Algebras of approximation sequences: Spectral and pseudospectral approximation of band-dominated operators

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Abstract

This paper is devoted to relations between the spectrum (or certain kinds of a generalized spectrum) of a band-dominated operator A and of the spectra of its approximations $A_n = P_n A P_n$, obtained by compressing A onto the ranges of the orthogonal projections P_n. Particular attention is paid to the asymptotic behaviour of the spectra (or its generalizations) of the operators A_n. These results will appear as special cases of some general theorems on spectral approximation.

1 Introduction

Given a Hilbert space H and a positive integer N, let E_N stand for the linear space of all sequences $f = (f(x))_{x \in \mathbb{Z}^N}$ with values in H such that

$$\|f\|^2 := \sum_{x \in \mathbb{Z}^N} \|f(x)\|_H^2 < \infty.$$

Clearly, E_N is a Hilbert space with respect to the inner product

$$\langle f, g \rangle := \sum_{x \in \mathbb{Z}^N} \langle f(x), g(x) \rangle_H.$$

For $k \in \mathbb{Z}^N$, we let V_k refer to the shift operator on E_N,

$$V_k : f \mapsto g \quad \text{where} \quad g(x) := f(x-k),$$

and for every bounded function $a : \mathbb{Z}^N \to L(H)$, we consider the operator M_a of multiplication by a,

$$M_a : E_N \to E_N, \quad (M_a f)(x) := a(x) f(x).$$

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A *band-dominated operator* on E_N is a norm limit of a sequence of *band operators* on E_N, i.e. of operators which are a finite sum of operators $M_a V_k$ where a and k are as above.

For $n \in \mathbb{N}$, let P_n refer to the projection operator on E_N which maps the sequence f to the sequence g where $g(x) = f(x)$ if $|x| \leq n$ and $g(x) = 0$ if $|x| > n$. The operators $A_n := P_n A P_n$ are called the *finite sections* of the band-dominated operator A. The sequence (A_n) is called *stable* if there is an n_0 such that the operators $A_n : \operatorname{Im} P_n \to \operatorname{Im} P_n$ are invertible for all $n \geq n_0$ and if the norms of their inverses are uniformly bounded. A stability criterion for the finite section method for a band-dominated operator A has been derived in [11].

The goal of the present paper is to examine the asymptotic behaviour of the spectra (and of some of their generalizations) of the operators $P_n A P_n$. These results will be obtained as special cases of some general theorems on spectral approximation. A convenient way to formulate these general theorems uses the language of C^*-algebras. Thus, the application of these results to a concrete approximation sequence such as $(P_n A P_n)$ requires not only a precise knowledge on the stability properties of the sequence $(P_n A P_n)$ itself, but for *every* sequence in a C^*-algebra of sequences which contains $(P_n A P_n)$.

The paper is organized as follows. We will start with recalling the stability results for the finite section method of band-dominated operators from [11] and with extending these results to sequences in a C^*-algebra $\mathcal{B}$ generated by finite sections sequences. The obtained generalization will be as follows: There is a family $\{W_t\}$ of *-homomorphisms from $\mathcal{B}$ into $L(E)$ such that a sequence $\mathbf{A} = (A_n)$ in $\mathcal{B}$ is stable if and only if all operators $W_t(\mathbf{A})$ are invertible *and if the norms of their inverses are uniformly bounded.*

The latter is an obvious difference to some of the C^*-algebraic stability results which were previously derived for several classes of approximation sequences (see [6, 7] for an overview). These former results are essentially of the following form: With every sequence $\mathbf{A}$ in a certain algebra of sequences, there is associated a family $\{W_t(\mathbf{A})\}$ of operators such that the sequence $\mathbf{A}$ is stable if and only if all operators $W_t(\mathbf{A})$ are invertible. Thus, the *uniform* boundedness of the inverses of the operators $W_t(\mathbf{A})$ is *not* required in these previous examples. In [7] we summarized some general results on the approximation of spectra which hold if the sequences satisfy the more restrictive form of the stability condition just mentioned. Thus, we will have to generalize the spectral approximation results from [7] to a context which allows us to include the finite section method for band-dominated operators. This will be done in Sections 3 – 5. These results in combination with the stability result will yield the wanted assertions on spectral approximation for finite sections of band-dominated operators almost at once.

On this occasion, we will also extend the results from [7] into another direction. The natural discretization parameters for the approximation methods considered in [7] are the positive integers. Thus, the discretization of an operator A by these methods leads to a *sequence* $(A_n)_{n \in \mathbb{N}}$ of discretized operators. But for operators which act on $L^2(\mathbb{R}^+)$, for instance, the natural discretization parameters might be the non-negative reals. Moreover, the discretization of an integral operator on $L^2(\mathbb{R}^+)$ is usually done in two steps: first one compresses the operator onto the space $L^2([0, t])$ with some $t > 0$, and then this compression is discretized by a standard quadrature or collocation method which finally leads to an finite linear system. Thus, the natural discretization parameters are the points in $\mathbb{R}^+ \times \mathbb{N}$ in this case. Similar situations appear when standard discretization procedures are combined with cutting off techniques (see Chapter 5 in [8] or Chapter 4 in [7] for first impressions and for further references), which also leads to discretization parameters in $\mathbb{N} \times \mathbb{N}$ or in $\mathbb{R}^+ \times \mathbb{N}$. Consequently, we will examine the spectral approximation problem for *nets* of approximation operators rather than for sequences, i.e. the points of a certain directed set will serve us as discretization parameters.

2 Algebras of finite sections sequences

The mentioned stability results for the finite sections method for band-dominated operators will be formulated in the language of limit operators. Let $A \in L(E_N)$, and let h be a sequence of points in $\mathbb{Z}^N$ which tends to infinity. The operator A_h is called the limit operator of A with respect to h if, for every $k \in \mathbb{N}$,

$$\|(V_{-h(n)}AV_{h(n)} - A_h)P_k\| \to 0 \quad \text{and} \quad \|P_k(V_{-h(n)}AV_{h(n)} - A_h)\| \to 0$$

as $n \to \infty$. We say that A is *rich* if every sequence g, which tends to infinity, possesses a subsequence h for which the limit operator A_h exists. If H is a finite-dimensional Hilbert space, then every band-dominated operator on $E_N = l^2(\mathbb{Z}^N, H)$ is rich. The class of all rich band-dominated operators on E_N will be denoted by $\mathcal{A}_N^{rich}$. This class is a closed and symmetric subalgebra of $L(E_N)$. For this and further results on limit operators, which are cited here without proof, we refer to [9, 10, 11].

In what follows, we will exclusively deal with the finite section method for *rich* band-dominated operators. It will be convenient to consider, instead of the one-sided sequence $(P_nAP_n)_{n\in\mathbb{N}}$ of operators acting on $\operatorname{Im} P_n$, the two-sided sequence $(P_nAP_n + Q_n)_{n\in\mathbb{Z}}$ of operators acting on E_N, where $P_n := 0$ if $n < 0$ and $Q_n := I - P_n$ for all n. Clearly, both sequences are simultaneously stable or not.

With every bounded sequence $\mathbf{A} = (A_n)_{n\in\mathbb{Z}}$ of operators on E_N, we associate an operator $\operatorname{Op}(\mathbf{A})$ on E_{N+1} as follows. For, we write every vector $x \in \mathbb{Z}^{N+1}$ as $x = (x', x_{N+1}) \in \mathbb{Z}^N \times \mathbb{Z}$ and, for every $m \in \mathbb{Z}$, we define the operator of restriction

$$R_m : E_{N+1} \to E_N, \quad (R_mf)(x') := f(x', m)$$

and the operator of embedding

$$S_m : E_N \to E_{N+1}, \quad (S_m f)(x', n) := \begin{cases} f(x') & \text{if } n = m \\ 0 & \text{if } n \neq m. \end{cases}$$

Finally, we set

$$(\mathrm{Op}\,(\mathbf{A})f)(x', m) := (A_m R_m f)(x').$$

It turns out that, if A is a rich band dominated operator on E_N, then $\mathrm{Op}\,(\mathbf{A})$ with $\mathbf{A} := (P_n A P_n + Q_n)_{n\in\mathbb{Z}}$ is a rich band-dominated operator on E_{N+1}. We let $\sigma_{stab}(\mathbf{A})$ stand for the set of all operators $R_m B S_m$ where $m \in \mathbb{Z}$ and where B is a limit operator of $\mathrm{Op}\,(\mathbf{A})$. In [11], the following is proved.

Theorem 2.1 *Let $A \in \mathcal{A}_N^{rich}$. Then the sequence $\mathbf{A} := (P_n A P_n + Q_n)$ is stable if and only if all operators in $\sigma_{stab}(\mathbf{A})$ are invertible and if the norms of their inverses are uniformly bounded.*

For $A \in \mathcal{A}_N^{rich}$, let $\mathcal{H}_A$ denote the set of all sequences h for which the limit operator A_h exists. If $h \in \mathcal{H}_A$, then the limit operators B_h exist for all operators B which belong to the smallest closed subalgebra $C^*(A)$ of $L(E_N)$ containing the operators A, A^* and the identity operator I. The algebra $C^*(A)$ is a unital C^*-algebra, and the mapping $B \mapsto B_h$ is a unital *-homomorphism from $C^*(A)$ into $L(E)$. We claim that

$$\sigma_{op}(B) = \{B_h : h \in \mathcal{H}_A\} \quad \text{for every operator } B \in C^*(A) \tag{1}$$

where $\sigma_{op}(B)$ refers to the set of all limit operators of B. Indeed, let $g \in \mathcal{H}$ be a sequence for which the limit operator B_g exists. Since A belongs to $\mathcal{A}_N^{rich}$, there is a subsequence h of g which lies in $\mathcal{H}_A$. Clearly, B_h exists and is equal to B_g.

With this observation, it is easy to check that the proof of Theorem 2.1 given in [11] carries over to the algebraic setting. For, we denote the algebra of all bounded sequences $(A_n)_{n\in\mathbb{Z}}$, provided with element-wise operations and with the supremum norm, by $\mathcal{F}$, and the smallest closed subalgebra of $\mathcal{F}$, which contains all sequences $(P_n A P_n + Q_n)$ with $A \in \mathcal{A}_N^{rich}$, by $\mathcal{B}_N^{rich}$.

Theorem 2.2 *A sequence $\mathbf{A} \in \mathcal{B}_N^{rich}$ is stable if and only if all operators in $\sigma_{stab}(\mathbf{A})$ are invertible and if the norms of their inverses are uniformly bounded.*

For an equivalent formulation which fits to our purposes, we agree upon the following definition.

Definition 2.3 *Let $\mathcal{B}$ be a C^*-algebra with identity element e and, for every element t of a set T, let $\mathcal{B}_t$ be a C^*-algebra with identity element e_t, and let W_t be a *-homomorphism from $\mathcal{B}$ into $\mathcal{B}_t$ with $W_t(e) = e_t$. We call $\{W_t\}_{t\in T}$ a* weakly sufficient family of homomorphisms for $\mathcal{B}$ *if the following assertions are equivalent for every $b \in \mathcal{B}$:*

(a) b is invertible in $\mathcal{B}$.
(b) $W_t(b)$ is invertible in $\mathcal{B}_t$ for every $t \in T$, and

$$\sup_{t \in T} \|(W_t(b))^{-1}\| < \infty.$$

We call $\{W_t\}_{t\in T}$ a sufficient family of homomorphisms *for $\mathcal{B}$ if, for every $b \in \mathcal{B}$, the assertion (a) is equivalent to*
(c) $W_t(b)$ is invertible in $\mathcal{B}_t$ for every $t \in T$.

Let $\mathcal{G}$ stand for the set of all sequences $(A_n) \in \mathcal{F}$ with $\lim \|A_n\| = 0$. This set forms a closed ideal of $\mathcal{F}$, and a sequence $\mathbf{A} \in \mathcal{F}$ is stable if and only if its coset $\mathbf{A} + \mathcal{G}$ is invertible in the quotient algebra $\mathcal{F}/\mathcal{G}$. Further, given a sequence $\mathbf{A} = (A_n) \in \mathcal{B}_N^{rich}$, let $C^*(\mathbf{A})$ denote the smallest closed subalgebra of $\mathcal{F}$ which contains the sequences $\mathbf{A}$, $\mathbf{A}^* := (A_n^*)$ and $\mathbf{I} := (I)$.

If $h \in \mathcal{H}_{\text{Op}(\mathbf{A})}$ and $m \in \mathbb{Z}$, then the limit operator $(\text{Op}(\mathbf{B}))_h$ exists for every sequence $\mathbf{B} \in C^*(\mathbf{A})$, and every limit operator of $\text{Op}(\mathbf{B})$ arises in this way. Thus, $\mathbf{B} \mapsto R_m(\text{Op}(\mathbf{B}))_h S_m$ is a *-homomorphism from $C^*(\mathbf{A})$ into $L(E)$. Since the ideal $C^*(\mathbf{A}) \cap \mathcal{G}$ lies in the kernel of that homomorphism, the mapping

$$W_{h,m} : C^*(\mathbf{A})/(C^*(\mathbf{A}) \cap \mathcal{G}) \to L(E), \quad \mathbf{B} + C^*(\mathbf{A}) \cap \mathcal{G} \mapsto R_m(\text{Op}(\mathbf{B}))_h S_m$$

is correctly defined for every sequence $h \in \mathcal{H}_{\text{Op}(\mathbf{A})}$ and every $m \in \mathbb{Z}$. The following is an immediate consequence of Theorem 2.2.

Theorem 2.4 *Let $\mathbf{A} \in \mathcal{B}_N^{rich}$. Then the set $\{W_{h,m}\}$ with $h \in \mathcal{H}_{\text{Op}(\mathbf{A})}$ and with m running through $\mathbb{Z}$ forms a weakly sufficient family of homomorphisms for the algebra $C^*(\mathbf{A})/(C^*(\mathbf{A}) \cap \mathcal{G})$.*

3 Weakly sufficient families and spectra

Now we turn over to a more general context. Throughout this section, let $\mathcal{B}$ be a C^*-algebra with identity element e and let T be a non-empty set. Further suppose that, for every $t \in T$, we are given a C^*-algebra $\mathcal{B}_t$ with identity e_t and a *-homomorphism W_t from $\mathcal{B}$ into $\mathcal{B}_t$ with $W_t(e) = e_t$.

It is our goal to realize how certain spectral quantities of $b \in \mathcal{B}$ can be expressed by the corresponding spectral quantities of the $W_t(b)$, provided that $\{W_t\}$ forms a weakly sufficient (a sufficient) family of homomorphisms. It will be convenient to use the following notation. Given a family $(M_t)_{t\in T}$ of subsets of $\mathbb{C}$, we set

$$\sup_{t\in T} M_t := \text{clos}\,(\cup_{t\in T} M_t),$$

and we call $\sup_{t\in T} M_t$ the *maximum* of the family (M_t) (and write $\max_{t\in T} M_t$ in that case) if $\cup_{t\in T} M_t$ is closed.

Norms. We start with a relation between the norms of b and of $W_t(b)$. (In the very weak sense that the spectrum of b is contained in the closed ball with radius $\|b\|$, the norms might also be considered as some kind of a spectral approximation.)

Theorem 3.1 *Let $\mathcal{B}$, $\mathcal{B}_t$ and W_t be as above. If $\{W_t\}_{t\in T}$ is a weakly sufficient family for $\mathcal{B}$, then*

$$\|b\| = \sup_{t\in T} \|W_t(b)\|_t \quad \textit{for every } b \in \mathcal{B}. \tag{2}$$

If the family $\{W_t\}_{t\in T}$ is sufficient, then the supremum in (2) is a maximum.

Proof. *-Homomorphisms are contractive. Hence,

$$\|b\| \geq \sup_{t\in T} \|W_t(b)\|_t \quad \text{for every } b \in \mathcal{B}.$$

Suppose there is a $b \in \mathcal{B}$ such that $\|b\| > \sup_{t\in T} \|W_t(b)\|_t$. Then, by the C^*-axiom,

$$\|b^*b\| > \sup_{t\in T} \|W_t(b^*b)\|_t.$$

Set $d := \|b^*b\| - \sup_{t\in T} \|W_t(b^*b)\|_t$. Then $\|W_t(b^*b)\| \leq \|b^*b\| - d < \|b^*b\|$ for every $t \in T$. Hence, all operators $W_t(b^*b - \|b^*b\|e) = W_t(b^*b) - \|b^*b\|e_t$ are invertible, and

$$\begin{aligned} \|(W_t(b^*b - \|b^*b\|e))^{-1}\| &= \sup_{x\in\sigma(W_t(b^*b))} |(x - \|b^*b\|)^{-1}| \\ \leq \sup_{x\in[0,\rho(W_t(b^*b))]} &|(x - \|b^*b\|)^{-1}| = (\|b^*b\| - \rho(W_t(b^*b)))^{-1} < 1/d \end{aligned}$$

where ρ denotes the spectral radius. Thus,

$$\sup_{t\in T} \|(W_t(b^*b - \|b^*b\|e))^{-1}\| \leq 1/d.$$

Since $\{W_t\}$ is a weakly sufficient family by assumption, the latter estimate implies the invertibility of $b^*b - \|b^*b\|e$, which is impossible. This shows (2). The result for sufficient families if Theorem 5.39 in [7]. ■

Spectra. As usual, we let

$$\sigma(b) = \sigma_{\mathcal{B}}(b) := \{\lambda \in \mathbb{C} : b - \lambda e \text{ is not invertible}\}.$$

Theorem 3.2 *Let $\mathcal{B}$, $\mathcal{B}_t$ and W_t be as above. If $\{W_t\}_{t\in T}$ is a weakly sufficient family for $\mathcal{B}$, then*

$$\sigma_{\mathcal{B}}(b) = \sup_{t\in T} \sigma_{\mathcal{B}_t}(W_t(b)) \tag{3}$$

for all normal elements b of $\mathcal{B}$. If the family $\{W_t\}_{t\in T}$ is sufficient, then the supremum in (3) is a maximum, and the assertion holds for every $b \in \mathcal{B}$.

Proof. The inclusion $\supseteq$ in (3) is trivial and holds also in the context of general Banach algebras. It remains to show that, for every normal element $b \in \mathcal{B}$,

$$\lambda \notin \sup_{t\in T} \sigma_{\mathcal{B}_t}(W_t(b)) \quad \Rightarrow \quad \lambda \notin \sigma_{\mathcal{B}}(b).$$

Without loss, let $\lambda = 0$. Then, since $\sup_{t\in T} \sigma_{\mathcal{B}_t}(W_t(b))$ is compact, there is a closed disk with center 0 and with positive radius r which has no points with $\sup_{t\in T} \sigma_{\mathcal{B}_t}(W_t(b))$ in common. Thus, $W_t(b)$ is invertible for every $t \in T$, and since $(W_t(b))^{-1}$ is normal, we get

$$\begin{aligned} \|(W_t(b))^{-1}\| &= \rho((W_t(b))^{-1}) \\ &= \sup\{\lambda \in \mathbb{C} : \lambda \in \sigma((W_t(b))^{-1})\} \\ &= \inf\{\lambda \in \mathbb{C} : \lambda \in \sigma(W_t(b))\}^{-1} < 1/r. \end{aligned}$$

The weak sufficiency of $\{W_t\}$ implies the invertibility of b. The assertion for sufficient families follows immediately from the definitions. ■

Pseudospectra. Let $\varepsilon > 0$. The ε-*pseudospectrum* of $b \in \mathcal{B}$ is the set

$$\sigma^\varepsilon(b) := \{\lambda \in \mathbb{C} : b - \lambda e \text{ is not invertible or } \|(b - \lambda e)^{-1}\| \geq 1/\varepsilon\}.$$

Pseudospectra are non-empty and compact, and for the pseudospectral radius of an element $b \in \mathcal{B}$ one has

$$\max\{|\lambda| : \lambda \in \sigma^\varepsilon(b)\} \leq \|b\| + \varepsilon. \tag{4}$$

Basic properties of pseudospectra can be found in [1, 3, 7, 12, 14, 15]. The following result is based on arguments from [1].

Theorem 3.3 *Let $\mathcal{B}$, $\mathcal{B}_t$ and W_t be as above, and let $\varepsilon > 0$. If $\{W_t\}_{t\in T}$ is a weakly sufficient family for $\mathcal{B}$, then*

$$\sigma^\varepsilon(b) = \sup_{t\in T} \sigma^\varepsilon(W_t(b)) \quad \textit{for every } b \in \mathcal{B}. \tag{5}$$

If the family $\{W_t\}_{t\in T}$ is sufficient, then the supremum in (5) *is a maximum.*

Again, one inclusion holds in a more general context. We formulate it separately.

Lemma 3.4 *Let $\mathcal{B}$ and $\mathcal{C}$ be unital Banach algebras and $W : \mathcal{B} \to \mathcal{C}$ be a unital and contractive homomorphism. Then*

$$\sigma^\varepsilon(W(b)) \subseteq \sigma^\varepsilon(b) \quad \textit{for every } b \in \mathcal{B}.$$

Proof. Let $\lambda \in \sigma^\varepsilon(W(b))$. If $W(b) - \lambda e = W(b - \lambda e)$ is not invertible, then $b - \lambda e$ is not invertible. Hence, $\lambda \in \sigma^\varepsilon(b)$ in this case. Let now $W(b - \lambda e)$ be invertible and $\|(W(b - \lambda e))^{-1}\| \geq 1/\varepsilon$. If $b - \lambda e$ is not invertible, then we have $\lambda \in \sigma^\varepsilon(b)$ again. If $b - \lambda e$ is invertible, then $(W(b - \lambda e))^{-1} = W((b - \lambda e)^{-1})$, whence $\|W((b - \lambda e)^{-1})\| \geq 1/\varepsilon$. Since W is a contraction, this shows that $\|(b - \lambda e)^{-1}\| \geq 1/\varepsilon$, i.e. $\lambda \in \sigma^\varepsilon(b)$. ■

In the proof of Theorem 3.3, we will employ the following result by Daniluk which states that the maximum principle (which, in general, does not hold for operator-valued analytic functions) holds for resolvent functions.

Theorem 3.5 *Let $\mathcal{B}$ be a C^*-algebra with identity e, and let $a \in \mathcal{B}$ be such that $a - ze$ is invertible for all z in some open subset U of the complex plane. If $\|(a - ze)^{-1}\| \leq C$ for all $z \in U$, then $\|(a - ze)^{-1}\| < C$ for all $z \in U$.*

A proof is in [7], Theorem 3.32.

Proof of Theorem 3.3. From the preceding lemma we conclude that

$$\sigma^\varepsilon(W_t(b)) \subseteq \sigma^\varepsilon(b) \quad \text{for every } b \in \mathcal{B} \text{ and } t \in T.$$

Since pseudospectra are closed, this implies

$$\sup_{t \in T} \sigma^\varepsilon(W_t(b)) \subseteq \sigma^\varepsilon(b) \quad \text{for every } b \in \mathcal{B}$$

for every family $\{W_t\}$ of *-homomorphisms. For the reverse inclusion, let $\{W_t\}$ be a weakly sufficient family of *-homomorphisms, and let $\lambda \in \sigma^\varepsilon(b)$. If there is a $t \in T$ such that $\lambda \in \sigma^\varepsilon(W_t(b))$, then nothing is to prove. So let us assume that all elements $W_t(b - \lambda e)$ are invertible and that $\|(W_t(b - \lambda e))^{-1}\| < 1/\varepsilon$. Then

$$\sup_{t \in T} \|(W_t(b - \lambda e))^{-1}\| \leq 1/\varepsilon.$$

Since $\{W_t\}$ is a weakly sufficient family, the element $b - \lambda e$ is invertible, and from Theorem 3.1 we conclude that $\|(b - \lambda e)^{-1}\| \leq 1/\varepsilon$. Since $\lambda \in \sigma^\varepsilon(b)$ by hypothesis, this shows that $\|(b - \lambda e)^{-1}\| = 1/\varepsilon$.

In every open neighborhood U of λ, there is a $\tilde{\lambda}$ such that $\|(b - \tilde{\lambda} e)^{-1}\| > 1/\varepsilon$. Indeed, otherwise we would have $\|(b - \tilde{\lambda} e)^{-1}\| \leq 1/\varepsilon$ for all $\tilde{\lambda} \in U$ whence, via Theorem 3.5, $\|(b - \tilde{\lambda} e)^{-1}\| < 1/\varepsilon$ for all $\tilde{\lambda} \in U$ including $\tilde{\lambda} = \lambda$.

Thus, if $k \in \mathbb{N}$ is sufficiently large, then there are $\lambda_k \in \mathbb{C}$ such that

$$|\lambda - \lambda_k| < 1/k \quad \text{and} \quad \|(b - \lambda_k e)^{-1}\| \geq \frac{1}{\varepsilon - 2/k}.$$

Further, again by Theorem 3.1, there are $t_k \in T$ such that

$$\|W_{t_k}(b - \lambda_k e))^{-1}\| = \|(W_{t_k}(b) - \lambda_k e_{t_k})^{-1}\| \geq \frac{1}{\varepsilon - 1/k}.$$

Since $\frac{1}{\varepsilon - 1/k} > \frac{1}{\varepsilon}$, we have $\lambda_k \in \sigma^\varepsilon(W_{t_k}(b))$, and since $\lambda_k \to \lambda$ as $k \to \infty$, we get $\lambda \in \sup_{t \in T} \sigma^\varepsilon(W_t(b))$. Thus, (5) is verified. The proof for sufficient families is in [7], Corollary 5.40. ■

Numerical ranges. Let $\mathcal{B}$ be a Banach algebra with identity e and $S(\mathcal{B})$ its *state space*, i.e. the set of all $f \in \mathcal{B}^*$ with $f(e) = 1$ and $\|f\| = 1$. The *numerical range* of $b \in \mathcal{B}$ is the set

$$N(b) := \{f(b) : f \in S(\mathcal{B})\}.$$

Numerical ranges are non-empty, compact and convex subsets of $\mathbb{C}$. For a bounded linear operator A on a Hilbert space H, one also considers its *spatial numerical range*

$$SN(A) := \{\langle Ax, x\rangle : x \in H, \|x\| = 1\}.$$

Let $A \in L(H)$. Then the spatial numerical range $SN_H(A)$ (where A is considered as a bounded linear operator on H) and the numerical range $N_{L(H)}(A)$ (where A is considered as an element of the C^*-algebra $L(H)$) are related by

$$N_{L(H)}(A) = \operatorname{clos} SN_H(A).$$

These and further properties of numerical ranges can be found in [4, 5, 7]. The proof of the following result on based on arguments from [13].

Theorem 3.6 *Let $\mathcal{B}$, $\mathcal{B}_t$ and W_t be as above. If $\{W_t\}_{t\in T}$ is a weakly sufficient family for $\mathcal{B}$, then*

$$N(b) = \operatorname{conv} \sup_{t\in T} N(W_t(b)) \quad \textit{for every } b \in \mathcal{B}. \tag{6}$$

One of the inclusions in (6) holds in the more general context of Banach algebras.

Lemma 3.7 *Let $\mathcal{B}$ and $\mathcal{C}$ be unital Banach algebras and $W : \mathcal{B} \to \mathcal{C}$ be a unital and contractive homomorphism. Then*

$$N(W(b)) \subseteq N(b) \quad \textit{for every } b \in \mathcal{B}.$$

Proof. Let $\lambda \in N(W(b))$, and let f be a state of $\mathcal{C}$ with $f(W(b)) = \lambda$. Since W is unital and contractive, one has $(f \circ W)(e) = 1$ and $\|f \circ W\| \le 1$, whence $\|f \circ W\| = 1$. Thus, $f \circ W$ is a state of $\mathcal{B}$, which implies that $\lambda \in N(b)$. ■

Proof of Theorem 3.6. From Lemma 3.7 we infer that

$$\cup_{t\in T} N(W_t(b)) \subseteq N(b).$$

Since $N(b)$ is a closed and convex set, this implies the inclusion $\supseteq$ in (6).

For the reverse inclusion, we think of each $\mathcal{B}_t$ as a C^*-algebra of linear bounded operators on a Hilbert space H_t (which is possible by the GNS-construction). Let $H := \oplus H_t$ refer to the orthogonal sum of the Hilbert spaces H_t, $t \in T$, and write W for the mapping from $\mathcal{B}$ into $L(H)$ which associates with every $b \in \mathcal{B}$ the operator

$$(x_t)_{t\in T} \mapsto (W_t(b)x_t)_{t\in T}.$$

This mapping is an isometry from $\mathcal{B}$ onto the C^*-subalgebra $W(\mathcal{B})$ of $L(H)$. Thus, $N_{\mathcal{B}}(b) = N_{W(\mathcal{B})}(W(b))$. It is further a simple consequence of the Hahn-Banach Theorem that $N_{W(\mathcal{B})}(W(b)) = N_{L(H)}(W(b))$, which implies that

$$N_{\mathcal{B}}(b) = N_{W(\mathcal{B})}(W(b)) = N_{L(H)}(W(b)) = \operatorname{clos} SN_H(W(b)).$$

Thus, given $\lambda \in N_{\mathcal{B}}(b)$ and $\varepsilon > 0$, there is a vector $(x_t)_{t\in T} \in H$ with norm 1 such that

$$|\lambda - \langle (W_t(b)x_t), (x_t)\rangle_H| = \left| \lambda - \sum_{t\in T} \langle W_t(b)x_t, x_t\rangle_{H_t} \right| < \varepsilon.$$

Let $\mathbb{M}$ denote the (at most countable) set of all $t \in T$ with $x_t \neq 0$ and, for $t \in \mathbb{M}$, set $y_t := x_t/\|x_t\|$. Then we have

$$\left| \lambda - \sum_{t\in\mathbb{M}} \langle W_t(b)x_t, x_t\rangle_{H_t} \right| = \left| \lambda - \sum_{t\in\mathbb{M}} \|x_k\|^2 \langle W_t(b)y_t, y_t\rangle_{H_t} \right| < \varepsilon.$$

Since $\|x_t\|^2 \geq 0$ and $\sum_{t\in\mathbb{M}} \|x_t\|^2 = \|(x_t)\|_H^2 = 1$, this shows that λ can be approximated by convex linear combinations of points $\langle W_t(b)y_t, y_t\rangle \in \cup_{t\in T} SN_{H_t}(W_t(b))$ as closely as desired. Hence,

$$\lambda \in \operatorname{clos}\operatorname{conv} \cup_{t\in T} SN_{H_t}(W_t(b)) \subseteq \operatorname{clos}\operatorname{conv} \cup_{t\in T} N_{L(H_t)}(W_t(b)),$$

which gives

$$\lambda \in \operatorname{clos}\operatorname{conv} \cup_{t\in T} N_{W_t(\mathcal{B})}(W_t(b)). \tag{7}$$

Since $\operatorname{clos}\operatorname{conv} M = \operatorname{conv}\operatorname{clos} M$ for every bounded subset M of the complex plane, (7) is just the assertion. ∎

4 Asymptotic behaviour of norms and spectra

Beginning with this section, we let $T = (T, \succ)$ be a directed set, i.e. $\succ$ is a partial order on T, and for each pair $s, t \in T$ there is a $u \in T$ such that $u \succ s$ and $u \succ t$. Further we assume that, for every $t \in T$, we are given a C^*-algebra $\mathcal{B}_t$ with identity e_t. By $\mathcal{F}$ we denote the set of all bounded functions a on T which take at $t \in T$ a value $a_t \in \mathcal{B}_t$. This set becomes a C^*-algebra with identity when provided with pointwise operations and with the supremum norm. The set $\mathcal{G}$ of all nets $(g_t)_{t\in T} \in \mathcal{F}$ with $\lim_{t\in T} \|g_t\| = 0$ forms a closed ideal of $\mathcal{F}$.

Let $a = (a_t)_{t\in T} \in \mathcal{F}$. The following results relate the asymptotic behaviour of the spectra (some kinds of generalized spectra) of the elements a_t with the spectrum (the generalized spectrum) of the coset of the net (a_t) modulo the ideal $\mathcal{G}$ and, hence, in terms of the stability spectrum of the sequence (A_n) in case $T = \mathbb{N}$. For, we define the limes superior of a family $(M_t)_{t\in T}$ of subsets of the complex plane by

$$\limsup_{t\in T} M_t := \cap_{t\in T} \sup_{s\succ t} M_s.$$

Norms. Again, we start with a result on the asymptotic behaviour of norms.

Proposition 4.1 *For all nets $a = (a_t) \in \mathcal{F}$,*

$$\|a + \mathcal{G}\|_{\mathcal{F}/\mathcal{G}} = \limsup_{t \in T} \|a_t\|.$$

Proof. Let $a = (a_t) \in \mathcal{F}$. For every net $g = (g_t) \in \mathcal{G}$,

$$\limsup \|a_t\| = \limsup \|a_t + g_t\| \leq \sup \|a_t + g_t\| = \|a + g\|_{\mathcal{F}},$$

whence the estimate $\limsup \|a_t\| \leq \|a + \mathcal{G}\|$. For the reverse inequality, let $\varepsilon > 0$, and choose $t_0 \in T$ such that $\|a_t\| \leq \limsup \|a_t\| + \varepsilon$ for all $t \succ t_0$. The net g, defined by

$$g_t := \begin{cases} 0 & \text{if } t \succ t_0 \\ -a_t & \text{if } t \not\succ t_0, \end{cases}$$

belongs to $\mathcal{G}$, and

$$\|a + \mathcal{G}\| \leq \|a + g\| = \sup_{t \succ t_0} \|a_t\| \leq \limsup \|a_t\| + \varepsilon.$$

Letting ε go to zero yields the desired result. ■

Combining this result (where now T is $\mathbb{N}$ and $\succ$ is $\geq$) with Theorems 3.1, 2.4 and 2.1, we get:

Theorem 4.2 *Let $\mathbf{A} = (A_n) \in \mathcal{B}_N^{rich}$. Then*

$$\limsup \|A_n\| = \|\mathbf{A} + \mathcal{G}\| = \sup\{\|A_h\| : A_h \in \sigma_{stab}(\mathbf{A})\}.$$

Proof. The first equality follows from the preceding proposition. The the second one is a consequence of Theorems 3.1 and 2.1 which follows since the operators in $\sigma_{stab}(\mathbf{A})$ are just the operators of the form $W_{h,m}(\mathbf{A})$ with $h \in \mathcal{H}_{\mathrm{Op}(\mathbf{A})}$ and with $m \in \mathbb{Z}$. ■

Spectra. Let $(a_t) \in \mathcal{F}$. It turns out that the limes superior $\limsup_{t \in T} \sigma(a_t)$ is related to some kind of stability which might be called 'spectral' stability. The net (a_t) is *spectrally stable* if there is a $t_0 \in T$ such that the a_t are invertible and the spectral radii $\rho(a_t^{-1})$ of their inverses are uniformly bounded for all $t \succ t_0$ (whereas the common notion of stability requires the invertibility of a_t for all $t \succ t_0$ and the uniform boundedness of the norms $\|a_t^{-1}\|$). Clearly, every stable net is also spectrally stable.

Theorem 4.3 *Let $(a_t) \in \mathcal{F}$. Then $\lambda \in \limsup_{t \in T} \sigma(a_t)$ if and only if the net $(a_t - \lambda e_t)$ fails to be spectrally stable.*

Proof. Let the net $(a_t - \lambda e_t)$ be spectrally stable, i.e. there is a $t_0 \in T$ such that

$$\sup_{t \succ t_0} \rho((a_t - \lambda e_t)^{-1}) =: m < \infty.$$

Then, for all $t \succ t_0$,

$$m \geq \sup\{|\mu| : \mu \in \sigma((a_t - \lambda e_t)^{-1})\} = (\inf\{|\mu| : \mu \in \sigma(a_t - \lambda e_t)\})^{-1},$$

whence

$$1/m \leq \inf\{|\mu| : \mu \in \sigma(a_t) - \lambda\} = \inf\{|\mu - \lambda| : \mu \in \sigma(a_t)\} = \text{dist}\,(\lambda, \sigma(a_t)).$$

Hence, λ cannot belong to $\limsup \sigma(a_t)$.

Let, conversely, $\lambda \notin \limsup \sigma(a_t)$. Then there is an $t_0 \in T$ such that λ does not belong to clos $\cup_{t \succ t_0} \sigma(a_t)$. The boundedness of the net (a_t) implies the compactness of this set, hence,

$$\text{dist}\,(\lambda,\, \text{clos}\, \cup_{t \succ t_0} \sigma(a_t)) =: m > 0.$$

Consequently,

$$\text{dist}\,(0,\, \sigma(a_t - \lambda e_t)) \geq m > 0 \quad \text{for all } t \succ t_0,$$

which implies that $a_t - \lambda e_t$ is invertible and $\rho((a_t - \lambda e_t)^{-1}) \leq 1/m$ for all $t \succ t_0$. Thus, the net (a_t) is spectrally stable. ■

As we have just seen, the determination of $\limsup \sigma(a_t)$ requires to investigate the spectral stability of the nets $(a_t - \lambda e_t)$. This can be easily done for nets for which stability and spectral stability coincide.

Corollary 4.4 *Let $a = (a_t) \in \mathcal{F}$ be a net of normal elements. Then*

$$\limsup \sigma(a_t) = \sigma_{\mathcal{F}/\mathcal{G}}(a + \mathcal{G}).$$

Proof. The spectral radius and the norm of a normal element coincide. Hence, the sequence $(a_t - \lambda e_t)$ is spectrally stable if and only if it is stable. The stability of $(a_t - \lambda e_t)$ is equivalent to the invertibility of the coset $(a_t - \lambda e_t) + \mathcal{G}$. ■

These results lead to the following theorem in a similar way as we derived Theorem 4.2.

Theorem 4.5 *Let $\mathbf{A} = (A_n) \in \mathcal{B}_N^{rich}$ be a sequence of normal operators. Then*

$$\limsup \sigma(A_n) = \sigma(\mathbf{A} + \mathcal{G}) = \sup \sigma(A_h)$$

where the supremum is taken over all operators $A_h \in \sigma_{stab}(\mathbf{A})$.

Pseudospectra and numerical ranges. Here are the analogous results for pseudospectra and numerical ranges. In case $T = \mathbb{N}$, these results can be derived in a similar way as in [7], Theorems 3.31 and 3.46. In the following sections we will propose an alternative approach which also works in the general case.

Theorem 4.6 *Let $a = (a_t) \in \mathcal{F}$ and $\varepsilon > 0$. Then*

$$\limsup \sigma^\varepsilon(a_t) = \sigma^\varepsilon_{\mathcal{F}/\mathcal{G}}(a + \mathcal{G}).$$

Combining the preceding theorem with Theorems 2.4 and 3.3 we get the following result for the pseudospectra of the finite sections of a band-dominated operator.

Theorem 4.7 *Let $\mathbf{A} = (A_n) \in \mathcal{B}_N^{rich}$ and $\varepsilon > 0$. Then*

$$\limsup \sigma^\varepsilon(A_n) = \sigma^\varepsilon(\mathbf{A} + \mathcal{G}) = \sup \sigma^\varepsilon(A_h)$$

where the supremum is taken over all operators $A_h \in \sigma_{stab}(\mathbf{A})$.

Theorem 4.8 *Let $a = (a_t) \in \mathcal{F}$. Then*

$$\text{conv } \limsup_{n\to\infty} N(a_t) = N_{\mathcal{F}/\mathcal{G}}(a + \mathcal{G}).$$

Observe that the limes superior of a net of convex sets need not to be convex again, which explains the conv operator on the left hand side. The implications of Theorem 4.8 (in combination with Theorems 2.4 and 3.6) for the finite section method are as follows.

Theorem 4.9 *Let $\mathbf{A} = (A_n) \in \mathcal{B}_N^{rich}$. Then*

$$\text{conv } \limsup N(A_n) = N_{\mathcal{F}/\mathcal{G}}(\mathbf{A} + \mathcal{G}) = \sup N(A_h)$$

where the supremum is taken over all operators $A_h \in \sigma_{stab}(\mathbf{A})$.

5 Theorems of Weyl type for concrete set sequences

In this section, we will derive theorems of Weyl type for some set functions. By a set function on an algebra $\mathcal{B}$, we mean a mapping from $\mathcal{B}$ into the set of all subsets of the complex plane. Then we say that a Weyl type theorem holds for two set functions $\Sigma_\mathcal{C}$ on an algebra $\mathcal{C}$ and $\Sigma_{\mathcal{C}/\mathcal{J}}$ on the quotient $\mathcal{C}/\mathcal{J}$ of $\mathcal{C}$ by its ideal $\mathcal{J}$ if

$$\Sigma_{\mathcal{C}/\mathcal{J}}(c + \mathcal{J}) = \cap_{j\in\mathcal{J}} \Sigma_\mathcal{C}(b + j) \quad \text{for every } c \in \mathcal{C}.$$

Spectra. If $\Sigma(b)$ is the spectrum of b, then there is no Weyl type theorem in general. To have an example, let $\mathcal{C} = L(l^2)$, $\mathcal{J}$ the ideal of the compact operators on l^2, and $c = V$ the forward shift operator on l^2. Then $c + j$ is a Fredholm operator with non-vanishing index for every $j \in \mathcal{J}$, whence $0 \in \sigma(c + j)$. But V is a Fredholm operator, hence, $c + \mathcal{J}$ is invertible and $0 \notin \sigma(c + \mathcal{J})$.

It turns out that there is a theorem of Weyl type for spectra in case of the algebra $\mathcal{F}$ and its ideal $\mathcal{G}$.

Theorem 5.1 *Let $\mathcal{F}$ and $\mathcal{G}$ be as above. Then, for every net $a \in \mathcal{F}$,*

$$\sigma_{\mathcal{F}/\mathcal{G}}(a + \mathcal{G}) = \cap_{g \in \mathcal{G}} \sigma_{\mathcal{F}}(a + g). \tag{8}$$

One of the inclusions in (8) holds in a more general setting.

Lemma 5.2 *Let $\mathcal{C}$ be a Banach algebra with identity and $\mathcal{J}$ a closed ideal of $\mathcal{C}$. Then, for every $c \in \mathcal{C}$,*

$$\sigma_{\mathcal{C}/\mathcal{J}}(c + \mathcal{J}) \subseteq \cap_{j \in \mathcal{J}} \sigma_{\mathcal{C}}(c + j).$$

Indeed, if $W : \mathcal{C} \to \mathcal{C}/\mathcal{J}$ denotes the canonical homomorphism, then

$$\sigma(c + \mathcal{J}) = \sigma(W(c)) = \sigma(W(c + j)) \subseteq \sigma(c + j) \quad \text{for every } j \in \mathcal{J}.$$

Proof of Theorem 5.1. The inclusion $\subseteq$ in (8) follows from Lemma 5.2. For the reverse inclusion, let $\lambda \notin \sigma(a + \mathcal{G})$, i.e. the coset $a - \lambda e + \mathcal{G}$ is invertible in $\mathcal{F}/\mathcal{G}$. Then there are nets $b \in \mathcal{F}$ and $g, h \in \mathcal{G}$ such that

$$(a - \lambda e)b = e + g \quad \text{and} \quad b(a - \lambda e) = e + h.$$

Choose $t_g, t_h \in T$ such that $\|g_t\| < 1/2$ for all $t \succ t_g$ and $\|h_t\| < 1/2$ for all $t \succ t_h$, and let $t_0 \in T$ be such that $t_0 \succ t_g$ and $t_0 \succ t_h$. Further, define $a', b' \in \mathcal{F}$ by

$$a'_t := \begin{cases} a_t & \text{if } t \succ t_0 \\ (1 + \lambda)e_t & \text{if } t \not\succ t_0 \end{cases} \quad \text{and} \quad b'_t := \begin{cases} b_t & \text{if } t \succ t_0 \\ e_t & \text{if } t \not\succ t_0. \end{cases}$$

Then $a - a'$ and $b - b'$ belong to $\mathcal{G}$, and

$$(a' - \lambda e)b' = e + g' \quad \text{and} \quad b'(a' - \lambda e) = e + h' \tag{9}$$

with

$$g'_t := \begin{cases} g_t & \text{if } t \succ t_0 \\ 0 & \text{if } t \not\succ t_0 \end{cases} \quad \text{and} \quad h'_t := \begin{cases} h_t & \text{if } t \succ t_0 \\ 0 & \text{if } t \not\succ t_0. \end{cases}$$

In particular, $\|g'\| \le 1/2$. Thus, $e + g'$ is invertible if $\mathcal{F}$, and its inverse is of the form $e + k$ with $k \in \mathcal{G}$. Multiplying the first equality of (9) by $e + k$ from the left hand side yields

$$(a + s - \lambda e)b' = e \quad \text{with} \quad s := ka' - (a - a') - \lambda k \in \mathcal{G}.$$

Repeating these arguments for the second equality in (9), we get the invertibility of b' and, hence, that of $a + s - \lambda e$. Thus, $\lambda \notin \sigma(a + s)$, whence finally $\lambda \notin \cap_{g \in \mathcal{G}} \sigma(a + g)$. ∎

Pseudospectra. The example from the preceding paragraph also indicates that, in general, one cannot expect a theorem of Weyl type for pseudospectra. But for the concrete algebra $\mathcal{F}$ and its ideal $\mathcal{G}$, there is again such a result.

Theorem 5.3 *Let $\mathcal{F}$ and $\mathcal{G}$ be as above, and let $\varepsilon > 0$. Then, for every net $a \in \mathcal{F}$,*

$$\sigma^{\varepsilon}_{\mathcal{F}/\mathcal{G}}(a + \mathcal{G}) = \cap_{g \in \mathcal{G}} \sigma^{\varepsilon}_{\mathcal{F}}(a + g). \tag{10}$$

Observe that again one inclusion in (10) holds in the context of Banach algebras, which follows easily from Lemma 3.4.

Lemma 5.4 *Let $\mathcal{C}$ be a Banach algebra with identity, $\mathcal{J}$ a closed ideal of $\mathcal{C}$, and $\varepsilon > 0$. Then, for every $c \in \mathcal{C}$,*

$$\sigma^{\varepsilon}_{\mathcal{C}/\mathcal{J}}(c + \mathcal{J}) \subseteq \cap_{j \in \mathcal{J}} \sigma^{\varepsilon}_{\mathcal{C}}(c + j).$$

Proof of Theorem 5.3. The inclusion $\subseteq$ is a consequence of the preceding lemma, and for the reverse inclusion we proceed as in the proof of the preceding theorem. If $\lambda \notin \sigma^{\varepsilon}(a + \mathcal{G})$, then there is a net $b \in \mathcal{F}$ with $\beta := \|b + \mathcal{G}\| < 1/\varepsilon$, and there are nets g, $h \in \mathcal{G}$ such that

$$(a - \lambda e)b = e + g \quad \text{and} \quad b(a - \lambda e) = e + h.$$

Further, since $\|b + \mathcal{G}\| < 1/\varepsilon$, there is a net $k \in \mathcal{G}$ with $\|b - k\| < (\beta + 1/\varepsilon)/2$. Let $0 < \delta < (1/\varepsilon - \beta)/2$ and choose $t_k \in T$ such that $\|k_t\| < \delta$ for all $t \succ t_k$. This choice implies that

$$\|b_t\| \le \|b_t - k_t\| + \|k_t\| \le (\beta + 1/\varepsilon)/2 + \delta < \varepsilon \tag{11}$$

for all $t \succ t_k$. Let further t_g and t_h be as in the proof of Theorem 5.1, and let $t_0 \in T$ be greater than both t_g, t_h and t_k. Then we define a', $b' \in \mathcal{F}$ by

$$a'_t := \begin{cases} a_t & \text{if } t \succ t_0 \\ (\lambda + 1/\beta)e_t & \text{if } t \not\succ t_0 \end{cases} \quad \text{and} \quad b'_t := \begin{cases} b_t & \text{if } t \succ t_0 \\ \beta e_t & \text{if } t \not\succ t_0. \end{cases}$$

Observe that

$$\|b'\| = \sup_{t \in T} \|b'_t\| = \max\{\sup_{t \succ t_0} \|b_t\|, \beta\} \le \max\{(\beta + 1/\varepsilon)/2 + \delta, \beta\} < 1/\varepsilon$$

due to (11). The remaining steps are as in the proof of Theorem 5.1. They show the existence of a net $s \in \mathcal{G}$ such that

$$(a + s - \lambda e)b' = b'(a + s - \lambda e) = e,$$

whence $\lambda \notin \sigma^{\varepsilon}(a + s)$. ∎

Let us mention that a similar result also holds (with obvious modifications in the proof) for the so-called *structured pseudospectra* or *spectral value sets* of an element a of a C^*-algebra $\mathcal{C}$, which are defined by

$$\sigma^{\varepsilon}_{b,c}(a) := \{\lambda \in \mathbb{C} : a - \lambda e \text{ is not invertible or } \|b(a-\lambda e)^{-1}c\| \geq 1/\varepsilon\}$$

where b, c are fixed elements of $\mathcal{C}$ and $\varepsilon > 0$. Let us also mention that in [2], there is proved a version of Theorem 3.5 which is then used to derive a modification of Theorem 3.3. This modification holds for structured pseudospectra of certain band matrices the entries of which are constant along each diagonal. Even though this class of matrices seems to be very special, the results from [2] are highly non-trivial, and there seems to be no hope to extend these results to more general classes of band and band-dominated operators.

Numerical ranges. Here the situation is much easier: one has a Weyl type theorem for every Banach algebra with identity and every closed ideal of that algebra.

Theorem 5.5 *Let $\mathcal{C}$ be a Banach algebra with identity and $\mathcal{J}$ a closed ideal of $\mathcal{C}$. Then, for every $c \in \mathcal{C}$,*

$$N_{\mathcal{C}/\mathcal{J}}(c + \mathcal{J}) \subseteq \cap_{j \in \mathcal{J}} N_{\mathcal{C}}(c + j).$$

A proof is in [5], Section 22, Lemma 3.

6 The limes superior of a family of set functions

Recall that a set function Σ on a Banach algebra $\mathcal{B}$ with identity e is called

- *bounded* if $\Sigma(a)$ is a bounded subset of $\mathbb{C}$ for every $a \in \mathcal{B}$.
- *semi-homogeneous* if $\Sigma(a+\lambda e) \subseteq \lambda + \Sigma(a)$ for every $a \in \mathcal{B}$ and every $\lambda \in \mathbb{C}$.
- *upper semi-continuous at* $a \in \mathcal{B}$ if, for every $\varepsilon > 0$, there is a $\delta > 0$ such that, for all $b \in \mathcal{B}$ with $\|b-a\| < \delta$, $\Sigma(b)$ lies in the ε-neighborhood of $\Sigma(a)$.

Let T, $\mathcal{B}_t$, $\mathcal{F}$ and $\mathcal{G}$ be as in the preceding sections. Assume further that, for every $t \in T$, we are given a set function Σ_t on $\mathcal{B}_t$. To the family $(\Sigma_t)_{t\in T}$, we associate two set functions Σ_{sup} and Σ_{limsup} on $\mathcal{F}$ by

$$\Sigma_{\text{sup}}(a) := \sup_{t\in T} \Sigma_t(a_t) = \text{clos} \cup_{t\in T} \Sigma_t(a_t)$$

and

$$\Sigma_{\text{limsup}}(a) := \limsup_{t\in T} \Sigma_t(a_t) = \cap_{t\in T} \sup_{s\succ t} \Sigma_s(a_s)$$

where $a = (a_t)$. Further we call the family $(\Sigma_t)_{t\in T}$ *uniformly bounded* if the set function Σ_{sup} is bounded, and we call this family *uniformly upper semi-continuous at* $a \in \mathcal{F}$ if, for every $\varepsilon > 0$, there is a $\delta > 0$ such that, for all $b \in \mathcal{F}$ with $\|b - a\| < \delta$ and for all $t \in T$, $\Sigma_t(b_t)$ lies in the ε-neighborhood of $\Sigma_t(a_t)$.

Theorem 6.1 *Let T, $\mathcal{B}_t$, Σ_t, $\mathcal{F}$ and $\mathcal{G}$ be as above, and let every set function Σ_t be semi-homogeneous and the family $(\Sigma_t)_{t\in T}$ be uniformly bounded and uniformly upper semi-continuous on $\mathcal{F}$. Then, for every net $a := (a_t)_{t\in T} \in \mathcal{F}$,*

$$\Sigma_{\text{limsup}}(a) = \cap_{g\in\mathcal{G}}\, \Sigma_{\text{sup}}(a+g) \tag{12}$$

or, equivalently,

$$\limsup_{t\in T} \Sigma_t(a_t) = \cap_{g\in\mathcal{G}} \sup_{t\in T} \Sigma_t(a_t+g_t).$$

Proof. Abbreviate the left and the right hand side of (12) by S_l and S_r, respectively. We first show the inclusion $S_r \subseteq S_l$. Let $\lambda \in \cap_{g\in\mathcal{G}} \sup_{t\in T} \Sigma_t(a_t+g_t)$ and $s \in T$. Choose $r \in \mathbb{C}$ such that

$$\text{dist}\,(\lambda,\; r + \sup_{t\not\succ s} \Sigma_t(0)) \geq 1$$

which is possible since $\sup_{t\not\succ s} \Sigma_t(0) \subseteq \sup_{t\in T} \Sigma_t(0) = \Sigma_{\text{sup}}(0)$ is bounded by assumption. Then define $g \in \mathcal{F}$ by

$$g_t := \begin{cases} 0 & \text{if } t \succ s \\ -a_t + re_t & \text{if } t \not\succ s. \end{cases}$$

The net g lies in $\mathcal{G}$. Consequently,

$$\begin{aligned} \lambda &\in \sup_{t\in T} \Sigma_t(a_t+g_t) \;=\; \text{clos}\,(\cup_{t\in T}\Sigma_t(a_t+g_t)) \\ &= \text{clos}\,(\cup_{t\succ s}\Sigma_t(a_t+g_t) \,\cup\, \cup_{t\not\succ s}\Sigma_t(a_t+g_t)) \\ &= \text{clos}\,(\cup_{t\succ s}\Sigma_t(a_t) \,\cup\, \cup_{t\not\succ s}\Sigma_t(re_t)) \\ &\subseteq \text{clos}\,(\cup_{t\succ s}\Sigma_t(a_t) \,\cup\, (r + \cup_{t\not\succ s}\Sigma_t(0))) \end{aligned}$$

due to the semi-homogeneity. The choice of r ensures that $\lambda \notin \text{clos}\,(r+\cup_{t\not\succ s}\Sigma_t(0))$. Hence,

$$\lambda \in \text{clos}\,(\cup_{t\succ s}\Sigma_t(a_t)) = \sup_{t\succ s} \Sigma_t(a_t)$$

whence $\lambda \in S_l$.

To prove the reverse inclusion, let $\lambda \in \limsup_{t\in T} \Sigma_t(a_t)$ and $g = (g_t) \in \mathcal{G}$. Due to the uniform upper semi-continuity at $a+g \in \mathcal{F}$, given $\varepsilon > 0$, there is a $\delta > 0$ such that, for all $b = (b_t) \in \mathcal{F}$ with $\|b - (a+g)\| < \delta$, $\Sigma_t(b_t)$ lies in the ε-neighborhood of $\Sigma_t(a_t+g_t)$. Choose $s \in T$ such that $\|g_t\| < \delta/2$ for all $t \succ s$, and define $b = (b_t) \in \mathcal{F}$ by

$$b_t := \begin{cases} a_t & \text{if } t \succ s \\ a_t + g_t & \text{if } t \not\succ s. \end{cases}$$

Then $\|b-(a+g)\| \le \delta/2 < \delta$, and from $\lambda \in S_l$ we conclude

$$\begin{aligned} \lambda &\in \sup_{t \succ s} \Sigma_t(a_t) = \sup_{t \succ s} \Sigma_t(b_t) = \operatorname{clos}(\cup_{t\succ s}\Sigma_t(b_t)) \\ &\subseteq \operatorname{clos} \cup_{t\succ s} \ (\varepsilon\text{-neighborhood of } \Sigma_t(a_t+g_t)) \\ &\subseteq \operatorname{clos} \cup_{t\in T} \ (\varepsilon\text{-neighborhood of } \Sigma_t(a_t+g_t)). \end{aligned}$$

Thus, for every $\varepsilon > 0$, there is a λ_ε in the union over t of all ε-neighborhoods of $\Sigma_t(a_t+g_t)$ for which $|\lambda-\lambda_\varepsilon| \le \varepsilon$. Consequently,

$$\lambda \in \operatorname{clos} \cup_{t\in T} \ \Sigma_t(a_t+g_t).$$

Since $g \in \mathcal{G}$ is arbitrary, this shows that $\lambda \in S_r$. ■

7 Applications to concrete set functions

Pseudospectra. Let us check that the assumptions of Theorem 6.1 are satisfied if the set functions Σ_t are specified to be the ε-pseudospectra. The semi-homogeneity is evident in this case, and the uniform boundedness follows from (4).

Proposition 7.1 *Let $\mathcal{B}_t$ and $\mathcal{F}$ be as above, and let $\varepsilon_0 > 0$. Let further $\Sigma_t = \sigma^{\varepsilon_0}$ for every $t \in T$. Then the family $(\Sigma_t)_{t\in T}$ is uniformly upper semi-continuous on $\mathcal{F}$.*

Proof. Suppose there is a net $a^{(0)} \in \mathcal{F}$ at which the family (Σ_t) is not uniformly upper semi-continuous. Thus, there is an $\varepsilon > 0$ such that, for all $n \in \mathbb{N}$, there is a net $a^{(n)} \in \mathcal{F}$ with $\|a^{(n)} - a^{(0)}\| < 1/n$, and there are points $t_n \in T$ and $\lambda_n \in \sigma^{\varepsilon_0}(a_{t_n}^{(n)})$ with

$$\operatorname{dist}(\lambda_n,\, \sigma^{\varepsilon_0}(a_{t_n}^{(0)})) \ge \varepsilon.$$

Since $\cup_n \sigma^{\varepsilon_0}(a_{t_n}^{(n)})$ is bounded, there are $n_1 < n_2 < \ldots$ such that the sequence (λ_{n_k}) converges to a $\lambda^* \in \mathbb{C}$. For simplicity, let $n_k = k$, i.e. let the sequence (λ_n) converge to λ^*. Consider

$$a_{t_n}^{(n)} - \lambda_n e_{t_n} = a_{t_n}^{(0)} - \lambda^* e_{t_n} + (\lambda^* - \lambda_n) e_{t_n} + a_{t_n}^{(n)} - a_{t_n}^{(0)}, \tag{13}$$

and choose n_0 such that $|\lambda^* - \lambda_n| < \varepsilon/2$ for all $n \ge n_0$. Set further $T^* := \{t_n : n \ge n_0\}$, and let $\mathcal{F}^*$ refer to the product of the algebras $\mathcal{B}_t$ with $t \in T^*$. Evidently, the family of homomorphisms

$$\mathcal{F}^* \to \mathcal{B}_t,\ a \to a_t \quad \text{with } t \in T^*$$

is weakly sufficient for $\mathcal{F}^*$. Thus, if we denote the restriction of $a^{(0)}$ onto T^* by $a^{(0)}$ again, and if we consider this restriction as an element of the algebra $\mathcal{F}^*$, then we conclude from

$$\operatorname{dist}(\lambda^*,\, \sigma^{\varepsilon_0}(a_{t_n}^{(0)})) \ge \operatorname{dist}(\lambda_n,\, \sigma^{\varepsilon_0}(a_{t_n}^{(0)})) - |\lambda^* - \lambda_n| \ge \varepsilon - \varepsilon/2 = \varepsilon/2$$

(which holds for all $n \geq n_0$) and from Theorem 3.3 that $\lambda^* \notin \sigma^{\varepsilon_0}_{\mathcal{F}^*}(a^{(0)})$. Thus, $a^{(0)} - \lambda^* e$ is ε_0-invertible in $\mathcal{F}^*$. Since the set of all ε_0-invertible elements of a Banach algebra is open, there is an $\varepsilon_1 > 0$ such that $a^{(0)} - \lambda^* e + s$ is ε_0-invertible for all $s \in \mathcal{F}^*$ with $\|s\| < \varepsilon_1$. Choose $n_1 > n_0$ such that

$$|\lambda^* - \lambda| + \|a^{(n)}_{t_n} - a^{(0)}_{t_n}\| < \varepsilon_1/2 \quad \text{for all } n \geq n_1.$$

Then the net

$$T^* \ni t_n \mapsto \begin{cases} a^{(0)}_{t_n} - \lambda^* e & \text{if } \ n < n_1 \\ a^{(0)}_{t_n} - \lambda^* e_{t_n} + (\lambda^* - \lambda_n) e_{t_n} + a^{(n)}_{t_n} - a^{(0)}_{t_n} & \text{if } \ n \geq n_1 \end{cases}$$

is ε_0-invertible in $\mathcal{F}^*$. Hence, the right hand side of (13) is ε_0-invertible for sufficiently large n, whereas the left hand side of (13) fails to be ε_0-invertible by construction. Contradiction. ■

Now the proof of Theorem 4.6 follows easily: the Weyl theorem (Theorem 5.3) implies that

$$\sigma^{\varepsilon}_{\mathcal{F}/\mathcal{G}}(a + \mathcal{G}) = \cap_{g \in \mathcal{G}} \sigma^{\varepsilon}_{\mathcal{F}}(a + g),$$

which is equal to $\cap_{g \in \mathcal{G}} \sup_{t \in T} \Sigma_t(a_t + g_t)$ by Theorem 3.3, applied to the family of all homomorphisms $a \mapsto a_t$. From Theorem 6.1 we finally conclude that

$$\cap_{g \in \mathcal{G}} \sup_{t \in T} \Sigma_t(a_t + g_t) = \limsup_{t \in T} \Sigma_t(a_t).$$

Numerical ranges. For the proof of Theorem 4.8 we need an auxiliary result.

Lemma 7.2 *Let $(T, \succ)$ be a directed set and let $(M_t)_{t \in T}$ be a bounded and monotonically decreasing (i.e. $M_s \subseteq M_t$ whenever $s \succ t$) net of non-empty closed subsets of $\mathbb{C}$. Then*

$$\text{conv } \cap_{t \in T} M_t = \cap_{t \in T} \text{conv } M_t.$$

Proof. Since $\cap_t M_t \subseteq \cap_t \text{conv } M_t$, and since the intersection of convex sets is convex again, the inclusion $\text{conv } \cap_t M_t \subseteq \cap_t \text{conv } M_t$ is evident.

Conversely, let $m \in \cap_t \text{conv } M_t$. Then, for every $t \in T$, there are elements m^t_1, m^t_2 in M_t as well as non-negative numbers λ^t_1, λ^t_2 with $\lambda^t_1 + \lambda^t_2 = 1$ such that

$$m = \lambda^t_1 m^t_1 + \lambda^t_2 m^t_2. \tag{14}$$

Due to the boundedness of the net (M_t), there exists a convergent subnet $(y^s_1)_{s \in S}$ of the net $(m^t_1)_{t \in T}$ with limit $m_1 \in \text{clos} \cup_t M_t$. (Recall that $(y^s_1)_{s \in S}$ is a subnet of $(m^t_1)_{t \in T}$ if S is a directed set and if there is a mapping $f : S \to T$ such that $y^s_1 = m^{f(s)}_1$ for all $s \in S$ and such that, given $t_0 \in T$, there is an $s_0 \in S$ with $f(s) \succ t_0$ for all $s \succ s_0$.) We claim that $m_1 \in M_t$ for every $t \in T$.

Let $t_0 \in T$ and $\varepsilon > 0$. Then there is an $s_0 \in S$ such that $|y_1^s - m_1| < \varepsilon$ for all $s \succ s_0$. Further we choose $s_1 \in S$ such that $f(s) \succ t_0$ for all $s \succ s_1$, and we choose $s^* \in S$ such that both $s^* \succ s_0$ and $s^* \succ s_1$. Then

$$|y_1^{s^*} - m_1| = |m_1^{f(s^*)} - m_1| < \varepsilon$$

and $m_1^{f(s^*)} \in M_{t_0}$ since $f(s^*) \succ t_0$. Thus, $\operatorname{dist}(m_1, M_{t_0}) < \varepsilon$ for all $\varepsilon > 0$. Since M_{t_0} is closed, this implies $m_1 \in M_{t_0}$ and proves our claim.

For $s \in S$, set $y_2^s := m_s^{f(s)}$. The same arguments as above yield the existence of a convergent subnet of $(y_2^s)_{s\in S}$ the limit m_2 of which belongs to M_t for every $t \in T$. Repeating these considerations for the nets (λ_1^t) and (λ_2^t) as well, we finally get convergent subnets $(z_1^u)_{u\in U}$, $(z_2^u)_{u\in U}$, $(\mu_1^u)_{u\in U}$, $(\mu_2^u)_{u\in U}$ of $(m_1^t)_{t\in T}$, $(m_2^t)_{t\in T}$, $(\lambda_1^t)_{t\in T}$, $(\lambda_2^t)_{t\in T}$ with limits $m_1, m_2 \in \cap_t M_t$ and $\mu_1, \mu_2 \in [0, 1]$, $\mu_1 + \mu_2 = 1$, respectively. From (14), we conclude that

$$m = \mu_1^u z_1^u + \mu_2^u z_2^u \quad \text{for all } u \in U,$$

whence $m \in \operatorname{conv} \cap_{t\in T} M_t$. ■

Proof of Theorem 4.8. Let $a = (a_t) \in \mathcal{F}$. From the Weyl theorem (Theorem 5.5) we know that

$$N(a + \mathcal{G}) = \cap_{g\in\mathcal{G}} N(a + g),$$

and Theorem 3.6 (applied to the weakly sufficient family for $\mathcal{F}$, consisting of all homomorphisms $a \mapsto a_t$) further yields

$$N(a + \mathcal{G}) = \cap_{g\in\mathcal{G}} \operatorname{conv} \sup_{t\in T} N(a_t + g_t). \tag{15}$$

Fix $s \in T$, choose $m_s \in N(a_s)$, and set

$$g_t^{(s)} := \begin{cases} 0 & \text{if } t \succ s \\ -a_t + m_s e_t & \text{if } t \not\succ s. \end{cases}$$

Then the net $g(s)$ belongs to the ideal $\mathcal{G}$ and, since $N(a_t + g_t^{(s)}) = \{m_s\} \subseteq N(a_s)$ for all $t \not\succ s$, we get

$$\sup_{t\in T} N(a_t + g_t^{(s)}) = \operatorname{clos} \cup_{t\succ s} N(a_t) = \sup_{t\succ s} N(a_t).$$

Together with equality (15), this implies that

$$N(a + \mathcal{G}) \subseteq \cap_{s\in T} \operatorname{conv} \sup_{t\succ s} N(a_t).$$

Applying Lemma 7.2 to the sets $M_s := \operatorname{clos} \cup_{t\succ s} N(a_s)$ we get

$$N(a + \mathcal{G}) \subseteq \operatorname{conv} \cap_{s\in T} \sup_{t\succ s} N(a_t) = \operatorname{conv} \limsup_{t\in T} N(a_t). \tag{16}$$

The proof of the reverse inclusion is based on Theorem 6.1, with the set functions Σ_t being the numerical ranges. The hypotheses of that theorem are evidently satisfied: since states are linear and unital, one has the homogenity, and since the norm of a state is 1, one has

$$\max\{|\lambda| : \lambda \in N(a)\} \leq \|a\|,$$

whence the uniform boundedness. Further, if f is a state, then $|f(a) - f(b)| \leq \|a - b\|$ which yields the uniform upper semi-continuity. Thus, Theorem 6.1 applies and yields the equality

$$\limsup_{t \in T} N(a_t) = \cap_{g \in \mathcal{G}} \sup_{t \in T} N(a_t + g_t).$$

The right hand side is obviously contained in $\cap_{g \in \mathcal{G}} \text{conv} \sup_{t \in T} N(a_t + g_t)$, and this set coincides with $\cap_{g \in \mathcal{G}} N(a + g)$ due to Theorem 3.6 and, hence, with $N(a + \mathcal{G})$ by the Weyl theorem. Thus, $\limsup_{t \in T} N(a_t) \subseteq N(a + \mathcal{G})$, and since numerical ranges are convex, this finally implies that

$$\text{conv} \limsup_{t \in T} N(a_t) \subseteq N(a + \mathcal{G}),$$

which finishes the proof of Theorem 4.8. ∎

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