Risk Capital Allocation by Coherent Risk Measures Based on One-Sided Moments

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Abstract

In this paper we propose differentiability properties for positively homogeneous risk measures. These properties ensure that the gradient can be applied for reasonable risk capital allocation on non-trivial portfolios. We show that the differentiability properties are fulfilled for a wide class of coherent risk measures based on the mean and the one-sided moments of a risky payoff. In contrast to quantile-based risk measures like Value-at-Risk, risk measures of this class allow allocation in portfolios of very general distributions, e.g. discrete ones. In an example we show how a particular risk measure of this class can be chosen by adapting it to the VaR of a certain portfolio. As a consequence, the risk capital corresponding to the VaR can be allocated by the gradient due to the adapted risk measure.

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1 Introduction

From the works Denault (2001) and Tasche (2000) it is known that differentiability of risk measures is crucial for risk capital allocation in portfolios. The reason is, that in the case of differentiable coherent or, more general, differentiable positively homogeneous risk measures the gradient due to asset weights has figured out to be the unique reasonable per-unit allocation principle. However, a result of this paper shows that at least in the coherent case differentiability on all portfolios is not desirable. As a solution

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we define weaker differentiabiliy properties. For positively homogeneous and, in particular, coherent risk measures these properties allow allocation by the gradient on all relevant portfolios. Excluded are portfolios that contain only one type of assets. However, in these cases the allocation problem is trivial. As an example for these weakened differentiability properties we introduce a wide class of coherent risk measures based on the mean and the one-sided moments of a risky payoff. In contrast to quantile-based risk measures like Value-at-Risk, risk measures of this class allow allocation in portfolios of very general distributions, e.g. discrete ones in the case of credit risk. In a numerical example we show, how a particular risk measure of this class can be chosen by adapting it to the VaR of a certain portfolio. As a consequence, the risk capital corresponding to the VaR can be allocated by the gradient due to the adapted risk measure.

Given a probability space $(\Omega, \mathcal{A}, \mathbb{Q})$ we will consider the vector space $L^p(\Omega, \mathcal{A}, \mathbb{Q})$, or just $L^p(\mathbb{Q})$, for $1 \leq p \leq \infty$. Even though $L^p(\mathbb{Q})$ consists of equivalence classes of p-integrable random variables, we will often treat its elements as random variables. Due to the context, no confusion should arise. The notation will be as follows. We have $||X||_p = (\mathbf{E}_{\mathbb{Q}}|X|^p)^{\frac{1}{p}}$ and $||X||_\infty =$ ess.sup$\{|X|\}$. Recall, that $L^p(\mathbb{Q}) \subset L^q(\mathbb{Q})$ if $1 \leq q < p \leq \infty$, since $||.||^q \leq ||.||^p$. X^- is defined as $\max\{-X, 0\}$. We denote $\sigma_p^-(X) = ||(X - \mathbf{E}_{\mathbb{Q}}[X])^-||_p$. Now, let $U \subset \mathbb{R}^n$ for $n \in \mathbb{N}_+ = \mathbb{N} \setminus \{0\}$ be open and positively homogeneous, i.e. for $u \in U$ we have $\lambda u \in U$ for all $\lambda > 0$. A *positively homogeneous* function is a function $f : U \to \mathbb{R}$, where $f(\lambda u) = \lambda f(u)$ for all $\lambda > 0$, $u \in U$. When f is also differentiable at every $u = (u_1, \ldots, u_n) \in U$, we obtain the well-known Euler Theorem

$$f(u) = \sum_{i=1}^{n} u_i \frac{\partial f}{\partial u_i}(u). \tag{1.1}$$

We consider a one-period framework, that means we have the present time 0 and a future time horizon T. Between 0 and T no trading is possible. We assume *risk* to be given by a risky payoff X, i.e. a random variable out of $L^p(\mathbb{Q})$ representing a cashflow at T. We want to consider a risk measure $\rho(X)$ to be the extra minimum cash added to X that makes the position acceptable for the holder or a regulator. For this reason, we state the following definition.

DEFINITION 1.1. *A* **risk measure** *on $L^p(\mathbb{Q})$, $1 \leq p \leq \infty$, is defined by a functional $\rho : L^p(\mathbb{Q}) \to \mathbb{R}$.*

We now give a definition of coherent risk measures. For a further motivation and interpretation of this axiomatic approach to risk measurement we refer to the article of ARTZNER ET AL. (1999).

DEFINITION 1.2. *A functional $\rho : L^p(\mathbb{Q}) \to \mathbb{R}$, where $1 \leq p \leq \infty$, is called a* **coherent risk measure (CRM)** *on $L^p(\mathbb{Q})$ if the following properties hold.*

(M) Monotonicity: If $X \geq 0$ then $\rho(X) \leq 0$.

(S) Subadditivity: $\rho(X+Y) \leq \rho(X) + \rho(Y)$.

(PH) Positive homogeneity: For $\lambda \geq 0$ we have $\rho(\lambda X) = \lambda\rho(X)$.

(T) Translation: For constants a we have $\rho(a+X) = \rho(X) - a$.

As we work without interest rates - in contrast to ARTZNER ET AL. (1999) - there is no discounting factor in definition 1.2. A generalization of CRM to the space of all random variables on a probability space can be found in DELBAEN (2000a). However, having $p \geq 1$ prevents us from being forced to allow infinitely high risks. See DELBAEN (2000a) for details on this topic.

2 Risk capital allocation by the gradient

Let us consider the payoff $X(u) := \sum_{i=1}^n u_i X_i \in L^p(\mathbb{Q})$ of a portfolio $u = (u_i)_{1\leq i \leq n} \in \mathbb{R}^n$ consisting of assets (or subportfolios) with payoffs $X_i \in L^p(\mathbb{Q})$.

DEFINITION 2.1. *A* **portfolio base** *in $L^p(\mathbb{Q})$ is a vector $B \in (L^p(\mathbb{Q}))^n$, $n \in \mathbb{N}_+$. The components of B do not have to be linearly independent.*

Having $B = (X_1, \dots, X_n)$, a risk measure ρ on the payoffs $L^p(\mathbb{Q})$ implies a risk measure ρ_B on the portfolios $\mathbb{R}^n$. In particular, we define $\rho_B : \mathbb{R}^n \to \mathbb{R}$ by $\rho_B : u \mapsto \rho(X(u))$. If ρ_B is obtained from a CRM ρ on $L^p(\mathbb{Q})$ and X_n is the only constant component in B and not equal zero, ρ_B is also called coherent (cf. DENAULT (2001)). If ρ fulfills axiom (S) and (PH) in definition 1.2, ρ_B is subadditive and positively homogenous on $\mathbb{R}^n$.

Due to diversification effects (or subadditivity of the risk measure) the total risk of a portfolio is usually assumed to be less then the sum of the risks of each subportfolio, i.e. we often have $\rho_B(u) < \sum_{i=1}^n \rho_B(u_i e_i)$, where e_i is the i-th canonical unit vector in $\mathbb{R}^n$. For this reason it is important to know how risk capital should be allocated to the subportfolios or single assets, and hence how the subportfolios should benefit from the diversification.

DEFINITION 2.2. *Given a portfolio base B and a risk measure ρ_B on $\mathbb{R}^n$ a* **per-unit allocation** *in $u \in \mathbb{R}^n$ is a vector $(a_i(\rho_B, u))_{1\le i\le n}$, such that*

$$\sum_{i=1}^{n} u_i a_i(\rho_B, u) = \rho_B(u). \tag{2.1}$$

In Denault (2001) the author drives the attention of the reader to a result of Aubin in the theory of coalitional games with fractional players. Aubin's theorem states, that in the case of a positively homogeneous, convex and differentiable cost function, the core of such a game (Aubin uses the prefix *fuzzy*) consists of one element: the gradient of the cost function due to the normed weights of the players (Aubin (1979)). From this result it is immediate, that in the case of a subadditive and positively homogeneous risk measure (e.g. a coherent one), which is differentiable at a portfolio $u \in \mathbb{R}^n$, the gradient $(\frac{\partial \rho_B}{\partial u_i}(u))_{1\le i\le n}$ is the *unique fair* per-unit allocation. To derive this statement from Aubin's result, the notion of cost functions in game theory has to be replaced by our notion of a risk measure. The players of the game are given by the certain $u_i X_i$, coalitions of fractional players are given by portfolios v, with $0 \le v \le u$, where the given portfolio u can without loss of generality be assumed to be positive. Note, that under positive homogeneity, convexity and subadditivity are equivalent. The core of such a game is build up of all per-unit allocations $(a_i(\rho_B, u))_{1\le i\le n}$, such that for all coalitions v with $0 \le v \le u$ we have $\sum_{i=1}^{n} v_i a_i(\rho_B, u) \le \rho_B(v)$. That means, no subcoalition v of u features less standalone risk than the risk, the coalition v would have been charged by the respective per-unit allocation due to u. In this sense, the elements of the core are fair allocations. For CRM Denault proves, that the Aumann-Shapley value, which is the above gradient, features certain coherence properties (Denault (2001)). For a deeper study of the connections between the theory of convex games and coherent risk measures we refer to Delbaen (2000b).

In the case of just positively homogeneous risk measures the theory of convex games is no longer suitable to model the allocation problem. However, it is still possible to talk about reasonable allocations. Tasche (2000) considers the so-called return on risk-adjusted capital (RORAC) of the payoff $X(u)$ of a portfolio u. He defines the risk-adjusted return function $f(u) = \mathbf{E}_{\mathbb{Q}}[X(u)]/\rho_B(u)$. Note, that what we called risk measure is denoted economic capital by Tasche, whereas he defines risk as fluctuation risk from the mean. Now, the idea is to call a per-unit allocation *reasonable for performance measurement* with ρ_B, when $(a_i(\rho_B, u))_{1\le i\le n}$ gives the right signals for changes in the portfolio. More precise, if $\mathbf{E}_{\mathbb{Q}}[X_i]/a_i(\rho_B, u) > f(u)$, there should be an $\varepsilon_0 > 0$, such that for all $\varepsilon \in (0, \varepsilon_0)$ we have $f(u - \varepsilon e_i) < f(u) < f(u + \varepsilon e_i)$. Analogously, for $\mathbf{E}[X_i]/(a_i(\rho_B, u)) < f(u)$ we demand $f(u - \varepsilon e_i) > f(u) > f(u + \varepsilon e_i)$. Tasche shows, that in the case of differentiable positively homogeneous risk measures the unique per-unit allocation $(a_i(\rho_B, u))_{1 \le i \le n}$ that is continuous on $\mathbb{R}^n$ and suitable for performance measurement due to the risk adjusted return function is the gradient $(\frac{\partial \rho_B}{\partial u_i}(u))_{1 \le i \le n}$ (Tasche (2000)).

In both approaches, Denault's and Tasche's, the relationship between total risk and risk contribution per unit is established by the Euler Theorem (cf. (1.1)), i.e. $\rho_B(u) = \sum_{i=1}^n u_i \frac{\partial \rho_B}{\partial u_i}(u)$. Hence, concerning risk capital allocation due to a (subadditive) positively homogeneous risk measure on $L^p(\mathbb{Q})$, it would be desirable to have ρ_B to be differentiable on $\mathbb{R}^n$ for every portfolio base $B \in (L^p(\mathbb{Q}))^n$ for all $n \in \mathbb{N}_+$. Considering the initial ρ on $L^p(\mathbb{Q})$, this implies the existence of Gâteaux-derivatives, i.e. derivatives due to directions on $L^p(\mathbb{Q})$.

PROPOSITION 2.3. *For a coherent risk measure ρ on $L^p(\mathbb{Q})$, $1 \le p \le \infty$, the following properties are equivalent: (i) ρ is Gâteaux-differentiable on $L^p(\mathbb{Q})$, (ii) ρ is linear, (iii) ρ is minimal, i.e. there is no CRM $\rho' \ne \rho$ with $\rho'(X) \le \rho(X)$ for all $X \in L^p(\mathbb{Q})$. Differentiability of ρ on $L^p(\mathbb{Q})$ implies (i), (ii) and (iii).*

COROLLARY 2.4. *A continuous coherent risk measure ρ on $L^p(\mathbb{Q})$ is Gâteaux-differentiable on $L^p(\mathbb{Q})$, $1 < p < \infty$, if and only if there exists a probability measure $\mathbb{Q}_\rho \sim \mathbb{Q}$ on Ω, such that $\rho(X) = -\mathbf{E}_{\mathbb{Q}_\rho}[X]$.*

The proof of 2.3 is omitted since equivalence of (i) and (ii) can be shown by a simple application of the coherence axioms, and as CRM are sublinear functionals, the well-known proof for equivalence of (ii) and (iii) in the general sublinear case can easily be adapted to the CRM case. The corollary follows from the duality of the $L^p(\mathbb{Q})$ spaces.

As the two statements are also true for subspaces of $L^p(\mathbb{Q})$, we face the following problem: If ρ_B is a differentiable CRM on $\mathbb{R}^n$, then it is easy to show that ρ_B is linear. Therefore ρ_B features no diversification effects. We also obtain that ρ is linear on the linear span $\langle B \rangle$ of the components of B, which implies that ρ is a minimal CRM on $\langle B \rangle$. Hence, differentiability on the whole $\mathbb{R}^n$ might be not useful.

Now, consider a portfolio base $B = (X_1, \ldots, X_n)$ and a portfolio $u = u_i e_i = (0, \ldots, 0, u_i, 0, \ldots, 0)$, $u_i \in \mathbb{R}$, $1 \le i \le n$. In this case the allocation problem is trivial, since by (2.1) the risk capital allocated to X_i - which is the only asset - is simply $\rho_B(u)/u_i$. The following definition is motivated by this consideration.

DEFINITION 2.5. *Consider a portfolio base* $B = (X_1, \dots, X_n) \in (L^p(\mathbb{Q}))^n$*,* $n \in \mathbb{N}_+$*,* $1 \le p \le \infty$*, and a portfolio* $u \in \mathbb{R}^n$*. Define* $U_e = \bigcup_{i=1}^n \langle e_i \rangle$*, where* $\langle e_i \rangle \subset \mathbb{R}^n$ *is the linear span of* e_i*. We propose to call a (subadditive) positively homogeneous risk measure* ρ *on* $L^p(\mathbb{Q})$ **suitable for risk capital allocation** *by the gradient due to the portfolio base* B *if the function* $\rho_B : \mathbb{R}^n \to \mathbb{R}$ *with* $\rho_B : u \mapsto \rho(X(u))$ *is differentiable on the open set* $\mathbb{R}^n \setminus U_e$*.*

Having a quantile-based risk measure ρ like VaR, it is known that ρ_B is not differentiable on $\mathbb{R}^n \setminus U_e$ in general. Roughly speaking, for differentiability at least one of the X_i has to posses a continuous density (Tasche (2000)). Hence, it is a problem to deal with discrete spaces $(\Omega, \mathcal{A}, \mathbb{Q})$ like e.g. in the case of credit portfolios or digital options. It will be shown below, that the step to moment based risk measures avoids this difficulty. Beside the differentiability difficulties, it is know, that VaR is not subadditive (Artzner et al. (1999)). As diversification is not rewarded, this is a major drawback.

3 A class based on one-sided moments

We define a class of coherent risk measures which depend on the mean and the one-sided higher moments of a risky position.

LEMMA 3.1. *Given a payoff* $X \in L^p(\mathbb{Q})$*, where* $1 \le p \le \infty$ *and* $0 \le a \le 1$*, the risk measure* $\rho_{p,a}$ *with*

$$\rho_{p,a}(X) = -\mathbf{E}_{\mathbb{Q}}[X] + a \cdot \sigma_p^-(X) = -\mathbf{E}_{\mathbb{Q}}[X] + a \cdot ||(X - \mathbf{E}_{\mathbb{Q}}[X])^-||_p \qquad (3.1)$$

is coherent on $L^p(\mathbb{Q})$*.*

Delbaen (2000b) shows that these risk measures can be obtained by the set of probability measures (also called *generalized scenarios*, compare Artzner et al. (1999)) $P = \{1 + a(g - \mathbf{E}[g]) \mid g \ge 0; ||g||_q \le 1\}$, where $q = p/(p-1)$ and probability measures are identified with their densities. In Delbaen (2000a) we find another type of risk measures that are connected to higher moments.

Proof of lemma 3.1. The L^p-norm on the right side of (3.1) is finite, since $X \in L^p(\mathbb{Q})$. Axiom (T) and (PH) are obvious. From Minkowski's inequality and the inequality $(a+b)^- \le a^- + b^-$ for $a, b \in \mathbb{R}$, we obtain axiom (S). Axiom (M): Let $X \ge 0$. We have $X - \mathbf{E}_{\mathbb{Q}}[X] \ge -\mathbf{E}_{\mathbb{Q}}[X]$, therefore $(X - \mathbf{E}_{\mathbb{Q}}[X])^- \le \mathbf{E}_{\mathbb{Q}}[X]$ and hence $||(X - \mathbf{E}_{\mathbb{Q}}[X])^-||_\infty = \text{ess.sup}\{(X - \mathbf{E}_{\mathbb{Q}}[X])^-\} \le \mathbf{E}_{\mathbb{Q}}[X]$. Since $||(X - \mathbf{E}_{\mathbb{Q}}[X])^-||_p \le ||(X - \mathbf{E}_{\mathbb{Q}}[X])^-||_\infty$ for $p \in [1, \infty]$, we get $||(X - \mathbf{E}_{\mathbb{Q}}[X])^-||_p \le \mathbf{E}_{\mathbb{Q}}[X]$. Remembering $0 \le a \le 1$, this completes the proof. □

The L^p-norms imply that $\rho_{q,a} \le \rho_{p,a}$ if $q < p$. The following result is on weighted sums of coherent risk measures and generalizes the well-known fact that convex sums of CRM are again CRM.

LEMMA 3.2. *Let $I \subset \mathbb{R}$ be an index set and $(\rho_i)_{i \in I}$ be a family of coherent risk measures respectively defined on $L^{p(i)}(\mathbb{Q})$, where $p : I \to [1, \infty]$. Let $(\rho_i)_{i \in I}$ be pointwise uniformly bounded on $L^{\sup p(I)}(\mathbb{Q})$ in the sense that there is a function $b : L^{\sup p(I)}(\mathbb{Q}) \to \mathbb{R}_0^+$ such that for each $X \in L^{\sup p(I)}(\mathbb{Q})$ we have $|\rho_i(X)| \le b(X)$ for all $i \in I$. Let R be a random variable with range I that is defined on a probability space Ω' with measure $\mathbb{P}$. Now, if for all $X \in L^{\sup p(I)}(\mathbb{Q})$ the mapping $\rho_{R(.)}(X) : \Omega' \to \mathbb{R}$ is measurable,*

$$\rho(X) = \mathbf{E}_{\mathbb{P}}[\rho_R(X)] \tag{3.2}$$

defines a coherent risk measure on $L^{\sup p(I)}(\mathbb{Q})$.

Proof. ρ is welldefined, since for each $X \in L^{\sup p(I)}(\mathbb{Q})$ we know from $|\rho_i(X)| \le b(X)$ and the measurability assumption, that $\rho_R(X)$ is a bounded random variable and therefore $\mathbb{P}$-integrable. Now, the coherence axioms are obvious by the properties of $\mathbf{E}_{\mathbb{P}}$. □

Using lemma 3.2, the result of lemma 3.1 can be generalized.

PROPOSITION 3.3. *Let $X \in L^p(\mathbb{Q})$ be a risky payoff, $1 \le p \le \infty$ and $0 \le a \le 1$. Let P be a random variable on a probability space $(\Omega', \mathbb{P})$ with range $P(\Omega') \subset [1, p]$. The risk measure*

$$\rho(X) = -\mathbf{E}_{\mathbb{Q}}[X] + a \cdot \mathbf{E}_{\mathbb{P}}[\sigma_P^-(X)] \tag{3.3}$$

is coherent on $L^p(\mathbb{Q})$. We have $-\mathbf{E}_{\mathbb{Q}}[X] \le \rho(X) \le \text{ess.sup}\{-X\}$.

Proof. Due to lemma 3.1 we consider a family $(\rho_{i,a})_{i \in [1,p]}$ of coherent risk measures given by (3.1), respectively defined on $L^i(\mathbb{Q})$. Now, let $b(X) = |\mathbf{E}_{\mathbb{Q}}[X]| + ||(X - \mathbf{E}_{\mathbb{Q}}[X])^-||_p$. Clearly, $|\rho_i(X)| \le b(X)$ for all $1 \le i \le p$. For all $X \in L^p(\mathbb{Q})$ the mapping $\rho_{P(.),a}(X) : \Omega' \to \mathbb{R}$ is measurable, since $P(.)$ is measurable and for all $Y \in L^p(\mathbb{Q})$ the mapping $q \mapsto ||Y||_q$ is measurable on $P(\Omega')$, as it is continuous due to the relative topology on $P(\Omega')$ in $\mathbb{R} \cup \{\infty\}$ with the canonical topology. We obtain coherence of (3.3) by lemma 3.2. The last statement follows from $||.||_p \leq ||.||_\infty$ and $\sigma_\infty^- = \text{ess.sup}\{(X - \mathbf{E}_\mathbb{Q}[X])^-\} = \text{ess.sup}\{-X + \mathbf{E}_\mathbb{Q}[X]\}$. □

An immediate consequence of the proof is that for a particular X, ρ can be chosen such that $\rho(X)$ equals any value in $[-\mathbf{E}_\mathbb{Q}[X], \text{ess.sup}\{-X\}]$.

EXAMPLE 3.4. $\rho(X) = -\mathbf{E}_\mathbb{Q}[X] + a_1\sigma_1^- + a_2\sigma_2^- + \ldots + a_\infty\sigma_\infty^-$, where $a_p \geq 0$ for $p \in \{1, 2, 3, \ldots, \infty\}$ and $a_\infty + \sum_{p=1}^\infty a_p \leq 1$ is a coherent risk measure on $L^q(\mathbb{Q})$, where $q := \sup\{p | a_p > 0\}$ (we use the convention $0 \cdot (\pm\infty) = (\pm\infty) \cdot 0 = 0$). In particular, $a_2 = a_\infty = \frac{1}{2}$ could be interpreted as a *coherent mixture* of the (μ, σ)- and the maximum-loss-principle.

DEFINITION 3.5. *For $B \in (L^p(\mathbb{Q}))^n$, $n \in \mathbb{N}_+$, $1 < p < \infty$, the set $U_C(B)$ denotes the set of all $u \in \mathbb{R}^n$, for which $\sum_{i=1}^n u_i X_i \equiv const.$*

LEMMA 3.6. *The set $\mathbb{R}^n \setminus U_C(B)$ is open in $\mathbb{R}^n$.*

Proof. The linear mapping $X(.) : \mathbb{R}^n \to L^p(\mathbb{Q})$, where $u \mapsto X(u)$, is bounded, since $||X(u)||_p \leq \sum_{i=1}^n |u_i| \cdot ||X_i||_p \leq ||u|| \cdot \sum_{i=1}^n ||X_i||_p$. Hence, $X(.)$ is continuous on $\mathbb{R}^n$. The set C of all constant elements of $L^p(\mathbb{Q})$ is closed, since $L^p(\mathbb{Q})$ is a Banach-space due to the theorem of Riesz-Fischer and every Cauchy-sequence of constant elements in $L^p(\mathbb{Q})$ converges to a constant limit in $L^p(\mathbb{Q})$ (due to L^p-norm). Since $X(.)$ is continuous, $[X(.)]^{-1}(C) = U_C(B)$ is closed and $\mathbb{R}^n \setminus U_C(B)$ open. □

We can now state a result on differentiability of the class of coherent risk measures that was introduced in proposition 3.3.

PROPOSITION 3.7. *Assume $B \in (L^p(\mathbb{Q}))^n$, $n \in \mathbb{N}_+$, $1 < p < \infty$ and $0 \leq a \leq 1$. Let $1 < P \leq p$ be a random variable on a probability space with measure $\mathbb{P}$. The risk measures ρ_B implied by* (3.3) *are differentiable on $\mathbb{R}^n \setminus U_C(B)$. The partial derivatives are*

$$\begin{aligned} \frac{\partial \rho_B}{\partial u_i}(u) &= -\mathbf{E}_\mathbb{Q}[X_i] + a \cdot \mathbf{E}_\mathbb{P}[\sigma_P^-(X(u))^{1-P} \cdot \\ &\qquad \mathbf{E}_\mathbb{Q}[(-X_i + \mathbf{E}_\mathbb{Q}[X_i]) \cdot ((X(u) - \mathbf{E}_\mathbb{Q}[X(u)])^-)^{P-1}]]. \end{aligned} \tag{3.4}$$

The proof of proposition 3.7 is rather technical and therefore given in the appendix. We want to show, that the risk measures (3.3) actually can not be differentiable at some $u \in U_C(B)$. We suppose $u \in U_C(B)$, $a > 0$ and the risk measure defined by (3.1), which is the special case $P \equiv const$. We have $\rho_{p,a}(u) = -\mathbf{E}_{\mathbb{Q}}[X(u)]$, since $X(u) \equiv \mathbf{E}_{\mathbb{Q}}[X(u)]$. Easily we obtain the two different one-sided partial derivatives $-\mathbf{E}_{\mathbb{Q}}[X_i] + a \cdot ||(\pm X_i \mp \mathbf{E}_{\mathbb{Q}}[X_i])^-||_p$ in u, but $||(X_i - \mathbf{E}_{\mathbb{Q}}[X_i])^-||_p \neq ||(-X_i + \mathbf{E}_{\mathbb{Q}}[X_i])^-||_p$ in general. So, we have no differentiability in general.

COROLLARY 3.8. *Under the assumptions of 3.7, the risk measures ρ implied by* (3.3) *are suitable for risk capital allocation by the gradient due to the portfolio base B if the components $X_1, \ldots, X_n$ of B are linearly independent and $X_n \not\equiv 0$ is constant. The per-unit allocations are explicitly given by* (3.4).

Proof. $U_C(B) = \langle(0, \ldots, 0, 1)\rangle \subset U_e$. □

Corollary 3.8 is the main result on risk capital allocation by the considered class of coherent risk measures. We did not make any assumptions on the underlying probability space $(\Omega, \mathcal{A}, \mathbb{Q})$, e.g. discrete spaces can be taken into consideration. The assumption of linear independence is quite weak, as it should be no problem to find a vector base in a real market. Even the particular choice of the portfolio base B is not important, as the gradient is an aggregation invariant allocation principle (Denault (2001)). That means, if we have two different portfolio bases B and B' as given in corollary 3.8, with $\langle B\rangle = \langle B'\rangle$, there exists a linear isomophism A on $\mathbb{R}^n$, such that we have $X(u) \equiv X'(u')$ and $\rho_B(u) = \rho_{B'}(u')$ for every $u = Au' \in \mathbb{R}^n$. We therefore obtain from standard analysis for any two equivalent portfolios v and v' with $v = Av'$

$$\sum_{i=1}^{n} v_i' \frac{\partial \rho_{B'}}{\partial u_i'}(u') = \sum_{i=1}^{n} v_i \frac{\partial \rho_B}{\partial u_i}(u). \tag{3.5}$$

So, the risk capital allocated to equivalent subportfolios, i.e. subportfolios with the same payoff in $L^p(\mathbb{Q})$, is identical.

4 Numerical example

In this example we want to show how risk capital obtained by the Value-at-Risk can be allocated using the risk measures from section 3. In particular, we use a risk measure of type $\rho_{p,1}(X) = -\mathbf{E}_{\mathbb{Q}}[X] + \sigma_p^-(X)$ as given in (3.1). We define the Value-at-Risk by

$$\mathrm{VaR}_\alpha(X) = -\inf\{x : \mathbb{Q}(X \le x) > \alpha\}. \tag{4.1}$$

Now, suppose two stochastically independent payoff variables X_1, X_2 with distributions as given in table 1. The portfolio base is given by $B = (X_1, X_2, 1)$. X_1 and X_2 could be interpreted as one unit of a credit engagement. Obviously, X_1 bears higher risks as losses are more probable. We consider the portfolio $u = (u_1, u_2) = (1000, 1000)$. Easily we compute $\mathrm{VaR}_{0.05}(X(u)) = 500$. To allocate the given risk capital, we adjust $\rho_B(u)$ by choosing p, such that $\rho_{p,1}(X(u)) = \mathrm{VaR}_{0.05}(X(u)) = 500$. We obtain $p \approx 2.9157$. From the discrete version of (3.4) ($|\Omega| = 9$, $P \equiv p$, $a = 1$) we obtain $\frac{\partial \rho_B}{\partial u_1}(u) \approx 0.31504$ and $\frac{\partial \rho_B}{\partial u_2}(u) \approx 0.18496$. The risk capital allocated to u_1X_1 is 315.04, for u_2X_2 it is 184.96. To check what happens for a more conservative VaR, we compute $\mathrm{VaR}_{0.01}(X(u))$, which is 1000. We obtain $p \approx 9.4355$ and the risk capital allocated to u_1X_1 is 477.98, for u_2X_2 it is 522.02. It is interesting that in the second case more risk capital is allocated to X_2, which seems to bear less risk. However, the relative difference is quite small compared to the first case. This seems to be reasonable as we have $\mathrm{VaR}_{0.01}(u_1X_1) = \mathrm{VaR}_{0.01}(u_2X_2) = 1000$.

x	$\mathbb{Q}(X_1 = x)$	$\mathbb{Q}(X_2 = x)$
0.0	0.78	0.96
-0.5	0.20	0.02
-1.0	0.02	0.02

Table 1: Distribution of X_1, X_2

A Appendix

The proof of proposition 3.7 needs the following technical lemmas.

LEMMA A.1. *Let U be an open subset of $\mathbb{R}^n$, $n \in \mathbb{N}_+$, and $f : U \times \Omega \to \mathbb{R}$ be a function with following properties:*

a) $\omega \mapsto f(u, \omega)$ is $\mathbb{Q}$-integrable for all $u \in U$.

b) $u \mapsto f(u, \omega)$ is in any $u \in U$ partial differentiable with respect to u_i.

c) There exists a $\mathbb{Q}$-integrable function $h_U \ge 0$ on Ω with $\left|\frac{\partial f}{\partial u_i}(u, \omega)\right| \le h_U(\omega)$ for all $(u, \omega) \in U \times \Omega$.

The function $\varphi(u) = \int f(u,\omega) d\mathbb{Q}(\omega)$ *on* U *is partially differentiable with respect to* u_i. *The mapping* $\omega \mapsto \frac{\partial f}{\partial u_i}(u,\omega)$ *is* $\mathbb{Q}$*-integrable and for* $u \in U$

$$\frac{\partial \varphi}{\partial u_i}(u) = \int \frac{\partial f}{\partial u_i}(u,\omega) d\mathbb{Q}(\omega). \tag{A.1}$$

Proof. By the dominated convergence theorem. □

LEMMA A.2. *Define* $U = \triangle u_1 \times \cdots \times \triangle u_n \subset \mathbb{R}^n$, *where for all* $i \in \{1,\dots,n\}$ $\triangle u_i$ *is a nonempty, bounded and open interval in* $\mathbb{R}$. *Let* $X(u) = \sum_{i=1}^n u_i X_i$ *be a sum of real-valued random variables* $X_i \in L^p(\mathbb{Q})$ *with* $u = (u_1,\dots,u_n) \in U$, $n \in \mathbb{N}_+$ *and* $1 < p < \infty$. *Let* $y(u)$ *be a real-valued function that is differentiable, bounded and for which* $y(u) < \text{ess.sup}\{-X(u)\}$ *on* U. *The partial derivatives* $\frac{\partial y}{\partial u_i}(u)$ *are also assumed to be bounded on* U. *Under this assumptions,* $||(X(u)+y(u))^-||_p$ *is differentiable on* U.

Proof. Define $g(u,\omega) = (X(u,\omega) + y(u))^-$. For $1 \le i \le n$ we will prove existence and continuity of the partial derivatives of $||g(u)||_p$.
Existence: We have $||g(u)||_p = \left(\int g(u,\omega)^p \, d\mathbb{Q}(\omega)\right)^{1/p}$. Now, if we can apply lemma A.1 to g^p (where f from A.1 corresponds to g^p) and if $g(u)$ is *not* constant 0 for every $u \in U$, we obtain for every i

$$\frac{\partial ||g(u)||_p}{\partial u_i}(u) = \int \frac{\partial g^p}{\partial u_i}(u) \, d\mathbb{Q} \cdot \frac{1}{p} \cdot \left(\int g(u)^p \, d\mathbb{Q}\right)^{\frac{1}{p}-1}. \tag{A.2}$$

Note, that for $u \in U$ we have $g(u) > 0$ on a set of measure greater 0, since $y(u) < \text{ess.sup}\{-X(u)\}$. Therefore the right integral in (A.2) is greater 0 (no division by zero!). We are going to check the points a) to c) from lemma A.1. Ad a). $\omega \mapsto g(u,\omega)^p$ is $\mathbb{Q}$-integrable, since $X(u) \in L^p(\mathbb{Q})$ and $y(u) \in \mathbb{R}$. Ad b). First, we consider the function $[(.)^-]^p : \mathbb{R} \to \mathbb{R}_0^+$, $x \mapsto (x^-)^p$. Clearly, this function is differentiable for $1 < p < \infty$. Now, $g(u,\omega)^p = [(\sum_{i=1}^n u_i X_i(\omega) + y(u))^-]^p$ - as a combination of a differentiable and a partially differentiable function - is partially differentiable at u_i. We obtain

$$\frac{\partial g^p}{\partial u_i}(u,\omega) \quad = \quad -\left(X_i(\omega) + \frac{\partial y}{\partial u_i}(u)\right) \cdot p \cdot g(u,\omega)^{p-1}. \tag{A.3}$$

Ad c). There exist positive constants a and b, such that for all $j \in \{1,\dots,n\}$ we have $|\frac{\partial y}{\partial u_j}(u)| \le a$ and $|y(u)| \le b$ on U. Now, define

$$u_{\max}(U) = \sup\{|u'_j| : u'_j \in \triangle u_j, \, j \in \{1,\dots,n\}\}, \tag{A.4}$$

which is finite, and

$$k_U(\omega) = n \cdot u_{\max}(U) \cdot \max_j \{|X_j(\omega)|\} + b. \tag{A.5}$$

Clearly, $k_U(\omega) \geq g(u, \omega)$. Now define

$$h_U(\omega) = (|X_i(\omega)| + a) \cdot p \cdot (k_U(\omega))^{p-1}. \tag{A.6}$$

Comparing this to (A.3), we clearly obtain

$$0 \leq \left| \frac{\partial g^p}{\partial u_i}(u, \omega) \right| \leq h_U(\omega) \tag{A.7}$$

for all $(u, \omega) \in U \times \Omega$. Concerning integrability of (A.6) we know that $(|X_i(\omega)| + a) \cdot p$ is p-integrable, since X_i is. We also know that $(k_U(\omega))^{p-1}$ is $\frac{p}{p-1}$-integrable. The latter statement follows from the fact that every single $|X_j(\omega)|$ is p-integrable and therefore $k_U(\omega)$ - as a multiple of the maximum plus a constant - is p-integrable. We further have $1/p + (p-1)/p = 1$. As an immediate consequence of Hölder's inequality the product $h_U(\omega)$ of $(|X_i(\omega)| + a) \cdot p$ and $(k_U(\omega))^{p-1}$ is integrable.

Continuity: Consider a sequence $(u_n)_{n \in \mathbb{N}}$ with $\lim_{n \to \infty} u_n = u$ in $U = \triangle u_1 \times \cdots \times \triangle u_n$. Now, substitute u by u_n in (A.2). For fix $\omega \in \Omega$ it follows from the definition of $g(u)$ and (A.3) that the substituted expressions under the integrals in (A.2) converge (pointwise in ω) to the original expressions (in u). Now have in mind, that h_U (A.7) dominates the left integrand of (A.2) and $(k_U)^p$ (A.5) dominates the right one. As h_U and $(k_U)^p$ are integrable it follows from the dominated convergence theorem that the substituted integrals themselves converge to the original integrals. Hence, (A.2) is continuous in u. □

LEMMA A.3. *Assume $B \in (L^p(\mathbb{Q}))^n$, $n \in \mathbb{N}_+$, $1 < p < \infty$. Suppose $0 \leq a \leq 1$. The risk measures $\rho_B(u)$ implied by (3.1) are differentiable on $\mathbb{R}^n \setminus U_C(B)$. The partial derivatives are*

$$\begin{aligned} \frac{\partial \rho_B}{\partial u_i}(u) \quad = \quad & -\mathbf{E}_{\mathbb{Q}}[X_i] + a \cdot \sigma_p^-(X(u))^{1-p} \cdot \\ & \mathbf{E}_{\mathbb{Q}}[(-X_i + \mathbf{E}_{\mathbb{Q}}[X_i]) \cdot ((X(u) - \mathbf{E}_{\mathbb{Q}}[X(u)])^-)^{p-1}]. \end{aligned} \tag{A.8}$$

Proof. As $\mathbb{R} \setminus U_C(B)$ is open, it can be seen as union of bounded n-dimensional open intervals U. We focus on the $L^p(\mathbb{Q})$-norm expression in $\rho_B(u)$. Define $y(u) = -\mathbf{E}_{\mathbb{Q}}[X(u)]$. Now, the requirements of lemma A.2 are satisfied, since

$-\mathbf{E}_{\mathbb{Q}}[X(u)] < \text{ess.sup}\{-X(u)\}$ as long as $X(u) \not\equiv$ *const*. We obtain that the risk measure is differentiable in U and

$$\frac{\partial \rho_B}{\partial u_i}(u) = -\mathbf{E}_{\mathbb{Q}}[X_i] + \int \frac{\partial g^p}{\partial u_i}(u) d\mathbb{Q} \cdot a \cdot \frac{1}{p} \cdot ||g(u)||_p^{1-p} \quad . \tag{A.9}$$

As (A.9) does not depend on the choice of the particular $U \subset \mathbb{R}^n \setminus U_C(B)$, $\rho_B(u)$ is differentiable on $\mathbb{R}^n \setminus U_C(B)$. Since by definition $||g(u)||_p = \sigma_p^-(X(u))$, we obtain (A.8) by combining (A.3) with (A.9). □

Proof of proposition 3.7. We use the notation from the proofs of the lemmas A.2 and A.3. Assume $U = \triangle u_1 \times \cdots \times \triangle u_n$ to be a bounded nonempty n-dimensional open interval in $\mathbb{R}^n \setminus U_C(B)$, where for all $i \in \{1, \ldots, n\}$ $\triangle u_i$ is an open interval. Consider equation (3.3). We have

$$\mathbf{E}_{\mathbb{P}}[\sigma_P^-(X(u))] = \int ||g(u)||_{P(\omega')} d\mathbb{P}(\omega') \; . \tag{A.10}$$

We prove the existence and continuity of the partial derivatives of (A.10). *Existence*: Again, we are going to check the points a) to c) from lemma A.1 (f corresponds to $||g(u)||_{P(\omega')}$). Ad a). $\omega' \mapsto ||g(u)||_{P(\omega')}$ is integrable, since $||g(u)||_{P(\omega')} \leq ||g(u)||_p < \infty$. Ad b). Since $P(\omega')$ is fix, it follows from the proof of lemma A.3 (equation (A.9)), that $u \mapsto ||g(u)||_{P(\omega')}$ is in every point $u \in U$ partially differentiable with respect to u_i. Ad c). From (A.9) we get

$$\frac{\partial f}{\partial u_i}(u, \omega') \;\; = \;\; \int \frac{\partial g^{P(\omega')}}{\partial u_i}(u) d\mathbb{Q} \cdot \frac{a}{P(\omega')} \cdot ||g(u)||_{P(\omega')}^{1-P(\omega')}. \tag{A.11}$$

From (A.3) we obtain

$$\frac{\partial g^{P(\omega')}}{\partial u_i}(u, \omega) = -(X_i(\omega) - \mathbf{E}_{\mathbb{Q}}[X_i]) \cdot P(\omega') \cdot g(u, \omega)^{P(\omega')-1}. \tag{A.12}$$

As $g(u, \omega)^{P(\omega')-1}$ is $\frac{P(\omega')}{P(\omega')-1}$-integrable, we get from Hölder's inequality

$$\begin{aligned} \left| \int \frac{\partial g^{P(\omega')}}{\partial u_i}(u, \omega) d\mathbb{Q} \right| \;\; &\leq \;\; \left|\left| \frac{\partial g^{P(\omega')}}{\partial u_i}(u, \omega) \right|\right|_1 \qquad\qquad (A.13) \\ &\leq \;\; ||(X_i - \mathbf{E}_{\mathbb{Q}}[X_i])||_{P(\omega')} \cdot P(\omega') \cdot ||g(u)||_{P(\omega')}^{P(\omega')-1}. \end{aligned}$$

Combining this with (A.11) we obtain

$$\begin{aligned} \left| \frac{\partial f}{\partial u_i}(u, \omega') \right| \;\; &\leq \;\; ||(X_i - \mathbf{E}_{\mathbb{Q}}[X_i])||_{P(\omega')} \cdot a \qquad\qquad (A.14) \\ &\leq \;\; ||(X_i - \mathbf{E}_{\mathbb{Q}}[X_i])||_p \cdot a \equiv const. \end{aligned}$$

Choosing $h_U(\omega') = ||(X_i - \mathbf{E}_{\mathbb{Q}}[X_i])||_p \cdot a$, this completes the proof of c). From the arbitrariness of $U \subset \mathbb{R}^n \setminus U_C(B)$, we obtain partial differentiability of ρ on $\mathbb{R}^n \setminus U_C(B)$. Equation (3.4) follows from the combination of lemma (A.1) with the result (A.8) of lemma A.3.

Continuity: As we know from the proof of lemma A.3, expression (A.11) is continuous on $\mathbb{R}^n \setminus U_C(B)$. By (A.14), dominated convergence proves continuity of the partial derivatives. □

References

[1] Artzner, P., Delbaen, F., Eber, J.-M., Heath, D. (1999) - *Coherent measures of risk*, Mathematical Finance, Vol. 9, No. 3, 203-228

[2] Aubin, J.-P. (1979) - *Mathematical Methods of Game and Economic Theory*, North-Holland Publishing Co., Amsterdam

[3] Delbaen, F. (2000a) - *Coherent risk measures on general probability spaces*, ETH Zürich, http://www.math.ethz.ch/~delbaen/ftp/preprints/RiskMeasuresGeneralSpaces.pdf

[4] Delbaen, F. (2000b) - *forthcoming notes of the tutorial at the Scuola Normale Superiore*, Pisa

[5] Denault, M. (2001) - *Coherent Allocation of Risk Capital*, Journal of Risk, Vol. 4, No. 1, 1-34.

[6] Tasche, D. (2000) - *Risk contributions and performance measurement*, working paper, TU München, http://www-m4.mathematik.tu-muenchen.de/m4/pers/tasche/riskcon.pdf

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