Local theory of the Fredholmness of band-dominated operators with slowly oscillating coefficients

Vladimir S. Rabinovich, Steffen Roch *

Dedicated to the sixtieth birthday of Bernd Silbermann.

Abstract

A band-dominated operators on an l^p-space of vector-valued functions is an (in a generalized sense) Fredholm operator if and only if all of its limit operators are invertible and if their inverses are uniformly bounded (see [6]). We show that the limit operators approach is also compatible with the local Fredholmness of band-dominated operators with respect to localization over the maximal ideal space of the algebra of the slowly oscillating scalar-valued functions. A corollary of this result is that the uniform boundedness condition is redundant for band-dominated operators with slowly oscillating operator-valued coefficients.

1 Introduction

Let X be a complex Banach space. For $p \in (1, \infty)$ and N a positive integer, consider the Banach spaces $l^p(\mathbb{Z}^N, X)$ and $l^\infty(\mathbb{Z}^N, X)$ of all functions f which are defined on $\mathbb{Z}^N$ and take values in X such that

$$\|f\|_p^p := \sum_{x \in \mathbb{Z}^N} \|f(x)\|_X^p < \infty \quad \text{and} \quad \|f\|_\infty := \sup_{x \in \mathbb{Z}^N} \|f(x)\|_X < \infty,$$

respectively. Further, $c_0(\mathbb{Z}^N, X)$ refers to the closed subspace of $l^\infty(\mathbb{Z}^N, X)$ consisting of all functions f with

$$\lim_{x \to \infty} \|f(x)\|_X = 0.$$

In case $X = \mathbb{C}$, we will simply write $l^p(\mathbb{Z}^N)$ and $c_0(\mathbb{Z}^N)$, and we let E stand for one of the spaces $l^p(\mathbb{Z}^N, X)$ with $p \in (1, \infty)$.

Every function $a \in l^\infty_{L(X)} := l^\infty(\mathbb{Z}^N, L(X))$ gives rise to a multiplication operator on E on defining

$$(af)(x) = a(x)f(x), \quad x \in \mathbb{Z}^N.$$

We denote this operator by aI. Evidently, $aI \in L(E)$ and $\|aI\|_{L(E)} = \|a\|_\infty$. Finally, for $\alpha \in \mathbb{Z}^N$, let V_α refer to the shift operator

$$(V_\alpha f)(x) = f(x - \alpha), \quad x \in \mathbb{Z}^N,$$

which also belongs to $L(E)$ and has norm 1.

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Definition 1.1 *A* band operator *is a finite sum of the form* $\sum_\alpha a_\alpha V_\alpha$ *where* $\alpha \in \mathbb{Z}^N$ *and* $a_\alpha \in l^\infty(\mathbb{Z}^N, L(X))$. *A* band-dominated operator *is the uniform limit of a sequence of band operators.*

The band-dominated operators on E form a closed subalgebra of $L(E)$ which we denote by $\mathcal{A}_E$. (For this and the following facts we refer to the papers [5, 6].)

Given $m \in \mathbb{Z}^N$, let s_m stand for the function on $\mathbb{Z}^N$ which is $I \in L(X)$ at m and 0 at all other points. The operator of multiplication by s_m will be denoted by S_m. For $n \geq 0$, define $P_n := \sum_{|m| \leq n} S_m$ and $Q_n := I - P_n$, and let $\mathcal{P}$ refer to the family (P_n).

Definition 1.2 *An operator* $K \in L(E)$ *is* $\mathcal{P}$-compact *if*

$$\|KQ_n\| \to 0 \quad \textit{and} \quad \|Q_n K\| \to 0 \quad \textit{as } n \to \infty.$$

By $K(E, \mathcal{P})$ *we denote the set of all* $\mathcal{P}$*-compact operators on* E*, and by* $L(E, \mathcal{P})$ *the set of all operators* $A \in L(E)$ *for which both* AK *and* KA *are* $\mathcal{P}$*-compact whenever* K *is* $\mathcal{P}$*-compact.*

It turns out that $L(E, \mathcal{P})$ is a closed subalgebra of $L(E)$, $K(E, \mathcal{P})$ is a closed two-sided ideal of $L(E, \mathcal{P})$, and $K(E, \mathcal{P}) \subset \mathcal{A}_E \subset L(E, \mathcal{P})$. Operators $A \in L(E, \mathcal{P})$ for which the coset $A + K(E, \mathcal{P})$ is invertible in the quotient algebra $L(E, \mathcal{P})/K(E, \mathcal{P})$ are called $\mathcal{P}$-Fredholm. If X is a finite-dimensional space, then $L(E, \mathcal{P}) = L(E)$, $K(E, \mathcal{P})$ is the ideal of the compact operators on E, and the $\mathcal{P}$-Fredholm operators are just the Fredholm operators in the common sense. Let further stand $\mathcal{H}$ for the set of all sequences $h = (h(m))_{m=0}^\infty \subset \mathbb{Z}^N$ which tend to infinity.

Definition 1.3 *Let* $A \in L(E, \mathcal{P})$ *and* $h \in \mathcal{H}$. *The operator* $A_h \in L(E)$ *is called* limit operator of A with respect to h *if*

$$\lim_{n \to \infty} \|(V_{-h(n)} A V_{h(n)} - A_h) P_m\| = \lim_{n \to \infty} \|P_m (V_{-h(n)} A V_{h(n)} - A_h)\| = 0 \qquad (1)$$

for every $P_m \in \mathcal{P}$. *The set* $\sigma_{op}(A)$ *of all limit operators of* A *is called the* operator spectrum *of* A.

We let finally refer $\mathcal{A}_E^{rich}$ to the set of all operators $A \in \mathcal{A}_E$ enjoying the following property: every sequence h tending to infinity possesses a subsequence g for which the limit operator A_g exists. Then the main result of [6] can be stated as follows:

Theorem 1.4 *An operator* $A \in \mathcal{A}_E^{rich}$ *is* $\mathcal{P}$*-Fredholm if and only if all of its limit operators are invertible and if*

$$\sup\{\|(A_h)^{-1}\| : A_h \in \sigma_{op}(A)\} < \infty. \qquad (2)$$

It is the main goal of this paper to discuss and weaken the uniform invertibility condition (2). To reach this goal, we examine several local theories of $\mathcal{P}$-Fredholmness. To describe some typical ideas and results we have to introduce some more notations. Let S^{N-1} denote the unit sphere $\{\eta \in \mathbb{R}^N : |\eta|_2 = 1\}$ where $|\eta|_2$ stands for the Euklidean norm of η. Given a 'radius' $R > 0$, a 'direction' $\eta \in S^{N-1}$, and a neighborhood $U \subseteq S^{N-1}$ of η, define

$$W_{R,U} := \{z \in \mathbb{Z}^N : |z| > R \text{ and } z/|z| \in U\}. \qquad (3)$$

We will call $W_{R,U}$ a *neighborhood at infinity of* η. If h is a sequence which tends to infinity, then we say that h tends into the direction of $\eta \in S^{N-1}$ if, for every neighborhood at infinity $W_{R,U}$ of η, there is an m_0 such that

$$h(m) \in W_{R,U} \quad \text{for all } m \geq m_0.$$

Definition 1.5 *Let $\eta \in S^{N-1}$ and $A \in L(E)$.*

(a) The local operator spectrum $\sigma_\eta(A)$ of A at η *is the set of all limit operators A_h of A with respect to sequences h tending into the direction of η.*

(b) The operator A is locally invertible at η *if there are operators $B, C \in L(E)$ and a neighborhood at infinity W of η such that*

$$BA\hat{\chi}_W I = \hat{\chi}_W AC = \hat{\chi}_W I$$

where $\hat{\chi}_W$ refers to the characteristic function of W.

The following theorem and its corollary (which is also partially based on Theorem 6.5 below) have been shown in [5, 6].

Theorem 1.6 *Let $A \in \mathcal{A}_E^{rich}$ and $\eta \in S^{N-1}$. Then the operator A is locally invertible at η if and only if all limit operators in $\sigma_\eta(A)$ are invertible and if*

$$\sup\{\|(A_h)^{-1}\| : A_h \in \sigma_\eta(A)\} < \infty.$$

Corollary 1.7 *An operator $A \in \mathcal{A}_E^{rich}$ is $\mathcal{P}$-Fredholm if and only if all of its limit operators are invertible, and if*

$$\sup\{\|(A_h)^{-1}\| : A_h \in \sigma_\eta(A)\} < \infty \quad \textit{for all}\ \eta \in S^{N-1}.$$

Observe that this is a true generalization of Theorem 1.4 since it is not required in the corollary that the suprema are uniformly bounded with respect to η.

In the present paper we will show that an analogous result holds if the sphere S^{N-1} is replaced by the fiber $M^\infty(SO)$ at infinity of the maximal ideal space of the algebra of the slowly oscillating functions on $\mathbb{Z}^N$. This fiber is much larger than S^{N-1}, hence, the resulting localization is much finer, and this localization will provide a further essential improvement of Theorem 1.4. It should be also noted that the localization over $M^\infty(SO)$ is, in some sense, the finest possible.

It is due to the topological properties of the maximal ideal space of the algebra of the slowly oscillating functions that we have to replace sequences tending to infinity by general nets tending to infinity. This requires some additional work which is done in the Sections 2 – 5. In particular, we will derive a version of Cantor's diagonalization procedure for nets in place of sequences. The Sections 6 and 7 are devoted to the proof of the local Fredholm criterion and of one of its consequences, which states that a band-dominated operator with rich slowly oscillating coefficients is $\mathcal{P}$-Fredholm if and only if all of its limit operators are invertible (Theorem 7.2). Thus, for these operators, the uniform invertibility of the inverses of the limit operators is not needed to guarantee their $\mathcal{P}$-Fredholmness, which is a second main result of the present paper. In the course of the proof we will also see that the method of limit operators is compatible with another local theory, the so-called local principle by Allan (Theorems 6.5 and 6.7 below). The final section contains an alternative proof of Theorem 7.2 which borrows some arguments from the symbol calculus for pseudodifferential operators, and which sheds new light upon the properties of band-dominated operators with slowly oscillating coefficients.

2 Slowly oscillating functions

A function $a \in l^\infty_{L(X)}$ is *slowly oscillating* if

$$\lim_{x \to \infty} (a(x + k) - a(x)) = 0 \quad \text{for all } k \in \mathbb{Z}^n. \tag{4}$$

We denote the class of all slowly oscillating functions in $l^\infty_{L(X)}$ by $SO_{L(X)}$ and write SO instead of $SO_{L(\mathbb{C})}$ for brevity. Trivial examples of slowly oscillating functions are provided by the continuous functions on $\mathbb{Z}^N$ which possess a limit at infinity, whereas $\mathbb{Z} \to \mathbb{C} : x \mapsto \sin \sqrt{|x|}$ is an example of a slowly oscillating function which does not have this property.

It follows essentially from the definition of the class SO that a function a is slowly oscillating if and only if the operator $V_{-k}aV_k - aI$ is $\mathcal{P}$-compact for every $k \in \mathbb{Z}^N$ or, equivalently, if and only if the commutator $aV_k - V_k aI = V_k(V_{-k}aV_k - aI)$ is $\mathcal{P}$-compact for every k. Since $K(E, \mathcal{P})$ is a closed ideal of $L(E, \mathcal{P})$, we conclude that $SO_{L(X)}$ is a closed subalgebra of $l^\infty_{L(X)}$. If, moreover, the slowly oscillating function a is scalar-valued, then the operator of multiplication by a also commutes with every multiplication operator. Summarizing we get:

Proposition 2.1 *If $f \in SO$ and $A \in \mathcal{A}_E$, then the operator $fA - AfI$ is $\mathcal{P}$-compact on E. If, conversely, $f \in l^\infty_{L(X)}$ is a function for which $fA - AfI$ is $\mathcal{P}$-compact for every $A \in \mathcal{A}_E$, then $f \in SO$.*

Thus, SO (more precisely, the image of SO in $L(E, \mathcal{P})/K(E, \mathcal{P})$ under the canonical embedding) is *the* natural candidate for localizing the algebra $\mathcal{A}_E/K(E, \mathcal{P})$ by means of the local principle by Allan. We will pursue this idea in Section 6.

Another special feature of slowly oscillating functions concerns the limit operators of their multiplication operators.

Proposition 2.2 *Let $a \in SO_{L(X)}$. Then every limit operator of aI is a multiplication operator in $\mathbb{C}_{L(X)}$, i.e. an operator of multiplication by a constant function with values in $L(X)$.*

Proof. Let $a \in SO_{L(X)}$. From (4) we conclude that

$$\lim_{k \to \infty} (a(x' + h(k)) - a(x'' + h(k))) = 0$$

for all sequences h tending to infinity and for all $x', x'' \in \mathbb{Z}^n$. Hence, if h is a sequence such that the limit operator $(aI)_h$ exists, then $\lim_{k\to\infty} a(x + h_k)$ is independent of $x \in \mathbb{Z}^n$, i.e. $(aI)_h = AI$ with an operator $A \in L(X)$. ∎

3 Local invertibility with respect to $M^\infty(SO)$

Let $M(SO)$ denote the maximal ideal space of the commutative C^*-algebra SO, and write $M^\infty(SO)$ for the fiber of $M(SO)$ consisting of all characters $\eta \in M(SO)$ such that $\eta(a) = 0$ whenever $a \in c_0$. Every $m \in \mathbb{Z}^N$ defines a character of SO by $f \mapsto f(m)$. In this sense, $\mathbb{Z}^N$ is embedded into $M(SO)$, and $M(SO)$ is the union of its disjoint subsets $\mathbb{Z}^N$ and $M^\infty(SO)$.

Theorem 3.1 *$\mathbb{Z}^N$ is densely and homeomorphically embedded into $M(SO)$ with respect to the Gelfand topology.*

This is a special case of a general result on compactifications of topological spaces, see [3], Chapter I, Theorem 8.2.

We will run into a lot of trouble when trying to realize the simple and natural idea of localizing the algebra $\mathcal{A}_E/K(E, \mathcal{P})$ over SO. The main reason for this is the following observation.

Proposition 3.2 *Let $\eta \in M^\infty(SO)$. Then $\eta \in \mathrm{clos}_{M(SO)}\mathbb{Z}^N$, but there is* no sequence *in $\mathbb{Z}^N$ which tends to η with respect to the Gelfand topology of $M(SO)$.*

Proof. We know from Theorem 3.1 that η is in $\mathrm{clos}_{M(SO)}\mathbb{Z}^N$ and that, hence, there is a *net* with values in $\mathbb{Z}^N$ which converges to η. Assume there is a *sequence* h with values in $\mathbb{Z}^N$ and with limit η in the Gelfand topology. Since every subsequence of h also converges to η, we can assume without loss that

$$|h(n+1)| \geq |h(n)| + 2^{n+2} \quad \text{for all } n.$$

Let $\varphi_0 : \mathbb{R}^N \to [0, 1]$ be a continuous function with support in $\{t \in \mathbb{R}^N : |t| \leq 1\}$ and with $\varphi_0(0) = 1$, and set $\varphi_n(t) := \varphi(t/2^n)$ for $n \geq 1$. Then the function

$$\varphi(t) := \sum_{n \geq 0}^{\infty} \hat{\varphi}_{2n}(t - h(2n))$$

is slowly oscillating, and $\varphi(h(2n)) = 1$ and $\varphi(h(2n+1)) = 0$ for all n. The assumed convergence of h to η implies that both sequences $(\varphi(h(2n)))$ and $(\varphi(h(2n+1)))$ converge to $\varphi(\eta)$. Contradiction. ∎

Consequently, if $h \in \mathcal{H}$, then the closure $\overline{h}$ of the set $\{h(m) : m \in \mathbb{Z}^N\}$ of the values of h in the Gelfand topology cannot consist of a single point of $M^\infty(SO)$ only. Nevertheless, the sequences in $\mathcal{H}$ separate the points of $M^\infty(SO)$ in the following sense.

Proposition 3.3 *Given $\eta, \theta \in M^\infty(SO)$, there is a function $h \in \mathcal{H}$ such that $\eta \in \overline{h}$ and $\theta \notin \overline{h}$.*

Proof. Choose disjoint neighborhoods U_η and U_θ of η and θ in $M(SO)$, and let $h \in \mathcal{H}$ be a sequence such that

$$\{h(m) : m \in \mathbb{Z}^N\} = U_\eta \cap \mathbb{Z}^N.$$

(Recall that the intersection $U_\eta \cap \mathbb{Z}^N$ is not empty by Theorem 3.1 and, hence, countable. Thus, h can be even chosen as a bijection from $\mathbb{Z}^N$ onto $U_\eta \cap \mathbb{Z}^N$.) Since $\mathbb{Z}^N$ is dense in $M(SO)$, it is clear that $\eta \in \overline{U_\eta \cap \mathbb{Z}^N} = \overline{h}$, but θ cannot belong to $\overline{h}$ since

$$\theta \in U_\theta \subseteq M(SO) \setminus \overline{U_\eta} = M(SO) \setminus \overline{h},$$

i.e. θ is an interior point of the complement of $\overline{h}$. ∎

The Proposition 3.3 suggests the following definition.

Definition 3.4 *Let $\eta \in M^\infty(SO)$ and $A \in L(E)$. The* local operator spectrum of A at η *is the set*

$$\sigma_\eta(A) := \{A_h : h \in \mathcal{H}_A \text{ and } \eta \in \overline{h}\}.$$

Above we observed that, if h is a sequence, there are many η's in $\overline{h}$. We will see now that, nevertheless, local spectra of operators of multiplication by slowly oscillating functions are singletons, thus giving another justification for the proposed definition of a local operator spectrum.

Proposition 3.5 *Let $\eta \in M^\infty(SO)$.*
(a) If $A = aI$ with $a \in SO$, then $\sigma_\eta(A) = \{a(\eta)\}$ (where we use the same notation for a function in SO and its Gelfand transform).
(b) If $A = aI$ with $a \in SO_{L(X)}$, then $\sigma_\eta(A)$ contains at most one operator.

Proof. (a) Let $h \in \mathcal{H}$ be a sequence such that $\eta \in \overline{h}$ and such that the limit operator $(aI)_h$ exists. By Proposition 2.2, $(aI)_h = \alpha I$ with the complex number $\alpha := \lim a(h(n))$. We claim that $\alpha = a(\eta)$.

Let $\varepsilon > 0$. Since a is continuous at η, there is an open neighborhood U of η such that

$$|a(\eta) - a(\theta)| < \varepsilon/2 \quad \text{for all } \theta \in U.$$

Further, since $\eta \in \overline{h}$, there is an infinite subsequence g of h the values of which are in U. Choose m such that $|a(g(m)) - \alpha| < \varepsilon/2$. Then

$$|a(\eta) - \alpha| \leq |a(\eta) - a(g(m))| + |a(g(m)) - \alpha| < \varepsilon.$$

This estimate holds for arbitrary $\varepsilon > 0$; hence, $a(\eta) = \alpha$.

(b) Suppose there are sequences $h_1, h_2 \in \mathcal{H}$ such that $\eta \in \overline{h_1} \cap \overline{h_2}$ and that the limit operators $(aI)_{h_1}$ and $(aI)_{h_2}$ exist, but that $(aI)_{h_1} \neq (aI)_{h_2}$. By Proposition 2.2, $(aI)_{h_1}$ and $(aI)_{h_2}$ are the operators of multiplication by the constant functions $x \mapsto A_1$ and $x \mapsto A_2$ with $A_1, A_2 \in L(X)$. Since $A_1 \neq A_2$, there is a functional $\varphi \in L(X)^*$ such that $\varphi(A_1) \neq \varphi(A_2)$. Consider the function $\hat{a} : \mathbb{Z}^N \to \mathbb{C} : x \mapsto \varphi(a(x))$. This function is in SO:

$$|\hat{a}(x+k) - \hat{a}(x)| \leq \|\varphi\| \, \|a(x+k) - a(x)\|_{L(X)} \to 0 \quad \text{as } x \to \infty.$$

From $\|a(h_i(m)) - A_i\| \to 0$ for $i = 1, 2$ we conclude that

$$\|\hat{a}(h_i(m)) - \varphi(A_i)\| \to 0 \quad \text{for } i = 1, 2.$$

Hence, both $\varphi(A_1)I$ and $\varphi(A_2)I$ are limit operators of $\hat{a}I$ at η. This contradicts assertion (a) of this proposition, stating that $\sigma_\eta(\hat{a}I)$ is a singleton. ■

If $h \in \mathcal{H}$, then the intersection $\text{clos}_{M(SO)}\{h(m) : m \in \mathbb{Z}^N\} \cap M^\infty(SO)$ is non-empty by Theorem 3.1. Consequently,

$$\sigma_{op}(A) = \cup_{\eta \in M^\infty(SO)} \sigma_\eta(A) \quad \text{for every } A \in L(E).$$

Let $\eta \in M^\infty(SO)$, and let U be a neighborhood of η in $M(SO)$ with respect to the Gelfand topology. Then we agree upon calling the intersection $U \cap \mathbb{Z}^N$ a *neighborhood at infinity of* η.

Definition 3.6 *Let $\eta \in M^\infty(SO)$ and $A \in L(E)$. The operator A is* locally invertible at η *if there are operators $B, C \in L(E)$ and a neighborhood at infinity W of η such that*

$$BA\hat{\chi}_W I = \hat{\chi}_W AC = \hat{\chi}_W I$$

where $\hat{\chi}_W$ refers to the characteristic function of W.

The following result, which states the analogue of Theorem 1.6 with respect to the much finer localization over points in $M^\infty(SO)$ instead of points in S^{N-1}, is the main outcome of this section.

Theorem 3.7 *Let $A \in \mathcal{A}_E^{rich}$ and $\eta \in M^\infty(SO)$. Then the operator A is locally invertible at η if and only if all limit operators in $\sigma_\eta(A)$ are invertible and if*

$$\sup\{\|(A_h)^{-1}\| : A_h \in \sigma_\eta(A)\} < \infty.$$

The proof will be given in Section 6. To prepare this proof we recall and provide some facts about nets and about limit operators with respect to nets in the following two sections.

4 Preliminaries on nets

Nets and subnets. A set T is *directed* if there is a binary relation $\geq$ on T such that

$\forall t \in T :$	$t \geq t$	(reflexivity),
$\forall r, s, t \in T :$	$r \geq s,\ s \geq t \Rightarrow s \geq t$	(transitivity),
$\forall r, s \in T\ \exists t \in T :$	$t \geq r$ and $t \geq s$	(inductivity),

A mapping x from a directed set T into a topological space X is called a *net* , and this net *converges to a point* $x^* \in X$ if, for every neighborhood U of x^*, there is a $t_0 \in T$ such that $x(t) \in U$ for all $t \geq t_0$. The net $x : T \to X$ is sometimes also denoted by $(x_t)_{t\in T}$ where $x_t = x(t)$. Accordingly, if $x : T \to X$ converges to x^*, we will write

$$\lim_{t\in T} x_t = x^* \quad \text{or} \quad x_t \to x^* \text{ with respect to } T.$$

A net $(y_s)_{s\in S}$ is a *subnet* of the net $(x_t)_{t\in T}$ if there is a mapping $F : S \to T$ such that

$$\begin{aligned} &\forall s \in S: && y_s = x_{F(s)}, \\ &\forall t \in T\; \exists\, s_0 \in S: && F(s) \geq t \quad \text{for all } s \geq s_0. \end{aligned}$$

A subset S of a directed set T is called *cofinal* if

$$\forall t \in T\; \exists\, s \in S: \quad s \geq t.$$

Every cofinal subset S of a directed set T is again a directed set with respect to the restriction of the order relation $\geq$ onto S. If S is a cofinal subset of T, and if $(x_t)_{t\in T}$ is a net, then the restriction of $(x_t)_{t\in T}$ onto S is a subnet of $(x_t)_{t\in T}$. We will be mainly interested in subnets which do *not* arise in this simple manner.

Nets tending to infinity. In what follows we will only be concerned with nets in $\mathbb{Z}^N$. A net $(x_t)_{t\in T}$ with values in $\mathbb{Z}^N$ is said to *converge to infinity* if

$$\forall k \in \mathbb{N}\; \exists\, t_0 \in T: \quad |x_t| \geq k \quad \text{for all } t \geq t_0.$$

Let $\mathcal{N}$ denote the set of all nets in $\mathbb{Z}^N$ which converge to infinity.

Lemma 4.1 (a) *For every net* $(x_t)_{t\in T} \in \mathcal{N}$*, the set* $\{x_t : t \in T\}$ *of its values is countably infinite.*
(b) *If* $h : \mathbb{N} \to \mathbb{Z}^N$ *is injective, then the sequence* h *belongs to* $\mathcal{N}$.

Proof. (a) Since $\mathbb{Z}^N$ is countable, $(x_t)_{t\in T} \subseteq \mathbb{Z}^N$ is an at most countable set, and since $(x_t)_{t\in T}$ tends to infinity, this set cannot be finite.
(b) Suppose the sequence h does not converge to infinity. Then

$$\exists\, k \in \mathbb{N}\; \forall n_0 \in \mathbb{N}\; \exists\, n \geq n_0 : |h(n)| \leq k.$$

Repeating this argument we get an infinite sequence $n_0 < n_1 < n_2 < \ldots$ such that $|h(n_r)| \leq k$ for all r. But h is injective. Thus, $h(n_r) \neq h(n_s)$ whenever $r \neq s$. So we have infinitely many points in $\{z \in \mathbb{Z}^N : |z| \leq k\}$ which is nonsense. ∎

Lemma 4.2 *Let* $x \in \mathcal{N}$ *be a net, and let* h *be a bijection from* $\mathbb{N}$ *onto the set of the values of* x*. Then* x *is a subnet of the sequence* h*. In particular, every net* $x \in \mathcal{N}$ *is a subnet of a sequence* $h \in \mathcal{H}$.

Proof. Let $x = (x_t)_{t\in T} \in \mathcal{N}$, and let $h : \mathbb{N} \to \{x_t : t \in T\}$ be a bijection. Such bijections exist by Lemma 4.1.

To show that x is a subnet of h, define $F : T \to \mathbb{N}$ by $F(t) := h^{-1}(x_t)$. Then, clearly, $x_t = h_{F(t)}$ for every $t \in T$, and it remains to check whether

$$\forall n \in \mathbb{N}\; \exists\, t_0 \in T: \quad F(t) \geq n \quad \text{for all } t \geq t_0. \tag{5}$$

Given $n \in \mathbb{N}$, set $k := \max\{|h_1|, \ldots, |h_n|\}$. Since $(x_t)_{t\in T}$ belongs to $\mathcal{N}$, there is a $t_0 \in T$ such that

$$|x_t| \geq k + 1 \quad \text{for all } t \geq t_0.$$

By the definition of F, this implies $F(t) \geq n$ for all $t \geq t_0$ which gives (5). Hence, x is a subnet of h, and this sequence belongs to $\mathcal{H}$ due to Lemma 4.1 (b). ∎

A version of Cantor's diagonalization procedure. The following result can be regarded as a substitute for the well-known diagonalization argument for sequences due to Cantor.

Theorem 4.3 *Let Z be a set, and let $(f_n)_{n\geq 1}$ be a sequence of functions $f_n : Z \to \mathbb{R}^+$ which converges uniformly on Z to a function $f : Z \to \mathbb{R}^+$. Assume further that $(x^0_{t_0})_{t_0\in T_0}$ is a net with values in Z and with the property that, for every $n \geq 1$, there is a subnet $(x^n_{t_n})_{t_n\in T_n}$ of $(x^{n-1}_{t_{n-1}})_{t_{n-1}\in T_{n-1}}$ such that*

$$\lim_{t_n\in T_n} f_n(x^n_{t_n}) = 0. \tag{6}$$

Then there is a subnet $(y_w)_{w\in W}$ of $(x^0_{t_0})_{t_0\in T_0}$ with $\lim_{w\in W} f(y_w) = 0$.

Proof. We split the proof into several steps and emphasize some partial results as lemmas. Our starting point is a net $(x^0_{t_0})_{t_0\in T_0}$ in Z and, for every $n \geq 1$, a subnet $(x^n_{t_n})_{t_n\in T_n}$ of $(x^{n-1}_{t_{n-1}})_{t_{n-1}\in T_{n-1}}$ with (6). In particular, we have mappings $F_n : T_n \to T_{n-1}$ with $x^n_{t_n} = x^{n-1}_{F_n(t_n)}$ for all $t_n \in T_n$ and such that

$$\forall t_{n-1} \in T_{n-1}\ \exists\, t^0_n \in T_n : \quad F(t_n) \geq t_{n-1} \quad \text{for all } t_n \geq t^0_n. \tag{7}$$

Step 1. *We show that the directed sets $T_0, T_1, \ldots$ can be replaced be one and the same directed set S.*

Indeed, set $S := T_0 \times T_1 \times T_2 \times \ldots$ and provide S with the order

$$(s_0,\, s_1,\, s_2,\, \ldots) \geq (s'_0,\, s'_1,\, s'_2,\, \ldots) \quad \Longleftrightarrow \quad s_k \geq s'_k \text{ for all } k$$

which makes S to a directed set. Further, there are canonical mappings

$$G_n : S \to T_n, \quad (s_0,\, s_1,\, s_2,\, \ldots) \mapsto s_n.$$

For every $n \in \mathbb{N}$, define a net $(y^n_s)_{s\in S}$ by $y^n_s := x^n_{G_n(s)}$.

Lemma 4.4 (a) *For all $n \geq 0$, $(y^n_s)_{s\in S}$ is a subnet of $(x^n_{t_n})_{t_n\in T_n}$.*
(b) *For all $n \geq 1$, $(y^n_s)_{s\in S}$ is a subnet of $(y^{n-1}_s)_{s\in S}$.*

Proof of Lemma 4.4. (a) By the definition of y^n_s, what we have to check is whether

$$\forall t_n \in T_n\ \exists\, s^0 \in S : \quad G_n(s) \geq t_n \quad \text{for all } s \geq s^0.$$

But this is obvious: Set $s^0 := (t_0,\, t_1,\, t_2,\, \ldots) \in S$. Then, for $s \geq s^0$, one indeed has $G_n(s) \geq t_n$.
(b) For $n \geq 1$, define

$$H_n : S \to S,\ (s_0,\, s_1,\, s_2,\, \ldots) \mapsto (s_0,\, \ldots,\, s_{n-2},\, F_n(s_n),\, s_n,\, s_{n+1},\, \ldots)$$

with the $F_n(s_n)$ standing at the $n-1$ th position. Then, for all $s = (s_0,\, s_1,\, s_2,\, \ldots) \in S$ and all $n \geq 1$,

$$y^n_s = x^n_{G_n(s)} = x^n_{s_n} = x^{n-1}_{F_n(s_n)} = x^{n-1}_{G_{n-1}(H_n(s))} = y^{n-1}_{H_n(s)}, \tag{8}$$

and it remains to show that

$$\forall \hat{s} \in S\ \exists s^0 \in S : \quad H_n(s) \geq \hat{s} \quad \text{for all } s \geq s^0. \tag{9}$$

Let $\hat{s} = (\hat{s}_0,\, \hat{s}_1,\, \hat{s}_2,\, \ldots) \in S$. For $k \neq n$, set $s^0_k := \hat{s}_k$. In case $k = n$, we first choose $s^{00}_n \in T_n$ such that

$$\forall s_n \geq s^{00}_n : \quad F_n(s_n) \geq \hat{s}_{n-1} \tag{10}$$

(which is possible due to (7)), and then we choose $s_n^0 \in T_n$ such that both $s_n^0 \geq s_n^{00}$ and $s_n^0 \geq \hat{s}_n$. Define $s^0 := (s_0^0, s_1^0, s_2^0, \ldots) \in S$. Then, for all $s = (s_0, s_1, s_2, \ldots) \geq s^0$, we have

$$\begin{array}{lllll} s_k & \geq & s_0^k & = & \hat{s}_k \quad \text{for all } 0 \leq k \leq n-2, \\ s_n & \geq & s_n^0 & \geq & s_n^{00}, \quad \text{whence } F_n(s_n) \geq \hat{s}_{n-1} \text{ due to (10)}, \\ s_n & \geq & s_n^0 & \geq & \hat{s}_n, \\ s_k & \geq & s_k^0 & = & \hat{s}_k \quad \text{for all } k \geq n+1. \end{array}$$

Consequently,

$$\begin{aligned} H_n(s_0, s_1, s_2, \ldots) &= (s_0, \ldots, s_{n-2}, F_n(s_n), s_n, s_{n+1}, \ldots) \\ &\geq (\hat{s}_0, \ldots, \hat{s}_{n-2}, \hat{s}_{n-1}, \hat{s}_n, \hat{s}_{n+1}, \ldots) = \hat{s}. \end{aligned}$$

This proves (9) and the lemma. ■

Step 2. *Choice of the diagonal net.*

Let $\Omega := S \times \mathbb{N}$. This set becomes directed by the order relation

$$(s, n) \geq (s', n') \quad \Longleftrightarrow \quad s \geq s' \text{ and } n \geq n'.$$

Consider the net

$$y : \Omega \to \mathbb{Z}^N, \quad y_{(s,n)} := y_s^n. \tag{11}$$

Of course (and as in the standard diagonalization procedure for sequences) one cannot expect that $(y_{(s,n)})_{(s,n)\in\Omega}$ is a subnet of $(y_s^n)_{s\in S}$. But (also as for standard diagonalization) one has the following result where we write $\Omega_{n_0} := \{(s, n) \in \Omega : n > n_0\}$ for brevity. Clearly, Ω_{n_0} is a cofinal subset of Ω for every $n_0 \in \mathbb{N}$.

Lemma 4.5 *For all $n_0 \in \mathbb{N}$, $(y_{(s,n)})_{(s,n)\in\Omega_{n_0}}$ is a subnet of $(y_s^{n_0})_{s\in S}$.*

Proof of Lemma 4.5. For all $s \in S$ and all $n > n_0$, we have

$$y_{(s,n)} = y_s^n = y_{H_n(s)}^{n-1} = y_{H_{n-1}(H_n(s))}^{n-2} = \ldots = y_{(H_{n_0+1}\circ H_{n_0+2}\circ\ldots\circ H_n)(s)}^{n_0}$$

(compare (8)). This equality suggests to define

$$K_{n_0} : \Omega_{n_0} \to S, \quad (s, n) \mapsto (H_{n_0+1} \circ H_{n_0+2} \circ \ldots \circ H_n)(s).$$

Then, obviously,

$$y_{(s,n)} = y_{K_{n_0}(s,n)}^{n_0} \quad \text{for all } (s, n) \in \Omega_{n_0},$$

and what remains to verify is

$$\forall \hat{s} \in S\ \exists (\tilde{s}, \tilde{n}) \in \Omega_{n_0} : \quad K_{n_0}(s,n) \geq \hat{s} \quad \text{for all } (s, n) \geq (\tilde{s}, \tilde{n}).$$

Set $\tilde{n} := n_0 + 1$ and construct $\tilde{s} := (\tilde{s}_0, \tilde{s}_1, \ldots)$ successively as follows. Let $\hat{s} = (\hat{s}_0, \hat{s}_1, \ldots) \in S$. We set $\tilde{s}_k := \hat{s}_k$ for $k \leq n_0$. Further, by (7), given $\hat{s}_{n_0} \in T_{n_0}$,

$$\exists\, \overline{s}_{n_0+1} \in T_{n_0+1} : \quad F_{n_0+1}(s) \geq \hat{s}_{n_0} \quad \forall s \geq \overline{s}_{n_0+1}.$$

Then choose $\tilde{s}_{n_0+1}$ both larger than $\overline{s}_{n_0+1}$ and $\hat{s}_{n_0+1}$.

For $\tilde{s}_{n_0+1} \in T_{n_0+1}$, we choose $\overline{s}_{n_0+2} \in T_{n_0+2}$ such that

$$\forall s \geq \overline{s}_{n_0+2} : \quad F_{n_0+2}(s) \geq \tilde{s}_{n_0+1} \ (\geq \hat{s}_{n_0+1})$$

and, hence,

$$F_{n_0+1}(F_{n_0+2}(s)) \geq \hat{s}_{n_0}.$$

Then choose $\tilde{s}_{n_0+2}$ both larger than $\overline{s}_{n_0+2}$ and $\hat{s}_{n_0+2}$.

We proceed in this way, i.e. we choose $\overline{s}_{n_0+3} \in T_{n_0+3}$ such that

$$\forall s \geq \overline{s}_{n_0+3} : \quad F_{n_0+3}(s) \geq \tilde{s}_{n_0+2} \ (\geq \hat{s}_{n_0+2})$$

which implies that

$$F_{n_0+2}(F_{n_0+3}(s)) \geq \hat{s}_{n_0+1}$$

and, hence,

$$F_{n_0+1}(F_{n_0+2}(F_{n_0+3}(s))) \geq \hat{s}_{n_0}.$$

Then choose $\tilde{s}_{n_0+3}$ larger than $\overline{s}_{n_0+3}$ and $\hat{s}_{n_0+3}$.

Thus we have fixed $\tilde{s}$. Let now $s = (s_0, s_1, \ldots) \geq \tilde{s}$. Then, due to our construction,

$$\begin{array}{rcll} s_k & \geq & \hat{s}_k & \text{for all } k \leq n_0 - 1, \\ (F_{n_0+1} \circ F_{n_0+2} \circ \ldots \circ F_n)(s_n) & \geq & \hat{s}_{n_0}, & \\ (F_{n_0+2} \circ F_{n_0+3} \circ \ldots \circ F_n)(s_n) & \geq & \hat{s}_{n_0+1}, & \\ & \vdots & & \\ F_n(s_n) & \geq & \hat{s}_{n-1}, & \\ s_k \ \geq \ \tilde{s}_k & \geq & \hat{s}_k & \text{for all } k \geq n. \end{array}$$

This shows that

$$K_{n_0}(s, n) = (H_{n_0+1} \circ \ldots \circ H_n)(s) \geq \hat{s}$$

since

$$H_n(s) = (s_0, \ldots, s_{n-2}, F_n(s_n), s_n, s_{n+1}, \ldots),$$

$$(H_{n-1} \circ H_n)(s) = (s_0, \ldots, s_{n-3}, F_{n-1}(F_n(s_n)), F_n(s_n), s_n, s_{n+1}, \ldots),$$

$$(H_{n-2} \circ H_{n-1} \circ H_n)(s) = \\ (s_0, \ldots, s_{n-4}, F_{n-2}(F_{n-1}(F_n(s_n))), F_{n-1}(F_n(s_n)), F_n(s_n), s_n, s_{n+1}, \ldots),$$

and so on. This finishes the proof of Lemma 4.5. ■

Step 3. *Let $W := \Omega_0$. Then $(y_w)_{w \in W}$ is the net we are looking for.*

It is obvious from the above construction that $(y_w)_{w \in W}$ is a subnet of $(x^0_{t_0})_{t_0 \in T_0}$. So we are left with verifying that $\lim_{w \in W} f(y_w) = 0$.

Given $\varepsilon > 0$, choose and fix $n \geq 1$ such that $\|f - f_n\| < \varepsilon/2$. Then, by hypothesis,

$$\lim_{t_n \in T_n} f_n(x^n_{t_n}) = 0.$$

Since $(y_w)_{w \in \Omega_n}$ is a subnet of $(x^n_{t_n})_{t_n \in T_n}$, we also have $\lim_{w \in \Omega_n} f_n(y_w) = 0$, whence the existence of an $w_n \in \Omega_n$ with

$$|f_n(y_w)| < \varepsilon/2 \quad \text{for all } w \geq w_n. \tag{12}$$

Let now $w \in W$ with $w \geq w_n$. Then, evidently, $w \in \Omega_n$, and from (12) we conclude

$$|f(y_w)| \leq |f(y_w) - f_n(y_w)| + |f_n(y_w)| \leq \|f - f_n\|_\infty + |f_n(y_w)| < \varepsilon.$$

Hence, $\lim_{w \in W} f(y_w) = 0$ which finishes the proof of Theorem 4.3. ■

5 Limit operators with respect to nets

Now we return to band-dominated operators on one of the sequence spaces E. If $y := (y_w)_{w \in W}$ is a net in $\mathcal{N}$, then we call the operator A_y the *limit operator* of the operator $A \in L(E)$ with respect to y if

$$\lim_{n \to \infty} \|(V_{-y_w} A V_{y_w} - A_y) P_m\| = \lim_{n \to \infty} \|P_m (V_{-y_w} A V_{y_w} - A_y)\| = 0$$

for every $P_m \in \mathcal{P}$. Roughly speaking, the properties of limit operators with respect to sequences (as derived in [5, 6]), remain valid without changes also for limit operators with respect to nets. We will illustrate this fact by two results for which the Cantor diagonalization procedure for nets is employed.

Theorem 5.1 *Let $A = aI \in L(E)$ be a rich multiplication operator. Then every net $(x_t)_{t \in T} \in \mathcal{N}$ possesses a subnet $y := (y_w)_{w \in W}$ such that the limit operator A_y exists.*

Proof. Recall that A_y is a limit operator of A with respect to the net y if and only if

$$\lim_{w \in W} \|(V_{-y_w} A V_{y_w} - A_y) S_k\| = 0 \quad \text{for every } k \in \mathbb{Z}^N$$

where, as before, S_k refers to the operator of multiplication by the function which is I at $k \in \mathbb{Z}^N$ and 0 at all other points.

Set $(x^0_{t_0})_{t_0 \in T_0} := (x_t)_{t \in T}$ and choose a bijection $m : \mathbb{N} \to \mathbb{Z}^N$. Since A is rich we find, for every $n \geq 1$, a subnet $(x^n_{t_n})_{t_n \in T_n}$ of $(x^{n-1}_{t_{n-1}})_{t_{n-1}} \in T_{n-1}$ as well as an operator $B_n \in L(\operatorname{Im} S_{m(n)})$ such that

$$\|(V_{-x^n_{t_n}} A V_{x^n_{t_n}} - B_n) S_{m(n)}\| \to 0. \tag{13}$$

Let B stand for the operator of multiplication by the function

$$\mathbb{Z}^N \to L(X), \quad k \mapsto B_{m^{-1}(k)}.$$

We claim that B is the limit operator of A with respect to the net y. For, we reify Cantor's scheme (= Theorem 4.3) as follows. Set $Z := \mathbb{Z}^N$. For $n \geq 1$ and $z \in \mathbb{Z}^N$, define

$$f_n(z) := \sum_{k=1}^{n} 2^{-k} \|(V_{-z} A V_z - B) S_{m(k)}\|,$$

and let

$$f(z) := \sum_{k=1}^{\infty} 2^{-k} \|(V_{-z} A V_z - B) S_{m(k)}\|.$$

Then, obviously, $\|f_n - f\|_\infty \to 0$. Further, by (13), we have $\lim_{t_n \in T_n} f_n(x^n_{t_n}) = 0$. Now we conclude from Theorem 4.3 that there is a subnet $(y_w)_{w \in W}$ of $(x_t)_{t \in T}$ such that $\lim_{w \in W} f(y_w) = 0$. This immediately implies the $\mathcal{P}$-strong convergence of the net $(V_{-y_w} A V_{y_w})_{w \in W}$ to B, whence $B = A_y$. ■

Of course, a similar result holds for rich band operators. For another application of Theorem 4.3, consider the set of all operators $A \in L(E)$ having the following property: every net $(x_t)_{t \in T} \in \mathcal{N}$ possesses a subnet $y := (y_w)_{w \in W}$ such that the limit operator A_y exists. We denote this class by $\mathcal{L}_E^{nets}$ for a moment. As we have just remarked, every rich band operator belongs to $\mathcal{L}_E^{nets}$.

Theorem 5.2 *$\mathcal{L}_E^{nets}$ is norm-closed.*

Proof. Let $(A_n)_{n\geq 1} \subseteq \mathcal{L}_E^{nets}$ be a sequence with norm limit $A \in L(E)$, and let $(x^0_{t_0})_{t_0\in T_0} \in \mathcal{N}$. By hypothesis, for every $n \geq 1$, there exists a subnet $x^n := (x^n_{t_n})_{t_n\in T_n}$ of $(x^{n-1}_{t_{n-1}})_{t_{n-1}\in T_{n-1}}$ such that the limit operator A_{n,x^n} of A_n with respect to x^n exists. If $n \geq m$, then $(x^n_{t_n})_{t_n\in T_n}$ is a subnet of $(x^m_{t_m})_{t_m\in T_m}$, thus, the limit operator A_{m,x^n} also exists, and it coincides with A_{m,x^m}. Since $\|A_h\| \leq \|A\|$ for every limit operator A_h of A, we obtain

$$\|A_{n,x^n} - A_{m,x^m}\| = \|A_{n,x^n} - A_{m,x^n}\| = \|(A_n - A_m)_{x^n}\| \leq \|A_n - A_m\|$$

for all $n \geq m$. Hence, the sequence (A_{n,x^n}) converges in the norm, and we let B denote its norm limit.

Now define for all $n \geq 1$ and $z \in \mathbb{Z}^N$ (with the notations S_k and m as in the proof of Theorem 5.1)

$$f_n(z) := \sum_{k=1}^{\infty} 2^{-k}\, \|(V_{-z}A_nV_z - A_{n,x^n})S_{m(k)}\|$$

and

$$f(z) := \sum_{k=1}^{\infty} 2^{-k}\, \|(V_{-z}AV_z - B)S_{m(k)}\|.$$

Then again $\|f_n - f\| \to 0$ and $\lim_{t_n\in T_n} f_n(x^n_{t_n}) = 0$, whence via Theorem 4.3 the existence of a subnet $y = (y_w)_{w\in W}$ of $(x^0_{t_0})_{t_0\in T_0}$ such that $\lim_{w\in W} f(y_w) = 0$. Thus, $B = A_y$. ∎

As a consequence we get $\mathcal{A}_E^{rich} \subseteq \mathcal{L}_E^{nets}$. Now one might ask whether one gets something new when considering limit operators with respect to nets instead of sequences. The next theorem says that the answer is *no* in some sense: every limit operator, which is defined with respect to a net, can also be reached by a sequence! (Nevertheless, limit operators with respect to nets *are* useful as we will point out in the next sections when we will apply them to study the local invertibility of band-dominated operators at points in $M^\infty(SO)$.)

Theorem 5.3 *Let $A \in L(E)$, and let $y = (y_w)_{w\in W} \in \mathcal{N}$ be a net for which the limit operator A_y of A exists. Then there is a sequence $z = (z_n)_{n\in\mathbb{N}} \in \mathcal{H}$ for which the limit operator A_z of A exists, and $A_z = A_y$. Moreover, z can be chosen such that there is a cofinal subset $\tilde{W}$ of W for which $(y_w)_{w\in\tilde{W}}$ is a subnet of z.*

Proof. Let $y = (y_w)_{w\in W}$ be a net for which the limit operator A_y of A exists, and define a function $f : \mathbb{Z}^N \to \mathbb{R}^+$ by

$$f(z) := \sum_{k=1}^{\infty} 2^{-k}\, \|(V_{-z}AV_z - A_y)S_{m(k)}\|$$

with the notations being as in the proof of Theorem 5.1. Then $\lim_{w\in W} f(y_w) = 0$. For every $n \in \mathbb{N}$, choose $w_n \in W$ such that

$$0 \leq f(y_w) < 1/n \quad \text{for all } w \geq w_n. \tag{14}$$

Further set $W_{\mathbb{N}} := \{w_n : n \in \mathbb{N}\}$, $W_\infty := \{w \in W : w \geq w_n \text{ for every } n\}$, and $\tilde{W} := W_{\mathbb{N}} \cup W_\infty$.

The set $\tilde{W}$ is cofinal in W. Indeed, let $w \in W$. Then, either, there is a w_n with $w_n \geq w$, or $w \geq w_n$ for every n. In the first case, choose $w^* := w_n$, in the second $w^* := w$. Thus, in any case, $w^* \in \tilde{W}$ and $w^* \geq w$.

Consequently, $(y_w)_{w\in\tilde{W}}$ is a subnet of $(y_w)_{w\in W}$, whence $\lim_{w\in\tilde{W}} f(y_w) = 0$, and this subnet takes the values $f(y_{w_n}) \in [0, 1/n)$ and 0 only. The latter happens if $w \in W_\infty$, in which case $0 \leq f(y_w) < 1/n$ for all n, hence $f(y_w) = 0$.

Now construct a bijection z from $\mathbb{N}$ onto the set $\{y_w : w \in \tilde{W}\}$ of the values of $(y_w)_{w\in\tilde{W}}$ as follows. If the set $\{y_w : w \in W_\infty \setminus W_\mathbb{N}\}$ is infinite (hence, countable), let $z_{2k} := y_{w_k}$ for $k \geq 1$, and let $z|_{2\mathbb{N}+1}$ be any bijection from $2\mathbb{N}+1$ onto $\{y_w : w \in W_\infty \setminus W_\mathbb{N}\}$. If $\{y_w : w \in W_\infty \setminus W_\mathbb{N}\}$ is finite and consists of n elements, we set $z_k := y_{k-n}$ for $k > n$, and we let $z|_{\{1,2,\ldots,n\}}$ be any bijection from $\{1, 2, \ldots, n\}$ onto $\{y_w : w \in W_\infty \setminus W_\mathbb{N}\}$. In any case, we get a sequence z which has $(y_w)_{w\in\tilde{W}}$ as its subnet by Lemma 4.2.

It is further evident from the definition of z that $\lim_{n\to\infty} f(z_n) = 0$. Hence, A_y is also the limit operator of A with respect to the sequence z. ■

6 Local invertibility at points in $M^\infty(SO)$

Now we will provide the proof of Theorem 3.7 and discuss some of its consequences. The proof will follow the line of the proof of Theorem 1.6, and we will pay our attention mainly to the differences which are involved by the topology of $M(SO)$ and, hence, by the need of using nets instead of sequences.

A basic step is the specification of Proposition 14 from [5] resp. Proposition 2.17 from [6] to the present context. For, we need some more notations. Let $\varphi : \mathbb{R} \to [0, 1]$ be a continuous function with

$$\varphi(x) \begin{cases} = 1 & \text{for } |x| \leq 1/3 \\ > 0 & \text{for } |x| < 2/3 \\ = 0 & \text{for } |x| \geq 2/3. \end{cases} \tag{15}$$

We further suppose that the family $\{\varphi_\alpha^2\}_{\alpha\in\mathbb{Z}}$ with $\varphi_\alpha(x) := \varphi(x - \alpha)$ forms a partition of unity on $\mathbb{R}$ in the sense that

$$\sum_{\alpha\in\mathbb{Z}} \varphi_\alpha(x)^2 = 1 \quad \text{for all } x \in \mathbb{R}.$$

This choice of φ can always be forced as follows: If $f : \mathbb{R} \to [0, 1]$ is a continuous function satisfying (15) in place of φ, then the function

$$\varphi(x) := \frac{f(x)^2}{\sum_{\alpha\in\mathbb{Z}} f(x-\alpha)^2}, \quad x \in \mathbb{R}.$$

has the desired properties. This definition makes sense since the series $\sum f(x-\alpha)^2$ is strictly positive and has only finitely many non-vanishing terms for each fixed x.

Given $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$, $\alpha \in \mathbb{Z}^N$, and $R > 0$, define $\varphi^{(N)}(x) := \varphi(x_1)\ldots\varphi(x_N)$, $\varphi_\alpha^{(N)}(x) := \varphi^{(N)}(x-\alpha)$ and $\varphi_{\alpha,R}^{(N)}(x) := \varphi_\alpha(x/R)$. Further, let $\psi : \mathbb{R} \to [0, 1]$ be a continuous function which also satisfies (15) in place of φ, but with the constants $1/3$ and $2/3$ being replaced by $3/4$ and $4/5$, respectively. For this function, we define $\psi_{\alpha,R}^{(N)}$ analogously. Clearly, $\varphi_{\alpha,R}^{(N)}\psi_{\alpha,R}^{(N)} = \varphi_{\alpha,R}^{(N)}$ for all α and R. The family $\{\varphi_{\alpha,R}\}$ is a partition of unity on $\mathbb{R}^N$ for every fixed R (but observe that the family $\{\psi_\alpha\}$ is not required to form a partition of unity. With these notations, the announced analogue of Proposition 14 from [5] reads as follows.

Proposition 6.1 *Let $A \in \mathcal{A}_E$, $\eta \in M^\infty(SO)$. Suppose there is a constant $M > 0$ such that, for all positive integers R, there is a neighborhood at infinity U of η such that, for all $\alpha \in U$, there are operators $B_{\alpha,R}$ and $C_{\alpha,R}$ with $\|B_{\alpha,R}\|_{L(E)} \leq M$, $\|C_{\alpha,R}\|_{L(E)} \leq M$ and*

$$B_{\alpha,R}\, A\, \hat{\psi}_{\alpha,R} I = \hat{\psi}_{\alpha,R}\, A\, C_{\alpha,R} = \hat{\psi}_{\alpha,R} I.$$

Then the operator A is locally invertible at η, i.e. there are operators $B, C \in \mathcal{A}_E$ and a neighborhood at infinity W of η such that

$$BA\hat{\chi}_W I = \hat{\chi}_W AC = \hat{\chi}_W I. \tag{16}$$

Proof. We follow exactly the proof of Proposition 14 from [5] where we replace the condition $|\alpha| \geq \rho(R)$ by $\alpha \in U$. What results is the existence of a positive integer R such that

$$(I + T_R)^{-1} B_R A = I - (I + T_R)^{-1} \sum_{\alpha \in \mathbb{Z}^N \setminus U} \hat{\varphi}_{\alpha,R} I.$$

The assertion follows once we have shown that there is a neighborhood at infinity W of η such that $\sum_{\alpha \in \mathbb{Z}^N \setminus U} \hat{\varphi}_{\alpha,R} \chi_W = 0$. This will be done in Proposition 6.4 below. ∎

To fill the gap in the preceding proof requires more precise knowledge on subsets of $M(SO)$. The following definition as well as Theorem 6.3 and its proof are taken from [4].

Definition 6.2 *(a) A subset $V \subseteq \mathbb{Z}^N$ is called* growing *if, for every bounded set $D \subset \mathbb{Z}^N$, there is an $x \in \mathbb{Z}^N$ such that $x + D \subseteq V$.*

(b) An unbounded subset V_0 of a growing set V is called a center *if, for every bounded set $D \subset \mathbb{Z}^N$, there is a bounded set M such that $(V_0 \setminus M) + D \subseteq V$.*

Theorem 6.3 *Let W be an unbounded subset of $\mathbb{Z}^N$ and $\eta \in \overline{W} \cap M^\infty(SO)$ (where the bar refers to the closure with respect to the Gelfand topology on $M(SO)$), and let $U \subseteq M(SO)$ be a neighborhood of η. Then $V := U \cap \mathbb{Z}^N$ is a growing set, and there is a neighborhood $U_0 \subseteq U$ of η such that $V_0 := U_0 \cap \mathbb{Z}^N$ is contained in W and a center of V.*

Proof. By Uryson's lemma, there is a continuous function $f : M(SO) \to [0, 1]$ which is 0 at η and 1 on $M(SO) \setminus U$. Since f is continuous on $M(SO)$, the restriction of f onto $\mathbb{Z}^N$ is a slowly oscillating function. Set

$$U_0' := \{x \in M(SO) : f(x) < 1/2\} \quad \text{and} \quad U_0 := U_0' \cap W,$$

and define $V := U \cap \mathbb{Z}^N$ and $V_0 := U_0 \cap \mathbb{Z}^N$. Then $V_0 \subseteq W \cap V$. Moreover, since U_0' is a neighborhood of η, the set V_0 is unbounded. We claim that, for every bounded set M, there is a bounded set D such that $(V_0 \setminus D) + M \subseteq V$. The claim implies that V is growing and that V_0 is a center of V.

Assume the claim is wrong. Then there exists a bounded set M such that $(V_0 \setminus D) + M \not\subseteq V$, hence, $V_1 := (V_0 + M) \setminus V$ is an unbounded set. So it makes sense to consider the limes superior of $|f(x)|$ when $x \in V_1$ tends to infinity. Since $V_1 \subseteq V_0 + M$ and f is slowly oscillating, we get

$$\begin{aligned} \limsup_{x \in V_1, x \to \infty} |f(x)| &\leq \limsup_{y \in V_0, y \to \infty} \max_{m \in M} |f(y+m)| \\ &\leq \limsup_{y \in V_0, y \to \infty} \max_{m \in M} |f(y+m) - f(y)| + \limsup_{y \in V_0, y \to \infty} |f(y)| \leq 0 + 1/2 = 1/2. \end{aligned}$$

This is impossible since V_1 is in the complement of U and, hence, f is 1 on V_1. ∎

Proposition 6.4 *Let R a positive integer, $\eta \in M^\infty(SO)$ and $U \subseteq M(SO)$ a neighborhood of η. Then there exists a neighborhood at infinity $\tilde{U}$ of η such that $\sum_{\alpha \in \mathbb{Z}^N \setminus U} \hat{\varphi}_{\alpha,R} \chi_{\tilde{U}} = 0$.*

Proof. We apply Theorem 6.3 (with the W in that theorem being $\mathbb{Z}^N$) to obtain: $V := U \cap \mathbb{Z}^N$ is a growing set, and there is a neighborhood $U_0 \subseteq U$ of η such that $V_0 := U_0 \cap \mathbb{Z}^N$ is a center of V.

The support of every function $\hat{\varphi}_{\alpha,R}$ is contained in a smallest ball with center αR and with a radius r which depends on R but not on α. From V, we remove all points z for which the ball with center z and radius r is not completely contained in V. What we get is a set $\tilde{V}$, and we set $\tilde{U} := V_0 \cap \tilde{V}$.

We claim that $\tilde{V}$ is a growing set and that $\tilde{U}$ is one of its centers. Let $D \subset \mathbb{Z}^N$ be bounded, and let B be the ball with center 0 and radius r. Then $D + B$ is a bounded set, and since V_0 is a center of V, there is a bounded set M such that

$$(V_0 \setminus M) + (D + B) \subseteq V.$$

Then, of course, $(V_0 \setminus M) + D \subseteq \tilde{V}$, whence

$$(\tilde{U} \setminus M) + D \subseteq \tilde{V}. \tag{17}$$

Analogously, there is a bounded set N such that $(V_0 \setminus N) + B \subseteq V$. Thus, all points in $V_0 \setminus N$ belong to $\tilde{V}$ and, consequently, also to $\tilde{U}$. This shows that $\tilde{U}$ and V_0 differ by a bounded set only:

$$V_0 \setminus N \subseteq \tilde{U} \subseteq V_0. \tag{18}$$

A first consequence of (18) is that $\tilde{U}$ is an unbounded set. Together with (17) this implies that $\tilde{V}$ is a growing set, and that $\tilde{U}$ is a center of $\tilde{V}$. As another consequence of (18) we observe that, since V_0 is a neighborhood at infinity of η, also $\tilde{U}$ is a neighborhood at infinity of η. This finishes the proof since the support of every function $\hat{\varphi}_{\alpha,R}$ with $\alpha \in \mathbb{Z}^N \setminus U$ is contained in the complement of $\tilde{V}$, hence in the complement of $\tilde{U}$. ∎

Proof of Theorem 3.7. We will only prove that the uniform invertibility of the operators in $\sigma_\eta(A)$ implies the local invertibility of A at η. Let $A \in \mathcal{A}_E^{rich}$ be an operator with

$$M_A := \sup \{\|A_h^{-1}\| : A_h \in \sigma_\eta(A)\} < \infty,$$

but suppose A is not locally invertible at η. Then, by Proposition 6.1, there is a net $(y_t)_{t\in T}$ with values in $\mathbb{Z}^N$ which converges to η in the topology of $M(SO)$ and which has the property that

$$BA\hat{\psi}_{y_t,R}I \neq \hat{\psi}_{y_t,R}I \tag{19}$$

for all $t \in T$ and all B with $\|B\| \leq M_A$. Since A belongs to $\mathcal{A}_E^{rich} \subseteq \mathcal{L}_E^{nets}$, the net $(y_t)_{t\in T}$ possesses a subnet $x = (x_s)_{s\in S}$ such that the limit operator A_x exists. Clearly, the net $(x_s)_{s\in S}$ still converges to η, and

$$BA\hat{\psi}_{x_s,R}I \neq \hat{\psi}_{x_s,R}I \tag{20}$$

for all $s \in S$ and all B with $\|B\| \leq M_A$. From Theorem 5.3 we conclude: there is a cofinal subset $\tilde{S}$ of S and a *sequence* $(z_n)_{n\in\mathbb{N}}$ such that the limit operator A_z exists and coincides with A_x, and such that $(x_s)_{s\in\tilde{S}}$ is a subnet of (z_n). Since $(x_s)_{s\in\tilde{S}}$ converges to η, and since the nets $(x_s)_{s\in\tilde{S}}$ and (z_n) take the same values, it is clear that

$$\eta \in \operatorname{clos}\{z_n : n \in \mathbb{N}\}, \quad \text{whence} \quad A_y = A_z \in \sigma_\eta(A).$$

By hypothesis, A_y is invertible, and $\|A_y^{-1}\| \leq M_A$. This yields a contradiction in the very same way as in the proof of Theorem 1.4 by using Proposition 15 from [5]. ∎

Our next goal is to point out the connections between local invertibility at η and localization by means of the local principle. For the reader's convenience, we state this principle here. Let $\mathcal{B}$ be a unital Banach algebra. By a *central* subalgebra $\mathcal{C}$ of $\mathcal{B}$ we mean a closed subalgebra of the center of $\mathcal{B}$ which contains the identity element. Thus, every element of $\mathcal{C}$ commutes with every element from $\mathcal{B}$, and $\mathcal{C}$ is a commutative Banach algebra with maximal ideal space $M(\mathcal{B})$. To each maximal ideal x of $\mathcal{C}$, we associate the smallest closed two-sided ideal $\mathcal{I}_x$ of $\mathcal{B}$ which contains x, and we let Φ_x refer to the canonical homomorphism from $\mathcal{B}$ onto the quotient algebra $\mathcal{B}/\mathcal{I}_x$. Notice that, in contrast to the commutative setting, the quotient algebras $\mathcal{B}/\mathcal{I}_x$ can differ from each other in dependence on $x \in M(\mathcal{C})$. Moreover, it may happen that $\mathcal{I}_x = \mathcal{B}$ for some points x. In this case we *define* that $\Phi_x(a)$ is invertible in $\mathcal{B}/\mathcal{I}_x$ and that $\|\Phi_x(a)\| = 0$ for each $a \in \mathcal{B}$.

Theorem 6.5 (Allan) *Let $\mathcal{C}$ be a central subalgebra of the unital Banach algebra $\mathcal{B}$ which contains the identity element. Then an element $a \in \mathcal{B}$ is invertible if and only if the cosets $\Phi_x(a)$ are invertible in $\mathcal{B}/\mathcal{I}_x$ for every $x \in M(\mathcal{C})$.*

We have seen in Proposition 2.1 that the algebra $\mathcal{C}$ of all cosets $aI + K(E, \mathcal{P})$ with $a \in SO$ lies in the center of the quotient algebra $\mathcal{A}_E/K(E, \mathcal{P})$. From the isomorphy

$$\mathcal{C} \cong (SO \cdot I + K(E, \mathcal{P}))/K(E, \mathcal{P}) \cong SO \cdot I/(SO \cdot I \cap K(E, \mathcal{P})) \cong SO/c_0$$

we conclude that the maximal ideal space of the algebra $\mathcal{C}$ is homeomorphic to the fiber $M^\infty(SO)$. Given $\eta \in M^\infty(SO)$, we denote the local algebra of $\mathcal{A}_E/K(E, \mathcal{P})$ which is associated with η by $\mathcal{A}_{E,\eta}$, and we write π_η for the canonical homomorphism from $\mathcal{A}_E$ onto $\mathcal{A}_{E,\eta}$. Applying Theorem 6.5 to the current situation yields:

Theorem 6.6 *An operator $A \in \mathcal{A}_E$ is $\mathcal{P}$-Fredholm if and only if the cosets $\pi_\eta(A)$ are invertible for all $\eta \in M^\infty(SO)$.*

The following theorem relates the invertibility of the coset $\pi_\eta(A)$ with the local invertibility of A at η and can be proved in the very same way as Proposition 23 in [5].

Theorem 6.7 *Let $A \in \mathcal{A}_E$ and $\eta \in M^\infty(SO)$. The coset $\pi_\eta(A)$ is invertible in $\mathcal{A}_{E,\eta}$ if and only if A is locally invertible at η.*

Together with Allan's local principle and with Theorem 3.7, this results implies a further and essential refinement of Theorem 1.4 and Corollary 1.7.

Corollary 6.8 *An operator $A \in \mathcal{A}_E^{rich}$ is $\mathcal{P}$-Fredholm if and only if all of its limit operators are invertible, and if*

$$\sup\{\|(A_h)^{-1}\| : A_h \in \sigma_\eta(A)\} < \infty \quad \textit{for every } \eta \in M^\infty(SO).$$

7 Fredholmness of band-dominated operators with slowly oscillating coefficients

We will now specify Corollary 6.8 to band-dominated operators with slowly oscillating coefficients. Let $SO^{rich}_{L(X)}$ refer to the class of all slowly oscillating functions with values in $L(X)$ for which the associated multiplication operator is rich. Further we let $\mathcal{A}_E(SO_{L(X)})$ (resp. $\mathcal{A}_E(SO^{rich}_{L(X)})$) stand for the smallest closed subalgebra of $\mathcal{A}_E$ which contains all band operators $\sum_{|\alpha| \le k} a_\alpha V_\alpha$ with $a_\alpha \in SO_{L(X)}$ (resp. $a_\alpha \in SO^{rich}_{L(X)}$). For the limit operators of operators with slowly oscillating coefficients we have the following.

Proposition 7.1 *(a) If $A \in \mathcal{A}_E(SO_{L(X)})$, then every limit operator of A belongs to $\mathcal{A}_E(\mathbb{C}_{L(X)})$.*
(b) For $A \in \mathcal{A}_E(SO^{rich}_{L(X)})$, every local operator spectrum $\sigma_\eta(A)$ with $\eta \in M^\infty(SO)$ is a singleton.

Proof. (a) Limit operators of shift operators are shift operators and, hence, in $\mathcal{A}_E(\mathbb{C}_{L(X)})$. By Proposition 2.2, the same is true for operators of multiplication by slowly oscillating functions.
(b) If $a \in SO^{rich}_{L(X)}$, then $\sigma_\eta(aI)$ is not empty since $\mathcal{A}^{rich}_E \subseteq \mathcal{L}^{nets}_E$ (see Section 5), and this spectrum is a singleton by Proposition 3.5 (b). With Proposition 1 from [5] we conclude first that every local spectrum of a band operator with coefficients in $SO^{rich}_{L(X)}$ is a singleton, too, and get then the assertion also in the general case. ■

Now we can formulate and prove the $\mathcal{P}$-Fredholm criterion for operators with rich slowly oscillating coefficients. It turns out that the uniform boundedness condition is redundant.

Theorem 7.2 *Operators in $\mathcal{A}_E(SO^{rich}_{L(X)})$ are $\mathcal{P}$-Fredholm if and only if all of their limit operators are invertible.*

Proof. Since $\sigma_\eta(A)$ is a singleton, the assertion follows immediately from Corollary 6.8. ■

8 An alternative proof of Theorem 7.2

This section is devoted to an alternative proof of the preceding theorem which works under more restrictive assumptions only, but which also has its own merits, and which sheds new light upon the properties of band-dominated operators with slowly oscillating coefficients. We let H be a Hilbert space and $E := l^2(\mathbb{Z}^N, H)$. Further, we again write $SO_{L(H)}$ and $SO^{rich}_{L(H)}$ for the algebra of all slowly oscillating functions $\mathbb{Z}^N \to L(H)$ and for the algebra of all slowly oscillating functions $\mathbb{Z}^N \to L(H)$ for which the associated multiplication operator is rich, respectively, and we let $\mathcal{A}_E(SO_{L(H)})$ and $\mathcal{A}_E(SO^{rich}_{L(H)})$ stand for the closures in $L(E)$ of the algebra of the band operators with coefficients in $SO_{L(H)}$ and in $SO^{rich}_{L(H)}$.

Generating functions. The alternative proof is based on the notion of the generating function of a band-dominated operator. This notion is borrowed from the pseudodifferential operator calculus (where the generating functuion is usually referred to as the symbol of the operator) and adapted for our purposes.

We start with defining the generating function of a band operator. For

$$A = \sum_{|\alpha| \le M} a_\alpha V_\alpha \quad \text{with } a_\alpha \in SO_{L(H)}, \tag{21}$$

let the *generating function* of A be

$$\mathrm{gen}_A : \mathbb{Z}^N \times \mathbb{T}^N \to L(H), \quad (x, t) \mapsto \sum_{|\alpha| \le M} a_\alpha(x) t^\alpha \tag{22}$$

where $t^\alpha := t_1^{\alpha_1} \dots t_n^{\alpha_n}$. There is a one-to-one correspondence between band operators and their generating functions.

We denote by $C_b(\mathbb{Z}^N \times \mathbb{T}^N, L(H))$ the set of all continuous functions on $\mathbb{Z}^N \times \mathbb{T}^N$ with values in $L(H)$. Provided with pointwisely defined operations and the supremum norm, this set becomes a C^*-algebra, and the set $c_0(\mathbb{Z}^N \times \mathbb{T}^N, L(H))$ of all functions $a \in C_b(\mathbb{Z}^N \times \mathbb{T}^N, L(H))$ with

$$\lim_{x\to\infty} \sup_{t\in\mathbb{T}^N} \|a(x,t)\|_{L(H)} = 0$$

is a closed ideal of $C_b(\mathbb{Z}^N \times \mathbb{T}^N, L(H))$. The quotient algebra C_b/c_0 will be abbreviated to $\widehat{C}_b$, and the coset which contains $a \in C_b(\mathbb{Z}^N \times \mathbb{T}^N, L(H))$ to $\widehat{a}$. Notice that

$$\|\widehat{a}\|_0 := \limsup_{x\to\infty} \sup_{t\in\mathbb{T}^N} \|a(x,t)\|_{L(H)}$$

is just the canonical quotient norm of the coset $\widehat{a}$ in the quotient algebra $\widehat{C}_b$.

Evidently, if A is a band operator of the form (21), then its generating function belongs to $C_b(\mathbb{Z}^N \times \mathbb{T}^N, L(H))$.

Proposition 8.1 *Let A be as in* (21). *Then* $\|\widehat{\mathrm{gen}_A}\|_0 \le \|A\|$.

Proof. Choose a sequence $(x_n) \subset \mathbb{Z}^N$ tending to infinity, a sequence $(t_n) \in \mathbb{T}^N$, and a sequence (v_n) of unit vectors in H, such that

$$\|\widehat{\mathrm{gen}_A}\|_0 = \lim_{n\to\infty} \|\mathrm{gen}_A(x_n, t_n)v_n\|_H.$$

Since $\mathbb{T}^N$ is compact, we can moreover assume that (t_n) is a convergent sequence with limit $t_0 \in \mathbb{T}^N$. The assertion follows once we have shown that, given $\varepsilon > 0$, there is an n_0 such that

$$\|\mathrm{gen}_A(x_n, t_n)v_n\|_H \le \|A\| + \varepsilon \tag{23}$$

for all $n \ge n_0$.

Given vectors $v \in H$ and $u = (u_k)_{k\in\mathbb{Z}^N} \in l^2$, let $v \otimes u$ denote the sequence $(u_k v)_{k\in\mathbb{Z}^N}$ in $E = l^2(\mathbb{Z}^N, H)$. Let further $A_{n,n}$, A_n and B_n stand for the band operators with generating functions

$$(x,t) \mapsto \mathrm{gen}_A(x_n, t_n), \quad (x,t) \mapsto \mathrm{gen}_A(x + x_n, t), \quad (x,t) \mapsto \mathrm{gen}_A(x_n, t),$$

respectively. Then we have, for every unit vector $u \in l^2$,

$$\begin{aligned} \|\mathrm{gen}_A(x_n, t_n)v_n\|_H &= \|A_{n,n}(v_n \otimes u)\|_E \\ &\le \|(A_{n,n} - B_n)(v_n \otimes u)\|_E \\ &\quad + \|(B_n - A_n)(v_n \otimes u)\|_E + \|A_n(v_n \otimes u)\|_E. \end{aligned} \tag{24}$$

Since $A_n = V_{-x_n} A V_{x_n}$, we get

$$\|A_n(v_n \otimes u)\|_E = \|V_{-x_n} A V_{x_n}(v_n \otimes u)\|_E \le \|A\|$$

for the last term in (24). The middle term on the right hand side of (24) is not greater than $\|B_n - A_n\|$, which goes to zero as $n \to \infty$ since the coefficients of A are slowly oscillating. Thus, this middle becomes less than $\varepsilon/2$ uniformly with respect to u and v_n if only n is large enough.

To estimate the first term, choose $\delta > 0$ such that

$$\sup_{x\in\mathbb{Z}^N} \|\mathrm{gen}_A(x,t) - \mathrm{gen}_A(x,t_0)\| \le \varepsilon/4 \quad \text{for all } |t - t_0| < \delta,$$

and choose the unit vector $u = (u_k)_{k\in\mathbb{Z}^N}$ in l^2 such that the u_k are the Fourier coefficients of a continuous function $\hat{u}$ on $\mathbb{T}^N$ with support in $\{t \in \mathbb{T}^N : |t - t_0| < \delta\}$.

Since $A_{n,n} - B_n$ is the operator of convolution by the function $\text{gen}_{A_{n,n}} - \text{gen}_{B_n}$, we get

$$\begin{aligned}
\|(A_{n,n} - B_n)(v_n \otimes u)\|^2_{l^2(\mathbb{Z}^N,H)} &= \|(\text{gen}_{A_{n,n}} - \text{gen}_{B_n})(\hat{u}v_n)\|^2_{L^2(\mathbb{T}^N,H)} \\
&= \int_{\mathbb{T}^N} \|(\text{gen}_A(x_n,t_n) - \text{gen}_B(x_n,t))\hat{u}(t)v_n\|^2_H \, dt \\
&\le \sup_{|t-t_0|<\delta} \|\text{gen}_A(x_n,t_n) - \text{gen}_A(x_n,t)\|^2_{L(H)} \, \|v_n \otimes u\|^2.
\end{aligned}$$

Due to the choice of δ, this term becomes less than $\varepsilon/2$ if n becomes large. ■

This proposition allows us to associate with every operator A in $\mathcal{A}_E(SO_{L(H)})$ a uniquely determined coset in $\widehat{C}_b$ which we denote by $\Gamma(A)$.

In what follows, we will make use of the notion of the main diagonal of a band-dominated operator. If A is the band operator $\sum_{|\alpha|\le k} a_\alpha V_\alpha$, then its main diagonal is, by definition, the function $D(A) := a_0$. Since

$$\|D(A)\|_\infty = \sup_k \|a_0(k)\| = \sup_k \|S_k A S_k\| \le \|A\|$$

(where the S_k are as in the introduction), we can extend the mapping D by continuity onto the set of all band-dominated operators. For a band-dominated operator A, we call $D(A)$ its main diagonal and $D(AV_{-\alpha})$ its αth diagonal.

Proposition 8.2 *Γ is a *-homomorphism from $\mathcal{A}_E(SO^{rich}_{L(H)})$ into $\widehat{C}_b$ with kernel $K(E,\mathcal{P})$.*

Proof. It is elementary to check that Γ acts as a *-homomorphism on the algebra of all band operators with slowly oscillating coefficients. Since this algebra is dense in $\mathcal{A}_E(SO_{L(H)})$, and since Γ is continuous on this algebra by the preceding proposition, this proves the first assertion. It is further evident that the ideal $K(E,\mathcal{P})$ lies in the kernel of Γ. Let, finally, A be an operator in $\mathcal{A}_E(SO_{L(H)})$ with $\Gamma(A) = 0$. We have to show that A lies in $K(E,\mathcal{P})$.

Let (A_n) be a sequence of band operators in $\mathcal{A}_E(SO^{rich}_{L(H)})$ which converges to A. Then, trivially, $\|\Gamma(A_n)\| \to 0$. For $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{Z}^N$, consider the functions $a^{(n)}_\alpha : \mathbb{Z}^N \to L(H)$ which take at $x \in \mathbb{Z}^N$ the value

$$\frac{1}{(2\pi)^N} \int_0^{2\pi} \ldots \int_0^{2\pi} \text{gen}_{A_n}(x, (e^{is_1}, \ldots, e^{is_N}))\, e^{-i\alpha_1 s_1} \ldots e^{-i\alpha_N s_N} \, ds_1 \ldots ds_N. \quad (25)$$

If the band operator A_n is of the form $\sum b^{(n)}_\alpha V_\alpha$, then its αth diagonal $b^{(n)}_\alpha$ just coincides with the function $a^{(n)}_\alpha$ given by (25). From (25) we immediately conclude that

$$\|a^{(n)}_\alpha(x)\|_{L(H)} \le \sup_{t\in\mathbb{T}^N} \|\text{gen}_{A_n}(x,t)\|_\infty$$

whence, in particular,

$$\limsup_{x\to\infty} \|a^{(n)}_\alpha(x)\| \le \limsup_{x\to\infty} \sup_{t\in\mathbb{T}^N} \|\text{gen}_{A_n}(x,t)\|_\infty = \|\Gamma(A_n)\| \to 0$$

as $n \to \infty$. Thus, if a_α denotes the αth diagonal of A, then

$$\begin{aligned}
\limsup_{x\to\infty} \|a_\alpha(x)\| &\le \sup_{x\in\mathbb{Z}^N} \|a_\alpha(x) - a^{(n)}_\alpha(x)\| + \limsup_{x\to\infty} \|a^{(n)}_\alpha(x)\| \\
&\le \|A - A_n\| + \|\Gamma(A_n)\|. \qquad (26)
\end{aligned}$$

This shows that every diagonal of A lies in $c_0(\mathbb{Z}^N, L(H))$, which on its hand implies that all limit operators of A are 0: Indeed, let h be a sequence for which the limit operator A_h exists. Then the operators $S_i V_{-h(n)} A V_{h(n)} S_j$ converge in the norm to $S_i A_h S_j$ for every pair of indices $i, j \in \mathbb{Z}^N$. Since

$$\lim_{n\to\infty} \|S_i V_{-h(n)} A V_{h(n)} S_j\| = \lim_{n\to\infty} \|a_{i-j}(i + h(n))\| = 0$$

due to (26), this shows that $S_i A_h S_j = 0$ for all i and j, whence $A_h = 0$. But a rich band-dominated operator having 0 as its only limit operator lies in $K(E, \mathcal{P})$ due to Theorems 2.24 and 2.24 in [6]. ∎

Now we can present an alternative proof of Theorem 7.2.

Theorem 8.3 *The following assertions are equivalent for* $A \in \mathcal{A}_E(SO^{rich}_{L(H)})$*:*

(a) *A is $\mathcal{P}$-Fredholm.*
(b) *$\Gamma(A)$ is invertible in $\widehat{C}_b$.*
(c) *All limit operators of A are invertible.*

Proof. The equivalence of (a) and (b) is quite obvious: If the coset $A + K(E, \mathcal{P})$ is invertible in $L(E, \mathcal{P})/K(E, \mathcal{P})$, then it is also invertible in $\mathcal{A}_E(SO^{rich}_{L(H)})/K(E, \mathcal{P})$ (inverse closedness of C^*-algebras). Hence, there are operators $B \in \mathcal{A}_E(SO^{rich}_{L(H)})$ and $K_1, K_2 \in K(E, \mathcal{P})$ such that $AB = I + K_1$ and $BA = I + K_2$. Applying the homomorphism Γ to these equalities yields invertibility of $\Gamma(A)$. If, conversely, $\Gamma(A)$ is invertible in $\widehat{C}_b$, then it is also invertible in the image of $\mathcal{A}_E(SO^{rich}_{L(H)})$ under the mapping Γ (again by the inverse closedness of C^*-algebras). Thus, one can find a $B \in \mathcal{A}_E(SO^{rich}_{L(H)})$ with $\Gamma(A)\Gamma(B) = \Gamma(B)\Gamma(A) = 1$, showing that $AB - I$ and $BA - I$ belong to $\ker \Gamma = K(E, \mathcal{P})$.

Since (a) obviously implies (c) (see also the first lines of the proof of Theorem 1 in [5]), we are left with the implication $(c) \Rightarrow (b)$. Assume that all limit operators of $A \in \mathcal{A}_E(SO^{rich}_{L(H)})$ are invertible, but that $\Gamma(A)$ is not invertible in $\widehat{C}_b$. If A is not a band operator, then we let gen_A be any function in the coset $\Gamma(A)$.

We define the lower norm of an operator $C \in L(H)$ by $\nu(C) := \inf_{x\neq 0} \|Cx\|/\|x\|$. It is well known that C is invertible if both $\nu(C)$ and $\nu(C^*)$ are positive and that, conversely, invertibility of C implies $\nu(C) = \nu(C^*) = 1/\|A^{-1}\|$. Thus, if both

$$\lim_{R\to\infty} \inf_{|x|\geq R,\, t\in\mathbb{T}^N} \nu(\text{gen}_A(x, t)) > 0 \tag{27}$$

and

$$\lim_{R\to\infty} \inf_{|x|\geq R,\, t\in\mathbb{T}^N} \nu(\text{gen}_A(x, t)^*) > 0, \tag{28}$$

then the function gen_A is invertible in C_b modulo functions in c_0. Since $\Gamma(A)$ is non-invertible by assumption, one of the conditions (27) and (28) must be violated, say the first one for definiteness. Then there exist a sequence $x = (x_m)_{m\geq 1} \subset \mathbb{Z}^N$ which tends to infinity, a sequence $(t_m)_{m\geq 1} \subset \mathbb{T}^N$ which we can also suppose to be convergent to a point $t_0 \in \mathbb{T}^N$, as well as a sequence $(v_m)_{m\geq 1}$ of unit vectors in H such that

$$\|\text{gen}_A(x_m, t_m)v_m\| \to 0 \quad \text{as } m \to \infty.$$

We will further suppose without loss that the limit operator A_x of A with respect to the sequence x exists.

Let $\varepsilon < 1/(4\,\|A_x^{-1}\|)$, and let A' be a band operator with coefficients in $SO^{rich}_{L(H)}$ such that $\|A - A'\| < \varepsilon$. Then $\|\Gamma(A) - \Gamma(A')\|_0 < \varepsilon$, which implies that

$$\limsup_{m\to\infty} \|\text{gen}_{A'}(x_m, t_m)v_m\|$$

$$
\begin{aligned}
&\leq \limsup_{m\to\infty} \|\mathrm{gen}_{A'}(x_m,t_m)v_m - \mathrm{gen}_A(x_m,t_m)v_m\| + \lim_{m\to\infty} \|\mathrm{gen}_A(x_m,t_m)v_m\| \\
&\leq \limsup_{m\to\infty} \sup_{t\in\mathbb{T}^N} \|\mathrm{gen}_{A'}(x_m,t) - \mathrm{gen}_A(x_m,t)\| \\
&= \|\Gamma(A) - \Gamma(A')\|_0 < \varepsilon.
\end{aligned}
$$

Hence, $\|\mathrm{gen}_{A'}(x_m,t_m)v_m\| < \varepsilon$ for all sufficiently large m. We further suppose without loss that the limit operator of A' with respect to the sequence x exists (otherwise Let $\varepsilon < 1/(4\,\|A_x^{-1}\|)$, and let A' be a band operator with coefficients in $SO^{rich}_{L(H)}$ such that $\|A - A'\| < \varepsilon$. Then $\|\Gamma(A) - \Gamma(A')\|_0 < \varepsilon$, which implies that

$$
\begin{aligned}
&\limsup_{m\to\infty} \|\mathrm{gen}_{A'}(x_m,t_m)v_m\| \\
&\quad\leq \limsup_{m\to\infty} \|\mathrm{gen}_{A'}(x_m,t_m)v_m - \mathrm{gen}_A(x_m,t_m)v_m\| + \lim_{m\to\infty} \|\mathrm{gen}_A(x_m,t_m)v_m\| \\
&\quad\leq \limsup_{m\to\infty} \sup_{t\in\mathbb{T}^N} \|\mathrm{gen}_{A'}(x_m,t) - \mathrm{gen}_A(x_m,t)\| \\
&\quad= \|\Gamma(A) - \Gamma(A')\|_0 < \varepsilon.
\end{aligned}
$$

Hence, $\|\mathrm{gen}_{A'}(x_m,t_m)v_m\| < \varepsilon$ for all sufficiently large m. We further suppose without loss that the limit operator of A' with respect to the sequence x exists (otherwise we pass to a suitable subsequence of x). As in the proof of Proposition 8.1, we can find a unit vector $u \in l^2$ such that

$$\|V_{-x_m} A' V_{x_m}(v_m \otimes u)\| < 2\varepsilon \quad \text{for all sufficiently large } m$$

and, according to the definition of limit operators, we further have

$$\|(V_{-x_m} A' V_{x_m} - A'_x)(v \otimes u)\| \to 0$$

uniformly with respect to the unit vectors v. Hence, $\|A'_x(v_m \otimes u)\| < 3\varepsilon$ for all sufficiently large m. Since $\|v_m \otimes u\| = 1$, we conclude that

$$\text{either } A'_x \text{ is not invertible or } \|(A'_x)^{-1}\| > 1/(3\varepsilon). \tag{29}$$

On the other hand,

$$\|A_x - A'_x\| \leq \|A - A'\| < \varepsilon < 1/(4\,\|A_x^{-1}\|).$$

Thus, by a Neumann series argument, A'_x is invertible, and

$$\|(A'_x)^{-1}\| \leq \frac{\|(A_x)^{-1}\|}{1 - \|(A_x)^{-1}\|\,\|A_x - A'_x\|} \leq \frac{\|(A_x)^{-1}\|}{1 - \varepsilon\,\|(A_x)^{-1}\|}.$$

Together with (29), this yields

$$\frac{1}{3\varepsilon} < \frac{\|(A_x)^{-1}\|}{1 - \varepsilon\,\|(A_x)^{-1}\|}$$

or, equivalently, $\varepsilon > 1/(4\,\|(A_x)^{-1}\|$. The obtained estimate contradicts the choice of ε. ∎

In a similar way, the following refinement of the local Fredholm criterion (Theorem 1.6 and its corollary) can be derived.

Theorem 8.4 *The following assertions are equivalent for $A \in \mathcal{A}_E(SO^{rich}_{L(H)})$:*

(*a*) *A is locally invertible at $\eta \in S^{N-1}$.*
(*b*) *The local coset $\pi_\eta(A)$ is invertible.*
(*c*) *All operators in local operator spectrum $\sigma_\eta(A)$ of A are invertible.*

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Author's addresses:

V. S. Rabinovich, Instituto Politechnico National, ESIME-Zacatenco, Ed.1, 2-do piso, Av.IPN, Mexico, D.F., 07738

email:rabinov@maya.esimez.ipn.mx

S. Roch, Technical University Darmstadt, Faculty of Mathematics, Schlossgartenstrasse 7, 64289 Darmstadt, Germany

email: roch@mathematik.tu-darmstadt.de