

$$G = (\exp \mathfrak{g})^2$$

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More than one hundred years ago, F. ENGEL and E. STUDY considered the question of the surjectivity of the exponential function of Lie groups ([2],[3], compare also the remarks in [9]). Interestingly, a general solution of this problem is not found yet though there exist equivalent criteria for the surjectivity of special classes of Lie groups (compare [8]). So it is amazing that there exists a short proof for a conjecture of M. MOSKOWITZ and R. SACKSTEDER that every real connected Lie group is equal to  $(\exp \mathfrak{g})^2$ . We present this proof in the paper in hand.

In the whole article,  $G$  denotes a real connected Lie group. First, let us assume that  $G$  is semisimple. A subgroup  $K \subseteq G$  is called *compactly embedded* if  $\text{Ad}(K)$  is relatively compact in  $\text{GL}(\mathfrak{g})$ . It is shown in [7] that there exists maximal compactly embedded subgroups and that all maximal compactly embedded subgroups are closed and connected. The last property ensures that all maximal compactly embedded subgroups are conjugate to each other. Moreover, it is well-known that every compactly embedded connected Lie group has surjective exponential function.

First, let us assume that  $G$  is semisimple. The Iwasawa decomposition says that every semisimple Lie group can be written as  $KAM$  where  $K$  is maximal compactly embedded,  $A$  is abelian connected and consists of  $\text{Ad}$ -semisimple elements with  $\text{spec ad}(a) \subseteq \mathbb{R}$  for every  $a \in \mathfrak{a}$ ,  $M$  is nilpotent connected, and  $AM$  is solvable. By DIXMIER's and SAITO's Theorem ([1],[6]),  $S := AM = \exp(\mathfrak{a} + \mathfrak{n})$ .

Now we recall the Levi decomposition: If  $G$  is a real connected Lie group, then for every maximal semisimple subgroup (called *Levi factor*)  $L$  we have  $G = L \text{Rad}(G)$  and  $L \cap \text{Rad}(G)$  discrete. All Levi factors are conjugate to each other.

A *Cartan subalgebra*  $\mathfrak{h} \subset \mathfrak{g}$  of a real finite dimensional Lie algebra  $\mathfrak{g}$  is a nilpotent subalgebra which equals its own normalizer  $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h})$ . Cartan subgroups are defined in various ways which in case of connected Lie groups are all equivalent (compare [5]). We will use the following definition:  $H \subseteq G$  is a Cartan subgroup if  $\mathfrak{h}$  is a Cartan subalgebra and for all  $g \in H$  and all  $x \in \mathfrak{h}$  we have  $\text{Ad}(g) \text{ad}(x)_s = \text{ad}(x)_s \text{Ad}(g)$  where  $\text{ad}(x)_s$  is the semisimple part of the Jordan decomposition of  $\text{ad}(x)$ .

Theorem 1.9 (ii) of [10] states that for all Cartan subgroups  $H$  the intersection  $H \cap \text{Rad}(G)$  is connected. Since  $H \cap \text{Rad}(G)$  is nilpotent, we have  $H \cap \text{Rad}(G) = \exp(\mathfrak{h} \cap \mathfrak{rad}(\mathfrak{g}))$ . Theorem 1.11 of [10] says that for each Cartan subgroup  $H$  of  $G$  there exists a Levi complement  $L$  such that  $H \cap L$  is a Cartan subgroup of  $L$ ,  $H = (H \cap L)(H \cap \text{Rad}(G))$ , and  $H \cap \text{Rad}(G) \subseteq Z_G(L)$ . This leads to the following:

**Lemma.** If  $\mathfrak{h}$  is a Cartan subalgebra of a real finite-dimensional Lie algebra  $\mathfrak{g}$  and  $\mathfrak{n}$  the nilradical of  $\mathfrak{g}$ , then  $\mathfrak{rad}(\mathfrak{g}) = (\mathfrak{h} \cap \mathfrak{rad}(\mathfrak{g})) + \mathfrak{n}$ . If  $H$  is a Cartan subgroup of a real Lie group  $G$  and  $N$  the nilradical, then  $\text{Rad}(G) = (H \cap \text{Rad}(G))N$ .

*Proof.* We choose  $\mathfrak{l}$  such that  $\mathfrak{h} = (\mathfrak{h} \cap \mathfrak{l}) \oplus (\mathfrak{h} \cap \mathfrak{rad}(\mathfrak{g}))$  (compare Theorem 1.8 of [10]). We observe that  $\mathfrak{g}' \subseteq \mathfrak{l} + \mathfrak{n}$ . Moreover,  $\mathfrak{g} = \mathfrak{h} + \mathfrak{g}' = (\mathfrak{h} \cap \mathfrak{rad}(\mathfrak{g})) + \mathfrak{n} + \mathfrak{l}$ , hence  $\mathfrak{rad}(\mathfrak{g}) = (\mathfrak{h} \cap \mathfrak{rad}(\mathfrak{g})) + \mathfrak{n}$ . The proof for Lie groups works equivalently.

Thus, with the above notation we gain  $G = KS(H \cap \text{Rad}(G))N = K(H \cap \text{Rad}(G))SN = \exp \mathfrak{k} \exp(\mathfrak{h} \cap \mathfrak{rad}(\mathfrak{g})) \exp \mathfrak{s} \exp \mathfrak{n}$ . Since  $[\mathfrak{k}, \mathfrak{h} \cap \mathfrak{rad}(\mathfrak{g})] = \{0\}$ , we get  $\exp \mathfrak{k} \exp(\mathfrak{h} \cap \mathfrak{rad}(\mathfrak{g})) = \exp(\mathfrak{k} + (\mathfrak{h} \cap \mathfrak{rad}(\mathfrak{g})))$ . Moreover,  $SN$  is a solvable connected Lie group and  $\text{spec ad}_{\mathfrak{g}}(s) \subseteq \mathbb{R}$  for each  $s \in \mathfrak{s}$  by Proposition 3.7 of [11]. Again by Dixmier's and Saito's Theorem we get  $SN = \exp(\mathfrak{s} + \mathfrak{n})$ . By Proposition 3.7 of [11], every element  $x \in \mathfrak{s} + \mathfrak{n}$  satisfies  $\text{spec ad}(x) \subseteq \mathbb{R}$ . So, we have proved the following theorem:

**Theorem.** If  $G$  is a real connected Lie group, then  $G = (\exp \mathfrak{g})^2$ . Moreover, for every element  $g \in G$  there is an  $\exp$ -regular element  $x$  and an element  $y$  in  $\mathfrak{g}$  with  $g = \exp x \exp y$ .

**Definition.** An element  $x$  of a real finite-dimensional Lie algebra is called *exp-regular* if  $\text{spec ad } x \cap$

$2\pi i\mathbb{Z} = \{0\}$ . The set of all exp-regular elements of  $\mathfrak{g}$  is denoted by  $\text{reg exp}$ .

**Corollary.** If  $G$  is a connected real Lie group and for every  $g \in G$  there is an exp-regular  $x \in \mathfrak{g}$  and a  $y \in \mathfrak{g}$  with  $g = \exp x \exp y$ , then there are exp-regular  $u, w \in \mathfrak{g}$  with  $g = \exp u \exp w$ .

*Proof.* Since  $\text{reg exp}$  is dense in  $\mathfrak{g}$  (for example by Lemma 2 of [4]), the image  $\exp(\text{reg exp})$  is dense in  $\exp \mathfrak{g}$ . Since  $\exp$  is regular at each  $x \in \mathfrak{g}$ , the set  $\exp(\text{reg exp})$  is open in  $G$ , hence also in  $\exp \mathfrak{g}$ . The openness of  $\exp(\text{reg exp})$  implies that there is a symmetric 1-neighborhood  $U$  such that  $\exp x \cdot U \subseteq \exp(\text{reg exp})$ . The density implies that  $\exp y \cdot U \cap \text{reg exp} \neq \emptyset$ . So there is a  $u \in U$  such that  $(\exp x)u^{-1}$  and  $u \exp y$  are in  $\exp(\text{reg exp})$ . This implies the assertion.

## References

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