Remarks on the Quasistatic Problem of Viscoelasticity

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Existence, Uniqueness and Homogenization

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Abstract
This article is devoted to the following quasistatic problem of viscoelasticity:

$$-\operatorname{div}_x\{L(x)(E(\nabla_x r) - u)\} = b(x,t), \qquad r|_{x\in\partial\Omega} = r_\Gamma(x,t). \tag{0.1a}$$

$$\partial_t u + \dot{U}(-L(x)(E(\nabla_x r) - u) + \Lambda(x)u) \ni 0, \qquad u|_{t=0} = u_0(x). \tag{0.1b}$$

$$E(w) := \frac{1}{2}(w + w^T).$$

In (0.1) r describes the displacement, u describes the the plastic strain, L describes the elastic modulus, and Λ describes the plastic modulus. It turns out that (0.1) can be rewritten in the following way:

$$\partial_t u + A(\Phi(u) + \varphi(t)) \ni 0, \qquad u(0) = u_0. \tag{0.2}$$

In (0.2) A denotes a maximal monotone operator on some real Hilbert space $\mathcal{H}$. In section 2 we consider the abstract problem (0.2). We prove existence, uniqueness and stability of solutions with respect to the data (Φ, φ, f, u_0). In section 3 we apply our abstract results to the viscoelastic problem (0.1). First we prove existence and uniqueness of solutions to the n–dimensional problem. Next we consider the 1–dimensional case where $\Omega = (0,1)$. Therefore, we divide the interval $(0,1)$ into a grid of gridlength $\frac{1}{n}$. Our basic assumption is that in each grid point the corresponding elastic modulus L_n admits one of the values $\underline{l}$ and $\overline{l}$ with probability p and $1-p$ respectively where $\underline{l} < \overline{l}$. Applying the stability result of section 2 we show that the expectation values $\mathrm{Ex}[u_n]$ of the corresponding solutions u_n converge to some limit function u as $n \longrightarrow \infty$, and that u is the solution to a homogenized problem corresponding to some constant elastic modulus L. We close our discussion with some remarks on the homogenization of the n–dimensional problem.

Contents

1 Introduction

This article is devoted to the quasistatic problem of viscoelasticity. We assume that at a given reference point $x \in \Omega$ and at a given time $t \in [0,T]$ the balance of force and the evolution of the plastic strain read as follows:

$$-\operatorname{div}_x \Big\{ L(x)\Big(E(\nabla_x r(x,t)) - u(x,t)\Big)\Big\} = b(x,t), \qquad r(x,t)\Big|_{x\in\partial\Omega} = r_\Gamma(x,t). \tag{1.1a}$$

$$\partial_t u(x,t) + \dot{U}\Big(- L(x)\Big(E(\nabla_x r(x,t)) - u(x,t)\Big) + \Lambda(x)u(x,t)\Big) \ni 0,$$

$$u(x,0) = u_0(x). \tag{1.1b}$$

$$E(\nabla_x r(x,t)) := \frac{1}{2}\Big\{\nabla_x r(x,t) + \Big(\nabla_x r(x,t)\Big)^T\Big\}. \tag{1.1c}$$

In (1.1) r describes the displacement, u describes the the plastic strain, L describes the elastic modulus, Λ describes the plastic modulus, and $\dot{U}$ describes the (negative of the) plastic strain rate. We assume that the moduli L and Λ are linear symmetric positive definite mapings defined on the set $\mathbb{R}^{n\times n}_{\text{sym}}$ of symmetric n by n matrices, and that $\dot{U}$ is a maximal monotone subset of $\mathbb{R}^{n\times n}_{\text{sym}} \times \mathbb{R}^{n\times n}_{\text{sym}}$ containing $(0,0)$. Our strategy is to solve the linear elliptic boundary value problem (1.1a) for r first and to insert the respective solution operator into the initial value problem (1.1b) for u. We make the following definition:

$$\mathcal{H} := L^2(\Omega, \mathbb{R}^{n\times n}_{\text{sym}}). \tag{1.2}$$

Now the initial value problem (1.1b) reads:

$$\partial_t u(t) + A\Big(\Phi\big(u(t)\big) + \varphi(t)\Big) \ni 0, \qquad u(0) = u_0. \tag{1.3}$$

By construction A is a maximal monotone subset of $\mathcal{H} \times \mathcal{H}$ containing $(0,0)$, Φ is a linear continuous symmetric positive definite maping defined on $\mathcal{H}$ and φ is a given function depending on the data b and r_Γ.

In section 2 we consider the following abstract initial value problem given in some Hilbert space $\mathcal{H}$:

$$\partial_t u(t) + A\Big(\Phi\big(u(t)\big) + \varphi(t)\Big) \ni f(t), \qquad u(0) = u_0. \tag{1.4}$$

In (1.4) A is a maximal monotone subset of $\mathcal{H} \times \mathcal{H}$ containing $(0,0)$, and Φ is a linear continuous symmetric positive definite maping defined on $\mathcal{H}$. We can choose different approaches in order to prove existence and uniqueness of solutions u to problem (1.4). On the one hand we can rewrite (1.4) as a monotone initial value problem and make use of the well known theory of monotone sets, cf. [1] (Alber). Therefore, we make the following definitions:

$$v(t) := \Phi\big(u(t)\big) + \varphi(t), \qquad g(t) := \Phi\big(f(t)\big) + \partial_t\varphi(t), \qquad B(x) := \Phi\big(A(x)\big). \tag{1.5}$$

Now problem (1.4) reads:

$$\partial_t v(t) + B\big(v(t)\big) \ni g(t), \qquad v(0) = \Phi(u_0) + \varphi(0). \tag{1.6}$$

By construction B is a maximal monotone subset of $\mathcal{H} \times \mathcal{H}$ with respect to the equivalent scalar product on $\mathcal{H}$ defined by Φ^{-1}. In this case the solution v to problem (1.6) is obtained by replacing B with its Yoshida approximation B_λ and passing to the limit $\lambda \longrightarrow 0+$. On the other hand, in the article at hand we choose a different approach, cf. [5] (Chelminski). Following the lines of [11] (Pazy) we directly prove existence and uniqueness of solutions u to problem (1.4). In this case the solution u to problem (1.4) is obtained by replacing A with its Yoshida approximation A_λ and passing to the limit $\lambda \longrightarrow 0+$. This program is carried out in subsections 2.1 and 2.2. In subsection 2.3 we prove stability of solutions u to (1.4) with respect to the data (Φ, φ, f, u_0). We note that in view of our application to the homogenization of the quasistatic viscoelastic problem (1.3) the stability of solutions with respect to Φ is of particular interest. Moreover, we note that in (1.6) both, the monotone set B as well as the respective scalar product on $\mathcal{H}$, explicitly depend on Φ, whereas in (1.4) both, the monotone set A as well as the scalar product on $\mathcal{H}$, are independent of Φ. Consequently, in view of the stability of solutions the initial value problem (1.4) is easier to handle than the initial value problem (1.6). Finally, we note that our definition of a solution requires Lipschitz continuity with respect to t. Consequently, by Rademacher's theorem a solution is differentiable in the classical sense almost everywhere, see [11] (Pazy) and compare with [4] (Brezis).

In section 3 we consider the quasistatic viscoelastic problem (1.1). In subsection 3.1 we first give a precise formulation of the problem and the underlying assumptions. Next we rewrite (1.1) in the form (1.3). Applying the results of section 2 we obtain existence and uniqueness of solutions u to the n–dimensional problem (1.1). In subsection 3.2 we consider the 1–dimensional case $\Omega = (0,1)$ of problem (1.1):

$$
\begin{aligned}
& -\partial_x\Big\{L(x)\Big(\partial_x r(x,t) - u(x,t)\Big)\Big\} = b(x,t), \\
& r(0,t) = \rho_0(t), \qquad r(1,t) = \rho_1(t).
\end{aligned} \tag{1.7a}
$$

$$
\begin{aligned}
& \partial_t u(x,t) + \dot{U}\Big(- L(x)\Big(\partial_x r(x,t) - u(x,t)\Big) + \Lambda(x)u(x,t)\Big) \ni 0, \\
& u(x,0) = u_0(x).
\end{aligned} \tag{1.7b}
$$

First we show existence and uniqueness of solutions u to problem (1.7) analogous to the previous subsection. Next we turn to the question of statistic homogenization of problem (1.7). Therefore we divide the interval $(0,1)$ into a grid of gridlength $\frac{1}{n}$. Our basic assumption is that in each grid point the corresponding elastic modulus $\underline{L_n}$ admits one of the values $\underline{l}$ and $\bar{l}$ with probability p and $1-p$ respectively where $\underline{l} < \bar{l}$. Applying the stability result of subsection 2.3 we show that the expectation values $\mathrm{Ex}[u_n]$ of the corresponding solutions u_n converge to some limit function u as $n \longrightarrow \infty$, and that u is the solution to a homogenized problem corresponding to some constant elastic modulus L. We note that this is essentially a consequence of the law of large numbers. We close our discussion in subsection 3.3 with some remarks on homogenization of the n–dimensional problem (1.1). Our basic assumption is that the elastic modulus is now given by some rapidly oscillating function

$$
L^{\varepsilon}(x) := L\Big(\frac{x}{\varepsilon}\Big) \tag{1.8}
$$

where L is $[0,1]^n$–periodic. Unlike in the 1–dimensional case a formal argument shows that we cannot expect that the corresponding solutions u^{ε} converge to some limit function u as $\varepsilon \longrightarrow 0+$, and that u is the solution to a homogenized problem corresponding to some constant elastic modulus L^0. More precisely, we expect this to be true if and only if L satisfies a structural condition of the following form:

$$
\Big(L(y)^{-1}\Big)_{ijkl} = \frac{1}{2}\Big(\partial_{y^i}\phi_{jkl}(y) + \partial_{y^j}\phi_{ikl}(y)\Big) = \frac{1}{2}\Big(\partial_{y^k}\phi_{lij}(y) + \partial_{y^l}\phi_{kij}(y)\Big). \tag{1.9}
$$

An example shows that at least for Lame's law of linear elasticity the structural condition (1.9) is satisfied. However, homogenization for the general case remains an open problem and is subject to recent research, cf. [2] (Alber).

2 Abstract Theory

Let $\mathcal{H}$ be a Hilbert space over $\mathbb{R}$, and let $T > 0$. We consider the following initial value problem:

$$\partial_t u(t) + A\Big(\Phi\Big(u(t)\Big) + \varphi(t)\Big) \ni f(t). \tag{2.1a}$$

$$u(0) = u_0. \tag{2.1b}$$

Throughout this section we make the following assumptions:

1. Let $A \subset \mathcal{H} \times \mathcal{H}$ be a monotone set with domain $\mathcal{D}(A) \subset \mathcal{H}$.
2. Let $\Phi : \mathcal{H} \longrightarrow \mathcal{H}$ be linear, continuous, symmetric and positive definit.
3. Let $\varphi \in \mathcal{C}^2([0,T],\mathcal{H})$, and let $f \in \mathcal{C}^1([0,T],\mathcal{H})$.
4. Let $\Phi(u_0) + \varphi(0) \in \mathcal{D}(A)$.

We say that u is a *solution* to the initial boundary value problem (2.1) if the following statements hold:

1. $u \in \mathcal{C}^{0,1}([0,T],\mathcal{H})$.
2. $\Phi\Big(u(t)\Big) + \varphi(t) \in \mathcal{D}(A) \ \forall\ 0 \le t \le T$.
3. $u(t)$ satisfies the PDE (2.1a) in the classical sense $\forall\ 0 \le t \le T$ almost everywhere.
4. $u(0)$ satisfies the initial condition (2.1b).

REMARK (Rademacher's theorem)
Let $u \in \mathcal{C}^{0,1}([0,T],\mathcal{H})$. Then the following statements hold:

(a) u is differentiable in the classical sense $\forall\ 0 \le t \le T$ almost everywhere.

(b) Let $0 \le a \le b \le T$. Then the following statement holds:

$$u(b) - u(a) = \int_a^b \partial_t u(t)\,\mathrm{d}t. \tag{2.2}$$

In order to simplify the initial value problem (2.1) we make the following definitions:

$$\overline{u}(t) := u(t) + \Phi^{-1}\Big(\varphi(t)\Big). \tag{2.3a}$$

$$\overline{f}(t) := f(t) + \Phi^{-1}\Big(\partial_t\varphi(t)\Big). \qquad \Longrightarrow \qquad \overline{f} \in \mathcal{C}^1([0,T],\mathcal{H}). \tag{2.3b}$$

$$\overline{u}_0 := u_0 + \Phi^{-1}\Big(\varphi(0)\Big). \qquad \Longrightarrow \qquad \Phi(\overline{u}_0) \in \mathcal{D}(A). \tag{2.3c}$$

Now the initial value problem (2.1) reads:

$$\partial_t \overline{u}(t) + A\Big(\Phi\Big(\overline{u}(t)\Big)\Big) \ni \overline{f}(t). \tag{2.4a}$$

$$\overline{u}(0) = \overline{u}_0. \tag{2.4b}$$

REMARK
The following statements are equivalent:

1. u is a solution to the initial value problem (2.1).
2. $\overline{u}$ is a solution to the initial value problem (2.4).

We define an equivalent scalar product $\langle \cdot \mid \cdot \rangle_\Phi$ on $\mathcal{H}$ by:

$$\langle y \mid x \rangle_\Phi := \langle y \mid \Phi(x) \rangle\,. \tag{2.5}$$

We define the corresponding equivalent norm $\|\cdot\|_\Phi$ on $\mathcal{H}$ by:

$$\|x\|_\Phi := \sqrt{\langle x \mid x \rangle_\Phi}. \tag{2.6}$$

By construction Φ is invertible. By the bounded inverse theorem Φ^{-1} is linear and continuous. By construction Φ^{-1} is symmetric and positive definit. This yields:

$$\|x\|^2 = \left\langle \Phi^{-1}(x) \mid x \right\rangle_\Phi \leq \left\|\Phi^{-1}(x)\right\|_\Phi \|x\|_\Phi \leq \sqrt{\|\Phi^{-1}\|}\, \|x\|\, \|x\|_\Phi\,. \tag{2.7}$$

Consequently, the following estimates hold:

$$\|x\| \leq \sqrt{\|\Phi^{-1}\|}\, \|x\|_\Phi\,, \qquad \|x\|_\Phi \leq \sqrt{\|\Phi\|}\, \|x\|\,. \tag{2.8}$$

2.1 Uniqueness of Solutions

THEOREM 2.1 (Uniqueness of solutions)
Let $\overline{u}_i$ be solutions to the initial value problem (2.4) *corresponding to the data* $(\overline{f}_i, \overline{u}_{i0})$. *Then the following estimate holds* $\forall\ 0 \leq t \leq T$:

$$\|\overline{u}_1(t) - \overline{u}_2(t)\| \leq 2\sqrt{\|\Phi\|\, \|\Phi^{-1}\|}\Big(\, \|\overline{u}_{10} - \overline{u}_{20}\| + T\, \left\|\overline{f}_1 - \overline{f}_2\right\|_{\mathcal{C}^0([0,T],\mathcal{H})}\Big). \tag{2.9}$$

In particular, the initial value problem (2.4) *has at most one solution* $\overline{u}$.

PROOF
By construction $\exists\ v_i : [0,T] \longrightarrow \mathcal{H}$ with the following properties:

$$\Big(\Phi\Big(\overline{u}_i(t)\Big), v_i(t)\Big) \in A \qquad \forall\ 0 \leq t \leq T. \tag{2.10a}$$

$$\partial_t \overline{u}_i(t) + v_i(t) = \overline{f}_i(t) \qquad \forall\ 0 \leq t \leq T \text{ almost everywhere.} \tag{2.10b}$$

By assumption A is monotone. With the help of (2.10) we obtain $\forall\ 0 \leq t \leq T$ almost everywhere:

$$\begin{aligned}
&\partial_t \left\|\overline{u}_1(t) - \overline{u}_2(t)\right\|_\Phi^2 = 2\left\langle \partial_t \overline{u}_1(t) - \partial_t \overline{u}_2(t) \,\middle|\, \overline{u}_1(t) - \overline{u}_2(t)\right\rangle_\Phi \\
&= 2\left\langle \overline{f}_1(t) - \overline{f}_2(t) \,\middle|\, \overline{u}_1(t) - \overline{u}_2(t)\right\rangle_\Phi - 2\left\langle v_1(t) - v_2(t) \,\middle|\, \Phi\Big(\overline{u}_1(t)\Big) - \Phi\Big(\overline{u}_2(t)\Big)\right\rangle \\
&\leq 2\left\|\overline{f}_1(t) - \overline{f}_2(t)\right\|_\Phi \left\|\overline{u}_1(t) - \overline{u}_2(t)\right\|_\Phi \\
&\leq T\left\|\overline{f}_1(t) - \overline{f}_2(t)\right\|_\Phi^2 + \frac{1}{T}\left\|\overline{u}_1(t) - \overline{u}_2(t)\right\|_\Phi^2 .
\end{aligned} \tag{2.11}$$

With the help of (2.4b), (2.11) and Rademacher's theorem we obtain $\forall\ 0 \leq t \leq T$:

$$\begin{aligned}
&\left\|\overline{u}_1(t) - \overline{u}_2(t)\right\|_\Phi^2 = \left\|\overline{u}_1(0) - \overline{u}_2(0)\right\|_\Phi^2 + \int_0^t \partial_s \left\|\overline{u}_1(s) - \overline{u}_2(s)\right\|_\Phi^2 \,\mathrm{d}s \\
&\leq \left\|\overline{u}_{10} - \overline{u}_{20}\right\|_\Phi^2 + T\int_0^T \left\|\overline{f}_1(t) - \overline{f}_2(t)\right\|_\Phi^2 \,\mathrm{d}t + \frac{1}{T}\int_0^t \left\|\overline{u}_1(s) - \overline{u}_2(s)\right\|_\Phi^2 \,\mathrm{d}s.
\end{aligned} \tag{2.12}$$

With the help of (2.12) and Gronwall's lemma we obtain:

$$\left\|\overline{u}_1(t) - \overline{u}_2(t)\right\|_\Phi^2 \leq \exp\left(\frac{t}{T}\right)\left(\left\|\overline{u}_{10} - \overline{u}_{20}\right\|_\Phi^2 + T\int_0^T \left\|\overline{f}_1(t) - \overline{f}_2(t)\right\|_\Phi^2 \,\mathrm{d}t\right). \tag{2.13}$$

With the help of (2.8) and (2.13) we obtain (2.9).
□

THEOREM 2.2
Let $\overline{u}$ be a solution to the PDE (2.4a), let $0 \leq a < b < T$, and let $\overline{u}$ be differentiable in the classical sense at a and b. Then the following estimate holds:

$$\left\|\partial_t \overline{u}(b)\right\| \leq 2\sqrt{\|\Phi\|\,\|\Phi^{-1}\|}\left(\left\|\partial_t \overline{u}(a)\right\| + T\left\|\partial_t \overline{f}\right\|_{\mathcal{C}^0([0,T],\mathcal{H})}\right). \tag{2.14}$$

PROOF
Let $h > 0$. We make the following definitions:

$$\overline{u}_1(t) := \overline{u}(t + a + h), \qquad \overline{u}_2(t) := \overline{u}(t + a). \tag{2.15a}$$

$$\overline{u}_{10} := \overline{u}(a + h), \qquad \overline{u}_{20} := \overline{u}(a). \tag{2.15b}$$

$$\overline{f}_1(t) := \overline{f}(t+a+h), \qquad \overline{f}_2(t) := \overline{f}(t+a). \tag{2.15c}$$

By construction the $\overline{u}_i$ are solutions to the following initial value problems:

$$\partial_t \overline{u}_i(t) + A\Big(\Phi\Big(\overline{u}_i(t)\Big)\Big) \ni \overline{f}_i(t), \qquad \overline{u}_i(0) = \overline{u}_{i0}. \tag{2.16}$$

With the help of theorem 2.1 we obtain:

$$\begin{aligned}
&\|\overline{u}(b+h) - \overline{u}(b)\| = \|\overline{u}_2(b-a) - \overline{u}_1(b-a)\| \\
&\le 2\sqrt{\|\Phi\|\,\|\Phi^{-1}\|}\Big(\,\|\overline{u}_{10} - \overline{u}_{20}\| + T\,\big\|\overline{f}_1 - \overline{f}_2\big\|_{\mathcal{C}^0([0,T-a-h],\mathcal{H})}\Big) \\
&= 2\sqrt{\|\Phi\|\,\|\Phi^{-1}\|}\Big(\,\|\overline{u}(a+h) - \overline{u}(a)\| + T\,\big\|\overline{f}(\cdot+h) - \overline{f}\big\|_{\mathcal{C}^0([a,T-h],\mathcal{H})}\Big).
\end{aligned} \tag{2.17}$$

This yields (2.14).
□

2.2 Existence of Solutions

We make the following additional assumption:

1.a Let A be maximal monotone.

LEMMA 2.3
Let $\mathcal{X}$ be a Hilbert space over $\mathbb{R}$, let $x_i \in \mathcal{X}$, and let $\lambda_i > 0$ with the following properties:

$$\langle x_i - x_j \mid \lambda_i x_i - \lambda_j x_j \rangle_{\mathcal{X}} \le 0. \tag{2.18}$$

Then the following statements hold:

(a) *Let λ_i be monotone increasing as $i \longrightarrow \infty$. Then the following statements hold:*

  (i) *$\|x_i\|_{\mathcal{X}}$ is monotone decreasing as $i \longrightarrow \infty$.*

  (ii) *$x_i \xrightarrow{i\to\infty} x$ strongly in $\mathcal{X}$.*

(b) *Let λ_i be monotone decreasing as $i \longrightarrow \infty$. Then the following statements hold:*

  (i) *$\|x_i\|_{\mathcal{X}}$ is monotone increasing as $i \longrightarrow \infty$.*

  (ii) *If $\lim\limits_{i\to\infty} \|x_i\|_{\mathcal{X}} < \infty$ then $x_i \xrightarrow{i\to\infty} x$ strongly in $\mathcal{X}$.*

PROOF
By assumption we have:

$$(\lambda_i + \lambda_j)\,\|x_i - x_j\|^2_{\mathcal{X}} + (\lambda_i - \lambda_j)\Big(\,\|x_i\|^2_{\mathcal{X}} - \|x_j\|^2_{\mathcal{X}}\Big)$$

$$
\begin{aligned}
&= 2\Big(\lambda_i \|x_i\|^2_{\mathcal{X}} - (\lambda_i + \lambda_j) \langle x_i \mid x_j \rangle_{\mathcal{X}} + \lambda_j \|x_i\|^2_{\mathcal{X}}\Big) \\
&= 2 \langle x_i - x_j \mid \lambda_i x_i - \lambda_j x_j \rangle_{\mathcal{X}} \\
&\leq 0. \qquad (2.19)
\end{aligned}
$$

This yields:

$$
(\lambda_i - \lambda_j)\Big(\|x_i\|^2_{\mathcal{X}} - \|x_j\|^2_{\mathcal{X}} \Big) \leq 0. \qquad (2.20a)
$$

$$
\|x_i - x_j\|^2_{\mathcal{X}} \leq \left| \frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j} \right| \left| \|x_i\|^2_{\mathcal{X}} - \|x_j\|^2_{\mathcal{X}} \right|. \qquad (2.20b)
$$

With the help of (2.20a) we obtain (a) (i) and (b) (i). Now let the sequence $\{\|x_i\|_{\mathcal{X}}\}_{i=1}^{\infty}$ be monotone and bounded. Then $\{\|x_i\|_{\mathcal{X}}\}_{i=1}^{\infty}$ converges. With the help of (2.20b) we obtain (a) (ii) and (b) (ii).
□

THEOREM 2.4 (Existence of solutions)
The initial value problem (2.4) has at least one solution $\overline{u}$. In particular, the following estimate holds $\forall\ 0 \leq t \leq T$ almost everywhere:

$$
\begin{aligned}
&\|\partial_t \overline{u}(t)\| \\
&\leq 2\sqrt{\|\Phi\| \, \|\Phi^{-1}\|}\Big(\left\|A^0\Big(\Phi(\overline{u}_0)\Big)\right\| + \|\overline{f}\|_{\mathcal{C}^0([0,T],\mathcal{H})} + T \left\|\partial_t \overline{f}\right\|_{\mathcal{C}^0([0,T],\mathcal{H})} \Big). \qquad (2.21)
\end{aligned}
$$

In (2.21) $A^0(x)$ denotes the minimal selection of $A(x)$.

PROOF
Sei $\lambda > 0$. We define the the Yoshida approximation A_λ of A by:

$$
A_\lambda := \{(x + \lambda y, y) \mid (x, y) \in A\}. \qquad (2.22)
$$

From the general theory of monotone sets we know that $A_\lambda : \mathcal{H} \longrightarrow \mathcal{H}$ is single valued and Lipschitz continuous with Lipschitz constant $\frac{1}{\lambda}$. We consider the following approximated initial value problem:

$$
\partial_t \overline{u}_\lambda(t) + A_\lambda\Big(\Phi\Big(\overline{u}_\lambda(t)\Big)\Big) = \overline{f}(t), \qquad \overline{u}_\lambda(0) = u_0. \qquad (2.23)
$$

Since A_λ is globally Lipschitz continuous the initial value problem (2.23) has a unique classical solution:

$$
\overline{u}_\lambda \in \mathcal{C}^1([0,T],\mathcal{H}). \qquad (2.24)
$$

From the general theory of monotone sets we know the following estimate:

$$
\|A_\lambda(x)\| \leq \left\|A^0(x)\right\| \qquad \forall\ x \in \mathcal{H}. \qquad (2.25)
$$

With the help of theorem 2.2, (2.23) and (2.25) we obtain $\forall\ 0 \le t \le T$:

$$\begin{aligned}
&\|\partial_t \overline{u}_\lambda(t)\| \\
&\le 2\sqrt{\|\Phi\|\,\|\Phi^{-1}\|}\Big(\ \|\partial_t \overline{u}_\lambda(0)\| + T\,\big\|\partial_t \overline{f}\big\|_{\mathcal{C}^0([0,T],\mathcal{H})}\Big) \\
&\le 2\sqrt{\|\Phi\|\,\|\Phi^{-1}\|}\Big(\ \Big\|A_\lambda\Big(\Phi(\overline{u}_0)\Big)\Big\| + \big\|\overline{f}(0)\big\| + T\,\big\|\partial_t \overline{f}\big\|_{\mathcal{C}^0([0,T],\mathcal{H})}\Big) \\
&\le 2\sqrt{\|\Phi\|\,\|\Phi^{-1}\|}\Big(\ \Big\|A^0\Big(\Phi(\overline{u}_0)\Big)\Big\| + \big\|\overline{f}\big\|_{\mathcal{C}^0([0,T],\mathcal{H})} + T\,\big\|\partial_t \overline{f}\big\|_{\mathcal{C}^0([0,T],\mathcal{H})}\Big) \\
&=: L.
\end{aligned} \tag{2.26a}$$

$$\Big\|A_\lambda\Big(\Phi\Big(\overline{u}_\lambda(t)\Big)\Big)\Big\| \le \|\partial_t \overline{u}_\lambda(t)\| + \big\|\overline{f}(t)\big\| \le 2L. \tag{2.26b}$$

Let $\lambda_i \xrightarrow{i\to\infty} 0+$. We show that $\{\overline{u}_{\lambda_i}\}_{i=1}^{\infty}$ is a Cauchy sequence in $\mathcal{C}^0([0,T],\mathcal{H})$. We define the the resolvent J_λ of A by:

$$J_\lambda := \{(x+\lambda y, x)\,|\,(x,y)\in A\}\,. \tag{2.27}$$

From the general theory of monotone sets we know that $J_\lambda : \mathcal{H} \longrightarrow \mathcal{H}$ is single valued. With the help of (2.23) we obtain $\forall\ 0 \le t \le T$:

$$\begin{aligned}
&\big\|\overline{u}_{\lambda_i}(t) - \overline{u}_{\lambda_j}(t)\big\|_\Phi^2 \\
&= -2\int_0^t \Big\langle A_{\lambda_i}\Big(\Phi(\overline{u}_{\lambda_i})\Big) - A_{\lambda_j}\Big(\Phi(\overline{u}_{\lambda_j})\Big)\,\Big|\,\Phi(\overline{u}_{\lambda_i}) - \Phi(\overline{u}_{\lambda_j})\Big\rangle\,\mathrm{d}s \\
&= -2\int_0^t \Big\langle A_{\lambda_i}\Big(\Phi(\overline{u}_{\lambda_i})\Big) - A_{\lambda_j}\Big(\Phi(\overline{u}_{\lambda_j})\Big)\Big|\,\Big\{\Phi(\overline{u}_{\lambda_i}) - J_{\lambda_i}\Big(\Phi(\overline{u}_{\lambda_i})\Big)\Big\} \\
&\qquad\qquad -\Big\{\Phi(\overline{u}_{\lambda_j}) - J_{\lambda_i}\Big(\Phi(\overline{u}_{\lambda_j})\Big)\Big\}\Big\rangle\,\mathrm{d}s \\
&\quad - 2\int_0^t \Big\langle A_{\lambda_i}\Big(\Phi(\overline{u}_{\lambda_i})\Big) - A_{\lambda_j}\Big(\Phi(\overline{u}_{\lambda_j})\Big)\,\Big|\,J_{\lambda_i}\Big(\Phi(\overline{u}_{\lambda_i})\Big) - J_{\lambda_i}\Big(\Phi(\overline{u}_{\lambda_j})\Big)\Big\rangle\,\mathrm{d}s.
\end{aligned} \tag{2.28}$$

From the general theory of monotone sets we know the following fact:

$$\Big(J_\lambda(x), A_\lambda(x)\Big) \in A \qquad \forall\ x \in \mathcal{H}. \tag{2.29}$$

By assumption A is monotone. This yields $\forall\ 0 \le t \le T$:

$$\Big\langle A_{\lambda_i}\Big(\Phi(\overline{u}_{\lambda_i})\Big) - A_{\lambda_j}\Big(\Phi(\overline{u}_{\lambda_j})\Big)\,\Big|\,J_{\lambda_i}\Big(\Phi(\overline{u}_{\lambda_i})\Big) - J_{\lambda_i}\Big(\Phi(\overline{u}_{\lambda_j})\Big)\Big\rangle \ge 0. \tag{2.30}$$

With the help of (2.8), (2.26b), (2.28) and (2.30) we obtain $\forall\ 0 \le t \le T$:

$$\big\|\overline{u}_{\lambda_i}(t) - \overline{u}_{\lambda_j}(t)\big\|^2 \le \big\|\Phi^{-1}\big\|\,\big\|\overline{u}_{\lambda_i}(t) - \overline{u}_{\lambda_j}(t)\big\|_\Phi^2$$

$$
\begin{aligned}
&\leq -2\left\|\Phi^{-1}\right\| \int_0^t \Big\langle A_{\lambda_i}\Big(\Phi(\overline{u}_{\lambda_i})\Big) - A_{\lambda_j}\Big(\Phi(\overline{u}_{\lambda_j})\Big) \Big| \Big\{\Phi(\overline{u}_{\lambda_i}) - J_{\lambda_i}\Big(\Phi(\overline{u}_{\lambda_i})\Big)\Big\} \\
&\qquad\qquad - \Big\{\Phi(\overline{u}_{\lambda_j}) - J_{\lambda_i}\Big(\Phi(\overline{u}_{\lambda_j})\Big)\Big\}\Big\rangle \, \mathrm{d}s \\
&= -2\left\|\Phi^{-1}\right\| \int_0^t \Big\langle A_{\lambda_i}\Big(\Phi(\overline{u}_{\lambda_i})\Big) - A_{\lambda_j}\Big(\Phi(\overline{u}_{\lambda_j})\Big) \Big| \\
&\qquad\qquad \lambda_i A_{\lambda_i}\Big(\Phi(\overline{u}_{\lambda_i})\Big) - \lambda_j A_{\lambda_j}\Big(\Phi(\overline{u}_{\lambda_j})\Big)\Big\rangle \, \mathrm{d}s \\
&\leq 2(\lambda_i + \lambda_j)\left\|\Phi^{-1}\right\| \int_0^t \Big(\Big\|A_{\lambda_i}\Big(\Phi(\overline{u}_{\lambda_i})\Big)\Big\| + \Big\|A_{\lambda_j}\Big(\Phi(\overline{u}_{\lambda_j})\Big)\Big\|\Big)^2 \, \mathrm{d}s \\
&\leq 32(\lambda_i + \lambda_j)\left\|\Phi^{-1}\right\| T L^2.
\end{aligned} \tag{2.31}
$$

Consequently $\{\overline{u}_{\lambda_i}\}_{i=1}^{\infty}$ is a Cauchy sequence in $\mathcal{C}^0([0,T],\mathcal{H})$. This yields:

$$
\overline{u}_{\lambda_i} \xrightarrow{i\to\infty} \overline{u} \qquad \text{in } \mathcal{C}^0([0,T],\mathcal{H}). \tag{2.32}
$$

This yields $\forall\ 0 \leq t \leq T$:

$$
\overline{u}_{\lambda_i}(t) \xrightarrow{i\to\infty} \overline{u}(t) \qquad \text{strongly in } \mathcal{H}. \tag{2.33}
$$

In particular, $\overline{u}$ satisfies the initial condition (2.4b). By (2.26a) the $\overline{u}_{\lambda_i}$ are Lipschitz continuous with Lipschitz constant L. Consequently $\overline{u}$ is Lipschitz continuous with Lipschitz constant L. With the help of Rademacher's theorem we obtain (2.21).

Let $0 \leq t \leq T$ be fixed, and let $\lambda_i \xrightarrow{i\to\infty} 0+$. We show that $\Phi\Big(u(t)\Big) \in \mathcal{D}(A)$. With the help of (2.26b) we obtain:

$$
\Big\|\Phi\Big(\overline{u}_{\lambda_i}(t)\Big) - J_{\lambda_i}\Big(\Phi\Big(\overline{u}_{\lambda_i}(t)\Big)\Big)\Big\| = \lambda_i \Big\|A_{\lambda_i}\Big(\Phi\Big(\overline{u}_{\lambda_i}(t)\Big)\Big)\Big\| \leq 2\lambda_i L \xrightarrow{i\to\infty} 0. \tag{2.34}
$$

With the help of (2.33) and (2.34) we obtain:

$$
J_{\lambda_i}\Big(\Phi\Big(\overline{u}_{\lambda_i}(t)\Big)\Big) \xrightarrow{i\to\infty} \Phi\Big(u(t)\Big) \qquad \text{strongly in } \mathcal{H}. \tag{2.35}
$$

By (2.26b) the sequence $\Big\{A_{\lambda_i}\Big(\Phi\Big(\overline{u}_{\lambda_i}(t)\Big)\Big)\Big\}_{i=1}^{\infty}$ is bounded in $\mathcal{H}$. This yields:

$$
A_{\lambda_{i_j}}\Big(\Phi\Big(\overline{u}_{\lambda_{i_j}}(t)\Big)\Big) \xrightarrow{j\to\infty} v(t) \qquad \text{weakly in } \mathcal{H}. \tag{2.36}
$$

From the general theory of monotone sets we know that (2.29), (2.35) and (2.36) together imply:

$$
\Big(\Phi\Big(\overline{u}(t)\Big), v(t)\Big) \in A. \tag{2.37}
$$

Consequently $\Phi\Big(\overline{u}(t)\Big) \in \mathcal{D}(A)$.

We show that $\overline{u}$ satisfies the PDE (2.4a) in the classical sense $\forall\ 0 \leq t \leq T$ almost everywhere. Let $\lambda_i \xrightarrow{i\to\infty} 0+$ be monotone decreasing. We define:

$$\mathcal{X} := L^2([0,T],\mathcal{H}). \tag{2.38}$$

With the help of (2.23) and (2.30) we obtain:

$$\begin{aligned}
&\Big\langle A_{\lambda_i}\Big(\Phi(\overline{u}_{\lambda_i})\Big) - A_{\lambda_j}\Big(\Phi(\overline{u}_{\lambda_j})\Big)\,\Big|\,\lambda_i A_{\lambda_i}\Big(\Phi(\overline{u}_{\lambda_i})\Big) - \lambda_j A_{\lambda_j}\Big(\Phi(\overline{u}_{\lambda_j})\Big)\Big\rangle_{\mathcal{X}}\\
&= \int_0^T \Big\langle A_{\lambda_i}\Big(\Phi(\overline{u}_{\lambda_i})\Big) - A_{\lambda_j}\Big(\Phi(\overline{u}_{\lambda_j})\Big)\,\Big|\,\lambda_i A_{\lambda_i}\Big(\Phi(\overline{u}_{\lambda_i})\Big) - \lambda_j A_{\lambda_j}\Big(\Phi(\overline{u}_{\lambda_j})\Big)\Big\rangle\,\mathrm{d}t\\
&= \int_0^T \Big\langle \{A_{\lambda_i}\Big(\Phi(\overline{u}_{\lambda_i})\Big) - A_{\lambda_j}\Big(\Phi(\overline{u}_{\lambda_j})\Big)\Big|\,\Big\{\Phi(\overline{u}_{\lambda_i}) - J_{\lambda_i}\Big(\Phi(\overline{u}_{\lambda_i})\Big)\Big\}\\
&\qquad\qquad -\Big\{\Phi(\overline{u}_{\lambda_j}) - J_{\lambda_j}\Big(\Phi(\overline{u}_{\lambda_i})\Big)\Big\}\Big\rangle\,\mathrm{d}t\\
&\leq \int_0^T \Big\langle A_{\lambda_i}\Big(\Phi(\overline{u}_{\lambda_i})\Big) - A_{\lambda_j}\Big(\Phi(\overline{u}_{\lambda_j})\Big)\,\Big|\,\Phi(\overline{u}_{\lambda_i}) - \Phi(\overline{u}_{\lambda_j})\Big\rangle\,\mathrm{d}t\\
&= -\frac{1}{2}\left\|\overline{u}_{\lambda_i}(T) - \overline{u}_{\lambda_j}(T)\right\|_{\Phi}^2\\
&\leq 0.
\end{aligned} \tag{2.39}$$

With the help of (2.26b) we obtain:

$$\left\|A_{\lambda_i}\Big(\Phi(\overline{u}_{\lambda_i})\Big)\right\|_{\mathcal{X}}^2 = \int_0^T \left\|A_{\lambda_i}\Big(\Phi(\overline{u}_{\lambda_i})\Big)\right\|^2\,\mathrm{d}s \leq 4TL^2. \tag{2.40}$$

With the help of (2.39), (2.40) and lemma 2.3 (b) we obtain:

$$A_{\lambda_i}\Big(\Phi(\overline{u}_{\lambda_i})\Big) \xrightarrow{i\to\infty} \overline{v} \qquad \text{strongly in } \mathcal{X}. \tag{2.41}$$

This yields $\forall\ 0 \leq t \leq T$ almost everywhere:

$$A_{\lambda_{i_j}}\Big(\Phi\Big(\overline{u}_{\lambda_{i_j}}(t)\Big)\Big) \xrightarrow{j\to\infty} \overline{v}(t) \qquad \text{strongly in } \mathcal{H}. \tag{2.42}$$

With the help of (2.36), (2.37) and (2.42) we obtain $\forall\ 0 \leq t \leq T$ almost everywhere:

$$\overline{v}(t) = v(t) \in A\Big(\Phi\Big(u(t)\Big)\Big). \tag{2.43}$$

Now let $0 \leq t < t+h \leq T$. With the help of (2.23), (2.33) and (2.41) we obtain:

$$\begin{aligned}
&\overline{u}(t+h) - \overline{u}(t) = \lim_{i\to\infty}\Big(\overline{u}_{\lambda_i}(t+h) - \overline{u}_{\lambda_i}(t)\Big) = \lim_{i\to\infty}\int_t^{t+h} \partial_s \overline{u}_{\lambda_i}(s)\,\mathrm{d}s\\
&= \lim_{i\to\infty}\Big(\int_t^{t+h} \overline{f}(s)\,\mathrm{d}s - \int_t^{t+h} A_{\lambda_i}\Big(\overline{u}_{\lambda_i}(s)\Big)\,\mathrm{d}s\Big)
\end{aligned}$$

$$= \int_t^{t+h} \overline{f}(s)\,\mathrm{d}s - \int_t^{t+h} \overline{v}(s)\,\mathrm{d}s. \tag{2.44}$$

With the help of (2.26b) and (2.42) we obtain $\forall\ 0 \le t \le T$ almost everywhere:

$$\|\overline{v}(t)\| = \lim_{j\to\infty} \left\| A_{\lambda_{i_j}}\left(\Phi\left(\overline{u}_{\lambda_{i_j}}(t)\right)\right)\right\| \le 2L. \tag{2.45}$$

With the help of (2.44) and (2.45) we obtain $\forall\ 0 \le t \le T$ almost everywhere:

$$\frac{1}{h}\Big(\overline{u}(t+h) - \overline{u}(t)\Big) = \frac{1}{h}\int_t^{t+h} \overline{f}(s)\,\mathrm{d}s - \frac{1}{h}\int_t^{t+h} \overline{v}(s)\,\mathrm{d}s \xrightarrow{h\to 0+} \overline{f}(t) - \overline{v}(t). \tag{2.46}$$

This yields $\forall\ 0 \le t \le T$ almost everywhere:

$$\partial_t \overline{u}(t) + \overline{v}(t) = \overline{f}(t). \tag{2.47}$$

Consequently $\overline{u}$ satisfies the PDE (2.4a) in the classical sense $\forall\ 0 \le t \le T$ almost everywhere.
□

2.3 Stability of Solutions with Respect to the Data

We make the following additional assumption:

1.b Let A be maximal monotone, and let $(0,0) \in A$.

LEMMA 2.5
Let $\overline{u}$ be the solution to the initial value problem (2.4)*. Then the following estimates hold:*

$$\|\overline{u}\|_{\mathcal{C}^0([0,T],\mathcal{H})} \le 2\sqrt{\|\Phi\|\,\|\Phi^{-1}\|}\Big(\|\overline{u}_0\| + T\,\big\|\overline{f}\big\|_{\mathcal{C}^0([0,T],\mathcal{H})}\Big). \tag{2.48a}$$

$$\begin{aligned}&\|\partial_t \overline{u}\|_{L^\infty([0,T],\mathcal{H})}\\ &\le 2\sqrt{\|\Phi\|\,\|\Phi^{-1}\|}\Big(\left\|A^0\left(\Phi(\overline{u}_0)\right)\right\| + \big\|\overline{f}\big\|_{\mathcal{C}^0([0,T],\mathcal{H})} + T\,\big\|\partial_t\overline{f}\big\|_{\mathcal{C}^0([0,T],\mathcal{H})}\Big).\end{aligned} \tag{2.48b}$$

PROOF
By assumption $(0,0) \in A$. Consequently $\overline{u} = 0$ is the solution to the initial value problem (2.4) corresponding to the data $(\overline{f}, \overline{u}_0) = (0,0)$. With the help of theorem 2.1 we obtain (2.48a). Now, (2.48b) is an immediate consequence of theorem 2.4.
□

THEOREM 2.6 (Stability of solutions with respect to the data)
Let $\overline{u}_i$ be the solutions to the initial value problem (2.4) *corresponding to the data* $(\Phi_i, \overline{f}_i, \overline{u}_{i0})$*. Then the following estimate holds:*

$$\begin{aligned} &\|\overline{u}_1 - \overline{u}_2\|_{\mathcal{C}^0([0,T],\mathcal{H})} \\ &\leq C\Big\{ \|\Phi_1 - \Phi_2\|^{\frac{1}{2}} + \Big(\int_0^T \big\|\overline{f}_1(t) - \overline{f}_2(t)\big\| \, \mathrm{d}t\Big)^{\frac{1}{2}} + \|\overline{u}_{10} - \overline{u}_{20}\| \Big\}. \end{aligned} \tag{2.49a}$$

$$C = \hat{C}\Big(T, \|\Phi_i\|, \big\|\Phi_i^{-1}\big\|, \|\overline{f}_i\|_{\mathcal{C}^1([0,T],\mathcal{H})}, \|\overline{u}_{i0}\|, \Big\|A^0\Big(\Phi_i(\overline{u}_{i0})\Big)\Big\| \Big). \tag{2.49b}$$

PROOF
We define:

$$K_i := 2\sqrt{\|\Phi_i\| \, \big\|\Phi_i^{-1}\big\|}\Big(\|\overline{u}_{i0}\| + T \, \|\overline{f}_i\|_{\mathcal{C}^0([0,T],\mathcal{H})} \Big). \tag{2.50a}$$

$$L_i := 2\sqrt{\|\Phi_i\| \, \big\|\Phi_i^{-1}\big\|}\Big(\Big\|A^0\Big(\Phi_i(\overline{u}_{i0})\Big)\Big\| + \|\overline{f}_i\|_{\mathcal{C}^0([0,T],\mathcal{H})} + T \, \|\partial_t\overline{f}_i\|_{\mathcal{C}^0([0,T],\mathcal{H})} \Big). \tag{2.50b}$$

By construction $\exists \; v_i : [0,T] \longrightarrow \mathcal{H}$ with the following properties:

$$\Big(\Phi_i\Big(\overline{u}_i(t)\Big), v_i(t)\Big) \in A \qquad \forall \; 0 \leq t \leq T. \tag{2.51a}$$

$$\partial_t\overline{u}_i(t) + v_i(t) = \overline{f}_i(t) \qquad \forall \; 0 \leq t \leq T \text{ almost everywhere.} \tag{2.51b}$$

By assumption A is monotone. With the help of (2.51) and lemma 2.5 we obtain $\forall$ $0 \leq t \leq T$ almost everywhere:

$$\begin{aligned} &\partial_t \|\overline{u}_1(t) - \overline{u}_2(t)\|^2_{\Phi_1} = 2\Big\langle \partial_t\Big(\overline{u}_1(t) - \overline{u}_2(t)\Big) \Big| \, \Phi_1\Big(\overline{u}_1(t) - \overline{u}_2(t)\Big)\Big\rangle \\ &= 2\Big\langle \partial_t\overline{u}_1(t) - \partial_t\overline{u}_2(t) \Big| \, \Phi_1\Big(\overline{u}_1(t)\Big) - \Phi_2\Big(\overline{u}_2(t)\Big)\Big\rangle \\ &\quad - 2\Big\langle \partial_t\overline{u}_1(t) - \partial_t\overline{u}_2(t) \Big| \, \Phi_1\Big(\overline{u}_2(t)\Big) - \Phi_2\Big(\overline{u}_2(t)\Big)\Big\rangle \\ &= 2\Big\langle \overline{f}_1(t) - \overline{f}_2(t) \Big| \, \Phi_1\Big(\overline{u}_1(t)\Big) - \Phi_2\Big(\overline{u}_2(t)\Big)\Big\rangle \\ &\quad - 2\Big\langle v_1(t) - v_2(t) \Big| \, \Phi_1\Big(\overline{u}_1(t)\Big) - \Phi_2\Big(\overline{u}_2(t)\Big)\Big\rangle \\ &\quad - 2\Big\langle \partial_t\overline{u}_1(t) - \partial_t\overline{u}_2(t) \Big| \, \Phi_1\Big(\overline{u}_2(t)\Big) - \Phi_2\Big(\overline{u}_2(t)\Big)\Big\rangle \\ &\leq 2\Big(\|\Phi_1\| \, \|\overline{u}_1(t)\| + \|\Phi_2\| \, \|\overline{u}_2(t)\| \Big) \big\|\overline{f}_1(t) - \overline{f}_2(t)\big\| \\ &\quad + 2\,\|\Phi_1 - \Phi_2\| \, \|\overline{u}_2(t)\| \Big(\|\partial_t\overline{u}_1(t)\| + \|\partial_t\overline{u}_2(t)\| \Big) \end{aligned}$$

$$\leq 2\Big(\|\Phi_1\| K_1 + \|\Phi_2\| K_2 \Big) \left\|\overline{f}_1(t) - \overline{f}_2(t)\right\| + 2\|\Phi_1 - \Phi_2\| K_2(L_1 + L_2). \tag{2.52}$$

With the help of (2.8), (2.52) and Rademacher's theorem we obtain $\forall\ 0 \leq t \leq T$:

$$\begin{aligned}
&\|\overline{u}_1(t) - \overline{u}_2(t)\|^2 \leq \left\|\Phi_1^{-1}\right\| \|\overline{u}_1(t) - \overline{u}_2(t)\|_{\Phi_1}^2 \\
&= \left\|\Phi_1^{-1}\right\| \|\overline{u}_{10} - \overline{u}_{20}\|_{\Phi_1}^2 + \left\|\Phi_1^{-1}\right\| \int_0^t \partial_s \|\overline{u}_1(s) - \overline{u}_2(s)\|_{\Phi_1}^2 \, \mathrm{d}s \\
&\leq \|\Phi_1\| \left\|\Phi_1^{-1}\right\| \|\overline{u}_{10} - \overline{u}_{20}\|^2 \\
&\qquad + 2\left\|\Phi_1^{-1}\right\| \Big(\|\Phi_1\| K_1 + \|\Phi_2\| K_2 \Big) \int_0^T \left\|\overline{f}_1(t) - \overline{f}_2(t)\right\| \, \mathrm{d}t \\
&\qquad + 2T \|\Phi_1 - \Phi_2\| \left\|\Phi_1^{-1}\right\| K_2(L_1 + L_2).
\end{aligned} \tag{2.53}$$

Symmetrizing with respect to the indices 1 and 2 yields:

$$\begin{aligned}
&\|\overline{u}_1 - \overline{u}_2\|_{\mathcal{C}^0([0,T],\mathcal{H})}^2 \\
&\leq \frac{1}{2}\Big(\|\Phi_1\| + \|\Phi_2\| \Big)\Big(\left\|\Phi_1^{-1}\right\| + \left\|\Phi_2^{-1}\right\| \Big) \|\overline{u}_{10} - \overline{u}_{20}\|^2 \\
&\qquad + \Big(\|\Phi_1\| + \|\Phi_2\| \Big)\Big(\left\|\Phi_1^{-1}\right\| + \left\|\Phi_2^{-1}\right\| \Big)(K_1 + K_2) \int_0^T \left\|\overline{f}_1(t) - \overline{f}_2(t)\right\| \, \mathrm{d}t \\
&\qquad + T\|\Phi_1 - \Phi_2\| \Big(\left\|\Phi_1^{-1}\right\| + \left\|\Phi_2^{-1}\right\| \Big)(K_1 + K_2)(L_1 + L_2).
\end{aligned} \tag{2.54}$$

This yields (2.49).
□

COROLLARY 2.7 (Stability of solutions with respect to the data)
Let u_i be the solutions to the initial value problem (2.1) corresponding to the data $(\Phi_i, \varphi_i, f_i, u_{i0})$. Then the following estimate holds:

$$\begin{aligned}
&\|u_1 - u_2\|_{\mathcal{C}^0([0,T],\mathcal{H})} \\
&\leq C\Big\{ \|\Phi_1 - \Phi_2\|^{\frac{1}{2}} + \|\varphi_1 - \varphi_2\|_{\mathcal{C}^0([0,T],\mathcal{H})} + \Big(\int_0^T \|\partial_t\varphi_1(t) - \partial_t\varphi_2(t)\| \, \mathrm{d}t \Big)^{\frac{1}{2}} \\
&\qquad + \Big(\int_0^T \|f_1(t) - f_2(t)\| \, \mathrm{d}t \Big)^{\frac{1}{2}} + \|u_{10} - u_{20}\| \Big\}.
\end{aligned} \tag{2.55a}$$

$$\begin{aligned}
C = \hat{C}\Big(&T, \|\Phi_i\|, \left\|\Phi_i^{-1}\right\|, \|\varphi_i\|_{\mathcal{C}^2([0,T],\mathcal{H})} \|f_i\|_{\mathcal{C}^1([0,T],\mathcal{H})}, \|u_{i0}\|, \\
&\left\| A^0\Big(\Phi_i(u_{i0}) + \varphi_i(0)\Big) \right\| \Big).
\end{aligned} \tag{2.55b}$$

PROOF
We recall the following well known facts:

$$\left\|\Phi_1^{-1}(x_1) - \Phi_2^{-1}(x_2)\right\| \leq \left\| \Big(\Phi_1^{-1} - \Phi_2^{-1}\Big)(x_1) \right\| + \left\|\Phi_2^{-1}(x_1 - x_2)\right\|$$

$$\leq \left\|\Phi_1^{-1} - \Phi_2^{-1}\right\| \|x_1\| + \left\|\Phi_2^{-1}\right\| \|x_1 - x_2\| . \tag{2.56a}$$

$$\left\|\Phi_1^{-1} - \Phi_2^{-1}\right\| = \left\|\Phi_1^{-1}(\Phi_2 - \Phi_1)\Phi_2^{-1}\right\| \leq \|\Phi_1 - \Phi_2\| \left\|\Phi_1^{-1}\right\| \left\|\Phi_2^{-1}\right\|$$
$$\leq \|\Phi_1 - \Phi_2\|^{\frac{1}{2}} \Big(\|\Phi_1\| + \|\Phi_2\| \Big)^{\frac{1}{2}} \left\|\Phi_1^{-1}\right\| \left\|\Phi_2^{-1}\right\| . \tag{2.56b}$$

Now (2.55) is an immediate consequence of (2.3) and (2.49).
□

3 Application to Viscoelasticity

3.1 Existence and Uniqueness of Solutions

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary Γ, and let $T > 0$. Ω describes a material body, and $[0, T]$ describes the time interval of observation. We consider the following functions:

$$r : \overline{\Omega} \times [0, T] \longrightarrow \mathbb{R}^n : (x, t) \longmapsto r(x, t). \tag{3.1a}$$

$$u : \overline{\Omega} \times [0, T] \longrightarrow \mathbb{R}^{n \times n}_{\text{sym}} : (x, t) \longmapsto u(x, t). \tag{3.1b}$$

In (3.1) $\mathbb{R}^{n \times n}_{\text{sym}}$ denotes the set of symmetric n by n matrices. r describes the displacement, and u describes the plastic strain. We consider the following quasistatic problem of viscoelasticity:

$$-\operatorname{div}_x \Big\{ S\Big(E(\nabla_x r(x, t)), u(x, t), x\Big) \Big\} = b(x, t), \qquad r(x, t)\Big|_{x \in \Gamma} = r_\Gamma(x, t). \tag{3.2a}$$

$$\partial_t u(x, t) + \dot{U}\Big(\Sigma\Big(E(\nabla_x r(x, t)), u(x, t), x\Big)\Big) \ni 0, \qquad u(x, 0) = u_0(x). \tag{3.2b}$$

(3.2a) describes the balance of force, and (3.2b) describes the evolution of the plastic strain. We make the following assumptions:

1. Let $b \in \mathcal{C}^2([0, T], H^{-1}(\Omega, \mathbb{R}^n))$.
2. Let $r_\Gamma \in \mathcal{C}^2([0, T], H^{\frac{1}{2}}(\partial\Omega, \mathbb{R}^n))$.
3. Let $u_0 \in L^2(\Omega, \mathbb{R}^{n \times n}_{\text{sym}})$.
4. Let $\dot{U} \subset \mathbb{R}^{n \times n}_{\text{sym}} \times \mathbb{R}^{n \times n}_{\text{sym}}$ be a maximal monotone set, and let $(0, 0) \in \dot{U}$.
5. Let $L \in L^\infty(\Omega, (\mathbb{R}^{n \times n}_{\text{sym}} \otimes \mathbb{R}^{n \times n}_{\text{sym}})_{\text{sym}})$, let $0 < \underline{l} \leq \overline{l}$, and let the following estimate hold in the sense of symmetric tensors $\forall\, x \in \Omega$ almost everywhere:

$$\underline{l} \leq L(x) \leq \overline{l}. \tag{3.3}$$

6. Let $\Lambda \in L^\infty(\Omega, (\mathbb{R}^{n\times n}_{\text{sym}} \otimes \mathbb{R}^{n\times n}_{\text{sym}})_{\text{sym}})$, let $0 < \underline{\lambda} \leq \overline{\lambda}$, and let the following estimate hold in the sense of symmetric tensors $\forall\, x \in \Omega$ almost everywhere:

$$\underline{\lambda} \leq \Lambda(x) \leq \overline{\lambda}. \tag{3.4}$$

In the above assumptions $(\mathcal{X} \otimes \mathcal{X})_{\text{sym}}$ denotes the set of symmetric tensors on a Hilbert space $\mathcal{X}$. b describes the body force, r_Γ describes the displacement at the boundary, u_0 describes the plastic strain at the initial time, $\dot{U}$ describes the (negative of the) plastic strain rate, L describes the elastic modulus, and Λ describes the plastic modulus. We make the following definitions:

$$E : \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}^{n\times n}_{\text{sym}} : \qquad E(w) := \frac{1}{2}(w + w^T). \tag{3.5a}$$

$$V : \mathbb{R}^{n\times n}_{\text{sym}} \times \mathbb{R}^{n\times n}_{\text{sym}} \times \overline{\Omega} \longrightarrow \mathbb{R} :$$

$$V(w, z, x) := \frac{1}{2} \langle w - z \,|\, L(x)(w - z)\rangle_{\mathbb{R}^{n\times n}_{\text{sym}}} + \frac{1}{2} \langle z \,|\, \Lambda(x) z\rangle_{\mathbb{R}^{n\times n}_{\text{sym}}} . \tag{3.5b}$$

$$S : \mathbb{R}^{n\times n}_{\text{sym}} \times \mathbb{R}^{n\times n}_{\text{sym}} \times \overline{\Omega} \longrightarrow \mathbb{R}^{n\times n}_{\text{sym}} :$$

$$S(w, z, x) := \frac{\partial V}{\partial w}(w, z, x) = L(x)(w - z). \tag{3.5c}$$

$$\Sigma : \mathbb{R}^{n\times n}_{\text{sym}} \times \mathbb{R}^{n\times n}_{\text{sym}} \times \overline{\Omega} \longrightarrow \mathbb{R}^{n\times n}_{\text{sym}} :$$

$$\Sigma(w, z, x) := \frac{\partial V}{\partial z}(w, z, x) = -S(w, z, x) + \Lambda(x) z. \tag{3.5d}$$

E describes the strain, V describes the free energy, S describes the elastic stress, and Σ describes the plastic stress.

Our goal is to rewrite the quasistatic viscoelastic problem (3.2) *as an initial value problem for u and to apply the abstract theory developed in section 2.*

We proceed in several steps:

1. We make the following definitions:

$$\mathcal{H} := L^2(\Omega, \mathbb{R}^{n\times n}_{\text{sym}}). \tag{3.6a}$$

$$A := \left\{ (w, z) \in \mathcal{H} \times \mathcal{H} \,\middle|\, z(x) \in \dot{U}(w(x)) \ \forall\, x \in \Omega \text{ almost everywhere} \right\}. \tag{3.6b}$$

From the general theory of monotone sets we know that $A \subset \mathcal{H} \times \mathcal{H}$ is maximal monotone. By construction $(0, 0) \in A$.

2. We consider the following boundary value problem:

$$-\operatorname{div}_x \Big\{ S\Big(E(\nabla_x r_1(x,t)), 0, x\Big)\Big\} = b(x,t), \qquad r_1(x,t)\Big|_{x\in\Gamma} = r_\Gamma(x,t). \tag{3.7}$$

From the general theory of linear elliptic boundary value problems (Lax–Milgram's lemma, Korn's inequality) we know that (3.7) has a unique weak solution:

$$r_1 \in \mathcal{C}^2([0,T], H^1(\Omega, \mathbb{R}^n)). \tag{3.8}$$

We make the following definition:

$$\varphi(x,t) := -S\Big(E(\nabla_x r_1(x,t)), 0, x\Big) \qquad \Longrightarrow \qquad \varphi \in \mathcal{C}^2([0,T], \mathcal{H}). \tag{3.9}$$

3. Let $w \in \mathcal{H}$. We consider the following boundary value problem:

$$-\operatorname{div}_x \Big\{ S\Big(E(\nabla_x r_2(x)), w(x), x\Big)\Big\} = 0, \qquad r_2(x)\Big|_{x\in\Gamma} = 0. \tag{3.10}$$

From the general theory of linear elliptic boundary value problems (Lax–Milgram's lemma, Korn's inequality) we know that (3.10) has a unique weak solution:

$$r_2[w] \in H_0^1(\Omega, \mathbb{R}^n). \tag{3.11}$$

We make the following definition:

$$\hat{S}[w](x) := S\Big(E(\nabla_x r_2[w](x)), w(x), x\Big). \tag{3.12}$$

By construction r_2 and $\hat{S}$ are linear continuous mapings:

$$r_2 : \mathcal{H} \longrightarrow H_0^1(\Omega, \mathbb{R}^n), \qquad \hat{S} : \mathcal{H} \longrightarrow \mathcal{H}. \tag{3.13}$$

4. We define an equivalent scalar product $\langle \cdot \mid \cdot \rangle_L$ on $\mathcal{H}$ by:

$$\langle z \mid w \rangle_L := \int_\Omega \langle z(x) \mid L(x) w(x) \rangle_{\mathbb{R}^{n\times n}_{\text{sym}}} \, \mathrm{d}x. \tag{3.14}$$

We make the following definition:

$$\mathcal{H}_0 := \big\{ E(\nabla_x \rho) \,\big|\, \rho \in H_0^1(\Omega, \mathbb{R}^n) \big\} \subset \mathcal{H}. \tag{3.15}$$

From general functional analysis (Poincare's inequality, Korn's inequality) we know that $\mathcal{H}_0$ is a closed subspace of $\mathcal{H}$. We make the following definition:

$$P_L : \mathcal{H} \longrightarrow \mathcal{H}_0 \qquad \text{orthorgonal projection w.r.t. } \langle \cdot \mid \cdot \rangle_L. \tag{3.16}$$

5. Now the weak formulation of the boundary value problem (3.10) reads:

$$\int_\Omega \Big\langle E(\nabla_x \rho(x)) \,\Big|\, S\Big(E(\nabla_x r_2(x)), w(x), x\Big)\Big\rangle_{\mathbb{R}^{n\times n}_{\mathrm{sym}}} = 0$$
$$\forall\ \rho \in H^1_0(\Omega, \mathbb{R}^n). \tag{3.17}$$

$\Longleftrightarrow$

$$\langle z \,|\, E(\nabla_x r_2) - w\rangle_L = 0 \qquad \forall\ z \in \mathcal{H}_0. \tag{3.18}$$

This yields:

$$E(\nabla_x r_2[w]) = P_L[w]. \tag{3.19a}$$

$$\hat{S}[w](x) = L(x)\Big(P_L - I\Big)[w](x) = -L(x)P_L^{\perp}[w](x). \tag{3.19b}$$

Consequently $\hat{S}$ is symmetric and negative semidefinit w.r.t. the original scalar product on $\mathcal{H}$. We make the following definition:

$$\Phi[w](x) := -\hat{S}[w](x) + \Lambda(x)w(x). \tag{3.20}$$

By construction Φ is a linear continuous symmetric positive definit maping:

$$\Phi : \mathcal{H} \longrightarrow \mathcal{H}. \tag{3.21}$$

Now the quasistatic viscoelastic problem (3.2) reads as follows:

$$r(x,t) = r_1(x,t) + r_2[u(t)](x). \tag{3.22a}$$

$$\partial_t u(t) + A\Big(\Phi\Big(u(t)\Big) + \varphi(t)\Big) \ni 0, \qquad\qquad u(0) = u_0. \tag{3.22b}$$

By construction the following statements hold:

1. $r_1 \in \mathcal{C}^2([0,T], H^1(\Omega, \mathbb{R}^n))$.
2. $r_2 : \mathcal{H} \longrightarrow H^1_0(\Omega, \mathbb{R}^n)$ is linear and continuous.
3. $A \subset \mathcal{H} \times \mathcal{H}$ is maximal monotone with $(0,0) \in A$
4. $\Phi : \mathcal{H} \longrightarrow \mathcal{H}$ is linear, continuous, symmetric and positive definit.
5. $\varphi \in \mathcal{C}^2([0,T], \mathcal{H})$.

With the help of (3.22), theorem 2.1 and theorem 2.4 we immediately obtain the following theorem.

THEOREM 3.1 (Existence and uniqueness of solutions)
Let the following additional assumption hold:

$$\Phi(u_0) + \varphi(0) \in \mathcal{D}(A). \tag{3.23}$$

Then the quasistatic viscoelastic problem (3.2) *has a unique solution:*

$$r \in \mathcal{C}^{0,1}([0,T], H^1(\Omega, \mathbb{R}^n)), \qquad u \in \mathcal{C}^{0,1}([0,T], \mathcal{H}). \tag{3.24}$$

3.2 Statistic Homogenization of the 1–Dimensional Problem

We consider the 1–dimensional case $\Omega = (0,1)$ of the quasistatic viscoelastic problem (3.2):

$$-\partial_x \Big\{ L(x) \Big(\partial_x r(x,t) - u(x,t) \Big) \Big\} = b(x,t). \tag{3.25a}$$

$$r(0,t) = \rho_0(t), \qquad r(1,t) = \rho_1(t). \tag{3.25b}$$

$$\partial_t u(x,t) + \dot{U}\Big(- L(x) \Big(\partial_x r(x,t) - u(x,t) \Big) + \Lambda(x) u(x,t) \Big) \ni 0. \tag{3.25c}$$

$$u(x,0) = u_0(x). \tag{3.25d}$$

We make the following assumptions:

1. Let $b \in \mathcal{C}^2([0,T], \mathcal{C}^0([0,1]))$.
2. Let $\rho_1, \rho_1 \in \mathcal{C}^2([0,T])$.
3. Let $u_0 \in \mathcal{C}^0([0,1]))$.
4. Let $\gamma : (-1,1) \longrightarrow \mathbb{R}^2$ be a continuous curve with the following properties:

   $$\gamma(0) = (0,0), \qquad \lim_{s \to \pm 1} \|\gamma(s)\|_{\mathbb{R}^2} = \infty. \tag{3.26}$$

   Let $\dot{U} := \mathrm{graph}(\gamma) \subset \mathbb{R} \times \mathbb{R}$ be a monotone set. From the general theory of monotone sets we know that $\dot{U}$ is also maximal monotone.
5. Let $L \in \mathcal{C}^0([0,1])$, let $0 < \underline{l} \leq \bar{l}$, and let the following estimate hold $\forall\ x \in [0,1]$:

   $$\underline{l} \leq L(x) \leq \bar{l}. \tag{3.27}$$

6. Let $\Lambda \in \mathcal{C}^0([0,1])$, let $0 < \underline{\lambda} \leq \overline{\lambda}$, and let the following estimate hold $\forall\, x \in [0,1]$:

$$\underline{\lambda} \leq \Lambda(x) \leq \overline{\lambda}. \tag{3.28}$$

According to the previous subsection we make the following definitions:

$$\mathcal{H} := L^2((0,1)). \tag{3.29a}$$

$$A := \Big\{(w,z) \in \mathcal{H} \times \mathcal{H} \,\Big|\, z(x) \in \dot{U}(w(x)) \ \forall\, x \in (0,1) \text{ almost everywhere}\Big\}. \tag{3.29b}$$

$$\begin{aligned} r_1(x,t) := \rho_0(t) &- \int_0^x \frac{1}{L(x_1)} \int_0^{x_1} b(x_2,t)\,\mathrm{d}x_2\,\mathrm{d}x_1 \\ &+ \Big(\int_0^x \frac{1}{L(x_1)}\,\mathrm{d}x_1\Big)\Big(\int_0^1 \frac{1}{L(x_1)}\,\mathrm{d}x_1\Big)^{-1} \\ &\times \Big(\rho_1(t) - \rho_0(t) + \int_0^1 \frac{1}{L(x_1)} \int_0^{x_1} b(x_2,t)\,\mathrm{d}x_2\,\mathrm{d}x_1\Big). \end{aligned} \tag{3.29c}$$

$$\begin{aligned} &r_2[z](x) \\ &:= \int_0^x z(x_1)\,\mathrm{d}x_1 - \Big(\int_0^x \frac{1}{L(x_1)}\,\mathrm{d}x_1\Big)\Big(\int_0^1 \frac{1}{L(x_1)}\,\mathrm{d}x_1\Big)^{-1}\Big(\int_0^1 z(x_1)\,\mathrm{d}x_1\Big). \end{aligned} \tag{3.29d}$$

$$\Phi[z](x) := \Big(\int_0^1 \frac{1}{L(x_1)}\,\mathrm{d}x_1\Big)^{-1}\Big(\int_0^1 z(x_1)\,\mathrm{d}x_1\Big) + \Lambda(x)z(x). \tag{3.29e}$$

$$\begin{aligned} \varphi(x,t) := \int_0^x b(x_1,t)\,\mathrm{d}x_1 &- \Big(\int_0^1 \frac{1}{L(x_1)}\,\mathrm{d}x_1\Big)^{-1} \\ &\times \Big(\rho_1(t) - \rho_0(t) + \int_0^1 \frac{1}{L(x_1)} \int_0^{x_1} b(x_2,t)\,\mathrm{d}x_2\,\mathrm{d}x_1\Big). \end{aligned} \tag{3.29f}$$

We make the following additional assumption:

7. Let $\alpha > 0$, and let the following estimate hold:

$$\int_0^1 \Big|\dot{U}^0\Big(\Phi[u_0](x) + \varphi(x,0)\Big)\Big|^2\,\mathrm{d}x \leq \alpha^2. \tag{3.30}$$

   In (3.30) $\dot{U}^0$ denotes the minimal selection of $\dot{U}$.

Now according to the previous subsection the 1–dimensional quasistatic viscoelastic problem (3.25) reads as follows:

$$r(x,t) = r_1(x,t) + r_2[u(t)](x). \tag{3.31a}$$

$$\partial_t u(t) + A\Big(\Phi\Big(u(t)\Big) + \varphi(t)\Big) \ni 0, \qquad\qquad u(0) = u_0. \tag{3.31b}$$

According to the previous subsection and (3.30) the following statements hold:

1. $r_1 \in \mathcal{C}^2([0,T], H^1((0,1)))$.
2. $r_2 : \mathcal{H} \longrightarrow H_0^1((0,1))$ is linear and continuous.
3. $A \subset \mathcal{H} \times \mathcal{H}$ is maximal monotone with $(0,0) \in A$
4. $\Phi : \mathcal{H} \longrightarrow \mathcal{H}$ is linear, continuous, symmetric and positive definit.
5. $\varphi \in \mathcal{C}^2([0,T], \mathcal{H})$.
6. $\Phi(u_0) + \varphi(0) \in \mathcal{D}(A)$. In particular, the following estimate holds:
$$\Big\|A^0\Big(\Phi(u_0) + \varphi(0)\Big)\Big\| \le \alpha. \tag{3.32}$$

With the help of (3.31), theorem 2.1 and theorem 2.4 we immediately obtain the following theorem.

THEOREM 3.2 (Existence and uniqueness of solutions)
The 1–dimensional quasistatic viscoelastic problem (3.25) *has a unique solution:*

$$r \in \mathcal{C}^{0,1}([0,T], H^1((0,1))), \qquad\qquad u \in \mathcal{C}^{0,1}([0,T], \mathcal{H}). \tag{3.33}$$

We turn to the question of statistic homogenization of the 1–dimensional quasistatic viscoelastic problem (3.25).

Our goal is to formulate a probabilistic model for a 2–component material, and to derive the homogenized limit problem.

We proceed in several steps:

1. Let $(\Omega, \mathcal{F}, W)$ be a propability space, let $0 < p < 1$, let ξ_i be random variables, and let the following statements hold:
$$W(\xi_i = 1) = p, \qquad\qquad W(\xi_i = 0) = 1 - p. \tag{3.34a}$$
$$\mathrm{Ex}[\xi_i \xi_j] = \mathrm{Ex}[\xi_i]\mathrm{Ex}[\xi_j] = p^2 \qquad \forall\ i \neq j. \tag{3.34b}$$
In (3.34b) Ex denotes the expectation value w.r.t. W.

2. Let $\psi \in \mathcal{C}_0^\infty(\mathbb{R})$ with the following properties:

$$0 \le \psi(x) \le 1, \qquad \psi(x) = \psi(-x), \qquad \sum_{i\in\mathbb{Z}} \psi(x-i) = 1. \tag{3.35a}$$

$$\text{supp}(\psi) = [-1,1]. \tag{3.35b}$$

We make the following definition:

$$\psi_{ni}(x) := \psi(nx - i). \tag{3.36}$$

By construction $\{\psi_{ni}\}_{i=0}^n$ is a partition of unity on $[0,1]$.

3. Let $\omega \in \Omega$. We make the following definitions:

$$\frac{1}{L} := \frac{p}{\underline{l}} + \frac{1-p}{\overline{l}}. \tag{3.37a}$$

$$\frac{1}{L_n[\omega](x)} := \sum_{i=0}^n \psi_{ni}(x)\Big(\frac{\xi_i(\omega)}{\underline{l}} + \frac{1-\xi_i(\omega)}{\overline{l}}\Big). \tag{3.37b}$$

THEOREM 3.3 (Statistic homogenization)
Let $(r_n[\omega], u_n[\omega])$ and (r,u) be the solutions to the 1–dimensional quasistatic viscoelastic problem (3.25) w.r.t. the data $(r_{1n}[\omega], r_{2n}[\omega], \Phi_n[\omega], \varphi_n[\omega])$ and $(r_1, r_2, \Phi, \varphi)$ respectively where $(r_{1n}[\omega], r_{2n}[\omega], \Phi_n[\omega], \varphi_n[\omega])$ and $(r_1, r_2, \Phi, \varphi)$ are defined by (3.29) w.r.t. $L_n[\omega]$ and L. Then the following statemens hold:

$$\text{Ex}\Big[\, \|r_n - r\|_{\mathcal{C}^0([0,T],H_0^1((0,1)))} \Big] \xrightarrow{n\to\infty} 0. \tag{3.38a}$$

$$\text{Ex}\Big[\, \|u_n - u\|_{\mathcal{C}^0([0,T],\mathcal{H})} \Big] \xrightarrow{n\to\infty} 0. \tag{3.38b}$$

PROOF
Let $\omega \in \Omega$. With the help of (3.27), (3.28), (3.29) and (3.32) we obtain the following estimates:

$$\|r_{2n}[\omega]\|_{\mathcal{L}(\mathcal{H},H_0^1((0,1)))} \le 2\Big(1 + \frac{\overline{l}}{\underline{l}}\Big). \tag{3.39a}$$

$$\|\Phi_n[\omega]\|_{\mathcal{L}(\mathcal{H},\mathcal{H})} \le \overline{l} + \overline{\lambda}, \qquad \big\|\Phi_n[\omega]^{-1}\big\|_{\mathcal{L}(\mathcal{H},\mathcal{H})} \le \underline{\lambda}. \tag{3.39b}$$

$$\|\varphi_n[\omega]\|_{\mathcal{C}^2([0,T],\mathcal{H})} \le \Big(1+\frac{\bar{l}}{\underline{l}}\Big)\,\|b\|_{\mathcal{C}^2([0,T],\mathcal{H})} + \bar{l}\,\|\rho_1-\rho_0\|_{\mathcal{C}^2([0,T])}\,. \tag{3.39c}$$

$$\Big\|A^0\Big(\Phi_n[\omega](u_0)+\varphi_n[\omega](0)\Big)\Big\| \le \alpha. \tag{3.39d}$$

In (3.39) $\mathcal{L}(\mathcal{X},\mathcal{Y})$ denotes the set of linear continuous mapings $\mathcal{X}\longrightarrow\mathcal{Y}$. With the help of (3.31), (3.39) and theorem 2.7 we find a constant $C>0$ such that the following estimates hold $\forall\ \omega\in\Omega$:

$$\begin{aligned}
&\|r_n[\omega]-r\|_{\mathcal{C}^0([0,T],H_0^1((0,1)))}\\
&\le \|r_{1n}[\omega]-r_1\|_{\mathcal{C}^0([0,T],H_0^1((0,1)))} + \|r_{2n}[\omega](u_n[\omega])-r_2(u)\|_{\mathcal{C}^0([0,T],H_0^1((0,1)))}\\
&\le \|r_{1n}[\omega]-r_1\|_{\mathcal{C}^0([0,T],H_0^1((0,1)))} + \|r_{2n}[\omega](u_n[\omega]-u)\|_{\mathcal{C}^0([0,T],H_0^1((0,1)))}\\
&\qquad + \|r_{2n}[\omega](u)-r_2(u)\|_{\mathcal{C}^0([0,T],H_0^1((0,1)))}\,.\\
&\le \|r_{1n}[\omega]-r_1\|_{\mathcal{C}^0([0,T],H_0^1((0,1)))} + 2\Big(1+\frac{\bar{l}}{\underline{l}}\Big)\,\|u_n[\omega]-u\|_{\mathcal{C}^0([0,T],\mathcal{H})}\\
&\qquad + \|r_{2n}[\omega](u)-r_2(u)\|_{\mathcal{C}^0([0,T],H_0^1((0,1)))}\,.
\end{aligned} \tag{3.40a}$$

$$\begin{aligned}
\|u_n[\omega]-u\|_{\mathcal{C}^0([0,T],\mathcal{H})} \le C\Big\{\,&\|\Phi_n[\omega]-\Phi\|^{\frac12}_{\mathcal{L}(\mathcal{H},\mathcal{H})} + \|\varphi_n[\omega]-\varphi\|_{\mathcal{C}^0([0,T],\mathcal{H})}\\
&+\Big(\int_0^T \|\partial_t\varphi_n[\omega](t)-\partial_t\varphi(t)\|\,\mathrm{d}t\Big)^{\frac12}\Big\}.
\end{aligned} \tag{3.40b}$$

Let $g\in\mathcal{C}^0([0,1])$, and let $0\le x\le 1$. With the help of (3.27) and (3.37) we obtain:

$$\begin{aligned}
&\Big|\int_0^x \Big(\frac{1}{L_n[\omega](x_1)}-\frac{1}{L}\Big)g(x_1)\,\mathrm{d}x_1\Big|\\
&\le \Big|\int_0^{\frac1n[nx]} \Big(\frac{1}{L_n[\omega](x_1)}-\frac{1}{L}\Big)g(x_1)\,\mathrm{d}x_1\Big| + \frac{\|g\|_{\mathcal{C}^0([0,1])}}{n}\Big(\frac{1}{\underline{l}}-\frac{1}{\bar{l}}\Big)\\
&= \Big|\sum_{k=1}^{[nx]}\sum_{i=k-1}^{k}(\xi_i(\omega)-p)\Big(\frac{1}{\underline{l}}-\frac{1}{\bar{l}}\Big)\int_{\frac{k-1}{n}}^{\frac{k}{n}}\psi_{ni}(x)g(x_1)\,\mathrm{d}x_1\Big| + \frac{\|g\|_{\mathcal{C}^0([0,1])}}{n}\Big(\frac{1}{\underline{l}}-\frac{1}{\bar{l}}\Big)\\
&\le \Big\{\Big|\sum_{i=1}^{[nx]-1}(\xi_i(\omega)-p)g_{ni}\Big| + \frac{3\,\|g\|_{\mathcal{C}^0([0,1])}}{n}\Big\}\Big(\frac{1}{\underline{l}}-\frac{1}{\bar{l}}\Big).
\end{aligned} \tag{3.41a}$$

$$g_{ni} := \int_{\frac{i-1}{n}}^{\frac{i+1}{n}} \psi_{ni}(x)g(x_1)\,\mathrm{d}x_1. \tag{3.41b}$$

With the help of the Cauchy–Schwartz inequality and (3.34) we obtain the following law of large numbers:

$$
\begin{aligned}
&\mathrm{Ex}\Big[\Big|\sum_{i=1}^{[nx]-1}(\xi_i-p)g_{ni}\Big|\Big] \le \mathrm{Ex}\Big[\Big(\sum_{i=1}^{[nx]-1}(\xi_i-p)g_{ni}\Big)^2\Big]^{\frac{1}{2}}\\
&=\Big(\sum_{i=1}^{[nx]-1}\mathrm{Ex}\Big[\Big((\xi_i-p)g_{ni}\Big)^2\Big]\Big)^{\frac{1}{2}}\\
&\le 2\,\|g\|_{\mathcal{C}^0([0,1])}\sqrt{\frac{p(1-p)}{n}}.
\end{aligned}
\tag{3.42}
$$

With the help of (3.41) and (3.42) we obtain:

$$
\begin{aligned}
&\mathrm{Ex}\Big[\Big|\int_0^x\Big(\frac{1}{L_n(x_1)}-\frac{1}{L}\Big)g(x_1)\,\mathrm{d}x_1\Big|\Big]\\
&\le \|g\|_{\mathcal{C}^0([0,1])}\Big\{2\sqrt{\frac{p(1-p)}{n}}+\frac{3}{n}\Big\}\Big(\frac{1}{\underline{l}}-\frac{1}{\overline{l}}\Big).
\end{aligned}
\tag{3.43}
$$

With the help of (3.27), (3.29) and (3.43) we obtain:

$$
\mathrm{Ex}\Big[\|r_{1n}-r_1\|_{\mathcal{C}^0([0,T],H_0^1((0,1)))}\Big]=O\Big[\frac{1}{\sqrt{n}}\Big].
\tag{3.44a}
$$

$$
\mathrm{Ex}\Big[\|r_{2n}(u)-r_2(u)\|_{\mathcal{C}^0([0,T],H_0^1((0,1)))}\Big]=O\Big[\frac{1}{\sqrt{n}}\Big].
\tag{3.44b}
$$

$$
\mathrm{Ex}\Big[\|\Phi_n-\Phi\|_{\mathcal{L}(\mathcal{H},\mathcal{H})}\Big]=O\Big[\frac{1}{\sqrt{n}}\Big].
\tag{3.44c}
$$

$$
\mathrm{Ex}\Big[\|\varphi_n-\varphi\|_{\mathcal{C}^1([0,T],\mathcal{H})}\Big]=O\Big[\frac{1}{\sqrt{n}}\Big].
\tag{3.44d}
$$

With the help of (3.40), (3.44) and the Cauchy–Schwartz inequality we obtain (3.38).
□

3.3 A Remark on Homogenization of the n–Dimensional Problem

We reconsider the n–dimensional quasistatic viscoelastic problem (3.2). For simplicity we restrict ourselves to the linear case $\dot{U}=I$:

$$
-\operatorname{div}_x\Big\{L(x)\Big(E(\nabla_x r(x,t))-u(x,t)\Big)\Big\}=b(x,t),\qquad r(x,t)\Big|_{x\in\Gamma}=r_\Gamma(x,t).
\tag{3.45a}
$$

$$\partial_t u(x,t) - L(x)\Big(E(\nabla_x r(x,t)) - u(x,t)\Big) + \Lambda(x)u(x,t) = 0,$$

$$u(x,0) = u_0(x). \tag{3.45b}$$

Now we turn to the question of homogenization of the quasistatic viscoelastic problem (3.45). Therefore, let $\varepsilon > 0$. We make the following additional assumption:

5.a Let $L : \mathbb{R}^n \longrightarrow (\mathbb{R}^{n\times n}_{\rm sym} \otimes \mathbb{R}^{n\times n}_{\rm sym})_{\rm sym}$ be $[0,1]^n$–periodic .

We make the following definition:

$$L^\varepsilon(x) := L\Big(\frac{x}{\varepsilon}\Big). \tag{3.46}$$

L^ε describes a periodic distribution of two different components in a material.

We consider the quasistatic viscoelastic problem (3.45) *corresponding to* L^ε. *Our goal is to derive the homogenized limit problem as* $\varepsilon \longrightarrow 0$ *and to formulate structural conditions for the elastic modulus* L.

According to subsection 3.1 we can solve the linear elliptic boundary value problem (3.45a) separately and insert the solution operator into the linear initial value problem (3.45b). Now the quasistatic viscoelastic problem corresponding to L^ε reads:

$$r^\varepsilon(x,t) = r_1^\varepsilon(x,t) + r_2^\varepsilon[u^\varepsilon(t)](x). \tag{3.47a}$$

$$\partial_t u^\varepsilon(x,t) - L^\varepsilon(x)\Big\{E\Big(\nabla_x r_2^\varepsilon[u^\varepsilon(t)](x)\Big) - u^\varepsilon(x,t)\Big\} + \Lambda(x)u^\varepsilon(x,t) = 0,$$

$$u^\varepsilon(x,0) = u_0(x). \tag{3.47b}$$

According to the general theory of homogenization of linear elliptic boundary value problems we assume that the solution operators in (3.47a) admit the following asymptotic expansions:

$$r_1^\varepsilon(x,t) = r_1^0(x,t) + \varepsilon r_1^1(x,y,t)\Big|_{y=\frac{x}{\varepsilon}} + O[\varepsilon^2]. \tag{3.48a}$$

$$r_2^\varepsilon[w](x) = r_2^0[w](x) + \varepsilon r_2^1[w](x,y)\Big|_{y=\frac{x}{\varepsilon}} + O[\varepsilon^2]. \tag{3.48b}$$

We note that from the general theory of homogenization we know that r_1^ε and r_2^ε admit the asymptotic expansions (3.48) w.r.t. the L^2–norm, and that r_1^0 and r_2^0 are the solution operators to respective linear elliptic boundary value problems corresponding to some constant elastic modulus $L^0 \in (\mathbb{R}^{n\times n}_{\rm sym} \otimes \mathbb{R}^{n\times n}_{\rm sym})_{\rm sym}$. Our additional assumption is that this is also true for the H^1–norm. Moreover, we assume that the solution u^ε to the linear initial boundary value problem (3.47b) admits the following asymptotic expansion:

$$u^\varepsilon(x,t) = u^0(x,t) + O[\varepsilon]. \tag{3.49}$$

We insert (3.48) and (3.49) into (3.47). This yields:

$$r^0(x,t) = r_1^0(x,t) + r_2^0[u^0(t)](x). \tag{3.50a}$$

$$\begin{aligned} &\partial_t u^0(x,t) - \Big\{L(y)\Big(E\Big(\nabla_x r_1^0(x,t) + \nabla_x r_2^0[u^0(t)](x)\Big) \\ &+ E\Big(\nabla_y r_1^1(x,y,t) + \nabla_y r_2^1[u^0(t)](x,y)\Big) - u^0(x,t)\Big)\Big\}\Big|_{y=\frac{x}{\varepsilon}} + \Lambda(x)u^0(x,t) \\ &= 0. \end{aligned} \tag{3.50b}$$

$$u^0(x,0) = u_0(x). \tag{3.50c}$$

We see that (3.50) becomes a consistent homogenized limit problem for (3.47) if and only if $\exists$ some maping $\sigma : \mathcal{H} \times [0,T] \longrightarrow \mathcal{H}$ with the following properties:

1. The following statement holds $\forall\ w \in \mathcal{H}\ \forall\ y \in \mathbb{R}^n$ almost everywhere:

$$\begin{aligned} &L(y)\Big\{E\Big(\nabla_x r_1^0(x,t) + \nabla_x r_2^0[w](x)\Big) + E\Big(\nabla_y r_1^1(x,y,t) + \nabla_y r_2^1[w](x,y)\Big) - w(x)\Big\} \\ &= \sigma[w,t](x). \end{aligned} \tag{3.51}$$

2. The following maping is continuous linear and onto $\forall\ 0 \le t \le T$:

$$\mathcal{H} \longrightarrow \mathcal{H} : w \longmapsto \sigma(w,t). \tag{3.52}$$

Now let some σ with the above properties be given. We make the following definition:

$$\begin{aligned} &\rho[w,t](x,y) \\ &:= \Big(\nabla_x r_1^0(x,t) + \nabla_x r_2^0[w](x) - w(x)\Big)y + r_1^1(x,y,t) + r_2^1[w](x,y). \end{aligned} \tag{3.53}$$

With the help of (3.51) we obtain:

$$L(y)^{-1}\sigma[w,t](x) = E\Big(\nabla_y \rho[w,t](x,y)\Big). \tag{3.54}$$

By construction $L(y)^{-1}$ has the following symmetry:

$$L(y)^{-1} \in (\mathbb{R}^{n\times n}_{\mathrm{sym}} \otimes \mathbb{R}^{n\times n}_{\mathrm{sym}})_{\mathrm{sym}}. \tag{3.55}$$

With the help of (3.52), (3.54) and (3.55) we find that $\exists$ some maping $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}^n \otimes \mathbb{R}^{n\times n}_{\mathrm{sym}}$ such that the following structural condition holds:

$$\Big(L(y)^{-1}\Big)_{ijkl} = \frac{1}{2}\Big(\partial_{y^i}\phi_{jkl}(y) + \partial_{y^j}\phi_{ikl}(y)\Big) = \frac{1}{2}\Big(\partial_{y^k}\phi_{lij}(y) + \partial_{y^l}\phi_{kij}(y)\Big). \tag{3.56}$$

In view of our formal analysis the structural condition (3.56) *is a necessary assumption on order to pass to the limit* $\varepsilon \longrightarrow 0$ *in the quasistatic viscoelastic problem* (3.47). *The corresponding homogenized limit problem is given by* (3.50).

The following example shall show that the set of physically admissible elastic moduli L satisfying the structural condition (3.56) is not empty.

EXAMPLE
Let $\lambda, \mu > 0$. We consider Lame's law of linear elasticity:

$$L := \lambda I_{\mathbb{R}^{n\times n}_{\mathrm{sym}}} + 2\mu I_{\mathbb{R}^n} \otimes I_{\mathbb{R}^n}. \tag{3.57}$$

A straightforeward calculation yields:

$$L^{-1} = \alpha I_{\mathbb{R}^{n\times n}_{\mathrm{sym}}} + 2\beta I_{\mathbb{R}^n} \otimes I_{\mathbb{R}^n}. \tag{3.58a}$$

$$\alpha := \frac{1}{\lambda}, \qquad \beta := -\frac{\mu}{\lambda(\lambda + 2n\mu)}. \tag{3.58b}$$

In particular, we have:

$$(L^{-1})_{ijkl} = \frac{\alpha}{2}\Big(\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il}\Big) + 2\beta\delta_{ij}\delta_{kl}. \tag{3.59}$$

We make the following definition:

$$\phi_{jkl}(y) := \frac{\alpha}{2}\Big(y^k\delta_{jl} + y^l\delta_{jk}\Big) + 2\beta y^j\delta_{kl}. \tag{3.60}$$

A straightforward calculation yields (3.56).

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