The exponential function of locally connected compact abelian groups

by

Karl H. Hofmann, Sidney A. Morris, and Detlev Poguntke

Abstract. It is shown that the following four conditions are equivalent for a compact connected abelian group G: (i) the exponential function of G is open onto its image; (ii) G has arbitrarily small connected direct summands N such that G/N is a finite dimensional torus; (iii) the arc component G_a of the identity is locally arcwise connected; (iv) the character group $\widehat{G}$ is a torsion free group in which every finite rank pure subgroup is free and is a direct summand.

Introduction

A morphism of topological groups $f: G \to H$ is said to be *open onto its image* if for any open subset U of G its image $f(U)$ is open in the subgroup $f(G)$ of H. This is tantamount to saying that the morphism $f': G/\ker f \to f(G)$ induced by f via $f'(g \ker f) = f(g)$ is an isomorphism of topological groups. Equivalently, this property can be expressed by saying that the corestriction $f: G \to f(G)$ to its image is a quotient morphism.

We denote by $\mathbb{T}$ the additively written circle group $\mathbb{R}/\mathbb{Z}$. If G is a torus group $\mathbb{T}^X$ for an arbitrary set X, then the exponential function of G, which we may identify with the canonical quotient $\mathbb{R}^X \to \mathbb{T}^X$, and which therefore is readily seen to be open, and thus, in particular, open onto its image. It is surprising that there are compact connected groups which are not torus groups but for which the exponential function is open onto its image. The image of the exponential function $\exp: \mathfrak{L}(G) \to G$ of a compact connected abelian group G is precisely the arc component G_a of the identity. (See [3], p. 389, Theorem 8.30(ii).) We shall investigate when the corestriction $\exp': \mathfrak{L}(G) \to G_a$ is open. In fact, we shall show that a strong form of local connectivity is characteristic for this property. The character group G of the discrete group $\mathbb{Z}^{\mathbb{N}}$ is an example of a compact connected abelian group of weight $2^{\aleph_0}$ for which $\exp': G \to G_a$ is open, while $G_a \neq G$.

It is well known that complete topological abelian groups may have incomplete quotients in manifold ways [5]. Yet it is noteworthy that this example will show that the product $\mathbb{R}^{2^{\aleph_0}}$ of a continuum cardinality of factors of the real line has an incomplete quotient whose completion is compact; this example is quite relevant in the theory of projective limits of finite dimensional Lie groups of which $\mathbb{R}^{2^{\aleph_0}}$ is the simplest example which is not locally compact. (See [4]).

We begin by focussing on a class of torsion free abelian groups A characterized by the property that every finite subset of A is contained if a free direct summand; the members of this class we shall call S-groups. The class of S groups contains the class of free groups but is properly smaller than the class of abelian groups whose countable subgroups are free. A compact connected abelian group G will eventually turn out to have an exponential function which is open onto its image ifand only if its character group $\widehat{G}$ is an S-group.

1. Separable Abelian Groups

We shall write abelian groups additively. A subgroup G of an abelian group G is said to *split* if there is a subgroup H of A such that $A = G \oplus H$. Likewise we say that a short exact sequence

$$(E) \qquad 0 \to G \xrightarrow{j} A \to B \to 0$$

is split if $j(G)$ is a split subgroup of A. This is tantamount to saying that the equivalence class of (E) in $\mathrm{Ext}(B, G)$ is zero. (See e.g. [3], pp. 643 ff.) A subgroup G of an abelian group A splits automatically if it is divisible or A/G is free. (See e.g. [3], pp. 637 and 6.27.)

A subgroup P of an abelian group A is called a *pure subgroup* if the following condition is satisfied.
(i) $(\forall p \in P,\ a \in A, n \in \mathbb{N})\, n{\cdot}a = p \Rightarrow (\exists x \in P)\, n{\cdot}x = p.$
One observes directly (see also [3], p. 630) that for a torsion free group A, this condition is equivalent to any of the following conditions:
(ii) The factor group A/G is torsion free.
(iii) $(\forall n \in \mathbb{N}, a \in A)\, n{\cdot}a \in G \Rightarrow a \in G.$

In a torsion free abelian group A every subgroup G is contained in a unique pure subgroup $[G] = \{a \in A : (\exists n \in \mathbb{N})\, n{\cdot}a \in G\}$. (See [3], p. 630) Moreover, the following conditions are equivalent (see [3], p. 652)
(a) Every finite rank pure subgroup of A is free.
(b) Every countable subgroup of A is free.
An abelian group satisfying these conditions is said to be $\aleph_1$-*free.*

We are interested in a subclass of the class of $\aleph_1$-free groups.

Proposition 1.1. *Let A be a torsion free abelian group. Then the following conditions are equivalent.*
(i) $(\forall a \in A)\, [\mathbb{Z}{\cdot}a]$ *is free and splits.*
(ii) *Every rank one pure subgroup is free and splits.*
(iii) *Every finite rank pure subgroup is free and splits.*

Proof. Clearly, (iii)⇒(ii)⇔(i). We must prove (i)⇒(iii).
Step 1. We first note that if A satisfies (i) then every pure subgroup B of A satisfies (i): Indeed let $b \in B$; since B is pure the pure subgroup $[\mathbb{Z}{\cdot}b]$ generated by b in B is pure in A, and as a subgroup of the free pure subgroup generated by b in A it is free. Since A satisfies (i), we have $A = [\mathbb{Z}{\cdot}b] \oplus K$ with a suitable

subgroup K. By the modular law, $B = [\mathbb{Z}{\cdot}b] \oplus (K \cap B)$. Setting $H = K \cap B$ we obtain $B = [\mathbb{Z}{\cdot}b] \oplus H$ as claimed.

Step 2. We claim that every finite rank pure subgroup is free; in other words we show that A is $\aleph_1$-free. Let G be a finite rank pure subgroup of A and assume that G is a counterexample of minimal rank n. From (i) we know $n > 1$. Let $0 \neq g \in G$. Since G satisfies (i), $G = [\mathbb{Z}{\cdot}g] \oplus H$ and $[\mathbb{Z}{\cdot}g]$ is free. Since the subgroup H has smaller rank than G, it is not a counterexample and is therefore free. Hence G is free and thus can't be a counterexample.

Step 3. Now we prove that every finite rank pure subgroup G splits. By Step 2, $G = \bigoplus_{m=1}^{N} \mathbb{Z}{\cdot}e_m$, where $N \in \mathbb{N}$. By (i) there is a subgroup H_1 of A such that $A = \mathbb{Z}{\cdot}e_1 \oplus H_1$ and $G = \mathbb{Z}{\cdot}e_1 \oplus (H_1 \cap G)$. Assume that $H_1 \supseteq H_2 \supseteq \cdots \supseteq H_n$, $n < N$ has been constructed in such a fashion that $A = \mathbb{Z}{\cdot}e_1 \oplus \cdots \oplus \mathbb{Z}{\cdot}e_m \oplus H_m$ and $G = \mathbb{Z}{\cdot}e_1 \oplus \cdots \oplus \mathbb{Z}{\cdot}e_m \oplus (H_m \cap G)$, $m = 1, 2, \ldots, n$. By the first step we may apply (i) to the pure subgroup H_n and find a subgroup H_{n+1} of H_n such that $H_n = \mathbb{Z}{\cdot}e_{n+1} \oplus H_{n+1}$ and $H_n \cap G = \mathbb{Z}{\cdot}e_{n+1} \cap (H_{n+1} \cap G)$. This yields a descending family of subgroups H_n such that $A = \bigoplus_{m=1}^{n} \mathbb{Z}{\cdot}e_m \oplus H_n$ and $e_m \in H_n$ for $m > n$. We set $H = H_N$. Then $A = \bigoplus_{m=1}^{N} \mathbb{Z}{\cdot}e_m \oplus H_N = G \oplus H$, as was to be shown. □

Definition 1.2. We say that an abelian group A is an S-group if it satisfies the equivalent conditions of Proposition 1.1. □

The S-groups have been called *separable* [1] which is not an advisable terminology here because we will deal with topological abelian groups for which the adjective separable refers to groups having a dense countable subset, and this is entirely different. One might have called S-groups *strongly* $\aleph_1$*-free*; our terminology reflects the "strongly" as well.

Every free group is an S-group, since every subgroup of a free group is free, and the quotient of a free group modulo a pure subgroup is free (see e.g. [3], p. 632, Proposition A1.24(ii)); thus every pure subgroup splits. A *Whitehead group* is an abelian group A such that $\mathrm{Ext}(A, \mathbb{Z}) = \{0\}$, that is, every extension

$$0 \to \mathbb{Z} \to G \to A \to 0$$

splits.

Example 1.3. The group $A = \mathbb{Z}^{\mathbb{N}}$ has the following properties:
(i) A is an S-group.
(ii) A is not a Whitehead group.
(iii) The subgroup $\mathbb{Z}^{(\mathbb{N})}$ of A is a countable free pure subgroup which does not split.

Proof. (i) and (ii): The group $\mathbb{Z}^{\mathbb{N}}$ is an $\aleph_1$-free group which is not a Whitehead group: see e.g. [3], p. 652, Example A1.65.

We verify Condition 1.1(ii) for $A = \mathbb{Z}^{\mathbb{N}}$. Let P be a rank one pure subgroup of A. Since A is $\aleph_1$-free, P is free and thus $P = \mathbb{Z}{\cdot}k$ with an element $k = (k_n)_{n\in\mathbb{N}}$. If $d = \text{g.c.d.}\{k_n : n \in \mathbb{N}\} > 0$ denotes the greatest common divisor of $\{k_n : n \in \mathbb{N}\}$, then $(k_n/d)_{n\in\mathbb{N}} \in P$; then there is an $m \in \mathbb{Z}$ such that $k_n/d = mk_n$ for all $n$and thus $dm = 1$, that is, $d = 1$. The decreasing sequence g.c.d$\{k_1, k_2, \ldots, k_n\}$ is eventually constant; that is, there is a natural number N such that the integers $k_1, \ldots, k_N$ have the greatest common divisor 1. Then the subgroup P_N generated in the finitely generated free group $\mathbb{Z}^N$ by $(k_1, \ldots, k_N)$ is is pure. By the Elementary Divisor Theorem (see e.g. [3], p.623) applied to $\mathbb{Z}^N$, after choosing a new basis, we may assume that $P_N = \mathbb{Z} \times \{0\} \times \cdots \times \{0\}$. It is therefore no loss of generality to assume that $(k_n)_{n\in\mathbb{N}} = (1, 0, \ldots, 0, k_{N+1}, k_{N+2}, \ldots)$. Set $G = \{0\} \times \mathbb{Z} \times \mathbb{Z} \times \cdots$. Then $\mathbb{Z}^{\mathbb{N}} = P \oplus G$ and $\mathbb{Z}^{\mathbb{N}}$ is an S-group as asserted.

(iii): If $m{\cdot}(k_n)_{n\in\mathbb{N}} \in \mathbb{Z}^{(\mathbb{N})}$ for some $m \in \mathbb{Z}$ then $mk_n = 0$ for all but a finite number of $n \in \mathbb{N}$. Then $(k_n)_{n\in\mathbb{N}}$. Thus $\mathbb{Z}^{(\mathbb{N})}$ is a pure subgroup of $Z^{\mathbb{N}}$ which is obviously countable and free. The group $\mathbb{Z}^{\mathbb{N}}/\mathbb{Z}^{(\mathbb{N})}$ is a torsion free algebraically compact group and contains a copy $\mathbb{Z}_p$ of the p-adic integers for each prime as a direct summand. (See e.g. [2], p. 176, 42.2 and p. 169, 40.4.) Since $\mathbb{Z}_p$ contains countable groups which are not free (e.g. $\frac{1}{q^\infty}\mathbb{Z}$ for any prine q different from p), and since $\mathbb{Z}^{\mathbb{N}}$ is $\aleph_1$-free, $\mathbb{Z}^{\mathbb{N}}$ cannot contain a subgroup isomorphic to $A/\mathbb{Z}^{(\mathbb{N})}$. □

Example 1.4. There is an abelian group B with a subgroup $C \cong \mathbb{Z}$ such that $B/C \cong \mathbb{Z}^{\mathbb{N}}$ and that every morphism $B \to \mathbb{Z}$ annihilates C. The group B is an $\aleph_1$-free group which is not an S-group.

Proof. In [3], pp. 653, 654, the following lemmas are proved:

Lemma A. *Let* $E = [0 \to C \hookrightarrow B \to X \to 0]$ *be any extension of* $C \cong \mathbb{Z}$ *by an abelian group* X*. Then there is a homomorphism* $f\colon B \to \mathbb{Z}$ *whose restriction to* C *is nontrivial if and only* E *represents an element of finite order in* $\mathrm{Ext}(X, \mathbb{Z})$. □

Lemma B. $\mathrm{Ext}(\mathbb{Z}^{\mathbb{N}}, \mathbb{Z})$ *contains* $2^{(2^{\aleph_0})}$ *elements of infinite order.* □

Taken together, these Lemmas yield the existence of a torsion free group B and a cyclic subgroup C such that $B/C \cong \mathbb{Z}^{\mathbb{N}}$, and that every morphism $B \to \mathbb{Z}$ vanishes on C. The subgroup C is a subgroup of rank 1 which does not split, and since B/C is torsion free, C is a pure subgroup of B. Thus B is not an S-group. If P is a finite rank pure subgroup of B, then $\left[\frac{P+C}{C}\right]$ is a finite rank pure subgroup of $B/C \cong A$ and is therefore finitely generated free; its full inverse image P' in B is a finitely generated torsion free group and is, therefore, free. Thus P as a subgroup of a free group is free. Hence B is an $\aleph_1$-free group. □

In this area of the theory of abelian groups, $\mathbb{Z}^{\mathbb{N}}$ is a universal test example. For instance, Proposition 1.1 cannot be complemented by another equivalent condition which would say: *Every countable pure subgroup splits.* The example shows, in particular, that the class of S-groups is properly smaller than that of $\aleph_1$-free groups and is not contained in the class of Whitehead groups and thus is properly bigger than the class of free groups.

$\aleph_1$-free groups
|
S-groups
|
free groups

2. Strengthening Local Connectivity for Locally Compact Abelian Groups

A locally compact abelian group G is completely characterized by its Pontryagin dual $\widehat{G} = \mathrm{Hom}(G, \mathbb{T})$. (See e.g. [3], Chapters 7 and 8.) A topological group G is locally connected if and only if its identity component G_0 is open in G and is locally connected; a connected locally compact abelian group G contains a unique characteristic maximal compact subgroup C and a subgroup $V \cong \mathbb{R}^n$ such that the morphism $(v, c) \mapsto v + c : V \times C \to G$ is an isomorphism of topological groups. (See e.g. [3], p. 348, Theorem 7.57.) In discussing local connectivity of a locally compact abelian group G, it is no loss of generality to assume that G is compact and connected. A locally compact abelian group G is compact and connected if and only if its character group $\widehat{G}$ is discrete and torsion free. (See e.g. [3], p. 297, Proposition 7.5(i), and p. 369, Corollary 8.5.) Local connectivity of a compact connected abelian group is characterized as follows:

Proposition 2.1. *For a compact connected abelian group G, the following statements are equivalent:*

(i) *There are arbitrarily small compact connected subgroups N such that G/N is a finite dimensional torus group.*
(ii) *The character group $\widehat{G}$ is the directed union of pure finitely generated free subgroups.*
(iii) *$\widehat{G}$ is $\aleph_1$-free.*
(iv) *G is locally connected.*

Proof. For a proof see [3], p. 396, Theorem 8.36. □

We compare this proposition with the following

Proposition 2.2. *For a compact connected abelian group G, the following statements are equivalent:*

(i) *There are arbitrarily small compact connected subgroups N for which there is a finite dimensional torus subgroup T_N of G such that $(n, t) \mapsto n + t : N \times T_N \to G$ is an isomorphism of topological groups.*
(ii) *The character group $\widehat{G}$ is the directed union of finitely generated free* split *subgroups.*

(iii) $\widehat{G}$ *is an* S-*group.*

Proof. The equivalence of (i) and (ii) follows at once from duality.
(ii)⇒(iii): By (ii), the abelian group $\widehat{G}$ is torsion free. Let P be a rank one pure subgroup of $\widehat{G}$. The $P = [\mathbb{Z}{\cdot}a]$ for some $a \in \widehat{G}$. By (ii) there is a finitely generated free split subgroup F of $\widehat{G}$ containing a. Since a direct summand is a pure subgroup we have $P = [\mathbb{Z}{\cdot}a] \subseteq F$. As a pure subgroup of a finitely generated torsion free group, P is a direct summand of F, and since F is a direct summand of $\widehat{G}$, P is a split subgroup of $\widehat{G}$.
(iii)⇒(ii): As a torsion free abelian group, $\widehat{G}$ is the directed union of all of its finite rank pure subgroups P; by (iii), every such P is split and free, and thus (ii) follows. □

The comparison of Propositions 2.1 and 2.2 justifies the following definition:

Definition 2.3. A locally compact abelian group is said to be *strongly locally connected* if its identity component is open and its unique maximal compact connected subgroup satisfies the equivalent conditions of Proposition 2.2.

In particular, a compact connected abelian group is strongly locally connected if and only if its character group is an S-group. □

Example 2.4. Let $G \stackrel{\text{def}}{=} \widehat{\mathbb{Z}^{\mathbb{N}}}$. Then G is a strongly locally connected and connected but not arcwise connected compact abelian group.

There is a compact connected, locally connected, but not strongly locally connected group H of weight $2^{\aleph_0}$ containing G such that H/G is a circle group.

G has a metric torus group quotient which is not a homomorphic retract.

Proof. A compact connected abelian group H is arcwise connected if and only if its character group $\widehat{H}$ is a Whitehead group. (See e.g. [3], pp. 389, 390, Theorem 8.30(iv).) The claim thus follows by duality from Examples 1.3 and 1.4. □

The class of connected strongly locally connected compact abelian groups is properly larger than that of torus groups and properly smaller than that of connected and locally connected compact abelian groups.

3. The Exponential Function of Strongly Locally Connected Groups

We shall investigate when the exponential function $\exp\colon \mathfrak{L}(G) \to G$ of a compact connected abelian group G in the present context. For a detailed exposition of the exponential function of compact abelian groups we refer to [3], notably Chapters 7 and 8. We need to know here that $\mathfrak{L}(G) = \mathrm{Hom}(\mathbb{R}, G)$ is the topological vector space of all one parameter subgroups, i.e. continuous group morphisms $X\colon \mathbb{R} \to G$, where $\mathrm{Hom}(\mathbb{R}, G)$ is given the topology of uniform convergence on compact sets. The exponential function is given by evaluation via $\exp X = X(1)$. By duality, $\mathfrak{L}(G)$ may also be viewed as the vector space $\mathrm{Hom}(\widehat{G}, R)$ with the topology of pointwise convergence. We note that $\mathfrak{L}(G)$ is isomorphic to a product of copies of $\mathbb{R}$ (see [3], p. 355, Theorem 7.66 (i) and pp. 325, 326, Theorem 7.30(ii)). For any compact abelian group G, the exponential function $\exp_G\colon \mathfrak{L}(G) \to G$ (cf. [3], pp. 355, 356, Theorem 7.66) is a morphism of abelian topological groups. Let $G_a = \mathrm{im} \exp_G$ denote the arc component of the zero element 0. (See [3], p. 389, 390, Theorem 8.30.) We note the exact sequence:

$$\text{(exp)} \qquad 0 \to \mathfrak{K}(G) \to \mathfrak{L}(G) \xrightarrow{\exp_G} G \to \mathrm{Ext}(\widehat{G}, \mathbb{Z}) \to 0$$

where $\mathfrak{K}(G) = \ker \exp_G$ The corestriction $\exp'_G\colon \mathfrak{L}(G) \to G_a$ of the exponential function to its image is a surjective morphism of topological groups. A surjective morphism between topological groups is open if and only if it is a quotient morphism. Thus $\exp'_G$ is open iff the induced bijective morphism of topological groups $\mathfrak{L}(G)/\mathfrak{K}(L) \to G_a$ is an isomorphism of topological groups.

Proposition 3.1. *Let G be a connected and strongly locally connected compact abelian group. Then $\exp'_G\colon \mathfrak{L}(G) \to G_a$ is open.*

Proof. Let $\mathcal{P}$ denote the set of pure finite rank subgroups of $\widehat{G}$. then $P \in \mathcal{P}$ is a finitely generated free split subgroup of $\widehat{G}$. We select a subgroup $S_P \subseteq \widehat{G}$ such that $\widehat{G} = P \oplus S_P$. Let $N_P \overset{\text{def}}{=} P^\perp$ denote the annihilator of P in G and $T_P = S_P^\perp$ the annihilator of S_P. By duality $(n, t) \mapsto n + 1 : N_P \times T_P \to G$ is an isomorphism of topological groups, that is, $G = N_P \oplus T_P$ algebraically and topologically. The groups T_P, G/N_P and $\widehat{P}$ are naturally isomorphic by the Annihilator Mechanism (see [3], p. 352, Theorem 7.64) and thus are finite dimensional torus groups. The morphism $\exp_{N_P} \times \exp_{T_P} : \mathfrak{L}(N_P) \times \mathfrak{L}(T_P) \to N_P \times T_P$ is naturally equivalent to the exponential function of $N_P \times T_P$. Let $\mathcal{U}_P$ be the set of arcwise connected open zero-neighborhoods U of $\mathfrak{L}(T_P)$ mapped homoemorphically onto an open zero neighborhood V of T_P by $\exp_{T_P}$; such neighborhoods U and V exist as T_P is a Lie group. Then $(\exp_{N_P} \times \exp_{T_P})(\mathfrak{L}(N_P) \times U) = (N_P)_a \times V = (N_P \times V) \cap (N_P \times T_P)_a$. It follows that $\exp_G(\mathfrak{L}(N_P) \oplus U)$ is a an identity neighborhood of G_a in the topology induced from that of G. We claim that $\{\mathfrak{L}(N_P) \oplus U : P \in \mathcal{P}, U \in \mathcal{U}_P\}$ is a basis for the open zero-neighborhoods of $\mathfrak{L}(G)$, where $\mathfrak{L}(N_P)$ is naturally considered as a cofinite dimensional vector subspace of $\mathfrak{L}(G)$. One this claim is established, $\exp'_G\colon \mathfrak{L}(G) \to G_a$ is open and the proof of the proposition will be complete.

Since $\widehat{G} = \bigcup \mathcal{P} = \mathrm{colim}_{P \in \mathcal{P}} P$ by duality, we have $G = \lim_{P \in \mathcal{P}} G/N_P$ The functor $\mathfrak{L} = \mathrm{Hom}(\mathbb{R}, -)$ preserves projective limits (see e.g. [3], p. 336, Proposition 7.38(iv); in fact $\mathfrak{L}$ preserves *all* limits). Furthermore, $\mathfrak{L}$ preserves quotients (see [3], pp. 355, 356, Theorem 7.66(iv)). Hence $\mathfrak{L}(G) = \lim_{P \in \mathcal{P}} \mathfrak{L}(G)/\mathfrak{L}(N_P)$. Let $p_P\colon \mathfrak{L}(G) \to \mathfrak{L}(G)/\mathfrak{L}(N_P)$ denote the quotient morphism. For any zero neighborhood W of $L(G)$, by basic properties of the limit, there is a $P \in \mathcal{P}$ and a zero neighborhood $U \in \mathcal{U}_P$ in $\mathfrak{L}(T_P) \cong \mathfrak{L}(G)/\mathfrak{L}(N_P)$ such that $p_P^{-1}\big((U + \mathfrak{L}(N_P))/\mathfrak{L}(N_P)\big) \subseteq$

W. (See e.g. [3], p. 21, Proposition 1.33(i)(a); this part of 1.33 has nothing to do with compactness). Since $p_P^{-1}\big((U+\mathfrak{L}(N_P)\big) = \mathfrak{L}(N_P)\oplus U$, the claim is proved. □

Corollary 3.2. (i) *The uncountable product* $V \stackrel{\rm def}{=} \mathbb{R}^{\mathbb{R}}$ *has a closed totally disconnected algebraically free subgroup* K *of countable rank such that the quotient* V/K *is incomplete and its completion* G *is a compact connected and strongly locally connected abelian group of continuum weight.*

(ii) *Let* $q: V \to V/K$ *denote the quotient map and* $\gamma_{V/K}: V/K \to G$ *be the completion map. There is a morphism* $f: V \to C$ *into a compact hence complete group whose kernel is* K *and which has the property that the factorisation map* $f': G \to C$ *determined uniquely by* $f = f' \circ \gamma_{V/K} \circ q$ *is* not *injective.*

Proof. (i): Let $G = \widehat{\mathbb{Z}^{\mathbb{N}}}$ be the compact abelian group of 2.4. The rank of $\mathbb{Z}^{\mathbb{N}}$ agrees with the cardinal of $\mathbb{Z}^{\mathbb{N}}$ and that is the cardinal $2^{\mathbb{N}}$ of the continuum. Then $\mathfrak{L}(G) = \mathrm{Hom}(\mathbb{R}, G) = \mathrm{Hom}(\mathbb{Z}^{\mathbb{N}}, \mathbb{R}) \cong \mathbb{R}^{2^{\mathbb{N}}} \cong \mathbb{R}^{\mathbb{R}}$. Thus we take for V the additive group of $\mathfrak{L}(G)$ and $K = \mathfrak{K}(G)$ and know that $\varepsilon: V/K \to G_a$ is an isomorphism of topological groups by Proposition 3.1. The completion of G_a is G, and $w(G) = \mathrm{card}\,\mathbb{Z}^{\mathbb{N}} = 2^{\aleph_0}$. The kernel $\mathfrak{K}(G)$ is algebraically isomorphic to $\mathrm{Hom}(\mathbb{Z}^{\mathbb{N}}, \mathbb{Z})$ (see [3], p. 355, Theorem 7.66(ii)). But $\mathrm{Hom}(\mathbb{Z}^{\mathbb{N}}, \mathbb{Z}) \cong \mathbb{Z}^{(\mathbb{N})}$ (see [1], p. 61, Corollary 2.5). Thus K is free of countable rank.

(ii): In [3], p. 652, Example A1.65 one finds the construction of a character $\chi: \mathbb{Z}^{\mathbb{N}} \to \mathbb{T}$ of order 2 (i.e., $2{\cdot}\chi = 0$ in additive notation) which does not factor in the form

$$\mathbb{Z}^{\mathbb{N}} \xrightarrow{\ \varphi\ } \mathbb{R} \xrightarrow{\ p\ } \mathbb{T}, \quad p(r) = r + \mathbb{Z}.$$

Then, as an element of $G = \mathrm{Hom}(\mathbb{Z}^{\mathbb{N}}, \mathbb{T})$, the character χ is not in the image G_a of $\mathrm{Hom}(\mathbb{Z}^{\mathbb{N}}, p) = \exp_G: \mathrm{Hom}(\mathbb{Z}^{\mathbb{N}}, \mathbb{R}) = \mathfrak{L}(G) \to G$. Set $Z = \{0, \chi\}$. Then Z is a closed subgroup of G such that $G_a \cap Z = \{0\}$. Put $C = G/Z$ and let $f': G \to C$ be the quotient morphism whose kernel is Z. The restriction $F: G_a \to C$ is injective. Let $f: V \to C$ be defined by $f(X) = F(\exp_G X)$. By (i) the corestriction $q: V = \mathfrak{L}(G) \to G_a$ of the exponential function is a quotient morphism, and $F = f \circ q$ is the canonical epic-monic factorisation of F. Since G is isomorphic to the completion G of G_a and the inclusion $G_a \to G$ is the completion morphism $\gamma_{G_a}: G_a \to G$, the assertion follows. □

The significance of 3.2(ii) is as follows: In the category of all complete topological abelian groups, the completion of a quotient plays the role of a quotient in the category as it has the expected universal properties; nevertheless, it will in general fail to have familiar properties as 3.2(ii) illustrates. The possible incompleteness of quotients plays a somewhat disturbing role in a general theory of projective limits of finite dimensional Lie groups. The simplest nontrivial projective limits of finite dimensional Lie groups are the products $\mathbb{R}^X$. The product $\mathbb{R}^{\mathbb{N}}$ is metrizable and complete, hence every quotient is complete. Corollary 3.2 shows that the "next largest product", $\mathbb{R}^{\mathbb{R}}$ already has incomplete quotients, and it is remarkable that there are such quotients whose completion is compact.

Proposition 3.3. *Assume that the corestriction* $\exp'_G: \mathfrak{L}(G) \to G_a$ *of the exponential function of a compact connected abelian group is a quotient morphism. Then*

(i) G_a *has arbitrarily small open arcwise connected identity neighborhoods in the topology induced from that of* G.

(ii) G *is locally connected.*

(iii) $\widehat{G}$ *is* $\aleph_1$*-free.*

Proof. (i) A quotient morphism is open. But $\mathfrak{L}(G)$ is a locally convex topological vector space and thus has arbitrarily small arcwise connected neighborhoods of zero which are mapped onto open identity neighborhoods of G_a by $\exp'_G$.

(ii) Let W be an identity neighborhood of G. Then there is an identity neighborhood V such that $V^2 \subseteq W$. By (i), G_a has an open arcwise connected identity neighborhood U satisfying $U \subseteq V$. Then the closure $\overline{U}$ of U in G is contained in $\overline{V} subseteq VV \subseteq W$. There is an open set U_G in G such that $U = U_G \cap G_a$, and since G_a is dense in G we have $\overline{U} = \overline{U_G}$. Also, since U is arcwise connected, $\overline{U}$ is connected. is an identity neighborhood in G. Thus $\overline{U}$ is a connected identity neighborhood in G which is contained in W. Thus G is locally connected.

(iii) This follows from 2.1 and (ii) above. □

Thus the exponential function of a compact connected abelian group can be open onto its image only if the group is locally connected.

Proposition 3.4. *If the arc component* G_a *of the zero element of a compact connected abelian group* G *is locally arcwise connected, then the corestriction*

$$\exp'_G: \mathfrak{L}(G) \to G_a$$

of the exponential function is open.

Proof. On the group G there is a filterbasis $\mathcal{N}(G)$ converging to 1 and consisting of closed compact subgroups N such that G/N is a finite dimensional torus. By [3], p. 355, 356, Theorem 7.66(iv), there are arbitrarily small identity neighborhoods of G of the form $N \oplus V$ where $V = \exp_G U$ for an open n-cell neighborhood U in a finite dimensional vector subspace $\mathfrak{F}$ of $\mathfrak{L}(G)$ such that $(n, X) \mapsto n + \exp_G X: N \times U \to N \oplus V$ is a homeomorphism. Now $N_a \oplus V = (N \oplus V)_a$ is the arc component of 0 in $G_a \cap (N \oplus V)$. Since G_a is locally arcwise connected, arc componens of open sets of G_a are open in G_a, and thus $N_a \oplus V$ is open in G_a. Therefore G_a has arbitrarily small identity neighborhoods of the form $N_a \oplus V$. However, these are of the form $N_a \oplus V = \exp_N \mathfrak{L}(N) \oplus \exp_G U = \exp_G(\mathfrak{L}(N) \oplus U) = \exp'_G(\mathfrak{L}(N) \oplus U)$. Now we know that $G = \lim_{N \in \mathcal{N}(G)} G/N$ and just as in the proof of 3.1 we conclude that therefore $\mathfrak{L}(G) = \lim_{N \in \mathcal{N}(G)} \mathfrak{L}(G)/\mathfrak{L}(N)$ and that $\mathfrak{L}(G)$ has arbitrarily small neighborhoods of the form $\mathfrak{L}(N) \oplus U$. This proves that $\exp'_G$ is an open morphism. □

4. A Characterisation of Strong Local Connectivity

We have seen that the corestriction of the exponential function $\exp'_G\colon \mathfrak{L}(G) \to G_a$ of a compact connected abelian group G is open if G is strongly locally connected. We shall show in this section that the converse holds as well.

In order to form a topological intuition, let us briefly consider the case of a finite dimensional compact abelian group. A compact connected group G is n-dimensional for $n = 0, 1, \ldots$ iff and only if its character group $\widehat{G}$ has rank n, that is, $\dim_{\mathbb{Q}} \mathbb{Q} \otimes \widehat{G} = n$ (see e.g. [3], p. 650); this is the case iff $\dim \mathfrak{L}(G) = n$ (see [3], p. 382, Theorem 8.22). If $\exp'\colon \mathfrak{L}(G) \to G_a$ is open, then G_a is a quotient group of the locally compact group $\mathfrak{L}(G) \cong \mathbb{R}^n$ and thus is locally compact. Hence it is a closed subgroup of G, (see e.g. [3], p. 777); since it is also dense in G (see e.g. [3], p. 359, Theorem 7.71) we have $G_a = G$. Every compact quotient group of $\mathbb{R}^n$ is a torus (see e.g. [3], p. 625, Theorem A1.12). Thus we record:

Lemma 4.1. *If the exponential function is open onto its image in a finite-dimensional compact connected group G, then G it is a torus.* □

In order to work towards the general case, we let **1** denote the one element group. In the following numbers 4.2 and 4.3, commutativity plays no role and thus we use the multiplicative notation. All topological groups we consider are assumed to be Hausdorff.

Definition 4.2. A sequence of Hausdorff topological groups and continuous morphisms

$$\mathbf{1} \longrightarrow G_1 \xrightarrow{\ j\ } G_2 \xrightarrow{\ p\ } G_3 \longrightarrow \mathbf{1}$$

is said to be *topologically exact* if it is exact algebraically and if j is a topological embedding and p is a quotient map. □

The following lemma will be crucial.

Lemma 4.3. *Assume that the following is a commutative diagram of topological groups*

$$\begin{array}{ccccccccc}
 & & & & K_2 & \xrightarrow{\ p\ } & K_3 & & \\
 & & & & {\scriptstyle j_2}\big\downarrow & & \big\downarrow{\scriptstyle j_k} & & \\
\mathbf{1} & \longrightarrow & L_1 & \longrightarrow & L_2 & \xrightarrow{\ q\ } & L_3 & \longrightarrow & \mathbf{1} \\
 & & {\scriptstyle e_1}\big\downarrow & & {\scriptstyle e_2}\big\downarrow & & \big\downarrow{\scriptstyle e_3} & & \\
\mathbf{1} & \longrightarrow & G_1 & \longrightarrow & G_2 & \xrightarrow[\ r\ ]{} & G_3 & \longrightarrow & \mathbf{1}.
\end{array}$$

Hypotheses:

(a) *The horizontal sequences are topologically exact.*

(b) *j_2, and j_3 are the inclusions of the kernels of e_n.*

(c) *e_1 has a dense image.*

(d) *e_2 is open onto its image.*

Conclusions:

(i) *p has a dense image. If, in addition, K_3 is discrete, then p is surjective.*

(ii) *The morphism e_3 is open onto its image.*

Proof. (i): Clearly, if p has dense image, then the discreteness of its range implies its surjectivity. We shall show now that p has a dense image. By way of contradiction we suppose that there is an open subset U of $L_3\setminus\overline{p(K_2)}$ which meets K_3. From (d) we know that $e_2\big(q^{-1}(U)\big)$ is open in $e_2(L_2)$. Thus $e_2\big(q^{-1}(U)\big)\cap G_1$ is open in $e_2(L_2)\cap G_1$. Let $u\in U\cap K_3$. By (a) there is an $x\in L_2$ such that $q(x)=u$. Then $e_2(x)\in e_2\big(q^{-1}(U)\big)$, and $p\big(e_2(x)\big)=e_3\big(q(x)\big)=e_3(u)\in e_3(K_3)=\{\mathbf{1}\}$, that is, $e_2(x)\in L_1$. Thus $e_2\big(q^{-1}(U)\big)\cap L_1\neq\emptyset$. Then by (c) we have an $a\in L_1$ such that $e_2(a)=e_1(a)\in e_2\big(q^{-1}(U)\big)\cap L_1$. Thus there is an element $b\in L_2$ such that $q(b)\in U$ and $e_2(b)=e_2(a)\in L_1=\ker q$. Hence $ba^{-1}\in\ker e_2=K_2$ and $q(ba^{-1})=q(b)\in U$. Thus $U\cap q(K_2)\neq\emptyset$. This is a contradiction which proves the (i).

(ii): We have to show that the corestriction $e_3'\colon L_3\to e_3(L_3)$ is a quotient morphism, that is, if $X\subseteq L_3$ is such that $XK_3=X$ is closed in L_3, then $\overline{e_3(X)}\cap e_3(L_3)=e_3(X)$. For this last conclusion, in view of homogeneity, it suffices to show that $1\in\overline{e_3(X)}$ implies $1\in e_3(X)$. The subset $Y\overset{\rm def}{=}p^{-1}(X)$ of L_2 is closed and K_2-saturated, that is, it satisfies $YK_2=Y$, since $p(YK_2)=XK_3=X$. We may assume that $L_1\subseteq L_2$. The preimage $r^{-1}\big(e_3(X)\big)$ of $e_3(X)\subseteq G_2$ in G_1 is the saturated set $e_2(Y)G_1$. Since r is a quotient map, and $1\in\overline{e_3(X)}$, by (c) we conclude that $1\in\overline{e_2(Y)G_1}=\overline{e_2(Y)\overline{e_1(L_1)}}=\overline{e_2(Y)e_1(L_1)}=\overline{e_2(YL_1)}=\overline{e_2(Y)}$. By (d), the map e_2 is a quotient morphism onto its image, and $Y=YL_1$ is closed in L_2; thus $e_2(Y)$ is closed in $e_2(L_2)$, i.e. $\overline{e_2(Y)}\cap e_2(L_2)=e_2(Y)$. We conclude $1\in e_2(Y)$ and thus $1\in r\big(e_2(Y)\big)=e_3(X)$ which we had to show. ∎

We apply this to a compact connected abelian group G with a closed subgroup N. Then

$$0\to N\to G\to G/N\to 0$$

is a topologically exact sequence, as is

$$0\to\mathfrak{L}(N)\to\mathfrak{L}(G)\to\mathfrak{L}(G/N)\to 0$$

by [3], pp. 355, 356, Theorem 7.66(iii). Write $T\overset{\rm def}{=}G/N$ and let $f\colon G\to T$ be the quotient morphism. Then we have a commutative diagram of abelian topological groups with topologically exact rows.

$$(*)\qquad\begin{array}{ccccccccc}
 & & & & \mathfrak{K}(G) & \xrightarrow{\mathfrak{K}(f)} & \mathfrak{K}(T) & & \\
 & & & & {\scriptstyle j_2}\big\downarrow & & \big\downarrow{\scriptstyle j_k} & & \\
0 & \longrightarrow & \mathfrak{L}(N) & \longrightarrow & \mathfrak{L}(G) & \xrightarrow{\mathfrak{L}(f)} & \mathfrak{L}(T) & \longrightarrow & 0\\
 & & {\scriptstyle \exp_N}\big\downarrow & & {\scriptstyle \exp_G}\big\downarrow & & \big\downarrow{\scriptstyle \exp_T} & & \\
0 & \longrightarrow & N & \longrightarrow & G & \xrightarrow[f]{} & T & \longrightarrow & 0.
\end{array}$$

Proposition 4.4. *Let G be a connected compact abelian group such that $\exp'_G\colon \mathfrak{L}(G) \to G_a$ is open. Then G is strongly locally connected.*

Proof. In view of Definition 2.3 we have to show that $\widehat{G}$ is an S-group. Let P be a rank one pure subgroup; we must show that P splits as a direct summand.

Since $\widehat{G}$ is $\aleph_1$-free by Proposition 3.3, there is an isomorphism $\pi\colon P \to \mathbb{Z}$, and so $\widehat{P} \cong \mathbb{T}$. Set $N = P^\perp$, the anniliator of P in G. Since $\widehat{G}/P \cong \widehat{N}$ is torsion free, N is a closed connected subgroup such that $G/N \cong \widehat{P}$ is a one-torus; We shall write $f\colon G \to T \overset{\text{def}}{=} \widehat{P}$ for the quotient map and we shall identify $\widehat{T}$ with P by considering $\widehat{f}\colon \widehat{T} \to \widehat{G}$ as the inclusion map incl: $P \to \widehat{G}$.

Now we check the hypotheses of Lemma 4.3. The preceding comments show that (a) and (b) are satisfied, and (d) is our hypothesis. Hypothesis (c) concerns the exponential function $\exp_N\colon \mathfrak{L}(N) \to N$; since N is connected, $N_a = \operatorname{im}\exp_N$ is dense in N (see [3], p. 359, Theorem 7.71). Thus hypothesis 4.3(c) is satisfied as well. The exponential function $\exp_T\colon \mathfrak{L}(T) \to T$ of a one dimensional torus is equivalent to the quotient map $\mathbb{R} \to \mathbb{T}$ and therefore has a discrete kernel isomorphic to $\mathbb{Z}$. Thus Lemma 4.3(i) proves that $\mathfrak{K}(f)\colon \mathfrak{K}(G) \to \mathfrak{K}(t)$ is surjective. In view of [3], p. 355, Theorem 7.66(ii), we have a commutative diagram

$$\begin{array}{ccc}
\operatorname{Hom}(\widehat{G}, P) & \xrightarrow{\operatorname{Hom}(\mathrm{incl}, P)} & \operatorname{Hom}(P, P) \\
{\scriptstyle \operatorname{Hom}(\widehat{G},\pi)}\big\downarrow & & \big\downarrow{\scriptstyle \operatorname{Hom}(\widehat{T},\pi)} \\
\operatorname{Hom}(\widehat{G}, \mathbb{Z}) & \xrightarrow{\operatorname{Hom}(\widehat{f},\mathbb{Z})} & \operatorname{Hom}(\widehat{T}, \mathbb{Z}) \\
{\scriptstyle \cong}\big\downarrow & & \big\downarrow{\scriptstyle \cong} \\
\mathfrak{K}(G) & \xrightarrow[\mathfrak{K}(f)]{} & \mathfrak{K}(T),
\end{array}$$

where $\operatorname{Hom}(\mathrm{incl}, P)$ is the restriction $\alpha \mapsto \alpha|P : \operatorname{Hom}(\widehat{G}, P) \to \operatorname{Hom}(P, P)$. This map we now know to be surjective. Hence there is a morphism $p\colon \widehat{G} \to P$ such that $p|P$ is the identity morphism of P. Thus p is a homomorphic retraction and thus P splits. ☐

5. Summary: The Main Result. Generalisations

Propositions 3.1, 3.3, 3.4, 4.4 yield the following main result:

Theorem 5.1. *For a compact connected abelian group G and its zero arc-component G_a, the following conditions are equivalent:*

(i) *G is strongly locally connected.*
(ii) *The exponential function $\exp_G\colon \mathfrak{L}(G) \to G$ is open onto its image.*
(iii) *G_a is locally arcwise connected.*
(iv) *$\widehat{G}$ is an S-group, that is, every finite rank pure subgroup of $\widehat{G}$ is free and is a direct summand.* ☐

Corollary 5.2. *Let G be a metric compact connected abelian group. Then the following statements are equivalent:*
(i) *G is strongly locally connected.*
(ii) *The exponential function* $\exp_G: \mathfrak{L}(G) \to G$ *is open onto its image.*
(iii) *G_a is locally arcwise connected.*
(iv) *$\widehat{G}$ is free and countable.*
(v) *G is a torus.*
(vi) *G is arcwise connected.*
(vii) *G is locally connected.*

Proof. (vii)$\Rightarrow$(i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii)$\Leftrightarrow$
(iv$'$) *$\widehat{G}$ is a countable* S-*group.*
is clear from Theorem 5.1. and the fact that G is metric iff the weight of G is countable iff is countable (see [3], p. 772, Theorem A4.16, p. 361, Theorem 7.76. A countable abelian S-group os a countable $\aleph_1$-free group and is therefore free, that is, (iv$'$)$\Rightarrow$(iv). By duality, $\widehat{G}$ is free iff G is a torus. By [3], pp. 404, 405, Theorem 8.46(iii), Conditions (v), (vi), (vii) are equivalent for metric compact abelian groups. □

Corollary 5.2 shows that the concepts emerging in Theorem 5.1 attain their true significance in the case of nonmetric compact connected groups. By Example 2.4, the character group of the discrete group $\mathbb{Z}^{\mathbb{N}}$ is group satisfying the conditions of Theorem 5.1 which is not arcwise connected. By 5.2, this example is minimal if we accept the continuum hypothesis. There are locally connected connected compact abelian groups of weight $2^{\aleph_0}$ which are not strongly locally connected.

Even though we have dealt with compact connected abelian groups, in view of the known structure of locally compact groups, the results of this note generalize to arbitrary locally compact groups. A topological group G is locally connected if and only if its identity component G_0 is open in G and is locally connected. The inclusion morphism $G_0 \to G$ induces a homeomorphism of pointed topological spaces $\mathrm{Hom}(\mathbb{R}, G_0) \to \mathrm{Hom}(\mathbb{R}, G)$. If G is, say, a locally compact group, more generally, if G is any topological group *having a Lie algebra* (see [4], Chapter 2), then this homeomorphism is in fact an isomorphism of topological Lie-algebras $\mathfrak{L}(G_0) \to \mathfrak{L}(G)$. Thus, in particular, for a locally compact abelian group G, the exponential function $\exp_G: \mathfrak{L}(G) \to G$ is open onto its image if and only the exponential function $\exp_{G_0}: \mathfrak{L}(G_0) \to G_0$ is open onto its image.

A connected locally compact abelian group G is isomorphic to $\mathbb{R}^n \times K$ with a unique compact connected compact subgroup K of G by the Vector Group Splitting Theorem (see [3], p. 348, Theorem 7.57(iii)), and $\mathfrak{L}(G) \cong \mathbb{R}^n \times \mathfrak{L}(K)$ such that the exponential function may be written in the form $\exp_G: \mathbb{R}^n \times \mathfrak{L}(K) \to \mathbb{R}^n \times K$, $\exp_G(v, X) = (v, \exp_K X)$. Hence $\exp_G$ is open onto its image if and only if $\exp_K$ is open onto its image.

We finally note that by Pontryagin Duality, the largest compact connected subgroup K of a locally compact abelian group $\widehat{G}$ is the annihilator subgroup of the smallest open subgroup M of the character group $\widehat{G}$ such that $\widehat{G}/M$ is torsion free.

With these observations, the Main Theorem 5.1 yields at once

Theorem 5.3. *For a locally compact abelian group G, the following statements are equivalent.*

(O) $\exp'_G\colon \mathfrak{L}(G) \to G_a$ *is open.*

(LC) G_0 *is strongly locally connected.*

(LC_a) G_a *is locally arcwise connected.*

(S) *The character group $\widehat{K}$ of the unique largest compact connected subgroup K of G is an* S*-group.*

(S′) *If M denotes the smallest open subgroup of $\widehat{G}$ such that $\widehat{G}/M$ is torsion free, then $\widehat{G}/M$ is an S-group.* □

We realize that this may be extended appropriately to locally compact not necessarily abelian groups; however, the exponential function then ceases to be a morphism and condition (O) above has to be replaced by
(LO) $\exp_G\colon \mathfrak{L}(G) \to G$ is open at 1. One needs to know the Iwasawa Splitting Theorem saying that a locally compact connected group G is locally isomorphic to $L \times K$ where L is a connected finite dimensional Lie group and K a compact normal subgroup of G. Then one needs to know structural results on the compact connected group K_0 which are available through [3], Chapter 9.

6. References

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Authors' addresses:

Karl H. Hofmann
Fachbereich Mathematik
TU Darmstadt
Schlossgartenstr. 7
D-64289 Darmstadt
Germany
hofmann@mathematik.
tu-darmstadt.de

Sidney A. Morris
School of Information Technology
and Mathematical Sciences
University of Ballarat
P.O. Box 663, Ballarat
Ballarat, Victoria 3353,
Australia
s.morris@ballarat.edu.au

Detlev Poguntke
Fakultät für Mathematik
Universität Bielefeld
Postfach 100131
D-33501 Bielefeld
Germany
poguntke@mathematik.
uni-bielefeld.de