Nancy Lectures on Infinite-Dimensional Lie Groups

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Abstract. These are lecture note of a course given in Februaru and March 2002 in Nancy. The main purpose of this course was to present some of the main ideas of infinite-dimensional Lie theory and to explain how it differs from the finite-dimensional classical theory. After the introduction where we present some of the main types of infinite-dimensional Lie groups: lineare Lie groups associated to continuous inverse algebras, groups of maps and diffeomorphism groups, we turn in more detail to manifolds modeled on locally convex spaces. In Section III we present some of the basic Lie theory of locally convex Lie groups, including a discussion of the exponential function and the non-existence of groups for Lie algebras. In the final Section IV we discuss the topology of the main classes of infinite-dimensional Lie groups with an emphasis on their homotopy groups.

I. Introduction

Lie groups arise most naturally as symmetry groups or automorphism groups of algebraic or geometric structures. This is true for finite-dimensional Lie groups and remains valid for infinite-dimensional Lie groups. Moreover, it is well known from finite-dimensional Lie theory that not every automorphism group of an algebraic or geometric structure is a Lie group. Limitations of this type remain valid for infinite-dimensional Lie groups as well, although many important groups which are not finite-dimensional Lie groups have a natural structure as an infinite-dimensional Lie group.

In this introduction we discuss several classes of infinite-dimensional Lie groups without going into details. The main purpose is to give an impression of the enormous variety of infinite-dimensional Lie groups and to explain some of the differences to the finite-dimensional theory.

The concept of an infinite-dimensional Lie group

Our general idea of a Lie group is that it should be a manifold G (defined suitably in an infinite-dimensional context) which carries a group structure for which multiplication and inversion are smooth maps. Therefore the concept of an infinite-dimensional Lie group relies very much on the corresponding concept of an infinite-dimensional manifold.

The concept of a Banach–Lie group, i.e., a Lie group modeled on a Banach space, has been introduced by G. Birkhoff in [Bi38]. The step to more general classes of infinite-dimensional Lie groups modeled on complete locally convex spaces occurs first in an article of Marsden and Abraham [MA70] in the context of hydrodynamics. This Lie group concept has been worked out by J. Milnor in his Les Houches lecture notes [Mil83] which provide many basic results of the general theory. The observation that the completeness condition on the underlying locally convex space can be omitted for the basic theory is due to H. Glöckner ([Gl01a]). This is important for quotient constructions because quotients of complete locally convex spaces need not be complete.

There are other, weaker, concepts of Lie groups, resp., infinite-dimensional manifolds. One is based on the "convenient setting" for global analysis developed by Fröhlicher, Kriegl and Michor ([FK88] and [KM97]). In the context of Fréchet manifolds this setting does not differ from the one mentioned above, but for more general model spaces it provides a concept of a smooth map which does not necessarily imply continuity, hence leads to Lie groups which are not topological groups. Another approach is based on the concept of a diffeological space due to J.-M. Souriau ([So85]) which can be used to study spaces like quotients of $\mathbb{R}$ by non-discrete subgroups in a differential geometric context.

Throughout these notes $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and all vector spaces are real or complex.

I.1. Linear Lie groups

In finite-dimensional Lie theory, a natural approach to Lie groups is via matrix groups, i.e., subgroups of the group $\mathrm{GL}_n(\mathbb{R})$ of invertible real $n \times n$-matrices. Since every finite-dimensional algebra can be embedded into a matrix algebra, this is equivalent to considering subgroups of the unit groups $A^\times := \{a \in A : (\exists b \in A) ab = ba = \mathbf{1}\}$ of finite-dimensional unital associative algebras A. The advantage of this approach is that one can define the exponential function quite directly and thus take a shortcut to several deeper results on Lie groups. This approach also works quite well in the context of Banach-Lie groups. Here the linear Lie groups are subgroups of the unit groups $A^\times$ of Banach algebras A. To get some feeling for this context, let us take a look at some types of Banach algebras.

Examples I.1.1. (a) If X is a Banach space, then the space $B(X)$ of all continuous operators on X is a unital Banach algebra with respect to the operator norm

$$\|A\| := \sup\{\|A.x\| : x \in X, \|x\| \leq 1\}.$$

Conversely, if A is a unital Banach algebra, then we have an embedding $L : A \hookrightarrow B(A)$ given by the left regular representation $L(a).b := ab$ of A on itself. Therefore Banach algebras are algebras of operators on Banach spaces which are closed in the operator norm.
(b) If A is a unital Banach algebra, then the same holds for all the matrix algebras $M_n(A)$, $n \in \mathbb{N}$. To see this, we may w.l.o.g. assume that A is a closed subalgebra of some $B(X)$, X a Banach space. We endow the space X^n with the norm

$$\|(x_1, \ldots, x_n)\| := \max(\|x_1\|, \ldots, \|x_n\|)$$

and consider on $M_n(A)$ the operator norm coming from the embedding $M_n(A) \hookrightarrow B(X^n)$. This turns $M_n(A)$ into a unital Banach algebra.

So far this works also in a finite-dimensional context, but in general we can also consider the Banach space

$$X^\infty := l^\infty(\mathbb{N}, X) := \{(x_n)_{n\in\mathbb{N}} : \|x\|_\infty := \sup_{n\in\mathbb{N}} \|x_n\| < \infty\}$$

of all bounded X-valued sequences. Then we have for each $n \in \mathbb{N}$ an isometric embedding

$$\eta_n : M_n(A) \hookrightarrow B(X^\infty), \quad \eta_n(a).(x_i)_{i\in\mathbb{N}} := \Big(\sum_{j=1}^n a_{ij} x_j\Big)_{i\in\mathbb{N}}.$$

Based on this observation, we identify $M_n(A)$ with a closed subalgebra of $B(X^\infty)$ and define

$$M_\infty(A) := \overline{\bigcup_n M_n(A)} \subseteq B(X^\infty).$$

The elements of $M_\infty(A)$ can be viewed as infinite matrices $a = (a_{ij})_{i,j\in\mathbb{N}}$ with entries in A, where the matrix coefficients a_{ij} tend to zero for increasing i and j. Note that the completion $M_\infty(A)$ depends on the choice of the norm on the spaces X^n. If we take a norm of the type $\|x\|_p := \big(\sum_{j=1}^n \|x_j\|^p\big)^{\frac{1}{p}}$, $1 \leq p < \infty$, then we obtain a different completion.

(c) If X is a compact space and B is a Banach algebra, then the space $C(X,B)$ is a Banach algebra with respect to the sup-norm

$$\|f\| := \sup\{\|f(x)\|: x \in X\}.$$

(d) If S is a semigroup and B is a Banach algebra, then the Banach space

$$A := l^1(S,B) := \Big\{f: S \to B: \|f\|_1 := \sum_{s\in S} \|f(s)\| < \infty\Big\}$$

is a Banach algebra with respect to the convolution product

$$(f * g)(u) := \sum_{s,t\in S, st=u} f(s)g(t).$$

Note that the finiteness of $\|f\|_1$ implies that only countably many values of the function f are non-zero. In the case where S is a group, we can write the convolution product also in the form

$$(f * g)(u) := \sum_{s\in S} f(s)g(s^{-1}u).$$ ■

As the preceding discussion shows, there are many types of Banach algebras A, and their unit groups $A^\times$ are basic examples of Banach–Lie groups.

Examples I.1.2. Further examples of Banach–Lie groups which are more like finite-dimensional classical groups can be obtained as follows.
(a) If X is a Banach space and $\beta: X \times X \to \mathbb{K}$ is a continuous bilinear form, then the corresponding *orthogonal group*

$$\mathrm{O}(X,\beta) := \{g \in \mathrm{GL}(X): (\forall x,y \in X)\beta(g.x,g.y) = \beta(x,y)\}$$

is a Banach–Lie group.

If β is skew-symmetric and non-degenerate in the sense that $\beta(x,X) = \{0\}$ implies $x = 0$, then we call (X,β) a *symplectic Banach space* and

$$\mathrm{Sp}(X,\beta) := \mathrm{O}(X,\beta)$$

the corresponding *symplectic group*.
(b) If H is a complex Hilbert space, then the *unitary group*

$$\mathrm{U}(H) := \{g \in \mathrm{GL}(H): (\forall x,y \in H)\langle g.x,g.y\rangle = \langle x,y\rangle\}$$

is an important example of a Banach–Lie group.
(c) If A is a Banach algebra, then its automorphism group $\mathrm{Aut}(A)$ is a Banach–Lie group. ■

For an associative algebra A we write A_+ for the algebra $A \times \mathbb{K}$ with the multiplication

$$(a,s)(b,t) := (ab + sb + ta, st).$$

This is a unital algebra with unit $\mathbf{1} = (0,1)$. For many purposes it is natural to extend the concept of a Banach algebra to the more general concept of a *continuous inverse algebra* (c.i.a.). These are locally convex algebras A with continuous multiplication such that the group $A_+^\times$ of units of the algebra A_+, endowed with the product topology of $A \times \mathbb{K}$, is open and the inversion is a continuous map $A_+^\times \to A_+$.

For each c.i.a. A the matrix algebras $M_n(A)$ are also c.i.a. (see [Bos90]). Further each closed Lie subalgebra $\mathfrak{g} \subseteq M_n(A)$ corresponds to some analytic subgroup G of $\mathrm{GL}_n(A^+)$ ([Gl01c]). In the context of infinite-dimensional Lie theory over locally convex spaces, these groups form the natural generalizations of linear Lie groups.

Examples I.1.3. (a) Each Banach algebra is a continuous inverse algebra.
(b) If B is a c.i.a. and M is a compact manifold, then the algebra $C^\infty(M,B)$ is a continuous inverse algebra.
(c) Let B be a Banach algebra and $\alpha: G \times B \to B$ a strongly continuous action of the finite-dimensional Lie group G on B by isometric automorphisms. Then the space B^∞ of smooth vectors for this action is a dense subalgebra and a Fréchet c.i.a. ([Bos90, Prop. A.2.9]). ■

I.2. Groups of continuous and smooth maps

In the context of Banach–Lie groups one constructs Lie groups of mappings as follows. For a compact space X and a Banach–Lie group K the group $C(X,K)$ of continuous maps is a Banach–Lie group with Lie algebra $C(X,\mathfrak{k})$, where $\mathfrak{k}$ is the Lie algebra of K.

In the larger context of locally convex Lie groups one also obtains for each Lie group K and a compact smooth manifold M a Lie group structure on the group $C^\infty(M,K)$ of smooth maps from M to K. This is a Fréchet–Lie group if K is a Fréchet–Lie group. Its Lie algebra is the space $C^\infty(M,\mathfrak{k})$.

The passage from continuous maps to smooth maps is made necessary by the behavior of central extensions of these groups. The groups $C^\infty(M,K)$ have much more central extensions as the groups $C(M,K)$, hence exhibit a richer geometric structure.

A larger class of groups is obtained as *gauge groups* of principal bundles. If a smooth map $q\colon P \to B$ defines a principal K-bundle, then we consider the associated bundle $q_K\colon P_K \to B$, where P_K is the space of K-orbits in the space $P \times K$ with respect to the action given by $k.(p,x) := (pk^{-1}, kx)$ for $k,x \in K$ and $p \in P$. The gauge group $\mathrm{Gau}(P)$ is the group of smooth sections of the bundle P_K. If the bundle P is trivial, then $P_K \cong B \times K$ and $q_K(b,k) = b$, so that $\mathrm{Gau}(P) \cong C^\infty(B,K)$.

I.3. Groups of homeomorphisms and diffeomorphisms

Once a geometric structure on a space is given, one considers its group of automorphisms. In the spirit of Felix Klein's Erlangen Program, one may even say that the geometry or the geometric structure is given by the corresponding group of automorphisms.

I.3.1. For a compact topological space X we have the C^*-algebra $C(X,\mathbb{C})$ of continuous complex valued functions. From Gelfand's duality theory of commutative C^*-algebras we obtain

$$X \cong \mathrm{Hom}_{\mathrm{alg}}(C(X,\mathbb{R}),\mathbb{R}) \setminus \{0\}$$

in the sense that every non-zero algebra homomorphism $C(X,\mathbb{R}) \to \mathbb{R}$ is given by a point evaluation $\delta_p(f) = f(p)$. This implies that the space X can be recovered from the Banach algebra $C(X,\mathbb{R})$ if we endow $\mathrm{Hom}_{\mathrm{alg}}(C(X,\mathbb{R}),\mathbb{R})$ with the topology of pointwise convergence.

We conclude that the Lie group $\mathrm{Aut}(C(X,\mathbb{R}))$ of automorphisms of this algebra, endowed with the uniform operator topology, can be identified with the group $\mathrm{Homeo}(X)$ of homeomorphisms of X acting on $C(X,\mathbb{R})$ by

$$(\gamma.f)(x) := f(\gamma^{-1}.x).$$

We claim that the uniform topology turns $\mathrm{Homeo}(X)$ into a discrete group. In fact, if γ is a non-trivial homeomorphism of X and $p \in X$ is moved by γ, then there exists a continuous function $f \in C(X,\mathbb{R})$ with $\|f\| = 1$, $f(p) = 0$ and $f(\gamma^{-1}(p)) = 1$. Then $\|\gamma.f - f\| \geq 1$ implies that $\|\gamma - \mathbf{1}\| \geq 1$. Therefore the group $\mathrm{Homeo}(X)$ is discrete with respect to the topology inherited from the Banach algebra $B(C(X,\mathbb{R}))$.

Nevertheless, one considers continuous actions of connected Lie groups G on X, where the continuity of the action means that the action map $\alpha\colon G \times X \to X$ is continuous. But this does not mean that the corresponding homomorphism $G \to \mathrm{Homeo}(X)$ is continuous. We will see that this phenomenon, i.e., that certain automorphism groups are endowed with Lie group structures which are too fine for many purposes, reoccurs at many levels of the theory[1].

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[1] There are other reasonable topologies on the group $\mathrm{Homeo}(X)$ which are coarser and therefore more suitable to study transformation groups. A quite natural one is obtained as the initial topology with respect to the map $\mathrm{Homeo}(X) \to C(X,X)^2, g \mapsto (g,g^{-1})$ with respect to the compact open topology on $C(X,X)$.

I.3.2. Now let M be a compact smooth manifold and consider the Fréchet algebra $A := C^\infty(M,\mathbb{R})$ of smooth functions on M. Again one can show that

$$M \cong \mathrm{Hom}(C^\infty(M,\mathbb{R}),\mathbb{R}) \setminus \{0\}$$

in the sense that every non-zero algebra homomorphism $C^\infty(M,\mathbb{R}) \to \mathbb{R}$ is given by a point evaluation $\delta_p(f) := f(p)$ for some $p \in M$ (see Lemma I.3.5 below). Moreover, the smooth structure on M is completely determined by the requirement that the maps $M \to \mathbb{R}, p \mapsto \delta_p(f)$ are smooth. This implies that the group $\mathrm{Aut}(C^\infty(M,\mathbb{R}))$ of automorphisms of $C^\infty(M,\mathbb{R})$ can be identified with the group $\mathrm{Diff}(M)$ of all diffeomorphisms of M.

In sharp contrast to the topological context, the group $\mathrm{Diff}(M)$ has a non-trivial structure as a Lie group modeled on the space $\mathcal{V}(M)$ of (smooth) vector fields on M, which then is the Lie algebra of this group. Moreover, for a finite-dimensional Lie group G, smooth actions $\alpha: G \times M \to M$ correspond to Lie group homomorphisms $G \to \mathrm{Diff}(M)$. For $G = \mathbb{R}$ we obtain in particular the correspondence between smooth flows on M, smooth vector fields on M, and one-parameter subgroups of $\mathrm{Diff}(M)$.

If $X \in \mathcal{V}(M)$ is a vector field and $\mathrm{Fl}_X: \mathbb{R} \to \mathrm{Diff}(M)$ the corresponding flow, then

$$\exp: \mathcal{V}(M) \to \mathrm{Diff}(M), \quad X \mapsto \mathrm{Fl}_X(1)$$

is the exponential function of the Fréchet–Lie group $\mathrm{Diff}(M)$.

Other important groups of diffeomorphisms arise as subgroups of $\mathrm{Diff}(M)$. Of particular importance is the stabilizer subgroup $\mathrm{Diff}(M,\mu)$ of a volume form μ on M (if M is orientable), and the stabilizer $\mathrm{Sp}(M,\omega)$ of a symplectic form ω if (M,ω) is symplectic (cf. [KM97]).

I.3.3. If M is a non-compact σ-compact smooth manifold, then we still have

$$M \cong \mathrm{Hom}(C^\infty(M,\mathbb{R}),\mathbb{R}) \setminus \{0\} \quad \text{and} \quad \mathrm{Diff}(M) \cong \mathrm{Aut}(C^\infty(M,\mathbb{R})),$$

but then there is no natural Lie group structure on $\mathrm{Diff}(M)$ such that smooth actions of Lie groups G on M correspond to Lie group homomorphisms $G \to \mathrm{Diff}(M)$.

Nevertheless, in the framework of the "convenient setting" ([KM97]), one can turn $\mathrm{Diff}(M)$ into a Lie group with Lie algebra $\mathcal{V}_c(M)$, the Lie algebra of all smooth vector fields with compact support. If M is compact, this yields the natural Lie group structure on $\mathrm{Diff}(M)$, but if M is not compact, then the corresponding topology on $\mathrm{Diff}(M)$ is so fine that the global flow generated by a vector field whose support is not compact, does not lead to a continuous homomorphism $\mathbb{R} \to \mathrm{Diff}(M)$.

More recent investigations in this direction show that, at least for $M = \mathbb{R}^n$, the natural manifold structure on the group $\mathrm{Diff}_c(M)$ of all diffeomorphisms φ which coincide with id_M outside a compact set has a natural Lie group structure with Lie algebra $\mathcal{V}_c(M)$ ([Gl02]). Here we do not have to refer to the convenient setting with the advantage that $\mathrm{Diff}_c(M)$ is a topological group. This Lie group structure on $\mathrm{Diff}_c(M)$ can then be used to define a Lie group structure on $\mathrm{Diff}(M)$ for which $\mathrm{Diff}_c(M)$ is an open subgroup. This contrasts the results of Tatsuuma, Shimomura and Hirai, stating that the natural direct limit topology with respect to the subgroups

$$\mathrm{Diff}_K(M) := \{\varphi \in \mathrm{Diff}(M): \varphi|_{M\setminus K} = \mathrm{id}_{M\setminus K}\},$$

K a compact subset of M, does not turn $\mathrm{Diff}_c(M)$ into a topological group because the multiplication is not continuous.

I.3.4. The situation for non-compact manifolds is similar to the situation we encounter in the theory of unitary group representations. Let H be a Hilbert space and $U(H)$ its unitary group. This group has two natural topologies. The uniform topology on $U(H)$ inherited from the Banach algebra $B(H)$ turns it into a Banach–Lie group, but this topology is rather fine. The strong operator topology (the topology of pointwise convergence) turns $U(H)$ into a topological group such that continuous unitary representations of a topological group G correspond to continuous group homomorphisms $G \to U(H)$. If G is a finite-dimensional Lie group, then a continuous unitary representation is continuous with respect to the uniform topology on $U(H)$ if and only if all operators of the derived representation are bounded, but this implies already that the representation factors through a Lie group with compact Lie algebra (cf. [Si52], [Gu80]). In some sense the condition that the operators of the derived representation are bounded is analogous to the requirement that the vector fields corresponding to a smooth action on a manifold have compact support. In this sense the uniform topology on $U(H)$ shows similarities to the Lie group structure on $\mathrm{Diff}(M)$ if M is non-compact (see I.3.3). The case of a compact manifold M corresponds to the case of a finite-dimensional Hilbert space H, for which the two topologies on $U(H)$ coincide.

Lemma I.3.5. *If M is a compact manifold and $\chi: C^\infty(M,\mathbb{R}) \to \mathbb{R}$ a non-zero algebra homomorphism, then there exists $p \in M$ with $\chi(f) = f(p)$ for all $f \in C^\infty(M,\mathbb{R})$.*

Proof. Let $N := \ker\chi$. If there exists $p \in M$ such that all functions in N vanish at p, then $C^\infty(M,\mathbb{R}) = N \oplus \mathbb{R}\mathbf{1}$ implies $\chi = \delta_p$. Let us assume that this is not the case. Then there exists for each $p \in M$ a smooth function $f_p \in N$ with $f_p(p) \neq 0$. Then the open sets $f_p^{-1}(\mathbb{R}^\times)$ form an open covering of M, and we find finitely many points $p_1, \ldots, p_n$ such that M is covered by the sets $f_{p_j}^{-1}(\mathbb{R}^\times)$, which means that the function $f := \sum_{j=1}^n f_{p_j}^2 \in N$ vanishes nowhere. We conclude that the ideal N contains a unit and therefore $N = C^\infty(M,\mathbb{R})$, contradicting our assumption that χ is non-zero. ■

Remark I.3.6. If M is non-compact, then one has to modify the argument in the proof of Lemma I.3.5 as follows. First we observe that, since N is an ideal, we may assume that the support of the functions f_p is contained in a given neighborhood U_p of p because we may multiply f_p by any function supported in U_p and not vanishing at p.

Let $\chi: C^\infty(M,\mathbb{R}) \to \mathbb{R}$ be a continuous algebra homomorphism (with respect to the topology defined in Example II.1.4(b) below) and assume that $\chi \neq \delta_p$ for each $p \in M$. Hence the ideal $\ker\chi$ contains for each $p \in M$ a function not vanishing at p. We choose compact subsets $K_n \subseteq M$ with $\bigcup_n K_n = M$ and $K_n \subseteq K_{n+1}^0$. For $p \in K_n \setminus K_{n-1}$ we pick a function $f_p \in \ker\chi$ in such a way that $\mathrm{supp}(f_p) \subseteq K_{n+1} \setminus K_{n-1}$ and $f_p(p) \neq 0$. Now we choose the points $p_1, \ldots, p_{k_1} \in K_1$ such that $\sum_{j=1}^{k_1} f_{p_j}^2$ is positive on K_1, then $p_{k_1+1}, \ldots, p_{k_2}$ such that $\sum_{j=1}^{k_2} f_{p_j}^2$ is positive on K_2, and so on. The precautions from above ensure a that the series $f := \sum_{j=1}^\infty f_{p_j}^2$ converges in $C^\infty(M,\mathbb{R})$ because on each set K_n it is eventually constant. For each continuous character $\chi: C^\infty(M,\mathbb{R}) \to \mathbb{R}$ which is not a point evaluation we thus obtain an invertible function $f \in \ker\chi$, which implies $\chi = 0$. Here the continuity of χ is needed to ensure that $\ker\chi$ is closed and hence that $f \in \ker\chi$. ■

II. Infinite-dimensional manifolds

In this section $\mathbb{K}$ always stands for $\mathbb{R}$ or $\mathbb{C}$ and V is a $\mathbb{K}$-vector space.

II.1. Locally convex spaces

Definition II.1.1. (a) If p is a seminorm on a $\mathbb{K}$-vector space V, then $N_p := p^{-1}(0)$ is a subspace of V, and $V_p := V/N_p$ is a normed space with $\|v + N_p\| := p(v)$. Let $\alpha_p: V \to V_p$ denote the corresponding quotient map.
(b) We call a set $\mathcal{P}$ of seminorms on V *separating* if $p(v) = 0$ for all $p \in \mathcal{P}$ implies $v = 0$.
(c) If X is a set and $f_j: X \to X_j$, $j \in J$, mappings into topological spaces, then the coarsest topology on X for which all these maps are continuous is called the *initial topology on X with respect to the family $(f_j)_{j\in J}$*. This topology is generated by the inverse images of open subsets of the spaces X_j under the maps f_j.

(d) To each separating family $\mathcal{P}$ of seminorms on V we associate the initial topology $\tau_{\mathcal{P}}$ on V defined by the maps $\alpha_p: V \to V_p$ to the normed spaces V_p. We call it the *locally convex topology* on V defined by $\mathcal{P}$.

Since the family $\mathcal{P}$ is separating, V is a Hausdorff space. Further it is easy to show that V is a topological vector space in the sense that addition and scalar multiplication on V are continuous maps.

(e) A locally convex space V is called a *Fréchet space* if its topology can be defined by a countable family $\mathcal{P} = \{p_n : n \in \mathbb{N}\}$ of seminorms and if V is complete with respect to the compatible metric

$$d(x,y) := \sum_{n\in\mathbb{N}} 2^{-n} \frac{p_n(x-y)}{1+p_n(x-y)}. \qquad \blacksquare$$

Exercise II.1. Let $(V, \tau_{\mathcal{P}})$ be a locally convex space.

(1) Show that a seminorm q on V is continuous if and only if there exists a $\lambda > 0$ and $p_1, \ldots, p_n \in \mathcal{P}$ such that
$$q \leq \lambda \max(p_1, \ldots, p_n).$$

(2) Two sets $\mathcal{P}_1$ and $\mathcal{P}_2$ of seminorms on V define the same locally convex topology if and only if all seminorms in $\mathcal{P}_2$ are continuous w.r.t. $\tau_{\mathcal{P}_1}$ and vice versa. ■

Remark II.1.2. (a) A sequence $(x_n)_{n\in\mathbb{N}}$ in a locally convex space V is said to be a *Cauchy sequence* if each sequence $\alpha_p(x_n)$, $p \in \mathcal{P}$, is a Cauchy sequence in V_p. We say that V is *sequentially complete* if every Cauchy sequence in V converges.

(b) One has a natural notion of completeness of locally convex spaces (every Cauchy filter converges). Complete locally convex spaces then correspond to closed subspaces of products of Banach spaces [1]. ■

Examples II.1.3. (a) Let X be a topological space. For each compact subset $K \subseteq X$ we obtain a seminorm p_K on $C(X, \mathbb{R})$ by

$$p_K(f) := \sup\{|f(x)| : x \in K\}.$$

The family $\mathcal{P}$ of these seminorms defines on $C(X,\mathbb{R})$ the locally convex topology of uniform convergence on compact subsets of X.

If X is compact, then we may take $K = X$ and obtain a norm on $C(X,\mathbb{R})$ which defines the topology; all other seminorms p_K are redundant. In this case $C(X,\mathbb{R})$ is a Banach space.

(b) The preceding example can be generalized to the space $C(X,V)$, where X is a topological space and V is a locally convex space. Then we define for each compact subset $K \subseteq X$ and each continuous seminorm q on V a seminorm

$$p_{K,q}(f) := \sup\{q(f(x)) : x \in K\}.$$

The family of these seminorms defines a locally convex topology on $C(X,V)$ which again coincides with the topology of uniform convergence on compact subsets of X.

(c) If X is locally compact and σ-compact, then there exists a sequence $(K_n)_{n\in\mathbb{N}}$ of compact subsets of X with $\bigcup_n K_n$ and $K_n \subseteq K_n^0$. Then each compact subset of X lies in some K_n, so that each seminorm p_K is dominated by some p_{K_n}. This implies that $C(X,\mathbb{R})$ is metrizable, and since it is also complete, it is a Fréchet space. ■

[1] In §31.6 of Köthe's book [Kö69] one finds an example of a complete locally convex space X and a closed subspace $Y \subseteq X$ for which the quotient space X/Y is not complete. This does not happen if X is metrizable and complete, i.e., an F-space. Then all quotients of X by closed subspaces are complete.

Example II.1.4. (a) Let $U \subseteq \mathbb{R}^n$ be an open subset and consider the algebra $C^\infty(U, \mathbb{R})$. For each multiindex $m = (m_1, \ldots, m_n) \in \mathbb{N}_0$ with $|m| := m_1 + \ldots + m_n$ we consider the differential operator

$$D^m := D_1^{m_1} \cdots D_n^{m_n} := \frac{\partial^{|m|}}{\partial_1^{m_1} \cdots \partial_n^{m_n}}.$$

We now obtain for each m and each compact subset $K \subseteq U$ a seminorm on $C^\infty(U, \mathbb{R})$ by

$$p_{K,m}(f) := \sup\{|D^m f(x)|: x \in K\}.$$

The family of all these seminorms defines a locally convex topology on $C^\infty(U, \mathbb{R})$. Since U is locally compact and σ-compact (exercise), the topology on $C^\infty(U, \mathbb{R})$ can be defined by a countable set of seminorms. Moreover, it is not hard to see that $C^\infty(U, \mathbb{R})$ is complete with respect to the corresponding metric, hence a Fréchet space.
(b) Let M be a smooth n-dimensional manifold and consider the vector space $C^\infty(M, \mathbb{R})$. To introduce a topology on this algebra, for each compact subset $K \subseteq M$ for which there exists a chart $\varphi: U \to \mathbb{R}^n$ with $K \subseteq U$ and for each multiindex $m \in \mathbb{N}_0^n$ we define a seminorm by

$$p_{K,m}(f) := \sup\{|D^m(f \circ \varphi^{-1})(x)|: x \in \varphi(K)\}.$$

We thus obtain a natural Fréchet topology on $C^\infty(M, \mathbb{R})$ which is called the *topology of local uniform convergence of all partial derivatives.*
(c) If M is a complex manifold, then we consider the algebra $\mathrm{Hol}(M, \mathbb{C})$ of holomorphic functions on M as a subspace of $C(M, \mathbb{C})$, endowed with the topology of uniform convergence on compact subsets of M (Example II.1.3). This topology turns $\mathrm{Hol}(M, \mathbb{C})$ into a Fréchet space. Moreover, one can show that the injective map $\mathrm{Hol}(M, \mathbb{C}) \hookrightarrow C^\infty(M, \mathbb{C})$is also a topological embedding.■

Definition II.1.5. Let V be a vector space and $\alpha_j: V_j \to V$ linear maps, defined on locally convex spaces V_j. We consider the system $\mathcal{P}$ of all those seminorms p on V for which all compositions $p \circ \alpha_j$ are continuous seminorms on the spaces V_j. By means of $\mathcal{P}$, we obtain on V a locally convex topology called the *final locally convex topology* defined by the mappings $(\alpha_j)_{j \in J}$.

This locally convex topology has the universal property that a linear map $\varphi: V \to W$ into a locally convex space W is continuous if and only if all the maps $\varphi \circ \alpha_j$, $j \in J$, are continuous.■

Example II.1.6. (a) Let X be a locally compact space and $C_c(X, \mathbb{R})$ the space of compactly supported continuous functions. For each compact subset $K \subseteq X$ we then have a natural inclusion

$$\alpha_K: C_K(X, \mathbb{R}) := \{f \in C_c(X, \mathbb{R}): \mathrm{supp}(f) \subseteq K\} \hookrightarrow C_c(X, \mathbb{R}).$$

Each space $C_K(X, \mathbb{R})$ is a Banach space with respect to the norm

$$\|f\|_\infty := \sup\{|f(x)|: x \in X\}.$$

We endow $C_c(X, \mathbb{R})$ with the final locally convex topology defined by the maps α_K.
(b) Let M be a smooth manifold and consider the space $C_c^\infty(M, \mathbb{R})$ of smooth functions with compact support. For each compact subset $K \subseteq M$ we then have a natural inclusion

$$\alpha_K: C_K^\infty(M, \mathbb{R}) := \{f \in C_c^\infty(M, \mathbb{R}): \mathrm{supp}(f) \subseteq K\} \hookrightarrow C_c^\infty(M, \mathbb{R}).$$

We endow each space $C_K^\infty(M, \mathbb{R})$ with the subspace topology inherited from $C^\infty(M, \mathbb{R})$, which turns it into a Fréchet space. On $C_c^\infty(M, \mathbb{R})$ we now obtain the final locally convex topology defined by the maps α_K. ■

II.2. Calculus on locally convex spaces

In this section we explain briefly how calculus works in locally convex spaces. The main point is that one uses an appropriate notion of differentiability which for the special case of Banach spaces differs from Fréchet differentiability but which is more convenient in the setup of locally convex spaces. Our basic references are [Ha82] and [Gl01a], where one finds detailed proofs. One readily observes that once one has the Fundamental Theorem of Calculus, then the proofs of the Fréchet case carry over to a more general setup where one still requires smooth maps to be continuous (cf. also [Mil83]). A different approach to differentiability in infinite-dimensional spaces in the so-called convenient setting can be found in [FK88] and [KM97a]. A central feature of this approach is that smooth maps are no longer required to be continuous, but for calculus over Fréchet spaces one finds the same class of smooth maps described by Hamilton and Milnor. Another approach which also gives up the continuity of smooth maps and requires only their continuity on compact sets is discussed by E. G. F. Thomas in [Th96]. The concept of a diffeological space due to J.-M. Souriau ([So85]) goes even one step further. It is primarily designed to study spaces like quotients of $\mathbb{R}$ by non-discrete subgroups in a differential geometric context.

Definition II.2.1. Let X and Y be topological vector spaces, $U \subseteq X$ open and $f\colon U \to Y$ a map. Then the *derivative of f at x in the direction of h* is defined as

$$df(x)(h) := \lim_{t\to 0} \frac{1}{t}\big(f(x+th) - f(x)\big)$$

whenever the limit exists. The function f is called *differentiable at x* if $df(x)(h)$ exists for all $h \in X$. It is called *continuously differentiable or C^1* if it is differentiable at all points of U and

$$df\colon U \times X \to Y, \quad (x,h) \mapsto df(x)(h)$$

is a continuous map. It is called a C^n-map if df is a C^{n-1}-map, and C^∞ (or *smooth*) if it is C^n for all $n \in \mathbb{N}$. This is the notion of differentiability used in [Mil83], [Ha82], [Gl01a] and [Ne01].
(b) If X and Y are complex vector spaces, then the map f is called *holomorphic* if it is C^1 and for all $x \in U$ the map $df(x)\colon X \to Y$ is complex linear (cf. [Mil83, p. 1027]). We will see below that the maps $df(x)$ are always real linear (Lemma II.2.3).
(c) Higher derivatives are defined for C^n-maps by

$$d^n f(x)(h_1,\ldots,h_n) := \lim_{t\to 0} \frac{1}{t}\big(d^{n-1}f(x+th_n)(h_1,\ldots,h_{n-1}) - d^{n-1}f(x)(h_1,\ldots,h_{n-1})\big). \quad \blacksquare$$

Remark II.2.2. (a) If X and Y are Banach spaces, then the notion of continuous differentiability is weaker than the usual notion of continuous Fréchet-differentiability in Banach spaces, which requires that the map $x \mapsto df(x)$ is continuous with respect to the operator norm. Nevertheless, one can show that a C^2-map in the sense defined above is C^1 in the sense of Fréchet differentiability, so that the two concepts lead to the same class of C^∞-functions (cf. [Ne01, I.6 and I.7]).
(b) We also note that the existence of linear maps which are not continuous shows that the continuity of f does not follow from the differentiability of f because each linear map $f\colon X \to Y$ is differentiable at each $x \in X$ in the sense of Definition II.2.1(a). ■

Now we recall the precise statements of the most fundamental facts needed in the following.

Lemma II.2.3. *Let X and Y be locally convex spaces and $U \subseteq X$ an open subset. The following assertions hold:*

(i) *If* $f: U \to Y$ *is* C^1 *and* $x \in U$, *then* $df(x): X \to Y$ *is a linear map and* f *is continuous. If moreover* $x + th \in U$ *holds for all* $t \in [0,1]$, *then*

$$f(x+h) = f(x) + \int_0^1 df(x+uh)(h)\,du.$$

In particular f *is locally constant if and only if* $df = 0$.

(ii) *If* f *is* C^n, *then the functions* $(h_1, \ldots, h_n) \mapsto d^n f(x)(h_1, \ldots, h_n)$, $x \in U$, *are symmetric* n*-linear maps. For each* $x \in U$ *and* $v \in X$ *with* $x + tv \in U$ *for* $t \in [0,1]$ *we have the Taylor formula*

$$f(x+v) = f(x) + df(x)(v) + \ldots + \frac{1}{(n-1)!} d^{n-1} f(x)(v, \ldots, v) + \frac{1}{(n-1)!} \int_0^1 (1-t)^{n-1} d^n f(x+tv)(v, \ldots, v)\,dt.$$

Proof. (i) The first part is [Ha82, Th. 3.2.5] and the integral representation is [Ha82, Th. 3.2.2] for Fréchet spaces. For the refinement to locally convex spaces see [Gl01a]. This is based on the observation that, although the integral $\int_0^1 df(x+uh)(h)\,du$ exists a priori only in the completion of Y, the fact that it equals the difference $f(x+h) - f(x)$ implies that it is contained in Y. Therefore no completeness condition on Y is needed to ensure the existence of the integral.

To see that f is continuous, let p be a continuous seminorm on Y and $\varepsilon > 0$. Then there exists a balanced 0-neighborhood $U_1 \subseteq X$ with $x + U_1 \subseteq U$ and $p\big(df(x+uh)(h)\big) < \varepsilon$ for $u \in [0,1]$ and $h \in U_1$. Hence

$$p\big(f(x+h) - f(x)\big) \leq \int_0^1 p\big(df(x+uh)(h)\big)\,du \leq \varepsilon,$$

and thus f is continuous.
(ii) follows from [Ha82, Th. 3.6.2] and by iteration of (i). ■

Proposition II.2.4. (The chain rule) *If* X, Y *and* Z *are locally convex spaces,* $U \subseteq X$ *and* $V \subseteq Y$ *are open, and* $f_1: U \to V$, $f_2: V \to Z$ *are* C^1, *then* $f_2 \circ f_1: U \to Z$ *is* C^1 *with*

$$d(f_2 \circ f_1)(x) = df_2\big(f_1(x)\big) \circ df_1(x) \quad \text{for} \quad x \in U.$$

Proof. [Ha82, Th. 3.3.4] ■

Proposition II.2.5. *If* X_1, X_2 *and* Y *are locally convex spaces,* $X = X_1 \times X_2$, $U \subseteq X$ *is open, and* $f: U \to Y$ *is continuous, then the partial derivatives*

$$d_1 f(x_1, x_2)(h) := \lim_{t \to 0} \frac{1}{t}\big(f(x_1 + th, x_2) - f(x_1, x_2)\big)$$

and

$$d_2 f(x_1, x_2)(h) := \lim_{t \to 0} \frac{1}{t}\big(f(x_1, x_2 + th) - f(x_1, x_2)\big)$$

exist and are continuous if and only if df *exists and is continuous. In this case we have*

$$df(x_1, x_2)(h_1, h_2) = d_1 f(x_1, x_2)(h_1) + d_2 f(x_1, x_2)(h_2).$$

Proof. [Ha82, Th. 3.4.3] ■

Remark II.2.6. (a) If $f: X \to Y$ is a continuous linear map, then f is smooth with

$$df(x)(h) = f(h)$$

for all $x, h \in X$, and $d^n f = 0$ for $n \geq 2$.
(b) From (a) and Proposition II.2.5 it follows that a continuous k-linear map $m: X_1 \times \ldots \times X_k \to Y$ is continuously differentiable with

$$dm(x)(h_1, \ldots, h_k) = m(h_1, x_2, \ldots, x_k) + \cdots + m(x_1, \ldots, x_{k-1}, h_k).$$

Inductively one obtains that m is smooth with $d^{k+1}m = 0$.
(c) If $f: U \to Y$ is C^{n+1}, then Lemma II.2.3(ii) and Proposition II.2.5 imply that

$$\begin{aligned} d(d^n f)(x, h_1, \ldots, h_n)(y, k_1, \ldots, k_n) &= d^{n+1} f(x)(h_1, \ldots, h_n, y) \\ &+ d^n f(x)(k_1, h_2, \ldots, h_n) + \ldots + d^n f(x)(h_1, \ldots, h_{n-1}, k_n). \end{aligned}$$

It follows in particular that, whenever f is C^n, then f is C^{n+1} if and only if $d^n f$ is C^1.
(d) If $f: U \to Y$ is holomorphic, then the finite-dimensional theory shows that for each $h \in X$ the function $U \to Y, x \mapsto df(x)(h)$ is holomorphic. Hence $d^2 f(x)$ is complex bilinear and therefore $d(df)$ is complex linear. Thus $df: U \times X \to Y$ is also holomorphic. ■

Example II.2.7. In the definition of C^1-maps we have not required the underlying topological vector spaces to be locally convex and one may wonder whether this assumption is made for convenience or if there are some serious underlying reasons. The following example shows that local convexity is crucial to have a calculus with the properties discussed in Lemma II.2.3.

Let V denote the space of measurable functions $f: [0,1] \to \mathbb{R}$ for which

$$|f| := \int_0^1 |f(x)|^{\frac{1}{2}}\, dx$$

is finite and observe that $d(f,g) := |f - g|$ defines a metric on this space because the function $x \mapsto \sqrt{x}$ is subadditive on $\mathbb{R}^+$. We thus obtain a topological vector space (V, d).

For a subset $E \subseteq [0,1]$ let χ_E denote its characteristic function. Consider the curve

$$\gamma: [0,1] \to V, \quad \gamma(t) := \chi_{[0,t]}.$$

Then

$$|h^{-1}\big(\gamma(t+h) - \gamma(t)\big)| = |h|^{-\frac{1}{2}}|h| \to 0$$

for each $t \in [0,1]$ as $h \to 0$. Hence γ is C^1 with $d\gamma = 0$. Since γ is not constant, the Fundamental Theorem of Calculus does not hold in V.

The defect in this example is caused by the non-local convexity of V. In fact, one can even show that all continous linear functionals on V vanish. ■

Remark II.2.8. In the context of Banach spaces one has an Inverse Function Theorem and also an Implicit Function Theorem ([La99]). Such results cannot be expected in general for Fréchet spaces (cf. the exponential functions of certain Fréchet groups). Nevertheless, the recent paper [Hi99] contains an implicit function theorem for maps of the type $f: E \times F \to F$, where F is a Banach space and E is Fréchet. ■

Remark II.2.9. (Pathologies of linear ODEs in Fréchet spaces)
(a) First we give an example of a linear ODE for which solutions to initial value problems exist, but are not unique. We consider the Fréchet space $V := C^\infty([0,1], \mathbb{R})$ and the continuous linear operator $Lf := f'$ on this space. We are asking for solutions of the initial value problem

$$\gamma'(t) = L\gamma(t), \quad \gamma(0) = \gamma_0. \tag{2.1}$$

Let us assume that $\operatorname{supp}(\gamma_0)$ is a compact subset of $]0,1[$, so that γ_0 permits smooth extensions to a function on $\mathbb{R}$. Let h be such a function and consider

$$\gamma\colon \mathbb{R} \to V, \quad \gamma(t)(x) := h(t+x).$$

Then $\gamma(0) = h|_{[0,1]} = \gamma_0$ and $\gamma'(t)(x) = h'(t+x) = (L\gamma(t))(x)$. It is clear that these solutions of (2.1) depend on the choice of the extension h of γ_0. Different choices lead to different extensions.

Does every smooth function on $[0,1]$ have a smooth extension to $\mathbb{R}$?

(b) Now we consider the space $V := C^\infty(\mathbb{S}^1, \mathbb{C})$ which we identify with the space of 2π-periodic smooth functions on the real line. We consider the linear operator $Lf := -f''$ and the equation (2.1), which in this case is the heat equation with reversed time. It is easy to analyze this equation in terms of the Fourier expansion of γ. So let

$$\gamma(t)(x) = \sum_{n\in\mathbb{Z}} a_n(t) e^{inx}$$

be the Fourier expansion of $\gamma(t)$. Then (2.1) implies $a_n'(t) = n^2 a_n(t)$ for each $n \in \mathbb{Z}$, so that $a_n(t) = a_n(0)e^{tn^2}$ holds for any solution γ of (2.1). If the Fourier coefficients $a_n(0)$ of γ_0 do not satisfy

$$\sum_n |a_n(0)| e^{\varepsilon n^2} < \infty$$

for some $\varepsilon > 0$ (which need not be the case for a smooth function γ_0), then (2.1) does not have a solution on $[0, \varepsilon]$. ■

Remark II.2.10. (a) We briefly recall the basic definitions underlying the convenient calculus in [KM97]. Let E be a locally convex space. The *c^∞-topology* on E is the final topology with respect to the set $C^\infty(\mathbb{R}, E)$. We call E *convenient* if for each smooth curve $c_1\colon \mathbb{R} \to E$ there exists a smooth curve $c_2\colon \mathbb{R} \to E$ with $c_2' = c_1$ (cf. [KM97, p.20]).

Let $U \subseteq E$ be an open subset and $f\colon U \to F$ a function, where F is a locally convex space. Then we call f *conveniently smooth* if

$$f \circ C^\infty(\mathbb{R}, U) \subseteq C^\infty(\mathbb{R}, F).$$

This concept quite directly implies nice cartesian closedness properties for smooth maps (cf. [KM97, p.30]).

(b) If E is a sequentially complete locally convex (s.c.l.c.) space, then it is convenient because the sequential completeness implies the existence of Riemann integrals of continuous E-valued functions on compact intervals ([KM97, Th. 2.14]). If E is a Fréchet space, then the c^∞-topology coincides with the original topology ([KM97, Th. 4.11]).

Moreover, for an open subset U of a Fréchet space, a map $f\colon U \to F$ is conveniently smooth if and only if it is smooth in the sense of Definition II.2.1. This can be shown as follows. Since $C^\infty(\mathbb{R}, E)$ is the same space for both notions of differentiability, the chain rule shows that smoothness in the sense of Definition II.2.1 implies smoothness in the sense of convenient calculus. Now we assume that $f\colon U \to F$ is conveniently smooth. Then the derivative $df\colon U \times E \to F$ exists and defines a conveniently smooth map $df\colon U \to L(E,F) \subseteq C^\infty(E,F)$ ([KM97, Th. 3.18]). Hence $df\colon U \times E \to F$ is also conveniently smooth, and thus continuous with respect to the c^∞-topology. As $E \times E$ is a Fréchet space, it follows that df is continuous. Therefore f is C^1 in the sense of Definition II.2.1, and now one can iterate the argument. ■

II.3. Differentiable manifolds

Since we have a chain rule for C^1-maps between locally convex spaces, we can define smooth manifolds as one defines them in the finite-dimensional case (cf. [Ha82], [Mil83], [Gl01a]).

Let M be a Hausdorff topological space and X a locally convex space. An X-*chart* of an open subset $U \subseteq M$ is a homeomorphism $\varphi: U \to \varphi(U) \subseteq X$ onto an open subset $\varphi(U)$ of X. We denote such a chart as a pair (φ, U). Two charts (φ, U) and (ψ, V) are said to be *smoothly compatible* if the map

$$\psi \circ \varphi^{-1}|_{\varphi(U\cap V)}: \varphi(U \cap V) \to \psi(U \cap V)$$

is smooth. From the chain rule it follows right aways that compatibility of charts is an equivalence relation on the set of all X-charts of M. An X-*atlas of* M is a family $\mathcal{A} := (\varphi_i, U_i)_{i\in I}$ of pairwise compatible X-charts of M for which $\bigcup_i U_i = M$. A *smooth* X-*structure on* M is a maximal X-atlas and a smooth X-manifold is a pair $(M, \mathcal{A})$, where $\mathcal{A}$ is a maximal X-atlas on M.

Locally convex spaces are *regular* in the sense that each point has a neighborhood base consisting of closed sets, and this property is inherited by manifolds modeled on these spaces (cf. [Mil83]).

One defines the tangent bundle $\pi: TM \to M$ as follows. Let $\mathcal{A} := (\varphi_i, U_i)_{i\in I}$ be an X-atlas of M. On the disjoint union of the set $\varphi(U_i) \times X$ we define an equivalence relation by

$$(x, v) \sim \big((\varphi_j \circ \varphi_i^{-1})(x), d(\varphi_j \circ \varphi_i^{-1})(x)(v)\big)$$

for $x \in \varphi_i(U_i \cap U_j)$ and write $[x, v]$ for the equivalence class of (x, v). Let $p \in U_i$. Then the equivalence classes of the form $[\varphi_i(p), v]$ are called *tangent vectors in* p. Since all the differentials $d(\varphi_j \circ \varphi_i^{-1})(x)$ are invertible linear maps, it easily follows that the set $T_p(M)$ of all tangent vectors in p forms a vector space isomorphic to X under the map $X \to T_p(M), v \mapsto [x, v]$. Now we turn the *tangent bundle*

$$TM := \bigcup_{p\in M} T_p(M)$$

into a manifold by the charts

$$\psi_i: TU_i \to \varphi(U_i) \times X, \quad [\varphi_i(x), v] \mapsto (\varphi_i(x), v).$$

It is easy to see that for each open subset U of a locally convex space X we have $TU \cong U \times X$ and in particular $TU_j \cong U_j \times X$ in the setting from above.

We will call a manifold modeled on a l.c. space, resp., Fréchet space, resp., Banach space a *locally convex*, resp., *Fréchet*, resp., *Banach manifold.*

Note that it is far more subtle to define a cotangent bundle because this requires a locally convex topology on the dual space E' of the underlying vector space E and therefore depends on this topology.

Let M and N be smooth manifolds modeled on locally convex spaces and $f: M \to N$ a smooth map. We write $Tf: TM \to TN$ for the corresponding map induced on the level of tangent vectors. Locally this map is given by

$$Tf(x, h) = \big(f(x), df(x)(h)\big),$$

where $df(p): T_p(M) \to T_{f(p)}(N)$ denotes the differential of f at p. In view of Remark II.2.6(c), the tangent map Tf is also smooth if f is smooth. In the following we will always identify M with the zero section in TM. In this sense we have $Tf|_M = f$ with $Tf(M) \subseteq N \subseteq TN$.

A *vector field* X on M is a smooth section of the tangent bundle $TM \to M$. We write $\mathcal{V}(M)$ for the space of all vector fields on M. If $f \in C^\infty(M, \mathbb{C})$ is a smooth function on M and $X \in \mathcal{V}(M)$, then we obtain a smooth function on M via

$$(X.f)(p) := df(p)\big(X(p)\big).$$

Since locally $X(p) = \big(p, \widetilde{X}(p)\big)$, where $\widetilde{X}$ is a smooth function, we have $X.f = df \circ X$. Therefore the smoothness of $X.f$ follows from the smoothness of the maps $df: TM \to \mathbb{C}$ and $X: M \to TM$.

Lemma II.3.1. *If $X, Y \in \mathcal{V}(M)$, then there exists a vector field $[X,Y] \in \mathcal{V}(M)$ which is uniquely determined by the property that on each open subset $U \subseteq M$ we have*

$$[X,Y].f = X.(Y.f) - Y.(X.f) \tag{3.1}$$

for all $f \in C^\infty(U, \mathbb{C})$.

Proof. Locally the vector fields X and Y are given as $X(p) = \big(p, \widetilde{X}(p)\big)$ and $Y(p) = \big(p, \widetilde{Y}(p)\big)$. We define a vector field by

$$[X,Y]\widetilde{\ }(p) := d\widetilde{Y}(p)\big(\widetilde{X}(p)\big) - d\widetilde{X}(p)\big(\widetilde{Y}(p)\big). \tag{3.2}$$

Then the smoothness of the right hand side follows from the chain rule. The requirement that (3.1) holds on continuous linear functionals f determines $[X,Y]\widetilde{\ }$ uniquely. An easy calculation shows that (3.2) defines in fact a smooth vector field on M (cf. Lemma II.3.3 below). Now the assertion follows because locally (3.1) is a consequence of the chain rule. ■

Proposition II.3.2. $(\mathcal{V}(M), [\cdot,\cdot])$ *is a Lie algebra.*

Proof. The crucial part is to check the Jacobi identity. This follows from the observation that if $U \subseteq X$ is an open subset of a locally convex space, then the mapping

$$\Phi\colon \mathcal{V}(U) \to \operatorname{Der}\big(C^\infty(U,\mathbb{C})\big), \quad \Phi(X)(f) = X.f$$

is injective and satisfies $\Phi([X,Y]) = [\Phi(X), \Phi(Y)]$. Therefore the Jacobi identity in $\mathcal{V}(U)$ follows from the Jacobi identity in the associative algebra $\operatorname{End}\big(C^\infty(U,\mathbb{C})\big)$. ■

For the applications to Lie groups we will need the following lemma.

Lemma II.3.3. *Let M and N be smooth manifolds and $\varphi\colon M \to N$ a smooth map. Suppose that $X_N, Y_N \in \mathcal{V}(N)$ and $X_M, Y_M \in \mathcal{V}(M)$ satisfy $X_N \circ \varphi = T\varphi \circ X_M$ and $Y_N \circ \varphi = T\varphi \circ Y_M$. Then $[X_N, Y_N] \circ \varphi = T\varphi \circ [X_M, Y_M]$.*

Proof. It suffices to perform a local calculation. Therefore we may w.l.o.g. assume that $M \subseteq F$ is open, where F is a locally convex space and that N is a locally convex space. Then

$$[X_N, Y_N]\widetilde{\ }\big(\varphi(p)\big) = d\widetilde{Y}_N\big(\varphi(p)\big).\widetilde{X}_N\big(\varphi(p)\big) - d\widetilde{X}_N\big(\varphi(p)\big).\widetilde{Y}_N\big(\varphi(p)\big).$$

Next we note that our assumption implies that $\widetilde{Y}_N \circ \varphi = d\varphi \circ (\mathrm{id}_F \times \widetilde{Y}_M)$. Using the chain rule we obtain

$$d\widetilde{Y}_N\big(\varphi(p)\big) d\varphi(p) = d(d\varphi)\big(p, \widetilde{Y}_M(p)\big) \circ \big(\mathrm{id}_F, d\widetilde{Y}_M(p)\big)$$

which, in view of Remark II.2.6(c), leads to

$$\begin{aligned} d\widetilde{Y}_N\big(\varphi(p)\big).\widetilde{X}_N\big(\varphi(p)\big) &= d\widetilde{Y}_N\big(\varphi(p)\big) d\varphi(p).\widetilde{X}_M(p) \\ &= d(d\varphi)\big(p, \widetilde{Y}_M(p)\big) \circ \big(\mathrm{id}_F, d\widetilde{Y}_M(p)\big).\widetilde{X}_M(p) \\ &= d^2\varphi(p)\big(\widetilde{Y}_M(p), \widetilde{X}_M(p)\big) + d\varphi(p)\big(d\widetilde{Y}_M(p).\widetilde{X}_M(p)\big). \end{aligned}$$

Now the symmetry of the second derivative (Lemma II.2.3(ii)) implies that

$$[X_N, Y_N]\widetilde{\ }\big(\varphi(p)\big) = d\varphi(p)\big(d\widetilde{Y}_M(p).\widetilde{X}_M(p) - d\widetilde{X}_M(p).\widetilde{Y}_M(p)\big) = d\varphi(p)\big([X_M, Y_M]\widetilde{\ }(p)\big).$$ ■

Differential forms

Definition II.3.4. If M is a differentiable manifold and V a locally convex space, then a *V-valued k-form* ω on M is a function ω which associates to each $p \in M$ a k-linear alternating map $T_p(M)^k \to V$ such that in local coordinates the map $(p, v_1, \ldots, v_k) \mapsto \omega(p)(v_1, \ldots, v_k)$ is smooth. We write $\Omega^k(M, V)$ for the space of smooth k-forms on M with values in V. The differentials

$$d\colon \Omega^k(M, V) \to \Omega^{k+1}(M, V)$$

and the wedge products

$$\wedge\colon \Omega^k(M, \mathbb{C}) \times \Omega^l(M, \mathbb{C}) \to \Omega^{k+l}(M, \mathbb{C})$$

are defined by the same formulas as in the finite-dimensional case. ■

The assumption that V is sequentially complete is crucial in the following lemma to ensure the existence of the Riemann integral defining φ.

Lemma II.3.5. (Poincaré Lemma) *Let E be locally convex, V an s.c.l.c. space and $U \subseteq E$ an open subset which is star-shaped with respect to 0. Let $\omega \in \Omega^{k+1}(U, V)$ be a V-valued closed $(k+1)$-form. Then ω is exact. Moreover, $\omega = d\varphi$ for some $\varphi \in \Omega^k(U, V)$ with $\varphi(0) = 0$ given by*

$$\varphi(x)(v_1, \ldots, v_k) = \int_0^1 t^k \omega(tx)(x, v_1, \ldots, v_k)\, dt.$$

Proof. For the case of Fréchet spaces Remark II.2.10 implies that the assertion follows from [KM97, Lemma 33.20]. On the other hand, one can prove it directly in the context of locally convex spaces by using the fact that one may differentiate under the integral a function of the type $\int_0^1 H(t, x)\, dt$, where H is a smooth function $]-\varepsilon, 1+\varepsilon[\times U \to V$ (cf. [KM97, p.32]). The existence of the integrals follows from the sequential completeness of V. For the calculations needed for the proof we refer to [La99, Th. V.4.1]. ■

Remark II.3.6. (a) The Poincaré Lemma is the first step to de Rham's Theorem. To obtain de Rham's Theorem for finite-dimensional manifolds, one makes heavy use of smooth partitions of unity which do not always exist for infinite-dimensional manifolds, not even for Banach manifolds.
(b) We call a smooth manifold M *smoothly paracompact* if every open cover has a subordinated smooth partition of unity. De Rham's Theorem holds for every smoothly paracompact Fréchet manifold ([KM97,Thm. 34.7]). Smoothly Hausdorff second countable manifolds modeled on a smoothly regular space are smoothly paracompact ([KM97, 27.4]). Typical examples of smoothly regular spaces are nuclear Fréchet spaces ([KM97, Th. 16.10]).
(c) Examples of Banach spaces which are not smoothly paracompact are $C([0,1], \mathbb{R})$ and $l^1(\mathbb{N}, \mathbb{R})$. On these spaces there exists no non-zero smooth function supported in the unit ball ([KM97, 14.11]). ■

Proposition II.3.7. *Let M be a connected manifold, V an s.c.l.c. space and $\alpha \in \Omega^1(M, V)$ a closed 1-form. Then there exists a connected covering $q\colon \widehat{M} \to M$ and a smooth function $f\colon \widehat{M} \to V$ with $df = q^*\alpha$.*

Proof. (Sketch) We consider the product set $P := M \times V$ with the two projection maps $F\colon P \to V$ and $q\colon P \to M$. We define a topology on P as follows. For each pair (U, f) consisting of an open subset $U \subseteq M$ and a smooth function $f\colon U \to V$ with $df = \alpha \mid_U$ the graph $\Gamma(f, U) := \{(x, f(x))\colon x \in U\}$ is a subset of P. These sets form a basis for a topology τ on P.

With respect to this topology the mapping $q\colon P \to M$ is a covering map. To see this, let $x \in M$. Since M is a manifold, there exists a neighborhood U of x which is diffeomorphic to a convex subset of a locally convex space. Then the Poincaré Lemma implies for each $v \in V$

the existence of a smooth function f_v on U with $df_v = \alpha\,|_U$ and $f_v(x) = v$. Since U is connected, the function f_v is uniquely determined by its value at x, so that $f_v = f_0 + v$. Now $q^{-1}(U) = U \times V = \bigcup_{v\in V} \Gamma(f_v, U)$ is a disjoint union of open subsets of P (here we use the connectedness of U), and therefore q is a covering. We conclude that P carries a natural manifold structure for which q is a local diffeomorphism. For this manifold structure the function $F\colon P \to V$ is smooth with $dF = q^*\alpha$.

Now we fix a point $x_0 \in M$ and an element $v_0 \in V$. Then the connected component $\widehat{M}$ of (x_0, v_0) in P is a connected covering manifold of M with the required properties. ■

Corollary II.3.8. *If M is a simply connected manifold and V an s.c.l.c. space, then $H^1_{\rm dR}(M,V)$ vanishes.*

Proof. Let α be a closed V-valued 1-form on M. Using Proposition II.3.7, we find a covering $q\colon \widehat{M} \to M$ and a smooth function $f\colon \widehat{M} \to V$ with $df = q^*\alpha$. Since M is simply connected, the covering q is trivial, hence a diffeomorphism. Therefore α is exact. ■

Theorem II.3.9. *Let M be a connected manifold, V an s.c.l.c. space, $x_0 \in M$, and $\pi_1(M) := \pi_1(M, x_0)$. Then we have an inclusion*

$$\zeta\colon H^1_{\rm dR}(M,V) \hookrightarrow \mathrm{Hom}(\pi_1(M), V)$$

which is given on a piecewise differentiable loop $\gamma\colon [0,1] \to M$ in x_0 for $\alpha \in Z^1_{\rm dR}(M,V)$ by

$$\zeta(\alpha)(\gamma) := \zeta([\alpha])([\gamma]) := \int_\gamma \alpha := \int_0^1 \gamma^*\alpha.$$

The homomorphism $\zeta([\alpha])$ can also be calculated as follows: Let $q\colon \widetilde{M} \to M$ be the universal covering map, and write $\widetilde{M} \times \pi_1(M) \to \widetilde{M}, (x,g) \mapsto \mu_g(x)$ for the right action of $\pi_1(M)$ on $\widetilde{M}$. Further pick $f_\alpha \in C^\infty(\widetilde{M}, V)$ with $df_\alpha = q^*\alpha$. Then the function $f_\alpha \circ \mu_g - f_\alpha$ is constant equal to $\zeta([\alpha])(g)$.

Proof. (cf. Theorem XIV.1.7 in [God71]) Let $q\colon \widetilde{M} \to M$ be a simply connected covering manifold and $y_0 \in q^{-1}(x_0)$. In view of Corollary II.3.8, for each closed 1-form α on M, the closed 1-form $q^*\alpha$ on $\widetilde{M}$ is exact. Let $f_\alpha \in C^\infty(\widetilde{M}, V)$ with $\widetilde{f}_\alpha(y_0) = 0$ and $d\widetilde{f}_\alpha = q^*\alpha$.

Let $\widetilde{M} \times \pi_1(M) \to \widetilde{M}, (y,g) \mapsto \mu_g(y) := y.g$ denote the action of $\pi_1(M)$ on $\widetilde{M}$ by deck transformations. We put

$$\zeta(\alpha)(g) := f_\alpha(y_0.g).$$

Then $\zeta(\alpha)(\mathbf{1}) = 0$ and

$$\begin{aligned}\zeta(\alpha)(g_1g_2) &= f_\alpha(y_0.g_1g_2) = f_\alpha(y_0.g_1g_2) - f_\alpha(y_0.g_1) + f_\alpha(y_0.g_1)\\ &= f_\alpha(y_0.g_1g_2) - f_\alpha(y_0.g_1) + \zeta(\alpha)(g_1).\end{aligned}$$

For each $g \in \pi_1(M)$ the function $h := \mu_g^*f_\alpha - f_\alpha$ satisfies $h(y_0) = \zeta(\alpha)(g) = f_\alpha(y_0.g)$ and

$$dh = \mu_g^*df_\alpha - df_\alpha = \mu_g^*q^*\alpha - q^*\alpha = (q\circ\mu_g)^*\alpha - q^*\alpha = q^*\alpha - q^*\alpha = 0.$$

Therefore h is constantly $\zeta(\alpha)(g)$, and we obtain $\zeta(\alpha)(g_1g_2) = \zeta(\alpha)(g_2) + \zeta(\alpha)(g_1)$. This proves that $\zeta(\alpha) \in \mathrm{Hom}(\pi_1(M), V)$.

Suppose that $\zeta(\alpha) = 0$. Then $\mu_g^*f_\alpha - f_\alpha = 0$ holds for each $g \in \pi_1(M)$, showing that the function f_α factors through a smooth function $f\colon M \to V$ with $f\circ q = f_\alpha$. Now $q^*df = df_\alpha = q^*\alpha$ implies $df = \alpha$, so that α is exact. Conversely, if α is exact, then the function f_α is invariant under $\pi_1(M)$, and we see that $\zeta(\alpha) = 0$. Therefore $\zeta\colon Z^1_{\rm dR}(M,V) \to \mathrm{Hom}(\pi_1(M), V)$ factors through an inclusion $H^1_{\rm dR}(M,V) \hookrightarrow \mathrm{Hom}(\pi_1(M), V)$.

Finally, let $[\gamma] \in \pi_1(M)$, where $\gamma\colon [0,1] \to M$ is piecewise smooth. Let $\widetilde{\gamma}\colon [0,1] \to \widetilde{M}$ be a lift of γ with $\widetilde{\gamma}(0) = y_0$. Then

$$\begin{aligned}\zeta([\alpha])([\gamma]) &= f_\alpha([\gamma]) = f_\alpha(\widetilde{\gamma}(1)) = f_\alpha(\widetilde{\gamma}(0)) + \int_0^1 df_\alpha(\widetilde{\gamma}(t))\big(\widetilde{\gamma}'(t)\big)\,dt\\ &= f_\alpha(y_0) + \int_0^1 (q^*\alpha)(\widetilde{\gamma}(t))\big(\widetilde{\gamma}'(t)\big)\,dt = \int_0^1 \alpha(\gamma(t))\big(\gamma'(t)\big)\,dt = \int_0^1 \gamma^*\alpha = \int_\gamma \alpha.\end{aligned}$$

■

The following lemma shows that exactness of a vector-valued 1-form can be tested by looking at the associated scalar-valued 1-forms.

Lemma II.3.10. *Let $\alpha \in \Omega^1(M,V)$ be a closed 1-form. If for each continuous linear functional λ on V the 1-form $\lambda \circ \alpha$ is exact, then α is exact.*

Proof. If $\lambda\circ\alpha$ is exact, then the group homomorphism $\zeta(\alpha)\colon \pi_1(M) \to V$ satisfies $\lambda\circ\zeta(\alpha) = 0$ (Theorem II.3.9). If this holds for each $\lambda \in V'$, then the fact that the continuous linear functionals on the locally convex space V separate points implies that $\zeta(\alpha) = 0$ and hence that α is exact. ■

To see that the map ζ is surjective, one needs smooth paracompactness which is not always available, not even for Banach manifolds. For an infinite-dimensional version of de Rham's Theorem for smoothly paracompact manifolds we refer to [KM97, Thm. 34.7] (cf. Remark II.3.6(b)). The following proposition is a particular consequence:

Proposition II.3.11. *If M is a connected smoothly paracompact manifold, then the inclusion map $\zeta\colon H^1_{\rm dR}(M,V) \to \mathrm{Hom}(\pi_1(M),V)$ is bijective.* ■

Proposition II.3.12. *Let M be a connected manifold, V an s.c.l.c. space and $\Gamma \subseteq V$ a discrete subgroup. Then V/Γ carries a natural manifold structure such that the tangent space at every element of V/Γ can be canonically identified with V. For a smooth function $f\colon M \to V/\Gamma$ we can thus identify the differential df with a V-valued 1-form on M. For a closed V-valued 1-form α on M the following conditions are equivalent:*

(1) *There exists a smooth function $f\colon M \to V/\Gamma$ with $df = \alpha$.*

(2) $\zeta(\alpha)\big(\pi_1(M)\big) \subseteq \Gamma$.

Proof. Let $q\colon \widetilde M \to M$ denote the universal covering map and fix a point $x_0 \in \widetilde M$. Then the closed 1-form $q^*\alpha$ on $\widetilde M$ is exact (Theorem II.3.9), so that there exists a unique smooth function $\widetilde f\colon \widetilde M \to V$ with $d\widetilde f = q^*\alpha$ and $\widetilde f(x_0) = 0$. In Theorem II.3.9 we have seen that for each $g \in \pi_1(M)$ we have

$$\mu_g^*\widetilde f - \widetilde f = \zeta(\alpha)(g). \tag{3.3}$$

(1) $\Rightarrow$ (2): Let $p\colon V \to V/\Gamma$ denote the quotient map. We may w.l.o.g. assume that $f\big(q(x_0)\big) = p(0)$. The function $p\circ \widetilde f\colon \widetilde M \to V/\Gamma$ satisfies $d(p\circ\widetilde f) = q^*\alpha$, and the same is true for $f\circ q\colon \widetilde M \to V/\Gamma$. Since both have the same value at x_0, we see that $p\circ\widetilde f = f\circ q$. This proves that $p\circ\widetilde f$ is invariant under $\pi_1(M)$, and therefore (3.3) shows that $\zeta(\alpha)\big(\pi_1(M)\big) \subseteq \Gamma$.

(2) $\Rightarrow$ (1): If (2) is satisfied, then (3.3) implies that the function $p\circ\widetilde f\colon \widetilde M \to V/\Gamma$ is $\pi_1(M)$-invariant, hence factors through a function $f\colon M \to V/\Gamma$ with $f\circ q = p\circ\widetilde f$. Then f is smooth and satisfies $q^*df = d\widetilde f = q^*\alpha$, which implies that $df = \alpha$. ■

Smoothly non-trivial bundles

Remark II.3.13. Another remarkable pathology occurring already for Banach spaces is that there exists a closed subspace F of a Banach space E such that the quotient map $q\colon E \to E/F$ has no smooth sections. The existence of a smooth local section $\sigma\colon U \to E$ around $0 \in E/F$ would imply the existence of a closed complement $\mathrm{im}(d\sigma(0)) \cong E/F$ to F in E, but such a space does not exist. A simply example is the subspace $c_0(\mathbb{N},\mathbb{R})$ in $l^\infty(\mathbb{N},\mathbb{R})$ ([Wer95, Satz IV.6.5]).

Nevertheless, the map $q\colon E \to E/F$ defines the structure of a topological F-principal bundle over E/F which has a continuous global section by Michael's Selection Theorem ([Mi59]). ■

III. Infinite-dimensional Lie groups

III.1. Infinite-dimensional Lie groups and their Lie algebras

Definition III.1.1. A *locally convex Lie group* G is a locally convex manifold endowed with a group structure such that the multiplication map and the inversion map are smooth. ■

In our treatment of Lie groups we basically follow [Mil83]. Throughout this subsection G denotes a locally convex Lie group. For $g \in G$ we write $\lambda_g: G \to G, x \mapsto gx$ for the *left multiplication* by g and $\rho_g: G \to G, x \mapsto xg$ for the *right multiplication* by g. Both are diffeomorphisms of G. Moreover, we write $m: G \times G \to G, (x,y) \mapsto xy$ for the *multiplication map* and $\eta: G \to G, x \mapsto x^{-1}$ for the *inversion*.

Definition III.1.2. Let G be a Lie group. Then for each $g \in G$ the map

$$c_g: G \to G, \quad x \mapsto gxg^{-1},$$

is a smooth automorphism, hence induces a continuous linear automorphism

$$\mathrm{Ad}(g) := dc_g(\mathbf{1}): \mathfrak{g} \to \mathfrak{g}.$$

We thus obtain an action $G \times \mathfrak{g} \to \mathfrak{g}, (g, X) \mapsto \mathrm{Ad}(g).X$ called the *adjoint action* of G on $\mathfrak{g}$. ■

Proposition III.1.3. *For a Lie group G the following assertions hold:*

(i) $dm(g_1, g_2)(X_1, X_2) = d\rho_{g_2}(g_1).X_1 + d\lambda_{g_1}(g_2).X_2$ *and in particular we have*

$$dm(\mathbf{1}, \mathbf{1})(X_1, X_2) = X_1 + X_2.$$

(ii) $d\eta(\mathbf{1}).X = -X$.

(iii) *The mapping $Tm: TG \times TG \to TG$ defines a Lie group structure on TG with identity element $0 \in T_{\mathbf{1}}(G)$ and inversion $T\eta$.*

(iv) *Let $\mathfrak{g} := T_{\mathbf{1}}(G)$ denote the tangent space at the identity. Then the mapping*

$$\Phi: G \times \mathfrak{g} \to TG, \quad (g, X) \mapsto d\lambda_g(\mathbf{1}).X$$

is a diffeomorphism. Multiplication and inversion in TG are given by

$$\Phi(g_1, X_1) \cdot \Phi(g_2, X_2) = \Phi\big(g_1 g_2, \mathrm{Ad}(g_2)^{-1}.X_1 + X_2\big)$$
$$\Phi(g, X)^{-1} = \Phi\big(g^{-1}, -\mathrm{Ad}(g).X\big).$$

Proof. (i) We have

$$dm(g_1, g_2)(X_1, X_2) = dm(g_1, g_2)(X_1, 0) + dm(g_1, g_2)(0, X_2) = d\rho_{g_2}(g_1).X_1 + d\lambda_{g_1}(g_2).X_2.$$

(ii) From $m \circ (\mathrm{id}_G \times \eta) = \mathbf{1}$, we derive $0 = dm(\mathbf{1}, \mathbf{1})\big(X, d\eta(\mathbf{1}).X\big) = X + d\eta(\mathbf{1}).X$ and hence the assertion.

(iii) First we note that for a product of two smooth manifolds M and N we have a canonical diffeomorphism $T(M \times N) \to TM \times TN$. Since the multiplication map $m: G \times G \to G$ is smooth, the same holds for its tangent map

$$Tm: T(G \times G) \cong TG \times TG \to TG.$$

Let $\varepsilon: G \to \{\mathbf{1}\}$ denote the constant map and $u: \{\mathbf{1}\} \to G$ the group homomorphism representing the identity element. Then the group axioms for G are encoded in the relations $m \circ (m \times \mathrm{id}) = m \circ (\mathrm{id} \times m)$ (associativity), $m \circ (\eta \times \mathrm{id}) = m \circ (\mathrm{id} \times \eta) = \varepsilon$ (inversion), and $m \circ (u \times \mathrm{id}) = m \circ (\mathrm{id} \times u) = \mathrm{id}$ (unit element). Using the functorial properties of T, we see that these properties carry over to the corresponding maps on TG and show that TG is a Lie group with multiplication Tm, inversion $T\eta$, and unit element $\Phi(\mathbf{1}, 0)$.
(iv) The smoothness of Φ follows from the smoothness of Tm and $\Phi(g, X) = Tm(g, X)$ for $(g, X) \in G \times T_{\mathbf{1}}(G) \subseteq T(G) \times T(G)$ and the fact that the restriction of Tm to $G \times T_{\mathbf{1}}(G) \subseteq TG \times TG$ is smooth.

To see that Φ^{-1} is also smooth, let $\pi: TG \to G$ denote the canonical projection. Then

$$\Phi^{-1}: TG \to G \times \mathfrak{g}, \quad v \mapsto \big(\pi(v), d\lambda_{\pi(v)^{-1}}\big(\pi(v)\big).v\big) = (\pi(v), \pi(v)^{-1}.v)$$

and the smoothness of the group operations on TG imply the smoothness of Φ^{-1}.

To derive an explicit formula for the multiplication in terms of the trivialization given by Φ, we calculate

$$\begin{aligned}\Phi(g_1, X_1) \cdot \Phi(g_2, X_2) &= dm(g_1, g_2)\big(d\lambda_{g_1}(\mathbf{1}).X_1, d\lambda_{g_2}(\mathbf{1}).X_2\big) \\ &= d\rho_{g_2}(g_1) d\lambda_{g_1}(\mathbf{1}).X_1 + d\lambda_{g_1}(g_2) d\lambda_{g_2}(\mathbf{1}).X_2 \\ &= d\lambda_{g_1 g_2}(\mathbf{1})\big(d\lambda_{g_2}^{-1}(g_2) d\rho_{g_2}(\mathbf{1}).X_1 + X_2\big) \\ &= \Phi\big(g_1 g_2, \mathrm{Ad}(g_2)^{-1}.X_1 + X_2\big).\end{aligned}$$

The formula for the inversion follows directly from this formula. ■

One of the main consequences of Proposition III.1.3(iv) is that the tangent bundle of a Lie group is trivial, so that we can identify $\mathcal{V}(G)$ with $C^\infty(G, \mathfrak{g})$. We write $\mathcal{V}(G)^l \subseteq \mathcal{V}(G)$ for the subspace of *left invariant* vector fields, i.e., those satisfying

$$X(g) = d\lambda_g(\mathbf{1}).X(\mathbf{1}) \tag{1.1}$$

for all $g \in G$ or, equivalently, $X \circ \lambda_g = T(\lambda_g) \circ X$ if we consider X as a section $X : G \to TG$ of the tangent bundle TG. These are the vector fields that correspond to constant functions $G \to \mathfrak{g}$. We see in particular that each left invariant vector field is smooth, so that the mapping

$$\mathcal{V}(G)^l \to \mathfrak{g}, \quad X \mapsto X(\mathbf{1})$$

is a bijection. Moreover, Lemma II.3.3 implies that for $X, Y \in \mathcal{V}(G)^l$ we have

$$[X, Y] \circ \lambda_g = T(\lambda_g) \circ [X, Y],$$

i.e., that $[X, Y] \in \mathcal{V}(G)^l$. Hence there exists a unique Lie bracket $[\cdot, \cdot]$ on $\mathfrak{g}$ satisfying

$$[X, Y](\mathbf{1}) = [X(\mathbf{1}), Y(\mathbf{1})]$$

for all left invariant vector fields on G.

Definition III.1.4. The Lie algebra $\mathbf{L}(G) := (\mathfrak{g}, [\cdot, \cdot]) := (T_{\mathbf{1}}(G), [\cdot, \cdot])$ is called *the Lie algebra of* G. ■

Proposition III.1.5. *For a Lie group G the following assertions hold:*

(i) *If $X_l: G \to TG$ is a left invariant vector field with $X_l(\mathbf{1}) = X$, then $X_r: g \mapsto -X_l(g)^{-1}$ is a right-invariant vector field with $X_r(\mathbf{1}) = X$. The assignment $\mathfrak{g} \to \mathcal{V}(G)^r, X \mapsto X_r$ is an antiisomorphism of Lie algebras.*

(ii) *If $\sigma: G \times M \to M$ is a smooth action of G on the smooth manifold M, then the map $T\sigma: TG \times TM \to TM$ is a smooth action of TG on TM. The assignment*

$$\dot\sigma: \mathfrak{g} \to \mathcal{V}(M), \quad \textit{with} \quad \dot\sigma(X)(p) := -d\sigma(\mathbf{1}, p)(X, 0)$$

defines a homomorphism of Lie algebras.

Proof. (i) In view of Proposition III.1.3(ii), we have

$$X_r(g) = -d\eta(g^{-1}).X_l(g^{-1}) = -d\eta(g^{-1})d\lambda_{g^{-1}}(\mathbf{1}).X = -d\rho_g(\mathbf{1})d\eta(\mathbf{1}).X = d\rho_g(\mathbf{1}).X$$

and this proves the first part. The second part follows from Lemma II.3.3 which shows that

$$[X_r, Y_r](g) = d\eta(g^{-1}).[X_l, Y_l](g^{-1}) = d\eta(g^{-1}).[X, Y]_l(g^{-1}) = -[X, Y]_r(g).$$

(ii) That $T\sigma$ defines an action of TG on TM follows in the same way as in (iii) above by applying T to the commutative diagrams defining a group action.

For the second part we pick $p \in M$ and write $\varphi_p: G \to M, g \mapsto g.p$ for the smooth orbit map of p. Then the equivariance of φ_p means that $\varphi_p \circ \rho_g = \varphi_{g.p}$. From this we derive

$$-d\varphi_p(g).X_r(g) = -d\varphi_p(g)d\rho_g(\mathbf{1}).X = -d\varphi_{g.p}(\mathbf{1}).X = \dot\sigma(X)(g.p).$$

Therefore Lemma II.3.3 and (i) imply that

$$\dot\sigma([X, Y])(p) = -d\varphi_p(\mathbf{1})[X, Y]_r(\mathbf{1}) = d\varphi_p(\mathbf{1})[X_r, Y_r](\mathbf{1}) = [\dot\sigma(X), \dot\sigma(Y)](p).$$ ■

Proposition III.1.6. *The adjoint action* $\mathrm{Ad}: G \times \mathfrak{g} \to \mathfrak{g}, (g, x) \mapsto \mathrm{Ad}(g).x$ *is smooth. The operators*

$$\mathrm{ad}\, x: \mathfrak{g} \to \mathfrak{g}, \quad \mathrm{ad}\, x(y) := d\,\mathrm{Ad}(\mathbf{1}, y)(x, 0)$$

satisfy

$$\mathrm{ad}\, x(y) = [x, y].$$

In particular the bracket in $\mathfrak{g}$ *is continuous.*

Proof. The smoothness of the adjoint action of G on $\mathfrak{g}$ follows directly from the smoothness of the multiplication of the Lie group TG (Proposition III.1.3).

To calculate the linear maps $\mathrm{ad}\, x: \mathfrak{g} \to \mathfrak{g}$, we consider a local chart $\varphi: V \to \mathfrak{g}$ of G, where $V \subseteq G$ is an open $\mathbf{1}$-neighborhood and $\varphi(\mathbf{1}) = 0$. Let $W \subseteq V$ be an open symmetric $\mathbf{1}$-neighborhood with $WW \subseteq V$. Then we have on the open set $\varphi(W) \subseteq \mathfrak{g}$ the smooth multiplication

$$x * y := \varphi(\varphi^{-1}(x)\varphi^{-1}(y)), \quad x, y \in \varphi(W).$$

From $Tm(\mathbf{1}, \mathbf{1})(v, w) = v + w$ we immediately see that the Taylor series of $*$ is given by

$$x * y = x + y + b(x, y) + R(x, y),$$

where $R(x, y)$ is a smooth function whose derivatives up to order 2 vanish at $(0, 0)$, and $b: \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is a continuous bilinear map.

For $x \in W$ let $\lambda_x^*: W \to W, y \mapsto x * y$. Then the left invariant vector field corresponding to $v \in \mathfrak{g}$ is given on $\varphi(W)$ by

$$v_l(x) = d\lambda_x^*(0).v,$$

and in 0 the first and second order term of its Taylor series is $v + b(x, v)$. Therefore

$$[v, w] = [v_l, w_l](0) = dw_l(0).v_l(0) - dv_l(0).w_l(0) = dw_l(0).v - dv_l(0).w = b(v, w) - b(w, v).$$

This implies that the Lie bracket on $\mathfrak{g}$ is continuous.

For $x \in \varphi(W)$ we write $x^{-1} = \alpha_1(x) + \alpha_2(x) + S(x)$, where α_1 is linear, α_2 is quadratic and $S(x)$ stands for terms of order at least 3. Now

$$0 = x * x^{-1} = x + \alpha_1(x) + \alpha_2(x) + b(x, \alpha_1(x)) + \ldots$$

and by comparing terms of order 1 and 2, we get $\alpha_1(x) = -x$ and $\alpha_2(x) = -b(x, -x) = b(x, x)$. Therefore

$$\begin{aligned}(x * y) * x^{-1} &= \big(x + y + b(x, y)\big) + \big(-x + b(x, x)\big) + b(x + y, -x) + \cdots \\ &= y + b(x, y) - b(x, y) + \cdots,\end{aligned}$$

and by taking the derivative w.r.t. x in 0 in the direction z, we eventually get

$$\mathrm{ad}\, z.y = b(z, y) - b(y, z) = [z, y].$$ ■

From topological groups to Lie groups

The following lemma is helpful to obtain Lie group structures on topological groups.

Lemma III.1.7. *Let G be a connected topological group and $K = K^{-1}$ be an open $\mathbf{1}$-neighborhood in G. We further assume that K is a smooth manifold such that the inversion is smooth on K and there exists an open $\mathbf{1}$-neighborhood $V \subseteq K$ with $V^2 \subseteq K$ such that the group multiplication $m: V \times V \to K$ is smooth. Then there exists a unique structure of a Lie group on G for which the inclusion map $K \hookrightarrow G$ induces a diffeomorphism on open neighborhoods of $\mathbf{1}$.*

Proof. (cf. [Ch46, §14, Prop. 2] or [Ti83, p.14] for the finite-dimensional case) After shrinking V and K, we may assume that there exists a diffeomorphism $\varphi: K \to \varphi(K) \subseteq E$, where E is a locally convex space, that V satisfies $V = V^{-1}$, $V^4 \subseteq K$, and that $m: V^2 \times V^2 \to K$ is smooth. For $g \in G$ we consider the maps

$$\varphi_g: gV \to E, \quad \varphi_g(x) = \varphi(g^{-1}x)$$

which are homeomorphisms of gV onto $\varphi(V)$. We claim that $(\varphi_g, gV)_{g\in G}$ is a smooth atlas of G.

Let $g_1, g_2 \in G$ and put $W := g_1V \cap g_2V$. If $W \neq \emptyset$, then $g_2^{-1}g_1 \in VV^{-1} = V^2$. The smoothness of the map

$$\psi := \varphi_{g_2} \circ \varphi_{g_1}^{-1}|_{\varphi_{g_1}(W)}: \varphi_{g_1}(W) \to \varphi_{g_2}(W)$$

given by

$$\psi(x) = \varphi_{g_2}(\varphi_{g_1}^{-1}(x)) = \varphi_{g_2}(g_1\varphi^{-1}(x)) = \varphi(g_2^{-1}g_1\varphi^{-1}(x))$$

follows from the smoothness of the multiplication $V^2 \times V \to K$. This proves that the charts $(\varphi_g, gK)_{g\in G}$ form an atlas of G. Moreover, the construction implies that all left translations of G are smooth maps.

The construction also shows that for each $g \in V$ the conjugation $c_g: G \to G, x \mapsto gxg^{-1}$ is smooth in a neighborhood of $\mathbf{1}$. Since the set of all these g is a submonoid of G containing V, it contains V^n for each $n \in \mathbb{N}$, hence all of G because G is connected and thus generated by V. Therefore all conjugations and hence all right multiplications are smooth. The smoothness of the inversion follows from its smoothness on V and the fact that left and right multiplications are smooth. Finally the smoothness of the multiplication follows from the smoothness at $\mathbf{1} \times \mathbf{1}$ because

$$m_G(g_1x, g_2y) = g_1xg_2y = g_1g_2c_{g_2^{-1}}(x)y = g_1g_2m_G(c_{g_2^{-1}}(x), y).$$

The uniqueness of the Lie group structure is clear because each locally diffeomorphic bijective homomorphism between Lie groups is a diffeomorphism. ∎

III.2. Homomorphisms of Lie groups and Lie algebras

In this section we study the interplay between homomorphisms of Lie groups and Lie algebras. This is very much in the spirit of differentiation and integration in elementary calculus. In the finite-dimensional Lie theory one has three basic facts on homomorphisms between Lie groups:

(1) Every homomorphism $\varphi: G \to H$ between Lie groups induces a Lie algebra homomorphism $\mathbf{L}(\varphi): \mathbf{L}(G) \to \mathbf{L}(H)$.

(2) If G is connected, then φ is determined uniquely by $\mathbf{L}(\varphi)$.

(3) If G is simply connected, then every Lie algebra homomorphism $\mathbf{L}(G) \to \mathbf{L}(H)$ is an $\mathbf{L}(\varphi)$ for some group homomorphism $\varphi: G \to H$.

These results are still morally true for infinite-dimensional Lie groups, but one has to refine the assumptions. There are no problems with (1). Also (2) is still true, which is slightly remarkable because solutions of ordinary differential equations on Fréchet spaces are in general not uniquely determined by an initial condition (Remark II.2.9). Nevertheless, we will see that this uniqueness of solutions holds for the ODEs that we have to consider for (2) because we can reduce it to the fact that a C^1-map with vanishing derivative is locally constant (Lemma II.2.3). Property (3) is more subtle. For that we will assume that H is a regular Lie group (defined below). All Banach–Lie groups and all Lie groups which are quotients of s.c.l.c. spaces modulo discrete subgroups are regular, and, moreover, no non-regular Lie group modeled on an s.c.l.c. space is known.

From Lie group homomorphisms to Lie algebra homomorphisms

Lemma III.2.1. *Let $\varphi: G \to H$ be a homomorphism of Lie groups. Then*

$$\mathbf{L}(\varphi) := d\varphi(\mathbf{1}): \mathbf{L}(G) \to \mathbf{L}(H)$$

is a homomorphism of their Lie algebras.

Proof. Let $x, y \in \mathbf{L}(G) = T_{\mathbf{1}}(G)$ and x_l, y_l the corresponding left invariant vector fields. Then $\varphi \circ \lambda_g = \lambda_{\varphi(g)} \circ \varphi$ for each $g \in G$ implies that

$$T\varphi \circ x_l = \mathbf{L}(\varphi)(x)_l \circ \varphi \quad \text{and} \quad T\varphi \circ y_l = \mathbf{L}(\varphi)(y)_l \circ \varphi$$

and therefore

$$T\varphi \circ [x_l, y_l] = [\mathbf{L}(\varphi)(x)_l, \mathbf{L}(\varphi)(y)_l] \circ \varphi$$

(Lemma II.3.3). Evaluating at $\mathbf{1}$, we obtain

$$\mathbf{L}(\varphi).[x, y] = [\mathbf{L}(\varphi)(x), \mathbf{L}(\varphi)(y)].$$ ■

Remark III.2.2. The preceding lemma implies that the assignment $G \mapsto \mathbf{L}(G)$ and $\varphi \mapsto \mathbf{L}(\varphi)$ defines a functor $\mathbf{L}$ from the category of locally convex Lie groups to the category of locally convex Lie algebras. ■

Definition III.2.3. (a) Let G be a Lie group and $I \subseteq \mathbb{R}$ an interval. For a smooth curve $\gamma: I \to G$ we define its *left logarithmic derivative* $\delta^l(\gamma): I \to \mathfrak{g}$ by

$$\delta^l(\gamma)(t) := \gamma(t)^{-1}.\gamma'(t) = d\lambda_{\gamma(t)^{-1}}(\gamma(t)).\gamma'(t),$$

where $\gamma(t)^{-1}.\gamma'(t)$ has to be read in the group TG (cf. Proposition III.1.3).

The *right logarithmic derivative* of γ is likewise defined by

$$\delta^r(\gamma)(t) := \gamma'(t).\gamma(t)^{-1} = d\rho_{\gamma(t)^{-1}}(\gamma(t)).\gamma'(t).$$

(b) Let M be a smooth manifold. The notion of logarithmic derivative generalizes naturally to smooth maps $\gamma: M \to G$. We define the *left logarithmic derivative* $\delta^l(\gamma) \in \Omega^1(M, \mathfrak{g})$ (see Definition II.3.4) by

$$\delta^l(\gamma)(x) := \gamma(x)^{-1}.d\gamma(x), \quad T_x M \to \mathfrak{g}$$

and the *right logarithmic derivative* by

$$\delta^r(\gamma)(x) := d\gamma(x).\gamma(x)^{-1}, \quad T_x M \to \mathfrak{g}$$ ■

Lemma III.2.4. *For smooth functions $\gamma_i: M \to G$, $i = 1, 2$, we have*

$$\delta^r(\gamma_1\gamma_2) = \delta^r(\gamma_1) + \mathrm{Ad}(\gamma_1) \circ \delta^r(\gamma_2)$$

and

$$\delta^l(\gamma_1\gamma_2) = \delta^l(\gamma_2) + \mathrm{Ad}(\gamma_2)^{-1} \circ \delta^l(\gamma_1).$$

Proof. This follows from a straightforward verification. ■

The following lemma provides a uniqueness result for the equation

$$\delta^l(\gamma) = f, \quad f \in \Omega^1(M, \mathfrak{g}).$$

Lemma III.2.5. *If two smooth functions $\gamma_1, \gamma_2 : M \to G$ have the same left logarithmic derivative and M is connected, then there exists $g \in G$ with $\gamma_1 = \lambda_g \circ \gamma_2$.*

Proof. We have to show that the function $x \mapsto \gamma_1(x)\gamma_2(x)^{-1}$ is locally constant, hence constant because M is connected. First we obtain with Lemma III.2.4

$$\delta^l(\gamma_1\gamma_2^{-1}) = \delta^l(\gamma_2^{-1}) + \mathrm{Ad}(\gamma_2)\delta^l(\gamma_1) = \delta^l(\gamma_2^{-1}) + \mathrm{Ad}(\gamma_2)\delta^l(\gamma_2) = \delta^l(\gamma_2\gamma_2^{-1}) = 0.$$

This implies that $d(\gamma_1\gamma_2^{-1})$ vanishes, and hence that $\gamma_1\gamma_2^{-1}$ is locally constant. ∎

Lemma III.2.6. *If $f : M \to G$ is a smooth map and $\varphi : G \to H$ is a homomorphism of Lie groups, then*

$$\delta^l(\varphi \circ f) = \mathbf{L}(\varphi) \circ \delta^l(f) \quad \textit{and} \quad \delta^r(\varphi \circ f) = \mathbf{L}(\varphi) \circ \delta^r(f).$$

Proof. Let $x \in M$. Then $\varphi \circ \lambda_g = \lambda_{\varphi(g)} \circ \varphi$ implies that

$$T\varphi \circ T\lambda_g = T\lambda_{\varphi(g)} \circ T\varphi : TG \to TH.$$

Applying $T\varphi$ to the map $df = f.\delta^l(f) : TM \to TG$, we thus obtain

$$d(\varphi \circ f) = (\varphi \circ f).\big(\mathbf{L}(\varphi) \circ \delta^l(f)\big)$$

and therefore

$$\delta^l(\varphi \circ f) = \mathbf{L}(\varphi) \circ \delta^l(f).$$

The corresponding assertion for the right logarithmic derivative is proved in a similar way. ∎

Proposition III.2.7. *Let G be a connected Lie group and $\varphi_1, \varphi_2 : G \to H$ two Lie group homomorphisms for which the corresponding Lie algebra homomorphisms $\mathbf{L}(\varphi_1)$ and $\mathbf{L}(\varphi_2)$ coincide. Then $\varphi_1 = \varphi_2$.*

Proof. ([Mil83, Lemma 7.1]) Let $g \in G$. Since G is connected, there exists a smooth curve $\gamma : [0,1] \to G$ with $\gamma(0) = \mathbf{1}$ and $\gamma(1) = g$. Let $\varphi_1, \varphi_2 : G \to H$ be two Lie group homomorphisms with $\mathbf{L}(\varphi_1) = \mathbf{L}(\varphi_2)$. Then Lemma III.2.6 implies that the two curves $\eta_i := \varphi_i \circ \gamma : [0,1] \to G$ have the same left logarithmic derivative. Since both curves have the value $\mathbf{1}$ in 0, they coincide by Lemma III.2.5. Therefore

$$\varphi_1(g) = \eta_1(1) = \eta_2(1) = \varphi_2(g),$$

which proves that $\varphi_1 = \varphi_2$. ∎

Corollary III.2.8. *If G is a connected Lie group, then* $\ker \mathrm{Ad} = Z(G)$.

Proof. Let $c_g(x) = gxg^{-1}$. In view of Lemma III.1.14, for $g \in G$ the conditions $c_g = \mathrm{id}_G$ and $\mathbf{L}(c_g) = \mathrm{Ad}(g) = \mathrm{id}_{\mathfrak{g}}$ are equivalent. This implies the assertion. ∎

Regular Lie groups

Definition III.2.9. A Lie group G is called *regular* if for each closed interval $I \subseteq \mathbb{R}$, $0 \in I$, and $X \in C^\infty(I, \mathbf{L}(G))$ the initial value problem (IVP)

$$\gamma(0) = \mathbf{1}, \quad \delta^l(\gamma) = X, \tag{2.1}$$

has a solution $\gamma_X \in C^\infty(I, G)$ and the evolution map

$$\mathrm{evol}_G : C^\infty(\mathbb{R}, \mathbf{L}(G)) \to G, \quad X \mapsto \gamma_X(1)$$

is smooth.

For a regular Lie group G we define the *exponential function*

$$\exp: \mathbf{L}(G) \to G \quad \text{by} \quad \exp(X) := \gamma_X(1),$$

where $X \in \mathbf{L}(G)$ is considered as a constant function $\mathbb{R} \to \mathbf{L}(G)$. As a restriction of the smooth function evol_G, the exponential function is smooth. ■

For a general Lie group G we call a smooth function $\exp_G : \mathfrak{g} \to G$ an *exponential function for* G if for each $X \in \mathfrak{g}$ the curve $\gamma_X(t) := \exp(tX)$ is a solution of the IVP (2.1). According to Lemma III.2.5, such a solution is unique whenever it exists. Therefore a Lie group has at most one exponential function.

Remark III.2.10. (a) As a direct consequence of the existence of solutions to ordinary differential equations on open domains of Banach spaces and their smooth dependence on initial values and parameters, every Banach–Lie group is regular.
(b) All known Lie groups modeled on s.c.l.c. spaces are regular.

Let $A \subseteq C([0,1], \mathbb{C})$ denote the subalgebra of all rational functions endowed with the induced norm $\|f\| := \sup_{0 \le t \le 1} |f(t)|$. In [Gl01c, Sect. 7] it is shown that the unit group $A^\times$ of the algebra A is a Lie group but that its exponential function is only defined on the subspace $\mathbb{C}\mathbf{1}$ of $\mathbf{L}(A^\times) = A$.
(c) If V is an s.c.l.c. vector space, then V is a regular Lie group because the Fundamental Theorem of Calculus holds for curves in V. The smoothness of the evolution map is trivial in this case because it is a continuous linear map. Regularity is trivially inherited by all Lie groups $Z = V/\Gamma$, where $\Gamma \subseteq V$ is a discrete subgroup.
(d) If, conversely, Z is a regular Fréchet–Lie group, then the exponential function $\exp: V \to Z_0$ is a universal covering homomorphism of the identity component Z_0 of Z. Hence $Z_0 \cong V/\Gamma$, where $\Gamma := \ker \exp \cong \pi_1(Z)$ ([MT99]). ■

One of the main points of the notion of regularity is provided by the following theorem.

Theorem III.2.11. *If H is a regular Lie group, G is a simply connected Lie group, and $\varphi: \mathfrak{g} \to \mathfrak{h}$ is a continuous homomorphism of Lie algebras, then there exists a unique Lie group homomorphism $\alpha: G \to H$ with $d\alpha(\mathbf{1}) = \varphi$.*

Proof. This is Theorem 8.1 in [Mil83] (see also [KM97, Th. 40.3]). The uniqueness assertion follows from Proposition III.2.7. The idea is to proceed as follows. Since G is connected, there exists for each $g \in G$ a smooth function $\gamma: [0,1] \to G$ with $g = \gamma(1)$ and $\gamma(0) = \mathbf{1}$. Then the regularity of H implies the existence of a solution $\eta: [0,1] \to H$ of the IVP

$$\eta(0) = \mathbf{1} \quad \text{and} \quad \delta^l(\eta) = \varphi \circ \delta^l(\gamma).$$

We now want to define $\alpha(g) := \eta(1)$. It remains to verify that α is well defined and a smooth Lie group homomorphism.

First we need the relation

$$\varphi \circ \mathrm{Ad}(\gamma(t)) = \mathrm{Ad}(\eta(t)) \circ \varphi, \quad 0 \le t \le 1. \tag{2.2}$$

To obtain this relation, we first observe that the curve $\gamma_v(t) := \mathrm{Ad}(\gamma(t))^{-1}.v$ satisfies the differential equation

$$\gamma_v'(t) = -[\delta^l(\gamma)(t), \mathrm{Ad}(\gamma(t))^{-1}.v] = -[\delta^l(\gamma(t)), \gamma_v(t)].$$

Hence $\beta(t) := \mathrm{Ad}(\eta(t)).\varphi(\gamma_v(t))$ satisfies

$$\beta'(t) = -\mathrm{Ad}(\eta(t)) \circ \varphi.[\delta^l(\gamma(t)), \gamma_v(t)] + \mathrm{Ad}(\eta(t)).[\delta^l(\eta(t)), \varphi \circ \gamma_v(t)] = 0.$$

We conclude that $\beta(t) = \beta(0)$, which implies $\mathrm{Ad}(\eta(t)) \circ \varphi = \varphi \circ \mathrm{Ad}(\gamma(t))$.

Now we can show that the definition of α attempted above will define a group homomorphism. For curves η_i, $i = 1, 2$, with $\delta^l(\eta_i) = \varphi \circ \delta^l(\gamma_i)$ we use (2.2) to get

$$\delta^l(\eta_1\eta_2) = \delta^l(\eta_2) + \mathrm{Ad}(\eta_2)^{-1}\delta^l(\eta_1) = \varphi \circ \big(\delta^l(\gamma_2) + \mathrm{Ad}(\gamma_2)^{-1}\delta^l(\gamma_1)\big) = \varphi \circ \delta^l(\gamma_1\gamma_2),$$

so that $\eta_1\eta_2$ corresponds to the product curve $\gamma_1\gamma_2$.

For the remaining arguments including that α is well defined, we refer to [Mil83]. ■

Corollary III.2.12. *Let G be a simply connected Lie group, V an s.c.l.c. space, and $\alpha\colon \mathfrak{g} \to V$ a continuous Lie algebra homomorphism. Then there exists a unique smooth group homomorphism $f\colon G \to V$ with $df(\mathbf{1}) = \alpha$.*

Proof. Since every s.c.l.c. vector space V is a regular Lie group (Remark III.2.10), the assertion follows from Theorem III.2.11. Alternatively we can argue with Proposition II.3.12. ■

Lemma III.2.13. *If G is a Lie group with exponential function $\exp\colon \mathfrak{g} \to G$, then*

$$d\exp(0) = \mathrm{id}_{\mathfrak{g}}.$$

Proof. For $X \in \mathfrak{g}$ we have $\exp(X) = \gamma_X(1)$, where γ_X is a solution of the IVP

$$\gamma(0) = \mathbf{1}, \quad \delta^l(\gamma) = X.$$

This implies in particular that $\exp(tX) = \gamma_{tX}(1) = \gamma_X(t)$ and hence

$$d\exp(X) = d\gamma_X(0) = X. \qquad ■$$

The preceding lemma is not so useful in the infinite-dimensional context as it is in the finite-dimensional or Banach context. For Banach–Lie groups it follows from the Inverse Function Theorem that exp restricts to a diffeomorphism of some open 0-neighborhood in $\mathfrak{g}$ to an open $\mathbf{1}$-neighborhood in G, so that we can use the exponential function to obtain charts around $\mathbf{1}$. We will see below that this conclusion does not work for Fréchet–Lie groups because in this context there is no general Inverse Function Theorem. This observation also implies that to integrate Lie algebra homomorphisms to group homomorphisms it will in general not be enough to start with the prescription $\alpha(\exp_G x) := \exp_H \varphi(x)$ in the context of Theorem III.2.11 because the image of $\exp_G$ need not contain an identity neighborhood in G.

III.3. Some classes of examples

Linear Lie groups

Proposition III.3.1. *If A is a continuous inverse algebra, then its unit group $A^\times$ is a Lie group with Lie algebra A.*

Proof. Since $A^\times$ is an open subset of A, it carries a natural manifold structure. Moreover, the multiplication on A is bilinear and continuous, hence a smooth map. Therefore it remains to see that the inversion $\eta\colon A^\times \to A^\times$ is smooth. The assumptions on a c.i.a. imply that η is continuous.

For $a, b \in A^\times$ we have

$$b^{-1} - a^{-1} = a^{-1}(a-b)b^{-1},$$

which implies that for $t \in \mathbb{R}$ we get

$$\eta(a+th) - \eta(a) = (a+th)^{-1} - a^{-1} = a^{-1}(-th)a^{-1} = -ta^{-1}ha^{-1}.$$

Therefore η is everywhere differentiable with

$$d\eta(a)(h) = -a^{-1}ha^{-1}.$$

Now the continuity of η implies that $d\eta\colon A^\times \times A \to A$ is continuous, hence that η is a C^1-map. Iterating this argument, we conclude from the chain rule that η is smooth. ■

Remark III.3.2. (a) If A is a unital Banach algebra, then A is a continuous inverse algebra and therefore $A^\times$ is a Lie group. This applies in particular to the group $\mathrm{GL}(X) = B(X)^\times$ for a Banach space X.

(b) If A is a unital c.i.a., so is $M_n(A)$, and therefore $\mathrm{GL}_n(A) := M_n(A)^\times$ is a Lie group.

(c) If M is a compact manifold and B is a c.i.a., then $A := C^\infty(M,B)$ is a c.i.a. with unit group $A^\times = C^\infty(M,B^\times)$. For $B = M_n(C)$ for a c.i.a. C we obtain in particular that $C^\infty(M,\mathrm{GL}_n(C)) \cong \mathrm{GL}_n(C^\infty(M,C))$ is a Lie group. ■

Current groups

Definition III.3.3. If X is a topological space and K a topological group, then we consider $C(X,K)$ with the group structure given by pointwise multiplication:

$$(fg)(x) := f(x)g(x), \quad x \in X.$$

For a compact subset C of X and an identity neighborhood $U \subseteq K$ we define

$$W(C,U) := \{f \in C(X,K)\colon f(C) \subseteq U\}.$$

The sets $W(C,U)$ form a neighborhood basis for a group topology on $C(X,K)$ called the *topology of uniform convergence on compact subsets of* X. In this sense $C(X,K)$ carries a natural structure of a topological group. ■

Definition III.3.4. Let M be a finite-dimensional manifold and K a Lie group. Then we obtain a natural topology on the group $G := C^\infty(M,K)$ as follows.

Let $\mathfrak{k} := \mathbf{L}(K)$ denote the Lie algebra of K. Then the tangent bundle TK of K is a Lie group isomorphic to $\mathfrak{k} \rtimes K$, where K acts on $\mathfrak{k}$ by the adjoint representation (Proposition III.2.2). Iterating this procedure, we obtain a Lie group structure on all higher tangent bundles $T^n K$, which are diffeomorphic to $\mathfrak{k}^{2^n-1} \times K$.

For each $n \in \mathbb{N}_0$ we obtain topological groups $C(T^n M, T^n K)$ by using the topology of uniform convergence on compact subsets of $T^n M$ (Definition III.3.3). Therefore the canonical inclusion map

$$C^\infty(M,K) \hookrightarrow \prod_{n\in\mathbb{N}_0} C(T^n M, T^n K)$$

leads to a natural topology on $C^\infty(M,K)$ turning it into a topological group.

For compact manifolds M these groups can even be turned into Lie groups with Lie algebra $C^\infty(M,\mathfrak{k})$. Here $C^\infty(M,\mathfrak{k})$ is endowed with the topology defined above if we consider $\mathfrak{k}$ as an additive Lie group. The charts of G can be obtained easily from those of K as follows. If $\varphi: U \to \mathfrak{k}$ is a chart of K, i.e., a diffeomorphism of an open subset $U \subseteq K$ onto an open subset $\varphi(U)$ of $\mathfrak{k}$, then the set $U_M := \{f \in G : f(M) \subseteq U\}$ is an open subset of G and the maps

$$\varphi_M: U_M \to \mathfrak{g} := C^\infty(M,\mathfrak{k}), \quad f \mapsto \varphi \circ f$$

define an atlas of G. For details we refer to [Gl01b].

If $\exp_K: \mathfrak{k} \to K$ is an exponential function of K, then we immediately obtain an exponential function

$$\exp_G: \mathfrak{g} = C^\infty(M,\mathfrak{k}) \to G = C^\infty(M,K), \quad \xi \mapsto \exp_K \circ \xi. \qquad \blacksquare$$

Diffeomorphism groups

In this subsection we discuss the diffeomorphism group $\mathrm{Diff}(M)$ of a compact manifold M. We will explain how this group can be turned into a Lie group with Lie algebra $\mathfrak{g} = \mathcal{V}(M)$, the Lie algebra of smooth vector fields on M.

One difficulty arising for diffeomorphism groups is that, although they have an exponential function, this exponential function is not a local diffeomorphism of a 0-neighborhood in $\mathfrak{g}$ onto an identity neighborhood in G. Therefore we cannot use the exponential function to define charts for G. But there is an easy way around this problem.

Let g be a Riemannian metric on M and

$$\mathrm{Exp}: TM \to M$$

be its exponential function, which assigns to $v \in T_p(M)$ the point $\gamma(1)$, where $\gamma: [0,1] \to M$ is the geodesic segment with $\gamma(0) = p$ and $\gamma'(0) = v$. We then obtain a smooth map

$$\Phi: TM \to M \times M, \quad v \mapsto (p, \mathrm{Exp}\, v), \quad v \in T_p(M).$$

There exists an open neighborhood $U \subseteq TM$ of the zero section such that Φ maps U diffeomorphically onto an open neighborhood of the diagonal in $M \times M$. Now

$$U_\mathfrak{g} := \{X \in \mathcal{V}(M) : X(M) \subseteq U\}$$

is an open subset of the Fréchet space $\mathcal{V}(M)$, and we define a map

$$\varphi: U_\mathfrak{g} \to C^\infty(M,M), \quad \Phi(X)(p) := \mathrm{Exp}(X(p)).$$

It is clear that $\varphi(0) = \mathrm{id}_M$. It is not hard to show that after shrinking $U_\mathfrak{g}$, we may w.l.o.g. assume $\varphi(U_\mathfrak{g}) \subseteq \mathrm{Diff}(M)$. To see that $\mathrm{Diff}(M)$ carries a Lie group structure for which φ is a chart, one has to verify that the group operations are smooth in a 0-neighborhood when transfered to $U_\mathfrak{g}$ via φ. Then Lemma III.1.7 applies after $\mathrm{Diff}(M)$ is endowed with a group topology for which φ is a homeomorphism.

Remark III.3.5. (a) If M and N are compact manifolds, then the mapping space $C^\infty(M,N)$ has a natural manifold structure for which the tangent space $T_f(C^\infty(M,N))$ coincides with the space of smooth sections of the bundle $f^*TN \to M$.
(b) For a compact manifold the group $\mathrm{Diff}(M)$ is open in the space $C^\infty(M,M)$, so that one can also use (a) to get a natural manifold structure on $\mathrm{Diff}(M)$. To verify that $\mathrm{Diff}(M)$ is open, one picks a Riemannian metric g on M and defines

$$\delta(f) := \inf\Big\{\frac{d(f(x),f(y))}{d(x,y)} : x \neq y \in M\Big\}.$$

Then one shows that δ is continuous on $C^\infty(M,M)$ and that

$$\mathrm{Diff}(M) = \{f \in C^\infty(M,M) : \delta(f) > 0\}. \qquad \blacksquare$$

Below we show that the exponential function

$$\exp\colon \mathcal{V}(\mathbb{S}^1) \to \mathrm{Diff}(\mathbb{S}^1)$$

is not a local diffeomorphism by proving that every identity neighborhood of $\mathrm{Diff}(\mathbb{S}^1)$ contains elements which do not lie on a one-parameter group, hence are not contained in the image of $\exp$.

Let $G := \mathrm{Diff}_+(\mathbb{S}^1)$ denote the group of orientation preserving diffeomorphisms of $\mathbb{S}^1$, i.e., the identity component of $\mathrm{Diff}(\mathbb{S}^1)$. To get a better picture of this group, we first construct its universal covering group $\widetilde{G}$. Let

$$\widetilde{G} := \{\varphi \in \mathrm{Diff}(\mathbb{R}) : (\forall x \in \mathbb{R})\varphi(x+2\pi) = \varphi(x), \varphi' > 0\}.$$

We consider the map

$$q\colon \mathbb{R} \to \mathbb{S}^1 := \mathbb{R}/2\pi\mathbb{Z}, \quad x \mapsto x + 2\pi\mathbb{Z}$$

as the universal covering map of $\mathbb{S}^1$. Then every diffeomorphism $\psi \in \mathrm{Diff}(\mathbb{S}^1)$ lifts to a diffeomorphism $\widetilde{\psi}$ of $\mathbb{R}$ commuting with the translation action of the group $2\pi\mathbb{Z} \cong \pi_1(\mathbb{S}^1)$, which means that $\widetilde{\psi}(x+2\pi) = \widetilde{\psi}(x) + 2\pi$ for each $x \in \mathbb{R}$. The diffeomorphism $\widetilde{\psi}$ is uniquely determined by the choice of an element in $q^{-1}(\psi(q(0)))$. Moreover, ψ is orientation preserving means that $(\widetilde{\psi})' > 0$. Hence we have a surjective homomorphism

$$q_G\colon \widetilde{G} \to G, \quad q_G(\varphi)(q(x)) := q(\varphi(x))$$

with kernel isomorphic to $\mathbb{Z}$.

The Lie group structure of $\widetilde{G}$ is rather simple. It can be defined by a global chart. Let $C^\infty_{2\pi}(\mathbb{R},\mathbb{R})$ denote the Fréchet space of 2π-periodic smooth functions on $\mathbb{R}$, which is considered as a closed subspace of the Fréchet space $C^\infty(\mathbb{R},\mathbb{R})$. In this space

$$U := \{\varphi \in C^\infty_{2\pi}(\mathbb{R},\mathbb{R}) : \varphi' > -1\}$$

is an open convex subset and the map

$$\Phi\colon U \to \widetilde{G}, \quad \Phi(f)(x) := x + f(x)$$

is a bijection.

In fact, let $f \in U$. Then $\Phi(f)(x+2\pi) = \Phi(f)(x) + 2\pi$ follows directly from the requirement that f is 2π-periodic, and $\Phi(f)' > 0$ follows from $f' > -1$. Therefore $\Phi(f)$ is strictly increasing, hence a diffeomorphism of $\mathbb{R}$ onto the interval $\Phi(f)(\mathbb{R})$. As the latter interval is invariant under translation by 2π, we see that $\Phi(f)$ is surjective and therefore $\Phi(f) \in \widetilde{G}$. Conversely, it is easy to see that $\Phi^{-1}(\psi)(x) = \psi(x) - x$ yields an inverse of Φ. We define the manifold structure on $\widetilde{G}$ by declaring Φ to be a global chart. With respect to this chart, the group operations in $\widetilde{G}$ are given by

$$m(f,g)(x) := f(g(x)+x) - x \quad \text{and} \quad \eta(f)(x) = (f + \mathrm{id}_{\mathbb{R}})^{-1}(x) - x,$$

which can be shown directly to be smooth maps. We thus obtain on $\widetilde{G}$ the structure of a Lie group such that $\Phi\colon U \to \widetilde{G}$ is a diffeomorphism. In particular $\widetilde{G}$ is contractible and therefore simply connected, so that the map $q_G\colon \widetilde{G} \to G$ turns out to be the universal covering map of G.

Theorem II.3.7. *Every identity neighborhood in* $\mathrm{Diff}(\mathbb{S}^1)$ *contains elements not contained in the image of the exponential function.*

Proof. First we construct certain elements in $\widetilde{G}$ which are close to the identity. For $0 < \varepsilon < \frac{1}{n}$ we consider the function

$$f: \mathbb{R} \to \mathbb{R}, \quad x \mapsto x + \frac{\pi}{n} + \varepsilon \sin^2(nx)$$

and observe that $f \in \widetilde{G}$ follows from $f'(x) = 1 + 2\varepsilon n \sin(nx)\cos(nx) = 1 + \varepsilon n \sin(2nx) > 0$.

Step 1. For n large fixed and $\varepsilon \to 0$ we get elements in $\widetilde{G}$ which are arbitrarily close to $\mathrm{id}_{\mathbb{R}}$.

Step 2. $q_G(f)$ has a unique periodic orbit of order $2n$ on $\mathbb{S}^1$: Under $q_G(f)$ the point $q(0) \in \mathbb{S}^1$ is mapped to $\frac{\pi}{n}$ etc., so that we obtain the orbit

$$q(0) \to q(\tfrac{\pi}{n}) \to q(\tfrac{2\pi}{n}) \to \ldots \to q(\tfrac{(2n-1)\pi}{n}) \to q(0).$$

For $0 < x_0 < \frac{\pi}{n}$ we have for $x_1 := f(x_0)$:

$$x_0 + \frac{\pi}{n} < x_1 < \frac{2\pi}{n},$$

and for $x_n := f(x_{n-1})$ the relations

$$0 < x_0 < x_1 - \frac{\pi}{n} < x_2 - \frac{2\pi}{n} < \cdots < \frac{\pi}{n}.$$

Therefore $x_k - x_0 \notin 2\pi\mathbb{Z}$ for each $k \in \mathbb{N}$, and hence the orbit of $q(x_0)$ under $q_G(f)$ is not finite. This proves that $q_G(f)$ has a unique periodic orbit and that the order of this orbit is $2n$.

Step 3. $q_G(f) \neq g^2$ for all $g \in \mathrm{Diff}(\mathbb{S}^1)$: We analyze the periodic orbits. Every perodic point of g is a periodic point of g^2 and vice versa. If the period of x under g is odd, then the period of x under g and g^2 is the same. If the period of x is $2m$, then its orbit under g breaks up into two orbits under g^2, each of order m. Therefore g^2 can never have a single periodic orbit of even order, and this proves that $q_G(f)$ has no square root in $\mathrm{Diff}(\mathbb{S}^1)$. It follows in particular that $q_G(f)$ does not lie on any one-parameter subgroup, i.e., $q_G(f) \neq \exp X$ for each $X \in \mathcal{V}(M)$. ■

Remark III.3.8. (a) If M is a compact manifold, then one can show that the identity component $\mathrm{Diff}(M)_0$ of $\mathrm{Diff}(M)$ is a simple group (Epstein, Hermann and Thurston; see [Ep70]). Being normal in $\mathrm{Diff}(M)_0$, the subgroup $\langle \exp \mathcal{V}(M)\rangle$ coincides with $\mathrm{Diff}(M)_0$. Hence every diffeomorphism homotopic to the identity is a finite product of exponentials. This observation is due to D. McDuff.

(b) Although $\mathrm{Diff}(M)_0$ is a simple Lie group, its Lie algebra $\mathcal{V}(M)$ is far from being simple. For each subset $K \subseteq M$ the set $\mathcal{V}_K(M)$ of all vector fields supported in the set K is a Lie algebra ideal which is proper if K is not dense. ■

III.4. Non-enlargible Lie algebras

Definition III.4.1. We call a locally convex Lie algebra $\mathfrak{g}$ with continuous Lie bracket $[\cdot,\cdot]$ *enlargible* if there exists a Lie group with Lie algebra $\mathfrak{g}$. ■

Examples III.4.2. If $\mathfrak{g}$ is a finite-dimensional Lie algebra, endowed with its unique locally convex topology, then $\mathfrak{g}$ is enlargible. This is Lie's Third Theorem. One possibility to prove this is first to use Ado's Theorem to find an embedding $\mathfrak{g} \hookrightarrow \mathfrak{gl}_n(\mathbb{R})$ and then to endow the group $G := \langle \exp \mathfrak{g}\rangle \subseteq \mathrm{GL}_n(\mathbb{R})$ with a Lie group structure such that $\mathbf{L}(G) = \mathfrak{g}$. ■

Example III.4.3. To construct an example of a non-enlargible Banach–Lie algebra, we proceed as follows.

Let H be an infinite-dimensional complex Hilbert space and $U(H)$ its unitary group. This is a Banach–Lie group with Lie algebra

$$\mathbf{L}(U(H)) = \mathfrak{u}(H) := \{X \in B(H): X^* = -X\}.$$

The center of this Lie algebra is given by

$$\mathfrak{z}(\mathfrak{u}(H)) = \mathbb{R}i\mathbf{1}.$$

We consider the Banach–Lie algebra

$$\mathfrak{g} := \big(\mathfrak{u}(H) \oplus \mathfrak{u}(H)\big)/\mathbb{R}i(\mathbf{1}, \sqrt{2}\mathbf{1}).$$

We claim that $\mathfrak{g}$ is not enlargible. Let us assume to the contrary that G is a connected Lie group with Lie algebra $\mathfrak{g}$. Let

$$q: \mathfrak{u}(H) \oplus \mathfrak{u}(H) \to \mathfrak{g}$$

denote the quotient homomorphism. According to Kuiper's Theorem (Theorem IV.3.1 below), the group $U(H)$ and hence the group $G_1 := U(H) \times U(H)$ is contractible and therefore in particular simply connected. Hence there exists a unique Lie group homomorphism

$$f: G_1 \to G \quad \text{with} \quad \mathbf{L}(f) = q.$$

We then have $\exp_G \circ q = f \circ \exp_{G_1}$, and in particular

$$\exp \ker q \subseteq \ker f.$$

As $Z(G_1) \cong \mathbb{T}^2$ is a two-dimensional torus and $\exp \ker q$ is a dense one-parameter subgroup of $Z(G_1)$, the continuity of f further implies that $Z(G_1) \subseteq \ker f$ and hence that $\mathfrak{z}(\mathfrak{g}_1) \subseteq \ker \mathbf{L}(f) = \ker q$, which is a contradiction. ■

The first systematic discussion of the non-enlargibility problem for Banach–Lie algebras is given in [EK64], based on earlier results of van Est ([Est62]).

Theorem III.4.4. (van Est–Korthagen, 1964) *Let $\mathfrak{g}$ and $\mathfrak{h}$ be Banach–Lie algebras. If $\mathfrak{h}$ is enlargible and $\varphi: \mathfrak{g} \hookrightarrow \mathfrak{h}$ is injective, then $\mathfrak{g}$ is enlargible.*

Proof. (Idea) Let H be a Lie group with Lie algebra $\mathfrak{h}$. The main idea of the proof is to endow the subgroup $G := \langle \exp \varphi(\mathfrak{g}) \rangle$ of H with a Lie group topology for which $\mathbf{L}(G) = \mathfrak{g}$. This is much more complicated than in the finite-dimensional case because it is harder to control the behavior of analytic subgroups, especially when the image of φ is not closed. ■

Corollary III.4.5. *If $\mathfrak{g}$ is a Banach–Lie algebra, then $\mathfrak{g}/\operatorname{ad}\mathfrak{z}(\mathfrak{g}) \cong \operatorname{ad}\mathfrak{g}$ is enlargible.*

Proof. The adjoint representation $\operatorname{ad}: \mathfrak{g} \to \operatorname{der}\mathfrak{g}$ factors through an injective homomorphism $\mathfrak{g}/\mathfrak{z}(\mathfrak{g}) \hookrightarrow \operatorname{der}\mathfrak{g}$, and

$$\operatorname{der}\mathfrak{g} := \{D \in B(\mathfrak{g}): (\forall x, y \in \mathfrak{g})\ D([x,y]) = [D(x), y] + [x, D(y)]\}.$$

is the Lie algebra of the Banach–Lie group $\operatorname{Aut}(\mathfrak{g})$. ■

The preceding corollary reduces the enlargibility problem for Banach–Lie groups to the question when a central extension of an enlargible Lie algebra is again enlargible. In this context a central extension is a surjective morphism $q: \widehat{\mathfrak{g}} \to \mathfrak{g}$ of Banach–Lie algebras for which $\mathfrak{z} := \ker q$ is central in $\widehat{\mathfrak{g}}$. The Open Mapping Theorem implies that $\mathfrak{g} \cong \widehat{\mathfrak{g}}/\mathfrak{z}$ as Banach–Lie algebras. Now the question is the following: given a connected Lie group G with Lie algebra $\mathfrak{g}$, when is there a central group extension $Z \hookrightarrow \widehat{G} \to G$ "integrating" the corresponding Lie algebra extension? Without going too much into details, we cite the following theorem which points into a direction which can be followed with success for general Lie groups (see [Ne02a]).

Theorem III.4.6. (van Est-Korthagen) *Let G be a simply connected Banach–Lie group and $\mathfrak{g}$ its Lie algebra. Then one can associate to each central Banach–Lie algebra extension $\mathfrak{z} \hookrightarrow \widehat{\mathfrak{g}} \to \mathfrak{g}$ a singular cohomology class $c \in H^2(G, \mathfrak{z}) \cong \mathrm{Hom}(\pi_2(G), \mathfrak{z})$ which we interpret as a period map*

$$\mathrm{per}_c: \pi_2(G) \to \mathfrak{z}.$$

Then a corresponding central extension $Z \hookrightarrow \widehat{G} \twoheadrightarrow G$ exists for a Lie group Z with Lie algebra $\mathfrak{z}$ if and only if $\mathrm{im}(\mathrm{per}_c) \subseteq \mathfrak{z}$ is discrete. ■

Remark III.4.7. (a) Let $\mathfrak{g}$ be a Banach–Lie algebra and G_{ad} a simply connected Lie group with Lie algebra $\mathfrak{g}/\mathfrak{z}(\mathfrak{g})$ (Corollary III.4.5). Then the preceding theorem implies in particular that $\mathfrak{g}$ is enlargible if and only if the period homomorphism $\mathrm{per}_{\mathfrak{g}}: \pi_2(G_{\mathrm{ad}}) \to \mathfrak{z}(\mathfrak{g})$ associated to the central extension $\mathrm{ad}: \mathfrak{g} \to \mathfrak{g}/\mathfrak{z}(\mathfrak{g})$ has discrete image.

The problem with this characterization is that in general it might be quite hard to determine the image of the period homomorphism.

(b) If $\mathfrak{g}$ is enlargible and G is a simply connected Lie group with Lie algebra $\mathfrak{g}$, then the long exact homotopy sequence associated to the homomorphism $q: G \to G_{\mathrm{ad}}$ with kernel $Z(G)_0$ induces a surjective connecting homomorphism

$$\pi_2(G_{\mathrm{ad}}) \to \pi_1(Z(G))$$

(cf. Remark IV.1.1 below) and by identifying the universal covering group of $Z(G)_0$ with $(\mathfrak{z}(\mathfrak{g}), +)$, one can show that this connecting homomorphism coincides with the period map. Its image is the group $\pi_1(Z(G))$, considered as a subgroup of $\mathfrak{z}$. With this picture in mind one may think that the non-enlargibility on a Banach–Lie algebra $\mathfrak{g}$ is caused by the non-existence of a Lie group Z with Lie algebra $\mathfrak{z}(\mathfrak{g})$ and fundamental group $\mathrm{im}(\mathrm{per}_{\mathfrak{g}})$.

(c) If $\mathfrak{g}$ is finite-dimensional, then G_{ad} is also finite-dimensional, and therefore $\pi_2(G_{\mathrm{ad}})$ vanishes by a theorem of E. Cartan (Remark IV.1.3). Hence the period homomorphism $\mathrm{per}_{\mathfrak{g}}$ is trivial for every finite-dimensional Lie algebra $\mathfrak{g}$. ■

Example III.4.8. We consider the Lie algebra

$$\mathfrak{g} := \big(\mathfrak{u}(H) \oplus \mathfrak{u}(H)\big)/\mathbb{R}i(\mathbf{1}, \sqrt{2}\mathbf{1})$$

from Example III.4.3. Then $\mathfrak{z}(\mathfrak{g}) \cong i\mathbb{R}$ and one can show that the image of the period map is given by

$$2\pi i(\mathbb{Z} + \sqrt{2}\mathbb{Z}) \subseteq i\mathbb{R}$$

which is not discrete. ■

Proposition III.4.9. *Let G be a connected complex Lie group with Fréchet–Lie algebra $\mathfrak{g}$. Then each closed ideal of $\mathfrak{g}$ is invariant under $\mathrm{Ad}(G)$.*

Proof. Let $\mathfrak{a} \trianglelefteq \mathfrak{g}$ be a closed ideal. Since G is assumed to be connected, it suffices to show that there exists a $\mathbf{1}$-neighborhood $U \subseteq G$ with $\mathrm{Ad}(U).\mathfrak{a} \subseteq \mathfrak{a}$. We may w.l.o.g. assume that U is diffeomorphic to an open convex 0-neighborhood in $\mathfrak{g}$. Then we find for every $g \in U$ a connected open subset $V \subseteq \mathbb{C}$ and a holomorphic map $p: V \to G$ with $p(0) = \mathbf{1}$ and $p(1) = g$.

Let $w_0 \in \mathfrak{a}$ and $w(t) := \mathrm{Ad}(p(t)).w_0$ for $t \in V$. We have to show that $w(1) = \mathrm{Ad}(g).w_0 \in \mathfrak{a}$. For the right logarithmic derivative $v := \delta^r(p): V \to \mathfrak{g}$ we obtain the differential equation

$$w'(t) = \mathrm{Ad}(p(t)).[p^{-1}(t).p'(t), w_0] = [\delta^r(p)(t), w(t)] = [v(t), w(t)].$$

Since the maps v and w are holomorphic, their Taylor expansions in 0 converge:

$$v(t) = \sum_n v_n t^n \quad \text{and} \quad w(t) = \sum_n w_n t^n$$

for t close to 0 in V. Then the differential equation for w can be written as

$$\sum_n (n+1)w_{n+1}t^n = w'(t) = [v(t), w(t)] = \sum_n t^n \sum_{k=0}^n [v_k, w_{n-k}].$$

Comparing coefficients now leads to

$$w_{n+1} = \frac{1}{n+1}\sum_{k=0}^n [v_k, w_{n-k}],$$

so that we obtain inductively $w_n \in \mathfrak{a}$ for each $n \in \mathbb{N}$. Since $\mathfrak{a}$ is closed, it follows that $w(t) \in \mathfrak{a}$ for t close to 0. Applying the same argument in other points $t_0 \in V$, we see that the set $w^{-1}(\mathfrak{a})$ is an open closed subset of V, and therefore that $a(1) \in \mathfrak{a}$ because $a(0) \in \mathfrak{a}$ and V is connected.■

The preceding proposition can be generalized to the larger class of real analytic Lie groups, which we have not defined in these notes. Then this result can be used to conclude that the Lie group $\mathrm{Diff}(M)$ does not possess an analytic Lie group structure. Indeed for each non-dense subset $K \subseteq M$ the subspace

$$\mathfrak{a}_K := \{X \in \mathcal{V}(M) : X|_K = 0\}$$

is a closed ideal of $\mathcal{V}(M)$ not invariant under $\mathrm{Diff}(M)$ because $\mathrm{Ad}(\varphi).\mathfrak{a}_K = \mathfrak{a}_{\varphi(K)}$ for $\varphi \in \mathrm{Diff}(M)$.

Theorem III.4.10. (Lempert) *Let M be a compact manifold, $\mathfrak{g} := \mathcal{V}(M)$ the Lie algebra of smooth vector fields on M and $\mathfrak{g}_{\mathbb{C}}$ its complexification. Then $\mathfrak{g}_{\mathbb{C}}$ is not enlargible to a regular Lie group.*

Proof. (Sketch; see [Mil83]) For each subset $K \subseteq M$ the subspace

$$\mathfrak{a}_K := \{X \in \mathfrak{g}_{\mathbb{C}} : X|_K = 0\}$$

is a closed ideal of $\mathfrak{g}_{\mathbb{C}}$.

Let G be a regular Lie group with Lie algebra $\mathfrak{g}$ and let $q: D \to \mathrm{Diff}(M)_0$ denote the universal covering homomorphism of $\mathrm{Diff}(M)_0$. Then the inclusion homomorphism $\mathfrak{g} \hookrightarrow \mathfrak{g}_{\mathbb{C}}$ can be integrated to a Lie group homomorphism $\varphi: D \to G$. For $g \in D$ we then have

$$\mathrm{Ad}(\varphi(g)).\mathfrak{a}_K = \mathfrak{a}_{\varphi(g)(K)},$$

contradicting the invariance of $\mathfrak{a}_K$ under $\mathrm{Ad}(G)$. ■

Remark III.4.11. In [Omo81] Omori shows that for any non-compact smooth manifold M the Lie algebra $\mathcal{V}(M)$ is not enlargible. ■

IV. The topology of infinite-dimensional Lie groups

There are several methods to study the topology of infinite-dimensional Lie groups which are adapted to the different classes of groups considered above. We are mainly interested in the first three homotopy groups of a Lie group G, namely $\pi_0(G)$ (the group of connected components), $\pi_1(G)$ (the fundamental group), and $\pi_2(G)$. The importance of $\pi_0(G)$ is clear because one wants to know whether a concretely given group is connected or not. Information on the fundamental group is important for the integration of Lie algebra homomorphisms to group homomorphisms and hence in particular for representation theory. The interest in $\pi_2(G)$ comes from the crucial role this group plays for enlargibility of Lie algebras and for central extensions of G.

IV.1. Finite-dimensional Lie groups

Let G be a connected finite-dimensional Lie group with finitely many connected components and $K \subseteq G$ a maximal compact subgroup. Then $G \cong K \times \mathbb{R}^d$ as smooth manifolds holds for some $d \in \mathbb{N}_0$. This implies in particular that the inclusion map $K \hookrightarrow G$ is a homotopy equivalence, hence induces isomorphisms $\pi_k(K) \to \pi_k(G)$ for each $k \in \mathbb{N}_0$. This reduces all questions on the topology of finite-dimensional Lie groups to compact groups.

Remark IV.1.1. A crucial tool to analyze homotopy groups of Lie groups and their homogeneous spaces is the long exact homotopy sequence of fiber bundles. If $q\colon P \to B$ defines a K-principal bundle and the spaces B and P are connected, then the long exact homotopy sequence reads as follows:

$$\ldots \pi_3(B) \to \pi_2(K) \to \pi_2(P) \to \pi_2(B) \to \pi_1(K) \to \pi_1(P) \to \pi_1(B) \twoheadrightarrow \pi_0(K). \qquad \blacksquare$$

Lemma IV.1.2. *If X is a semilocally simply connected arcwise connected space and $q\colon \widetilde{X} \to X$ is the universal covering of X, then q induces isomorphisms*

$$\pi_k(q)\colon \pi_k(\widetilde{X}) \to \pi_k(X), \quad k \geq 2.$$

Proof. We consider $q\colon \widetilde{X} \to X$ as a principal bundle for the discrete group $K := \pi_1(X)$ and apply the exact homotopy sequence (Remark IV.1.1). Since K is discrete, we have $\pi_k(K) = \mathbf{1}$ for $k \geq 1$, and the assertion follows from the exactness of the sequence. ■

Remark IV.1.3. We recall some results on the homotopy groups of compact Lie groups K. First we have Cartan's Theorem

$$\pi_2(K) = \mathbf{1}$$

([Mi95, Th. 3.7]), and further Bott's Theorem that for a compact connected simple Lie group K we have

$$\pi_3(K) \cong \mathbb{Z}$$

([Mi95, Th. 3.9]).

In [Mi95, pp. 969-970] one also finds a table with $\pi_k(K)$ up to $k = 15$, showing that

$$\pi_4(K) \cong \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{for } K = \mathrm{SO}(4) \\ \mathbb{Z}_2 & \text{for } K = \mathrm{Sp}(n), \mathrm{SU}(2), \mathrm{SO}(3), \mathrm{SO}(5) \\ \mathbf{1} & \text{for } K = \mathrm{SU}(n),\ n \geq 3 \text{ and } \mathrm{SO}(n),\ n \geq 6 \\ \mathbf{1} & \text{for } K = G_2, F_4, E_6, E_7, E_8. \end{cases}$$

$$\pi_5(K) \cong \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{for } K = \mathrm{SO}(4) \\ \mathbb{Z}_2 & \text{for } K = \mathrm{Sp}(n), \mathrm{SU}(2), \mathrm{SO}(3), \mathrm{SO}(5) \\ \mathbb{Z} & \text{for } K = \mathrm{SU}(n),\ n \geq 3 \text{ and } \mathrm{SO}(6) \\ \mathbf{1} & \text{for } K = \mathrm{SO}(n),\ n \geq 7,\ G_2, F_4, E_6, E_7, E_8. \end{cases}$$

■

Remark IV.1.4. (a) Let K be a connected compact Lie group, $K_1, \ldots, K_m$ the connected simple normal subgroups of K, and $Z(K)$ its center. Then the multiplication map

$$Z(K)_0 \times K_1 \times \ldots \times K_m \to K$$

has finite kernel, hence is a covering map. Therefore we obtain for each $k > 1$ from Lemma IV.1.2

$$\pi_k(K) \cong \prod_{j=1}^{m} \pi_k(K_j)$$

because $Z(K)_0$ is a torus, so that all its homotopy groups of degree ≥ 2 vanish.
(b) If K is compact and simple, then a generator of $\pi_3(K)$ can be obtained from a homomorphism $\eta\colon \mathrm{SU}(2) \cong \mathbb{S}^3 \to K$. More precisely, let α be a long root in the root system $\Delta_{\mathfrak{k}}$ of $\mathfrak{k}$ and $\mathfrak{k}(\alpha) \subseteq \mathfrak{k}$ the corresponding $\mathfrak{su}(2)$-subalgebra. Then the corresponding homomorphic inclusion $\mathrm{SU}(2) \cong \mathbb{S}^3 \to K$ represents a generator of $\pi_3(K)$ ([Bo58]).
(c) A fundamental result in topology states that the spheres $\mathbb{S}^d$ carry a Lie group structure if and only if $d \in \{0, 1, 3\}$.

For finite-dimensional Lie groups this has the nice consequence that for $d = 1$ and $d = 3$ each homotopy class $\mathbb{S}^d \to K$ can be represented by a group homomorphism. For $d = 3$ this follows from (b) and for $d = 1$ it follows from the fact that for a maximal torus $T \subseteq K$ the homomorphism

$$\mathrm{Hom}(\mathbb{T}, T) \cong \pi_1(T) \to \pi_1(K)$$

is surjective.
(d) For a topological group G and $k \geq 1$ the groups $\pi_k(G)$ are abelian. The groups

$$\pi_k^{\mathbb{Q}}(G) := \mathbb{Q} \otimes \pi_k(G)$$

are called the *rational homotopy groups of* G. For most purposes, including applications to the period maps arising for central extensions, it suffices to know the rational homotopy groups because each homomorphism from $\pi_k(G)$ to a rational vector space factors through the natural map $\eta_k\colon \pi_k(G) \to \pi_k^{\mathbb{Q}}(G)$ which kills the torsion subgroup of $\pi_k(G)$.

We have seen above that a finite-dimensional connected Lie group is homotopy equivalent to a compact connected Lie group, hence, up to a finite covering to a product of a torus and finitely many compact simple Lie groups. For a simply connected simple compact Lie group it is known that its rational homotopy groups are the same as those of a product of odd-dimensional spheres whose dimensions can be computed from the corresponding root system. The rational homotopy groups of the sphere are known to be

$$\pi_k^{\mathbb{Q}}(\mathbb{S}^{2d+1}) \cong \begin{cases} \mathbb{Q} & \text{for } k = 2d+1 \\ \mathbf{0} & \text{otherwise} \end{cases} \quad \text{and} \quad \pi_k^{\mathbb{Q}}(\mathbb{S}^{2d}) \cong \begin{cases} \mathbb{Q} & \text{for } k = 2d \text{ and } k = 4d-1 \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

We therefore have complete information on the rational homotopy groups of finite-dimensional Lie groups. In particular we note that if K is a finite-dimensional Lie group, then $\pi_2(K)$ vanishes and $\pi_4(K)$ is a torsion group because the rational homotopy of K is the same as of a product of odd-dimensional spheres. ■

IV.2. Linear Lie groups

In this section we briefly discuss the unit group $A^\times$ of a unital continuous inverse algebra (c.i.a.) A (Proposition III.3.1). It is quite hard to get direct access to the homotopy groups $\pi_k(\mathrm{GL}_n(A))$ for a fixed n, but the situation becomes much better if we let n tend to infinity and study the direct limit of the homotopy groups for increasing n. In this sense we look at a "stable" picture. The natural inclusions

$$\mathrm{GL}_n(A) \hookrightarrow \mathrm{GL}_{n+1}(A), \quad a \mapsto \begin{pmatrix} a & 0 \\ 0 & \mathbf{1} \end{pmatrix} \tag{2.1}$$

lead to a sequence of inclusions

$$A^\times = \mathrm{GL}_1(A) \to \ldots \to \mathrm{GL}_n(A) \to \ldots$$

Definition IV.2.1. For $i \in \mathbb{N}$ we define the *topological* K*-groups of* A by

$$K_i(A) := \varinjlim \pi_{i-1}(\mathrm{GL}_n(A)),$$

where the connecting maps

$$\pi_{i-1}(\mathrm{GL}_n(A)) \to \pi_{i-1}(\mathrm{GL}_{n+1}(A))$$

are the group homomorphisms induced by the natural inclusions (2.1).

The definition of the group $K_0(A)$ is a bit more complicated. For a topological associative algebra B let $\mathrm{Idem}(B)$ denote the set of idempotents in B and $\pi_0(\mathrm{Idem}(B))$ the set of arc-components of $\mathrm{Idem}(B)$ with respect to the subspace topology induced by B. Now let

$$P(A) := \varinjlim \pi_0(\mathrm{Idem}(M_n(A)))$$

as a set, and observe that this set admits a monoid structure given by

$$[e] + [f] := [e \oplus f],$$

where for $e \in M_n(A)$ and $f \in M_m(A)$ the idempotent $e \oplus f \in M_{n+m}(A)$ is represented by the matrix $\begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}$. The *free group* or *Grothendieck group* $G(P(A))$ over the monoid $P(A)$ is a group with a monoid morphism $\iota\colon P(A) \to G(P(A))$ which has the universal property that for each monoid morphism $f\colon P(A) \to G$, G a group, there exists a unique group homomorphism $f_G\colon G(P(A)) \to G$ with $f_G \circ \iota = f$. We define

$$K_0(A) := G(P(A)).$$

A more algebraic approach is to define $P(A)$ directly as the set of isomorphism classes of finitely generated projective A-modules, which leads to the same object. ■

The use of K-theory for the topology of the unit groups of algebras is obvious from the following theorem.

Theorem IV.2.2. (Bott periodicity) *For a complex unital c.i.a. the following assertions hold:*
(1) $K_i(A) \cong K_{i+2}(A)$ *for* $i \in \mathbb{N}_0$.
(2) $K_{i+1}(A) \cong \pi_i(\mathrm{GL}_\infty(A))$ *if* A *is a Banach algebra.*

Proof. [Bos90, Prop. A.1.5]. ■

A major point of the K-groups of an algebra A is that K-theory provides tools like exact sequences which can be used to get information on the groups $K_0(A)$ and $K_1(A)$ of a c.i.a.. All other K-groups are redundant for a complex c.i.a. by Bott periodicity.

Directly relevant for the topology of $A^\times$ are the homomorphisms

$$\pi_0(A^\times) \to K_1(A), \quad \pi_1(A^\times) \to K_0(A) \quad \text{and} \quad \pi_2(A^\times) \to K_1(A).$$

Remark IV.2.3. The definition of the K-groups implies almost directly that they are *stable* in the sense that the inclusion $A \hookrightarrow M_n(A), a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ induces isomorphisms

$$K_i(A) \to K_i(M_n(A))$$

for each $n \in \mathbb{N}$. ■

Examples IV.2.4. (a) If $A = B(H)$ is the algebra of bounded operators on an infinite-dimensional complex Hilbert space, then we have for each $n \in \mathbb{N}$ the relations

$$\mathrm{GL}_n(B(H)) = M_n(B(H))^\times \cong B(H^n)^\times \cong \mathrm{GL}(H^n),$$

and all these groups are contractible by Kuiper's Theorem IV.3.1 below. Therefore $K_i(B(H)) = \mathbf{0}$ for each i.

(b) For $A = \mathbb{C}$ we have

$$K_0(\mathbb{C}) \cong \lim_{\longrightarrow} \pi_1(\mathrm{GL}_n(\mathbb{C})) \cong \pi_1(\mathbb{C}^\times) \cong \mathbb{Z} \quad \text{and} \quad K_1(\mathbb{C}) \cong \lim_{\longrightarrow} \pi_0(\mathrm{GL}_n(\mathbb{C})) = \mathbf{0}.$$

For $A = M_n(\mathbb{C})$ Remark IV.2.3 now leads to

$$K_0(A) \cong K_0(\mathbb{C}) \cong \mathbb{Z} \quad \text{and} \quad K_1(A) \cong K_1(\mathbb{C}) = \mathbf{0}.$$

(c) If X is a compact space and $A = C(X, \mathbb{C})$ with its natural Banach algebra structure, then $K_0(A) = K^0(X)$ and $K_1(A) = K^{-1}(X)$ are the K-groups of the topological space X defined by topological K-theory via vector bundles on X.

In particular we have for the circle $\mathbb{S}^1$ and, more generally, for tori $\mathbb{T}^d$:

$$K_0(C(\mathbb{S}^1, \mathbb{C})) \cong K_1(C(\mathbb{S}^1, \mathbb{C})) \cong \mathbb{Z} \quad \text{and} \quad K_0(C(\mathbb{T}^d, \mathbb{C})) \cong K_1(C(\mathbb{T}^d, \mathbb{C})) \cong \mathbb{Z}^{2^{d-1}}. \qquad \blacksquare$$

Remark IV.2.5. (a) If $\varphi: A \to B$ is a continuous morphism of c.i.a.'s with dense range, then $K_j(\varphi): K_j(A) \to K_j(B)$ is an isomorphism for each j.
(b) Let B be a complex Banach algebra and $\alpha: \mathbb{R} \times B \to B$ a continuous isometric action of $\mathbb{R}$ on B by automorphisms. Let $I \subseteq \mathbb{R}$ be a compact interval containing 0 and write $B(I) \subseteq B$ for the subalgebra of all those elements for which the orbit map $\mathbb{R} \to B$ extends to a continuous map $\mathbb{R} + iI \to B$ holomorphic on $\mathbb{R} + iI^0$. Then $B(I)$ is a dense subalgebra of B and the inclusion $B(I) \hookrightarrow B$ induces an isomorphism in K-theory ([Bos90, Th. 1.1.1]).

Let $0 < r < 1 < R$ and consider the annulus

$$A_{r,R} := \{z \in \mathbb{C} : r \leq |z| \leq R\}.$$

We write $\mathcal{O}(A_{r,R})$ for the Banach algebra of continuous functions on $A_{r,R}$ which are holomorphic on its interior. For $B := C(\mathbb{S}^1, \mathbb{C})$ and for the action of $\mathbb{R}$ on B given by $(t.f)(z) := f(ze^{it})$, the preceding result implies that the restriction map $\mathcal{O}(A_{r,R}) \hookrightarrow C(\mathbb{S}^1, \mathbb{C})$ induces an isomorphism in K-theory. This leads to

$$K_0(\mathcal{O}(A_{r,R})) \cong K_0(C(\mathbb{S}^1, \mathbb{C})) \cong K^0(\mathbb{S}^1) \cong \mathbb{Z}$$

and

$$K_1(\mathcal{O}(A_{r,R})) \cong K_1(C(\mathbb{S}^1, \mathbb{C})) \cong \pi_0(\mathrm{GL}(C(\mathbb{S}^1, \mathbb{C}))) \cong \pi_1(\mathrm{GL}(\mathbb{C})) \cong \mathbb{Z}. \qquad \blacksquare$$

IV.3. Groups of operators on Hilbert spaces

Theorem IV.3.1. (Kuiper's Theorem for general Hilbert spaces) *If H is an infinite-dimensional Hilbert space over $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, then the group $\mathrm{GL}(H, \mathbb{K})$ of invertible $\mathbb{K}$-linear automorphisms of H is contractible.* $\blacksquare$

Kuiper's Theorem can be used to prove that many "classical" groups of operators on a Hilbert space are contractible. Below we briefly discuss these applications.

Definition IV.3.2. (a) If H is a Hilbert space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, then we define

$$\mathrm{U}(H, \mathbb{K}) := \{g \in \mathrm{GL}(H, \mathbb{K}) : g^* g = gg^* = \mathbf{1}\}$$

as the unitary part of this group. We also write

$$\mathrm{O}(H) := \mathrm{U}(H, \mathbb{R}), \quad \mathrm{U}(H) := \mathrm{U}(H, \mathbb{C}) \quad \text{and} \quad \mathrm{Sp}(H) := \mathrm{U}(H, \mathbb{H}).$$

(b) Let H be a complex Hilbert space and I be an antilinear isometry with $I^2 \in \{\pm\mathbf{1}\}$. Then

$$\mathrm{GL}(H,I) := \{g \in \mathrm{GL}(H) \colon I g^* I^{-1} = g^{-1}\}$$

is a complex Lie subgroup of $\mathrm{GL}(H)$. For $I^2 = \mathbf{1}$ we then have

$$\mathrm{U}(H,I) := \mathrm{U}(H) \cap \mathrm{GL}(H,I) \cong \mathrm{O}(H_{\mathbb{R}}) \quad \text{with} \quad H_{\mathbb{R}} := \{x \in H \colon I.x = x\},$$

and for $I^2 = -\mathbf{1}$ we have

$$\mathrm{U}(H,I) \cong \mathrm{U}(H,\mathbb{H}) \cong \mathrm{Sp}(H),$$

where the quaternionic structure on H is given by the subalgebra $\mathbb{C}\mathbf{1} + \mathbb{C}I \cong \mathbb{H}$ of $B(H,\mathbb{R})$, the real linear endomorphisms of H.

(c) (Hermitian groups) Let H be a complex Hilbert space and $H = H_+ \oplus H_-$ be an orthogonal decomposition. Further let $T = T^* \in B(H)$ with $H_\pm = \ker(T \mp \mathbf{1})$. We define the corresponding *pseudo-unitary group*

$$\mathrm{U}(H_+,H_-) := \{g \in \mathrm{GL}(H) \colon T g^* T^{-1} = g^{-1}\}.$$

We define $\Omega(x,y) := \mathrm{Im}\langle x,y\rangle$ and write $H^{\mathbb{R}}$ for the real Hilbert space underlying H. Then

$$\mathrm{Sp}(H,\Omega) := \{g \in \mathrm{GL}(H^{\mathbb{R}},\mathbb{R}) \colon (\forall v,w \in H^{\mathbb{R}})\, \Omega(g.v,g.w) = \Omega(v,w)\}$$

is called the *symplectic group of* H. If we start with the real Hilbert space $H^{\mathbb{R}}$ and consider an isometric complex structure I on $H^{\mathbb{R}}$, then we can define

$$\Omega(x,y) := -\langle I.x,y\rangle = \langle x, I.y\rangle$$

and put

$$\mathrm{Sp}(H^{\mathbb{R}},I) := \{g \in \mathrm{GL}(H^{\mathbb{R}},\mathbb{R}) \colon (\forall v,w \in H^{\mathbb{R}})\, \Omega(g.v,g.w) = \Omega(v,w)\}.$$

It is easy to see that both constructions lead to isomorphic groups $\mathrm{Sp}(H^{\mathbb{R}},I) \cong \mathrm{Sp}(H,\Omega)$.

Now let I be a conjugation on the complex Hilbert space H and $H_+ \subseteq H$ a subspace for which we get an orthogonal decomposition $H = H_+ \oplus H_-$ with $H_- := I.H_+$. Then we define

$$\mathrm{O}^*(H,I) := \mathrm{U}(H,I) \cap \mathrm{U}(H_+,H_-). \qquad \blacksquare$$

Theorem IV.3.3. *If H is an infinite-dimensional Hilbert space over $\mathbb{K} \in \{\mathbb{R},\mathbb{C},\mathbb{H}\}$, then the following groups are contractible:*

(i) *the group of $\mathbb{K}$-linear automorphisms $\mathrm{GL}(H,\mathbb{K})$.*

(ii) *the group of isometric $\mathbb{K}$-linear automorphisms $\mathrm{U}(H,\mathbb{K})$, and in particular the groups $\mathrm{O}(H) = \mathrm{U}(H,\mathbb{R})$, $\mathrm{U}(H) = \mathrm{U}(H,\mathbb{C})$ and $\mathrm{Sp}(H) = \mathrm{U}(H,\mathbb{H})$.*

(iii) *the group $\mathrm{GL}(H,I)$ if H is complex and I an antilinear isometry with $I^2 \in \{\pm\mathbf{1}\}$. Moreover, $\mathrm{GL}(H,I)$ has a smooth polar decomposition.*

(iv) *the hermitian groups $\mathrm{U}(H_+,H_-)$, where $H = H_+ \oplus H_-$ is an orthogonal decomposition with two infinite-dimensional summands, $\mathrm{Sp}(H,\Omega)$, and $\mathrm{O}^*(H,I)$.*

Proof. (i) is Theorem IV.3.1.

(ii) follows from (i) and the polar decomposition $\mathrm{GL}(H,\mathbb{K}) \cong \mathrm{U}(H,\mathbb{K}) \times \mathrm{Herm}(H,\mathbb{K})$ of the group $\mathrm{GL}(H,\mathbb{K})$ with the unitary part $\mathrm{U}(H,\mathbb{K})$.

(iii) In view of Definition IV.3.2(b), the group $\mathrm{U}(H,I)$ is contractible, because it is one of the groups in (ii). Hence the assertion follows from the polar decomposition of $\mathrm{GL}(H,I)$ which can be obtained as follows. We consider the automorphism $\tau(g) := I(g^*)^{-1}I^{-1}$ of $\mathrm{GL}(H)$ and write $\tau_{\mathfrak{g}}(x) := -Ix^*I^{-1}$ for the corresponding antilinear automorphism of its Lie algebra $\mathfrak{gl}(H)$. Then

$$\mathrm{GL}(H,I) = \mathrm{GL}(H)^\tau := \{g \in \mathrm{GL}(H) \colon \tau(g) = g\}.$$

Let $g = ue^x$ be the polar decomposition of $g \in \mathrm{GL}(H)$. Then $\tau(g) = \tau(u)e^{\tau_{\mathfrak g}(x)}$ is the polar decomposition of $\tau(g)$, so that the uniqueness of this decomposition implies that $\tau(g) = g$, is equivalent to $\tau(u) = u$ and $\tau_{\mathfrak g}(x) = x$, i.e., $u \in \mathrm{U}(H, I)$ and $x \in \mathrm{Herm}(H, I)$.
(iv) For the hermitian groups we will see below that they have polar decompositions with

$$\mathrm{U}(H_+, H_-) \cap \mathrm{U}(H) \cong \mathrm{U}(H_+) \times \mathrm{U}(H_-), \quad \mathrm{Sp}(H, \Omega) \cap \mathrm{O}(H^{\mathbb R}) \cong \mathrm{U}(H)$$

and

$$\mathrm{O}^*(H, I) \cap \mathrm{U}(H) \cong \mathrm{U}(H_+),$$

where $H \cong H_+ \oplus I.H_+$ as in Definition IV.3.2(c). Therefore (ii) implies that all these groups are contractible.

To prove the polar decomposition of $\mathrm{U}(H_+, H_-)$, let $g \in \mathrm{GL}(H)$ with polar decomposition $g = ue^x$, $u \in \mathrm{U}(H)$ and $x = x^*$. For T as in Definition IV.3.2(c) we consider the automorphism $\tau(g) := T(g^*)^{-1}T^{-1}$ of $\mathrm{GL}(H)$ and write $\tau_{\mathfrak g}(x) := -Tx^*T^{-1}$ for the corresponding antilinear automorphism of its Lie algebra $\mathfrak{gl}(H)$. Then $\tau(g) = \tau(u)e^{\tau_{\mathfrak g}(x)}$ is the polar decomposition of $\tau(g)$, so that the uniqueness of this decomposition implies that $\tau(g) = g$ is equivalent to $\tau(u) = u$ and $\tau_{\mathfrak g}(x) = x$. Therefore $g \in \mathrm{U}(H_+, H_-)$ if and only if

$$u \in \mathrm{U}(H_+, H_-) \cap \mathrm{U}(H) \cong \mathrm{U}(H_+) \times \mathrm{U}(H_-) \quad \text{and} \quad x \in \mathfrak{u}(H_+, H_-).$$

To see that $\mathrm{Sp}(H, \Omega)$ is adapted to the polar decomposition, we observe that

$$\Omega(x, y) = \mathrm{Im}\langle x, y\rangle = \mathrm{Re}\langle x, iy\rangle = (x, Jy),$$

where $(\cdot,\cdot) := \mathrm{Re}\langle\cdot,\cdot\rangle$ denotes the real scalar product on $H^{\mathbb R}$. Therefore $g \in \mathrm{Sp}(H, \Omega)$ is equivalent to $g^\top Jg = J$, i.e., $g = \tau(g) := J(g^\top)^{-1}J^{-1}$. Then τ is an involutive automorphism of $\mathrm{GL}(H^{\mathbb R})$ and $\tau_{\mathfrak g}(x) := -Jx^\top J^{-1}$ is the corresponding Lie algebra automorphism. Let $g = ue^x$ be the polar decomposition of $g \in \mathrm{GL}(H^{\mathbb R})$, where $u \in \mathrm{O}(H^{\mathbb R})$ and $x^\top = x$. Then $\tau(g) = \tau(u)e^{\tau_{\mathfrak g}(x)}$ is the polar decomposition of $\tau(g)$ because ue^{-x} is the polar decomposition of $(g^\top)^{-1}$. Therefore $g \in \mathrm{Sp}(H, \Omega)$ is equivalent to $\tau(u) = u$, i.e., $u \in \mathrm{U}(H)$, and to $Jx = -xJ$, i.e., x is antilinear.

The argument for the group $\mathrm{O}^*(H, I)$ is similar. ■

IV.4. Current groups

Let K be a Lie group and M a compact connected manifold. We write $C^\infty(M, K)$ for the corresponding *current group*. In M we fix a base point x_M and in any group we consider the unit element $\mathbf{1}$ as the base point. We write $C^\infty_*(M, K) \subseteq C^\infty(M, K)$ for the subgroup of base point-preserving maps.

We then have

$$C^\infty(M, K) \cong C^\infty_*(M, K) \rtimes K$$

as Lie groups, where we identify K with the subgroup of constant maps. This relation already leads to

$$\pi_k(C^\infty(M, K)) \cong \pi_k(C^\infty_*(M, K)) \times \pi_k(K), \quad k \in \mathbb{N}_0. \tag{4.1}$$

For topological spaces X and Y we write $[X, Y]$ for the set of homotopy classes of continuous maps $f\colon X \to Y$, and for pointed spaces (X, x_0) and (Y, y_0) we write $[X, Y]_*$ for the set of all pointed homotopy classes of continuous base-point-preserving maps. For two compact pointed spaces we define

$$X \vee Y := X \times \{y_0\} \cup \{x_0\} \times Y \subseteq X \times Y \quad \text{and} \quad X \wedge Y := X \times Y / X \vee Y.$$

We then have for each pointed topological space (Z, z_0) a natural bijection

$$C_*(X, C_*(Y, Z)) \cong C_*(X \wedge Y, Z).$$

Moreover,

$$\mathbb{S}^k \wedge \mathbb{S}^d \cong \mathbb{S}^{k+d}, \quad k, d \in \mathbb{N}_0,$$

so that

$$\pi_k(C_*(X, K)) \cong \pi_0(C_*(\mathbb{S}^k, C_*(X, K))) \cong \pi_0(C_*(\mathbb{S}^k \wedge X, K)).$$

Theorem IV.4.1. *If M is a compact manifold, then the inclusion map*

$$C^\infty(M,K) \hookrightarrow C(M,K)$$

is a weak homotopy equivalence, i.e., induces isomorphisms of all homotopy groups. Therefore we have for each k an isomorphism

$$\pi_k(C^\infty(M,K)) \cong \pi_k(C(M,K)) \cong \pi_k(C_*(M,K)) \times \pi_k(K) \cong [\mathbb{S}^k \wedge M, K]_* \times \pi_k(K). \qquad \blacksquare$$

Let X be a locally compact space and K a Lie group. Then we write $C_0(M,K)$ for the Lie group of all continuous maps $f\colon M \to K$ vanishing at infinity in the sense that for each $\mathbf{1}$-neighborhood $U \subseteq K$ there exists a compact subset $C \subseteq X$ with $f(X \setminus C) \subseteq U$. If $X_\omega := X \cup \{\omega\}$ denote the *one-point compactification of* X, then this means that

$$C_0(X,K) \cong C_*(X_\omega, K)$$

because $f \in C_0(X,K)$ is equivalent to the extendibility of f to a continuous map $X_\omega \to K$ mapping ω to $\mathbf{1}$.

Theorem IV.4.2. *If M is a non-compact σ-compact manifold, then the inclusion map*

$$C^\infty_c(M,K) \hookrightarrow C_0(M,K)$$

is a weak homotopy equivalence, and we obtain isomorphisms

$$\pi_k(C^\infty_c(M,K)) \cong \pi_k(C_0(M,K)) \cong \pi_k(C_*(M_\omega,K)) \cong [\mathbb{S}^k \wedge M_\omega, K]_*. \qquad \blacksquare$$

With the above results, many calculations of homotopy groups of currents groups can thus be transfered into the continuous context, where one can use tools from topology to get more explicit information.

Example IV.4.3. If $M = \mathbb{S}^d$ is a d-dimensional sphere, then we have

$$\pi_k(C_*(\mathbb{S}^d,K)) \cong [\mathbb{S}^k \wedge \mathbb{S}^d, K]_* \cong [\mathbb{S}^{k+d}, K]_* \cong \pi_{k+d}(K) \tag{4.2}$$

and therefore

$$\pi_k(C(\mathbb{S}^d,K)) \cong \pi_k(K) \times \pi_{k+d}(K). \qquad \blacksquare$$

Example IV.4.4. We consider the case where $M = \mathbb{T}^d$ is an d-dimensional torus. Then

$$C(\mathbb{T}^d,K) \cong C(\mathbb{T}, C(\mathbb{T}^{d-1},K)) \cong C_*\big(\mathbb{T}, \big(C(\mathbb{T}^{d-1},K)\big) \rtimes C(\mathbb{T}^{d-1},K)$$

implies that

$$\pi_k(C(\mathbb{T}^d,K)) \cong \pi_{k+1}(C(\mathbb{T}^{d-1},K)) \oplus \pi_k(C(\mathbb{T}^{d-1},K))$$

and by induction we obtain

$$\pi_k(C(\mathbb{T}^d,K)) \cong \sum_{j=0}^d \pi_{k+j}(K)^{\binom{d}{j}}.$$

For $d = 2$ we get in particular

$$\pi_k(C(\mathbb{T}^2,K)) \cong \pi_k(K) \oplus \pi_{k+1}(K)^2 \oplus \pi_{k+2}(K)$$

which also follows from the calculations for surfaces in the following section. We also obtain for general d:

$$\pi_2(C(\mathbb{T}^d,K)) \cong \pi_2(K) \oplus \pi_3(K)^d \oplus \pi_4(K)^{\binom{d}{2}} \oplus \ldots.$$

$\blacksquare$

Oriented surfaces

In this subsection Σ denotes an orientable compact surface of genus g and K is an arbitrary topological group.[1]

Remark IV.4.5. We recall that Σ can be described as a CW-complex by starting with a bouquet

$$A_g \cong \underbrace{\mathbb{S}^1 \vee \mathbb{S}^1 \vee \ldots \vee \mathbb{S}^1}_{2g}$$

of $2g$-circles. We write $a_1, b_1, \ldots, a_g, b_g$ for the corresponding generators of the fundamental group of A_g which is a free group on $2g$ generators. Then we consider the continuous map $\gamma\colon \mathbb{S}^1 \to A_g$ corresponding to

$$[a_1, b_1] \cdots [a_g, b_g] \in \pi_1(A_g),$$

where $[x, y] = xyx^{-1}y^{-1}$ denotes a commutator. Now Σ is homeomorphic to the space obtained by identifying the points in $\partial \mathbb{B}^2 \cong \mathbb{S}^1$ with their images in A_g under γ, i.e.,

$$\Sigma \cong A_g \cup_\gamma \mathbb{B}^2.$$

In this sense we can identify A_g with a subset of Σ. The most instructive picture is to view $\mathbb{B}^2$ as the interior of a regular polygon with $4g$ edges, where we identify certain points on the edges such that in counterclockwise order the sequence of edges corresponds to the loop

$$a_1 b_1 a_1^{-1} b_1^{-1} a_2 \cdots a_n^{-1} b_n^{-1}.$$

Now A_g corresponds to the polygon modulo these identifications.

This procedure shows that a continuous map $f\colon A_g \to Z$ into a topological space Z extends to a map $\Sigma \to Z$ if and only if the corresponding map $\partial \mathbb{B}^2 \to Z$ extends to the interior of $\mathbb{B}^2$, which in turn means that it is a zero-homotopic curve. Finally, this can be expressed by the condition that

$$\pi_1(f)\colon \pi_1(A_g) \cong \underbrace{\mathbb{Z} * \mathbb{Z} * \cdots * \mathbb{Z}}_{2g} \to \pi_1(Z)$$

annihilates the commutator $a_1 b_1 a_1^{-1} b_1^{-1} a_2 \cdots a_n^{-1} b_n^{-1}$, hence factors to a homomorphism $\pi_1(\Sigma) \to \pi_1(Z)$.

Conversely, if such a homomorphism is given, then we can lift it to a homomorphism $\pi_1(A_g) \to \pi_1(Z)$ which can be trivially represented by a continuous map $A_g \to Z$. As we have seen above, this map extends to Σ, showing that the map

$$C_*(\Sigma, Z) \to \mathrm{Hom}(\pi_1(\Sigma), \pi_1(Z)) \tag{4.3}$$

is surjective for any pointed space Z. ■

Theorem IV.4.6. *For each topological group K we have a homeomorphism*

$$C(\Sigma, K) \cong C_*(\mathbb{S}^2, K) \times C_*(\mathbb{S}^1, K)^{2g} \times K$$

and

$$\pi_k(C(\Sigma, K)) \cong \pi_{k+2}(K) \times \pi_{k+1}(K)^{2g} \times \pi_k(K) \quad \textit{for all} \quad k \in \mathbb{N}_0.$$

---

[1] This subsection is based on conversations with F. Wagemann and on some calculations in his dissertation for the case $K{=}\mathrm{SU}(2){\cong}\mathbb{S}^3$ ([Wa98, Lemma 3.1.1]).

Proof. Let $(\gamma_1, \ldots, \gamma_{2g})$ be the natural generators of $\pi_1(\Sigma)$ coming from the maps $\mathbb{S}^1 \to A_g \hookrightarrow \Sigma$ given by $a_1, b_1, \ldots, a_g, b_g$. From (4.3) we obtain for $Z = \mathbb{S}^1$ with $\pi_1(Z) \cong \mathbb{Z}$ and base point 1 pointed continuous maps $\chi_1, \ldots, \chi_{2g}: \Sigma \to \mathbb{S}^1$ with

$$\pi_1(\chi_j)(\gamma_i) \cong [\chi_j \circ \gamma_i] = \delta_{ij}.$$

We can even get maps with

$$\chi_j \circ \gamma_i = \begin{cases} 1 & \text{for } i \neq j \\ \mathrm{id}_{\mathbb{S}^1} & \text{for } i = j \end{cases}$$

if we start with the continuous maps $\chi_j^0: A_g \to \mathbb{S}^1$ with the required property and observe that all these maps extend continuously to Σ because $\pi_1(\chi_j^0): \pi_1(A_g) \to \pi_1(\mathbb{S}^1) \cong \mathbb{Z}$ annihilates all commutators since $\mathbb{Z}$ is abelian.

Now we obtain for each topological group K a nice splitting of the restriction map

$$R: C_*(\Sigma, K) \to C_*(A_g, K) \cong C_*(\mathbb{S}^1, K)^{2g}$$

by the extension map

$$E: C_*(\mathbb{S}^1, K)^{2g} \to C_*(\Sigma, K), \quad (\alpha_1, \ldots, \alpha_{2g}) \mapsto (\alpha_1 \circ \chi_1) \cdots (\alpha_{2g} \circ \chi_{2g}).$$

Then $RE = \mathrm{id}$ follows directly from the choice of the maps χ_j. We conclude that

$$C_*(\Sigma, K) \to \ker(R) \times C_*(\mathbb{S}^1, K)^{2g}, \quad f \mapsto (fE(R(f))^{-1}, R(f))$$

is a homeomorphism whose inverse is given by $(\alpha, \beta) \mapsto \alpha E(\beta)$. Next we observe that

$$\ker R \cong C_*(\Sigma/A_g, K) \cong C_*(\mathbb{S}^2, K),$$

so that we obtain a homeomorphism

$$C_*(\Sigma, K) \cong C_*(\mathbb{S}^2, K) \times C_*(\mathbb{S}^1, K)^{2g}$$

and hence a homeomorphism

$$C(\Sigma, K) \cong C_*(\mathbb{S}^2, K) \times C_*(\mathbb{S}^1, K)^{2g} \times K. \tag{4.4}$$

This implies that we have the group isomorphism

$$\pi_0(C(\Sigma, K)) \cong [\Sigma, K] = \pi_2(K) \times \pi_1(K)^{2g} \times \pi_0(K).$$

Combining (4.2) and (4.4) further leads to

$$\pi_k(C(\Sigma, K)) \cong \pi_{k+2}(K) \times \pi_{k+1}(K)^{2g} \times \pi_k(K) \quad \text{for all} \quad k \in \mathbb{N}_0.$$ ■

Remark IV.4.7. Suppose that $g \geq 1$. Then the universal covering space $\widetilde{\Sigma}$ of Σ is contractible, showing that the only non-trivial homotopy group of Σ is $\pi_1(\Sigma)$. This means that Σ is a $K(\pi_1(\Sigma), 1)$-space in the sense of Eilenberg–MacLane. The result above shows that the natural homomorphism

$$[\Sigma, K]_* \to \mathrm{Hom}(\pi_1(\Sigma), \pi_1(K)) \cong \pi_1(K)^{2g}$$

has a kernel isomorphic to $\pi_2(K)$, hence is not injective. This means that the homotopy classes of maps $\Sigma \to K$ are NOT classified by the sequence of homomorphisms $\pi_k(\Sigma) \to \pi_k(K)$, $k \in \mathbb{N}_0$. ■

Corollary IV.4.8. *If K is 2-connected, then $C(\Sigma, K)$ is arcwise connected.*

Proof. This follows directly from Theorem IV.4.6.

We also give a second direct proof. If K is 2-connected, i.e.,

$$\pi_0(K) = \pi_1(K) = \pi_2(K) = \mathbf{1},$$

the inclusion $\mathbf{1} \hookrightarrow K$ is a 2-equivalence in the sense of [Br93, Cor. 11.13]. Since Σ is two-dimensional, this implies that the map $[\Sigma, \mathbf{1}] \to [\Sigma, K]$ is surjective, and hence that $[\Sigma, K]$ is a singleton. This means that $C(\Sigma, K)$ is arcwise connected. ∎

Remark IV.4.9. Suppose that the topological group K is semilocally simply connected, so that it has a universal covering group $\widetilde{K}$. This condition is in particular satisfied if K is locally contractible.

Let $q_K: \widetilde{K} \to K$ denote the simply connected covering homomorphism. For an arcwise connected locally arcwise connected space X, a continuous map $f: X \to K$ lifts to a map $X \to \widetilde{K}$ if and only if the homomorphism $\pi_1(f): \pi_1(X) \to \pi_1(K)$ vanishes (cf. [tD91, Satz 6.12]). Therefore we have an exact sequence of groups

$$C_*(X, \widetilde{K}) \xrightarrow{(q_K)_*} C_*(X, K) \to \operatorname{Hom}(\pi_1(X), \pi_1(K)).$$

If $f \in C_*(X, K)_0$, then it is homotopic to a constant map, so that $\pi_1(f)$ vanishes, and therefore it is contained in the range of $(q_K)_*: h \mapsto q_K \circ h$. We thus obtain an exact sequence

$$\pi_0(C_*(X, \widetilde{K})) \to \pi_0(C_*(X, K)) \to \operatorname{Hom}(\pi_1(X), \pi_1(K)).$$ ∎

Holomorphic current groups

Let M be *Stein manifold*, i.e., a complex manifold which can be realized as a closed submanifold of some $\mathbb{C}^n$. Further let K be a Banach–Lie group, then the groups $C(M, K)$ and $\operatorname{Hol}(M, K)$ are metrizable topological groups with respect to the topology of uniform convergence on compact subsets of M (Example II.1.6). In general these groups are not Lie groups and it is an interesting open problem to characterize those Stein manifolds M for which they are. We have a natural inclusion map

$$\eta: \operatorname{Hol}(M, K) \hookrightarrow C(M, K),$$

and one can show that this inclusion is a homotopy equivalence. This is based on results of R. Palais which imply that under certain conditions (here the metrizability) weak homotopy equivalences are homotopy equivalences. The statement about the weak homotopy equivalence is then reduced to Oka's Principle which asserts that the inclusion η induces a bijection on the level of connected components. Further, one uses that for each $k \in \mathbb{N}$ the group $C(\mathbb{S}^k, K)$ is also a Banach–Lie group, so that Oka's Principle applies to the topological group $\operatorname{Hol}(M, C(\mathbb{S}^k, K)) \cong C(\mathbb{S}^k, \operatorname{Hol}(M, K))$ [1].

These results are of particular interest if $M = \Sigma \setminus F$, where Σ is a compact Riemann surface and $F \subseteq \Sigma$ a finite set.

IV.5. Diffeomorphism groups

In this section we briefly discuss the topology of the groups $\operatorname{Diff}(\mathbb{S}^d)$. For more details we refer to [Mil83].

[1] The author learned the trick of replacing the group K in this context by $C(\mathbb{S}^k, K)$ from Bernhard Gramsch.

For $M = \mathbb{S}^1$ we have already seen in Section II.3 that the universal covering group of $\mathrm{Diff}_+(\mathbb{S}^1)$ is contractible. This implies that

$$\pi_k(\mathrm{Diff}(\mathbb{S}^1)) \cong \begin{cases} \mathbb{Z}_2 & \text{for } k = 0 \\ \mathbb{Z} & \text{for } k = 1 \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

For $M = \mathbb{S}^d$, $d \geq 2$, the situation gets more complicated. Of course we have a natural inclusion $\mathrm{O}(d+1, \mathbb{R}) \hookrightarrow \mathrm{Diff}(\mathbb{S}^d)$ and one may ask for which dimension d this inclusion is a homotopy equivalence. For $d = 1, 2$ this has been proved by S. Smale in 1959 and conjectured by him for $d = 3$. This conjecture was proved in 1983 by Hatcher. For $d = 4$ the answer is not known to the author, and for $d > 4$ the inclusion is not a homotopy equivalence ([Mil83]).

For $d = 2$ this leads to the following information on the homotopy groups. As $\mathrm{O}(3, \mathbb{R}) \cong \mathrm{SO}(3, \mathbb{R}) \times \mathbb{Z}_2$ and the universal covering group $\mathrm{SU}(2, \mathbb{C})$ of $\mathrm{SO}(3, \mathbb{R})$ is homeomorphic to $\mathbb{S}^3$, we obtain from Remark IV.1.3:

$$\pi_k(\mathrm{Diff}(\mathbb{S}^2)) \cong \begin{cases} \mathbb{Z}_2 & \text{for } k = 0 \\ \mathbb{Z}_2 & \text{for } k = 1 \\ \mathbf{0} & \text{for } k = 2 \\ \mathbb{Z} & \text{for } k = 3 \\ \mathbb{Z}_2 & \text{for } k = 4 \\ \mathbb{Z}_2 & \text{for } k = 5. \end{cases}$$

For $d = 3$ we have $\mathrm{O}(4, \mathbb{R}) \cong \mathrm{SO}(4, \mathbb{R}) \rtimes \mathbb{Z}_2$ and the universal covering group of $\mathrm{SO}(4, \mathbb{R})$ is a two-fold covering by $\mathrm{SU}(2, \mathbb{C})^2$. This leads to

$$\pi_k(\mathrm{Diff}(\mathbb{S}^3)) \cong \begin{cases} \mathbb{Z}_2 & \text{for } k = 0 \\ \mathbb{Z}_2 & \text{for } k = 1 \\ \mathbf{0} & \text{for } k = 2 \\ \mathbb{Z}^2 & \text{for } k = 3 \\ \mathbb{Z}_2^2 & \text{for } k = 4 \\ \mathbb{Z}_2^2 & \text{for } k = 5. \end{cases}$$

The group $\pi_0(\mathrm{Diff}_+(\mathbb{S}^d))$, which is finite for $d \geq 5$, has a remarkable differential geometric interpretation. Its elements correspond to oriented diffeomorphism classes of smooth $(d+1)$-dimensional manifolds with the homotopy type of $\mathbb{S}^{d+1}$. For $d \neq 2$ this implies that they are homeomorphic to $\mathbb{S}^{d+1}$ by the Poincaré conjecture, which has been proved except for $d = 2$.

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