1,1}$-domain. Then the smooth functions $C^\infty(\Omega)^n \cap D(\mathcal{A}_{q,\omega})$ are dense in $D^{\alpha,q}_\omega(\Omega)$.*

**Proof:** Given $f \in C^\infty_{0,\sigma}(\Omega)$ and $1 < r < \infty$ standard regularity theory of the stationary Stokes equation ([10], Chapter IV, Theorem 4.2) yields the existence of a solution $u \in \bigcap_{k=1}^\infty W^{k,r}_{loc}(\Omega)^n \subset C^\infty(\Omega)^n$ of the equation $\mathcal{A}_r u = -u + f$. This means $(I + \mathcal{A}_r)^{-1}(C^\infty_{0,\sigma}(\Omega)) \subset C^\infty(\Omega)^n$. Then (24) yields $(I + \mathcal{A}_{q,\omega})^{-1}(C^\infty_{0,\sigma}(\Omega)) \subset C^\infty(\Omega)^n$.

Since $C^\infty_{0,\sigma}(\Omega)$ is dense in $L^q_{\omega,\sigma}(\Omega)$ and $(I + \mathcal{A}_{q,\omega})^{-1} : L^q_{\omega,\sigma}(\Omega) \longrightarrow D(\mathcal{A}_{q,\omega})$ is a topological isomorphism, $(I + \mathcal{A}_{q,\omega})^{-1}(C^\infty_{0,\sigma}(\Omega))$ is dense in $D(\mathcal{A}_{q,\omega})$ and therefore in $D^{\alpha,q}_\omega(\Omega)$. □

Next we use the complex interpolation method ([17], [21]) to compare the spaces $D^{\alpha,q}_\omega(\Omega)$ to the spaces $H^{s,q}_\omega(\Omega)$. The following properties of the domains of fractional powers are needed:

**Lemma 4.4** *Let $X$ be a Banach space, $-A : D(A) \subset X \to X$ a generator of a bounded analytic semigroup on $X$ and $\theta \in (0,1)$. Let $D(A^\theta)$ for $0 < \theta < 1$ be equipped with the graph norm. Then the continuous embeddings*

$$[D(A^\alpha), D(A^\beta)]_{\overline{\theta}} \hookrightarrow D(A^\gamma) \hookrightarrow [D(A^\alpha), D(A^\beta)]_{\underline{\theta}}$$

*hold for $0 \le \alpha \le \beta \le 1$, $0 \le \underline{\theta} < \theta < \overline{\theta} \le 1$ and $\gamma = \theta\beta + (1-\theta)\alpha$.*

**Proof:** In [3], Theorem 6.16, the imbeddings $[X, D(A)]_{\overline{\theta}} \hookrightarrow D(A^\theta)$ and $D(A^\theta) \hookrightarrow [X, D(A)]_{\underline{\theta}}$ are proven for all $0 \le \underline{\theta} < \theta < \overline{\theta} \le 1$. Since $-A^{\beta-\alpha}$ also generates a bounded analytic semigroup on $X$ (see [15], Section 10), we may apply this result to obtain $[X, D(A^{\beta-\alpha})]_{\overline{\theta}} \hookrightarrow D(A^{\gamma-\alpha})$. Since $A^\alpha$ is an isomorphism from $[D(A^\alpha), D(A^\beta)]_{\overline{\theta}}$ to $[X, D(A^{\beta-\alpha})]_{\overline{\theta}}$ and from $D(A^\gamma)$ to $D(A^{\gamma-\alpha})$, we get $[D(A^\alpha), D(A^\beta)]_{\overline{\theta}} \hookrightarrow D(A^\gamma)$. The second embedding is proved analogously. □

**Lemma 4.5** *For every $0 \le \alpha \le 1$, $0 \le \theta \le 1$ and every $\varepsilon > 0$ it holds*

*(1)* $D^{\alpha+\varepsilon,q}_\omega(\Omega) \hookrightarrow H^{2\alpha}_\omega(\Omega)$,

*(2)* $[L^q_{\omega,\sigma}(\Omega), D^{-\alpha,q}_\omega(\Omega)]_\theta \hookrightarrow D^{-\theta\alpha-\varepsilon,q}_\omega(\Omega)$.

**Proof:** (1) By Lemma 4.4 and Theorem 3.3

$$D^{\alpha+\varepsilon,q}_\omega(\Omega) \hookrightarrow [L^q_{\omega,\sigma}(\Omega), D(\mathcal{A}_{q,\omega})]_\alpha \hookrightarrow [L^q_\omega(\Omega), H^{2,q}_\omega(\Omega)]_\alpha = H^{2\alpha,q}_\omega(\Omega).$$

(2) Corollary 4.4 yields $D^{\theta\alpha+\varepsilon,q'}_{\omega'}(\Omega) \hookrightarrow [L^{q'}_{\omega',\sigma}(\Omega), D^{\alpha,q'}_{\omega'}(\Omega)]_\theta$ by duality. □

# 5 Estimates of the nonlinear term

Let $1 < q < \infty$, $\omega \in A_q$ and let $\Omega \subset \mathbb{R}^n$ be equal to $\mathbb{R}^n_+$ or to a bounded domain with $\partial\Omega \in C^{1,1}$. In this situation $P = P_{q,\omega}$ denotes the corresponding Helmholtz projection and $A = \mathcal{A}_{q,\omega}$ denotes the corresponding Stokes operator.

**Lemma 5.1** *For $1 \le j \le n$ and $\alpha > \frac{1}{2}$ there is a constant $C \in \mathbb{R}_+$ such that*

$$\|P_{q,\omega}\partial_j u\|_{D_\omega^{-\alpha,q}} \le C\|u\|_{q,\omega}$$

*for all $u \in C_0^\infty(\Omega)^n$, i.e., the operator $P\partial_j$ defined on $C_0^\infty(\Omega)^n$ extends to a bounded linear operator from $L^q_\omega(\Omega)^n$ to $D_\omega^{-\alpha,q}(\Omega)$.*

**Proof:** Since $\alpha > \frac{1}{2}$, Lemma 4.5 (1) implies that $\partial_j \tau(I + \mathcal{A}_{q',\omega'})^{-\alpha} : L^{q'}_{\omega',\sigma}(\Omega) \hookrightarrow L^{q'}_{\omega'}(\Omega)^n$ is bounded, where $\tau$ denotes the injection $L^{q'}_{\omega',\sigma}(\Omega) \subset L^{q'}_{\omega'}(\Omega)^n$. Since the adjoint operator of $\tau$ is $P_{q,\omega}$ the result follows from duality. □

We define for sufficiently regular vector fields $u$ and $v$ on $\Omega$

$$F(u,v) := P(u \cdot \nabla v) = P\Big(\sum_{j=1}^n u_j\, \partial_j v\Big).$$

**Theorem 5.1** *Let $0 < \beta < \beta_0$, $0 < \gamma < \gamma_0$ and assume there exists a constant $C > 0$ such that*

$$|Q|^{2(\gamma+\beta)\frac{q}{n}} \le C\,\omega(Q) \tag{29}$$

*for all cubes $Q \subset U$, where $U = \mathbb{R}^n_+$ if $\Omega = \mathbb{R}^n_+$ and $U$ is a neighbourhood of $\overline{\Omega}$ if $\Omega$ is a bounded $C^{1,1}$-domain. Then for all $\delta \in [0, \frac{1}{2}]$*

$$F : D_\omega^{\beta_0,q}(\Omega) \times D_\omega^{\gamma_0-\delta+\frac{1}{2},q}(\Omega) \to D_\omega^{-\delta,q}(\Omega)$$

*is bounded as a bilinear map.*

**Proof:** In this proof $C > 0$ denotes a generic constant. For $\frac{1}{q} = \frac{1}{s} + \frac{1}{r}$ and all $\omega \in A_q$ satisfying

$$|Q|^{\frac{2\beta}{n}}\omega(Q)^{\frac{1}{r}-\frac{1}{q}} = |Q|^{\frac{2\beta}{n}}\omega(Q)^{-\frac{1}{s}} \le C \quad \forall Q \subset U \tag{30}$$

$$|Q|^{\frac{2\gamma}{n}}\omega(Q)^{\frac{1}{s}-\frac{1}{q}} \le C \quad \forall Q \subset U \tag{31}$$

it follows from Corollary 3.2, Lemma 4.5 i) and the boundedness of $P$ in $L^q_\omega(\Omega)^n$ that

$$\begin{aligned}\|P(u\cdot\nabla u)\|_{q,\omega} &\le \|u\|_{r,\omega}\|\nabla v\|_{s,\omega} \\ &\le C\|u\|_{H_\omega^{2\beta,q}}\|\nabla v\|_{H_\omega^{2\gamma,q}} \le C\|u\|_{D_\omega^{\beta_0,q}}\|v\|_{D_\omega^{\bar\gamma+\frac{1}{2},q}}\end{aligned} \tag{32}$$

for $\gamma < \bar\gamma < \gamma_0$.

Choosing $s = \frac{(\gamma+\beta)q}{\beta}$ we get $-\frac{\gamma}{\beta}\frac{1}{s} = \frac{1}{s} - \frac{1}{q}$. Hence both (30) and (31) are equivalent to (29) for this choice of $s$.

Since $F$ is bilinear, it is sufficient to show the boundedness of $F$ on a dense subspace. By Lemma 4.3 the space $D(A) \cap C^\infty(\Omega)^n$ is dense in $D_\omega^{\alpha,q}(\Omega)$ for all

$\alpha \geq 0$. Hence we can assume additionally that $u, v \in C^\infty(\Omega)$. Then $\operatorname{div} u = 0$ implies $u \cdot \nabla v = \sum_{j=1}^n \partial_j(u_j v)$.

Under the same assumptions on $s, r$ and $\omega$ it follows from Lemma 5.1 for arbitrary $\varepsilon > 0$ that

$$\begin{aligned}
\|P(u\cdot\nabla v)\|_{D_\omega^{-\frac{1}{2}-\varepsilon,q}} &\leq \sum_{j=1}^n \|P\partial_j(u_j v)\|_{D_\omega^{-\frac{1}{2}-\varepsilon,q}} \\
&\leq C\| \, |u||v| \, \|_{q,\omega} \leq C\|u\|_{r,\omega}\|v\|_{s,\omega} \leq C\,\|u\|_{H_\omega^{2\beta,q}}\|v\|_{H_\omega^{2\gamma,q}} \\
&\leq C\,\|u\|_{D_\omega^{\beta_0,q}}\|v\|_{D_\omega^{\bar\gamma,q}}.
\end{aligned}$$

Hence for fixed $u \in D_\omega^{\beta_0,q}(\Omega)$ the operator $F_u : v \mapsto F(u,v)$ is linear and bounded from $D_\omega^{\bar\gamma+\frac{1}{2},q}(\Omega)$ to $L^q_{\omega,\sigma}(\Omega)$ and from $D_\omega^{\bar\gamma,q}(\Omega)$ to $D_\omega^{-\frac{1}{2}-\varepsilon,q}(\Omega)$. Interplotaion yields that

$$F_u : [D_\omega^{\bar\gamma+\frac{1}{2},q}(\Omega), D_\omega^{\bar\gamma,q}(\Omega)]_\theta \longrightarrow [L^q_{\omega,\sigma}(\Omega), D_\omega^{-\frac{1}{2}-\varepsilon,q}(\Omega)]_\theta$$

is linear and bounded with norm less or equal to $C\|u\|_{D_\omega^{\beta_0,q}}$. Lemma 4.4 implies for $0 \leq \delta \leq \frac{1}{2}$ the embeddings

$$D_\omega^{\gamma_0+\frac{1}{2}-\delta,q}(\Omega) \hookrightarrow [D_\omega^{\bar\gamma+\frac{1}{2},q}(\Omega), D_\omega^{\bar\gamma,q}(\Omega)]_\theta, \tag{33}$$

$$[L^q_{\omega,\sigma}(\Omega), D_\omega^{-\frac{1}{2}-\varepsilon,q}(\Omega)]_\theta \hookrightarrow D_\omega^{-\delta,q}(\Omega), \tag{34}$$

if $\bar\gamma - \frac{\theta}{2} < \gamma_0 - \delta$ and $\frac{\theta}{2} + \varepsilon\theta < \delta$, or if $\delta = \theta = 0$. For $\delta \in (0, \frac{1}{2}]$ these conditions are satisfied by some $\theta \in [0,1]$ if $\varepsilon > 0$ is small enough. Then $F_u : D_\omega^{\gamma_0+\frac{1}{2}-\delta,q}(\Omega) \to D_\omega^{-\delta,q}(\Omega)$ is bounded with norm $\leq C\,\|u\|_{D_\omega^{\beta_0,q}}$. Since $F(u,v) = F_u(v)$, the proof is complete. □

# 6 Abstract existence theorem

In this section we follow [22] the abstract existence theorem in [22]. However we introduce an additional Banach space and an inhomogeneity $f$. This implies a more general local existence result for strong solutions of the Navier-Stokes equations.

**Theorem 6.1** *Let $W, X, Y, Z$ and $G$ be Banach spaces, which are imbedded into a common topological vector space. Furthermore let $\{e^{tA}\}_{t\geq 0}$ be a $C_0$-semigroup on $X$ satisfying the following assumptions:*

*(I) For every $t > 0$ the operator $e^{tA}$ extends to a linear bounded operator from $W$ to $X$. There exists some $a \in (0,1)$ and positive constants $C$ and $T$ such that*

$$|e^{tA}u|_X \leq C\,t^{-a}|u|_W$$

*for all $u \in W$ und $t \in (0,T]$.*

(II) *For every $t > 0$ it holds $e^{tA} \in \mathcal{L}(X,Y) \cap \mathcal{L}(X,Z) \cap L(X,G)$. There exist $b > 0$, $c > 0$, $d \in (a,1)$ and positive constants $C$ and $T$ such that*

$$|e^{tA}u|_Y \le C\,t^{-b}|u|_X \tag{35}$$

$$|e^{tA}u|_Z \le C\,t^{-c}|u|_X \tag{36}$$

$$|e^{tA}u|_G \le C\,t^{-(d-a)}|u|_X \tag{37}$$

*for all $u \in X$ and $t \in (0,T]$. Furthermore $e^{tA}u \in C((0,T],Y) \cap C((0,T],Z) \cap C((0,T],G)$ and*

$$\lim_{t\to 0} t^b\,|e^{tA}u|_Y = \lim_{t\to 0} t^c\,|e^{tA}u|_Z = \lim_{t\to 0} t^{d-a}\,|e^{tA}u|_G = 0.$$

*for all $u \in X$.*

*Additionally assume that $a + b + c \le 1$.*

*Let*

$$F : Y \times Z \to W$$

*be a bilinear bounded mapping and $F(u) := F(u,u)$.*

*Then for every $u_0 \in X$ and $f \in C((0,T],W)$ for some $T > 0$ with $|f(t)|_W = o(t^{-b-c})$ for $t \to 0$ there exists some $T^* > 0$ and a unique curve $u : [0,T^*] \to X$ with the properties:*

a) *$u : [0,T^*] \to X$ is continuous and $u(0) = u_0$.*

b) *$u \in C((0,T^*],Y) \cap C((0,T^*],Z) \cap C((0,T^*],G)$ and*

$$\lim_{t\to 0} t^{d-a}|u(t)|_G = \lim_{t\to 0} t^b|u(t)|_Y = \lim_{t\to 0} t^c|u(t)|_Z = 0.$$

c) *$u(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A}(\,Fu(s) + f(s)\,)ds$ for all $t \in [0,T^*]$.*

*Furthermore $T^* > 0$ can be chosen independly of the choice of $G$.*

**Proof:** Let $u_0 \in X$ and $\alpha, \beta, T > 0$ be such that

$$|e^{tA}u_0|_X \le \alpha,\; t^b\,|e^{tA}u_0|_Y \le \beta,\; t^c\,|e^{tA}u_0|_Z \le \beta \tag{38}$$

for all $t \in (0,T]$. For $T \to 0$ the value $\beta > 0$ can be chosen arbitrarily small.

For $\alpha$, $\beta$ and $T$ satisfying (38) let $M = M(\alpha,\beta,T)$ be the set of all mappings $u : (0,T] \to X$ satisfying

$$u \in C((0,T],X) \cap C((0,T],Y) \cap C((0,T],Z) \cap C((0,T],G),$$
$$|u(t)|_X \le 2\alpha,\; t^b|u(t)|_Y \le 2\beta,\; t^c|u(t)|_Z \le 2\beta,\; \sup_{t\in(0,T]} t^{d-a}|u(t)|_G < \infty$$

for all $t \in (0,T]$. Let $\gamma := \sup_{t\in(0,T]} t^d \, |e^{tA}|_{L(W,G)}$. Then $M(\alpha,\beta,T)$ equipped with the metric

$$d(u,v) = \sup_{t\in(0,T]} (\gamma \, |u(t)-v(t)|_X + \gamma \, t^b |u(t)-v(t)|_Y + \gamma \, t^c |u(t)-v(t)|_Z + t^{d-a}|u(t)-v(t)|_G)$$

is a nonempty complete metric space. For $u \in M$ define the operator

$$\mathcal{F}u(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A}(\, Fu(s) + f(s)\,)ds.$$

In order to apply Banach's fixed point theorem we show that $\mathcal{F} : M \to M$ is a $q-$contraction with $q < 1$ if $T > 0$ is sufficiently small.

To show that $\mathcal{F}$ maps $M$ into itself, we only prove that $t^b|\mathcal{F}u(t)|_Y \le 2\beta$ for $\beta > 0$ and $T > 0$ sufficiently small. The remaining estimates are proved analogously:

$$\begin{aligned}
&t^b|\mathcal{F}u(t)|_Y \\
&\le t^b \int_0^t |e^{(t-s)A}|_{L(W,Y)}|Fu(s)|_W \, ds \\
&\quad + t^b \int_0^t |e^{(t-s)A}|_{L(W,Y)}|f(s)|_W \, ds + t^b|e^{tA}u_0|_Y \\
&\le C\, t^b \int_0^t (t-s)^{-b-a}|u(s)|_Y|u(s)|_Z \, ds + c(t)\, t^b \int_0^t (t-s)^{-b-a}s^{-b-c}ds + \beta \\
&\le (\, 4\, C\, \beta^2 + c(t)\,)\, t^b \int_0^t (t-s)^{-b-a}s^{-b-c}ds + \beta \\
&= \tilde{C}\,(\, \beta^2 + c(t)\,)\, t^{1-(a+b+c)} + \beta \le \; 2\beta,
\end{aligned}$$

where $\lim_{t\to 0} c(t) = 0$ and the last $\le$-sign holds for $t \in (0,T]$, if $T > 0$ and $\beta > 0$ are sufficiently small.

We show that for $T$ small enough $\mathcal{F}$ is a $q$-contraction with $q < 1$, i.e.,

$$d(\mathcal{F}u, \mathcal{F}v) \le q\, d(u,v) \quad \forall\, u,v \in M.$$

We prove only $t^{d-a}|\mathcal{F}u(t) - \mathcal{F}v(t)|_G \le q\, d(u,v)$, where $q < 1$ for $t \in (0,T]$ and $T > 0$ small enough. The remaining estimates are proved analogously.

$$\begin{aligned}
&t^{d-a}\, |\mathcal{F}u(t) - \mathcal{F}v(t)|_G \\
&\le t^{d-a} \int_0^t |e^{(t-s)A}|_{L(W,G)}|Fu(s) - Fv(s)|_W \, ds \\
&\le C\, t^{d-a} \int_0^t (t-s)^{-d}(|u(s)|_Y\, \gamma\, |u(s)-v(s)|_Z + |v(s)|_Z\, \gamma\, |u(s)-v(s)|_Y)\, ds \\
&\le 2\, C\, \beta\, t^{d-a} \int_0^t (t-s)^{-d}s^{-b-c}d(u,v)\, ds \\
&= \tilde{C}\, \beta\, t^{1-(a+b+c)}\, d(u,v) \le q\, d(u,v)
\end{aligned}$$

for $t \in (0,T]$ with some $q < 1$, if $T > 0$ and $\beta > 0$ are sufficiently small. Note that $\gamma$ may depend on the choice of the space $G$, but $\tilde{C} > 0$ is independent of $G$. Hence $T > 0$ can be chosen independently of the choice of $G$.

Choosing for $T$ and for $\beta$ small values $T^* > 0$ and $\beta^* > 0$, Banach's fixed point theorem implies the existence of a unique fixed point $u \in M(\alpha, \beta^*, T^*)$ of $\mathcal{F}$, i.e., c) is proved. In a) and b) it remains to investigate the behaviour of $u$ for $t \to 0$:

Let $\beta_1 \leq \beta^*$ and $T_1 \leq T^*$ be such that (38) is satisfied, and let $u_1$ and $u$ be the fixed points of $\mathcal{F}$ in $M(\alpha, \beta_1, T_1)$ and in $M(\alpha, \beta^*, T^*)$ respectively. Since $M(\alpha, \beta_1, T_1) \subset M(\alpha, \beta^*, T_1)$ the uniqueness of $u$ implies $u_1 = u\big|_{(0,T_1]}$.

We show a). Due to the assumptions

$$
\begin{aligned}
&|u(t) - u_0|_X \\
&\leq \int_0^t |e^{(t-s)A}|_{L(W,X)}(|Fu(s)|_W + |f(s)|_W)\, ds + |e^{tA}u_0 - u_0|_X \\
&\leq \tilde{C}\,(\,\beta_1^2 + c(t))t^{1-(a+b+c)} + |e^{tA}u_0 - u_0|_X
\end{aligned}
$$

for all $t \in (0, T_1]$ with $c(t) \to 0$ for $t \to 0$. Since $T_1 > 0$ and $\beta_1 > 0$ can be chosen arbitrarily small it follows $u(t) \to u_0$ for $t \to 0$ in $X$.

To show $\lim_{t\to 0} t^{d-a}|u(t)|_G = 0$ in b) note that

$$
\begin{aligned}
t^{d-a}|u(t)|_G &\leq t^{d-a}\int_0^t |e^{(t-s)A}|_{L(W,G)}(|Fu(s)|_W + |f(s)|_W)\, ds + t^{d-a}|e^{tA}u_0|_G \\
&\leq C\,\gamma\, t^{d-a}\int_0^t (t-s)^{-d}|u(s)|_Y|u(s)|_Z\, ds \\
&\quad + c(t)\,\gamma\, t^{d-a}\int_0^t (t-s)^{-d}s^{-b-c}ds + t^{d-a}|e^{tA}u_0|_G \\
&\leq (\,4\,C\,\beta_1^2 + c(t)\,)\,\gamma\, t^{d-a}\int_0^t (t-s)^{-d}s^{-b-c}ds + t^{d-a}|e^{tA}u_0|_G \\
&\leq \tilde{C}\,\gamma\,(\,\beta_1^2 + c(t))t^{1-(a+b+c)} + t^{d-a}|e^{tA}u_0|_G.
\end{aligned}
$$

for $t \in (0, T_1]$. Note that due to the assumptions $t^{d-a}|e^{tA}u_0|_G \to 0$ for $t \to 0$ and $\beta_1$ can be chosen arbitrarily small by choosing $T_1$ small. Moreover $c(t) \to 0$ for $t \to 0$. The remaining estimates are analogous and thus the claim follows b). □

An analysis of the proof of Theorem 6.1 shows that Banach's fixed point theorem is applicable if $\beta > 0$ and $\sup_{0<t<T^*} c(t)$ are sufficiently small. This was achieved by choosing $T^* > 0$ sufficiently small. The following corollary is based on the fact that the smallness of $\beta > 0$ and $\sup_{0<t<T^*} c(t)$ can also be guaranteed by choosing some norms of $u_0$ and $f$ small enough.

**Corollary 6.1** *i) Additionally to the assumptions of the preceding theorem let the semigroup $\{e^{tA}\}_{t\geq 0}$ be uniformly bounded in $L(Y) \cap L(Z)$ and let $R > 0$. Then $T^* > 0$ does not depend on the choice of an initial value $u_0 \in Y \cap Z$ with $\max\{|u_0|_Y, |u_0|_Z\} \leq R$ and on $f \in C([0,T], W)$ with $\|f\|_{C([0,T],W)} \leq R$.*

*ii) Additionally to the assumptions of Theorem 6.1 let $T = \infty$ and $a+b+c = 1$. Then there are constants $c_1 > 0$ and $c_2 > 0$ such that for all $u_0$ and $f$ satisfying $|u_0|_X \le c_1$ and $\sup_{0<t<\infty} t^{b+c}|f|_W \le c_2$ the solution $u$ from Theorem 6.1 exists on $(0,\infty)$, i.e., a)- c) hold for arbitrary $T^* > 0$.*

**Proof:** i) Since $e^{tA} \in L(Y) \cap L(Z)$ and $\max\{|u_0|_Y, |u_0|_Z\} \le R$,

$$t^b\,|e^{tA}u_0|_Y \le C\,R\,t^b \quad \text{and } t^c\,|e^{tA}u_0|_Z \le C\,R\,t^c.$$

Comparing with (38) we get that $\beta$ can be chosen arbitrarily small and independent of $u_0$ if $T^*$ tends to zero. Furthermore for $|f(t)|_W \le R$, $t \in [0,T]$ the quantity $c(t) = t^{b+c}|f(t)|_W$ only depends on $R$ but not on $f$.
ii) Let $\beta > 0$ be given. Because of (35)-(37) we get (38) for all $t > 0$, if $|u_0|_X$ is sufficiently small. Replace $c(t)$ by $c_2 := \sup_{0<t<\infty} t^{b+c}|f(t)|_W$. Choosing $c_2$ small enough and assuming that $\beta > 0$ is given sufficiently small, then $u(t)$ exists on $(0,\infty)$. □

# 7 Instationary Navier-Stokes equations

Let $\Omega$ be equal to the half space $\mathbb{R}^n_+$ or to a bounded domain with $C^{1,1}$-boundary. We show a local existence result for strong solutions of the instationary Navier-Stokes equations (1) in weighted Sobolev spaces. Our aim is to embed results from [22] and [12] into the weighted context.

**Proof of Theorem 1.1:** Choose $\beta = \frac{\alpha-\delta+1}{2}$ and $\gamma = \frac{\alpha+\delta}{2}$. Then $\beta > 0$, $\beta \in (\alpha, 1-\delta)$ and $\gamma \in (0, \frac{1}{2})$. Furthermore $\beta+\gamma = \frac{1}{2}+\alpha$. Since $0 < \bar\alpha+\frac{1}{2} < \alpha+\frac{1}{2} = \beta+\gamma$, we can choose $\bar\beta \in (0,\beta)$ and $\bar\gamma \in (0,\gamma)$ such that $\bar\beta+\bar\gamma = \bar\alpha+\frac{1}{2}$. Hence by (4)

$$|Q|^{2(\bar\beta+\bar\gamma)\frac{q}{n}} = |Q|^{(1+2\bar\alpha)\frac{q}{n}} \le C\,\omega(Q) \quad \forall Q \subset U.$$

Since $\beta = \gamma + \frac{1}{2} - \delta$, Theorem 5.1 yields that

$$F : D^{\beta,q}_\omega(\Omega) \times D^{\beta,q}_\omega(\Omega) \longrightarrow D^{-\delta,q}_\omega(\Omega) \tag{39}$$

is bilinear bounded. Then for $\mu \in (\alpha, 1-\delta)$ and

$$\begin{aligned} W = D^{-\delta,q}_\omega(\Omega), &\quad X = D^{\alpha,q}_\omega(\Omega) \\ Y = Z = D^{\beta,q}_\omega(\Omega), \quad , &\quad G = D^{\mu,q}_\omega(\Omega) \\ a = \alpha+\delta, \quad b = c = \beta-\alpha, \quad , &\quad d = \mu+\delta \end{aligned} \tag{40}$$

by Lemma 4.2 the assumptions of Theorem 6.1 are satisfied: It holds $a, b, c > 0$ and $d \in (a, 1)$ by the choice of $\beta$, $\gamma$, $\delta$ and $\mu$. Furthermore $a+b+c = \alpha+\delta+2(\beta-\alpha) = 1$ by the choice of $\beta$.

By Lemma 4.2 the assumptions (I) and (II) of Theorem 6.1 on the semigroup $e^{-tA}$ are satisfied. By (39) the assumptions of Theorem 6.1 on $F$ are satisfied. Since $b + c = 2(\beta-\alpha) = 1-\delta-\alpha$, we have

$$|f|_W = \|f\|_{D^{-\delta,q}_\omega(\Omega)} = o(t^{\delta+\alpha-1}) = o(t^{-b-c}).$$

Then an application of Theorem 6.1 yields the assertions a),b),c).

Concerning uniqueness, note that if a) and b) hold for some $\mu \geq \frac{\alpha-\delta+1}{2} = \beta$, then because of

$$\|u\|_{D_\omega^{\beta,q}} \leq C\,\|u\|^{\theta}_{D_\omega^{\alpha,q}} \|u\|^{1-\theta}_{D_\omega^{\mu,q}}$$

with $\theta = \frac{\mu-\beta}{\mu-\alpha}$ (see [16], Chapter 2, Theorem 6.10) it follows that b) is satisfied for $\mu = \beta$. Hence the claim follows from the uniqueness statement of Theorem 6.1, if we choose $\mu = \beta = \frac{\alpha-\delta+1}{2}$ in (40), i.e., $Y = Z = G = D_\omega^{\beta,q}(\Omega)$. □

**Remark:** For the proof of Theorem 1.1 we formally only need to assume that $-\frac{1}{2} < \alpha < 1$. But note that the condition (4) can hold for $\omega \in A_q$ only if

$$(1 + 2\overline{\alpha})\frac{q}{n} \geq 1. \tag{41}$$

The reason is that (4) implies for the mean value of $\omega$ on $Q$

$$|Q|^{(1+2\bar{\alpha})\frac{q}{n}-1} \leq C\,\frac{\omega(Q)}{|Q|}.$$

If $(1+2\bar{\alpha})\frac{q}{n} < 1$, then for $|Q| \to 0$ the left hand side tends to infinity. By Lebesgue's differentiation theorem this implies $\omega \equiv \infty$ a.e. on $\Omega$, which is impossible because of $\omega \in A_q \subset L^1_{loc}(\mathbb{R}^n)$. By $\alpha > \overline{\alpha}$ and (41) we obtain the condition

$$\frac{n}{2q} - \frac{1}{2} < \alpha. \tag{42}$$

This lower bound for $\alpha$ was also found in [12] in the case without weights, i.e., $\omega \equiv 1$.

Let $q$ and $\alpha$ be as in Theorem 1.1. Then in the case that $\Omega$ is a bounded $C^{1,1}$-domain examples of weight functions $\omega$ satisfying the assumption (4) are

$$\begin{aligned} &|x - x_0|^\beta && \text{for } -n < \beta < (1+2\alpha)q - n,\\ &d(x,\partial\Omega)^\beta && \text{for } -1 < \beta < \min\{q-1,\,(1+2\alpha)q-n\}, \end{aligned}$$

where $x_0 \in \mathbb{R}^n$ is arbitrary and $d(x,\partial\Omega)$ is the distance from $x$ to $\partial\Omega$.

If $\Omega = \mathbb{R}^n_+$, then (4) implies that the weight is not allowed to decrease for $|x| \to \infty$: More precisely, it follows from (4) that

$$\omega(Q_0 + x) \geq C(Q_0)$$

for $x \in \mathbb{R}^n$ and a fixed cube $Q_0$, i.e. $\omega(x) \to 0$ for $|x| \to \infty$ is not possible. Simple examples of weights satisfying the condition (4) for $\Omega = \mathbb{R}^n_+$ are

$$\begin{aligned} &|x - x_0|^\beta && \text{for } 0 \leq \beta < (1+2\alpha)q - n,\\ &1 + |x - x_0|^\beta && \text{for } -n < \beta < 0,\\ &x_n^\beta && \text{for } 0 \leq \beta < \min\{q-1,\,(1+2\alpha)q-n\},\\ &1 + x_n^\beta && \text{for } -1 < \beta < 0, \end{aligned}$$

where $x_0 \in \mathbb{R}^n$ is arbitrary and $x_n$ denotes the $n$th component of a vector $x = (x_1, \dots, x_n) \in \mathbb{R}^n_+$.

The way how to conclude from Theorem 1.1 the existence of local strong solutions of the Navier-Stokes equations (2) is standard and completely analogous to the case without weights (see [12]). Therefore we skip the proof of the following result and refer to [12] for details:

**Theorem 7.1** *Additionally to the assumptions of Theorem 1.1 let* $f : (0,T] \to L^q_{\omega,\sigma}(\Omega)$ *be Hölder continuous on* $[\varepsilon, T]$ *for every* $0 < \varepsilon < T$. *Then there exists a* $T^* > 0$ *and a local strong solution* $u \in C([0,T^*], D^{\alpha,q}_\omega(\Omega)) \cap C^1((0,T^*], L^q_{\omega,\sigma}(\Omega))$ *of the Navier-Stokes equations (2) on* $[0,T^*]$ *satisfying assertion b) of Theorem 1.1 and* $u(t) \in D(\mathcal{A}_{q,\omega})$ *for all* $t \in (0,T^*]$.

*The solution is unique if assertion b) of Theorem 1.1 holds for some* $\mu \geq \frac{1-\delta+\alpha}{2}$.

We conclude with a global existence result for small data: Let $\Omega$ be a bounded $C^{1,1}$-domain. Then the following corollary is an easy consequence of Corollary 6.1 and the fact that the constant $C$ in Lemma 4.2 ii) is independent of $T$ if $\Omega$ is bounded:

**Corollary 7.1** *Let* $\Omega$ *be a bounded* $C^{1,1}$*-domain. Then there are constants* $c_1 > 0$ *and* $c_2 > 0$ *such that the solution* $u$ *of the Navier-Stokes equations given by Theorem 7.1 exists globally on* $\mathbb{R}_+$ *if*

$$\|u_0\|_{D^{\alpha,q}_\omega} \leq c_1 \quad \textit{and} \quad \sup_{0<t<\infty} t^{1-\alpha-\delta} \|f(t)\|_{D^{-\delta,q}_\omega} \leq c_2.$$

# References


[1] S.-K. Chua, *Extension theorems for weighted Sobolev spaces*. Indiana Univ. Math. J. 41, 1027-1076, (1992).

[2] R. Farwig, H. Sohr, *Weighted $L^q$-theory for the Stokes resolvent in exterior domains*. J. Math. Soc. Japan 49, 251-288, (1997).

[3] M. Franzke, *Die Navier-Stokes-Gleichungen in Öffnungsgebieten*. Shaker Verlag (2000).

[4] A. Fröhlich, *Stokes- und Navier-Stokes-Gleichungen in gewichteten Funktionenräumen*. Shaker-Verlag, (2001).

[5] A. Fröhlich, *The Helmholtz decomposition of Weighted $L^q$-spaces for Muckenhoupt weights*. Ann. Univ. Ferrara (2000) - Sez. VII - Sc. Mat. Vol. XLVI, 11-19, (2000).

[6] A. Fröhlich, *The Stokes operator in weighted $L^q$-spaces I: Weighted estimates for the Stokes resolvent problem in a half space*. TU Darmstadt, Fachbereich Mathematik, Preprint Nr. 2172, (2001), to appear in J. Math. Fluid Mech.

[7] A. Fröhlich, *The Stokes operator in weighted $L^q$-spaces II: Weighted resolvent estimates and maximal $L^p$-regulartiy.* TU Darmstadt, Fachbereich Mathematik, Preprint Nr. 2173, (2001), to appear in Math. Ann.

[8] H. Fujita, T. Kato, *On the Navier-Stokes initial value problem I.* Arch. Rational Mech. Anal. 16, 269-315, (1964).

[9] J.Garcia-Cuerva, J.L. Rubio de Francia, *Weighted norm inequalities and related topics.* North Holland, Amsterdam, (1985).

[10] G. P. Galdi, *An Introduction to the Mathematical Theory of the Navier-Stokes equations, Vol. I: Linear Steady Problems.*

[11] Y. Giga, T. Miyakawa, *Solutions in $L^r$ of the Navier-Stokes Initial Value Problem.* Arch. Rat. Mech. Anal. 89, 269-281, (1985).

[12] Y. Giga, T. Miyakawa, *Solutions in $L^r$ of the Navier-Stokes Initial Value Problem.* Arch. Rational Mech. Anal. 89, 269-281, (1985).

[13] Y. Giga, *Domains of the fractional powers of the Stokes operator in $L_r$-spaces.* Arch. Rational Mech. Anal. 89, 251-265, (1985).

[14] T. Kato, H. Fujita, *On the nonstationary Navier-Stokes system.* Rend. Sem. Mat. Univ. Padova 32, 234-260, (1962).

[15] H. Komatsu, *Fractional powers of operators.* Pacific J. Math. 19, 285-346, (1966).

[16] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations.* Springer-Verlag, (1983).

[17] C. Sadosky, *Interpolation of Operators and Singular Integrals.* Monographs and textbooks in pure and applied mathematics 53, Marcel Dekker, Inc., (1979).

[18] E. Sawyer, R. L. Wheeden, *Weighted inequalities for fractional integrals on euclidean and homogeneous spaces.* Amer. J. Math. 114, 813-874, (1992).

[19] E. Stein, *Singular Integrals and Differentiability properties of Functions.* Princeton University Press, Princeton, (1970).

[20] M. E. Taylor, *Partial Differential Equations III.* Applied Math. Sciences 117, Springer-Verlag, (1996).

[21] H.Triebel, *Interpolation Theory, Function Spaces, Differential Operators.* North-Holland, (1978).

[22] F. B. Weissler, *The Navier-Stokes initial value problem in $L^p$.* Arch. Rational Mech. Anal. 74, 219-230, (1980).