The nonstationary Stokes and Navier-Stokes flows through an aperture

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Abstract

We consider the nonstationary Stokes and Navier-Stokes flows in aperture domains $\Omega \subset \mathbb{R}^n, n \geq 3$. We develop the L^q-L^r estimates of the Stokes semigroup and apply them to the Navier-Stokes initial value problem. As a result, we obtain the global existence of a unique strong solution, which satisfies the vanishing flux condition through the aperture and some sharp decay properties as $t \to \infty$, when the initial velocity is sufficiently small in the L^n space. Such a global existence theorem is up to now well known in the cases of the whole and half spaces, bounded and exterior domains.
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1 Introduction

In the present paper we study the global existence and asymptotic behavior of a strong solution to the Navier-Stokes initial value problem in an aperture domain $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$:

$$\begin{cases} \partial_t u + u \cdot \nabla u &= \Delta u - \nabla p \qquad (x \in \Omega,\ t > 0), \\ \nabla \cdot u &= 0 \qquad (x \in \Omega,\ t \geq 0), \\ u|_{\partial\Omega} &= 0 \qquad (t > 0), \\ u|_{t=0} &= a \qquad (x \in \Omega), \end{cases} \tag{1.1}$$

where $u(x,t) = (u_1(x,t), \cdots, u_n(x,t))$ and $p(x,t)$ denote the unknown velocity and pressure of a viscous incompressible fluid occupying Ω, respectively, while $a(x) = (a_1(x), \cdots, a_n(x))$ is a prescribed initial velocity. The aperture

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domain Ω is a compact perturbation of two separated half spaces $H_+ \cup H_-$, where $H_\pm = \{x = (x_1, \cdots, x_n) \in \mathbb{R}^n; \pm x_n > 1\}$; to be precise, we call a connected open set $\Omega \subset \mathbb{R}^n$ an aperture domain (with thickness of the wall) if there is a ball $B \subset \mathbb{R}^n$ such that $\Omega \setminus B = (H_+ \cup H_-) \setminus B$. Thus the upper and lower half spaces $H_\pm$ are connected by an aperture (hole) $M \subset \Omega \cap B$, which is a smooth $(n-1)$-dimensional manifold so that Ω consists of upper and lower disjoint subdomains $\Omega_\pm$ and M: $\Omega = \Omega_+ \cup M \cup \Omega_-$.

The aperture domain is a particularly interesting class of domains with noncompact boundaries because of the following remarkable feature, which was in 1976 pointed out by Heywood [31]: the solution is not uniquely determined by usual boundary conditions even for the stationary Stokes system in this domain and therefore, in order to single out a unique solution, we have to prescribe either the flux through the aperture M

$$\phi(u) = \int_M N \cdot u d\sigma,$$

or the pressure drop at infinity (in a sense) between the upper and lower subdomains $\Omega_\pm$

$$[p] = \lim_{|x| \to \infty, x \in \Omega_+} p(x) - \lim_{|x| \to \infty, x \in \Omega_-} p(x),$$

as an additional boundary condition. Here, N denotes the unit normal vector on M directed to Ω_- and the flux $\phi(u)$ is independent of the choice of M since $\nabla \cdot u = 0$ in Ω. Consider stationary solutions of (1.1); then one can formally derive the energy relation

$$\int_\Omega |\nabla u(x)|^2 dx = [p]\phi(u),$$

from which the importance of these two physical quantities stems. Later on, the observation of Heywood in the L^2 framework was developed by Farwig and Sohr within the framework of L^q theory for the stationary Stokes and Navier-Stokes systems [19] and also the (generalized) Stokes resolvent system [20], [16]. Especially, in the latter case, they clarified that the assertion on the uniqueness depends on the class of solutions under consideration. Indeed, the additional condition must be required for the uniqueness if $q > n/(n-1)$, but otherwise, the solution is unique without any additional condition; for more details, see Farwig [16], Theorem 1.2.

The results of Farwig and Sohr [20] are also the first step to discuss the nonstationary problem (1.1) in the L^q space. They showed the Helmholtz decomposition of the L^q space of vector fields (see also Miyakawa [49]) $L^q(\Omega) = L^q_\sigma(\Omega) \oplus L^q_\pi(\Omega)$ for $n \geq 2$ and $1 < q < \infty$, where $L^q_\sigma(\Omega)$ is the completion in $L^q(\Omega)$ of the class of all smooth, solenoidal and compactly supported vector fields, and $L^q_\pi(\Omega) = \{\nabla p \in L^q(\Omega); p \in L^q_{loc}(\overline{\Omega})\}$. The space $L^q_\sigma(\Omega)$ is characterized as ([20], Lemma 3.1)

$$L^q_\sigma(\Omega) = \{u \in L^q(\Omega); \nabla \cdot u = 0, \nu \cdot u|_{\partial\Omega} = 0, \phi(u) = 0\}, \qquad (1.2)$$

where ν is the unit outer normal vector on $\partial\Omega$. Here, the condition $\phi(u) = 0$ follows from the other ones and may be omitted if $q \leq n/(n-1)$, but otherwise, the element of $L^q_\sigma(\Omega)$ must possess this additional property. Using the projection P_q from $L^q(\Omega)$ onto $L^q_\sigma(\Omega)$ associated with the Helmholtz decomposition, we can define the Stokes operator $A = A_q = -P_q\Delta$ on $L^q_\sigma(\Omega)$ with a right domain as in section 2. Then the operator $-A$ generates a bounded analytic semigroup e^{-tA} in each $L^q_\sigma(\Omega), 1 < q < \infty$, for $n \geq 2$ ([20], Theorem 2.5).

Besides [31] and [19] cited above, there are some other studies on the stationary Stokes and Navier-Stokes systems in domains with noncompact boundaries including aperture domains. We refer to Galdi [26], Pileckas [50] and the references therein.

We are interested in strong solutions to the nonstationay problem (1.1). However, there are no results on the global existence of such solutions in the L^q framework unless $q = 2$, while a few local existence theorems are known. In the 3-dimensional case, Heywood [31], [32] first constructed a local solution to (1.1) with a prescribed either $\phi(u(t))$ or $[p(t)]$, which should satisfy some regularity assumptions with respect to the time variable, when $a \in H^2(\Omega)$ fulfills some compatibility conditions. Franzke [23] has recently developed the L^q theory of local solutions via the approach of Giga and Miyakawa [29], which is traced back to Fujita and Kato [24], with use of fractional powers of the Stokes operator. When a suitable $\phi(u(t))$ is prescribed, his assumption on initial data is for instance that $a \in L^q(\Omega), q > n$, together with some compatibility conditions. The reason why the case $q = n$ is excluded is the lack of informations about purely imaginary powers of the Stokes operator. In order to discuss also the case where $[p(t)]$ is prescribed, Franzke introduced another kind of Stokes operator associated with the pressure drop condition, which generates a bounded analytic semigroup on the space $\{u \in L^q(\Omega); \nabla \cdot u = 0, \nu \cdot u|_{\partial\Omega} = 0\}$ for $n \geq 3$ and $n/(n-1) < q < n$ (based on a resolvent estimate due to Farwig [16]). Because of this restriction on q, the L^q theory with $q \geq n$ is not available under the pressure drop condition and thus one cannot avoid a regularity assumption to some extent on initial data.

It is possible to discuss the L^2 theory of global strong solutions for an arbitrary unbounded domain (with smooth boundary) in a unified way since the Stokes operator is a nonnegative selfadjoint one in L^2_σ; see Heywood [33] ($n = 3$), Kozono and Ogawa [40] ($n = 2$), [41] ($n = 3$) and Kozono and Sohr [43] ($n = 4, 5$). Especially, from the viewpoint of the class of initial data, optimal results were given by [40], [41] and [43]. In fact, they constructed a global solution with various decay properties for small $a \in D(A_2^{n/4-1/2})$

(when $n = 2$, the smallness is not necessary). Here, we should recall the continuous embedding relation $D(A_2^{n/4-1/2}) \subset L^n_\sigma$. For the aperture domain Ω their solutions $u(t)$ should satisfy the hidden flux condition $\phi(u(t)) = 0$ on account of $u(t) \in L^2_\sigma(\Omega)$ together with (1.2). In his Doktorschrift [22] Franzke studied, among others, the global existence of weak and strong solutions in a 3-dimensional aperture domain when either $\phi(u(t))$ or $[p(t)]$ is prescribed (the global existence of the former for $n \geq 2$ is covered by Masuda [47] when $\phi(u(t)) = 0$). As for the latter, indeed, the local strong solution in the L^2 space constructed by himself [21] was extended globally in time under the condition that both $a \in H^1_0(\Omega)$ (with compatibility conditions) and the other data are small in a sense, however, his method gave no information about the large time behavior of the solution.

The purpose of the present paper is to provide the global existence theorem for a unique strong solution $u(t)$ of (1.1), which satisfies the flux condition $\phi(u(t)) = 0$ and some decay properties with definite rates that seem to be optimal, for instance,

$$\|u(t)\|_{L^\infty(\Omega)} + \|\nabla u(t)\|_{L^n(\Omega)} = o\left(t^{-1/2}\right),$$

as $t \to \infty$, when the initial velocity a is small enough in $L^n_\sigma(\Omega), n \geq 3$. The space L^n is now well known as a reasonable class of initial data, from the viewpoint of scaling invariance, to find a global strong solution within the framework of L^q theory. We derive further sharp decay properties of the solution $u(t)$ under the additional assumption $a \in L^1(\Omega) \cap L^n_\sigma(\Omega)$; for instance, the dacay rate given above is improved as $O(t^{-n/2})$. For the proof, as is well known, it is crucial to establish the L^q-L^r estimates of the Stokes semigroup

$$\|e^{-tA} f\|_{L^r(\Omega)} \leq C t^{-\alpha} \|f\|_{L^q(\Omega)}, \tag{1.3}$$

$$\|\nabla e^{-tA} f\|_{L^r(\Omega)} \leq C t^{-\alpha-1/2} \|f\|_{L^q(\Omega)}, \tag{1.4}$$

for all $t > 0$ and $f \in L^q_\sigma(\Omega)$, where $\alpha = (n/q - n/r)/2 \geq 0$. Recently for $n \geq 3$ Abels [1] has proved some partial results: (1.3) for $1 < q \leq r < \infty$ and (1.4) for $1 < q \leq r < n$. However, because of the lack of (1.4) for the most important case $q = r = n$, his results are not satisfactory for the construction of the global strong solution possessing various time-asymptotic behaviors as long as one follows the straightforward method of Kato [36] (without using duality arguments in [42], [6], [44], [45] and [34]). In this paper we consider the case $n \geq 3$ and prove

(1.3) for $1 \leq q \leq r \leq \infty$ $(q \neq \infty, r \neq 1)$, and

$$(1.4) \text{ for } 1 \le q \le r \le n \ (r \neq 1) \text{ and } 1 \le q < n < r < \infty;$$

here, when $q = 1$, f should be taken from $L^1(\Omega) \cap L^s_\sigma(\Omega)$ for some $s \in (1, \infty)$. Estimate (1.4) is thus available, in other words, for $r = n$ if $q = n$, for $r \in [q, \infty)$ if $q \in (1, n)$, and for $r \in (1, \infty)$ if $q = 1$.

Up to now we have the same global existence result as above for the whole space (Kato [36]), the half space (Ukai [58]), bounded domains (Giga and Miyakawa [29]) and exterior domains (Iwashita [35]) since the L^q-L^r estimates (1.3) and (1.4) are well established for these four types of domains. Let us give a brief survey on the literature concerning the L^q-L^r estimates. For the whole space the Stokes semgroup is essentially the same as the heat semigroup because the Laplace operator commutes with the Helmholtz projection. For the half space Ukai [58] explicitly wrote down a solution foumula of the Stokes system and derived (1.3) and (1.4) for $n \ge 2$ and $1 < q \le r < \infty$. See also Borchers and Miyakawa [3] for (1.3) with $1 \le q < r \le \infty$ and the following literarure concerning marginal cases, that is, (1.3) for $q = r = \infty$ and (1.4) for $q = r = 1$ or ∞: Giga, Matsui and Y. Shimizu [28], Y. Shimizu [54], Desch, Hieber and Prüss [15] and Shibata and S. Shimizu [53]. For bounded domains (1.3) and (1.4) are deduced from the result of Giga [27] on a characterization of the domains of fractional powers of the Stokes operator. In this case, moreover, an exponential decay property of the semigroup for large t is available. For exterior domains with $n \ge 3$, based on (1.3) for $q = r$ due to Borchers and Sohr [7], some partial results were given by Iwashita [35], Giga and Sohr [30] and Borchers and Miyakawa [4]; in particular, Iwashita proved (1.3) for $1 < q \le r < \infty$ and (1.4) for $1 < q \le r \le n$, which made it possible to construct a global solution. Later on, due to the following authors, (1.3) for $n \ge 2$, $1 \le q \le r \le \infty$ $(q \neq \infty, r \neq 1)$ and (1.4) for $n \ge 2$, $1 \le q \le r \le n$ $(r \neq 1)$ were also derived: Chen [11] $(n = 3)$, Shibata [52] $(n = 3)$, Borchers and Varnhorn [9] $(n = 2$, (1.3) for $q = r)$, Dan and Shibata [13], [14] $(n = 2)$, Dan, Kobayashi and Shibata [12] $(n = 2, 3)$ and Maremonti and Solonnikov [46] $(n \ge 2)$.

In the proof of the L^q-L^r estimates, it seems to be heuristically reasonable to combine some local decay properties near the aperture with the L^q-L^r estimates of the Stokes semigroup for the half space by means of a localization procedure since the aperture domain Ω is obtained from $H_+ \cup H_-$ by a perturbation within a compact region. Indeed, Abels [1] used this idea that was well developed by Iwashita [35] and, later, Kobayashi and Shibata [37] in the case of exterior domains. We should however note that the boundary $\partial\Omega$ is noncompact; thus, a difficulty is to deduce the sharp local energy decay estimate

$$\|e^{-tA}f\|_{W^{1,q}(\Omega_R)} \le Ct^{-n/2q}\|f\|_{L^q(\Omega)}, \quad t \ge 1, \tag{1.5}$$

for $f \in L^q_\sigma(\Omega), 1 < q < \infty$, where $\Omega_R = \{x \in \Omega; |x| < R\}$, but this is the essential part of our proof (Lemma 5.3). Estimate (1.5) improves the local energy decay given by Abels [1], in which a little slower rate $t^{-n/2q+\varepsilon}$ was shown. In [1], similarly to Iwashita [35], a resolvent expansion around the origin $\lambda = 0$ was derived in some weighted function spaces. To this end, Abels made use of the Ukai formula of the Stokes semigroup for the half space ([58]) and, in order to estimate the Riesz operator appearing in this formula, he had to introduce Muckenhoupt weights, which caused some restrictions although his analysis itself is of interest. On the other hand, Kobayashi and Shibata [37] refined the proof of Iwashita in some sense and obtained the L^q-L^r estimates of the Oseen semigroup for the 3-dimensional exterior domain. As a particular case, the result of [37] includes the estimates of the Stokes semigroup as well. In this paper we employ in principle the strategy developed by [37] (without using any weighted function space) and extend the method to general $n \geq 3$.

This paper consists of six sections. In the next section, after notation is fixed, we present the precise statement of our main results: Theorem 2.1 on the L^q-L^r estimates of the Stokes semigroup, Theorem 2.2 on the global existence and decay properties of the Navier-Stokes flow, and Theorem 2.3 on some further asymptotic behaviors of the obtained flow under an additional summability assumption on initial data. We obtain an information about a pressure drop as well in the last theorem.

Section 3 is devoted to the investigation of the Stokes resolvent for the half space $H = H_+$ or H_-. We derive some regularity estimates near the origin $\lambda = 0$ of $(\lambda + A_H)^{-1}P_H f$ when $f \in L^q(H)$ has a bounded support, where $A_H = -P_H\Delta$ is the Stokes operator for the half space H (for the notation, see section 2). Although the obtained estimates do not seem to be optimal compared with those shown by [37] for the whole space, the results are sufficient for our aim and the proof is rather elementary: in fact, we represent the resolvent $(\lambda + A_H)^{-1}$ in terms of the semigroup e^{-tA_H} and, with the aid of local energy decay properties of this semigroup, we have only to perform several integrations by parts and to estimate the resulting formulae. One needs neither Fourier analysis nor resolvent expansions.

In section 4, based on the results for the half space, we proceed to the analysis of the Stokes resolvent for the aperture domain Ω. To do so, in an analogous way to [35], [37] and [1], we first construct the resolvent $(\lambda + A)^{-1}Pf$ near the origin $\lambda = 0$ for $f \in L^q(\Omega)$ with bounded support by use of the operator $(\lambda + A_H)^{-1}P_H$, the Stokes flow in a bounded domain and a cut-off function together with the result of Bogovskiĭ [2] on the boundary value problem for the equation of continuity. And then, for the same f as above, we deduce essentially the same regularity estimates near the origin $\lambda = 0$ of $(\lambda + A)^{-1}Pf$ as shown in section 3.

In section 5 we prove (1.5) and thereby (1.4) for $q = r \in (1, n]$ as well as (1.3) for $r = \infty$, from which the other cases follow. Some of the estimates obtained in section 4 enable us to justify a representation formula of the semigroup $e^{-tA}Pf$ in $W^{1,q}(\Omega_R)$ in terms of the Fourier inverse transform of $\partial_s^m(is+A)^{-1}Pf$ when $f \in L^q(\Omega)$ has a bounded support, where $n = 2m+1$ or $n = 2m+2$ (see (5.3); we note that the formula is not valid for $n = 2$). We then appeal to the lemma due to Shibata ([51]; see also [37] and a recent development [53]), which tells us a relation between the regularity of a function at the origin and the decay property of its Fourier inverse image, so that we obtain another local energy decay estimate

$$\|e^{-tA}Pf\|_{W^{1,q}(\Omega_R)} \le Ct^{-n/2+\varepsilon}\|f\|_{L^q(\Omega)}, \quad t \ge 1, \tag{1.6}$$

for $f \in L^q(\Omega), 1 < q < \infty$, with bounded support, where $\varepsilon > 0$ is arbitrary (Lemma 5.1). Estimate (1.6) was shown in [1] only for solenoidal data $f \in L^q_\sigma(\Omega)$ with bounded support, from which (1.5) with the rate replaced by $t^{-n/2q+\varepsilon}$ follows through an interpolation argument. But it is crucial for the proof of (1.5) to use (1.6) even for data which are not solenoidal (so that the support of Pf is unbounded). In order to deduce (1.5) from (1.6), we develop the method in [35] and [37] based on a localization argument using a cut-off function. In fact, we regard the Stokes flow for the aperture domain Ω as the sum of the Stokes flows for the half spaces $H_\pm$ and a certain perturbed flow. Since the Stokes flow for the half space enjoys the L^q-L^∞ decay estimate with the rate $t^{-n/2q}$ (Borchers and Miyakawa [3]), our main task is to show (1.5) for the perturbation part. In contrast to the case of exterior domains, the support of the derivative of the cut-off function touches the boundary $\partial\Omega$; indeed, this difficulty occurs in all stages of localization procedures in the course of the proof (sections 4 and 5) and thus we have to carry out such procedures carefully. Furthermore, the remainder term arising from the above-mentioned localization argument involves the pressure of the nonstationary Stokes system in the half space and, therefore, does not belong to any solenoidal function space. Hence, in order to treat this term, (1.6) is necessary for non-solenoidal data, while that is not the case for the exterior problem.

Once Theorem 2.1 is established, one can prove the existence part of Theorem 2.2 along the lines of Kato [36] (see also [24] and [29]) and therefore the proof may be omitted. Thus, in the final section, we derive various decay properties of the global strong solution as $t \to \infty$ to prove the remaining part of Theorem 2.2 and Theorem 2.3. This will be done by applying effectively the L^q-L^r estimates. Recently Wiegner [59] has discussed in detail sharp decay properties of exterior Navier-Stokes flows. Our proof is somewhat different from his and seems to be elementary. When $a \in L^1(\Omega) \cap L^n_\sigma(\Omega)$, some decay rates are better than those shown by [59] since, unlike exterior Stokes flows, (1.4) is available for $1 \le q < n < r < \infty$.

Finally, we compare the result on ∇e^{-tA} with that for exterior Stokes flows from the viewpoint of coercive estimates of derivatives. For the proof of (1.4) there is another approach based on fractional powers of the Stokes operator. When Ω is an exterior domain ($n \geq 3$), Borchers and Miyakawa [4] developed such an approach and succeeded in the proof of

$$\|\nabla u\|_{L^q(\Omega)} \leq C\|A^{1/2}u\|_{L^q(\Omega)}, \quad u \in D(A_q^{1/2}), \tag{1.7}$$

for $1 < q < n$ (this restriction is optimal as pointed out by themselves [5]), which implies (1.4) for $q \leq r < n$. Independently, as mentioned, Iwashita [35] derived (1.4) for $q \leq r \leq n$ and, later, Maremonti and Solonnikov [46] showed that the restriction $r \leq n$ cannot be improved for exterior domains. In our case of aperture domains, we have (1.4) for $q < n < r < \infty$, which is a consequence of the estimate due to Farwig and Sohr ([20], Theorem 2.5)

$$\|\nabla^2 u\|_{L^q(\Omega)} \leq C\|Au\|_{L^q(\Omega)}, \quad u \in D(A_q), \tag{1.8}$$

for $1 < q < n$ together with an embedding property ([20], Lemma 3.1); we mention that (1.8) holds true for $n = 2$ as well. This argument does not work for the exterior problem because (1.8) is valid only for $1 < q < n/2$ ($n \geq 3$) as shown by Borchers and Sohr [7] (the restriction on q is again optimal by, for instance, [5]). Thus, as for (1.8), we have the better result. We wish we could expect (1.7) for every q, which would imply (1.4) for $1 < q \leq r < \infty$; however, so far, no attempts have been made at the boundedness of purely imaginary powers of the Stokes operator (see [27] and [30] for bounded and exterior domains) and, unless $q = 2$, estimate (1.7) remains open.

2 Results

Before stating our main results, we introduce notation used throught this paper. We denote upper and lower half spaces by $H_\pm = \{x \in \mathbb{R}^n; \pm x_n > 1\}$, and sometimes write $H = H_+$ or H_- to state some assertions for the half space. Set $B_R = \{x \in \mathbb{R}^n; |x| < R\}$ for $R > 0$. Let $\Omega \subset \mathbb{R}^n$ be a given aperture domain with smooth boundary $\partial\Omega$, namely, there is $R_0 > 1$ so that

$$\Omega \setminus B_{R_0} = (H_+ \cup H_-) \setminus B_{R_0};$$

in what follows we fix such R_0. Since Ω should be connected, there are some apertures and one can take two disjoint subdomains $\Omega_\pm$ and a smooth $(n-1)$-dimensional manifold M such that $\Omega = \Omega_+ \cup M \cup \Omega_-, \Omega_\pm \setminus B_{R_0} = H_\pm \setminus B_{R_0}$ and $M \cup \partial M = \partial\Omega_+ \cap \partial\Omega_- \subset \overline{B_{R_0}}$. We set $\Omega_R = \Omega \cap B_R$ and $H_R = H \cap B_R$, which is one of $H_{\pm,R} = H_\pm \cap B_R$, for $R > 1$.

For a domain $G \subset \mathbb{R}^n$, integer $j \geq 0$ and $1 \leq q \leq \infty$, we denote by $W^{j,q}(G)$ the standard L^q-Sobolev space with norm $\|\cdot\|_{j,q,G}$ so that $L^q(G) =$

$W^{0,q}(G)$ with norm $\|\cdot\|_{q,G}$. The space $W_0^{j,q}(G)$ is the completion of $C_0^\infty(G)$, the class of C^∞ functions having compact support in G, in the norm $\|\cdot\|_{j,q,G}$, and $W^{-j,q}(G)$ stands for its dual space with norm $\|\cdot\|_{-j,q,G}$. For simplicity, we use the abbreviations $\|\cdot\|_q$ for $\|\cdot\|_{q,\Omega}$ and $\|\cdot\|_{j,q}$ for $\|\cdot\|_{j,q,\Omega}$ when $G=\Omega$. We often use the same symbols for denoting the vector and scalar function spaces if there is no confusion. It is convenient to introduce a Banach space

$$L^q_{[R]}(G)=\{u\in L^q(G); \text{supp } u\subset \overline{G_R}\}, \quad G=\Omega \text{ or } H,$$

for $R>1$, where supp u denotes the support of the function u. For a Banach space X we denote by $B(X)$ the Banach space which consists of all bounded linear operators from X into itself.

Given $R\geq R_0$, we take (and fix) two cut-off functions $\psi_{\pm,R}$ satisfying

$$\psi_{\pm,R}\in C^\infty(\mathbb{R}^n;[0,1]), \quad \psi_{\pm,R}(x)=\begin{cases} 1 & \text{in } H_\pm\setminus B_{R+1}, \\ 0 & \text{in } H_\mp\cup B_R. \end{cases} \tag{2.1}$$

In some localization procedures with use of the cut-off functions above, the bounded domain of the form

$$D_{\pm,R}=\{x\in H_\pm; R<|x|<R+1\}$$

appears, and for this we need the following result of Bogovskiǐ [2] which provides a certain solution having an optimal regularity of the boundary value problem for $\nabla\cdot u=f$ with $u=0$ on the boundary (see also Borchers and Sohr [8], Theorem 2.4 (a)(b)(c) and Galdi [26], Chapter III): there is a linear operator $S_{\pm,R}$ from $C_0^\infty(D_{\pm,R})$ to $C_0^\infty(D_{\pm,R})^n$ such that for $1<q<\infty$ and integer $j\geq 0$

$$\|\nabla^{j+1}S_{\pm,R}f\|_{q,D_{\pm,R}}\leq C\|\nabla^j f\|_{q,D_{\pm,R}}, \tag{2.2}$$

with $C=C(R,q,j)>0$ independent of $f\in C_0^\infty(D_{\pm,R})$ (where ∇^j denotes all the j-th derivatives); and

$$\nabla\cdot S_{\pm,R}f=f,$$

for all $f\in C_0^\infty(D_{\pm,R})$ with $\int_{D_{\pm,R}}f(x)dx=0$. By (2.2) the operator $S_{\pm,R}$ extends uniquely to a bounded operator from $W_0^{j,q}(D_{\pm,R})$ to $W_0^{j+1,q}(D_{\pm,R})^n$.

For $G=\Omega, H$ and a smooth bounded domain ($n\geq 2$), let $C_{0,\sigma}^\infty(G)$ be the set of all solenoidal (divergence free) vector fields whose components belong to $C_0^\infty(G)$, and $L^q_\sigma(G)$ the completion of $C_{0,\sigma}^\infty(G)$ in the norm $\|\cdot\|_{q,G}$. If, in particular, $G=\Omega$, then the space $L^q_\sigma(\Omega)$ is characterized as (1.2). The space $L^q(G)$ of vector fields admits the Helmholtz decomposition

$$L^q(G)=L^q_\sigma(G)\oplus L^q_\pi(G), \quad 1<q<\infty,$$

with $L^q_\pi(G) = \{\nabla p \in L^q(G); p \in L^q_{loc}(\overline{G})\}$; see [25], [55] for bounded domains, [3], [48] for $G = H$ and [20], [49] for $G = \Omega$. Let $P_{q,G}$ be the projection operator from $L^q(G)$ onto $L^q_\sigma(G)$ associated with the decomposition above. Then the Stokes operator $A_{q,G}$ is defined by the solenoidal part of the Laplace operator, that is,

$$D(A_{q,G}) = W^{2,q}(G) \cap W^{1,q}_0(G) \cap L^q_\sigma(G), \quad A_{q,G} = -P_{q,G}\Delta,$$

for $1 < q < \infty$. The dual operator $A^*_{q,G}$ of $A_{q,G}$ coincides with $A_{q/(q-1),G}$ on $L^q_\sigma(\Omega)^* = L^{q/(q-1)}_\sigma(\Omega)$. We use, for simplicity, the abbreviations P_q for $P_{q,\Omega}$ and A_q for $A_{q,\Omega}$, and the subscript q is also often omitted if there is no confusion. The Stokes operator enjoys the parabolic resolvent estimate

$$\|(\lambda + A_G)^{-1}\|_{B(L^q_\sigma(G))} \le C_\varepsilon/|\lambda|, \tag{2.3}$$

for $|\arg\lambda| \le \pi - \varepsilon$ $(\lambda \ne 0)$, where $\varepsilon > 0$ is arbitrary; see [48], [3], [17], [18], [15] for $G = H$ and [20] for $G = \Omega$. Estimate (2.3) implies that the operator $-A_G$ generates a bounded analytic semigroup $\{e^{-tA_G}; t \ge 0\}$ of class (C_0) in each $L^q_\sigma(G), 1 < q < \infty$. We write $E(t) = e^{-tA_H}$, which is one of $E_\pm(t) = e^{-tA_{H_\pm}}$.

The first theorem provides the L^q-L^r estimates of the Stokes semigroup e^{-tA} for the aperture domain Ω.

Theorem 2.1 *Let $n \ge 3$.*

1. *Let $1 \le q \le r \le \infty$ $(q \ne \infty, r \ne 1)$. There is a constant $C = C(\Omega, n, q, r) > 0$ such that (1.3) holds for all $t > 0$ and $f \in L^q_\sigma(\Omega)$ unless $q = 1$; when $q = 1$, the assertion remains true if f is taken from $L^1(\Omega) \cap L^s_\sigma(\Omega)$ for some $s \in (1, \infty)$.*

2. *Let $1 \le q \le r \le n$ $(r \ne 1)$ or $1 \le q < n < r < \infty$. There is a constant $C = C(\Omega, n, q, r) > 0$ such that (1.4) holds for all $t > 0$ and $f \in L^q_\sigma(\Omega)$ unless $q = 1$; when $q = 1$, the assertion remains true if f is taken from $L^1(\Omega) \cap L^s_\sigma(\Omega)$ for some $s \in (1, \infty)$.*

3. *Let $1 < q < \infty$ and $f \in L^q_\sigma(\Omega)$. Then*

$$\|e^{-tA}f\|_r = o(t^{-\alpha}) \quad \begin{cases} \text{as } t \to 0 & \text{if } q < r \le \infty, \\ \text{as } t \to \infty & \text{if } q \le r \le \infty, \end{cases} \tag{2.4}$$

$$\|\nabla e^{-tA}f\|_r = o(t^{-\alpha-1/2}) \tag{2.5}$$

$$\begin{cases} \text{as } t \to 0 & \text{if } q \le r \le \infty, \\ \text{as } t \to \infty & \text{if } q \le r \le n, \ q < n < r < \infty, \end{cases}$$

where $\alpha = (n/q - n/r)/2$. Furthermore, for each precompact set K in $L^q_\sigma(\Omega)$ every convergence above is uniform with respect to $f \in K$.

Remark 2.1. Estimate (1.4) for large t is not proved in the following cases: (i) $n < q = r < \infty$, (ii) $n \le q < r < \infty$. For the case (i), the decay rate $t^{-n/2q}$ will be shown in Lemma 5.4. Since we have (1.4) for $q < n < r < \infty$, a better decay rate than $t^{-n/2q}$ can be derived for the case (ii) through an interpolation argument; however, we do not know optimal decay rates of ∇e^{-tA} in both the cases (i) and (ii). According to Maremonti and Solonnikov [46], the decay rate $t^{-n/2q}$ is optimal for exterior Stokes flows whenever $r > n$.

Remark 2.2. Let $1 \le q \le r \le \infty$ $(q \ne \infty, r \ne 1)$. The L^q-L^r estimate for $\partial_t e^{-tA}$ with the rate $t^{-\alpha-1}$ is nothing but a simple corollary to (1.3). In fact, for example,

$$\|\partial_t e^{-tA} f\|_\infty \le Ct^{-n/2s}\|Ae^{-(t/2)A} f\|_s \le Ct^{-n/2-1}\|f\|_1,$$

for $t > 0$ and $f \in L^1(\Omega) \cap L^s_\sigma(\Omega)$.

By use of the Stokes operator A, one can formulate the problem (1.1) subject to the vanishing flux condition

$$\phi(u(t)) = \int_M N \cdot u(t) d\sigma = 0, \quad t \ge 0, \tag{2.6}$$

as the Cauchy problem

$$\partial_t u + Au + P(u \cdot \nabla u) = 0, \quad t > 0; u(0) = a, \tag{2.7}$$

in $L^q_\sigma(\Omega)$. Given $a \in L^n_\sigma(\Omega)$ and $0 < T \le \infty$, a measurable function u defined on $\Omega \times (0, T)$ is called a strong solution of (1.1) with (2.6) on $(0, T)$ if u is of class

$$u \in C([0, T); L^n_\sigma(\Omega)) \cap C(0, T; D(A_n)) \cap C^1(0, T; L^n_\sigma(\Omega))$$

together with $\lim_{t\to 0} \|u(t) - a\|_n = 0$ and satisfies (2.7) for $0 < t < T$ in $L^n_\sigma(\Omega)$.

The next theorem tells us the global existence of a strong solution with several decay properties provided that $\|a\|_n$ is small enough.

Theorem 2.2 *Let $n \ge 3$. There is a constant $\delta = \delta(\Omega, n) > 0$ with the following property: if $a \in L^n_\sigma(\Omega)$ satisfies $\|a\|_n \le \delta$, then the problem (1.1) with (2.6) admits a unique strong solution $u(t)$ on $(0, \infty)$, which enjoys*

$$\|u(t)\|_r = o\left(t^{-1/2+n/2r}\right) \quad \text{for } n \le r \le \infty, \tag{2.8}$$

$$\|\nabla u(t)\|_n = o\left(t^{-1/2}\right), \tag{2.9}$$

$$\|\partial_t u(t)\|_n + \|Au(t)\|_n = o\left(t^{-1}\right), \tag{2.10}$$

as $t \to \infty$.

Remark 2.3. When one prescribes a nontrivial flux

$$\phi(u(t)) = F(t) \in C^{1,\theta}([0,T]),$$

with some $\theta > 0$ and $T > 0$, there is $T_* \in (0,T]$ such that the problem (1.1) with the flux condition admits a unique strong solution on $(0,T_*)$ provided that $a \in L^n(\Omega)$ satisfies the compatibility conditions $\nabla \cdot a = 0, \nu \cdot a|_{\partial\Omega} = 0$ and $\phi(a) = F(0)$. This improves a related result of Franzke [23] and can be proved in the same manner as the proof of Theorem 2.2 with the aid of the auxiliary function of Heywood ([31], Lemma 11), which is used for the reduction of the problem to an equivalent one with the vanishing flux condition (2.6). As is well known, (2.4) and (2.5) as $t \to 0$ play an important role for the construction of the above local solution.
Remark 2.4. The solution obtained in Theorem 2.2 is unique within the class

$$u \in C([0,\infty); L^n_\sigma(\Omega)), \quad \nabla u \in C(0,\infty; L^n(\Omega)),$$

without assuming any behavior near $t = 0$ as pointed out by Brezis [10]. For the proof, one needs the final assertion of Theorem 2.1 on the uniform behavior of the semigroup as $t \to 0$ on each precompact set K in $L^n_\sigma(\Omega)$ together with the theory of local strong solutions mentioned in the previous remark (with $\phi(u) = F = 0$). In fact, it follows from the above property of the semigroup that the length of the existence interval of the local solution can be taken uniformly with respect to $a \in K$ and that the convergence (6.4) of the local solution as $t \to 0$ is also uniform with respect to $a \in K$. These two facts combined with the classical uniqueness theorem of Fujita-Kato type [24] (assuming some behaviors in (6.4) near $t = 0$) imply the desired uniqueness result.
Remark 2.5. Consider the 3-dimensional stationary Navier-Stokes problem

$$w \cdot \nabla w = \Delta w - \nabla \pi, \quad \nabla \cdot w = 0,$$

in Ω subject to $w|_{\partial\Omega} = 0$ and a nontrivial flux condition $\phi(w) = \gamma \in \mathbb{R}$. When $|\gamma|$ is small enough, there is a unique solution w such that $w \in L^q(\Omega)$ for $3/2 < q \leq 6$ and $\nabla w \in L^r(\Omega)$ for $1 < r \leq 2$ with $\|\nabla w\|_2^2 = \gamma[\pi]$; see Galdi [26]. By use of Theorem 2.1 it is possible to show the asymptotic stability of the small stationary solution w of the class above for small initial disturbance in $L^3_\sigma(\Omega)$ in the sense that the disturbance $u(t)$ decays like (2.8) and (2.9) as $t \to \infty$. In fact, the above summability properties of ∇w allow us to deal with the term $P(w \cdot \nabla u + u \cdot \nabla w)$ as a simple perturbation of the Stokes operator, as was done by Chen [11] (Lemma 3.1) and Borchers and Miyakawa [6] (Theorem 3.13); see Remark 6.1.

The final theorem shows further decay properties of the global solution when we additionally impose L^1-summability on the initial data.

Theorem 2.3 *Let $n \geq 3$. There is a constant $\eta = \eta(\Omega, n) \in (0, \delta]$ with the following property: if $a \in L^1(\Omega) \cap L^n_\sigma(\Omega)$ satisfies $\|a\|_n \leq \eta$, then the solution $u(t)$ obtained in Theorem 2.2 and the associated pressure $p(t)$ enjoy*

$$\|u(t)\|_r = O\left(t^{-(n-n/r)/2}\right) \quad \text{for } 1 < r \leq \infty, \tag{2.11}$$

$$\|\nabla u(t)\|_r = O\left(t^{-(n-n/r)/2-1/2}\right) \quad \text{for } 1 < r < \infty, \tag{2.12}$$

$$\|\partial_t u(t)\|_r + \|Au(t)\|_r = O\left(t^{-(n-n/r)/2-1}\right) \quad \text{for } 1 < r < \infty, \tag{2.13}$$

$$\|\nabla^2 u(t)\|_r + \|\nabla p(t)\|_r = O\left(t^{-(n-n/r)/2-1}\right) \quad \text{for } 1 < r < n, \tag{2.14}$$

as $t \to \infty$. Moreover, for each $t > 0$ there exist two constants $p_\pm(t) \in \mathbb{R}$ such that $p(t) - p_\pm(t) \in L^r(\Omega_\pm)$ with

$$\|p(t) - p_\pm(t)\|_{r,\Omega_\pm} + |[p(t)]| = O\left(t^{-(n-n/r)/2-1/2}\right) \text{ for } n/(n-1) < r < \infty, \tag{2.15}$$

as $t \to \infty$, where $[p(t)] = p_+(t) - p_-(t)$.

Remark 2.6. Indeed $\nabla u(t) \in L^r(\Omega)$ for $r > n$ even in Theorem 2.2, but we have asserted nothing about their decay rates since they do not seem to be optimal; see Remark 2.1 for the Stokes flow. On the other hand, in Theorem 2.3 the decay rates of $\nabla u(t)$ in $L^r(\Omega)$ for $r > n$ are better than $t^{-n/2}$ for exterior Navier-Stokes flows shown by Wiegner [59]. Taking Theorem 5.1 of [15] for the Stokes flow in the half space into account, we would not expect $u(t) \in L^1(\Omega)$ in general. Thus the decay rates obtained in Theorem 2.3 seem to be optimal; that is, for example, $\|u(t)\|_\infty = o(t^{-n/2})$ would not hold true. Concerning the exterior problem, Kozono [38], [39] made it clear that the Stokes and/or Navier-Stokes flows possess L^1-summability and more rapid decay properties than (2.11) only in a special situation.

Remark 2.7. In Theorem 2.2 one could not define a pressure drop (see Farwig [16], Remark 2.2) since the solution never belongs to $L^r(\Omega)$ for $r < n$. Due to the additional summability assumption on the initial data, we obtain in Theorem 2.3 the pressure drop written in the form

$$[p(t)] = \int_\Omega (\partial_t u + u \cdot \nabla u - u)(t) \cdot w dx,$$

where $w \in W^{2,q}(\Omega)$, $n/(n-1) < q < \infty$, is a unique solution (given by [20]) of the auxiliary problem

$$w - \Delta w + \nabla \pi = 0, \quad \nabla \cdot w = 0,$$

in Ω subject to $w|_{\partial\Omega} = 0$ and $\phi(w) = 1$. In fact, the formula above is derived from the relations

$$\int_\Omega w \cdot \nabla p(t) dx = -[p(t)]\phi(w) = -[p(t)],$$

$$\int_\Omega u(t) \cdot \nabla \pi dx = -[\pi]\phi(u(t)) = 0.$$

3 The Stokes resolvent for the half space

The resolvent $v = (\lambda + A_H)^{-1} P_H f$ together with the associated pressure π solves the system

$$\lambda v - \Delta v + \nabla \pi = f, \quad \nabla \cdot v = 0,$$

in the half space $H = H_+$ or H_- subject to $v|_{\partial H} = 0$ for the external force $f \in L^q(H), 1 < q < \infty$, and $\lambda \in \mathbb{C} \setminus (-\infty, 0]$. In this section we are concerned with the analysis of v near $\lambda = 0$. Our method is quite different from Abels [1]. One needs the following local energy decay estimate of the semigroup $E(t) = e^{-tA_H}$, which is a simple consequence of (1.3) for $\Omega = H$.

Lemma 3.1 *Let $n \geq 2, 1 < q < \infty, d > 1$ and $R > 1$. For any small $\varepsilon > 0$ and integer $k \geq 0$ there is a constant $C = C(n, q, d, R, \varepsilon, k) > 0$ such that*

$$\|\nabla^j \partial_t^k E(t) P_H f\|_{q,H_R} \leq C t^{-j/2-k} (1+t)^{-n/2+\varepsilon} \|f\|_{q,H}, \tag{3.1}$$

for $t > 0, f \in L^q_{[d]}(H)$ and $j = 0, 1, 2$.

Proof. We make use of the estimate

$$\|\nabla^j u\|_{r,H} \leq C \|A_H^{j/2} u\|_{r,H}, \quad u \in D(A_{r,H}^{j/2}), \tag{3.2}$$

for $1 < r < \infty$ and $j = 1, 2$ (Borchers and Miyakawa [3]). For $1 < p \le q \le r < \infty$ it follows from (1.3) for $\Omega = H$, (3.2) and a property of the analytic semigroup that

$$\begin{aligned} \|\nabla^j \partial_t^k E(t) P_H f\|_{q,H_R} &\le C\|A_H^{j/2+k} E(t) P_H f\|_{r,H} \\ &\le Ct^{-j/2-k}\|E(t/2) P_H f\|_{r,H} \\ &\le Ct^{-j/2-k-(n/p-n/r)/2}\|f\|_{p,H} \\ &\le Ct^{-j/2-k-(n/p-n/r)/2}\|f\|_{q,H}, \end{aligned}$$

for $t > 0, f \in L^q_{[d]}(H)$ and $j = 0, 1, 2$. This estimate with $p = q = r$ implies (3.1) for $0 < t < 1$. We may assume that $0 < \varepsilon/n < \min\{1/q, 1 - 1/q\}$; and then one can take p and r so that $1 - 1/p = 1/r = \varepsilon/n$ and $p < q < r$. Then the estimate above yields (3.1) for $t \ge 1$. This completes the proof. □

Lemma 3.1 is sufficient for our analysis of the resolvent in this section, but the local energy decay estimate of the following form will be used in section 5.

Lemma 3.2 *Let $n \ge 2, 1 < q < \infty$ and $R > 1$. Then there is a constant $C = C(n, q, R) > 0$ such that*

$$\|E(t)f\|_{2,q,H_R} + \|\partial_t E(t)f\|_{q,H_R} \le C(1+t)^{-n/2q}\|f\|_{D(A_{q,H})}, \tag{3.3}$$

for $t \ge 0$ and $f \in D(A_{q,H})$.

Proof. The left hand side of (3.3) is bounded from above by

$$C(\|A_H E(t)f\|_{q,H} + \|E(t)f\|_{q,H}) \le C\|f\|_{D(A_{q,H})},$$

which implies (3.3) for $0 \le t < 1$. For $t \ge 1$ it follows from (1.3) for $\Omega = H$ with $r = \infty$ that

$$\|E(t)f\|_{q,H_R} \le C\|E(t)f\|_{\infty,H} \le Ct^{-n/2q}\|f\|_{q,H}.$$

The other terms

$$\|\nabla^j E(t)f\|_{q,H_R} \le C\|A_H^{j/2} E(t)f\|_{r,H} \le Ct^{-j/2}\|E(t/2)f\|_{r,H} \quad (j = 1, 2),$$

$$\|\partial_t E(t)f\|_{q,H_R} \le Ct^{-1}\|E(t/2)f\|_{r,H},$$

decay more rapidly since we can take $r \in (q, \infty)$ above as large as we want. The proof is complete. □

We next employ Lemma 3.1 to show some regularity estimates near $\lambda = 0$ of the Stokes resolvent in the localized space $W^{2,q}(H_R)$.

Lemma 3.3 *Let $n \geq 3, 1 < q < \infty, d > 1$ and $R > 1$. Given $f \in L^q_{[d]}(H)$, set $v(\lambda) = (\lambda + A_H)^{-1} P_H f$. For any small $\varepsilon > 0$ there is a constant $C = C(n, q, d, R, \varepsilon) > 0$ such that*

$$|\lambda|^{\beta} \|\partial_\lambda^m v(\lambda)\|_{2,q,H_R} + \sum_{k=0}^{m-1} \|\partial_\lambda^k v(\lambda)\|_{2,q,H_R} \leq C\|f\|_{q,H}, \tag{3.4}$$

for Re $\lambda \geq 0$ $(\lambda \neq 0)$ and $f \in L^q_{[d]}(H)$, where

$$m = \begin{cases} (n-1)/2 & \text{if } n \text{ is odd,} \\ n/2 - 1 & \text{if } n \text{ is even,} \end{cases}$$

$$\beta = \beta(\varepsilon) = 1 + m - \frac{n}{2} + \varepsilon = \begin{cases} 1/2 + \varepsilon & \text{if } n \text{ is odd,} \\ \varepsilon & \text{if } n \text{ is even.} \end{cases}$$

Furthermore, we have

$$\sup \left\{ \frac{\|v(\lambda) - w\|_{2,q,H_R}}{\|f\|_{q,H}};\ f \neq 0, f \in L^q_{[d]}(H) \right\} \to 0, \tag{3.5}$$

as $\lambda \to 0$ with Re $\lambda \geq 0$, where

$$w = \int_0^\infty E(t) P_H f dt.$$

Proof. We recall the formula

$$v(\lambda) = (\lambda + A_H)^{-1} P_H f = \int_0^\infty e^{-\lambda t} E(t) P_H f dt, \tag{3.6}$$

which is valid in $L^q_\sigma(H)$ for Re $\lambda > 0$ and $f \in L^q(H)$. In the other region $\{\lambda \in \mathbb{C} \setminus (-\infty, 0]; \text{Re } \lambda \leq 0\}$ we usually utilize the analytic extension of the semigroup $\{E(t); \text{Re } t > 0\}$ to obtain the similar formula. For the case Re $\lambda = 0$ $(\lambda \neq 0)$ which is important for us, however, thanks to the local energy decay property (3.1), the formula (3.6) remains valid in the localized space $L^q(H_R)$ for $f \in L^q_{[d]}(H)$ (the function w in (3.5) is well-defined in $L^q(H_R)$ by the same reasoning). We thus obtain from (3.1)

$$\|\nabla^j \partial_\lambda^k v(\lambda)\|_{q,H_R} \leq \int_0^\infty t^k \|\nabla^j E(t) P_H f\|_{q,H_R} dt \leq C\|f\|_{q,H},$$

provided that

$$\begin{array}{ll} j = 0, 1 \quad \text{if } k = 0; \quad j = 0, 1, 2 \quad \text{if } n \geq 5, \ 1 \leq k \leq m-1; \\ j = 2 \quad \text{if } k = m, n = 2m+1; \quad j = 1, 2 \quad \text{if } k = m, n = 2m+2. \end{array}$$

For $\{k, j\} = \{0, 2\}$ we have only to use (3.2) together with (2.3) to see that

$$\|\nabla^2 v(\lambda)\|_{q,H_R} \leq C\|A_H(\lambda + A_H)^{-1}P_H f\|_{q,H} \leq C\|f\|_{q,H}.$$

The remaining case $k = m$ is the most important part of (3.4). Since

$$\begin{aligned} \|\partial_\lambda^m v(\lambda)\|_{2,q,H_R} &\leq Cm!\|(\lambda + A_H)^{-(m+1)}P_H f\|_{D(A_{q,H})} \\ &\leq Cm!\{|\lambda|^{-m} + |\lambda|^{-(m+1)}\}\|f\|_{q,H}, \end{aligned}$$

we have the assertion for $|\lambda| \geq 1$. For $0 < |\lambda| < 1$ and odd n (resp. even n), we have already shown the estimate as above when $j = 2$ (resp. $j = 1, 2$). Thus, let $j = 0$ or 1 for $n = 2m+1$ and $j = 0$ for $n = 2m+2$. We divide the integral of (3.6) into two parts

$$\partial_\lambda^m v(\lambda) = \left\{ \int_0^{1/|\lambda|} + \int_{1/|\lambda|}^\infty \right\} e^{-\lambda t}(-t)^m E(t) P_H f \, dt = w_1(\lambda) + w_2(\lambda).$$

Then (3.1) implies

$$\|\nabla^j w_1(\lambda)\|_{q,H_R} \leq C|\lambda|^{-\beta + j/2}\|f\|_{q,H},$$

for $f \in L^q_{[d]}(H)$. On the other hand, by integration by parts we get

$$w_2(\lambda) = \frac{e^{-\lambda/|\lambda|}}{\lambda} \left(\frac{-1}{|\lambda|} \right)^m E\left(\frac{1}{|\lambda|} \right) P_H f + \int_{1/|\lambda|}^\infty \frac{e^{-\lambda t}}{\lambda} \partial_t \left[(-t)^m E(t) P_H f \right] dt,$$

in $L^q(H_R)$ since (3.1) implies $\lim_{t \to \infty} t^m \|E(t) P_H f\|_{q,H_R} = 0$. With the aid of (3.1) again we see that

$$\begin{aligned} &\|\nabla^j w_2(\lambda)\|_{q,H_R} \\ \leq\ & \frac{1}{|\lambda|^{m+1}} \|\nabla^j E(1/|\lambda|) P_H f\|_{q,H_R} + \frac{1}{|\lambda|} \int_{1/|\lambda|}^\infty \|\nabla^j \partial_t \left[t^m E(t) P_H f \right] \|_{q,H_R} dt \\ \leq\ & C|\lambda|^{-\beta + j/2}\|f\|_{q,H}, \end{aligned}$$

for $f \in L^q_{[d]}(H)$. Collecting the estimates above leads us to (3.4). We next show (3.5). Since $|e^{-\lambda t} - 1| \leq 2^{1-\theta}|\lambda|^\theta t^\theta$ for $\mathrm{Re}\,\lambda \geq 0$ and $\theta \in (0, 1]$, we have

$$\|\nabla^j (v(\lambda) - w)\|_{q,H_R} \leq 2^{1-\theta}|\lambda|^\theta \int_0^\infty t^\theta \|\nabla^j E(t) P_H f\|_{q,H_R} dt,$$

for $j = 0, 1, 2$. From (3.1) together with a suitable choice of θ (for instance, $\theta < 1/2$ for $n = 3$), we conclude (3.5). □

Remark 3.1. When $n = 2$, one can show $|\lambda|^\beta \|v\|_{2,q,H_R} \le C\|f\|_{q,H}$ (with $\beta = \varepsilon$) which corresponds to (3.4) with $m = 0$. However, this will not help us since our key formula (5.3) is not valid for $m = 0$.
Remark 3.2. The Green tensor associated with the Stokes semigroup $E(t)$ for the half space (as well as the projection P_H) was explicitly given by Solonnikov [56], Maremonti and Solonnikov [46] (Section 2). In view of the simple relation $\int_0^\infty (4\pi t)^{-n/2} e^{-|x|^2/4t} dt = \Gamma(n/2)/2(n-2)\pi^{n/2}|x|^{n-2}$ for $n \ge 3$, the function w in (3.5) is the solution written by the Green tensor for the stationary Stokes problem in H and, thereby, we know the class of w (for the latter Green tensor, see for instance [26]).

Finally, we derive further information on the regularity of the resolvent along the imaginary axis.

Lemma 3.4 *Let $n \ge 3, 1 < q < \infty, d > 1$ and $R > 1$. Set*

$$\Phi_H^{(k)}(s) = \partial_s^k (is + A_H)^{-1} P_H \quad (s \in \mathbb{R} \setminus \{0\},\ k = m \text{ or } m-1),$$

where $i = \sqrt{-1}$. Then, for any small $\varepsilon > 0$, there is a constant $C = C(n, q, d, R, \varepsilon) > 0$ such that

$$\|\Phi_H^{(m)}(s+h)f - \Phi_H^{(m)}(s)f\|_{2,q,H_R} \le C|h||s|^{-\beta-1}\|f\|_{q,H}, \tag{3.7}$$

$$\|\Phi_H^{(m-1)}(s+h)f - \Phi_H^{(m-1)}(s)f\|_{2,q,H_R} \le C|h||s|^{-\beta}\|f\|_{q,H}, \tag{3.8}$$

for $h \in \mathbb{R}, |s| > 2|h|$ and $f \in L^q_{[d]}(H)$, where m and $\beta = \beta(\varepsilon)$ are the same as in Lemma 3.3.

Proof. Estimate (3.8) is a direct consequence of (3.4). In fact, we see that

$$\begin{aligned} \|\Phi_H^{(m-1)}(s+h)f - \Phi_H^{(m-1)}(s)f\|_{2,q,H_R} &\le \left| \int_s^{s+h} \|\Phi_H^{(m)}(\tau)f\|_{2,q,H_R} d\tau \right| \\ &\le C\|f\|_{q,H} \left| \int_s^{s+h} |\tau|^{-\beta} d\tau \right|, \end{aligned}$$

which together with the relation $|s+h| \ge |s| - |h| \ge |s|/2$ implies (3.8). We next show (3.7). By (3.6) with Re $\lambda = 0$ in $L^q(H_R)$ we have

$$\Phi_H^{(m)}(s+h)f - \Phi_H^{(m)}(s)f$$

$$= (-i)^m \left\{ \int_0^{1/|s|} + \int_{1/|s|}^\infty \right\} e^{-ist}(e^{-iht} - 1) t^m E(t) P_H f dt = (-i)^m (w_1 + w_2).$$

For the convenience we introduce the function

$$F_k(t) = \partial_t^k [t^m E(t) P_H f], \quad k \ge 0.$$

We then deduce from (3.1)

$$\|F_k(t)\|_{2,q,H_R} \leq Ct^{-k+m-1}(1+t)^{-n/2+1+\varepsilon}\|f\|_{q,H}, \tag{3.9}$$

for $t > 0$ and $f \in L^q_{[d]}(H)$. Taking $|e^{-iht} - 1| \leq |h|t$ into account, we see from (3.9) that

$$\begin{aligned}
\|w_1\|_{2,q,H_R} &\leq |h| \int_0^{1/|s|} t\|F_0(t)\|_{2,q,H_R} dt \\
&\leq C|h|\|f\|_{q,H} \int_0^{1/|s|} t^{1+m-n/2+\varepsilon} dt \\
&\leq C|h||s|^{-\beta-1}\|f\|_{q,H},
\end{aligned}$$

for $f \in L^q_{[d]}(H)$. By integration by parts we split $w_2 = w_{21} + w_{22} + w_{23}$, where

$$w_{21} = \frac{ih}{s(s+h)} e^{-i(s+h)/|s|} F_0\left(\frac{1}{|s|}\right) - \frac{i}{s} e^{-is/|s|}(e^{-ih/|s|} - 1) F_0\left(\frac{1}{|s|}\right),$$

$$w_{22} = \frac{ih}{s(s+h)} \int_{1/|s|}^{\infty} e^{-i(s+h)t} F_1(t) dt,$$

$$w_{23} = \frac{-i}{s} \int_{1/|s|}^{\infty} e^{-ist}(e^{-iht} - 1) F_1(t) dt.$$

Since $1/|s(s+h)| \leq 2/|s|^2$ for $|s| > 2|h|$, it follows from (3.9) that

$$\begin{aligned}
\|w_{21}\|_{2,q,H_R} &\leq 3|h||s|^{-2}\|F_0(1/|s|)\|_{2,q,H_R} \\
&\leq C|h||s|^{-2-m+n/2-\varepsilon}(1+|s|)^{-n/2+1+\varepsilon}\|f\|_{q,H} \\
&\leq C|h||s|^{-\beta-1}\|f\|_{q,H},
\end{aligned}$$

and that

$$\begin{aligned}
\|w_{22}\|_{2,q,H_R} &\leq 2|h||s|^{-2} \int_{1/|s|}^{\infty} \|F_1(t)\|_{2,q,H_R} dt \\
&\leq C|h||s|^{-2}\|f\|_{q,H} \int_{1/|s|}^{\infty} t^{-1+m-n/2+\varepsilon} dt \\
&\leq C|h||s|^{-\beta-1}\|f\|_{q,H},
\end{aligned}$$

for $f \in L^q_{[d]}(H)$. We perform integration by parts once more to obtain $w_{23} = w_{231} + w_{232} + w_{233}$ with

$$w_{231} = \frac{h}{s^2(s+h)} e^{-i(s+h)/|s|} F_1\left(\frac{1}{|s|}\right) - \frac{1}{s^2} e^{-is/|s|}(e^{-ih/|s|} - 1) F_1\left(\frac{1}{|s|}\right),$$

$$w_{232} = \frac{h}{s^2(s+h)} \int_{1/|s|}^{\infty} e^{-i(s+h)t} F_2(t)dt,$$

$$w_{233} = \frac{-1}{s^2} \int_{1/|s|}^{\infty} e^{-ist}(e^{-iht} - 1)F_2(t)dt.$$

By the same way as in $w_{21} + w_{22}$ we find

$$\begin{aligned}
&\|w_{231} + w_{232}\|_{2,q,H_R} \\
\leq\ & 3|h||s|^{-3} \left\{ \|F_1(1/|s|)\|_{2,q,H_R} + \int_{1/|s|}^{\infty} \|F_2(t)\|_{2,q,H_R} dt \right\} \\
\leq\ & C|h||s|^{-\beta-1}\|f\|_{q,H},
\end{aligned}$$

for $f \in L^q_{[d]}(H)$. Finally, we use (3.9) again to get

$$\begin{aligned}
\|w_{233}\|_{2,q,H_R} &\leq |h||s|^{-2} \int_{1/|s|}^{\infty} t\|F_2(t)\|_{2,q,H_R} dt \\
&\leq C|h||s|^{-2}\|f\|_{q,H} \int_{1/|s|}^{\infty} t^{-1+m-n/2+\varepsilon} dt \\
&\leq C|h||s|^{-\beta-1}\|f\|_{q,H},
\end{aligned}$$

for $f \in L^q_{[d]}(H)$. We gather all the estimates above to conclude (3.7). □

Remark 3.3. Estimate (3.7) together with (3.4) implies

$$\int_{-\infty}^{\infty} \|\Phi_H^{(m)}(s+h)f - \Phi_H^{(m)}(s)f\|_{2,q,H_R} ds \leq C|h|^{1-\beta}\|f\|_{q,H},$$

for $h \in \mathbb{R}$ and $f \in L^q_{[d]}(H)$ (see Lemma 4.4 and its proof), which is related to the assumption of Lemma 5.2. In Lemma 4.4 we will deduce the same regularity of $\partial_s^m (is + A)^{-1}Pf$ for an aperture domain Ω as above when $f \in L^q(\Omega)$ has a bounded support. For the Oseen resolvent system in the 3-dimensional whole space, Kobayashi and Shibata [37] (Lemma 3.6) showed a sharper estimate; indeed, $|h|^{1-\beta}$ can be replaced by $|h|^{1/2}$. Their method is different from ours.

4 The Stokes resolvent

In this section, based on the results for the half space obtained in the previous section, we address ourselves to analogous regularity estimates near $\lambda = 0$ of the Stokes resolvent $u = (\lambda + A)^{-1}Pf$, which together with the associated pressure p satisfies the system

$$\lambda u - \Delta u + \nabla p = f, \quad \nabla \cdot u = 0,$$

in an aperture domain Ω subject to $u|_{\partial\Omega} = 0$ and $\phi(u) = 0$, where $f \in L^q(\Omega), 1 < q < \infty$ and $\lambda \in \mathbb{C} \setminus (-\infty, 0]$. To this end, as in [35], [37] and [1], we start with the construction of the resolvent near $\lambda = 0$ for $f \in L^q(\Omega)$ with bounded support. We fix a smooth bounded subdomain D so that $\Omega_{R_0+3} \subset D \subset \Omega$. Given $f \in L^q(\Omega)$, we set $v_0 = A_{q,D}^{-1} P_{q,D} f$ and take a pressure π_0 associated to v_0; they solve the Stokes system

$$-\Delta v_0 + \nabla \pi_0 = f, \quad \nabla \cdot v_0 = 0,$$

in D subject to $v_0|_{\partial D} = 0$, where f is understood as the restriction of f on D. We further set

$$v_\pm(x,\lambda) = (\lambda + A_{q,H_\pm})^{-1} P_{q,H_\pm}[\psi_{\pm,R_0} f],$$

where $\psi_{\pm,R_0}$ are the cut-off functions given by (2.1). One needs also the case $\lambda = 0$

$$v_\pm(x,0) = \int_0^\infty E_\pm(t) P_{q,H_\pm}[\psi_{\pm,R_0} f]dt,$$

which is the solution written by the Green tensor for the Stokes problem in $H_\pm$ (see Remark 3.2). We take the pressures $\pi_\pm$ in $H_\pm$ associated to $v_\pm$ so that

$$\int_{D_{\pm,R_0+1}} \{\pi_\pm(x,\lambda) - \pi_0(x)\}dx = 0, \tag{4.1}$$

for each λ. In this section, for simplicity, we use the abbreviations $\psi_\pm$ for the cut-off functions $\psi_{\pm,R_0+1}$ given by (2.1) and $S_\pm$ for the Bogovskiǐ operators $S_{\pm,R_0+1}$ introduced in section 2. With use of $\{v_\pm, \pi_\pm\}, \{v_0, \pi_0\}$ and $\psi_\pm$ together with $S_\pm$, we set

$$\left\{ \begin{array}{ll} v & = T(\lambda)f \\ & = \psi_+ v_+ + \psi_- v_- + (1 - \psi_+ - \psi_-)v_0 \\ & \quad -S_+[(v_+ - v_0)\cdot \nabla \psi_+] - S_-[(v_- - v_0)\cdot \nabla \psi_-], \\ \pi & = \psi_+ \pi_+ + \psi_- \pi_- + (1 - \psi_+ - \psi_-)\pi_0. \end{array} \right. \tag{4.2}$$

We here note that $\int_{D_{\pm,R_0+1}} (v_\pm - v_0)\cdot \nabla \psi_\pm dx = 0$ since $\nabla \cdot v_\pm = \nabla \cdot v_0 = 0$. An elementary calculation shows that the pair $\{v, \pi\}$ satisfies

$$\lambda v - \Delta v + \nabla \pi = f + Q(\lambda)f, \quad \nabla \cdot v = 0, \tag{4.3}$$

in Ω subject to $v|_{\partial\Omega} = 0$ and

$$\phi(v) = \int_M N \cdot v_0 d\sigma = \int_{\Omega_+ \cap D} \nabla \cdot v_0 dx = 0,$$

where

$$Q(\lambda)f = Q_1(\lambda)f + Q_2(\lambda)f \tag{4.4}$$

with

$$\begin{aligned} Q_1(\lambda)f \ &= \lambda(1-\psi_+-\psi_-)v_0 - 2\nabla\psi_+\cdot\nabla(v_+-v_0) - 2\nabla\psi_-\cdot\nabla(v_--v_0) \\ &\quad -(\Delta\psi_+)(v_+-v_0) - (\Delta\psi_-)(v_--v_0) \\ &\quad +(\nabla\psi_+)(\pi_+-\pi_0) + (\nabla\psi_-)(\pi_--\pi_0) \\ &\quad -\lambda S_+[(v_+-v_0)\cdot\nabla\psi_+] - \lambda S_-[(v_--v_0)\cdot\nabla\psi_-], \end{aligned}$$

and

$$Q_2(\lambda)f = \Delta S_+[(v_+-v_0)\cdot\nabla\psi_+] + \Delta S_-[(v_--v_0)\cdot\nabla\psi_-].$$

By (2.2) we have $S_\pm[(v_\pm - v_0)\cdot\nabla\psi_\pm] \in W_0^{2,q}(D_{\pm,R_0+1})$. But one can obtain the regularity of this term only up to $W_0^{2,q}$ (unlike the exterior problem) and this is the reason why the remaining term $Q(\lambda)$ has been divided into two parts.

We first derive the regularity estimates near $\lambda = 0$ of $T(\lambda)$ and $Q(\lambda)$.

Lemma 4.1 *Let $n \geq 3, 1 < q < \infty, d \geq R_0$ and $R \geq R_0$. For any small $\varepsilon > 0$ there are constants $C_1 = C_1(\Omega, n, q, d, R, \varepsilon) > 0$ and $C_2 = C_2(\Omega, n, q, d, \varepsilon) > 0$ such that*

$$|\lambda|^\beta \|\partial_\lambda^m T(\lambda)f\|_{2,q,\Omega_R} + \sum_{k=0}^{m-1} \|\partial_\lambda^k T(\lambda)f\|_{2,q,\Omega_R} \leq C_1\|f\|_q, \tag{4.5}$$

for Re $\lambda \geq 0$ $(\lambda \neq 0)$ and $f \in L^q_{[d]}(\Omega)$; and

$$|\lambda|^\beta \|\partial_\lambda^m Q(\lambda)f\|_q + \sum_{k=0}^{m-1} \|\partial_\lambda^k Q(\lambda)f\|_q \leq C_2\|f\|_q, \tag{4.6}$$

for Re $\lambda \geq 0$ with $0 < |\lambda| \leq 2$ and $f \in L^q_{[d]}(\Omega)$, where m and $\beta = \beta(\varepsilon)$ are the same as in Lemma 3.3.

Proof. In view of (4.2), we deduce (4.5) immediately from (3.4) together with (2.2). One can show (4.6) likewise, but it remains to estimate the pressures $\pi_\pm$ contained in (4.4). By (4.1) we have

$$\int_{D_{\pm,R_0+1}} \partial_\lambda^k \pi_\pm(x,\lambda)dx = 0, \quad 1 \leq k \leq m. \tag{4.7}$$

On the other hand, from the Stokes resolvent system we obtain

$$\lambda\partial_\lambda^k v_\pm + k\partial_\lambda^{k-1}v_\pm - \Delta\partial_\lambda^k v_\pm + \nabla\partial_\lambda^k\pi_\pm = 0, \quad 1\le k\le m,$$

in $H_\pm$. This combined with (4.7) gives

$$\begin{aligned} & \|(\nabla\psi_\pm)\partial_\lambda^k\pi_\pm(\lambda)\|_q \\ \le\ & C\|\nabla\partial_\lambda^k\pi_\pm(\lambda)\|_{-1,q,D_{\pm,R_0+1}} \\ \le\ & C\|\nabla\partial_\lambda^k v_\pm(\lambda)\|_{q,H_{\pm,R_0+2}} + C|\lambda|\|\partial_\lambda^k v_\pm(\lambda)\|_{q,H_{\pm,R_0+2}} \\ & +Ck\|\partial_\lambda^{k-1}v_\pm(\lambda)\|_{q,H_{\pm,R_0+2}}, \end{aligned}$$

for $1\le k\le m$. Similarly, for $k=0$, we use (4.1) to get

$$\begin{aligned} & \|(\nabla\psi_\pm)(\pi_\pm(\lambda)-\pi_0)\|_q \\ \le\ & C\|\nabla(\pi_\pm(\lambda)-\pi_0)\|_{-1,q,D_{\pm,R_0+1}} \\ \le\ & C\|\nabla v_\pm(\lambda)\|_{q,H_{\pm,R_0+2}} + C|\lambda|\|v_\pm(\lambda)\|_{q,H_{\pm,R_0+2}} + C\|f\|_q. \end{aligned}$$

It thus follows from (3.4) that

$$|\lambda|^\beta\|(\nabla\psi_\pm)\partial_\lambda^m\pi_\pm(\lambda)\|_q + \sum_{k=0}^{m-1}\|(\nabla\psi_\pm)\partial_\lambda^k(\pi_\pm(\lambda)-\pi_0)\|_q \le C\|f\|_q,$$

for Re $\lambda\ge 0$ with $0<|\lambda|\le 2$ and $f\in L^q_{[d]}(\Omega)$. This completes the proof. □

Let us consider the case $\lambda=0$ and simply write $v_\pm = v_\pm(x,0)$. Since

$$\|(v_\pm - v_0)\cdot\nabla\psi_\pm\|_{2,q}\le C\|f\|_q,$$

the operator $[f\mapsto (v_\pm - v_0)\cdot\nabla\psi_\pm] : L^q(\Omega)\to W_0^{1,q}(D_{\pm,R_0+1})$ is compact, which combined with (2.2) implies that so is the operator $Q_2(0) : L^q(\Omega)\to L^q_{[d]}(\Omega)$, where $d\ge R_0+2$. The other part $Q_1(0)f$ fulfills

$$\|Q_1(0)f\|_{1,q}\le C\|f\|_q,$$

from which the compactness of $Q_1(0) : L^q(\Omega)\to L^q_{[d]}(\Omega)$ follows; as a consequence, $Q(0)=Q_1(0)+Q_2(0)$ is a compact operator from $L^q_{[d]}(\Omega)$, $d\ge R_0+2$, into itself. We will show that $1+Q(0)$ is injective in $L^q_{[d]}(\Omega)$. Let $f\in L^q_{[d]}(\Omega)$ satisfy $(1+Q(0))f=0$. In view of (4.3), the pair $\{v,\pi\}$ given by (4.2) for such f should obey

$$-\Delta v+\nabla\pi = 0, \quad \nabla\cdot v = 0,$$

in Ω subject to $v|_{\partial\Omega}=0$ and $\phi(v)=0$. Since $f\in L^r_{[d]}(\Omega)$ for $1<r<\min\{n,q\}$, we have

$$\nabla^2 v, \nabla\pi\in L^r(\Omega), \quad \nabla v\in L^{nr/(n-r)}(\Omega), \quad v,\pi\in L^r_{loc}(\overline{\Omega});$$

especially, the summability of ∇v at infinity is implied by the boundedness of the support of f. It thus follows from Theorem 1.4 (i) of Farwig [16] that $v = \nabla\pi = 0$; here, it should be remarked that the uniqueness holds without any radiation condition (unlike the exterior problem discussed in [35] and [37]). We go back to (4.2) to see that $v_\pm = \nabla\pi_\pm = f = 0$ in $H_\pm \setminus B_{R_0+2}$ and that $v_0 = \nabla\pi_0 = f = 0$ in Ω_{R_0+1}. Set $U_\pm = (D \cup B_{R_0}) \cap H_\pm$. Both $\{v_\pm, \pi_\pm\}$ and $\{v_0, \pi_0\}$ then belong to $W^{2,q}(U_\pm) \times W^{1,q}(U_\pm)$ and are the solutions of the Stokes system in $U_\pm$ with zero boundary condition for the external force f. They thus coincide with each other and, in view of (4.2) again, we have $v_0 = \nabla\pi_0 = f = 0$ in D; after all, $f = 0$ in Ω. Owing to the Fredholm theorem, $1 + Q(0)$ has a bounded inverse $(1 + Q(0))^{-1}$ on $L^q_{[d]}(\Omega)$.

Set $\Sigma_\eta = \{\lambda \in \mathbb{C}; \text{Re } \lambda \geq 0, 0 < |\lambda| \leq \eta\}$ for $\eta > 0$. Since

$$\begin{aligned} & \|Q(\lambda)f - Q(0)f\|_q \\ \leq\ & C\|v_+(\lambda) - v_+(0)\|_{1,q,H_{+,R_0+2}} + C\|v_-(\lambda) - v_-(0)\|_{1,q,H_{-,R_0+2}} \\ & + C|\lambda|\{\|v_+(\lambda)\|_{q,H_{+,R_0+2}} + \|v_-(\lambda)\|_{q,H_{-,R_0+2}} + \|v_0\|_{q,D}\}, \end{aligned}$$

we obtain from (3.5)

$$\|Q(\lambda) - Q(0)\|_{B(L^q_{[d]}(\Omega))} \to 0,$$

as $\lambda \to 0$ with Re $\lambda \geq 0$, which implies the existence of a constant $\eta > 0$ such that $1 + Q(\lambda)$ has also a bounded inverse (in terms of the Neumann series) on $L^q_{[d]}(\Omega)$ with uniform bounds

$$\|(1 + Q(\lambda))^{-1}\|_{B(L^q_{[d]}(\Omega))} \leq C, \tag{4.8}$$

for $\lambda \in \Sigma_\eta \cup \{0\}$. Since the resolvent is uniquely determined, one can represent it for $\lambda \in \Sigma_\eta$ and $f \in L^q_{[d]}(\Omega), d \geq R_0 + 2$, as

$$(\lambda + A)^{-1}Pf = T(\lambda)(1 + Q(\lambda))^{-1}f. \tag{4.9}$$

We are in a position to show an analogous result for the resolvent to (3.4).

Lemma 4.2 *Let $n \geq 3, 1 < q < \infty, d \geq R_0$ and $R \geq R_0$. Given $f \in L^q_{[d]}(\Omega)$, set $u(\lambda) = (\lambda + A)^{-1}Pf$. For any small $\varepsilon > 0$ there is a constant $C = C(\Omega, n, q, d, R, \varepsilon) > 0$ such that*

$$|\lambda|^\beta \|\partial_\lambda^m u(\lambda)\|_{2,q,\Omega_R} + \sum_{k=0}^{m-1} \|\partial_\lambda^k u(\lambda)\|_{2,q,\Omega_R} \leq C\|f\|_q, \tag{4.10}$$

for Re $\lambda \geq 0$ $(\lambda \neq 0)$ and $f \in L^q_{[d]}(\Omega)$, where m and $\beta = \beta(\varepsilon)$ are the same as in Lemma 3.3.

Proof. The problem is only near $\lambda = 0$ because we have (2.3) for $G = \Omega$. We may also assume $d \geq R_0 + 2$ since $L^q_{[R_0]}(\Omega) \subset L^q_{[d]}(\Omega)$ for such d. It thus suffices to show (4.10) for $\lambda \in \Sigma_\eta$ by use of (4.9). For such λ and $0 \leq k \leq m$ we see that $\partial_\lambda^k (1 + Q(\lambda))^{-1} \in B(L^q_{[d]}(\Omega))$; furthermore,

$$|\lambda|^\beta \|\partial_\lambda^m (1 + Q(\lambda))^{-1} f\|_q + \sum_{k=0}^{m-1} \|\partial_\lambda^k (1 + Q(\lambda))^{-1} f\|_q \leq C\|f\|_q, \tag{4.11}$$

for $f \in L^q_{[d]}(\Omega)$. In fact, we have the representation

$$\begin{aligned} & \partial_\lambda^k (1 + Q(\lambda))^{-1} f \\ = \; & -(1 + Q(\lambda))^{-1} [\partial_\lambda^k Q(\lambda)] (1 + Q(\lambda))^{-1} f + L_k(\lambda)(1 + Q(\lambda))^{-1} f, \end{aligned} \tag{4.12}$$

for $k \geq 1$ and $f \in L^q_{[d]}(\Omega)$, where $L_1(\lambda) = 0$ and $L_k(\lambda)$ with $k \geq 2$ consists of finite sums of finite products of $(1 + Q(\lambda))^{-1}, \partial_\lambda Q(\lambda), \cdots, \partial_\lambda^{k-1} Q(\lambda)$. Consequently, (4.6) together with (4.8) implies (4.11). In view of

$$\partial_\lambda^k u(\lambda) = \sum_{j=0}^{k} \begin{pmatrix} k \\ j \end{pmatrix} \partial_\lambda^{k-j} T(\lambda) \; \partial_\lambda^j (1 + Q(\lambda))^{-1} f,$$

we conclude (4.10) from (4.5) and (4.11). □

In the last part of this section we will complete the regularity estimate of the resolvent. To this end, we employ Lemma 3.4 to show the following lemma.

Lemma 4.3 *Let $n \geq 3, 1 < q < \infty, d \geq R_0$ and $R \geq R_0$. Set*

$$T^{(k)}(s) = \partial_s^k T(is), \quad Q^{(k)}(s) = \partial_s^k Q(is) \quad (s \in \mathbb{R} \setminus \{0\},\ 0 \leq k \leq m).$$

For any small $\varepsilon > 0$ there is a constant $C = C(\Omega, n, q, d, R, \varepsilon) > 0$ such that

$$\|T^{(k)}(s+h)f - T^{(k)}(s)f\|_{2,q,\Omega_R} + \|Q^{(k)}(s+h)f - Q^{(k)}(s)f\|_q$$

$$\leq \begin{cases} C|h||s|^{-\beta-1}\|f\|_q & \text{if } k = m, \\ C|h||s|^{-\beta}\|f\|_q & \text{if } k = m-1, \\ C|h|\|f\|_q & \text{if } n \geq 5,\ 0 \leq k \leq m-2, \end{cases} \tag{4.13}$$

for $2|h| < |s| \leq 1$ and $f \in L^q_{[d]}(\Omega)$, where m and $\beta = \beta(\varepsilon)$ are the same as in Lemma 3.3. Concerning the first term of the left-hand side, (4.13) holds true for $h \in \mathbb{R}$ and $|s| > 2|h|$.

Proof. Set

$$v_{\pm}^{(k)}(s)=\partial_s^k v_{\pm}(is), \quad \pi_{\pm}^{(k)}(s)=\partial_s^k \pi_{\pm}(is) \quad (s\in\mathbb{R}\setminus\{0\},\ k=m \text{ or } m-1).$$

It then follows from (4.2) together with (2.2) that

$$\begin{aligned}
&\|T^{(m)}(s+h)f-T^{(m)}(s)f\|_{2,q,\Omega_R}\\
\le\;& C\|v_{+}^{(m)}(s+h)-v_{+}^{(m)}(s)\|_{2,q,H_{+,R}}+C\|v_{-}^{(m)}(s+h)-v_{-}^{(m)}(s)\|_{2,q,H_{-,R}}.
\end{aligned}$$

In order to estimate $Q^{(m)}$, let us investigate the pressures $\pi_{\pm}^{(m)}$. Similarly to the proof of Lemma 4.1 with the aid of (4.7), one can show

$$\begin{aligned}
&\|(\nabla\psi_{\pm})\{\pi_{\pm}^{(m)}(s+h)-\pi_{\pm}^{(m)}(s)\}\|_q\\
\le\;& C\|\nabla\pi_{\pm}^{(m)}(s+h)-\nabla\pi_{\pm}^{(m)}(s)\|_{-1,q,D_{\pm,R_0+1}}\\
\le\;& C\|\nabla v_{\pm}^{(m)}(s+h)-\nabla v_{\pm}^{(m)}(s)\|_{q,H_{\pm,R_0+2}}\\
&+C\|(s+h)v_{\pm}^{(m)}(s+h)-sv_{\pm}^{(m)}(s)\|_{q,H_{\pm,R_0+2}}\\
&+Cm\|v_{\pm}^{(m-1)}(s+h)-v_{\pm}^{(m-1)}(s)\|_{q,H_{\pm,R_0+2}}.
\end{aligned}$$

This combined with estimates on the other terms by use of (2.2) yields

$$\begin{aligned}
&\|Q^{(m)}(s+h)f-Q^{(m)}(s)f\|_q\\
\le\;& C\|v_{+}^{(m)}(s+h)-v_{+}^{(m)}(s)\|_{1,q,H_{+,R_0+2}}\\
&+C\|v_{-}^{(m)}(s+h)-v_{-}^{(m)}(s)\|_{1,q,H_{-,R_0+2}}\\
&+C|s|\|v_{+}^{(m)}(s+h)-v_{+}^{(m)}(s)\|_{q,H_{+,R_0+2}}\\
&+C|s|\|v_{-}^{(m)}(s+h)-v_{-}^{(m)}(s)\|_{q,H_{-,R_0+2}}\\
&+C|h|\|v_{+}^{(m)}(s+h)\|_{q,H_{+,R_0+2}}+C|h|\|v_{-}^{(m)}(s+h)\|_{q,H_{-,R_0+2}}\\
&+Cm\|v_{+}^{(m-1)}(s+h)-v_{+}^{(m-1)}(s)\|_{q,H_{+,R_0+2}}\\
&+Cm\|v_{-}^{(m-1)}(s+h)-v_{-}^{(m-1)}(s)\|_{q,H_{-,R_0+2}}.
\end{aligned}$$

Hence (3.7), (3.8) and (3.4) imply (4.13) for the case $k=m$. For $0\le k\le m-1$ we have

$$\|T^{(k)}(s+h)f-T^{(k)}(s)f\|_{2,q,\Omega_R}\le\left|\int_s^{s+h}\|T^{(k+1)}(\tau)f\|_{2,q,\Omega_R}d\tau\right|,$$

$$\|Q^{(k)}(s+h)f-Q^{(k)}(s)f\|_q\le\left|\int_s^{s+h}\|Q^{(k+1)}(\tau)f\|_q d\tau\right|,$$

which together with (4.5) and (4.6) respectively lead us to (4.13). The proof is thus complete. □

The regularity of the resolvent along the imaginary axis given by the following lemma plays a crucial role in the next section.

Lemma 4.4 *Let $n \geq 3, 1 < q < \infty, d \geq R_0$ and $R \geq R_0$. Set*

$$\Phi^{(m)}(s) = \partial_s^m (is + A)^{-1} P \qquad (s \in \mathbb{R} \setminus \{0\}).$$

For any small $\varepsilon > 0$ there is a constant $C = C(\Omega, n, q, d, R, \varepsilon) > 0$ such that

$$\int_{-\infty}^{\infty} \|\Phi^{(m)}(s+h)f - \Phi^{(m)}(s)f\|_{2,q,\Omega_R} ds \leq C|h|^{1-\beta}\|f\|_q, \tag{4.14}$$

for $|h| < h_0 = \min\{\eta/4, 1/2\}$ and $f \in L^q_{[d]}(\Omega)$. Here, m and $\beta = \beta(\varepsilon)$ are the same as in Lemma 3.3, and $\eta > 0$ is the constant such that (4.9) is valid for $\lambda \in \Sigma_\eta$.

Proof. We may assume $d \geq R_0 + 2$ (as in the proof of Lemma 4.2). Given h satisfying $|h| < h_0$, we divide the integral into three parts

$$\int_{-\infty}^{\infty} \|\Phi^{(m)}(s+h)f - \Phi^{(m)}(s)f\|_{2,q,\Omega_R} ds$$

$$= \int_{|s| \leq 2|h|} + \int_{2|h| < |s| \leq 2h_0} + \int_{|s| > 2h_0} = I_1 + I_2 + I_3.$$

With the aid of (4.10), we find

$$I_1 \leq 2\int_{|s| \leq 3|h|} \|\Phi^{(m)}(s)f\|_{2,q,\Omega_R} ds \leq C|h|^{1-\beta}\|f\|_q,$$

for $f \in L^q_{[d]}(\Omega)$. In order to estimate I_2, we use the representation

$$\Phi^{(m)}(s)f = \sum_{j=0}^{m} \begin{pmatrix} m \\ j \end{pmatrix} T^{(m-j)}(s)\; V^{(j)}(s)f,$$

where

$$V^{(j)}(s) = \partial_s^j (1 + Q(is))^{-1} \in B(L^q_{[d]}(\Omega)) \qquad (0 < |s| \leq \eta,\ 0 \leq j \leq m).$$

Then,

$$\begin{aligned} & \Phi^{(m)}(s+h)f - \Phi^{(m)}(s)f \\ = \; & \sum_{j=0}^{m} \begin{pmatrix} m \\ j \end{pmatrix} [T^{(m-j)}(s+h) - T^{(m-j)}(s)]\; V^{(j)}(s+h)f \\ & + \sum_{j=0}^{m} \begin{pmatrix} m \\ j \end{pmatrix} T^{(m-j)}(s)\; [V^{(j)}(s+h) - V^{(j)}(s)]f. \end{aligned}$$

We first show

$$\|V^{(j)}(s+h)f - V^{(j)}(s)f\|_q$$

$$\leq \begin{cases} C|h||s|^{-\beta-1}\|f\|_q & \text{if } j=m, \\ C|h||s|^{-\beta}\|f\|_q & \text{if } j=m-1, \\ C|h|\|f\|_q & \text{if } n\geq 5,\ 0\leq j\leq m-2, \end{cases} \tag{4.15}$$

for $2|h| < |s| \leq 2h_0$ and $f \in L^q_{[d]}(\Omega)$. Similarly to the proof of (4.13) for $0 \leq k \leq m-1$, (4.11) implies (4.15) for $0 \leq j \leq m-1$. As in (4.12), we have

$$V^{(m)}(s) = -V^{(0)}(s)Q^{(m)}(s)V^{(0)}(s) + W_m(s)V^{(0)}(s),$$

where $W_1(s) = 0$ and, for $m \geq 2$, $W_m(s) = i^m L_m(is)$ consists of finite sums of finite products of $V^0(s), Q^{(1)}(s), \cdots, Q^{(m-1)}(s)$. Therefore, we collect (4.6), (4.8), (4.13) and (4.15) for $j=0$ to arrive at (4.15) for $j=m$. It thus follows from (4.5), (4.11), (4.13) and (4.15) that

$$\|\Phi^{(m)}(s+h)f - \Phi^{(m)}(s)f\|_{2,q,\Omega_R} \leq C|h||s|^{-\beta-1}\|f\|_q,$$

for $2|h| < |s| \leq 2h_0$ and $f \in L^q_{[d]}(\Omega)$. As a consequence, we are led to

$$I_2 \leq C|h|\|f\|_q \int_{|s|>2|h|} |s|^{-\beta-1}ds \leq C|h|^{1-\beta}\|f\|_q,$$

for $f \in L^q_{[d]}(\Omega)$. Finally, to estimate I_3, one does not need any localization. In fact, since

$$\Phi^{(m)}(s+h)f - \Phi^{(m)}(s)f = (-i)^{m+1}(m+1)! \int_s^{s+h} (i\tau + A)^{-(m+2)}Pf\,d\tau,$$

(2.3) gives

$$\begin{aligned} \|\Phi^{(m)}(s+h)f - \Phi^{(m)}(s)f\|_{2,q,\Omega_R} &\leq C\|\Phi^{(m)}(s+h)f - \Phi^{(m)}(s)f\|_{D(A_q)} \\ &\leq C|h||s|^{-(m+1)}\|f\|_q, \end{aligned}$$

for $|s| > 2h_0$ $(> 2|h|)$ and $f \in L^q(\Omega)$. Therefore, we obtain

$$I_3 \leq C|h|\|f\|_q \int_{|s|>2h_0} |s|^{-(m+1)}ds \leq C|h|\|f\|_q,$$

for $f \in L^q(\Omega)$. Collecting the estimates above on I_1, I_2 and I_3, we conclude (4.14). □

5 L^q-L^r estimates of the Stokes semigroup

In this section we will prove Theorem 2.1. As explained in section 1, the first step is to derive (1.6) for non-solenoidal data with bounded support.

Lemma 5.1 *Let $n \geq 3, 1 < q < \infty, d \geq R_0$ and $R \geq R_0$. For any small $\varepsilon > 0$ there is a constant $C = C(\Omega, n, q, d, R, \varepsilon) > 0$ such that*

$$\|e^{-tA}Pf\|_{1,q,\Omega_R} \leq Ct^{-1/2}(1+t)^{-n/2+1/2+\varepsilon}\|f\|_q, \tag{5.1}$$

for $t > 0$ and $f \in L^q_{[d]}(\Omega)$.

For the proof, the following lemma due to Shibata is crucial since we know the regularity of the Stokes resolvent given by Lemmas 4.2 and 4.4.

Lemma 5.2 *Let X be a Banach space with norm $\|\cdot\|$ and $g \in L^1(\mathbb{R}; X)$. If there are constants $\theta \in (0,1)$ and $M > 0$ such that*

$$\int_{-\infty}^{\infty} \|g(s)\|ds + \sup_{h\neq 0} \frac{1}{|h|^\theta} \int_{-\infty}^{\infty} \|g(s+h) - g(s)\|ds \leq M,$$

then the Fourier inverse image

$$G(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ist} g(s)ds$$

of g enjoys

$$\|G(t)\| \leq CM(1+|t|)^{-\theta},$$

with some $C > 0$ independent of $t \in \mathbb{R}$.

Remark 5.1. The assumption of Lemma 5.2 is equivalent to

$$g \in \left(L^1(\mathbb{R}; X),\ W^{1,1}(\mathbb{R}; X)\right)_{\theta,\infty},$$

where $(\cdot,\cdot)_{\theta,\infty}$ denotes the real interpolation (the space to which g belongs is known as a Besov space).

Proof of Lemma 5.2. Although this lemma was already proved by Shibata [51], we give our different proof which seems to be simpler. Since $\|G(t)\| \leq M/2\pi$, it suffices to consider the case $|t| > 1$. It is easily seen that if $ht \neq 2j\pi$ $(j = 0, \pm 1, \pm 2, \cdots)$, then

$$G(t) = \frac{e^{iht}}{2\pi(1-e^{iht})} \int_{-\infty}^{\infty} e^{ist}(g(s+h) - g(s))ds,$$

from which the assumption leads us to

$$\|G(t)\| \leq \frac{M|h|^\theta}{2\pi|1-e^{iht}|}.$$

Taking $h = 1/t$ immediately implies the desired estimate. □
Proof of Lemma 5.1. Since

$$\|e^{-tA}Pf\|_{1,q} \leq C\|e^{-tA}Pf\|_{D(A_q)}^{1/2}\|e^{-tA}Pf\|_q^{1/2} \leq Ct^{-1/2}\|f\|_q, \tag{5.2}$$

for $0 < t < 1$ and $f \in L^q(\Omega)$, we will concentrate ourselves on the proof of (5.1) for $t \geq 1$, namely (1.6). Given $R \geq R_0$, we set $\psi = 1 - \psi_{+,R} - \psi_{-,R}$, where the cut-off functions $\psi_{\pm,R}$ are given by (2.1). One can justify the following representation formula of the semigroup for $f \in L^q_{[d]}(\Omega)$:

$$\psi e^{-tA}Pf = \frac{i^m}{2\pi t^m}\int_{-\infty}^{\infty} e^{ist}\psi\Phi^{(m)}(s)f\,ds, \tag{5.3}$$

where $\Phi^{(m)}(s) = \partial_s^m(is + A)^{-1}P$ and m is the same as in Lemma 3.3. In fact, starting from the standard Dunford integral representation, we perform m-times integrations by parts and then move the path of integration to the imaginary axis but avoid the origin $\lambda = 0$, so that

$$\begin{aligned}\psi e^{-tA}Pf = \;& \frac{i^m}{2\pi t^m}\left\{\int_{-\infty}^{-\delta} + \int_{\delta}^{\infty}\right\} e^{ist}\psi\Phi^{(m)}(s)f\,ds \\ & + \frac{(-1)^m}{2\pi i t^m}\int_{\Gamma_\delta} e^{\lambda t}\psi\partial_\lambda^m(\lambda + A)^{-1}Pf\,d\lambda,\end{aligned}$$

for any $\delta > 0$, where $\Gamma_\delta = \{\delta e^{i\theta}; -\pi/2 \leq \theta \leq \pi/2\}$ (this formula is valid for $f \in L^q(\Omega)$ without ψ). Owing to (4.10), the last integral vanishes in $L^q(\Omega)$ as $\delta \to 0$ for $f \in L^q_{[d]}(\Omega)$; thus, we arrive at (5.3). Now, it follows from (4.10) and (2.3) that

$$\int_{-\infty}^{\infty}\|\psi\Phi^{(m)}(s)f\|_{1,q}ds \leq C\int_{|s|\leq 1}\frac{\|f\|_q}{|s|^\beta}ds + C\int_{|s|>1}\frac{\|f\|_q}{|s|^{m+1/2}}ds \leq C\|f\|_q.$$

Further, (4.14) and the estimate above respectively imply that

$$\sup_{0<|h|<h_0}\frac{1}{|h|^{1-\beta}}\int_{-\infty}^{\infty}\|\psi\Phi^{(m)}(s+h)f - \psi\Phi^{(m)}(s)f\|_{1,q}ds \leq C\|f\|_q,$$

and that

$$\sup_{|h|\geq h_0}\frac{1}{|h|^{1-\beta}}\int_{-\infty}^{\infty}\|\psi\Phi^{(m)}(s+h)f - \psi\Phi^{(m)}(s)f\|_{1,q}ds$$

$$\leq \frac{2}{h_0^{1-\beta}} \int_{-\infty}^{\infty} \|\psi \Phi^{(m)}(s) f\|_{1,q} ds \leq C\|f\|_q.$$

Hence, we can apply Lemma 5.2 with $X = W^{1,q}(\Omega)$ and $g(s) = \psi\Phi^{(m)}(s)f$ to the formula (5.3); as a consequence, we obtain

$$\|e^{-tA}Pf\|_{1,q,\Omega_R} \leq \|\psi e^{-tA}Pf\|_{1,q} \leq Ct^{-m}(1+t)^{-1+\beta}\|f\|_q,$$

for $t > 0$, which implies (5.1) for $t \geq 1$ and $f \in L^q_{[d]}(\Omega)$. This completes the proof. □

Remark 5.2. It is possible to show the decay rate $t^{-n/2+\varepsilon}$ of the semigroup in $W^{2,q}(\Omega_R)$ as well. This follows immediately from the proof given above with $X = W^{2,q}(\Omega)$ for $n \geq 5$. When $n = 3$ or 4 (thus $m = 1$), as in Kobayashi and Shibata [37], we have to introduce a cut-off function $\rho \in C_0^\infty(\mathbb{R}; [0,1])$ with $\rho(s) = 1$ near $s = 0$; then one can employ Lemma 5.2 with $X = W^{2,q}(\Omega)$ and $g(s) = \rho(s)\psi\Phi^{(m)}(s)f$ to obtain the desired result since a rapid decay of the remaining integral far from $s = 0$ is derived via integration by parts. We did not follow this procedure because Lemma 5.1 is sufficient for the proof of Theorem 2.1.

The next step is to deduce the sharp local energy decay estimate (1.5) from Lemma 5.1.

Lemma 5.3 *Let $n \geq 3, 1 < q < \infty$ and $R \geq R_0$. Then there is a constant $C = C(\Omega, n, q, R) > 0$ such that*

$$\|e^{-tA}f\|_{1,q,\Omega_R} \leq Ct^{-n/2q}\|f\|_q, \tag{5.4}$$

for $t \geq 2$ and $f \in L^q_\sigma(\Omega)$; and

$$\|e^{-tA}f\|_{1,q,\Omega_R} + \|\partial_t e^{-tA}f\|_{q,\Omega_R} \leq C(1+t)^{-n/2q}\|f\|_{D(A_q)}, \tag{5.5}$$

for $t \geq 0$ and $f \in D(A_q)$.

Proof. We employ a localization procedure which is similar to [35] and [37]. Given $f \in L^q_\sigma(\Omega)$, we set $g = e^{-A}f \in D(A_q)$ and intend to derive the decay estimate of $u(t) = e^{-tA}g = e^{-(t+1)A}f$ in $W^{1,q}(\Omega_R)$ for $t \geq 1$. We denote by p the pressure associated to u. We make use of the cut-off functions given by (2.1) and the Bogovskiĭ operator introduced in section 2. Set

$$g_\pm = \psi_{\pm,R_0+1}\, g - S_{\pm,R_0+1}[g \cdot \nabla\psi_{\pm,R_0+1}],$$

and

$$v_\pm(t) = E_\pm(t)g_\pm.$$

Note that $\int_{D_{\pm,R_0+1}} g \cdot \nabla\psi_{\pm,R_0+1}dx = 0$ and that $g_\pm \in D(A_{q,H_\pm})$ with

$$\|g_\pm\|_{D(A_{q,H_\pm})} \le C\|g_\pm\|_{2,q,H_\pm} \le C\|g\|_{2,q} \le C\|g\|_{D(A_q)} \le C\|f\|, \tag{5.6}$$

by (2.2). We take the pressures $\pi_\pm$ in $H_\pm$ associated to $v_\pm$ in such a way that

$$\int_{D_{\pm,R_0}} \pi_\pm(x,t)dx = 0, \tag{5.7}$$

for each t. In the course of the proof of this lemma, for simplicity, we abbreviate $\psi_{\pm,R_0}$ to $\psi_\pm$ and $S_{\pm,R_0}$ to $S_\pm$. We now define $\{u_\pm, p_\pm\}$ by

$$u_\pm(t) = \psi_\pm v_\pm(t) - S_\pm[v_\pm(t)\cdot\nabla\psi_\pm], \quad p_\pm(t) = \psi_\pm\pi_\pm(t).$$

Then it follows from Lemma 3.2 together with (2.2) and (5.6) that

$$\|u_\pm(t)\|_{1,q,\Omega_R} \le C\|v_\pm(t)\|_{1,q,H_{\pm,L}} \le C(1+t)^{-n/2q}\|f\|_q, \tag{5.8}$$

for $t \ge 0$, where $L = \max\{R, R_0+1\}$. Thus, in order to estimate $u(t)$, let us consider

$$v(t) = u(t) - u_+(t) - u_-(t), \quad \pi(t) = p(t) - p_+(t) - p_-(t),$$

which should obey

$$\partial_t v - \Delta v + \nabla\pi = K, \quad \nabla\cdot v = 0,$$

in Ω subject to $v|_{\partial\Omega} = 0$, $\phi(v) = \phi(u) = 0$ and

$$v|_{t=0} = v_0 = g - g_+ - g_- \in L^q_{[R_0+2]}(\Omega) \cap D(A_q),$$

where

$$\begin{aligned} K = \; & 2\nabla\psi_+\cdot\nabla v_+ + 2\nabla\psi_-\cdot\nabla v_- + (\Delta\psi_+)v_+ + (\Delta\psi_-)v_- \\ & -\Delta S_+[v_+\cdot\nabla\psi_+] - \Delta S_-[v_-\cdot\nabla\psi_-] \\ & +S_+[\partial_t v_+\cdot\nabla\psi_+] + S_-[\partial_t v_-\cdot\nabla\psi_-] - (\nabla\psi_+)\pi_+ - (\nabla\psi_-)\pi_-; \end{aligned}$$

we here note that $\nabla\cdot K \ne 0$ as well as $K|_{\partial\Omega} \ne 0$ and we can obtain the regularity of K only up to L^q (in contrast to the exterior problem discussed in [35] and [37]). By (5.7) and in view of the Stokes system in $H_\pm$ we have

$$\begin{aligned} \|(\nabla\psi_\pm)\pi_\pm(t)\|_q & \le C\|\nabla\pi_\pm(t)\|_{-1,q,D_{\pm,R_0}} \\ & \le C\|\nabla v_\pm(t)\|_{q,H_{\pm,R_0+1}} + C\|\partial_t v_\pm(t)\|_{q,H_{\pm,R_0+1}}, \end{aligned}$$

which together with (2.2) implies $K(t) \in L^q_{[R_0+1]}(\Omega)$ and

$$\begin{aligned} \|K(t)\|_q \leq \; & C\|v_+(t)\|_{1,q,H_{+,R_0+1}} + C\|v_-(t)\|_{1,q,H_{-,R_0+1}} \\ & + C\|\partial_t v_+(t)\|_{q,H_{+,R_0+1}} + C\|\partial_t v_-(t)\|_{q,H_{-,R_0+1}}. \end{aligned}$$

Therefore, Lemma 3.2 and (5.6) yield

$$\|K(t)\|_q \leq C(1+t)^{-n/2q}\|f\|_q, \tag{5.9}$$

for $t \geq 0$. In order to estimate

$$v(t) = e^{-tA}v_0 + \int_0^t e^{-(t-\tau)A}PK(\tau)d\tau,$$

we employ Lemma 5.1. By (5.1) with a suitable $\varepsilon > 0$ and (5.6) we find

$$\|e^{-tA}v_0\|_{1,q,\Omega_R} \leq Ct^{-n/2+\varepsilon}\|v_0\|_q \leq Ct^{-n/2q}\|f\|_q,$$

for $t \geq 1$. We next combine (5.1) with (5.9) to get

$$\begin{aligned} & \int_0^t \|e^{-(t-\tau)A}PK(\tau)\|_{1,q,\Omega_R}d\tau \\ \leq \; & C\|f\|_q \int_0^t (t-\tau)^{-1/2}(1+t-\tau)^{-n/2+1/2+\varepsilon}(1+\tau)^{-n/2q}d\tau \\ = \; & C\|f\|_q(I_1+I_2), \end{aligned}$$

where $I_1 = \int_0^{t/2}$ and $I_2 = \int_{t/2}^t$. An elementary calculation gives

$$I_1 \leq \left\{ \begin{array}{ll} Ct^{-1/2}(1+t/2)^{-n/2-n/2q+3/2+\varepsilon} & \text{if } q > n/2 \\ Ct^{-1/2}(1+t/2)^{-n/2+1/2+\varepsilon}\log(1+t/2) & \text{if } q = n/2 \\ Ct^{-1/2}(1+t/2)^{-n/2+1/2+\varepsilon} & \text{if } q < n/2 \end{array} \right\} \leq Ct^{-n/2q},$$

for $t \geq 1$ and

$$I_2 \leq (1+t/2)^{-n/2q}\int_0^\infty \tau^{-1/2}(1+\tau)^{-n/2+1/2+\varepsilon}d\tau \leq C(1+t/2)^{-n/2q},$$

for $t > 0$. We collect the estimates above to obtain

$$\|v(t)\|_{1,q,\Omega_R} \leq Ct^{-n/2q}\|f\|_q, \tag{5.10}$$

for $t \geq 1$. From (5.8) and (5.10) we deduce

$$\|u(t)\|_{1,q,\Omega_R} = \|v(t) + u_+(t) + u_-(t)\|_{1,q,\Omega_R} \leq Ct^{-n/2q}\|f\|_q,$$

for $t \geq 1$ and $f \in L^q_\sigma(\Omega)$, which proves (5.4). Let $f \in D(A_q)$. Then we easily observe

$$\|e^{-tA}f\|_{1,q,\Omega_R} + \|\partial_t e^{-tA}f\|_{q,\Omega_R} \leq C\|e^{-tA}f\|_{D(A_q)} \leq C\|f\|_{D(A_q)},$$

for $t \geq 0$ and also we can estimate $\partial_t e^{-tA}f$ for large t; in fact, by virtue of (5.4) just proved we get

$$\|\partial_t e^{-tA}f\|_{q,\Omega_R} = \|e^{-tA}Af\|_{q,\Omega_R} \leq Ct^{-n/2q}\|Af\|_q,$$

for $t \geq 2$. This implies (5.5). □

We are interested in the L^q estimate of ∇e^{-tA} for large t, in particular, the L^n estimate is quite important for us.

Lemma 5.4 *Let $n \geq 3$ and $1 < q < \infty$. Then there is a constant $C = C(\Omega, n, q) > 0$ such that*

$$\|\nabla e^{-tA}f\|_q \leq Ct^{-\min\{1/2, n/2q\}}\|f\|_q, \tag{5.11}$$

for $t \geq 2$ and $f \in L^q_\sigma(\Omega)$.

Proof. We fix $R \geq R_0 + 1$. Since we have already known the decay rate $t^{-n/2q}$ of $\|\nabla e^{-tA}f\|_{q,\Omega_R}$ by Lemma 5.3, it suffices to derive the estimate outside Ω_R, that is,

$$\|\nabla e^{-tA}f\|_{q,\Omega_\pm \setminus \Omega_R} \leq Ct^{-\min\{1/2, n/2q\}}\|f\|_q, \tag{5.12}$$

for $t \geq 2$ and $f \in L^q_\sigma(\Omega)$. In an analogous way to [35], [37] and [1], we make use of the decay properties of the semigroup $E_\pm(t)$ for the half space. Given $f \in L^q_\sigma(\Omega)$, we set $g = e^{-A}f \in D(A_q)$ and then $u(t) = e^{-tA}g = e^{-(t+1)A}f$. We choose two pressures $p_\pm$ in Ω associated to u in such a way that

$$\int_{D_{\pm,R-1}} p_\pm(x,t)dx = 0, \tag{5.13}$$

for each t (p_+ and p_- will be used independently). With use of the cut-off functions given by (2.1) and the Bogovskiǐ operator introduced in section 2, we define $\{v_\pm, \pi_\pm\}$ by

$$v_\pm(t) = \psi_\pm u(t) - S_\pm[u(t)\cdot\nabla\psi_\pm], \quad \pi_\pm(t) = \psi_\pm p_\pm(t).$$

Here and in what follows, we use the abbreviations $\psi_\pm$ for $\psi_{\pm,R-1}$ and $S_\pm$ for $S_{\pm,R-1}$. Since $v_\pm = u$ for $x \in \Omega_\pm \setminus \Omega_R = H_\pm \setminus B_R$, we will show

$$\|\nabla v_\pm(t)\|_{q,H_\pm} \le Ct^{-\min\{1/2,n/2q\}}\|g\|_{D(A_q)}, \tag{5.14}$$

for $t \ge 1$, which combined with $\|g\|_{D(A_q)} \le C\|f\|_q$ implies (5.12) for $t \ge 2$. It is easily observed that $\{v_\pm, \pi_\pm\}$ satisfies

$$\partial_t v_\pm - \Delta v_\pm + \nabla \pi_\pm = Z_\pm, \quad \nabla \cdot v_\pm = 0,$$

in $H_\pm$ subject to $v_\pm|_{\partial H_\pm} = 0$ and

$$v_\pm|_{t=0} = a_\pm = \psi_\pm g - S_\pm[g \cdot \nabla \psi_\pm],$$

where

$$\begin{aligned} Z_\pm = \quad & -2\nabla\psi_\pm \cdot \nabla u - (\Delta\psi_\pm)u + \Delta S_\pm[u \cdot \nabla\psi_\pm] \\ & -S_\pm[\partial_t u \cdot \nabla\psi_\pm] + (\nabla\psi_\pm)p_\pm. \end{aligned}$$

Our task is now to estimate the gradient of

$$v_\pm(t) = E_\pm(t)a_\pm + \int_0^t E_\pm(t-\tau)P_{H_\pm}Z_\pm(\tau)d\tau. \tag{5.15}$$

By virtue of (5.13) we have

$$\begin{aligned} \|(\nabla\psi_\pm)p_\pm(t)\|_{q,H_\pm} & \le C\|\nabla p_\pm(t)\|_{-1,q,D_{\pm,R-1}} \\ & \le C\|\nabla u(t)\|_{q,\Omega_R} + C\|\partial_t u(t)\|_{q,\Omega_R}, \end{aligned}$$

from which together with (2.2) it follows that

$$\|Z_\pm(t)\|_{q,H_\pm} \le C\|u(t)\|_{1,q,\Omega_R} + C\|\partial_t u(t)\|_{q,\Omega_R}.$$

Hence, (5.5) implies

$$\|P_{H_\pm}Z_\pm(t)\|_{r,H_\pm} \le C\|Z_\pm(t)\|_{q,H_\pm} \le C(1+t)^{-n/2q}\|g\|_{D(A_q)}, \tag{5.16}$$

for $t \ge 0$ and $r \in (1,q]$ since $Z_\pm(t) \in L^q_{[R]}(H_\pm) \subset L^r_{[R]}(H_\pm)$ for such r. In view of (5.15), we deduce from (1.4) for $\Omega = H_\pm$ together with (5.16)

$$\begin{aligned} & \|\nabla v_\pm(t)\|_{q,H_\pm} \\ \le\ & Ct^{-1/2}\|a_\pm\|_{q,H_\pm} \\ & +C\|g\|_{D(A_q)}\int_0^t (t-\tau)^{-1/2}(1+t-\tau)^{-(n/r-n/q)/2}(1+\tau)^{-n/2q}d\tau \\ \le\ & Ct^{-1/2}\|g\|_q + C\|g\|_{D(A_q)}(I_1+I_2), \end{aligned}$$

for $r \in (1,q]$, where $I_1 = \int_0^{t/2}$ and $I_2 = \int_{t/2}^t$. We take r so that $1 < r < \min\{n/2, q\}$. Then we see that

$$I_1 \leq \left\{ \begin{array}{ll} Ct^{-1/2}(1+t/2)^{-n/2r+1} & \text{if } q > n/2 \\ Ct^{-1/2}(1+t/2)^{-n/2r+1}\log(1+t/2) & \text{if } q = n/2 \\ Ct^{-1/2}(1+t/2)^{-(n/r-n/q)/2} & \text{if } q < n/2 \end{array} \right\} \leq Ct^{-1/2},$$

for $t > 0$ and that

$$I_2 \leq \left\{ \begin{array}{ll} C(1+t/2)^{-n/2q} & \text{if } q > n, \\ C(1+t/2)^{-1/2} & \text{if } q \leq n, \end{array} \right.$$

for $t > 0$. Collecting the estimates above concludes (5.14). This completes the proof. □

The following lemma is concerned with the L^∞ estimate of the semigroup (the restriction $q > n$ will be removed later).

Lemma 5.5 *Let $3 \leq n < q < \infty$. There is a constant $C = C(\Omega, n, q) > 0$ such that*

$$\|e^{-tA}f\|_\infty \leq Ct^{-n/2q}\|f\|_q, \tag{5.17}$$

for $t > 0$ and $f \in L^q_\sigma(\Omega)$.

Proof. For fixed $R \geq R_0 + 1$, estimate (5.4) together with the Sobolev embedding property implies

$$\|e^{-tA}f\|_{\infty,\Omega_R} \leq Ct^{-n/2q}\|f\|_q,$$

for $t \geq 2$ and $f \in L^q_\sigma(\Omega)$ on account of $n < q < \infty$. Along the lines of the proof of Lemma 5.4, one can show

$$\|e^{-tA}f\|_{\infty,\Omega_\pm\setminus\Omega_R} \leq Ct^{-n/2q}\|f\|_q, \tag{5.18}$$

for $t \geq 2$. In fact, given $f \in L^q_\sigma(\Omega)$, we take the same $g, \{u, p_\pm\}$ and $\{v_\pm, \pi_\pm\}$, and apply the L^q-L^∞ estimate (1.3) for $\Omega = H_\pm$ to (5.15). Then, taking (5.16) into account, we get

$$\begin{aligned} & \|v_\pm(t)\|_{\infty,H_\pm} \\ \leq\; & Ct^{-n/2q}\|a_\pm\|_{q,H_\pm} \\ & +C\|g\|_{D(A_q)}\int_0^t (t-\tau)^{-n/2q}(1+t-\tau)^{-(n/r-n/q)/2}(1+\tau)^{-n/2q}d\tau, \end{aligned}$$

for $r \in (1,q]$; we now choose $r \in (1, n/2)$ to find

$$\|v_\pm(t)\|_{\infty,H_\pm} \le Ct^{-n/2q}\|g\|_{D(A_q)},$$

for $t \ge 1$, which proves (5.18) for $t \ge 2$. We thus obtain (5.17) for $t \ge 2$. For $0 < t < 2$, we recall (5.2) to see

$$\|e^{-tA}f\|_\infty \le C\|e^{-tA}f\|_{1,q}^{n/q}\|e^{-tA}f\|_q^{1-n/q} \le Ct^{-n/2q}\|f\|_q.$$

The proof is complete. □

We are now in a position to prove Theorem 2.1. Abels [1] showed (1.3) for $1 < q \le r < \infty$; when we use this result, the first step of the following proof will become shorter. However, in order to make the present paper self-contained, we do not rely on any result of [1]. We emphasize that our proof is based on (5.11) and (5.17), in other words, the other estimates follow from them.

Proof of Theorem 2.1. The proof is divided into four steps.

Step 1. First of all, we observe (1.4) for $q = r \in (1, n]$. Indeed, it follows from (5.2) for $0 < t < 2$ and (5.11) for $t \ge 2$ that

$$\|\nabla e^{-tA}f\|_q \le Ct^{-1/2}\|f\|_q, \tag{5.19}$$

for $t > 0$ and $f \in L^q_\sigma(\Omega)$ provided $1 < q \le n$. In this step we accomplish the proof of (1.3) for $1 < q \le r \le \infty$ $(q \ne \infty)$ and (1.4) for $1 < q \le r \le n$. We begin with the removal of the restriction $q > n$ in Lemma 5.5. In view of (5.19) and the Sobolev embedding property we have

$$\|e^{-tA}f\|_r \le Ct^{-1/2}\|f\|_q, \tag{5.20}$$

for $t > 0$ and $f \in L^q_\sigma(\Omega)$ when $1 < q < n$ and $1/r = 1/q - 1/n$. Let $n/(k+1) < q < n/k$ with $k = 1, 2, \cdots, n-1$. We put $\{q_j\}_{j=0}^k$ in such a way that $1/q_{j+1} = 1/q_j - 1/n$ $(j = 0, 1, \cdots, k-1)$ with $q_0 = q$. Since $n < q_k < \infty$, we make use of (5.17) with $q = q_k$ and (5.20) to obtain

$$\|e^{-tA}f\|_\infty \le Ct^{-n/2q_k}\|e^{-(t/2)A}f\|_{q_k} \le Ct^{-n/2q_k - k/2}\|f\|_q,$$

for $t > 0$, which proves (5.17) except for $q = n, n/2, \cdots, n/(n-1)$. But the exceptional cases can be also deduced via interpolation. Thus the L^q-L^∞ estimate (5.17) has been established for all $q \in (1, \infty)$. This together with the L^q boundedness (namely, (1.3) for $q = r$) immediately gives (1.3) for $1 < q \le r \le \infty$, from which combined with (5.19) we further obtain (1.4) for $1 < q \le r \le n$.

Step 2. In this step we prove (1.4) for $1 < q < n < r < \infty$, making use of (1.8) due to [20]. Given $r \in (n, \infty)$, we take $s \in (n/2, n)$ so that

$1/s = 1/r + 1/n$. When $1 < q \le s$, an embedding relation given by Lemma 3.1 of [20] together with (1.8) implies

$$\|\nabla e^{-tA} f\|_r \le C\|\nabla^2 e^{-tA} f\|_s \le C\|Ae^{-tA} f\|_s \le Ct^{-1}\|e^{-(t/2)A} f\|_s,$$

for $t > 0$, from which together with (1.3) we obtain (1.4). If $s < q < n$, which implies $r < q_*$ with $1/q_* = 1/q - 1/n$, then by the same reasoning as above

$$\|\nabla e^{-tA} f\|_r \le \|\nabla e^{-tA} f\|_{q_*}^{1-\theta} \|\nabla e^{-tA} f\|_q^{\theta} \le C\|Ae^{-tA} f\|_q^{1-\theta} \|\nabla e^{-tA} f\|_q^{\theta},$$

for $t > 0$, where $1/r = (1-\theta)/q_* + \theta/q = 1/q - (1-\theta)/n$. Therefore, (5.19) yields (1.4).

Step 3. Let $f \in L^1(\Omega) \cap L^s_\sigma(\Omega)$ for some $s \in (1, \infty)$. This step is devoted to the case $q = 1$, namely L^1-L^r estimate. Let $1 < r < \infty$. We apply a simple duality argument; in fact, the L^q-L^∞ estimate implies

$$|(e^{-tA} f, g)| = |(f, e^{-tA} g)| \le \|f\|_1 \|e^{-tA} g\|_\infty \le Ct^{-(n-n/r)/2} \|f\|_1 \|g\|_{r/(r-1)},$$

for $g \in L^{r/(r-1)}_\sigma(\Omega)$, which gives (1.3) for $q = 1 < r < \infty$. Combining this with (5.17) and (1.4), respectively, we obtain (1.3) for $q = 1 < r = \infty$ and (1.4) for $q = 1 < r < \infty$.

Step 4. Once the L^q-L^r estimates (1.3) and (1.4) are established, (2.4) and (2.5) can be proved by means of a standard approximation procedure. We show only the behavior as $t \to \infty$ (which is the main concern in the present paper). Let $1 < q < \infty$ and $f \in L^q_\sigma(\Omega)$. For any $\varepsilon > 0$ we take $f_\varepsilon \in C^\infty_{0,\sigma}(\Omega)$ such that $\|f_\varepsilon - f\|_q < \varepsilon$. It then follows from (1.3) that

$$\|e^{-tA} f\|_q \le C\varepsilon + Ct^{-(n-n/q)/2} \|f_\varepsilon\|_1,$$

for $t > 0$, which immediately yields

$$\lim_{t\to\infty} \|e^{-tA} f\|_q = 0, \tag{5.21}$$

since $\varepsilon > 0$ is arbitrary (one can give another proof by use of $\ker(A_q) = \{0\}$). Let K be a precompact set in $L^q_\sigma(\Omega)$. For any $\eta > 0$ there is a finite set $\{f_j\}_{j=1}^m \subset K$ so that $\{B_\eta(f_j)\}_{j=1}^m$ is a covering of K, where $B_\eta(f_j)$ denotes the open ball centered at f_j with radius η. Then we have

$$\sup_{f\in K} \|e^{-tA} f\|_q \le C\eta + \max_{1\le j\le m} \|e^{-tA} f_j\|_q.$$

Hence, from (5.21) we deduce

$$\lim_{t\to\infty} \sup_{f\in K} \|e^{-tA} f\|_q = 0. \tag{5.22}$$

All the other decay properties as $t \to \infty$ follow from (5.22) combined with (1.3) and (1.4). We have completed the proof. □

6 The Navier-Stokes flow

In this section we apply the developed L^q-L^r estimates of the semigroup to the Navier-Stokes initial value problem. In the proof of Theorems 2.2 and 2.3, we will not cite (1.3) and/or (1.4) if the application is evident. We first prove Theorem 2.2.

Proof of Theorem 2.2. One can construct a unique global solution $u(t)$ of the integral equation

$$u(t) = e^{-tA}a - \int_0^t e^{-(t-\tau)A} P(u \cdot \nabla u)(\tau) d\tau, \quad t > 0, \tag{6.1}$$

by means of a standard contraction mapping principle, in exactly the same way as in Kato [36], provided that $\|a\|_n \le \delta_0$, where $\delta_0 = \delta_0(\Omega, n) > 0$ is a constant. The solution $u(t)$ satisfies

$$\|u(t)\|_r \le Ct^{-1/2+n/2r}\|a\|_n \quad \text{for } n \le r \le \infty, \tag{6.2}$$

$$\|\nabla u(t)\|_n \le Ct^{-1/2}\|a\|_n, \tag{6.3}$$

for $t > 0$ together with the singular behavior

$$\|u(t)\|_r = o\left(t^{-1/2+n/2r}\right) \quad \text{for } n < r \le \infty; \ \|\nabla u(t)\|_n = o\left(t^{-1/2}\right), \tag{6.4}$$

as $t \to 0$. Furthermore, due to the Hölder estimate (6.9) below which is implied by (6.2) and (6.3), the solution $u(t)$ becomes actually a strong one of (1.1) with (2.6) (see [24], [29] and [57]). We now prove

$$\lim_{t\to\infty} \|u(t)\|_n = 0, \tag{6.5}$$

for still smaller $a \in L^n_\sigma(\Omega)$. To this end, we derive a certain decay property of $u(t)$, which is weaker than (2.11) but sufficient for the proof of (6.5), assuming additionally $a \in L^1(\Omega) \cap L^n_\sigma(\Omega)$ with small $\|a\|_n$. Given $\gamma \in (0, 1/2)$, we take $q \in (n/2, n)$ so that $\gamma = n/2q - 1/2$; then,

$$\|u(t)\|_n \le Ct^{-\gamma}\|a\|_q + C\int_0^t (t-\tau)^{-1/2}\|u(\tau)\|_n\|\nabla u(\tau)\|_n d\tau,$$

which together with (6.3) implies

$$t^\gamma\|u(t)\|_n \le C\|a\|_q + C\|a\|_n \sup_{0<\tau\le t} \tau^\gamma\|u(\tau)\|_n,$$

for $t > 0$. Hence, for any $\gamma \in (0, 1/2)$ there are constants $\delta_* = \delta_*(\Omega, n, \gamma) \in (0, \delta_0]$ and $C = C(\Omega, n, \gamma) > 0$ such that if $\|a\|_n \le \delta_*$, then $\|u(t)\|_n \le Ct^{-\gamma}\|a\|_q$ for $t > 0$, which together with (6.2) yields

$$\|u(t)\|_n \le C(1+t)^{-\gamma}(\|a\|_1 + \|a\|_n), \qquad (6.6)$$

for $t \ge 0$ (this decay rate is not sharp and will be improved in Theorem 2.3). From now on we fix $\gamma \in (0, 1/2)$ and set $\delta = \delta_*(\Omega, n, \gamma)/2$. Given $a \in L^n_\sigma(\Omega)$ with $\|a\|_n \le \delta$ and any $\varepsilon \in (0, \delta]$, we take $a_\varepsilon \in C^\infty_{0,\sigma}(\Omega)$ so that $\|a_\varepsilon - a\|_n < \varepsilon$. Since $\|a_\varepsilon\|_n \le \delta_*$, the corresponding global solution fulfills (6.6). We combine this fact with the continuous dependence: $L^n_\sigma(\Omega) \ni u(0) \mapsto u \in BC([0,\infty); L^n_\sigma(\Omega))$, where BC denotes the class of bounded continuous functions. As a consequence, the global solution $u(t)$ with $u(0) = a$ satisfies $\|u(t)\|_n \le C\varepsilon + C(1+t)^{-\gamma}$, which proves (6.5) (although the method above was mentioned in [36] and is well known, we gave the proof for completeness; see also Theorem 3 of Wiegner [59] for another proof). Combining (6.5) with (6.2) for $r = \infty$ immediately leads us to (2.8) for $n \le r < \infty$. We next prove (2.8) for $r = \infty$ and (2.9). As is standard, we rewrite the integral equation (6.1) in the form

$$u(t) = e^{-(t/2)A}u(t/2) - \int_{t/2}^t e^{-(t-\tau)A}P(u\cdot\nabla u)(\tau)d\tau, \quad t > 0. \qquad (6.7)$$

Then we obtain

$$\begin{aligned} & \|u(t)\|_\infty + \|\nabla u(t)\|_n \\ \le\ & Ct^{-1/2}\|u(t/2)\|_n + C\int_{t/2}^t (t-\tau)^{-3/4}\|u(\tau)\|_{2n}\|\nabla u(t)\|_n d\tau, \end{aligned}$$

from which together with (6.3) we at once deduce

$$t^{1/2}(\|u(t)\|_\infty + \|\nabla u(t)\|_n) \le C\|u(t/2)\|_n + C\|a\|_n \sup_{t/2\le\tau\le t}\tau^{1/4}\|u(\tau)\|_{2n},$$

for $t > 0$. Obviously, (6.5) and (2.8) for $r = 2n$ conclude both (2.8) for $r = \infty$ and (2.9). These immediately yield

$$\|P(u\cdot\nabla u)(t)\|_n \le C\|u(t)\|_\infty\|\nabla u(t)\|_n = o(t^{-1}), \qquad (6.8)$$

as $t \to \infty$, which will be used to show (2.10) below. Fix $\theta \in (0, 1/2)$ arbitrarily. Since

$$\|u(t) - u(\tau)\|_\infty + \|\nabla u(t) - \nabla u(\tau)\|_n \le C(t-\tau)^\theta\tau^{-1/2-\theta}\|a\|_n, \qquad (6.9)$$

for $0 < \tau < t$, one can deduce from (6.7) the representation

$$Au(t) = Ae^{-(t/2)A}u(t/2) + \{e^{-(t/2)A} - 1\}P(u \cdot \nabla u)(t) + Az(t), \qquad (6.10)$$

in $L^n_\sigma(\Omega)$, where

$$z(t) = \int_{t/2}^{t} e^{-(t-\tau)A}P\{(u \cdot \nabla u)(t) - (u \cdot \nabla u)(\tau)\}d\tau.$$

In fact, (6.9) implies

$$\begin{aligned} \|Az(t)\|_n &\le C\int_{t/2}^{t}(t-\tau)^{-1+\theta}\tau^{-1/2-\theta}(\|\nabla u(t)\|_n + \|u(\tau)\|_\infty)d\tau \\ &\le Ct^{-1/2}\|\nabla u(t)\|_n + Ct^{-1}\sup_{t/2\le\tau\le t}\tau^{1/2}\|u(\tau)\|_\infty, \end{aligned}$$

for $t > 0$. As a direct consequence of (2.8) for $r = \infty$ and (2.9), we see that $\|Az(t)\|_n = o(t^{-1})$, as $t \to \infty$. In view of (6.10), we collect (6.5), (6.8) and the above decay property of $Az(t)$ to obtain $\|Au(t)\|_n = o(t^{-1})$ as $t \to \infty$, which together with (6.8) again shows (2.10). The proof is complete. □

Remark 6.1. Consider briefly the 3-dimensional stability problem mentioned in Remark 2.5. The problem is reduced to the global existence and asymptotic behavior of the solution to

$$u(t) = e^{-tA}a - \int_0^t e^{-(t-\tau)A}P(u \cdot \nabla u + w \cdot \nabla u + u \cdot \nabla w)(\tau)d\tau, \quad t > 0,$$

where w is a stationary solution of class $\nabla w \in L^r(\Omega), 1 < r \le 2$, and $a \in L^3_\sigma(\Omega)$ is a given initial disturbance. Set

$$E(t) = \sup_{0<\tau\le t}\tau^{1/2}(\|u(\tau)\|_\infty + \|\nabla u(\tau)\|_3) + \sup_{0<\tau\le t}\tau^{1/4}\|u(\tau)\|_6,$$

and fix $r \in (1, 3/2)$ arbitrarily. Then the integral equation yields the a priori estimate

$$E(t) \le C\|a\|_3 + CE(t)^2 + C(\|\nabla w\|_r + \|\nabla w\|_2)E(t),$$

for $t > 0$, which gives an affirmative answer to the stability problem provided that both $\|\nabla w\|_r + \|\nabla w\|_2$ and $\|a\|_3$ are small enough. In fact, by following the argument of Chen [11], the above inequality for $E(t)$ is deduced from

$$\begin{aligned} &\|e^{-tA}P(w \cdot \nabla u + u \cdot \nabla w)\|_\infty + \|\nabla e^{-tA}P(w \cdot \nabla u + u \cdot \nabla w)\|_3 \\ \le\ & C(\|\nabla w\|_r + \|\nabla w\|_2)(\|u\|_\infty + \|\nabla u\|_3)\ t^{-3/4}(1+t)^{-3/2r+3/4}, \end{aligned}$$

and

$$\begin{aligned} &\|e^{-tA}P(w \cdot \nabla u + u \cdot \nabla w)\|_6 \\ \le\ & C(\|\nabla w\|_r + \|\nabla w\|_2)(\|u\|_\infty + \|\nabla u\|_3)\ t^{-1/2}(1+t)^{-3/2r+3/4}. \end{aligned}$$

We next assume that $a \in L^1(\Omega) \cap L^n_\sigma(\Omega)$ with $\|a\|_n \le \delta$. Let $u(t)$ be the global solution constructed in Theorem 2.2. Our particular concern is more rapid decay properties of $u(t)$. Starting from (6.2) and (6.3), we observe that $u(t) \in W^{1,r}(\Omega)$ for $1 < r < n$ and $t > 0$ (without going back to approximate solutions). In fact, there is a constant $M = M(\Omega, n, r, \|a\|_1, \|a\|_n, T) > 0$ such that

$$\begin{aligned}\|\nabla^j u(t)\|_r &\le Ct^{-j/2}\|a\|_r + C\int_0^t (t-\tau)^{-1+n/2r-j/2}\|u(\tau)\|_n\|\nabla u(\tau)\|_n d\tau \\ &\le Mt^{-j/2},\end{aligned}$$

for $n/2 \le r < n$, $j = 0, 1$ and $0 < t \le T$, where $T > 0$ is arbitrarily fixed; and then,

$$\begin{aligned}\|\nabla^j u(t)\|_r &\le Ct^{-j/2}\|a\|_r + C\int_0^t (t-\tau)^{-j/2}\|u(\tau)\|_{2r}\|\nabla u(\tau)\|_{2r} d\tau \\ &\le Mt^{-j/2},\end{aligned}$$

for $n/4 \le r < n/2$, $j = 0, 1$ and $0 < t \le T$. We repeat the process above to get $u(t) \in W^{1,r}(\Omega)$ for $1 < r < n$ with

$$\sup_{0<t\le T} (\|u(t)\|_r + t^{1/2}\|\nabla u(t)\|_r) \le M. \tag{6.11}$$

Remark 6.2. Following the argument of Kato [36], we see that the above constant M does not depend on $T > 0$ if $\|a\|_n$ is still smaller. However, we do not rely on his procedure because the smallness of initial data depends on $r > 1$. Note that the constant η in Theorem 2.3 is independent of $r > 1$.

As the first step of our proof of Theorem 2.3, we show the following lemma which gives a little slower decay rate than desired (later on, $\varepsilon > 0$ will be removed so that estimates will become sharp).

Lemma 6.1 *Let $n \ge 3$ and $a \in L^1(\Omega) \cap L^n_\sigma(\Omega)$. For any small $\varepsilon > 0$ there are constants $\eta_* = \eta_*(\Omega, n, \varepsilon) \in (0, \delta]$ and $C = C(\Omega, n, \|a\|_1, \|a\|_n, \varepsilon) > 0$ such that if $\|a\|_n \le \eta_*$, then the solution $u(t)$ obtained in Theorem 2.2 satisfies*

$$\|u(t)\|_{n/(n-1)} \le C(1+t)^{-1/2+\varepsilon}, \tag{6.12}$$

$$\|u(t)\|_{2n} \le Ct^{-1/4}(1+t)^{-n/2+1/2+\varepsilon}, \tag{6.13}$$

$$\|\nabla u(t)\|_n \le Ct^{-1/2}(1+t)^{-n/2+1/2+\varepsilon}, \tag{6.14}$$

for $t > 0$.

Proof. We make use of (1.3) for $r = \infty$ to obtain

$$\begin{aligned} & |(e^{-(t-\tau)A}P(u\cdot\nabla u)(\tau),\varphi)| = |((u\cdot\nabla u)(\tau), e^{-(t-\tau)A}\varphi)| \\ \le\ & C(t-\tau)^{-(n-n/q)/2}\|u(\tau)\|_{n/(n-1)}\|\nabla u(\tau)\|_n\|\varphi\|_{q/(q-1)}, \end{aligned}$$

for all $\varphi \in C^\infty_{0,\sigma}(\Omega)$, which gives

$$\begin{aligned} & \|\nabla^j e^{-(t-\tau)A}P(u\cdot\nabla u)(\tau)\|_q \\ \le\ & C(t-\tau)^{-(n-n/q)/2-j/2}\|u(\tau)\|_{n/(n-1)}\|\nabla u(\tau)\|_n, \end{aligned} \tag{6.15}$$

for $1 < q < \infty$, $j = 0, 1$ and $0 < \tau < t$ (the case $j = 1$ follows from (1.4) and the case $j = 0$). Given $\varepsilon > 0$, we take $p \in (1, n/(n-1))$ so that $1/p = 1 - 2\varepsilon/n$. From (6.15) with $q = n/(n-1)$ it follows that

$$\|u(t)\|_{n/(n-1)} \le Ct^{-1/2+\varepsilon}\|a\|_p + C\int_0^t (t-\tau)^{-1/2}\|u(\tau)\|_{n/(n-1)}\|\nabla u(\tau)\|_n d\tau.$$

In an analogous way to the deduction of (6.6), one can take a constant $\eta_0 = \eta_0(\Omega, n, \varepsilon) \in (0, \delta]$ such that if $\|a\|_n \le \eta_0$, then $\|u(t)\|_{n/(n-1)} \le Ct^{-1/2+\varepsilon}\|a\|_p$ for $t > 0$, which together with (6.11) gives (6.12). To show (6.13) and (6.14), we will derive

$$\|\nabla u(t)\|_r \le Ct^{-(n-n/r)/2-1/2+\varepsilon} \quad \text{for } r = n,\ 2n/3, \tag{6.16}$$

for $t > 0$. We divide the integral of (6.1) into two parts

$$\int_0^t e^{-(t-\tau)A}P(u\cdot\nabla u)(\tau)d\tau = \int_0^{t/2} + \int_{t/2}^t = v(t) + w(t); \tag{6.17}$$

then we obtain

$$\|\nabla u(t)\|_r \le Ct^{-(n-n/r)/2-1/2+\varepsilon}\|a\|_p + I_1 + I_2,$$

for $t > 0$ (p is the same as above) with

$$I_1 = \|\nabla v(t)\|_r \le C\int_0^{t/2}(t-\tau)^{-(n-n/r)/2-1/2}\|u(\tau)\|_{n/(n-1)}\|\nabla u(\tau)\|_n d\tau,$$

$$I_2 = \|\nabla w(t)\|_r \le C\int_{t/2}^t (t-\tau)^{-1/2}\|u(\tau)\|_\infty\|\nabla u(\tau)\|_r d\tau,$$

where (6.15) has been used in I_1. Using (6.12) together with (6.2) and (6.3), we see that

$$I_1 \le Ct^{-(n-n/r)/2-1/2+\varepsilon}\|a\|_n, \quad I_2 \le C\|a\|_n \sup_{t/2\le\tau\le t}\|\nabla u(t)\|_r,$$

for $t > 0$. Therefore, setting

$$E_r(t) = \sup_{0<\tau\le t} \tau^{(n-n/r)/2+1/2-\varepsilon}\|\nabla u(\tau)\|_r \quad \text{for } r = n,\ 2n/3,$$

we get $E_r(t) \le C\|a\|_p + C\|a\|_n + C_0\|a\|_n E_r(t)$ for $t > 0$, where $C_0 > 0$ is independent of a. As a consequence, there is a constant $\eta_* = \eta_*(\Omega, n, \varepsilon) \in (0, \eta_0]$ such that if $\|a\|_n \le \eta_*$, then $E_n(t) + E_{2n/3}(t) \le C$ for $t > 0$, which proves (6.16). This combined with the Sobolev embedding, (6.2) for $r = 2n$ and (6.3) imply (6.13) and (6.14). □

Based on Lemma 6.1, we supply the proof of Theorem 2.3, by which we conclude this paper.

Proof of Theorem 2.3. We fix $\varepsilon \in (0, 1/2)$ and put $\eta = \eta(\Omega, n) = \eta_*(\Omega, n, \varepsilon)$. Assuming $\|a\|_n \le \eta$, we first show (2.11). Since

$$\|e^{-tA}a\|_r \le Ct^{-(n-n/r)/2}\|a\|_1,$$

for $t > 0$, our task is to derive the required estimate of (6.17). By (6.15) together with (6.12) and (6.14) we have

$$\begin{aligned}
\|v(t)\|_r &\le C\int_0^{t/2} (t-\tau)^{-(n-n/r)/2}\|u(\tau)\|_{n/(n-1)}\|\nabla u(\tau)\|_n d\tau \\
&\le Ct^{-(n-n/r)/2}\int_0^\infty \tau^{-1/2}(1+\tau)^{-n/2+2\varepsilon}d\tau \\
&\le Ct^{-(n-n/r)/2},
\end{aligned}$$

for $1 < r \le \infty$ and $t > 0$; here, note that the case $r = \infty$ follows from the L^q-L^∞ estimate (1.3) together with (6.15). If $1 < r < n/(n-2)$, then the same estimate of the integrand as above works well on $w(t)$ too; as a result, we have

$$\|w(t)\|_r \le Ct^{-(n-n/r)/2-n/2+1/2+2\varepsilon},$$

for $t > 0$. For $r = \infty$, we make use of (6.13) and (6.14) to get

$$\|w(t)\|_\infty \le C\int_{t/2}^t (t-\tau)^{-3/4}\|u(\tau)\|_{2n}\|\nabla u(\tau)\|_n d\tau \le Ct^{-n+1/2+2\varepsilon},$$

for $t > 0$. We collect the estimates above to obtain (2.11) for $1 < r < n/(n-2)$ and $r = \infty$; and the remaining case $n/(n-2) \le r < \infty$ follows via interpolation as well.

We next show (2.12). Let $1 < r \le n$. In view of (6.7), we have

$$\|\nabla u(t)\|_r \le Ct^{-1/2}\|u(t/2)\|_r + \|\nabla w(t)\|_r,$$

for $t > 0$, where $w(t)$ is the same as above. By (2.11) the proof is reduced to the estimate of $\|\nabla w(t)\|_r$. If in particular $1 < r < n/(n-1)$, then from

(2.11), (6.14) and (6.15) we deduce

$$\begin{aligned} \|\nabla w(t)\|_r &\le C\int_{t/2}^t (t-\tau)^{-(n-n/r)/2-1/2}\|u(\tau)\|_{n/(n-1)}\|\nabla u(\tau)\|_n d\tau \\ &\le Ct^{-(n-n/r)/2-n/2+\varepsilon}, \end{aligned}$$

for $t > 0$. If $r = n$, then one appeals again to (2.11) and (6.14) to find

$$\|\nabla w(t)\|_n \le C\int_{t/2}^t (t-\tau)^{-1/2}\|u(\tau)\|_\infty\|\nabla u(\tau)\|_n d\tau \le Ct^{-n+1/2+\varepsilon},$$

for $t > 0$. We thus obtain (2.12) for $1 < r < n/(n-1)$ and $r = n$; and the case $n/(n-1) \le r < n$ also follows via interpolation. It remains to show the case $n < r < \infty$. From (1.4) for $1 < q < n < r < \infty$ we deduce

$$\|\nabla u(t)\|_r \le Ct^{-(n/q-n/r)/2-1/2}\|u(t/2)\|_q + \|\nabla w(t)\|_r,$$

for $t > 0$, and the first term possesses the desired decay property on account of (2.11). We take p in such a way that $1/n < 1/p < 1/n + 1/r$. Since we have already known (2.12) for $r = p$ as well as (2.11), we are led to

$$\begin{aligned} \|\nabla w(t)\|_r &\le C\int_{t/2}^t (t-\tau)^{-(n/p-n/r)/2-1/2}\|u(\tau)\|_\infty\|\nabla u(\tau)\|_p d\tau \\ &\le Ct^{-n+n/2r}, \end{aligned}$$

for $t > 0$, which proves (2.12) for $n < r < \infty$.

Finally, by use of (6.10), we show (2.13) and thereby (2.14) and (2.15). From (2.11) and (2.12) it follows that

$$\|Ae^{-(t/2)A}u(t/2)\|_r \le Ct^{-1}\|u(t/2)\|_r = O\left(t^{-(n-n/r)/2-1}\right), \tag{6.18}$$

as $t \to \infty$ and that

$$\|P(u\cdot\nabla u)(t)\|_r \le C\|u(t)\|_\infty\|\nabla u(t)\|_r = O\left(t^{-n+n/2r-1/2}\right), \tag{6.19}$$

as $t \to \infty$. We are thus going to estimate

$$\begin{aligned} \|Az(t)\|_r &\le C\|\nabla u(t)\|_r\int_{t/2}^t (t-\tau)^{-1}\|u(t)-u(\tau)\|_\infty d\tau \\ &\quad +C\int_{t/2}^t (t-\tau)^{-1}\|u(\tau)\|_\infty\|\nabla u(t)-\nabla u(\tau)\|_r d\tau = I_1 + I_2. \end{aligned}$$

With the aid of (6.9) and (2.12) we observe

$$I_1 = O\left(t^{-(n-n/r)/2-1}\right), \tag{6.20}$$

as $t \to \infty$. We need also a Hölder estimate of $\nabla u(t)$ in $L^r(\Omega)$, which implies the decay property of I_2 as well as $u \in C(0,\infty; D(A_r)) \cap C^1(0,\infty; L^r_\sigma(\Omega))$. To this end, let us consider

$$\begin{aligned}
& u(t) - u(\tau) \\
= \ & \{e^{-(t-\tau/2)A} - e^{-(\tau/2)A}\}u(\tau/2) - \int_\tau^t e^{-(t-s)A}P(u\cdot\nabla u)(s)ds \\
& -\int_{\tau/2}^{\tau}\{e^{-(t-s)A} - e^{-(\tau-s)A}\}P(u\cdot\nabla u)(s)ds \\
= \ & w_1(t,\tau) + w_2(t,\tau) + w_3(t,\tau),
\end{aligned}$$

for $0 < \tau < t$. By a standard calculation with use of (2.11) we have

$$\|\nabla w_1(t,\tau)\|_r \le C(t-\tau)^\theta \tau^{-(n-n/r)/2-1/2-\theta}, \tag{6.21}$$

where $0 < \theta < 1$. In order to estimate w_2 and w_3, we take $q \in (1, r]$ so that $0 < 1/q - 1/n < 1/r$; then we see from (6.19) in $L^q_\sigma(\Omega)$ that

$$\|\nabla w_2(t,\tau)\|_r \le C(t-\tau)^{1/2-(n/q-n/r)/2}\tau^{-n+n/2q-1/2}, \tag{6.22}$$

and that

$$\begin{aligned}
& \|\nabla w_3(t,\tau)\|_r \\
\le \ & C\int_{\tau/2}^{\tau}(t-\tau)^\theta(\tau-s)^{-(n/q-n/r)/2-1/2-\theta}\|P(u\cdot\nabla u)(s)\|_q ds \\
\le \ & C(t-\tau)^\theta\tau^{-n+n/2r-\theta},
\end{aligned} \tag{6.23}$$

where $0 < \theta < 1/2 - (n/q - n/r)/2$. Collecting (6.21), (6.22) and (6.23) together with (2.11) yields

$$I_2 = O\left(t^{-n+n/2r-1/2}\right), \tag{6.24}$$

as $t \to \infty$. From (6.18), (6.19), (6.20) and (6.24) we obtain (2.13). Due to (1.8) and in view of the equation (1.1), we deduce (2.14) immediately from (2.11), (2.12) and (2.13). By Lemma 3.1 of [20] there exist $p_\pm(t) \in \mathbb{R}$ such that $\|p(t) - p_\pm(t)\|_{r,\Omega_\pm} + |[p(t)]| \le C\|\nabla p(t)\|_q$ for $1 < q < n$ and $1/r = 1/q - 1/n$. Hence, (2.14) implies (2.15). The proof is complete. □

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