The Stokes operator in weighted L^q-spaces II: Weighted resolvent estimates and maximal L^p-regularity

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Abstract

In this paper we establish a general weighted L^q-theory of the Stokes operator $\mathcal{A}_{q,\omega}$ in the whole space, the half space and a bounded domain for general Muckenhoupt weights $\omega \in A_q$. We show weighted L^q-estimates for the Stokes resolvent system in bounded domains for general Muckenhoupt weights. These weighted resolvent estimates imply not only that the Stokes operator $\mathcal{A}_{q,\omega}$ generates a bounded analytic semigroup but even yield the maximal L^p-regularity of $\mathcal{A}_{q,\omega}$ in the respective weighted L^q-spaces for arbitrary Muckenhoupt weights $\omega \in A_q$.

This conclusion is archived by combining a recent characterisation of maximal L^p-regularity by $\mathcal{R}$-bounded families due to L. Weis [20] with the fact that for L^q-spaces $\mathcal{R}$-boundedness is implied by weighted estimates.

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1 Introduction

The aim of this paper is to establish a general weighted L^q-theory of the Stokes operator where the domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is the whole space, the half space or a bounded domain. For $1 < q < \infty$ we consider weighted L^q-spaces

$$L^q_\omega(\Omega) = \{u \in L^1_{loc}(\overline{\Omega}) \ : \ \|u\|^q_{q,\omega} = \int_\Omega |u|^q \omega \ dx < \infty\}$$

with weight functions ω of Muckenhoupt class A_q defined by the condition

$$A_q(\omega) := \sup_Q \Big(\frac{1}{|Q|}\int_Q \omega \, dx\Big)\Big(\frac{1}{|Q|}\int_Q \omega^{-\frac{1}{q-1}} \, dx\Big)^{q-1} < \infty$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ and $|Q|$ denotes the Lebesgue measure of Q.

In [11] the existence of the Helmholtz decomposition

$$L^q_\omega(\Omega)^n = L^q_{\omega,\sigma}(\Omega) \oplus G^q_\omega(\Omega)$$

is proved for $1 < q < \infty$ and all $\omega \in A_q$, where $L^q_{\omega,\sigma}(\Omega)$ is the closure of $C^\infty_{0,\sigma}(\Omega) = \{u \in C^\infty_0(\Omega)^n \ : \ \operatorname{div} u = 0\}$ in $L^q_\omega(\Omega)^n$ and $G^q_\omega(\Omega)$ are the gradient fields in $L^q_\omega(\Omega)^n$. With the bounded Helmholtz projection

$$P_{q,\omega} : L^q_\omega(\Omega)^n \to L^q_{\omega,\sigma}(\Omega)$$

we define the Stokes operator $\mathcal{A}_{q,\omega}$ in $L^q_{\omega,\sigma}(\Omega)$ by

$$D(\mathcal{A}_{q,\omega}) := W^{2,q}_\omega(\Omega)^n \cap L^q_{\omega,\sigma}(\Omega) \cap \{u \in W^{1,q}_\omega(\Omega)^n \,:\, u\big|_{\partial\Omega} = 0\} \tag{1}$$

$$\mathcal{A}_{q,\omega} := -P_{q,\omega}\Delta \qquad \text{on } D(\mathcal{A}_{q,\omega}). \tag{2}$$

The central result is the weighted resolvent estimate for the Stokes operator obtained for general A_q-weights. The Stokes resolvent problem was also investigated in [10] with A_q-weights in an exterior domain, but under the additional restriction that the weights are bounded from above and from below in a neighborhood of the boundary by positive constants. We emphasize that our results hold for general Muckenhoupt weights without any restriction near $\partial\Omega$.

Theorem 1.1 *Let $1 < q < \infty$, $\omega \in A_q$ and $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be equal to $\mathbb{R}^n$, $\mathbb{R}^n_+$ or a bounded domain with boundary of class $C^{1,1}$. Then for every $0 < \varepsilon < \frac{\pi}{2}$ the sector*

$$\Sigma_\varepsilon = \{\lambda \in \mathbb{C} \,:\, \lambda \neq 0,\, |arg\,\lambda| < \pi - \varepsilon\}$$

belongs to the resolvent set of $-\mathcal{A}_{q,\omega}$. There is a constant $C > 0$ depending only on q, Ω, ε and A_q-consistently increasing on $\omega \in A_q$ such that

$$\|\lambda(\lambda + \mathcal{A}_{q,\omega})^{-1} f\|_{q,\omega} \leq C\,\|f\|_{q,\omega} \tag{3}$$

for all $f \in L^q_{\omega,\sigma}(\Omega)$ and for all $\lambda \in \Sigma_\varepsilon$.

In the case $\Omega = \mathbb{R}^n$ Theorem 1.1 is proved by Farwig and Sohr [10] and for $\Omega = \mathbb{R}^n_+$ the Theorem follows from Theorem 1.1 in part I [12]. Thus it remains to prove the Theorem for a bounded $C^{1,1}$-domain; this will be done in the present paper.
It is well known that the uniform dependence of the constant $C > 0$ in (3) on $\lambda \in \Sigma_\varepsilon$ for $\varepsilon > 0$ implies that $-\mathcal{A}_{q,\omega}$ generates a bounded analytic semigroup. The assertion that $C > 0$ is A_q-consistently increasing means that for every $c_0 > 0$ it can be chosen uniformly for all $\omega \in A_q$ with $A_q(\omega) \leq c_0$ (see Definition 2.2 below). We will prove the remarkable fact that this kind of uniform dependence of $C > 0$ on the weight implies even maximal L^p-regularity:
More precisely, the validity of the resolvent estimate (3) for *all* Muckenhoupt weights $\omega \in A_q$ with an A_q-consistently increasing constant implies, in particular, that the operator family

$$\{s\,(\mathcal{A}_{q,\omega} + is)^{-1} \,:\, s \in \mathbb{R} \setminus \{0\}\}$$

is even $\mathcal{R}$-bounded (see Definition 4.1 below). This enables us to apply a recent characterisation due to L.Weis [20] of maximal L^p-regularity by $\mathcal{R}$-bounded families, which yields the maximal L^p-regularity of the Stokes operator $\mathcal{A}_{q,\omega}$, i.e. we obtain the following Theorem:

Theorem 1.2 *Let $1 < p < \infty$, $1 < q < \infty$, $\omega \in A_q$ and $\Omega \subset \mathbb{R}^n$ be the whole space, the half space or a bounded domain with boundary of class $C^{1,1}$. Then the Stokes operator $\mathcal{A}_{q,\omega}$ has maximal L^p-regularity in $X = L^q_{\omega,\sigma}(\Omega)$, i.e., for every $f \in L^p(\mathbb{R}_+, L^q_{\omega,\sigma}(\Omega))$ the mild solution of the instationary Stokes problem*

$$u_t + \mathcal{A}_{q,\omega} u = f, \quad u(0) = 0$$

satisfies the estimate

$$\|u_t\|_{L^p(R_+;X)} + \|\mathcal{A}_{q,\omega} u\|_{L^p(R_+;X)} \leq C\,\|f\|_{L^p(R_+;X)}.$$

We see that, roughly speaking, weighted resolvent estimates for *all* Muckenhoupt weights imply maximal L^p-regularity.
In the case without weights the assertion of Theorem 1.2 is well known and was proved by another method, namely by showing boundedness of purely imaginary powers of the Stokes operator (see [15]).
This paper is organized as follows: In section 2 we give a summary of some properties of Muckenhoupt weights. Then we investigate compact embeddings for weighted Sobolev spaces and weighted Poincaré type inequalities. A refined version of the compact embedding of $W^{1,q}_\omega(\Omega)$ into $L^q_\omega(\Omega)$ will be proved to guarantee that the constant $C = C(\omega)$ in (3) is A_q-consistently increasing.
In section 3 the weighted estimates for the Stokes resolvent system in the bended half space and in a bounded $C^{1,1}$-domain are proved.
After an introduction to $\mathcal{R}$-bounded families of operators we show in section 4 that, roughly speaking, $\mathcal{R}$-boundedness follows from weighted estimates. Finally this result is combined with the weighted resolvent estimates obtained in section 3 to prove Theorem 1.2.

2 Preliminaries

2.1 Muckenhoupt weights

Definition 2.1 (Muckenhoupt weights) *Let $1 < q < \infty$. A function $0 \le \omega \in L^1_{loc}(\mathbb{R}^n)$ is called A_q-weight iff*

$$A_q(\omega) := \sup_Q \left(\frac{1}{|Q|} \int_Q \omega \, dx \right) \left(\frac{1}{|Q|} \int_Q \omega^{-\frac{1}{q-1}} dx \right)^{q-1} < \infty \tag{4}$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ and $|Q|$ denotes the Lebesgue measure of Q. We call $A_q(\omega)$ the A_q-constant of ω. We use $\omega(A)$ as an abbreviation for $\int_A \omega \, dx$.

In the sequel it will be important to choose constants $C = C(\omega)$ in estimates depending on the weight function $\omega \in A_q$ uniformly whenever the A_q-constant $A_q(\omega)$ is bounded from above, i.e. $A_q(\omega) \le c < \infty$. We call this property of the constant A_q-consistency. More precisely:

Definition 2.2 *A mapping $C : A_q \to \mathbb{R}_+$ is called A_q-consistently increasing, iff*

$$\forall c \in \mathbb{R}_+ : \quad \sup \{ C(\omega) \, : \, \omega \in A_q, \, A_q(\omega) \le c \} < \infty.$$

A mapping $C : A_q \to \mathbb{R}_+$ is called A_q-consistently decreasing, iff $\frac{1}{C}$ is A_q-consistently increasing.

We list some properties of Muckenhoupt weights. The proofs in [13] (or [18]) yield the A_q-consistency of the constants appearing in properties (III) and (IV):

(I) $\forall \, 1 < q < \infty \, : \, \omega \in A_q \iff \omega' := \omega^{-\frac{1}{q-1}} \in A_{q'}$.

(II) $A_p \subset A_q$ for $1 < p \le q < \infty$.

(III) For all $\omega \in A_q$ there exists an A_q-consistently increasing constant $C \in \mathbb{R}$ and an A_q-consistently decreasing constant $\varepsilon_0 > 0$ such that the **reverse Hölder inequality**

$$\Big(\frac{1}{|Q|}\int_Q \omega^{1+\varepsilon}\,dx\Big)^{\frac{1}{1+\varepsilon}} \le C\frac{1}{|Q|}\int_Q \omega\,dx \quad \forall\, 0 \le \varepsilon \le \varepsilon_0$$

holds for all cubes $Q \subset \mathbb{R}^n$.

(IV) For all $\omega \in A_q$ there is an A_q-consistently decreasing $\varepsilon_0 > 0$ such that $\omega \in A_{q-\varepsilon}$ and $\omega^{1+\varepsilon} \in A_q$ for all $0 \le \varepsilon \le \varepsilon_0$.

Proof: (I) follows from the definition.
(II) [13], Chapter IV, Theorem 1.14
(III) [13], Chapter IV, Lemma 2.5 or [18], Chapter IX, Theorem 3.5, Prop. 4.5 ii).
(IV) [18], Chapter IX, Prop. 4.5 or [13], Chapter IV, Theorem 2.6 und Theorem 2.7. □

Lemma 2.1 *Let $1 < q < \infty$, $\omega \in A_q$ and let $\psi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a bijective and Lipschitz continuous function with Lipschitz continuous inverse. Then $\omega \circ \psi \in A_q$ with $A_q(\omega \circ \psi) \le c\, A_q(\omega)$, where $c \in \mathbb{R}$ is independent of ω.*

Proof: Since ψ is Lipschitz continuous, there is an $R > 0$ such that for every cube Q there is a cube $\tilde{Q}$ such that $\psi(Q) \subset \tilde{Q}$ and $|\tilde{Q}| \le R|Q|$. By Rademacher's Theorem (see [8], section 3.1.2, Theorem 2) ψ and ψ^{-1} are a.e. differentiable. Since the Jacobian $|det\nabla(\psi^{-1})|$ is essentially bounded and $\nabla(\psi^{-1})(\psi(x))\nabla\psi(x) = I$ a.e. (see [8] , section 3.1.2, Corollary 1 (ii)), we have $|det\nabla\psi(x)| \ge c > 0$ a.e. By the change of variables formula (see [8], section 3.3.3, Theorem 2) we get

$$\begin{aligned}
&\left(\frac{1}{|Q|}\int_Q \omega(\psi(y))\,dy\right)\left(\frac{1}{|Q|}\int_Q \omega(\psi(y))^{-\frac{1}{q-1}}dy\right)^{q-1}\\
&\le \frac{1}{c^q}\left(\frac{1}{|Q|}\int_Q \omega(\psi(y))|det\nabla\psi(y)|\,dy\right)\left(\frac{1}{|Q|}\int_Q \omega(\psi(y))^{-\frac{1}{q-1}}|det\nabla\psi(y)|dy\right)^{q-1}\\
&\le \frac{1}{c^q}\left(\frac{1}{|Q|}\int_{\psi(Q)} \omega(x)\,dx\right)\left(\frac{1}{|Q|}\int_{\psi(Q)} \omega(x)^{-\frac{1}{q-1}}dx\right)^{q-1}\\
&\le \frac{1}{c^q}\left(\frac{R}{|\tilde{Q}|}\int_{\tilde{Q}} \omega(x)\,dx\right)\left(\frac{R}{|\tilde{Q}|}\int_{\tilde{Q}} \omega(x)^{-\frac{1}{q-1}}dx\right)^{q-1} \le \frac{R^q}{c^q}A_q(\omega).
\end{aligned}$$

□

Lemma 2.2 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Then there exists an $\varepsilon_0 > 0$ such that $L^q_\omega(\Omega)$ is continuously embedded into $L^{1+\varepsilon}(\Omega)$ for all $0 \le \varepsilon \le \varepsilon_0$. Here $\varepsilon_0 > 0$ depends A_q-consistently decreasing on ω.*
Furthermore there is an $r \in (1,\infty)$ such that $L^r(\Omega)$ is continuously embedded into $L^q_\omega(\Omega)$.

Proof: Because of the property (IV) of A_q-weights there exists an $\varepsilon_0 > 0$ such that $\omega \in A_{\frac{q}{1+\varepsilon}}$ and by property (I) $\omega^{-\frac{1+\varepsilon}{q-1-\varepsilon}} \in A_{(\frac{q}{1+\varepsilon})'} \subset L^1_{loc}(\mathbb{R}^n)$ for all $0 \le \varepsilon \le \varepsilon_0$. Here $\varepsilon_0 > 0$ depends A_q-consistently increasing on ω. Hölder's inequality implies

$$\int_\Omega |f|^{1+\varepsilon}\,dx \le \Big(\int_\Omega \omega(x)^{-\frac{1+\varepsilon}{q-1-\varepsilon}}\,dx\Big)^{\frac{q-1-\varepsilon}{q}}\Big(\int_\Omega |f|^q\omega\,dx\Big)^{\frac{1+\varepsilon}{q}}.$$

To prove the second embedding note that there is an $\varepsilon > 0$ such that $\omega^{1+\varepsilon} \in A_q \subset L^1_{loc}(\mathbb{R}^n)$. Then for $r = \frac{q(1+\varepsilon)}{\varepsilon}$ and $f \in L^r(\Omega)$ we have

$$\int_\Omega |f|^q \omega\, dx \le \Big(\int_\Omega \omega^{1+\varepsilon} dx \Big)^{\frac{1}{1+\varepsilon}} \Big(\int_Q |f|^r dx \Big)^{\frac{\varepsilon}{1+\varepsilon}}.$$

□

Remark: The proof of Lemma 2.2 shows that for all $\omega \in A_q$ with

$$A_q(\omega) \le C < \infty \quad \text{and} \quad \omega(Q) \ge c > 0, \tag{5}$$

where Q denotes a cube with $\Omega \subset Q$, for the embedding $L^q_\omega(\Omega) \hookrightarrow L^{1+\varepsilon}(\Omega)$ not only $\varepsilon > 0$, but even the embedding constant can be chosen uniformly: Since $\omega \in A_{\frac{q}{1+\varepsilon}}$ we have

$$\Big(\int_Q \omega(x)^{-\frac{1+\varepsilon}{q-1-\varepsilon}} dx \Big)^{\frac{q-1-\varepsilon}{1+\varepsilon}} \le A_{\frac{q}{1+\varepsilon}}(\omega)\, |Q|^{\frac{q}{1+\varepsilon}}\, \omega(Q)^{-1} \le c^{-1} |Q|^{\frac{q}{1+\varepsilon}} A_{\frac{q}{1+\varepsilon}}.$$

Finally, the proof of Theorem 2.6, Chapter IV, in [13] shows that $A_{\frac{q}{1+\varepsilon}}(\omega) \le C_\varepsilon A_q(\omega)$, where C_ε is A_q-consistently increasing.

Define the Hardy-Littlewood maximal operator for $f \in L^1_{loc}(\mathbb{R}^n)$ by

$$Mf(x) := \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f|\, dy$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ that contain the point $x \in \mathbb{R}^n$.

Theorem 2.1 (Muckenhoupt) *Let $1 < q < \infty$ and $\omega \in A_q$. Then the Hardy-Littlewood maximal operator M is bounded on $L^q_\omega(\mathbb{R}^n)$. More precisely, there is an A_q-consistently increasing constant $C \in \mathbb{R}$ such that*

$$\|Mf\|_{q,\omega} \le C\, \|f\|_{q,\omega} \tag{6}$$

for all $f \in L^q_\omega(\mathbb{R}^n)$.

Proof: See [13], Chapter IV, Theorem. 2.1 and [18], Chapter IX, Theorem 2.8.

Lemma 2.3 *Let $\varphi(x)$ be nonnegative, integrable on $\mathbb{R}^n$, radial and radially decreasing, i.e., φ depends only on $|x|$ and is monotonically decreasing in $|x|$. Let $\varphi_\varepsilon(x) := \frac{1}{\varepsilon^n}\varphi(\frac{x}{\varepsilon})$ for $\varepsilon > 0$. Then for all measurable functions f*

$$\sup_{\varepsilon > 0} |(f * \varphi_\varepsilon)(x)| \le \|\varphi\|_{L^1(\mathbb{R}^n)} Mf(x) \qquad \forall x \in \mathbb{R}^n.$$

Proof: See [4], Lemma 3.1. □

2.2 Weighted Sobolev spaces

Recall the definition of the weighted Sobolev spaces for $1 < q < \infty$, $\omega \in A_q, k \geq 1$ and a domain $\Omega \subset \mathbb{R}^n$

$$W^{k,q}_\omega(\Omega) := \{f \in L^q_\omega(\Omega) \;:\; D^\alpha u \in L^q_\omega(\Omega),\ |\alpha| \leq k\},$$
$$\widehat{W}^{k,q}_\omega(\Omega) = \{f \in W^{k,1}_{loc}(\Omega) \;:\; D^\alpha u \in L^q_\omega(\Omega),\ |\alpha| = k\},$$

the definition of their respective norms $\|\cdot\|_{k,q,\omega}$ and seminorms $\|\nabla^k \cdot\|_{q,\omega}$ and the defintion of their dual spaces $W^{-k,q'}_{\omega'}(\Omega)$, $\widehat{W}^{-k,q'}_{\omega'}(\Omega)$ from [12].
In particular, in [12] the existence of a well defined trace operator γ in the case $\Omega = \mathbb{R}^n_+$ is shown. However, by standard methods (see e. g. [17]), it is possible to define the trace γ of functions from $W^{k,q}_\omega(\Omega)$ and $\widehat{W}^{k,q}_\omega(\Omega)$ also for a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$. The subspaces of functions $u \in W^{1,q}_\omega(\Omega)$ and $u \in \widehat{W}^{1,q}_\omega(\Omega)$ with $\gamma(u) = 0$ are denoted by $W^{1,q}_{0,\omega}(\Omega)$ and $\widehat{W}^{1,q}_{0,\omega}(\Omega)$ respectively. By standard arguments it follows from Corollary 4.6 in [12] that for bounded Lipschitz domains $C^\infty_0(\Omega)$ is dense in $(W^{1,q}_{0,\omega}(\Omega), \|\cdot\|_{1,q,\omega})$ and in $(\widehat{W}^{1,q}_{0,\omega}(\Omega), \|\nabla \cdot\|_{q,\omega})$.

The following extension theorem of Chua [4] will be used. For the definition of a (ε,∞)-domain see [4] or [12], Definition 3.1. It is well known that every bounded Lipschitz domain is an (ε,∞)-domain.

Theorem 2.2 *Let $1 < q < \infty$ and $\omega \in A_q$.*

i) Let $\Omega \subset \mathbb{R}^n$ be an unbounded (ε,∞)-domain and $k_1, ..., k_N \in \mathbb{N}_0$. Then there exists a linear extension operator $E : \bigcap_{i=1}^N \widehat{W}^{k_i,q}_\omega(\Omega) \to \bigcap_{i=1}^N \widehat{W}^{k_i,q}_\omega(\mathbb{R}^n)$ such that

$$\|\nabla^{k_i} Eu\|_{q,\omega,\mathbb{R}^n} \leq C_i \, \|\nabla^{k_i} u\|_{q,\omega,\Omega}$$

holds for all $i = 1, \dots, N$ and $u \in \bigcap_{i=1}^N \widehat{W}^{k_i,q}_\omega(\Omega)$.

ii) Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $k \in \mathbb{N}$. Then there exists a linear, bounded extension operator $E : W^{k,q}_\omega(\Omega) \to W^{k,q}_\omega(\mathbb{R}^n)$ and an A_q-consistently increasing constant $C > 0$ such that

$$\|Eu\|_{k,q,\omega,\mathbb{R}^n} \leq C \, \|u\|_{k,q,\omega,\Omega} \tag{7}$$

for all $u \in W^{k,q}_\omega(\Omega)$.

Proof: For i) we refer to [4], Theorem 1.5. For ii) we refer to the proof [19], Theorem 2.1.13. The estimate (7) follows from estimates for regular singular integral operators (see [12], Theorem 2.2) and for the maximal operator (see Theorem 2.1). Hence the constant in (7) is A_q-consistently increasing. □

2.3 Compact embeddings and weighted Poincaré inequalities

Since the translation operators $\tau_h : u \mapsto u(\cdot + h)$ for $h \in \mathbb{R}^n$ in general do not map $L^q_\omega(\mathbb{R}^n)$ to $L^q_\omega(\mathbb{R}^n)$, the classical compactness criterion of Kolmogorov characterising precompact subsets of $L^q(\Omega)$ can not be valid for the weighted spaces $L^q_\omega(\Omega)$ for arbitrary $\omega \in A_q$.

Thus the usual proof of the compact embedding $W^{1,q}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ (see e. g. [1] or [17]) does not tranfer to the weighted case.
However an idea of [2], which uses mollifiers instead of translations, is appropriate for the weighted situation.

Theorem 2.3 *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $1 < q < \infty$ and $\omega \in A_q$. Then the following compact embedding holds:*

$$W^{1,q}_\omega(\Omega) \hookrightarrow\hookrightarrow L^q_\omega(\Omega).$$

Proof: Cf. [11]. Let $(u_k) \subset W^{1,q}_{0,\omega}(\Omega) := \overline{C_0^\infty(\Omega)}^{W^{1,q}_\omega(\Omega)}$ such that $u_k \rightharpoonup u$ weakly in $W^{1,q}_\omega(\Omega)$. Extend u and every u_k by 0 to $\mathbb{R}^n$. Let $\phi \in C_0^\infty(\mathbb{R}^n)$ be nonnegative, radial and radially decreasing with $\int \phi = 1$ and let $\phi_\varepsilon(x) = \varepsilon^{-n}\phi(\frac{x}{\varepsilon})$ for $\varepsilon > 0$. We claim that

$$\phi_\varepsilon * (u_k - u) \to 0$$

for $k \to \infty$ uniformly on $\mathbb{R}^n$ for every fixed $\varepsilon > 0$. Note that, if $x_k \in \overline{B_\epsilon(\Omega)}$ and $x_k \to x$, we have $\phi_\varepsilon(x_k - \cdot) \to \phi_\varepsilon(x - \cdot)$ uniformly and thus also in in $L^{q'}_{\omega'}(\mathbb{R}^n)$. From this fact and the weak convergence $u_k \rightharpoonup u$ in $L^q_\omega(\Omega)$ we get

$$\phi_\varepsilon * u_k(x_k) \to \phi_\varepsilon * u(x) \quad \text{for } k \to \infty$$

proving the uniform convergence $\phi_\varepsilon * (u_k - u) \to 0$. Since $\omega(\Omega) < \infty$ it follows

$$\forall \varepsilon > 0: \quad \|\phi_\varepsilon * (u_k - u)\|_{q,\omega} \to 0 \quad \text{for } k \to \infty. \tag{8}$$

Now we show that there is a constant $C > 0$ such that for $v = u_k$ and $v = u$

$$\|\phi_\varepsilon * v - v\|_{q,\omega} \le C\,\varepsilon\,\|\nabla v\|_{q,\omega}. \tag{9}$$

In fact, by Lemma 2.3 and Theorem 2.1

$$\begin{aligned}
& \|\phi_\varepsilon * v - v\|_{q,\omega} \\
= \;& \|\int_{\mathbb{R}^n} \phi_\varepsilon(h)(v(x-h) - v(x))dh\|_{q,\omega} \\
= \;& \|\int_{\mathbb{R}^n} \phi_\varepsilon(h)\int_0^1 \nabla v(x-th)\cdot h\,dt\,dh\|_{q,\omega} \\
\le \;& C\,\varepsilon\,\|\int_0^1\int_{\mathbb{R}^n} \phi_{\varepsilon t}(th)|\nabla v(x-th)|t^n dhdt\|_{q,\omega} \\
= \;& C\,\varepsilon\,\|\int_0^1 \phi_{\varepsilon t} * |\nabla v|dt\|_{q,\omega} \\
\le \;& C\,\varepsilon\,\|\sup_{\delta>0} \phi_\delta * |\nabla v|\,\|_{q,\omega} \\
\le \;& C\,\varepsilon\,\|M(|\nabla v|)\|_{q,\omega} \le C\,\varepsilon\,\|\nabla v\|_{q,\omega}.
\end{aligned}$$

From (8) and (9) it follows $u_k \to u$ in $L^q_\omega(\Omega)$, since (∇u_k) is bounded in $L^q_\omega(\Omega)$. So far we proved that $W^{1,q}_{0,\omega}(\Omega) \hookrightarrow\hookrightarrow L^q_\omega(\Omega)$.

Finally, by Theorem ii) 2.2 there is a linear bounded extension operator $E : W^{1,q}_\omega(\Omega) \to W^{1,q}_\omega(\mathbb{R}^n)$. Let $B_R(0)$ be an open ball containing $\overline{\Omega}$ and $\psi \in C_0^\infty(B_R(0)$ with $\psi \equiv 1$ on Ω be a cut off function. Then

$$\psi E : W^{1,q}_\omega(\Omega) \to W^{1,q}_{0,\omega}(B_R(0)),$$

is a linear bounded extension operator. This proves the Theorem. □

To ensure in the sequel the A_q-consistency of constants in estimates, which are proved by contradiction and compactness arguments, we need a refinement of Theorem 2.3.

Theorem 2.4 *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Let $1 < q < \infty$ and (ω_m) be a sequence of A_q-weights such that*

$$\sup_m A_q(\omega_m) < \infty \quad \textit{and} \quad \omega_m(Q) = 1 \quad \forall m \in \mathbb{N}$$

for an open cube Q with $\overline{\Omega} \subset Q$. Then we have:

i) If $\sup_m \|u_m\|_{1,q,\omega_m} < \infty$, then there is a subsequence $(u_{m_k}) \subset (u_m)$ such that

$$\forall \varepsilon > 0 \ \exists \, k_0(\varepsilon), \, v_\varepsilon \in C^\infty(\overline{\Omega}) \ : \quad \|u_{m_k} - v_\varepsilon\|_{q,\omega_{m_k},\Omega} \le \varepsilon \quad \forall \, k \ge k_0(\varepsilon).$$

In particular, the weak convergence $u_m \rightharpoonup 0$ in $W^{1,s}(\Omega)$ for some $1 < s < \infty$ implies

$$\|u_m\|_{q,\omega_m,\Omega} \to 0.$$

ii) If $\sup_m \|u_m\|_{1,q',\omega'_m} < \infty$, then there is a subsequence $(u_{m_k}) \subset (u_m)$ such that

$$\forall \varepsilon > 0 \ \exists \, k_0(\varepsilon), \, v_\varepsilon \in C^\infty(\overline{\Omega}) \ : \quad \|u_{m_k} - v_\varepsilon\|_{q',\omega'_{m_k},\Omega} \le \varepsilon \quad \forall \, k \ge k_0(\varepsilon).$$

Proof: i) The proof follows from a careful analysis of the proof of Theorem 2.3: First let $u_m \in W^{1,q}_{0,\omega_m}(\Omega)$ for all $m \in \mathbb{N}$. Since $\sup_m A_q(\omega_m) < \infty$ and $\omega_m(Q) = 1$, Lemma 2.2 yields the existence of some $s \in (1,\infty)$ such that $(u_m) \subset W^{1,s}(\Omega)$ with

$$\sup_m \|u_m\|_{1,s} \le C \sup_m \|u_m\|_{1,q,\omega_m} < \infty.$$

Suppressing the notion of subsequences we can assume that $u_m \rightharpoonup u$ in $W^{1,s}(\Omega)$. With the same argumentation and notation as in the proof of Theorem 2.3 we get for every $\varepsilon > 0$ the uniform convergence $\phi_\varepsilon * (u_m - u) \to 0$. Since the support of $\phi_\varepsilon * (u_m - u)$ is contained in Q for all m if $\varepsilon > 0$ is sufficiently small, we have

$$\|\phi_\varepsilon * (u_m - u)\|_{q,\omega_m} \to 0 \quad \text{for } m \to \infty \tag{10}$$

because of $\omega_m(Q) = 1$. Furthermore as in the proof of Theorem 2.3

$$\|\phi_\varepsilon * u_m - u_m\|_{q,\omega_m} \le c\,\varepsilon \, \|M(|\nabla u_m|)\|_{q,\omega_m} \le C\,\varepsilon\,\|\nabla u_m\|_{q,\omega_m} \le \tilde{C}\,\varepsilon,$$

where C and $\tilde{C}$ are independent of m, since $\sup_m A_q(\omega_m) < \infty$ and the constant C in Theorem 2.1 is A_q-consistently increasing. If $\epsilon > 0$ is arbitrarily given, choose $\delta = \frac{1}{2}\tilde{C}^{-1}\epsilon$ and m so large that $\|\phi_\delta * (u_m - u)\|_{q,\omega_m} \le \frac{\epsilon}{2}$. Then with $v_\epsilon := (\phi_\delta * u)\big|_\Omega$

$$\|u_m - v_\epsilon\|_{q,\omega_m} \le \|u_m - \phi_\delta * u_m\|_{q,\omega_m} + \|\phi_\delta * (u_m - u)\|_{q,\omega_m} \le \tilde{C}\delta + \frac{\epsilon}{2} = \epsilon.$$

To show part i) in the general case $u_m \in W^{1,q}_{\omega_m}(\Omega), \forall m$, note that by Theorem 2.2 ii) and the cut-off technique applied at the end of the proof of Theorem 2.3 there are extension operators $E_m : W^{1,q}_{\omega_m}(\Omega) \to W^{1,q}_{0,\omega_m}(\mathcal{O})$ for all m, where $\mathcal{O}$ is a domain with $\Omega \subset\subset \mathcal{O} \subset\subset Q$, such that

$$\sup_m \|Eu_m\|_{1,q,\omega_m,\mathcal{O}} \le C \sup_m \|u_m\|_{1,q,\omega_m,\Omega} < \infty,$$

since $\sup_m A_q(\omega_m) < \infty$ and the constant C in Theorem 2.2 is A_q-consistently increasing. Using $\mathcal{O}$ instead of Ω in the argumentation above proves the first assertion of i).
These extension operators E_m have a common bounded extension E to a bounded extension operator from $W^{1,s}(\Omega)$ to $W^{1,s}_0(\mathcal{O})$ (see the explicit construction in [19]). Hence the additional assumption $u_m \rightharpoonup 0$ in $W^{1,s}(\Omega)$ yields $Eu_m \rightharpoonup 0$ in $W^{1,s}_0(\mathcal{O})$. Then in the proof above $u = 0$ and therefore $v_\epsilon = 0$, proving the last assertion in i).
ii) We first show that the assumptions on (ω_m) imply the existence of some $s \in (1, \infty)$ such that $L^{q'}_{\omega'_m}(\Omega)$ is continuously emdedded into $L^s(\Omega)$ for all $m \in \mathbb{N}$ with an embedding constant independent of m: By Hölder's inequality we obtain

$$\|f\|_s \le \Big(\int_Q \omega_m^{\frac{s}{q(1-s)+s}}\, dx\Big)^{\frac{q'-s}{q's}} \Big(\int_\Omega |f|^{q'} \omega'_m\, dx\Big)^{\frac{1}{q'}}. \tag{11}$$

Since $\frac{s}{q(1-s)+s} \to 1$ for $s \to 1$ we can apply the reverse Hölder inequality for $s > 1$ sufficiently close to 1 to the first term on the right hand of (11). Note that because of $\sup_m A_q(\omega_m) < \infty$ both $s > 1$ and the constant in the reverse Hölder inequality can be chosen independently of m. Since $\omega_m(Q) = 1 \, \forall m \in \mathbb{N}$ the embedding constant can be chosen independently of m.
Hence we obtain as in the proof of i) the weak convergence of a subsequence of (u_m) to a v in $W^{1,s}(\Omega)$ and thus the uniform convergence $\phi_{varepsilon} * (u_m - v) \to 0$ for every fixed $\varepsilon > 0$. Since

$$\omega'_m(Q)^{q-1} \le A_q(\omega_m)\, |Q|^q \, \omega_m(Q)^{-1} \le C|Q|^q$$

for every fixed $\varepsilon > 0$

$$\|\phi_\varepsilon * (u_m - v)\|_{q',\omega'_m,\Omega} \to 0 \quad \text{for } m \to \infty.$$

Noting that $\sup_m A_{q'}(\omega'_m) = \sup_m A_q(\omega_m) < \infty$ the proof can be completed as in i). □

Corollary 2.1 (Poincaré inequality with A_q-consistent constant) *Let $1 < q < \infty$, $\omega \in A_q$, let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and h a continuous seminorm on $W^{1,1}(\Omega)$ such that $h(c) = 0$ implies $c = 0$ for constant functions c. Then there is an A_q-consistently increasing constant $C \in \mathbb{R}$ such that*

$$\|u\|_{q,\omega} \le C \, \|\nabla u\|_{q,\omega}$$

for all $u \in W^{1,q}_\omega(\Omega)$ with $h(u) = 0$.

Proof: Assume the Corollary was wrong. Then there is a constant $0 < c_0 < \infty$, a sequence $(\omega_k) \subset A_q$ with $A_q(\omega_k) \le c_0$ and a sequence (u_k) with $u_k \in W^{1,q}_{\omega_k}(\Omega)$ and $h(u_k) = 0$ for all $k \in \mathbb{N}$ such that

$$\|u_k\|_{q,\omega_k} > k \, \|\nabla u_k\|_{q,\omega_k} \qquad \forall k \in \mathbb{N}. \tag{12}$$

To apply Theorem 2.4 we choose an open cube Q with $\overline{\Omega} \subset Q$. By multiplying ω_k with $\alpha_k := \omega_k(Q)^{-1}$ we archieve that $\widetilde{\omega}_k(Q) = 1$ for $\widetilde{\omega}_k := \alpha_k \omega_k$ without changing the A_q-constant, i.e. $A_q(\widetilde{\omega}_k) = A_q(\omega_k) \le c_0$. Multiplication of (12) with $\alpha_k^{\frac{1}{q}}$ yields

$$\|u_k\|_{q,\widetilde{\omega}_k} > k \, \|\nabla u_k\|_{q,\widetilde{\omega}_k}. \tag{13}$$

W.l.o.g we may assume that

$$\|u_k\|_{q,\widetilde{\omega}_k} = 1. \tag{14}$$

and consequently $\sup_k \|u_k\|_{1,q,\widetilde{\omega}_k} \le 2$. The properties $\sup_k A_q(\widetilde{\omega}_k) \le c_0$ and $\widetilde{\omega_k}(Q) = 1$ imply by Lemma 2.2 (and the remark after this Lemma) the extistence of some $\varepsilon > 0$ such that for all $k \in \mathbb{N}$ the embedding $L^q_{\omega_k}(\Omega) \hookrightarrow L^{1+\varepsilon}(\Omega)$ holds with an embedding constant independent of k. Hence the sequence (u_k) is bounded in $W^{1,1+\varepsilon}(\Omega)$. Because of the compact embedding $W^{1,1+\varepsilon}(\Omega) \hookrightarrow\hookrightarrow L^{1+\varepsilon}(\Omega)$ we may assume that (u_k) is a Cauchy sequence in $L^{1+\varepsilon}(\Omega)$. By (13), (14) also (∇u_k) is a Cauchy sequence in $L^{1+\varepsilon}(\Omega)$. It follows

$$u_k \to u \quad \text{in } W^{1,1+\varepsilon}(\Omega) \tag{15}$$

for some $u \in W^{1,1+\varepsilon}(\Omega)$. Since (13), (14) imply $\nabla u_k \to 0$ in $W^{1,1+\varepsilon}(\Omega)$, it follows $\nabla u = 0$. Hence u is constant. But $0 = h(u_k) \to h(u)$ implies $h(u) = 0$ and therefore $u = 0$. Inserting this information in (15) and noting (14), we conclude by Theorem 2.4 that $\|u_k\|_{q,\omega_k} \to 0$ as $k \to \infty$. This is a contradiction to (14). □

Conclusions: Under the assumptions of Theorem 2.3 we have:

i) $\forall u \in W^{1,q}_\omega(\Omega) \, : \, \|u - u_\Omega\|_{q,\omega} \le C \, \|\nabla u\|_{q,\omega}$ with $u_\Omega = \frac{1}{|\Omega|} \int_\Omega u dx$

ii) $\forall u \in W^{1,q}_{0,\omega}(\Omega) : \, \|u\|_{q,\omega} \le C \, \|\nabla u\|_{q,\omega}$.

iii) $\widehat{W}^{1,q}_\omega(\Omega) = W^{1,q}_\omega(\Omega)$.

Proof: i) Choose $h(u) := |u_\Omega| = |\frac{1}{|\Omega|} \int_\Omega u dx|$.
ii) Choose $h(u) := | \int_{\partial\Omega} u d\sigma|$.
iii) Follows from i) and an approximation with smooth functions in $\widehat{W}^{1,q}_\omega(\Omega)$ (see [12], Corollary 4.1). □

Corollary 2.2 *Let $1 < q < \infty$, $\omega \in A_q$ and let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Then there is an A_q-consistently increasing constant C such that*

$$\|u\|_{2,q,\omega} \le C \, \|\nabla^2 u\|_{q,\omega}$$

for all $u \in W^{2,q}_\omega(\Omega)$ with $\gamma(u) = 0$.

Proof: Assume the Corollary was wrong. Then there is a constant $c_0 \in \mathbb{R}_+$, a sequence $(\omega_k) \subset A_q$ with $A_q(\omega_k) \le c_0$ and a sequence (u_k) with $u_k \in W^{2,q}_{\omega_k}(\Omega)$ and $\gamma(u_k) = 0$ for all $k \in \mathbb{N}$ such that

$$\|u_k\|_{1,q,\omega_k} > k \|\nabla^2 u_k\|_{q,\omega_k} \qquad \forall k \in \mathbb{N}. \tag{16}$$

Proceeding as in the proof of Corollary 2.1 we may assume

$$\|u_k\|_{1,q,\omega_k} = 1 \quad \text{and} \quad \omega_k(Q) = 1 \tag{17}$$

for an open cube Q containing $\overline{\Omega}$. By (16) it follows $\sup_k \|u_k\|_{2,q,\omega_k} < \infty$. By (17) and the remark after Lemma 2.2 there is an $\varepsilon > 0$ such that $L^q_{\omega_k}(\Omega) \hookrightarrow L^{1+\varepsilon}(\Omega)$ with an embedding constant independent of k. Hence the sequence (u_k) is bounded in $W^{2,1+\varepsilon}(\Omega)$. Because of the compact embedding $W^{2,1+\varepsilon}(\Omega) \hookrightarrow\hookrightarrow W^{1,1+\varepsilon}(\Omega)$ we conclude that for some subsequence

$$u_k \to u \quad \text{in } W^{1,1+\varepsilon}(\Omega). \tag{18}$$

By (16) we have $\nabla^2 u_k \to 0$ in $L^{1+\varepsilon}(\Omega)$. Therefore $\nabla^2 u = 0$ and thus u is a polynomial of first degree on Ω. (18) implies $0 = \gamma(u_k) \to \gamma(u)$ in $L^{1+\varepsilon}(\partial\Omega)$. Hence $\gamma(u) = 0$. Choose $x_0 \in \Omega$ arbirtrary; then every straight line through x_0 intersects the boundary $\partial\Omega$ at least twice. Since $\gamma(u) = 0$ this implies $u = 0$. Hence $u_k \to 0$ in $W^{2,1+\varepsilon}(\Omega)$. Since

$$\sup_k \|u_k\|_{2,q,\omega_k} < \infty \quad \text{und} \quad u_k \to 0 \quad \text{in } W^{2,1+\varepsilon}(\Omega),$$

Theorem 2.4 applied to (u_k) and (∇u_k) yields the convergence $\|u_k\|_{1,q,\omega_k} \to 0$ as $k \to \infty$ contradicting (17). □

3 The Stokes resolvent problem

We investigate the Stokes resolvent system

$$\lambda u - \Delta u + \nabla p = f \quad \text{in } \Omega \tag{19a}$$
$$\operatorname{div} u = g \quad \text{in } \Omega \tag{19b}$$
$$\gamma(u) = 0 \quad \text{on } \partial\Omega. \tag{19c}$$

in weighted Sobolev spaces for a bounded domain Ω with boundary of class $C^{1,1}$.
Recall the following Theorem from [12].

Theorem 3.1 *Let $n \geq 2$, $1 < q < \infty$, $\omega \in A_q$, $0 < \varepsilon < \frac{\pi}{2}$ and $\Omega = \mathbb{R}^n$ or $\Omega = \mathbb{R}^n_+$.*

i) Then for every $f \in L^q_\omega(\Omega)^n$, $g \in W^{1,q}_\omega(\Omega) \cap \widehat{\mathcal{W}}^{-1,q}_\omega(\Omega)$ and $\lambda \in \Sigma_\varepsilon$ there is a unique solution $(u,p) \in W^{2,q}_\omega(\Omega)^n \times \widehat{\mathcal{W}}^{1,q}_\omega(\Omega)$ of the resolvent problem (19). This solution satisfies the estimate

$$|\lambda|\|u\|_{q,\omega} + \|\nabla^2 u\|_{q,\omega} + \|\nabla p\|_{q,\omega} \leq C\,(\,\|f\|_{q,\omega} + \|\nabla g\|_{q,\omega} + \|\lambda g\|_{\widehat{\mathcal{W}}^{-1,q}_\omega}), \tag{20}$$

where $C > 0$ depends only on n, q, ε and A_q-consistently increasing on ω.

ii) If for some $r \in (1,\infty)$ and some $v \in A_r$ additionally $f \in L^r_v(\Omega)^n$ and $g \in W^{1,r}_v(\Omega) \cap \widehat{\mathcal{W}}^{-1,r}_v(\Omega)$, then $(u,p) \in W^{2,r}_v(\Omega) \times \widehat{\mathcal{W}}^{1,r}_v(\Omega)$.

Proof: i) The proof for the case $\Omega = \mathbb{R}^n$ can be found in [10] S.270. For $\Omega = \mathbb{R}^n_+$ the Theorem is proved in [12].

3.1 The bended half space

We transfer the approach of [9] to the weighted case. For a Lipschitz continuous function $\sigma : \mathbb{R}^{n-1} \to \mathbb{R}$ the bended half space H_σ is defined by

$$H_\sigma := \{x = (x', x_n) \in \mathbb{R}^n \ : \ x_n > \sigma(x')\}.$$

Theorem 3.2 *Let $n \geq 2$, $1 < q < \infty$, $\omega \in A_q$, $0 < \varepsilon < \frac{\pi}{2}$ and $\sigma \in C^{0,1}(\mathbb{R}^{n-1}) \cap W^{2,1}_{loc}(\mathbb{R}^{n-1})$. Then there is an A_q-consistently decreasing constant $K = K(n, q, \omega, \varepsilon) > 0$ and an A_q-consistently increasing constant $\lambda_0 = \lambda_0(\sigma, n, q, \omega, \varepsilon) > 0$ with the property: If $\|\nabla'\sigma\|_\infty \leq K$ and $\|\nabla^2\sigma\|_\infty < \infty$, then for every $f \in L^q_\omega(H_\sigma)^n$, $g \in W^{1,q}_\omega(H_\sigma) \cap \widehat{\mathcal{W}}^{-1,q}_\omega(H_\sigma)$ and every $\lambda \in \Sigma_\varepsilon$ mit $|\lambda| \geq \lambda_0$ there exists a unique solution $(u, p) \in W^{2,q}_\omega(H_\sigma)^n \times \widehat{\mathcal{W}}^{1,q}_\omega(H_\sigma)$ of the Stokes resolvent problem (19) in $\Omega = H_\sigma$. This solution satisfies the estimate:*

$$|\lambda| \|u\|_{q,\omega} + \|\nabla^2 u\|_{q,\omega} + \|\nabla p\|_{q,\omega} \leq C \, (\, \|(f, \nabla g)\|_{q,\omega} + \|\lambda g\|_{\widehat{\mathcal{W}}^{-1,q}_\omega} \,)$$

with an A_q-consistently increasing constant $C = C(n, q, \sigma, \omega, \varepsilon) \in \mathbb{R}$.

Proof: We reduce the resolvent problem to a half space problem by the coordinate transformation $\phi : H_\sigma \to \mathbb{R}^n_+$ defined by $\tilde{x} = (\tilde{x}', \tilde{x_n}) = \phi(x) = (x', x_n - \sigma(x'))$. Then ϕ is a bijection with Jacobian equal to 1. For a function u on H_σ we define $\tilde{u}(\tilde{x}) := u(\phi^{-1}(\tilde{x}))$, $\tilde{x} \in \mathbb{R}^n_+$, and denote by $\tilde{\partial_i}$, $\tilde{\nabla}, \ldots$ the derivatives with respect to the variables $\tilde{x} \in \mathbb{R}^n_+$. Then we obtain

$$\partial_i u(x) = (\tilde{\partial_i} - (\partial_i \sigma)\tilde{\partial_n})\tilde{u}(\tilde{x}) \tag{21}$$

$$\partial_i \partial_j u(x) = (\tilde{\partial_i}\tilde{\partial_j} - (\partial_i\sigma)\tilde{\partial_j}\tilde{\partial_n} - (\partial_j\sigma)\tilde{\partial_i}\tilde{\partial_n} - (\partial_j\partial_i\sigma)\tilde{\partial_n} + (\partial_j\sigma)(\partial_i\sigma)\tilde{\partial_n}\tilde{u}(\tilde{x}), \tag{22}$$

where we used that $\partial_n \sigma = 0$.
Note that by Lemma 2.1 for $\omega \in A_q$ also $\tilde{\omega} \in A_q$ with A_q-constant $A_q(\tilde{\omega}) \leq c \, A_q(\omega)$. For $u \in W^{2,q}_\omega(H_\sigma)$ we have with a constant $c > 0$ depending only on n

$$\begin{aligned} \|u\|_{q,\omega,H_\sigma} &= \|\tilde{u}\|_{q,\tilde{\omega},\mathbb{R}^n_+} \\ \|\nabla u\|_{q,\omega,H_\sigma} &\leq c\,(1 + \|\nabla'\sigma\|_\infty)\|\tilde{\nabla}\tilde{u}\|_{q,\tilde{\omega},\mathbb{R}^n_+} \\ \|\nabla^2 u\|_{q,\omega,H_\sigma} &\leq c\,(1 + \|\nabla'\sigma\|_\infty)^2\|\tilde{\nabla}^2\tilde{u}\|_{q,\tilde{\omega},\mathbb{R}^n_+} + c\,\|({\nabla'}^2\sigma)\tilde{\partial_n}\tilde{u}\|_{q,\tilde{\omega},\mathbb{R}^n_+}. \end{aligned}$$

Since $\|\tilde{\nabla}\tilde{\varphi}\|_{q',\tilde{\omega'},\mathbb{R}^n_+} \leq c\,(1 + \|\nabla'\sigma\|_\infty)\|\nabla\varphi\|_{q',\omega',H_\sigma}$ and $\int_{H_\sigma} g\,\varphi\,dx = \int_{\mathbb{R}^n_+} \tilde{g}\,\tilde{\varphi}\,d\tilde{x}$ for all $\varphi \in \widehat{W}^{1,q'}_{\omega'}(H_\sigma)$ we obtain for $g \in W^{1,q}_\omega(H_\sigma) \cap \widehat{\mathcal{W}}^{-1,q}_\omega(H_\sigma)$ that

$$\|g\|_{\widehat{\mathcal{W}}^{-1,q}_\omega(H_\sigma)} \leq c\,\|\tilde{g}\|_{\widehat{\mathcal{W}}^{-1,q}_{\tilde{\omega}}(\mathbb{R}^n_+)}.$$

To apply a perturbation argument we consider the following Banach spaces

$$\begin{aligned} X &= \big(W^{2,q}_\omega(H_\sigma) \cap W^{1,q}_{0,\omega}(H_\sigma)\big)^n \times \widehat{\mathcal{W}}^{1,q}_\omega(H_\sigma), \quad \|(u,p)\|_X = \|(\lambda u, \nabla^2 u, \nabla p)\|_{L^q_\omega(H_\sigma)} \\ Y &= L^q_\omega(H_\sigma)^n \times \big(W^{1,q}_\omega(H_\sigma) \cap \widehat{\mathcal{W}}^{1,q}_\omega(H_\sigma)\big), \\ \|(f,g)\|_Y &= \|(f, \nabla g)\|_{q,\omega,H_\sigma} + \|\lambda g\|_{\widehat{\mathcal{W}}^{-1,q}_\omega(H_\sigma)} \end{aligned}$$

and the operator

$$S^\lambda_{q,\omega} : X \longrightarrow Y, \quad S^\lambda_{q,\omega}(u,p) = (\lambda u - \Delta u + \nabla p, -\operatorname{div} u).$$

Further we define $\tilde X$, $\tilde Y, \tilde S^\lambda_{q,\tilde\omega}$ by replacing $H_\sigma, u, \omega, \nabla$ etc. by $\mathbb{R}^n_+, \tilde u, \tilde\omega, \tilde\nabla$ etc. Using (21), (22) we get

$$S^\lambda_{q,\omega}(u,p)(x) = \tilde S^\lambda_{q,\tilde\omega}(\tilde u,\tilde p)(\tilde x) + R_{q,\tilde\omega}(\tilde u,\tilde p)(\tilde x),$$

where the remainder $R_{q,\tilde\omega}$ on $\tilde X$ is given by

$$\begin{aligned} R_{q,\tilde\omega}(\tilde u,\tilde p) = (&-|\nabla'\sigma|^2\tilde\partial_n^2\tilde u + 2(\nabla'\sigma,0)\cdot\tilde\nabla\tilde\partial_n\tilde u \\ &+(\Delta'\sigma)\tilde\partial_n\tilde u - (\nabla'\sigma,0)\tilde\partial_n\tilde p,\, (\nabla'\sigma,0)\cdot\tilde\partial_n\tilde u). \end{aligned} \tag{23}$$

Then $R_{q,\tilde\omega}(\tilde u,\tilde p)$ can be estimated in the norm of $\tilde Y$ as follows

$$\begin{aligned} \|R_{q,\tilde\omega}(\tilde u)\|_{\tilde Y} \le{}& 3\|\nabla'\sigma\|_\infty(1+\|\nabla'\sigma\|_\infty)\|\tilde\nabla^2\tilde u\|_{q,\tilde\omega,\mathbb{R}^n_+} + \|\nabla'\sigma\|_\infty\|\tilde\partial_n\tilde p\|_{q,\tilde\omega,\mathbb{R}^n_+} \\ &+ 2\|\nabla'^2\sigma\|_\infty\|\tilde\partial_n\tilde u\|_{q,\tilde\omega,\mathbb{R}^n_+} + |\lambda|\|(\nabla'\sigma,0)\cdot\tilde\partial_n\tilde u\|_{\widehat{\mathcal{W}}^{-1,q}_{\tilde\omega}(\mathbb{R}^n_+)}. \end{aligned} \tag{24}$$

For the estimate of the term $\|\nabla'^2\sigma\|_\infty\|\tilde\partial_n\tilde u\|_{q,\tilde\omega,\mathbb{R}^n_+}$ use Corollary 5.3 from [12] to get for $\delta\in(0,1)$

$$\begin{aligned} \|\nabla'^2\sigma\|_\infty\|\tilde\partial_n\tilde u\|_{q,\tilde\omega,\mathbb{R}^n_+} &\le C_1\Big(\frac{\|\nabla'^2\sigma\|^2_\infty}{\delta}\|\tilde u\|_{q,\tilde\omega,\mathbb{R}^n_+} + \delta\|\tilde\nabla^2\tilde u\|_{q,\tilde\omega,\mathbb{R}^n_+}\Big) \\ &= C_1\,\delta\,\big(\lambda_0\|\tilde u\|_{q,\tilde\omega,\mathbb{R}^n_+} + \|\nabla^2\tilde u\|_{q,\tilde\omega,\mathbb{R}^n_+}\big), \end{aligned} \tag{25}$$

if $\lambda_0 = \|\nabla'^2\sigma\|^2_\infty\delta^{-2}$. The constant C_1 depends A_q-consistently increasing on $\tilde\omega$. Since $A_q(\tilde\omega)\le c\,A_q(\Omega)$ by Lemma 2.1 , C_1 depends A_q-consistently increasing on ω.
To estimate $\|(\nabla'\sigma,0)\cdot\tilde\partial_n\tilde u\|_{\widehat{\mathcal{W}}^{-1,q}_{\tilde\omega}}$ we note that $\gamma_{\mathbb{R}^n_+}(\tilde u)=0$ and $\tilde\partial_n(\nabla'\sigma,0)=0$ and get by an integration by parts ([12], Theorem 3.2) that

$$\int_{\mathbb{R}^n_+}(\nabla'\sigma,0)\,\tilde\partial_n\tilde u\,\tilde\varphi\,d\tilde x = -\int_{\mathbb{R}^n_+}(\nabla'\sigma,0)\,\tilde u\,\tilde\partial_n\tilde\varphi\,d\tilde x$$

for $\varphi\in C_0^\infty(\overline{\mathbb{R}^n_+})$. Since $C_0^\infty(\overline{\mathbb{R}^n_+})$ is dense in $\widehat{\mathcal{W}}^{1,q'}_{\omega'}(\mathbb{R}^n_+)$ (see [11], Corollary 4.1)

$$|\lambda|\,\|(\nabla'\sigma,0)\cdot\tilde\partial_n\tilde u\|_{\widehat{\mathcal{W}}^{-1,q}_{\tilde\omega}} \le \|\nabla'\sigma\|_\infty\|\lambda\tilde u\|_{q,\tilde\omega}. \tag{26}$$

Inserting (25) and (26) into (24) we get

$$\begin{aligned} \|R_{q,\tilde\omega}(\tilde u,\tilde p)\|_{\tilde Y} &\le 3\|\nabla'\sigma\|_\infty(1+\|\nabla'\sigma\|_\infty)\|\tilde\nabla^2\tilde u\|_{q,\tilde\omega,\mathbb{R}^n_+} + \|\nabla'\sigma\|_\infty\|\tilde\partial_n\tilde p\|_{q,\tilde\omega,\mathbb{R}^n_+} \\ &\quad + C_1\delta\big(\lambda_0\|\tilde u\|_{q,\tilde\omega,\mathbb{R}^n_+} + \|\nabla^2\tilde u\|_{q,\tilde\omega,\mathbb{R}^n_+}\big) + \|\nabla'\sigma\|_\infty\|\lambda\tilde u\|_{q,\tilde\omega} \\ &\le C\,(\|\nabla'\sigma\|_\infty(1+\|\nabla'\sigma\|_\infty) + C_1\delta)\,\|\lambda\tilde u - \tilde\Delta\tilde u + \tilde\nabla\tilde p\|_{q,\tilde\omega,\mathbb{R}^n_+} \\ &\le k\,\|\tilde S^\lambda_{q,\tilde\omega}(\tilde u,\tilde p)\|_{\tilde Y} \end{aligned}$$

on $\tilde X$ with an A_q-consistently increasing constant $C = C(n,q,\omega,\varepsilon)$. Here $k<1$, if $\|\nabla'\sigma\|_\infty \le K = K(n,q,\omega,\varepsilon)$ and $K>0$ as well as $\delta>0$ are sufficiently small. Since

the constants C and C_1 in the above estimate depend A_q-consistently increasing on ω, we can choose $K = K(n,q,\omega,\varepsilon) > 0$ and $\delta > 0$ A_q-consistently decreasing. Hence $\lambda_0 = \|\nabla'^2\sigma\|_\infty^2 \delta^{-2}$ can be chosen A_q-consistently increasing.
Since by Theorem 3.1 the operator $\tilde{S}^\lambda_{q,\tilde{\omega}} : \tilde{X} \longrightarrow \tilde{Y}$ is an isomorphism it follows that (see [16], chapter IV, Theorem 1.16) $\tilde{S}^\lambda_{q,\tilde{\omega}} + R_{q,\tilde{\omega}}$ is an isomorphism from $\tilde{X}$ to $\tilde{Y}$. Hence $S^\lambda_{q,\omega}$ is an isomorphism from X to Y. Summarizing we get

$$\begin{aligned}\|(u,p)\|_X &\le c_1\|(\tilde{u},\tilde{p})\|_{\tilde{X}} \le c_2\|\tilde{S}^\lambda_{q,\tilde{\omega}}(\tilde{u},\tilde{p})\|_{\tilde{Y}}\\ &\le c_3\|(\tilde{S}^\lambda_{q,\tilde{\omega}} + R_{q,\tilde{\omega}})(\tilde{u},\tilde{p})\|_{\tilde{Y}} \le c_4\|S^\lambda_{q,\omega}(u,p)\|_Y\end{aligned}$$

for all $\lambda \in \Sigma_\varepsilon$ with $|\lambda| \ge \lambda_0$, where the constants c_1, c_2, c_3 und c_4 are A_q-consistently increasing on ω. □

3.2 The bounded domain

Let $1 < q < \infty$, $\omega \in A_q$ and let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary of class $C^{1,1}$. Since by Lemma 2.2 ii) there are exponents $1 < r,\ s < \infty$ such that the continuous embeddings

$$L^r(\Omega) \hookrightarrow L^q_\omega(\Omega) \hookrightarrow L^s(\Omega) \tag{27}$$

hold, it is easy to transfer uniqueness and partially also existence results from the case without weights to the weighted case.
Consider for $\lambda \in \Sigma_\varepsilon \cup \{0\}$, $0 < \varepsilon < \frac{\pi}{2}$, the operator

$$\begin{aligned}S^\lambda_{q,\omega} : D(S^\lambda_{q,\omega}) &\longrightarrow L^q_\omega(\Omega)^n \times (W^{1,q}_\omega(\Omega) \cap \widehat{W}^{-1,q}_\omega(\Omega))\\ D(S^\lambda_{q,\omega}) &:= (W^{2,q}_\omega(\Omega) \cap W^{1,q}_{0,\omega}(\Omega))^n \times \widehat{W}^{1,q}_\omega(\Omega)\\ (u,p) &\mapsto (\lambda u - \Delta u + \nabla p,\ -\operatorname{div} u)\end{aligned}$$

In [9] it is shown for arbitrary $1 < q < \infty$ and for $\omega \equiv 1$ that this operator is an isomorphism.

Lemma 3.1 *Let $1 < q < \infty$ and $\omega \in A_q$, $\Omega \subset \mathbb{R}^n$ a bounded domain with boundary of class $C^{1,1}$ and let $\lambda \in \Sigma_\varepsilon \cup \{0\}$, $0 < \varepsilon < \frac{\pi}{2}$. Then:*

i) $S^\lambda_{q,\omega}$ is injective.

ii) The range $R(S^\lambda_{q,\omega})$ of $S^\lambda_{q,\omega}$ is dense in $L^q_\omega(\Omega)^n \times (W^{1,q}_\omega(\Omega) \times \widehat{W}^{-1,q}_\omega(\Omega))$.

Proof: i) We choose $1 < s < \infty$ such that $L^q_\omega(\Omega) \subset L^s(\Omega)$. By [9] the operator S^λ_s (with $\omega \equiv 1$) is injective. Since $S^\lambda_{q,\omega}$ is equal to the restriction of S^λ_s to $D(S^\lambda_{q,\omega})$, the assertion i) is clear.
ii) Choose $1 < r < \infty$ such that $L^r(\Omega) \subset L^q_\omega(\Omega)$. By [9] the operator S^λ_r is surjective and $L^r(\Omega)^n \times (W^{1,r}(\Omega) \cap \widehat{W}^{-1,r}(\Omega))$ is dense in $L^q_\omega(\Omega)^n \times (W^{1,q}_\omega(\Omega) \cap \widehat{W}^{-1,q}_\omega(\Omega))$. □

It remains to show the resolvent estimate. First, we show the following weaker estimate:

Lemma 3.2 *Under the assumptions of Lemma 3.1 let* $(u,p) \in D(S^\lambda_{q,\omega})$ *and* $(f,-g) = S^\lambda_{q,\omega}(u,p)$. *Then there is an* A_q*-consistently increasing constant* $C = C(n,q,\omega,\varepsilon,\Omega) \in \mathbb{R}$ *such that*

$$|\lambda|\|u\|_{q,\omega} + \|\nabla^2 u\|_{q,\omega} + \|\nabla p\|_{q,\omega} \le C\,\Big(\,\|(f,\nabla g)\|_{q,\omega} + \|\lambda g\|_{\widehat{W}^{-1,q}_\omega(\Omega)} + \|u\|_{1,q,\omega} + \|p\|_{q,\omega} + \|\lambda u\|_{(W^{1,q'}_{\omega'}(\Omega))'}\Big)$$

for all $\lambda \in \Sigma_\varepsilon \cup \{0\}$. *Here* $\|\nabla^2 u\|_{q,\omega}$ *may be replaced by* $\|u\|_{2,q,\omega}$.

The proof rests on the well known localisation method, which reduces the problem to $\mathbb{R}^n$ and H_σ:
Since Ω is a bounded $C^{1,1}$-domain there is a covering of $\overline{\Omega}$ by a finite number of balls $B_1,\dots,B_m \subset \mathbb{R}^n$ and nonnegative cut-off functions $\psi_1,\dots,\psi_m \in C_0^\infty(\mathbb{R}^n)$ with support $\operatorname{supp}\psi_j \subset B_j$ and $\sum_{j=1}^m \psi_j = 1$ on Ω. Since $\partial\Omega \in C^{1,1}$ for every j with $B_j \cap \partial\Omega \neq \phi$ there exists a function $\sigma_j \in C^{1,1}(\mathbb{R}^{n-1})$ such that (after a suitable translation and rotation of the coordinate system)

$$B_j \cap \Omega \subset H_{\sigma_j} \quad \text{and} \quad B_j \cap \partial\Omega \subset \partial H_{\sigma_j}.$$

We can choose a sufficiently large number m of sufficiently small balls B_j such that σ_j satisfies the assumptions of Theorem 3.2 for all $j \in \{1,\dots,m\}$. Since the constant $K > 0$ of Theorem 3.2 depends A_q-consistently decreasing on $\omega \in A_q$ the number m can be chosen A_q-consistently, i.e., for every $c \in \mathbb{R}_+$ one can choose the same partition of unity $\{\psi_j\}_{j=1}^m$ for all $\omega \in A_q$ with $A_q(\omega) \le c$.
Further note that the translation and rotation a_j of the coordinate system yields the new weight $\omega_j := \omega \circ a_j \in A_q$ with an A_q-constant $A_q(\omega_j)$, which is comparable to $A_q(\omega)$, i.e., there are constants $c_1, c_2 > 0$ independent of ω such that $c_1 A_q(\omega) \le A_q(\omega_j) \le c_2 A_q(\omega)$ for $j = 1,\dots,m$ (see Lemma 2.1). Since the constant C in Theorem 3.2 is A_q-consistently increasing, the mapping a_j will not disturb the A_q-consistency of constants in the estimates. Furthermore, note that $\|v \circ a_j\|_{q,\omega_j,\mathbb{R}^n} = \|v\|_{q,\omega,\mathbb{R}^n}$, $\|\nabla(v \circ a_j)\|_{q,\omega_j,\mathbb{R}^n} \sim \|\nabla v\|_{q,\omega,\mathbb{R}^n}, \dots$. Thus we can suppress the transformation a_j in the following.
In the case $B_j \cap \partial\Omega = \phi$ let even $\overline{B_j} \subset \Omega$. Then there are two kinds of cut-off functions ψ_j, $j \in \{1,\dots,m\}$:

$$\begin{array}{lcll} \text{type } \mathbb{R}^n & : & \psi_j & \text{if } \overline{B_j} \subset \Omega \\ \text{type } H_\sigma & : & \psi_j & \text{if } B_j \cap \partial\Omega \neq \phi. \end{array}$$

For $(u,p) \in D(S^\lambda_{q,\omega})$ und $(f,-g) = S^\lambda_{q,\omega}(u,p)$ it holds for $j = 1,\dots,m$

$$\lambda(\psi_j u) - \Delta(\psi_j u) + \nabla(\psi_j p) = f_j, \quad \operatorname{div}(\psi_j u) = g_j \tag{28}$$

in Ω with

$$f_j = \psi_j f - 2(\nabla\psi_j)\nabla u - (\Delta\psi_j)u + (\nabla\psi_j)p \tag{29}$$

$$g_j = \psi_j g + (\nabla\psi_j)\cdot u. \tag{30}$$

Depending on the type of ψ_j we can interpret these equations as problems in $\mathbb{R}^n$ or H_{σ_j}, where we suppress the transformation of the coordinate system.

Proof of Lemma 3.2: Let $(u,p) \in D(S^\lambda_{q,\omega})$ and $(f,-g) = S^\lambda_{q,\omega}(u,p)$. As explained above we choose a partition of unity $\{\psi_j : j = 1,\dots,m\}$ and consider a cut-off function ψ_j of type H_σ (for a cut off function ψ_j of type $\mathbb{R}^n$ the proof is analogous if we use Theorem 3.1 instead of Theorem 3.2). Adding $\lambda_0\psi_j u$ on both sides of the equation (28) we obtain

$$\begin{aligned}(\lambda+\lambda_0)(\psi_j u) - \Delta(\psi_j u) + \nabla(\psi_j p) &= f_j + \lambda_0\psi_j u\\ \operatorname{div}(\psi_j u) &= g_j.\end{aligned}$$

We choose $\lambda_0 > 0$ so large that Theorem 3.2 can be applied. By Theorem 3.2 λ_0 can be chosen A_q-consistently increasing. Thus we get the estimate

$$\begin{aligned}&\|(\lambda+\lambda_0)\psi_j u\|_{q,\omega} + \|\nabla^2(\psi_j u)\|_{q,\omega} + \|\nabla(\psi_j p)\|_{q,\omega}\\ &\quad\le C(\|f_j\|_{q,\omega} + \|\lambda_0\psi_j u\|_{q,\omega} + \|\nabla g_j\|_{q,\omega} + \|(\lambda+\lambda_0)g_j\|_{\widehat{\mathcal{W}}_\omega^{-1,q}(H_{\sigma_j})}).\end{aligned}$$

From (29), (30) and the weighted Poincaré inequality Corollary 2.1 ($g \in \widehat{\mathcal{W}}_\omega^{-1,q}(\Omega)$ implies $\int_\Omega g = 0$) it follows

$$\begin{aligned}\|f_j\|_{q,\omega} &\le C(\psi_j)(\|f\|_{q,\omega} + \|u\|_{1,q,\omega} + \|p\|_{q,\omega}),\\ \|\nabla g_j\|_{q,\omega} &\le C(\psi_j)(\|\nabla g\|_{q,\omega} + \|u\|_{1,q,\omega}).\end{aligned}$$

To estimate g_j in $\widehat{\mathcal{W}}_\omega^{-1,q}(H_{\sigma_j})$ let $\varphi \in C_0^\infty(\overline{H_{\sigma_j}})$ and let $\tilde\varphi = \varphi - \frac{1}{|G_j|}\int_{G_j}\varphi\,dx$ with $G_j = \operatorname{supp}\nabla\psi_j$. Since $\int_{G_j}\tilde\varphi\,dx = 0$ the weighted Poincaré inequality yields $\|\nabla(\psi_j\tilde\varphi)\|_{q',\omega',\Omega} \le c_1\|\nabla\varphi\|_{q',\omega',H_{\sigma_j}}$ and $\|(\nabla\psi_j)\tilde\varphi\|_{W^{1,q'}_{\omega'}(\Omega)} \le c_2\|\nabla\varphi\|_{q',\omega',H_{\sigma_j}}$ with A_q-consistently increasing constants $c_1, c_2 > 0$ (see Corollary 2.1). Since

$$(g_j,\varphi) = (\operatorname{div}(u\psi_j),\varphi) = -(u\psi_j,\nabla\tilde\varphi) = -(u,\nabla(\psi_j\tilde\varphi) + (u,(\nabla\psi_j)\tilde\varphi)$$

it follows that

$$\begin{aligned}\|g_j\|_{\widehat{\mathcal{W}}_\omega^{-1,q}(H_{\sigma_j})} &= \sup_{0\neq\varphi\in C_0^\infty(\overline{H_{\sigma_j}})}\frac{|(g_j,\varphi)|}{\|\nabla\varphi\|_{q',\omega',\Omega}}\\ &\le c_1 \sup_{0\neq v\in\widehat{W}^{1,q'}_{\omega'}(\Omega)}\frac{|(u,\nabla v)|}{\|\nabla v\|_{q',\omega',\Omega}} + c_2\,\|u\|_{[W^{1,q'}_{\omega'}(\Omega)]'}\\ &= c_1\,\|g\|_{\widehat{\mathcal{W}}_\omega^{-1,q}(\Omega)} + c_2\,\|u\|_{[W^{1,q'}_{\omega'}(\Omega)]'}.\end{aligned}$$

Hence $\|\lambda g_j\|_{\widehat{\mathcal{W}}_\omega^{-1,q}(H_{\sigma_j})} \le c(\|\lambda g\|_{\widehat{\mathcal{W}}_\omega^{-1,q}(\Omega)} + \|\lambda u\|_{[W^{1,q'}_{\omega'}(\Omega)]'})$. Moreover, since $\int_\Omega g = 0$, we obtain by applying the weighted Poincaré inequality twice $\|g\|_{\widehat{\mathcal{W}}_\omega^{-1,q}(\Omega)} \le c\|\nabla g\|_{q,\omega}$, whence $\|\lambda_0 g_j\|_{\widehat{\mathcal{W}}_\omega^{-1,q}(H_{\sigma_j})} \le C(\|\nabla g\|_{q,\omega} + \|u\|_{[W^{1,q'}_{\omega'}(\Omega)]'})$. Summing up the inequalities obtained above for $j = 1,\dots m$ yields the desired estimate of Lemma3.2. Since $\gamma(u) = 0$ by Corollary 2.2 also the additional remark is clear. □

Lemma 3.3 *Under the assumptions of Lemma 3.1 let $(u,p) \in D(S^\lambda_{q,\omega})$ and $(f,-g) = S^\lambda_{q,\omega}(u,p)$. Then there is an A_q-consistently increasing constant $C = C(\Omega,q,\omega,\varepsilon) \in \mathbb{R}$ such that*

$$|\lambda|\|u\|_{q,\omega} + \|u\|_{2,q,\omega} + \|\nabla p\|_{q,\omega} \le C\,(\,\|(f,\nabla g)\|_{q,\omega} + \|\lambda g\|_{\widehat{\mathcal{W}}_\omega^{-1,q}(\Omega)}). \tag{31}$$

Proof: Assumme the Lemma was wrong. Then there is a $c_0 \in \mathbb{R}_+$, a sequence $(\omega_j) \subset A_q$ with $\sup_j A_q(\omega_j) \le c_0$, sequences $(u_j, p_j) \in (W^{2,q}_{\omega_j}(\Omega) \cap W^{1,q}_{0,\omega_j}(\Omega))^n \times \widehat{W}^{1,q}_{\omega_j}(\Omega)$ and $(\lambda_j) \subset \Sigma_\varepsilon \cup \{0\}$ such that

$$\|\lambda_j u_j\|_{q,\omega_j} + \|u_j\|_{2,q,\omega_j} + \|\nabla p_j\|_{q,\omega_j} > j\,(\|(f_j, \nabla g_j)\|_{q,\omega_j} + \|\lambda_j g_j\|_{\widehat{\mathcal{W}}^{-1,q}_{\omega_j}(\Omega)}) \tag{32}$$

for all $j \in \mathbb{N}$, where $(f_j, -g_j) = S^{\lambda_j}_{q,\omega_j}(u_j, p_j)$. We may assume w. l. o. g. that $\int_\Omega p_j\, dx = 0$ for all $j \in \mathbb{N}$ and that (λ_j) converges to some $\lambda \in \overline{\Sigma_\varepsilon} \cup \{\infty\}$.
Let Q be an open cube containing $\overline{\Omega}$. Then for $\widetilde{\omega}_j := \omega_j(Q)^{-1}\omega_j$ it holds both $\widetilde{\omega}_j(Q) = 1$ and $A_q(\widetilde{\omega}_j) = A_q(\omega_j) \le c_0$ for all $j \in \mathbb{N}$. Multipling (32) by $\omega_j(Q)^{-\frac{1}{q}}$ yields the same inequality with ω_j replaced by $\widetilde{\omega}_j$. We suppress the notation $\widetilde{\omega_j}$ and write again ω_j. Then we can assume w. l. o. g. that

$$\|\lambda_j u_j\|_{q,\omega_j} + \|u_j\|_{2,q,\omega_j} + \|\nabla p_j\|_{q,\omega_j} = 1 \quad \forall j \in \mathbb{N} \tag{33}$$

$$\|(f_j, \nabla g_j)\|_{q,\omega_j} + \|\lambda_j g_j\|_{\widehat{\mathcal{W}}^{-1,q}_{\omega_j}(\Omega)} \to 0 \quad \text{für } j \to \infty. \tag{34}$$

By Lemma 2.2 ii) and the remark after this Lemma there is an $\epsilon > 0$, such that $L^q_{\omega_j}(\Omega)$ is continuously embedded into $L^{1+\epsilon}(\Omega)$ with an embedding constant is independent of $j \in \mathbb{N}$. Hence the sequences $(\lambda_j u_j)$, (u_j) and (∇p_j) are bounded in $L^{1+\epsilon}(\Omega)^n$, $W^{2,1+\epsilon}(\Omega)^n$ and $L^{1+\epsilon}(\Omega)^n$ respectively. Suppressing the notion of subsequences we get the weak convergences

$$\lambda_j u_j \rightharpoonup v, \quad u_j \rightharpoonup u, \quad \nabla p_j \rightharpoonup \nabla p \tag{35}$$

in $L^{1+\epsilon}(\Omega)^n$, $W^{2,1+\epsilon}(\Omega)^n$ resp. $L^{1+\epsilon}(\Omega)^n$, where $p \in W^{1,1+\epsilon}(\Omega)$ is chosen in such a way that $\int_\Omega p\, dx = 0$. Since $\lambda_j u_j \rightharpoonup v$ weakly in $L^{1+\epsilon}(\Omega)^n$ and $\operatorname{div}(\lambda_j u_j) = \lambda_j g_j \to 0$ in $\widehat{\mathcal{W}}^{-1,1+\epsilon}(\Omega)$ and therefore also in $\mathcal{D}'(\Omega)$, it follows $\operatorname{div} v = 0$. Hence the trace $\gamma_N(v) := \gamma(v \cdot N) \in [T^{1,(1+\epsilon)'}(\partial\Omega)]' = W^{-\frac{\epsilon}{1+\epsilon},1+\epsilon}(\partial\Omega)$ of v in direction of the outer unit vector N on $\partial\Omega$ is well defined. Because of $\gamma(u_j) = 0$ we get

$$\begin{aligned} < \gamma_N(v), \phi > &= \int_\Omega v \nabla\Phi\, dx = \lim_{j\to\infty} \int_\Omega \lambda_j u_j \nabla\Phi\, dx \\ &= -\lim_{j\to\infty} \int_\Omega \lambda_j g_j \Phi\, dx = 0 \end{aligned}$$

for every $\phi \in T^{1,(1+\epsilon)'}(\partial\Omega)$ and every extension $\Phi \in W^{1,(1+\epsilon)'}(\Omega)$ such that $\gamma(\Phi) = \phi$. Hence

$$\begin{aligned} v - \Delta u + \nabla p = 0, \quad \nabla \operatorname{div} u = 0, \quad \operatorname{div} v = 0 \quad \text{in } \Omega, \\ \gamma_N(v) = 0 \quad \text{auf } \partial\Omega. \end{aligned} \tag{36}$$

From (35) it follows by the compact embedding that $u_j \to u$ in $W^{1,1+\epsilon}(\Omega)$. Since $\gamma(u_j) = 0$ also $\gamma(u) = 0$. Hence $\int_\Omega \operatorname{div} u = 0$ und because of (36) we conclude $\operatorname{div} u = 0$.
We now distinguish the cases $\lambda \ne \infty$ and $\lambda = \infty$:
Let $\lambda_j \to \lambda \ne \infty$. The weak convergences (35) imply $\lambda u = v$. Therefore $S^\lambda_{1+\epsilon}(u,p) = 0$ because of (36) and since $S^\lambda_{1+\epsilon}$ is injective, it follows $u = 0$ and $\nabla p = 0$. But since $\int_\Omega p\, dx = 0$, even $p = 0$. Inserting this into (35) we get

$$u_j \rightharpoonup 0, \quad \nabla p_j \rightharpoonup 0, \quad \lambda_j u_j \rightharpoonup 0 \tag{37}$$

in $W^{2,1+\epsilon}(\Omega)^n$, $L^{1+\epsilon}(\Omega)^n$ resp. $L^{1+\epsilon}(\Omega)^n$. The first convergence, (33), the properties of the sequence (ω_j) and Theorem 2.4 yield

$$\|u_j\|_{1,q,\omega_j} \to 0 \quad \text{für } j \to \infty. \tag{38}$$

Because of the second convergence in (37), $\int_\Omega p_j \, dx = 0$ and because of Poincaré's inequality we can assume that (p_j) converges weakly in $W^{1,1+\epsilon}(\Omega)$ to 0. By Theorem 2.4 we obtain

$$\|p_j\|_{L^q_{\omega_j}(\Omega)} \to 0 \quad \text{für } j \to \infty. \tag{39}$$

Finally we claim that

$$\|\lambda_j u_j\|_{[W^{1,q'}_{\omega'_j}(\Omega)]'} \to 0. \tag{40}$$

Choose a sequence $v_j \in W^{1,q'}_{\omega'_j}(\Omega), j = 1, 2, \dots$ such that $\|v_j\|_{1,q',\omega'_j} = 1$ and

$$\|\lambda_j u_j\|_{[W^{1,q'}_{\omega'_j}(\Omega)]'} \le \frac{1}{j} + |(\lambda_j u_j, v_j)|.$$

By Theorem 2.4 ii) for arbitrary $\delta > 0$ there exists a $j_0(\delta)$ and a $v_\delta \in C^\infty(\overline{\Omega})$, such that $\|v_j - v_\delta\|_{q',\omega'_j,\Omega} \le \delta$ for all $j \ge j_0(\delta)$. Because of

$$|(\lambda_j u_j, v_j)| \le |(\lambda_j u_j, v_\delta)| + \|\lambda_j u_j\|_{q,\omega_j,\Omega} \|v_j - v_\delta\|_{q',\omega'_j,\Omega},$$

(33) and (37), the claim (40) is clear.
By Lemma 3.2

$$\begin{aligned} |\lambda_j| \|u_j\|_{q,\omega_j} + \|u_j\|_{2,q,\omega_j} + \|\nabla p_j\|_{q,\omega_j} \le C \big(&\|(f_j, \nabla g_j)\|_{q,\omega_j} + \|\lambda_j g_j\|_{\widehat{W}^{-1,q}_{\omega_j}(\Omega)} \\ &+ \|u_j\|_{1,q,\omega_j} + \|p_j\|_{q,\omega_j} + \|\lambda u_j\|_{[W^{1,q'}_{\omega'_j}(\Omega)]'} \big), \end{aligned}$$

where C is independent of $j \in \mathbb{N}$ because of the A_q-consistence of the constant in Lemma 3.2 and $A_q(\omega_j) \le c_0$. Using (34),(33), (38), (39), (40), we get the contradiction $1 \le 0$ for $j \to \infty$.
In the case $\lambda_j \to \infty$ (33) yields the convergence

$$\|u_j\|_{q,\omega_j,\Omega} \to 0,$$

which implies the convergence of (u_j) to 0 in $L^{1+\epsilon}(\Omega)^n$. Thus $u = 0$. Then (36) is the unique Helmholtz decomposition in $L^{1+\epsilon}(\Omega)^n$ of the zero vectorfield. It follows $v = \nabla p = 0$. By analogous argumentation as in the previous case we obtain the same contradiction. □

Corollary 3.1 *Under the assumptions of Lemma 3.1 the operator $S^\lambda_{q,\omega}$ is surjective.*

Proof: By Lemma 3.1 the range of $S^\lambda_{q,\omega}$ is dense. The estimate in Lemma 3.3 implies that the range of $S^\lambda_{q,\omega}$ is closed. □

Summing up we obtain the following result:

Theorem 3.3 *Let $n \geq 2$, $1 < q < \infty$, $\omega \in A_q$, $0 < \varepsilon < \frac{\pi}{2}$, $\lambda \in \Sigma_\varepsilon \cup \{0\}$ and let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary of class $C^{1,1}$.*

- *i) Then the operator $S^\lambda_{q,\omega}$ is an isomorphism. More precisely: For all $f \in L^q_\omega(\Omega)^n$, $g \in W^{1,q}_\omega(\Omega) \cap \widehat{W}^{-1,q}_\omega(\Omega)$ there exists a unique solution $(u,p) \in W^{2,q}_\omega(\Omega)^n \times \widehat{W}^{1,q}_\omega(\Omega)$ of the Stokes resolvent problem (19). This solution satisfies with an A_q-consistently increasing constant $C = C(\Omega, q, \omega, \varepsilon) \in \mathbb{R}$ the estimate*

$$|\lambda| \|u\|_{q,\omega} + \|u\|_{2,q,\omega} + \|\nabla p\|_{q,\omega} \leq C\,(\,\|f\|_{q,\omega} + \|\nabla g\|_{q,\omega} + \|\lambda g\|_{\widehat{W}^{-1,q}_\omega}).$$

- *ii) If additionally $(f,g) \in L^r_v(\Omega)^n \times (W^{1,r}_v(\Omega) \cap \widehat{W}^{-1,r}_v(\Omega))$ for some $r \in (1,\infty)$ and some $v \in A_r$, then the solution (u,p) from i) is also contained in $W^{2,r}_v(\Omega)^n \times \widehat{W}^{1,r}_v(\Omega)$.*

The regularity assertion ii) follows immediately from the fact that there exists an $s \in (1,\infty)$ such that $L^q_\omega(\Omega) + L^r_v(\Omega) \subset L^s(\Omega)$ and the uniqueness assertion from part i) of the above Theorem applied to the case $\omega \equiv 1$ and $q := s \in (1,\infty)$.

4 The Stokes operator

Let $1 < q < \infty$, $\omega \in A_q$ and $\Omega = \mathbb{R}^n, \mathbb{R}^n_+$ or a bounded domain with boundary of class $C^{1,1}$. In [11] the Helmholtz decomposition of $L^q_\omega(\Omega)^n$ is proved:
The space $L^q_\omega(\Omega)^n$ has an algebraic and topological decomposition

$$L^q_\omega(\Omega)^n = L^q_{\omega,\sigma}(\Omega) \oplus \nabla \widehat{W}^{1,q}_\omega(\Omega)$$

where $L^q_{\omega,\sigma}(\Omega)$ is the closure of $\{u \in C^\infty_0(\Omega)^n \;:\; \operatorname{div} u = 0\}$ in $L^q_\omega(\Omega)^n$. In particular, the projection operator $P_{q,\omega} : L^q_\omega(\Omega)^n \to L^q_{\omega,\sigma}(\Omega)$ with null space $\nabla \widehat{W}^{1,q}_\omega(\Omega)$ is bounded.
Moreover the operator norm of $P_{q,\omega}$ in $\mathcal{L}(L^q_\omega(\Omega)^n)$, where $\mathcal{L}(X)$ means the space of bounded linear operators on a Banach space X, is A_q-consistently increasing: This follows for $\Omega = \mathbb{R}^n$ from the weighted version of Michlins multiplier theorem (see [12], Theorem 2.1). By the same argumentation as in [12] for $\Omega = \mathbb{R}^n_+$ and as in the present paper for a bounded domain we see that the reflection arguments and cut off techniques used to prove the result on $\mathbb{R}^n_+$ and bounded domains do not disturb the A_q-consistence.
Recall the definition (1), (2) of the Stokes operator $\mathcal{A}_{q,\omega}$. Then Theorem 3.1 and Theorem 3.3 yield the following result:

Theorem 4.1 *Let $1 < q < \infty$, $\omega \in A_q$ and $\Omega = \mathbb{R}^n$, $\mathbb{R}^n_+$ or a bounded $C^{1,1}$-domain.*

- *i) The Stokes operator $\mathcal{A}_{q,\omega} : D(\mathcal{A}_{q,\omega}) \subset L^q_{\omega,\sigma}(\Omega) \longrightarrow L^q_{\omega,\sigma}(\Omega)$ is densely defined and closed.*

- *ii) For every $f \in L^q_{\omega,\sigma}(\Omega)$ and $\lambda \in \Sigma_\varepsilon$, $0 < \varepsilon < \frac{\pi}{2}$ the resolvent problem*

$$\lambda u + \mathcal{A}_{q,\omega} u = f \tag{41}$$

has a unique solution $u \in D(\mathcal{A}_{q,\omega})$. This solution satisfies the estimate

$$|\lambda| \|u\|_{q,\omega} + \|\mathcal{A}_{q,\omega} u\|_{q,\omega} \leq C_\varepsilon \|f\|_{q,\omega}. \tag{42}$$

where $C_\varepsilon = C_\varepsilon(\omega)$ is A_q-consistently increasing. Moreover

$$(\lambda + \mathcal{A}_{q,\omega})^{-1} f = (\lambda + \mathcal{A}_{p,v})^{-1} f \tag{43}$$

for $1 < p < \infty$, $v \in A_p$ and $f \in L^q_{\omega,\sigma}(\Omega) \cap L^p_{v,\sigma}(\Omega)$.

If Ω is a bounded $C^{1,1}$-domain, then $\mathcal{A}_{q,\omega}$ is invertible.

iii) The Stokes operator $-\mathcal{A}_{q,\omega}$ generates a bounded analytic semigroup in $L^q_{\omega,\sigma}(\Omega)$.

4.1 R-bounded families

Definition 4.1 *Let X be Banach space. A subset $\mathcal{T} \subset \mathcal{L}(X)$ is called $\mathcal{R}$-bounded, if there exists a constant $C \in \mathbb{R}$ such that*

$$\int_0^1 \Big\| \sum_{j=1}^N r_j(u)\, T_j x_j \Big\|_X \, du \le C \int_0^1 \Big\| \sum_{j=1}^N r_j(u)\, x_j \Big\|_X \, du \tag{44}$$

for all $T_1, \dots, T_N \in \mathcal{T}$, $x_1, \dots, x_N \in X$ and $N \in \mathbb{N}$, where (r_j) is a sequence of independent, symmetric distributed, $\{-1, 1\}$-valued random variables defined on $[0, 1]$, e. g. the Rademacher functions. The smallest constant C such that (44) holds is called $\mathcal{R}$-bound of $\mathcal{T}$ and is denoted by $\mathcal{R}(\mathcal{T})$.

For $1 \le p < \infty$ one can replace the condition (44) in Definition 4.1 by

$$\int_0^1 \Big\| \sum_{j=1}^N r_j(u)\, T_j(x_j) \Big\|_X^p \, du \le C \int_0^1 \Big\| \sum_{j=1}^N r_j(u)\, x_j \Big\|_X^p \, du. \tag{45}$$

because of Kahane's inequality (see [6] Theorem 11.1). □

Using (45) with $p = 2$ it is easy to see that for every Hilbert space X a family $\mathcal{T} \subset \mathcal{L}(X)$ is $\mathcal{R}$-bounded if and only if $\mathcal{T}$ is bounded in $\mathcal{L}(X)$.

Lemma 4.1 (Khinchin's inequality) *Let $1 \le p < \infty$ and (r_j) as in Definition 4.1. Then there is a constant $0 < C_p < \infty$ such that*

$$C_p^{-1} \Big(\sum_{j=1}^N |a_j|^2 \Big)^{\frac{1}{2}} \le \Big(\int_0^1 \Big| \sum_{j=1}^N r_j(u)\, a_j \Big|^p du \Big)^{\frac{1}{p}} \le C_p \Big(\sum_{j=1}^N |a_j|^2 \Big)^{\frac{1}{2}}$$

for all $a_1, \dots, a_N \in \mathbb{C}$ and $N \in \mathbb{N}$.

Proof: See [6], Theorem 1.10. □

The following lemma is well known; we give its proof for the convenience of the reader.

Lemma 4.2 *Let (Ω, Σ, μ) be a measure space, $1 < q < \infty$ and $X = L^q(\Omega, \mu)$. Then $\mathcal{T} \subset L(X)$ is $\mathcal{R}$-bounded, if and only if there is a constant $C \in \mathbb{R}$, such that*

$$\Big\| \Big(\sum_{j=1}^N |T_j f_j|^2 \Big)^{\frac{1}{2}} \Big\|_{L^q(\Omega,\mu)} \le C \Big\| \Big(\sum_{j=1}^N |f_j|^2 \Big)^{\frac{1}{2}} \Big\|_{L^q(\Omega,\mu)} \tag{46}$$

for all $T_1, \dots, T_N \in \mathcal{T}$, $f_1, \dots, f_N \in X$ and $N \in \mathbb{N}$.

Proof: Assume that (46) holds. We show (45) with $p := q$, which implies the $\mathcal{R}$-boundedness. By the Theorem of Fubini and Khinchin's inequality we see that

$$\begin{aligned}
\int_0^1 \Big\| \sum_{j=1}^N r_j(u) T_j f_j \Big\|_{L^q(\Omega,\mu)}^q du &= \int_\Omega \int_0^1 \Big| \sum_{j=1}^N r_j(u) T_j f_j) \Big|^q du\, d\mu \\
\leq C_q \Big\| \Big(\sum_{j=1}^N |T_j f_j|^2 \Big)^{\frac{1}{2}} \Big\|_{L^q(\Omega,\mu)}^q &\leq C_q\, C \Big\| \Big(\sum_{j=1}^N |f_j|^2 \Big)^{\frac{1}{2}} \Big\|_{L^q(\Omega,\mu)}^q \\
\leq C_q^2\, C \int_\Omega \int_0^1 \Big| \sum_{j=1}^N r_j(u) f_j \Big|^q du\, d\mu &= C_q^2\, C \int_0^1 \Big\| \sum_{j=1}^N r_j(u) f_j \Big\|_{L^q(\Omega,\mu)}^q du.
\end{aligned}$$

The inverse implication is proved analogously. □

Lemma 4.3 *Let $1 < p < q < \infty$, $\frac{1}{s} = 1 - \frac{p}{q}$ and $\omega \in A_q$. Then for every function $0 \leq u \in L^s_\omega(\mathbb{R}^n)$ there is a function $0 \leq v \in L^s_\omega(\mathbb{R}^n)$ such that*

a) $u \leq v$ a. e. in $\mathbb{R}^n$,

b) $\|v\|_{s,\omega} \leq c\,\|u\|_{s,\omega}$ with an absolute constant c,

c) $v\omega \in A_p$ with A_p-constant depending only on the A_q-constant of ω - but not on u and v.

Proof: See [13], Chapter IV, Lemma 5.18. □

Theorem 4.2 (Extrapolation) *Let $1 \leq p < \infty$, $1 < q < \infty$. Let $\mathcal{T}$ be a family of linear operators with the property that for all $w \in A_p$ there is constant $C = C(A_p(w)) \in \mathbb{R}$ depending only on the A_p-constant of w such that*

$$\|Tf\|_{p,w} \leq C\,\|f\|_{p,w}$$

for all $f \in L^p_w(\mathbb{R}^n)$ and all $T \in \mathcal{T}$. Then every $T \in \mathcal{T}$ can be extended to a linear bounded Operator on $L^q_\omega(\mathbb{R}^n)$ for all $\omega \in A_q$. More precisely, for all $\omega \in A_q$ there is a constant $C \in \mathbb{R}$ depending only on the A_q-constant of ω such that

$$\|Tf\|_{q,\omega} \leq C\,\|f\|_{q,\omega}$$

for all $f \in L^q_\omega(\mathbb{R}^n)$ and all $T \in \mathcal{T}$.

Proof: See [13], Chapter IV, Proof of Theorem 5.19. □

The next Theorem is a generalisation of Theorem 6.4 in chapter V of [13].

Theorem 4.3 *Let $1 < p, q < \infty$, $\omega \in A_q$ and $\Omega \subset \mathbb{R}^n$ an open set. Moreover let $\mathcal{T} \subset \mathcal{L}(L^q_\omega(\Omega))$ be a familiy of linear operators with the property that for all $w \in A_p$ there is a constant $C = C(A_p(w)) \in \mathbb{R}$ depending only on the A_p-constant of w such that*

$$\|Tf\|_{L^p_w(\Omega)} \leq C\,\|f\|_{L^p_w(\Omega)} \tag{47}$$

for all $f \in L^q_\omega(\Omega) \cap L^p_w(\Omega)$ and all $T \in \mathcal{T}$. Then $\mathcal{T}$ is $\mathcal{R}$-bounded in $\mathcal{L}(L^q_\omega(\Omega))$.

Proof: First, we prove the Theorem for $\Omega = \mathbb{R}^n$. By the Extrapolation Theorem 4.2 we can assume that $p = 2$. To verify the condition (46) in Lemma 4.2 we distinguish between the cases $q > 2$, $q < 2$ and $q = 2$:
Let $q > 2$. Then we choose $s \in (1, \infty)$ with $\frac{1}{s} = 1 - \frac{2}{q}$. Thus $s' = \frac{q}{2} > 1$ and we have for all $T_1, \dots, T_N \in \mathcal{T}$, $f_1, \dots f_N \in L^q_\omega(\mathbb{R}^n)$ and $N \in \mathbb{N}$

$$\Big\| \Big(\sum_{j=1}^N |T_j f_j|^2 \Big)^{\frac{1}{2}} \Big\|_{q,\omega} = \Big\| \sum_{j=1}^N |T_j f_j|^2 \Big\|^{\frac{1}{2}}_{s',\omega}. \tag{48}$$

By the Theorem of Hahn-Banach there is a $0 \le u \in L^s_\omega(\mathbb{R}^n)$ with norm equal to 1 such that the right hand side of (48) is equal to

$$\Big(\sum_{j=1}^N \int_{\mathbb{R}^n} |T_j f_j|^2 \, u \, \omega \, dx \Big)^{\frac{1}{2}}. \tag{49}$$

By Lemma 4.3 there is a function $0 \le v \in L^s_\omega(\mathbb{R}^n)$ with the properties a), b) and c) of that Lemma for $p = 2$. Therefore and since (47) holds with $p = 2$ and with a constant depending only on the A_2-constant of $w \in A_2$, we can estimate (49) from above by

$$\begin{aligned} \Big(\sum_{j=1}^N \int_{\mathbb{R}^n} |T_j f_j|^2 \, v \, \omega \, dx \Big)^{\frac{1}{2}} &\le C \Big(\sum_{j=1}^N \int_{\mathbb{R}^n} |f_j|^2 \, v \, \omega \, dx \Big)^{\frac{1}{2}} \\ \le C \Big\| \sum_{j=1}^N |f_j|^2 \Big\|^{\frac{1}{2}}_{s',\omega} \|v\|^{\frac{1}{2}}_{s,\omega} &\le \sqrt{c}\, C \Big\| (\sum_{j=1}^N |f_j|^2)^{\frac{1}{2}} \Big\|_{q,\omega} \end{aligned}$$

where $C = C(n, q, \mathcal{T}, \omega) \in \mathbb{R}$.
In the case $q < 2$ we use a duality argument: Again let $T_1, \dots, T_N \in \mathcal{T}$, $f_1, \dots f_N \in L^q_\omega(\mathbb{R}^n)$ and $N \in \mathbb{N}$ be arbitrary. Then by the Theorem of Hahn-Banach there is $g = (g_1, \dots, g_N) \in L^{q'}_{\omega'}(\mathbb{R}^n)^N$ with norm $\Big\| (\sum_{j=1}^N |g_j|^2)^{\frac{1}{2}} \Big\|_{q',\omega'} = 1$, such that because of $q' > 2$ the previous case implies that

$$\begin{aligned} \Big\| \Big(\sum_{j=1}^N |T_j f_j|^2 \Big)^{\frac{1}{2}} \Big\|_{q,\omega} &= \sum_{j=1}^N \int_{\mathbb{R}^n} T_j f_j \, g_j \, dx = \int_{\mathbb{R}^n} \sum_{j=1}^N f_j \, (T_j)' g_j \, dx \\ &\le \Big\| \Big(\sum_{j=1}^N |f_j|^2 \Big)^{\frac{1}{2}} \Big\|_{q,\omega} \Big\| \Big(\sum_{j=1}^N |(T_j)' g_j|^2 \Big)^{\frac{1}{2}} \Big\|_{q',\omega'} \\ &\le C(\omega) \Big\| \Big(\sum_{j=1}^N |f_j|^2 \Big)^{\frac{1}{2}} \Big\|_{q,\omega}. \end{aligned}$$

The case $q = 2$ is trivial: In this case $L^2_\omega(\Omega)$ is a Hilbert space. Thus the $\mathcal{R}$-boundedness of the family $\mathcal{T}$ follows from the uniform boundedness in $\mathcal{L}(L^2_\omega(\Omega))$.
Thus by Lemma 4.2 the Theorem is proved completely for $\Omega = \mathbb{R}^n$.
Now, let $\Omega \subset \mathbb{R}^n$ be an arbitrary open set. Let $E : L^q_\omega(\Omega) \to L^q_\omega(\mathbb{R}^n)$ by the extension by 0 to $\mathbb{R}^n$ and $R : L^q_\omega(\mathbb{R}^n) \to L^q_\omega(\Omega)$ be the restriction operator. Then for $T \in \mathcal{T}$ we define $\tilde{T} := E \, T \, R \in \mathcal{L}(L^q_\omega(\mathbb{R}^n))$. Because of (47) the estimate

$$\|\tilde{T} f\|_{p,w,\mathbb{R}^n} = \|T \, Rf\|_{p,w,\Omega} \le C \, \|f\|_{L^p_w(\mathbb{R}^n)}$$

holds for all $w \in A_p$ and $f \in L^q_\omega(\mathbb{R}^n) \cap L^p_w(\mathbb{R}^n)$, where C only depends on the A_p-constant of w. For arbitrary $T_1, \dots, T_N \in \mathcal{T}$, $f_1, \dots f_N \in L^q_\omega(\Omega)$ and $N \in \mathbb{N}$ it follows with the estimate already proved in the case $\Omega = \mathbb{R}^n$ that

$$\Big\|\Big(\sum_{j=1}^N |T_j f_j|^2\Big)^{\frac12}\Big\|_{q,\omega,\Omega} = \Big\|\Big(\sum_{j=1}^N |\tilde{T}_j\, E f_j|^2\Big)^{\frac12}\Big\|_{q,\omega,\mathbb{R}^n}$$
$$\le C\Big\|\Big(\sum_{j=1}^N |E f_j|^2\Big)^{\frac12}\Big\|_{q,\omega,\mathbb{R}^n} = C\Big\|\Big(\sum_{j=1}^N |f_j|^2\Big)^{\frac12}\Big\|_{q,\omega,\Omega},$$

where C depends only on n, q, $\mathcal{T}$ and ω. Thus the Theorem is completely proved. □

4.2 Maximal L^p-Regularity

Let A be the generator of a bounded analytic semigroup in the Banach space X. We consider the abstract Cauchy problem

$$u_t - Au = f, \qquad u(0) = 0. \tag{50}$$

For $f \in L^1_{loc}([0,\infty); X)$ the mild solution on $J = \mathbb{R}_+$ is given by

$$u(t) = \int_0^t e^{(t-s)A} f(s)ds.$$

Definition 4.2 *Let $1 < p < \infty$, $J = [0,\infty)$ and $f \in L^p(J,X)$. Then A has maximal L^p-regularity, if for all $f \in L^p(J,X)$ the mild solution u of (50) is differentiable a.e. in X, has values in $D(A)$ a.e. and if there is a constant $C \in \mathbb{R}$ such that the estimate*

$$\|u_t\|_{L^p(0,T;X)} + \|Au\|_{L^p(0,T;X)} \le C\,\|f\|_{L^p(0,T;X)} \tag{51}$$

holds for all $0 < T \le \infty$.

The definition of maximal L^p-regularity is independent of $1 < p < \infty$, i.e. the maximal L^p-regularity for one $p \in (1,\infty)$ implies the maximal L^r-regularity for all $p \in (1,\infty)$ (see [7]).

Definition 4.3 *A Banach space X is called UMD space if the Hilbert transform*

$$Hf(t) = PV - \int_{\mathbb{R}} \frac{1}{t-s} f(s)\,ds, \quad f \in \mathcal{S}(\mathbb{R}, X),$$

extends to a bounded linear operator in $L^p(\mathbb{R}, X)$ for $1 < p < \infty$.

It is well known that all closed subspaces of $L^q(\Omega,\mu)$ for $1 < q < \infty$ are UMD spaces (see e.g. [3]).
We apply the following Theorem due to L. Weis [20].

Theorem 4.4 *Let A be the generator of a bounded analytic semigroup in the UMD space X. Then A has maximal L^p-regularity iff the operator family*

$$\{\lambda\,(\lambda - A)^{-1} \,:\, \lambda \in i\mathbb{R},\, \lambda \neq 0\}$$

is $\mathcal{R}$-bounded.

Proof: See [20] Theorem 4.2 3) and Corollary 4.4 i) or [21].

Theorem 4.5 *Let* $1 < q < \infty$, $\omega \in A_q$ *and* $\Omega = \mathbb{R}^n, \mathbb{R}^n_+$ *or a bounded domain with boundary of class* $C^{1,1}$. *Then the Stokes operator* $-\mathcal{A}_{q,\omega}$ *has maximal* L^p*-regularity in* $L^q_{\omega,\sigma}(\Omega)$.

Proof: By Theorem 4.1 ii) and by the fact that the operator norm of the Helmholtz projection $P_{q,v}$ is A_q-consistently increasing, there is an A_q-consistently increasing constant $C = C(v) \in \mathbb{R}$ such that

$$\|\lambda(\mathcal{A}_{q,v} + \lambda)^{-1} P_{q,v} f\|_{q,v} \leq C \, \|f\|_{q,v}$$

for all $v \in A_q$, $f \in L^q_v(\Omega)$ and $\lambda \in i\mathbb{R}\backslash\{0\}$. Since the constant $C = C(v)$ is A_q-consistently increasing this constant can be chosen in such a way that it depends only on the A_q-constant of v , i. e. $C = C(A_q(v))$.
Hence by Theorem 4.3 and by (43) the family

$$\{\lambda\,(\mathcal{A}_{q,\omega} + \lambda)^{-1} P_{q,\omega} \; : \; \lambda \in i\mathbb{R}\backslash\{0\}\}$$

is $\mathcal{R}$-bounded in $L^q_\omega(\Omega)^n$. Since $f = P_{q,\omega} f$ for $f \in L^q_{\omega,\sigma}(\Omega)$ it follows that

$$\{\lambda(\mathcal{A}_{q,\omega} + \lambda)^{-1} \; : \; \lambda \in i\mathbb{R}\backslash\{0\}\} \tag{52}$$

is $\mathcal{R}$-bounded in $L^q_{\omega,\sigma}(\Omega)$.
Note that the space $L^q_{\omega,\sigma}(\Omega)$ is a closed subspace of $L^q_\omega(\Omega)^n$ and therefore a UMD space (see [3]). By Theorem 4.4, Theorem 4.1 iv) and the $\mathcal{R}$-boundedness of the family (52) the maximal L^p-regularity of the Stokes operator $-\mathcal{A}_{q,\omega}$ in $L^q_{\omega,\sigma}(\Omega)$ is completely proved. □

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