Greechie diagrams of orthomodular partial algebras

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Introduction

Greechie diagrams are a well known graphical representation of orthomodular partial algebras, orthomodular posets and orthomodular lattices. In [K83] and [D84] some characterisations of Greechie diagrams of orthomodular posets and of orthomodular lattices are given under some assumptions, for example, that the family of blocks is pasted, or that the intersection of each pair of blocks contains less or equal than four elements. Now I am going to present a generalisation of these characterisations for orthomodular partial algebras (or equivalently orthomodular posets see [BM94]). Here we consider arbitrary hypergraphs with finite lines. A Greechie diagram can be seen as a special hypergraph: Different points of the hypergraph have different interpretations in the corresponding partial algebra $\underline{A} := (A; \oplus, ', 0)$ of type (2,1,0) and each line in the hypergraph has a maximal Boolean subalgebra as interpretation, in which the points are the atoms. A diagram is complete if each maximal Boolean subalgebra is induced by a line of the hypergraph. The characterisation theorems in chapter 2 provide conditions to check, whether a hypergraph is a complete diagram of an orthomodular partial algebra. This poperty can be checked without having to compute the interpretation. We just have to consider the lines in the hypergraph.

1 Blocks of orthomodular partial algebras

Definition 1 An *orthomodular partial algebra* (briefly: *OMA*) is a partial algebra $\underline{A} := (A; \oplus, ', 0)$ of type $(2, 1, 0)$ such that the following axioms hold in $\underline{A}$ (the term existence statement $t \overset{e}{=} t$ is written as $\mathbb{D}(t)$):

(A0) $\mathbb{D}(0)$

(A1) $x'' \overset{e}{=} x$

(A2) $x \oplus x' \overset{e}{=} 0'$

(A3) $x \oplus 0 \overset{e}{=} x$

(A4) $\mathbb{D}(x \oplus y) \Rightarrow x \oplus y \overset{e}{=} y \oplus x$

(A5) $\mathbb{D}((x \oplus y) \oplus z) \Rightarrow (x \oplus y) \oplus z \stackrel{e}{=} x \oplus (y \oplus z)$

(A6) $\mathbb{D}(x \oplus y) \wedge \mathbb{D}(y' \oplus z) \Rightarrow \mathbb{D}(x \oplus z)$

(A7) $\mathbb{D}(x \oplus y') \wedge \mathbb{D}(x' \oplus y) \Rightarrow x \stackrel{e}{=} y$

(A8) $\mathbb{D}(x \oplus y) \wedge \mathbb{D}(y \oplus z) \wedge \mathbb{D}(x \oplus z) \Rightarrow \mathbb{D}(x \oplus (y \oplus z))$

(A9) $\mathbb{D}(x \oplus y') \Rightarrow x \oplus (x \oplus y')' \stackrel{e}{=} y$

Note that axioms (A3) and (A6) are consequences of the other axioms (see [Pu94]). When we use different partial algebras $\underline{A}, \underline{B}, \ldots$ then sometimes we write the algebra as index of the operations $(\oplus_{\underline{A}}, \oplus_{\underline{B}}, \ldots)$ to make clear which operation is meant. In an OMA $\underline{A}$ we define $1 := 0'$. There exists a canonical bijection between the class of all OMAs and the class of all orthomodular posets (see [BM94]): For every OMA $\underline{A}$ the structure $(A, \leq, ', 0)$ with $x \leq y$ iff $x \oplus y'$ exists is an orthomodular poset and for every orthomodular poset $(B, \leq, ', 0)$ the structure $(B, \oplus, ', 0)$ with $x \oplus y = z$ iff $x \leq y'$ and $z = \sup(x, y)$ is an OMA. These transformations are invers to each other. We have $x \leq y$ iff there is an element $z \in A$ with $x \oplus z = y$. Note that if $x \oplus y$ exists for $x, y \in A$ then $\inf(x, y) = 0$ (see [BM94]). The induced order $\leq_{\underline{B}}$ of a subalgebra $\underline{B}$ (which is always an OMA because the axioms are open formulas) is the restriction of the order $\leq_{\underline{A}}$. An OMA $\underline{A}$ is called *Boolean* iff the corresponding orhomodular poset $(A, \leq, ', 0)$ is a Boolean lattice. For an OMA $\underline{A}$ let $atoms(\underline{A})$ be the set of all atoms of the induced orthomodular poset $(A, \leq, ', 0)$.

For a set M let $\underline{\mathcal{P}(M)} := (\mathcal{P}(M), \oplus, ', \emptyset)$ be the Boolean OMA with the powerset of M as carrier set, $E' = M \setminus E$ and $E \oplus F = G$ iff G is the disjoint union of E and F for $E, F, G \subseteq M$. Let $\underline{\mathcal{P}^{\text{cofin}}_{\text{fin}}(M)} := (\mathcal{P}^{\text{cofin}}_{\text{fin}}(M), \oplus, ', \emptyset)$ be the Boolean subalgebra of $\underline{\mathcal{P}(M)}$ with $\mathcal{P}^{\text{cofin}}_{\text{fin}}(M) = \{E \subseteq M | E \text{ is finite or } M \setminus E \text{ is finite } \}$. If M is finite then we have $\underline{\mathcal{P}^{\text{cofin}}_{\text{fin}}(M)} = \underline{\mathcal{P}(M)}$.

For a partial algebra $\underline{A} = (A, \oplus, ', 0)$ the *cardinality* $|\underline{A}|$ of $\underline{A}$ is defined as the cardinality of the carrierset A: $|\underline{A}| := |A|$. A maximal Boolean subalgebra of a partial algebra $\underline{A} = (A, \oplus, ', 0)$ is called *block*. The subalgebra which is generated by a subset $E \subseteq A$ is denoted by $< E >$.

For a family $(\underline{A_i})_{i \in I}$ of OMAs the coproduct $\underline{C} = \coprod_{i \in I} \underline{A_i}$ in the category of partial algebras is a "0-1-gluing", that means all zero elements of the OMAs are identified

$$in_i(0_{\underline{A_i}}) = 0_{\underline{C}} = in_j(0_{\underline{A_j}}) \text{ for } i, j \in I,$$

where $in_i : \underline{A_i} \to \underline{C}$ is the canonical injection into the coproduct, and all units of these OMAs are identified

$$in_i(1_{\underline{A_i}}) = 1_{\underline{C}} = in_j(1_{\underline{A_j}}) \text{ for } i, j \in I.$$

All other elements remain unequal: $in_i(a) \neq in_j(b)$ for all $i, j \in I, a \in \underline{A_i}$ and $b \in \underline{A_j}$ with $(i, a) \neq (j, b)$ and $0_{\underline{A_i}} \neq a \neq 1_{\underline{A_i}}$.

Now we show some properties of OMAs.

Lemma 2 *Let $\underline{A}$ be a Boolean OMA, $a \in atoms(\underline{A})$ and $b \in \underline{A}$. Then $a \leq b$ or $a \leq b'$ holds. If b is also an atom with $a \neq b$ then $a \oplus b$ exists.*

Proof. If $a \not\leq b$ then, because of the distributivity of a Boolean lattice,

$$a = \inf(a, \sup(b, b')) = \sup(\inf(a, b), \inf(a, b')) = \sup(0, \inf(a, b')) = \inf(a, b'),$$

so $a \leq b'$. If b is an atom with $a \neq b$ then $a \not\leq b$ and $a \leq b'$, so $a \oplus b$ exists. ■

The following theorem is a generalisation of a remark in [BM94]:

Theorem 3 *Let $\underline{A}$ be an OMA and $E \subseteq A$. Then the following conditions are equivalent:*

1. *$a \oplus b$ exists for all $a, b \in E$ with $a \neq b$.*
2. *There exists an isomorphism $\phi : \underline{P} := \underline{\mathcal{P}^{cofin}_{fin}(G)} \to < E >$ with*

$$\phi(F) = \bigoplus F \text{ and } \phi(G \setminus F) = (\bigoplus F)'$$

   for finite subsets $F \subseteq G$, where $G := (E \cup \{(\bigoplus E)'\}) \setminus \{0\}$ if E is finite and $G := E \setminus \{0\}$ if E is infinite.
3. *E generates a Boolean subalgebra of $\underline{A}$ with $E \subseteq atoms(< E >) \cup \{0\}$.*

Proof.
$1 \to 2$:
Because of the axioms (A4), (A5) and (A8) the sum $\bigoplus F$ exists and is welldefined for all finite subsets $F \subseteq E$. Therefore the set G in condition 2 is welldefined. Let $\phi : \underline{P} \to < E >$ be given with $\phi(F) := \bigoplus F$ and $\phi(G \setminus F) = (\bigoplus F)'$ for finite subsets $F \subseteq G$. The function ϕ is welldefined because if F and $G \setminus F$ are finite then

$$\bigoplus F \oplus \bigoplus (G \setminus F) = \bigoplus G = \bigoplus E \oplus (\bigoplus E)' = 1$$

and therefore $\bigoplus F = (\bigoplus (G \setminus F))'$ because of the uniqueness of the complement (see [BM94]). Obviosly $\phi(F) \in < E >$ holds for all $F \in P$. The mapping ϕ is compatible with $'$ and 0.
Compability with $\oplus$:
Let $F_1, F_2 \in P$ be such that $F_1 \oplus_{\underline{P}} F_2$ exists. Then $F_1 \cap F_2 = \emptyset$ and F_1 or F_2 must be finite. If both are finite then we have $\phi(F_1 \oplus_{\underline{P}} F_2) = \bigoplus (F_1 \cup F_2) = \phi(F_1) \oplus \phi(F_2)$.

Now assume that F_1 is finite and F_2 is infinite. Then $\bigoplus F_1 \oplus (\bigoplus(G \setminus F_2))'$ exists because $F_1 \subseteq G \setminus F_2$ holds, so

$$(\phi(F_1) \oplus \phi(F_2)) \oplus (\phi(F_1 \oplus_{\underline{P}} F_2))' =$$

$$\bigoplus F_1 \oplus (\bigoplus(G \setminus F_2))' \oplus \bigoplus(G \setminus (F_1 \oplus_{\underline{P}} F_2)) =$$

$$(\bigoplus(G \setminus F_2))' \oplus \bigoplus(G \setminus F_2) = 1$$

so we get $\phi(F_1) \oplus \phi(F_2) = \phi(F_1 \oplus_{\underline{P}} F_2)$ because of the uniqueness of the complement. Analogously for infinite F_1 and finite F_2. Therefore ϕ is a homomorphism.

Closedness of ϕ:

Let $F_1, F_2 \in P$ such that $\phi(F_1) \oplus \phi(F_2)$ exists. Then we have $\inf(\phi(F_1), \phi(F_2)) = 0$. Assume that there exists an element $a \in F_1 \cap F_2$. If F_1 is finite then we get

$$a \leq \bigoplus F_1 = \phi(F_1)$$

and if F_1 is infinite then we get

$$a \leq (\bigoplus(G \setminus F_1))' = \phi(F_1)$$

because of the existence of $a \oplus \bigoplus(G \setminus F_1)$. Analogously we get $a \leq \phi(F_2)$ which is a contradiction to $\inf(\phi(F_1), \phi(F_2)) = 0 \neq a$. Therefore $F_1 \cap F_2 = \emptyset$ holds and $F_1 \oplus_{\underline{P}} F_2$ exists.

Injectivity of ϕ:

Let $F_1, F_2 \in P$ with $\phi(F_1) = \phi(F_2)$.

Case 1: F_1 and F_2 are finite.

Let $a \in F_1$. Then $\bigoplus F_1 \oplus a$ does not exist and therefore $\bigoplus F_2 \oplus a$ does not exist because of $\bigoplus F_2 = \phi(F_2) = \phi(F_1) = \bigoplus F_1$. Therefore $a \in F_2$. So we get $F_1 \subseteq F_2$ and analogously $F_2 \subseteq F_1$, so we have $F_1 = F_2$.

Case 2: F_2 is infinite.

If F_1 is finite then we get

$$\bigoplus F_1 = \phi(F_1) = \phi(F_2) = (\bigoplus(G \setminus F_2))',$$

so $\bigoplus F_1 \oplus \bigoplus(G \setminus F_2) = 1$. But the set $F_1 \cup (G \setminus F_2)$ is finite, so there exists an element $a \in G$ with $a \notin F_1 \cup (G \setminus F_2)$, and with axiom (A8) the sum

$$\bigoplus(F_1 \cup (G \setminus F_2) \cup \{a\}) = 1 \oplus a$$

exists, which is a contradiction to $0 \notin G$. So F_1 must be infinite, and because of $\phi(G \setminus F_1) = \phi(F_1)' = \phi(F_2)' = \phi(G \setminus F_2)$ we get $G \setminus F_1 = G \setminus F_2$ like in case 1.

Case 3: F_1 is infinite.

Analogously Case 2.
Therefore ϕ is injective.
Surjectivity of ϕ:
$\phi(P)$ is a subalgebra of $\underline{A}$ because ϕ is a closed homomorphism.
We have $E \setminus \{0\} \subseteq \phi(P)$ and therefore $< E > \subseteq \phi(P)$, so ϕ is surjective and an isomorphism.
$2 \to 3$:
$\{g\}$ is an atom of the Boolean algebra $\underline{\mathcal{P}_{\text{fin}}^{\text{cofin}}(G)}$ for $g \in G$, so $\phi(\{g\}) = g$ is an atom of the Boolean algebra $\underline{< E >}$. Therefore $E \subseteq atoms(< E >) \cup \{0\}$ holds.
$3 \to 1$: Lemma 2 and axioms (A3) and (A4). ■

Note that if $E = atoms(\underline{B})$ holds for a Boolean subalgebra $\underline{B} \leq \underline{A}$ which is generated by $atoms(\underline{B})$ then we get $G = E$ in this theorem, so $\underline{B} \cong \underline{\mathcal{P}_{\text{fin}}^{\text{cofin}}(E)}$. This theorem is usefull to find Boolean subalgebras and blocks in an OMA. It also provides a characterisation of Boolean OMAs that are generated by the atoms:

Corollary 4 *Every Boolean OMA $\underline{A}$ which is generated by $atoms(\underline{A})$ is (up to isomorphy) of the form $\underline{\mathcal{P}_{fin}^{cofin}(E)}$ for a set E.*

Proof.
Use $E := atoms(\underline{A})$ in Theorem 3. ■

In the following theorem we use Theorem 3 to show that the atoms of an OMA in which each block is generated by the atoms are exactly the atoms of the blocks occuring in the OMA.

Theorem 5 *Let $\underline{A}$ be an OMA, such that each block $\underline{B} \leq \underline{A}$ is generated by $atoms(\underline{B})$. Then*

$$\bigcup \{atoms(\underline{B}) | \underline{B} \text{ block of } \underline{A}\} = atoms(\underline{A})$$

holds.

Proof.
"$\subseteq$":
Let $\underline{B}$ be a block of $\underline{A}$ and $x \leq z \in atoms(\underline{B})$. Now it will be shown that $x \in \{0, z\}$ holds. $y := (x \oplus z')'$ exists and $x \oplus y = z$ holds because of axiom (A9).
Let $E := (atoms(\underline{B}) \setminus \{z\}) \cup \{x, y\}$. For $e \in atoms(\underline{B}) \setminus \{z\}$ the sums $x \oplus e$ and $y \oplus e$ exist because of the axioms (A5) and (A4) and the existence (see Lemma 2) of $z \oplus e = (x \oplus y) \oplus e$. So for all $a, b \in E$ with $a \neq b$ the sum $a \oplus b$ exists, and because of Theorem 3 E generates a Boolean subalgebra $\underline{C}$. We have $atoms(\underline{B}) \setminus \{z\} \subseteq C$ and

$z = x \oplus y \in C$, so $B \subseteq C$ because B is generated by $atoms(\underline{B})$. Therefore $\underline{B} = \underline{C}$ holds because $\underline{B}$ is a maximal Boolean subalgebra. So we have $x, y \in \underline{B}$ and therefore $x \in \{0, z\}$, and z is an atom of $\underline{A}$.
"$\supseteq$":
Now assume $a \in atoms(\underline{A})$. Then $\{0, 1, a, a'\}$ is a Boolean subalgebra which contains a. With the lemma of Zorn there exists a maximal Boolean subalgebra $\underline{B} \leq \underline{A}$ which contains a. Of course $a \in atoms(\underline{B})$ holds because a is an atom of $\underline{A}$. So we get $a \in \bigcup\{atoms(\underline{B}) | \underline{B} \text{ block of } \underline{A}\}$. ∎

A consequence of this theorem is, that every OMA $\underline{A}$ in which each block is generated by its atoms, is generated by $atoms(\underline{A})$, because each element $a \in A$ is in a block $\underline{B} \leq \underline{A}$ and therefore a is generated by $atoms(\underline{B}) \subseteq atoms(\underline{A})$.

2 Greechie diagrams

Greechie diagrams are used as a graphical representation of OMAs.

Definition 6 A *hypergraph*[1] $D = (P, R)$ consists of a set P and a system $R \subseteq \mathcal{P}(P)$ of sets with $\bigcup R = P$ and $\emptyset \notin R$. The elements of P are called points, the elements of R are called lines. The hypergraph is called *nontrivial* if $P \neq \emptyset$. Two points $a, b \in P$ are called *connected* by the line $r \in R$ if $a, b \in r$ holds. Let $\underline{C} := \coprod_{r \in R} \underline{\mathcal{P}^{\text{cofin}}_{\text{fin}}(r)}$ be the coproduct of the Boolean OMAs in the category of partial algebras. For $r \in R$ let $in_r : \underline{\mathcal{P}^{\text{cofin}}_{\text{fin}}(r)} \to \underline{C}$ be the canonical injection into the coproduct. Define $\sim_D := \{(in_r(\{a\}), in_s(\{a\})) | r, s \in R \text{ and } a \in r \cap s\}$. Let $<\!\sim_D\!>$ be the smallest congruence relation on $\underline{C}$, which contains $\sim_D$. The *interpretation* of D is defined by $[\![D]\!] := \underline{C}/ <\!\sim_D\!>$.
For $a \in P$ the interpretation of the point a is defined by $[\![a]\!] := in_r(\{a\})/ <\!\sim_D\!>$ where r is a line which contains a. Because of $P = \bigcup R$ there always exists such a line r, and because of the definition of $\sim_D$ the interpretation $[\![a]\!]$ of a is welldefined.
For $r \in R$ the interpretation of the line r is defined by

$$[\![r]\!] := in_r(\mathcal{P}^{\text{cofin}}_{\text{fin}}(r))/ <\!\sim_D\!> := \{in_r(E)/ <\!\sim_D\!> | E \in \mathcal{P}^{\text{cofin}}_{\text{fin}}(r)\}.$$

A hypergraph D is called *abstract Greechie diagram* if the following three conditions hold:
(C1) $a \neq b$ implies $[\![a]\!] \neq [\![b]\!]$ for $a, b \in P$.
(C2) $nat_{<\sim_D>} \circ in_r : \underline{\mathcal{P}^{\text{cofin}}_{\text{fin}}(r)} \to [\![r]\!]$ is an isomorphism[2] for all $r \in R$.
(C3) $[\![r]\!]$ is a block of $[\![D]\!]$ for all $r \in R$.

[1] see [Bg76]

[2] note that this condition is equivalent to the property that $nat_{<\sim_D>} \circ in_r : \underline{\mathcal{P}^{\text{cofin}}_{\text{fin}}(r)} \to [\![D]\!]$ is injective and closed

A hypergraph D is called *complete*, if for every block $\underline{B} \leq [\![D]\!]$ there exists a line $r \in R$ such that $[\![r]\!] = B$.
A hypergraph (diagram) D is called *OMA-hypergraph* (*OMA-diagram* resp.), if $[\![D]\!]$ is an OMA.

In the graphical representation of a hypergraph each line $r \in R$ connects the points $a \in r$. To distinguish between one and two lines (for example $R = \{\{a, b, c, d, e\}\}$ contains one line, that connects the elements a, b, c, d and e, but $R = \{\{a, b, c\}, \{c, d, e\}\}$ contains two lines, that connect the same elements) in the graphical representation, we consider a line $r \in R$ as a line without a corner, that means a differentiable curve. If two lines contain the same point $a \in P$, then these lines have to have different tangents at this point.

There is another possibility to define the inrepretation of a hypergraph D: Instead of the congruence relation $<\sim_D>$ we can use the congruence relation $< \sigma >$ with $\sigma := \{(in_r(E), in_s(E)) | r, s \in R \text{ and } E \in \mathcal{P}^{\text{cofin}}_{\text{fin}}(r) \cap \mathcal{P}^{\text{cofin}}_{\text{fin}}(s)\}$. If we now define $[\![D]\!] := \underline{C}/ < \sigma >$ then we get a different interpretation of the diagram (see example 2 in chapter 3). If every line of a diagram is finite, then both definitions coincide, we just get a difference if there are infinite lines. All theorems, lemmas and corollaries which are proved in this paper also hold for this new definition, except Theorem 22. In this theorem we would get some problems to prove the isomorphy $[\![D]\!] \cong [\![Comp(D)]\!]$. We do not know a counterexample for this isomorphy (with $< \sigma >$ instead of $<\sim_D>$), but because of the properties of example 2 (see chapter 3) we think Theorem 22 is wrong for this new definition of $[\![D]\!]$.

In the following in a diagram $D = (P, R)$ an element $a \in P$ is identified with the corresponding element $[\![a]\!] \in [\![D]\!]$. Because of condition (C1) and the definition of $\sim_D$ we have $a = b$ iff $[\![a]\!] = [\![b]\!]$ for $a, b \in P$, so P can be seen as a subset of $[\![D]\!]$: $P = \{[\![a]\!] | a \in P\} = \{in_r(\{a\})/ <\sim_D> | a \in r \in R\} \subseteq [\![D]\!]$.

If the elements of R are disjoint then $<\sim_D> = \text{id}_{\underline{C}}$ and $[\![D]\!] \cong \underline{C}$ hold. The trivial hypergraph $D = (\emptyset, \emptyset)$ is the only hypergraph with $|[\![D]\!]| = 0$. Obviously this hypergraph is a complete diagram. There does not exist a diagram D with $|[\![D]\!]| = 1$, because for each hypergraph $D = (P, R)$ with $|[\![D]\!]| = 1$ we get $R \neq \emptyset$, and with the condition $\emptyset \notin R$ we get the existence of $r \in R$ with $|r| \geq 1$, so we have $|\mathcal{P}^{\text{cofin}}_{\text{fin}}(r)| \geq 2$ but $|[\![r]\!]| \leq |[\![D]\!]| = 1$, so condition (C2) does not hold.

There exists up to isomorphy exactly one diagram D with $|[\![D]\!]| = 2$. This is proved in the following lemma.

Lemma 7 *Let $D = (P, R)$ be a diagram. Then the following conditions are equivalent:*

1. $|[\![D]\!]| = 2$.
2. *There exists a line $r \in R$ with $|r| = 1$.*
3. $R = \{\{a\}\}$ *for an element $a \in P$.*

4. $|P| = 1$.

Proof.
$1 \to 2$:
Because of $R \neq \emptyset$ there is a line $r \in R$. Because of the definition of hypergraphs we have $r \neq \emptyset$, and because of condition (C2) and $|[\![D]\!]| = 2$ we get $|r| = 1$.
$2 \to 3$:
Let $r \in R$ with $|r| = 1$. Because of condition (C3) the set $\{0, 1\} = [\![r]\!]$ is a maximal Boolean subalgebra, but for $s \in R$ this subalgebra is contained in $[\![s]\!]$, so we get $[\![s]\!] = \{0, 1\}$. With condition (C2) we get $|s| = 1$ and with condition (C1) we get $r = s$.
$3 \to 4$:
$P = \bigcup R = \{a\}$.
$4 \to 1$:
We have $\emptyset \notin R$, and because of $P = \bigcup R$ we have $R = \{P\}$ and $[\![D]\!] = \{0, 1\}$. ■

So a line in a diagram with $|P| > 1$ is not a singelton: $|r| > 1$ for all $r \in R$.

To check whether the interpretation of a hypergraph is an OMA we do not need to test all axioms, because some axioms are satisfied in interpretations of all hypergraphs.

Theorem 8 *Let $D = (P, R)$ be a hypergraph with $P \neq \emptyset$. Then in $[\![D]\!]$ the axioms (A0), (A1), (A2), (A3), (A4) and (A9) hold. If D is a diagram then axiom (A7) holds too.*

Proof.
Proof of (A0):
Because of $R \neq \emptyset$ there exists a line $r \in R$, so we get the existence of the constant 0 in $\underline{\mathcal{P}^{\text{cofin}}_{\text{fin}}(r)}$ and therefore in $[\![D]\!]$, so axiom (A0) holds.
Proof of (A1):
Let $x \in [\![D]\!]$. Then there exist $r \in R$ and $E \in \mathcal{P}^{\text{cofin}}_{\text{fin}}(r)$ with $x = in_r(E)/ <\sim_D>$. Because of the homomorphism $nat_{<\sim_D>} \circ in_r$ we have $x = in_r(E'')/ <\sim_D> = x''$, so axiom (A1) holds.
Proof of (A2):
Let $x \in [\![D]\!], r \in R$ and $E \in \mathcal{P}^{\text{cofin}}_{\text{fin}}(r)$ with $x = in_r(E)/ <\sim_D>$. Then we have

$$x \oplus x' = in_r(E)/ <\sim_D> \oplus in_r(E')/ <\sim_D> = in_r(E \oplus E')/ <\sim_D> = in_r(r)/ <\sim_D> = 0'$$

so axiom (A2) holds.
Proof of (A3):
Let $x \in [\![D]\!]$, $r \in R$ and $E \in \mathcal{P}^{\text{cofin}}_{\text{fin}}(r)$ with $x = in_r(E)/ <\sim_D>$. Then we have

$$x \oplus 0 = in_r(E)/ <\sim_D> \oplus in_r(\emptyset)/ <\sim_D> = in_r(E \oplus \emptyset)/ <\sim_D> = x$$

so axiom (A3) holds.
Proof of (A4):
Let $x, y \in \llbracket D \rrbracket$ such that $x \oplus y$ exists. Then, by the definition of a coproduct, there exist $r \in R$ and $E, F \in \mathcal{P}_{\text{fin}}^{\text{cofin}}(r)$ with

$$x = in_r(E)/ <\sim_D>, \ y = in_r(F)/ <\sim_D>$$

such that $E \oplus F$ exists. Therefore we have

$$x \oplus y = in_r(E \oplus F)/ <\sim_D> = in_r(F \oplus E)/ <\sim_D> = y \oplus x$$

so axiom (A4) holds.
Proof of (A9):
Let $x, y \in \llbracket D \rrbracket$ such that $x \oplus y'$ exists. Then, as above, there exist $r \in R$ and $E, F \in \mathcal{P}_{\text{fin}}^{\text{cofin}}(r)$ with

$$x = in_r(E)/ <\sim_D>, \ y' = in_r(F)/ <\sim_D>$$

such that $E \oplus F$ exists. Therefore we have

$$x \oplus (x \oplus y')' = in_r(E \oplus (E \oplus F)')/ <\sim_D> = in_r(F')/ <\sim_D> = y$$

because $\underline{\mathcal{P}_{\text{fin}}^{\text{cofin}}(r)}$ is an OMA, so axiom (A9) holds.
Proof of (A7):
Now assume that D is a diagram. Let $x, y \in \llbracket D \rrbracket$ such that $x \oplus y'$ and $x' \oplus y$ exists. Then there exist $r \in R$ and $E, F \in \mathcal{P}_{\text{fin}}^{\text{cofin}}(r)$ with $x = in_r(E)/ <\sim_D>$ and $y' = in_r(F)/ <\sim_D>$ such that $E \oplus F$ exists. Then we have $E \cap F = \emptyset$ and $x' = in_r(r \setminus E)/ <\sim_D>$ and $y = in_r(r \setminus F)/ <\sim_D>$. Because of condition (C2) and the existence $x' \oplus y$ of we get $(r \setminus E) \cap (r \setminus F) = \emptyset$ and $E = r \setminus F$, so

$$x = in_r(E)/ <\sim_D> = in_r(r \setminus F)/ <\sim_D> = y$$

and axiom (A7) holds. ∎

Lemma 9 *Let $D = (P, R)$ be a diagram and $r, s \in R$ such that $r \cap s$ is finite and $r \neq s$ holds. Then $|r \setminus s| > 1$ holds.*

Proof. The set $\llbracket r \rrbracket$ cannot be a subset of $\llbracket s \rrbracket$, because then (C3) would imply $\llbracket r \rrbracket = \llbracket s \rrbracket$, and (C2) together with (C1) would then imply $r = s$. So we have $|r \setminus s| > 0$. Now assume $|r \setminus s| = 1$. Let a be the only element of $r \setminus s$. Then we get

$$\{\llbracket b \rrbracket | a \neq b \in r\} \subseteq \llbracket s \rrbracket$$
$$\llbracket a \rrbracket = (in_r(r \cap s)/ <\sim_D>)' = (in_s(r \cap s)/ <\sim_D>)' \in \llbracket s \rrbracket$$
$$\llbracket r \rrbracket \subseteq \llbracket s \rrbracket$$

This is a contradiction to what was mentioned above. ∎

So in a diagram with finite lines a line cannot contain all but one element of another line, in particular no line is contained in another line.

The following lemma gives an easier condition than (C2) to check whether a hypergraph is a diagram:

Lemma 10 *Let $D = (P, R)$ be a hypergraph that satisfies (C1) and (C3). Then D is a diagram iff for each $a \in r \in R$ the element $[\![a]\!]$ is an atom of the block $[\![r]\!]$.*

Proof.
$\rightarrow$: Trivial
$\leftarrow$:
Let $r \in R$. Then $[\![r]\!]$ is a Boolean OMA because of condition (C3). The element $in_r(\{a\})/ <\sim_D>$ is an atom of this OMA for $a \in r$, and all these elements are different because of condition (C1). And therefore, for $E := \{in_r(\{a\})/ <\sim_D> | a \in r\}$, we have $0 \notin E$. Note that if r is finite then $(\bigoplus E)' = 1' = 0$ holds. With Theorem 3 the function

$$\phi := nat_{<\sim_D>} \circ in_r : \underline{\mathcal{P}^{\text{cofin}}_{\text{fin}}(r)} \rightarrow \underline{[\![r]\!]}$$

is an isomorphism, so (C2) holds and D is a diagram. ∎

So in the graphical representation of a diagram $D = (P, R)$ the interpretation $[\![r]\!]$ of each line $r \in R$ is a block, in which the points are the atoms and all points have different interpretations. These conditions are sufficient and necessary for the property that a hypergraph is a diagram. Another method to prove this property is given in Theorem 12. First we need a lemma:

Lemma 11 *Let $D = (P, R)$ be a hypergraph such that $(in_r(\{a\}), in_s(E)) \in <\sim_D>$ implies $E = \{a\}$ for all $a \in r \in R$ and $s \in R$ and $E \in \mathcal{P}^{cofin}_{fin}(s)$. Let $\underline{B}$ a Boolean subalgebra of $[\![D]\!]$ and $in_r(\{a\})/ <\sim_D> \in \underline{B}$ for some $a \in r \in R$. Then $in_r(\{a\})/ <\sim_D> \in atoms(\underline{B})$ holds.*

Proof. Let $y \in \underline{B}$ with $y \leq in_r(\{a\})/ <\sim_D>$. Then there exists an element $z \in \underline{B}$ with $y \oplus z = in_r(\{a\})/ <\sim_D>$. There exist $s \in R$ and $E, F \in \mathcal{P}^{\text{cofin}}_{\text{fin}}(s)$ with $y = in_s(E)/ <\sim_D>$, $z = in_s(F)/ <\sim_D>$ and $E \cap F = \emptyset$. We get $(in_r(\{a\}), in_s(E \cup F)) \in <\sim_D>$ and $E \cup F = \{a\}$. Therefore $y \in \{0, in_r(\{a\})/ <\sim_D>\}$ holds and $in_r(\{a\})/ <\sim_D>$ is an atom of $\underline{B}$. ∎

Theorem 12 *Let $D = (P, R)$ be a hypergraph such that each line of D is finite and*

$$(in_r(\{a\}), in_s(E)) \in <\sim_D> \textit{ implies } E = \{a\}$$

for $a \in r \in R$ and $E \subseteq s \in R$. Then the following conditions are equivalent:

1. *D is a diagram.*
2. *Condition (C2) holds.*
3. *Condition (C3) holds.*

Proof.
$1 \to 2$: Trivial
$2 \to 3$:
Let $r \in R$. Because of (C2) the set $\llbracket r \rrbracket$ is a Boolean subalgebra of $\llbracket D \rrbracket$. Let $\underline{B}$ be another Boolean subalgebra of $\llbracket D \rrbracket$ with $\llbracket r \rrbracket \subseteq \underline{B}$ and let $x \in \underline{B}$. The line r is finite, so there exists a finite Boolean subalgebra $\underline{C} \leq \underline{B}$ which contains $\llbracket r \rrbracket$ and x (take for example the sublattice of $\underline{B}$ which is generated by $\llbracket r \rrbracket \cup \{x, x'\}$ with respect to the lattice operations). For all $a \in r$ the element $in_r(\{a\})/ <\sim_D>$ is an atom of $\underline{C}$ because of Lemma 11. We have $\bigoplus\{in_r(\{a\})/ <\sim_D> | a \in r\} = in_r(r)/ <\sim_D> = 1$, so $atoms(\underline{C}) = \{in_r(\{a\})/ <\sim_D> | a \in r\}$ and $x \in \underline{C} = \llbracket r \rrbracket$ which proves $\underline{B} = \llbracket r \rrbracket$, so (C3) holds.
$3 \to 1$:
(C1) is satisfied because

$$in_r(\{a\})/ <\sim_D> = in_s(\{b\})/ <\sim_D> \text{ implies } a = b$$

for $a \in r \in R$ and $b \in s \in R$.
Let $r \in R$. Then $\llbracket r \rrbracket$ is a Boolean subalgebra of $\llbracket D \rrbracket$ because of (C3) and the function

$$\phi := nat_{<\sim_D>} \circ in_r : \underline{\mathcal{P}^{\text{cofin}}_{\text{fin}}(r)} \to \underline{\llbracket r \rrbracket}$$

is an OMA-homomorphism between Boolean algebras and therefore a Boolean lattice homomorphism (see [BM98]). Of course ϕ is surjective. Because of

$$(in_r(\{a\}), in_r(\emptyset)) \notin <\sim_D>$$

for $a \in r$ the equivalence class $0/kern(\phi)$ in $\underline{\mathcal{P}^{\text{cofin}}_{\text{fin}}(r)}$ cannot contain an atom of $\underline{\mathcal{P}^{\text{cofin}}_{\text{fin}}(r)}$, so ϕ is injective. A bijective lattice homomorphism between Boolean OMAs is an OMA-isomorphism, so (C2) holds and D is a diagram. ∎

This theorem states that for every hypergraph $D = (P, R)$ with finite lines in which no point $a \in P$ is equivalent to a different set $E \in \mathcal{P}^{\mathrm{cofin}}_{\mathrm{fin}}(s)$ of points, we only have to check condition (C2) (which is easier than (C3)) to decide whether D is a diagram. Later it will be shown that for every OMA-diagram in which each block is generated by its atoms $(in_r(\{a\}), in_s(E)) \in <\sim_D>$ implies $E = \{a\}$ for all $a \in r \in R$ and $s \in R$ with $E \in \mathcal{P}^{\mathrm{cofin}}_{\mathrm{fin}}(s)$ (see Theorem 17).

Lemma 13 *Let $(D_i)_{i \in I} = (P_i, R_i)_{i \in I}$ be a directed family of hypergraphs, that means for each $i, j \in I$ there exists a $k \in I$ with $R_i \cup R_j \subseteq R_k$.*
Let $D := (P, R) := (\bigcup_{i \in I} P_i, \bigcup_{i \in I} R_i)$. Then the following properties hold:

(1) $<\sim_D> = \bigcup_{i \in I} <\sim_{D_i}>$

(2) For all $r, s \in R$ and $E \in \mathcal{P}^{cofin}_{fin}(r)$ and $F \in \mathcal{P}^{cofin}_{fin}(s)$ the equality

$$in_r(E)/<\sim_D> = in_s(F)/<\sim_D>$$

holds iff there exists an $i \in I$ such that

$$in_r(E)/<\sim_{D_i}> = in_s(F)/<\sim_{D_i}>$$

holds.

(3) For all $r, s, t \in R$ and $E \in \mathcal{P}^{cofin}_{fin}(r), F \in \mathcal{P}^{cofin}_{fin}(s), G \in \mathcal{P}^{cofin}_{fin}(t)$ the equality

$$in_r(E)/<\sim_D> \oplus in_s(F)/<\sim_D> = in_t(G)/<\sim_D>$$

holds iff there exists an $i \in I$ such that

$$in_r(E)/<\sim_{D_i}> \oplus in_s(F)/<\sim_{D_i}> = in_t(G)/<\sim_{D_i}>$$

holds.

Proof.
Proof of (1):
For $R_i \subseteq R_j$ we have

$$\sim_{D_i} \subseteq \sim_{D_j} \subseteq \sim_D \quad \text{and}$$
$$\coprod_{r \in R_i} \underline{\mathcal{P}^{\mathrm{cofin}}_{\mathrm{fin}}(r)} \subseteq \coprod_{r \in R_j} \underline{\mathcal{P}^{\mathrm{cofin}}_{\mathrm{fin}}(r)} \subseteq \coprod_{r \in R} \underline{\mathcal{P}^{\mathrm{cofin}}_{\mathrm{fin}}(r)}.$$

Therefore we get $<\sim_{D_i}>\subseteq<\sim_{D_j}>\subseteq<\sim_D>$. The union of a directed family of congruence relations is a congruence relation and we have

$$\sim_D = \bigcup_{i\in I} \sim_{D_i} \subseteq \bigcup_{i\in I} <\sim_{D_i}>,$$

so we get

$$<\sim_D> = \bigcup_{i\in I} <\sim_{D_i}> .$$

Proof of (2):
(2) follows from (1).
Proof of (3):
$\leftarrow$:
Let $i \in I$ and $r, s, t \in R_i$ and $E \in \mathcal{P}_{\text{fin}}^{\text{cofin}}(r), F \in \mathcal{P}_{\text{fin}}^{\text{cofin}}(s), G \in \mathcal{P}_{\text{fin}}^{\text{cofin}}(t)$ such that

$$in_r(E)/ <\sim_{D_i}> \oplus in_s(F)/ <\sim_{D_i}> = in_t(G)/ <\sim_{D_i}>$$

holds. Then there exist $u \in R_i$ and $E_2, F_2, G_2 \in \mathcal{P}_{\text{fin}}^{\text{cofin}}(u)$ with

$$E_2 \cap F_2 = \emptyset,$$

$$E_2 \cup F_2 = G_2,$$

$$(in_r(E), in_u(E_2)) \in <\sim_{D_i}> \subseteq <\sim_D>,$$

$$(in_s(F), in_u(F_2)) \in <\sim_{D_i}> \subseteq <\sim_D>,$$

$$(in_t(G), in_u(G_2)) \in <\sim_{D_i}> \subseteq <\sim_D> .$$

Therefore $in_r(E)/ <\sim_D> \oplus in_s(F)/ <\sim_D> = in_t(G)/ <\sim_D>$ holds.
$\rightarrow$:
Let $r, s, t \in R_i$ and $E \in \mathcal{P}_{\text{fin}}^{\text{cofin}}(r), F \in \mathcal{P}_{\text{fin}}^{\text{cofin}}(s), G \in \mathcal{P}_{\text{fin}}^{\text{cofin}}(t)$ such that

$$in_r(E)/ <\sim_D> \oplus in_s(F)/ <\sim_D> = in_t(G)/ <\sim_D>$$

holds. Then there exist $u \in R$ and $E_2, F_2, G_2 \in \mathcal{P}^{\text{cofin}}_{\text{fin}}(u)$ with

$$E_2 \cap F_2 = \emptyset,$$

$$E_2 \cup F_2 = G_2,$$

$$(in_r(E), in_u(E_2)) \in <\sim_D> = \bigcup_{i \in I} <\sim_{D_i}>,$$

$$(in_s(F), in_u(F_2)) \in <\sim_D> = \bigcup_{i \in I} <\sim_{D_i}>,$$

$$(in_t(G), in_u(G_2)) \in <\sim_D> = \bigcup_{i \in I} <\sim_{D_i}> .$$

The family is directed, so there is an element $i \in I$ with

$$in_r(E)/ <\sim_{D_i}> = in_u(E_2)/ <\sim_{D_i}>,$$

$$in_s(F)/ <\sim_{D_i}> = in_u(F_2)/ <\sim_{D_i}>,$$

$$in_t(G)/ <\sim_{D_i}> = in_u(G_2)/ <\sim_{D_i}>, \text{ and therefore}$$

$$in_r(E)/ <\sim_{D_i}> \oplus in_s(F)/ <\sim_{D_i}> = in_t(G)/ <\sim_{D_i}> .$$

■

These properties are helpful to analyse infinite hypergraphs: The finite subdiagrams form a directed family of diagrams, so we can use the properties of Lemma 13 to get informations about the structure of the whole diagram while only considering finite subdiagrams. These properties are used in the following theorem to show that a union of a directed family of diagrams with finite lines is again a diagram.

Theorem 14 *Let $(D_i)_{i \in I} = (P_i, R_i)_{i \in I}$ be a directed family of diagrams in which each line is finite. Then $D := (P, R) := (\bigcup_{i \in I} P_i, \bigcup_{i \in I} R_i)$ is a diagram.*

Proof.
Proof of (C1):
Condition (C1) follows from (2) of Lemma 13 because (C1) holds for D_i.
Proof of (C2):
For $r \in R$ the injectivity of $nat_{<\sim_D>} \circ in_r$ follows from (2) of Lemma 13. The closedness follows from (3) of Lemma 13.
Proof of (C3):
Let $r \in R$. Because of (C2) the set $[\![r]\!]$ is a Boolean subalgebra of $[\![D]\!]$. Let $\underline{B}$ be

another Boolean subalgebra of $[\![D]\!]$ with $[\![r]\!] \subseteq \underline{B}$ and let $x \in \underline{B}$. The line r is finite, so there exists a finite Boolean subalgebra $\underline{C} \leq \underline{B}$ which contains $[\![r]\!]$ and x. The family is directed, so because of Lemma 13 and the finiteness of $\underline{C}$ there exists a $k \in I$ such that the set $A := \{in_r(E)/ <\sim_{D_k}> | in_r(E)/ <\sim_D> \in \underline{C}\}$ is a subalgebra which is isomorphic to $\underline{C}$. The Boolean algebra $\underline{A}$ contains $in_r(\mathcal{P}^{\text{cofin}}_{\text{fin}}(r))/ <\sim_{D_k}>$ and we get $\underline{A} = in_r(\mathcal{P}^{\text{cofin}}_{\text{fin}}(r))/ <\sim_{D_k}>$ because D_k is a diagram, therefore $x \in [\![r]\!]$. So we get $\underline{B} = [\![r]\!]$ and (C3) holds in D. ■

The Union of a directed family of OMA-hypergraph is an OMA-hypergraph:

Theorem 15 *Let $(D_i)_{i\in I} = (P_i, R_i)_{i\in I}$ be a directed family of OMA-hypergraphs. Then $D := (P, R) := (\bigcup_{i\in I} P_i, \bigcup_{i\in I} R_i)$ is an OMA-hypergraph.*

Proof. The axioms (A0)-(A4) and (A9) hold in $[\![D]\!]$ because of Theorem 8.
Proof of (A8):
Let $x = in_r(E)/ <\sim_D> \in [\![D]\!], y = in_s(F)/ <\sim_D> \in [\![D]\!], z = in_t(G)/ <\sim_D> \in [\![D]\!]$ such that $x \oplus y, y \oplus z$ and $x \oplus z$ exist. The family is directed, so with Lemma 13 there exists an $i \in I$ such that

$$in_r(E)/ <\sim_{D_i}> \oplus in_s(F)/ <\sim_{D_i}>,$$

$$in_s(F)/ <\sim_{D_i}> \oplus in_t(G)/ <\sim_{D_i}>,$$

$$in_r(E)/ <\sim_{D_i}> \oplus in_t(G)/ <\sim_{D_i}>$$

exist, so $in_r(E)/ <\sim_{D_i}> \oplus (in_s(F)/ <\sim_{D_i}> \oplus in_r(E)/ <\sim_{D_i}>)$ exists because D_i is an OMA. With Lemma 13 we get the existence of $in_r(E)/ <\sim_D> \oplus (in_s(F)/ <\sim_D> \oplus in_r(E)/ <\sim_D>)$.
Proof of (A5) and (A7):
Analogously (A8).
(A6) follows from the other axioms, so D is an OMA-hypergraph. ■

The following theorem shows, that in an OMA-diagram, in which each block is generated by its atoms the interpretation of the set of points P is the set of the atoms, which occur in a block of $[\![D]\!]$.

Theorem 16 *Let $D = (P, R)$ be an OMA-diagram such that each block of $[\![D]\!]$ is generated by its atoms. Then $P = \bigcup\{atoms(\underline{B}) | \underline{B}$ block of $[\![D]\!]\} = atoms([\![D]\!])$ holds.*

Proof.

$\bigcup\{atoms(\underline{B})|\underline{B}$ block of $[\![D]\!]\} = atoms([\![D]\!])$ holds because of Theorem 5.
Because of the conditions (C2) and (C3) each equivalence class $in_r(\{a\})/ <\sim_D>$ for $a \in r \in R$ is an atom of the block $[\![r]\!]$, so we have

$$P \subseteq \bigcup\{atoms(\underline{B})|\underline{B} \text{ block of } [\![D]\!]\}.$$

For $x = in_r(E)/ <\sim_D> \in atoms([\![D]\!])$ we get $|E| = 1$ because $0 = in_r(\emptyset)/ <\sim_D> \notin atoms([\![D]\!])$ holds and $|E| > 1$ would imply that $x = y \oplus z$ for some $y, z \in [\![D]\!]$ with $y \neq 0 \neq z$ which is a contradiction to $x \in atoms([\![D]\!])$. Therefore $x \in P$, so $atoms([\![D]\!]) \subseteq P$ holds. ■

For every complete diagram $D = (P, R)$ we also get the equality

$$P = \bigcup\{atoms(\underline{B})|\underline{B} \text{ block of } [\![D]\!]\}$$

because each block $\underline{B}$ is induced by a line r, so the atoms of $\underline{B}$ are exactly the elements of r. If the diagram is not an OMA-diagram then we do not always have an order on $[\![D]\!]$, so it does not make sense to ask whether $P = atoms([\![D]\!])$ holds. If D is a complete OMA-diagram then each block $\underline{B} \leq [\![D]\!]$ is generated by its atoms because $\underline{B}$ is induced by a line $r \in R$, so we get $P = atoms([\![D]\!])$ with Theorem 16.

Theorem 17 *Let $D = (P, R)$ be an OMA-diagram such that each block of $[\![D]\!]$ is generated by its atoms. Let $a \in r \in R$ and $s \in R$ and $E \in \mathcal{P}_{fin}^{cofin}(s)$ with $(in_r(\{a\}), in_s(E)) \in <\sim_D>$. Then $E = \{a\}$ holds.*

Proof. $in_s(E)/ <\sim_D> = in_r(\{a\})/ <\sim_D> \in P = atoms([\![D]\!])$ holds, so $|E| = 1$ like in the proof of 16, and with condition (C1) we get $E = \{a\}$. ■

This theorem shows that a point of a OMA-diagram in which each block is generated by its atoms cannot be equivalent to another set of points. Together with the following lemma this theorem can be used to prove that every two points $a, b \in P$ for which the sum $a \oplus b$ exists are connected by a line.

Lemma 18 *Let $D = (P, R)$ be a hypergraph such that $(in_r(\{a\}), in_s(E)) \in <\sim_D>$ implies $E = \{a\}$ for all $a \in r \in R$ and $s \in R$ and $E \in \mathcal{P}_{fin}^{cofin}(s)$. Let $a, b \in P$ with $a \neq b$. Then $[\![a]\!] \oplus [\![b]\!]$ exists iff there exists a line $t \in R$ with $a, b \in t$.*

Proof.
$\rightarrow$:
Let $r, s \in R$ with $a \in r$ and $b \in s$. Because of the existence of the sum

$$in_r(\{a\})/ <\sim_D> \oplus in_s(\{b\})/ <\sim_D>$$

there exist $t \in R$ and $E, F \in \mathcal{P}^{\text{cofin}}_{\text{fin}}(t)$ with $E \cap F = \emptyset$ such that

$$(in_r(\{a\}), in_t(E)) \in <\sim_D> \text{ and } (in_s(\{b\}), in_t(F)) \in <\sim_D>$$

hold, so we have $E = \{a\}$ and $F = \{b\}$.
$\leftarrow$:
The function $nat_{<\sim_D>} \circ in_t$ is a homomorphism. ■

This leads to the following theorem.

Theorem 19 *Let $D = (P, R)$ be an OMA-diagram such that each block of $[\![D]\!]$ is generated by its atoms. For $E \subseteq P$ the following conditions are equivalent:*

1. *E generates a Boolean subalgebra $\underline{B} \leq [\![D]\!]$ with $E \subseteq atoms(\underline{B})$.*
2. *For all $a, b \in E$ with $a \neq b$ the sum $[\![a]\!] \oplus [\![b]\!]$ exists in $[\![D]\!]$.*
3. *For all $a, b \in E$ there exists a line $r \in R$ with $a, b \in r$.*

Proof.
$1 \to 2$: Lemma 2
$2 \to 1$: With Theorem 3 E generates a Boolean subalgebra $\underline{B}$ with $E \subseteq atoms(\underline{B}) \cup \{0\}$ and with condition (C2) we get $E \subseteq atoms(\underline{B})$.
$2 \leftrightarrow 3$: Theorem 17 and Lemma 18. ■

In a hypergraph $D = (P, R)$ a maximal subset $E \subseteq P$ in which each pair of points is connected by a line is called clique.

Theorem 20 *If D is an OMA-diagram, such that each block of $[\![D]\!]$ is generated by its atoms then the blocks of $[\![D]\!]$ are exactly the subalgebras, that are induced by a clique. $E = atoms < E >$ holds for every clique.*

Proof.
$\to$:
For a block $\underline{B} \leq [\![D]\!]$ take $E := atoms(\underline{B}) \subseteq P$, then each pair of points in E is connected by a line. Let $E \subseteq F \subseteq P$ such that each pair of points in F is connected by a line, then with Theorem 19 F generates a Boolean subalgebra $\underline{C} \leq [\![D]\!]$ with $B \subseteq C$. The Boolean subalgebra $\underline{B}$ is maximal, so $\underline{B} = \underline{C}$ and

$$F \subseteq atoms(\underline{C}) = atoms(\underline{B}) = E$$

hold, which proves that E is a clique.
$\leftarrow$:
Let E be a clique. With Theorem 19 E generates a Boolean subalgebra $\underline{B}$ with $E \subseteq atoms(\underline{B})$. The algebra $\underline{B}$ is contained in a block $\underline{C}$. With Theorem 16 the set $F := atoms(\underline{C})$ is contained in P and with Theorem 19 each pair of points of F is connected by a line. For $e \in E$ we have $e \in atoms(\underline{B}) \subseteq C$ and with Theorem 16 e is an atom of $[\![D]\!]$ and therefore $e \in atoms(\underline{C}) = F$. So we have $E \subseteq F$ and therefore $E = F$ because E is maximal. So $\underline{B} = \underline{C}$ is a block and $E = F = atoms(\underline{C}) = atoms < E >$ holds. ■

For a diagram $D = (P, R)$ the *completion* of D is defined by $Comp(D) := (P, R_c)$ where $R_c := \{E \subseteq P | \{[\![e]\!] | e \in E\}$ is the set of all atoms of a block of $[\![D]\!]\} = \{atoms(\underline{B}) | \underline{B}$ block of $[\![D]\!]$ with $atoms(\underline{B}) \subseteq P\}$.

Corollary 21 *If D is an OMA-diagram, such that each block of $[\![D]\!]$ is generated by its atoms then $R_c = \{E \subseteq P | E$ is a clique$\}$ holds.*

Proof. The atoms of each block are contained in P because of Theorem 16 and the cliques are exactly the atoms of blocks because of Theorem 20. ■

To analyse a Greechie diagram with respect to some properties (for example whether it is an OMA-diagram) it is sometimes better if the diagram is complete. There exists a canonical isomorphism between a diagram in which each block is generated by its atoms and the completion of the diagram, which is shown in the next theorem.

Theorem 22 *Let $D = (P, R)$ be a diagram such that each block of $[\![D]\!]$ is generated by its atoms. Then $(P, R_c) := Comp(D)$ is a diagram with $R \subseteq R_c$ and $[\![D]\!] \cong [\![Comp(D)]\!]$, where the isomorphism is defined by*

$$\phi : [\![D]\!] \to [\![Comp(D)]\!], in_r(E)/ <\sim_D> \mapsto in_r(E)/ <\sim_{Comp(D)}> .$$

If $atoms(\underline{B}) \subseteq P$ holds for each block $\underline{B} \leq [\![D]\!]$ then $Comp(D)$ is complete.

Proof. We have $\emptyset \notin R_c$ because each block of D is generated by its atoms. D is a diagram, so the set $\{[\![a]\!] | a \in r\}$ for $r \in R$ is the set of all atoms of the block $[\![r]\!]$, and we have $R \subseteq R_c$. First we prove the isomorphy $[\![D]\!] \cong [\![(\bigcup(R \cup T), R \cup T)]\!]$ for every finite set $T \subseteq R_c$. Define $\{r_1, r_2, \ldots, r_n\} := T \setminus R$ with $r_i \neq r_j$ for $i \neq j$, $R_j := R \cup \{r_1, r_2, \ldots, r_j\}$ and $D_j := (P, R_j)$ for $0 \leq j \leq n$. The conditions (C1),

(C2), (C3) for D_j and the isomorphy $[\![D]\!] \cong [\![D_j]\!]$ are proved by induction:
Let $0 < j \leq n$ such that D_{j-1} is a diagram and

$$\phi_{j-1} : [\![D]\!] \to [\![D_{j-1}]\!], in_r(E)/ <\sim_D> \mapsto in_r(E)/ <\sim_{D_{j-1}}>$$

is an isomorphism.
For $a \in P$ let $s_a \in R$ with $a \in s_a$. For $r \in R_c$ the set $\{in_{s_a}(\{a\})/ <\sim_D> |a \in r\}$ is the set of all atoms of a block in $[\![D]\!]$, and because of the isomorphism ϕ_{j-1} the set $\{in_{s_a}(\{a\})/ <\sim_{D_{j-1}}> |a \in r\}$ is the set of all atoms of a block in $[\![D_{j-1}]\!]$. For $r \in R_c$ define

$$\tau_r : \underline{\mathcal{P}^{\text{cofin}}_{\text{fin}}(r)} \to [\![D_{j-1}]\!] \text{ with}$$

$$\tau_r(E) := \bigoplus\{in_{s_e}(\{e\})/ <\sim_{D_{j-1}}> |e \in E\} \text{ and}$$

$$\tau_r(r \setminus E) := (\bigoplus\{in_{s_e}(\{e\})/ <\sim_{D_{j-1}}> |e \in E\})'$$

for finite sets $E \subseteq r$. This function is welldefined and an embedding because of Theorem 3 (with $\underline{A} :=< \{in_{s_a}(\{a\})/ <\sim_{D_{j-1}}> |a \in r\} >$ as Boolean OMA), so $E \neq F$ iff $\tau_r(E) \neq \tau_r(F)$ holds for $E, F \in \mathcal{P}^{\text{cofin}}_{\text{fin}}(r)$.
Define

$$\psi : [\![D_{j-1}]\!] \to [\![D_j]\!], in_r(E)/ <\sim_{D_{j-1}}> \mapsto in_r(E)/ <\sim_{D_j}>$$

and $\phi_j := \psi \circ \phi_{j-1}$. These functions are welldefined and compatible with the operations because of $R_{j-1} \subseteq R_j$ and $\sim_{D_{j-1}} \subseteq \sim_{D_j}$, so they are homomorphisms.
Surjectivity of ψ:
Let $x \in [\![D_j]\!]$. Then there exist $r \in R_j \subseteq R_c$ and $E \in \mathcal{P}^{\text{cofin}}_{\text{fin}}(r)$ with

$$in_r(E)/ <\sim_{D_j}> = x.$$

ψ is a homomorphism, so if E is finite then

$$\begin{aligned}
\psi(\tau_r(E)) &= \\
\psi(\bigoplus\{in_{s_e}(\{e\})/ <\sim_{D_{j-1}}> |e \in E\}) &= \\
\bigoplus\{\psi(in_{s_e}(\{e\})/ <\sim_{D_{j-1}}>)|e \in E\} &= \\
\bigoplus\{in_{s_e}(\{e\})/ <\sim_{D_j}> |e \in E\} &= \\
\bigoplus\{in_r(\{e\})/ <\sim_{D_j}> |e \in E\} &= \\
in_r(E)/ <\sim_{D_j}> &= x
\end{aligned}$$

holds and if E is infinite then $r \setminus E$ is finite so we get

$$\psi(\tau_r(E)) =$$

$$\psi(\tau_r((r \setminus E)')) =$$

$$(in_r(r \setminus E)/ <\sim_{D_j}>)' = x$$

Therefore ψ is surjective.
In the following each equivalence class $\tau_{r_j}(E)$ for $E \in \mathcal{P}_{\text{fin}}^{\text{cofin}}(r_j)$ is used as a subset of $\coprod_{r \in R_j} \underline{\mathcal{P}_{\text{fin}}^{\text{cofin}}(r)}$. Define $\rho := \rho_1 \cup \rho_2 \cup \rho_3 \cup \rho_4$, where
$\rho_1 :=<\sim_{D_{j-1}}>$,
$\rho_2 := \bigcup\{\{in_{r_j}(E)\} \times \tau_{r_j}(E) | E \in \mathcal{P}_{\text{fin}}^{\text{cofin}}(r_j)\}$,
$\rho_3 := \bigcup\{\tau_{r_j}(E) \times \{in_{r_j}(E)\} | E \in \mathcal{P}_{\text{fin}}^{\text{cofin}}(r_j)\}$,
$\rho_4 := \{(in_{r_j}(E), in_{r_j}(E)) | E \subseteq r_j\}$.
Now we prove, that ρ is a congruence relation on $\coprod_{r \in R_j} \underline{\mathcal{P}_{\text{fin}}^{\text{cofin}}(u)}$:
Reflexivity of ρ:
ρ is reflexive because every pair $(in_r(E), in_r(E))$ for $r \in R_j$, $E \in \mathcal{P}_{\text{fin}}^{\text{cofin}}(r)$ is an element of ρ_1 or ρ_4.
Symmetry of ρ:
ρ is symmetrical because ρ_1, $\rho_2 \cup \rho_3$ and ρ_4 are symmetrical.
Transitivity of ρ:
Let

$$(in_r(E), in_s(F)) \in \rho \text{ and } (in_s(F), in_t(G)) \in \rho$$

with $r, s, t \in R_j$, and $E \in \mathcal{P}_{\text{fin}}^{\text{cofin}}(r), F \in \mathcal{P}_{\text{fin}}^{\text{cofin}}(s), G \in \mathcal{P}_{\text{fin}}^{\text{cofin}}(t)$. If $(in_r(E), in_s(F)) \in \rho_4$ or $(in_s(F), in_t(G)) \in \rho_4$ holds then we have $(in_r(E), in_t(G)) \in \rho$ so we just have to consider the relations ρ_1, ρ_2 and ρ_3.
Case 1: $(in_r(E), in_s(F)) \in \rho_1$
If $(in_s(F), in_t(G)) \in \rho_1$ also holds, then $(in_r(E), in_t(G)) \in \rho_1 \subseteq \rho$ because ρ_1 is transitive.
If $(in_s(F), in_t(G)) \in \rho_2$ then $in_s(F) \in (\coprod_{u \in R_{j-1}} \underline{\mathcal{P}_{\text{fin}}^{\text{cofin}}(u)}) \cap in_{r_j}(\mathcal{P}_{\text{fin}}^{\text{cofin}}(r_j)) = \{0, 1\}$ holds. For $in_s(F) = 0$ we get $F = \emptyset$ and $in_t(G) \in \tau_{r_j}(F) = \{0\}$ because condition (C2) holds for D_{j-1}. If $in_s(F) = 1$ then $F = s$ and we have $in_t(G) \in \tau_{r_j}(F) = \{1\}$. So $in_s(F) = in_t(G)$ holds and $(in_r(E), in_t(G)) \in \rho$.
If $(in_s(F), in_t(G)) \in \rho_3$ holds, then $(in_r(E), in_t(G)) \in \rho_3 \subseteq \rho$.
Case 2: $(in_r(E), in_s(F)) \in \rho_2$
If $(in_s(F), in_t(G)) \in \rho_1$ holds then $(in_r(E), in_t(G)) \in \rho_2 \subseteq \rho$.
If $(in_s(F), in_t(G)) \in \rho_2$ holds then $in_s(F) = in_t(G) \in \{0, 1\}$ and therefore we get $(in_r(E), in_t(G)) \in \rho$.
If $(in_s(F), in_t(G)) \in \rho_3$ holds then $(in_r(E), in_t(G)) \in \rho_4 \subseteq \rho$ because τ_{r_j} is injective.
Case 3: $(in_r(E), in_s(F)) \in \rho_3$

If $(in_s(F), in_t(G)) \in \rho_1 \cup \rho_3$ holds then $in_s(F) = in_t(G) \in \{0,1\}$ and therefore we get $(in_r(E), in_t(G)) \in \rho$.
If $(in_s(F), in_t(G)) \in \rho_2$ holds then $(in_r(E), in_t(G)) \in \rho_1 \subseteq \rho$.
Therefore ρ is transitive.
Compatibility with $'$:
ρ_1 and ρ_4 are compatible with $'$.
Let $(in_r(E), in_s(F)) \in \rho_2$. Then $r = r_j$ and $in_s(F) \in \tau_{r_j}(E)$ hold and therefore $in_s(F)' \in \tau_{r_j}(r \setminus E)$ because τ_{r_j} is an homomorphism. So we have

$$(in_r(E)', in_s(F)') = (in_r(r \setminus E), in_s(F)') \in \rho_2 \subseteq \rho$$

and analogously for ρ_3, so ρ is compatible with the operation $'$.
Compatibility with $\oplus$:
Let $(in_r(E), in_s(F)) \in \rho$ and $(in_t(G), in_u(H)) \in \rho$ such that $in_r(E) \oplus in_t(G)$ and $in_s(F) \oplus in_u(H)$ exist.
Case 1: $(in_r(E), in_s(F)) \in \rho_1$
If $(in_t(G), in_u(H)) \in \rho_1$ holds then $(in_r(E) \oplus in_t(G), in_s(F) \oplus in_u(H)) \in \rho_1 \subseteq \rho$.
If $(in_t(G), in_u(H)) \in \rho_2$ then $in_r(E) = 0$ or $in_t(G) = 0$ because $in_r(E) \oplus in_t(G)$ exists. Therefore we get

$$(in_r(E), in_s(F)) = (0,0) \text{ or } (in_t(G), in_u(H)) = (0,0).$$

So $(in_r(E) \oplus in_t(G), in_s(F) \oplus in_u(H)) \in \rho$.
Analogiously for $(in_t(G), in_u(H)) \in \rho_3 \cup \rho_4$.
Case 2: $(in_r(E), in_s(F)) \in \rho_2$
If $(in_t(G), in_u(H)) \in \rho_1 \cup \rho_3 \cup \rho_4$ we have

$$(in_r(E), in_s(F)) = (0,0) \text{ or } (in_t(G), in_u(H)) = (0,0)$$

so $(in_r(E) \oplus in_t(G), in_s(F) \oplus in_u(H)) \in \rho$.
If $(in_t(G), in_u(H)) \in \rho_2$ then $r = r_j = t, E \cap G = \emptyset$ and $in_s(F) \in \tau_{r_j}(E)$ and $in_u(H) \in \tau_{r_j}(G)$ and therefore $in_s(F) \oplus in_u(H) \in \tau_{r_j}(E \oplus G)$ because τ_{r_j} is an homomorphism. So we have $(in_r(E) \oplus in_t(G), in_s(F) \oplus in_u(H)) \in \rho_2 \subseteq \rho$.
Case 3: $(in_r(E), in_s(F)) \in \rho_3$
Analogously case 2.
Case 4: $(in_r(E), in_s(F)) \in \rho_4$
If $(in_t(G), in_u(H)) \in \rho_1 \cup \rho_2 \cup \rho_3$ we have

$$(in_r(E), in_s(F)) = (0,0) \text{ or } (in_t(G), in_u(H)) = (0,0),$$

so $(in_r(E) \oplus in_t(G), in_s(F) \oplus in_u(H)) \in \rho$.
If $(in_t(G), in_u(H)) \in \rho_4$ then $(in_r(E) \oplus in_t(G), in_s(F) \oplus in_u(H)) \in \rho_4 \subseteq \rho$ hold.
So ρ is a congruence relation on $\coprod_{r \in R_j} \underline{\mathcal{P}_{\text{fin}}^{\text{cofin}}(r)}$.

Let $(in_r(\{a\}), in_s(\{a\})) \in\sim_{D_j}$.
If $r \neq r_j \neq s$ holds, then $(in_r(\{a\}), in_s(\{a\})) \in \rho_1 \subseteq \rho$.
If $r = r_j \neq s$ holds, then $(in_r(\{a\}), in_s(\{a\})) \in \rho_2 \subseteq \rho$.
If $r \neq r_j = s$ holds, then $(in_r(\{a\}), in_s(\{a\})) \in \rho_3 \subseteq \rho$.
If $r = r_j = s$ holds, then $(in_r(\{a\}), in_s(\{a\})) \in \rho_4 \subseteq \rho$.
So we have $\sim_{D_j} \subseteq \rho$ and therefore $<\sim_{D_j}> \subseteq \rho$.
Injectivity of ψ:
Let $in_r(E)/ <\sim_{D_{j-1}}>, in_s(F)/ <\sim_{D_{j-1}}> \in [\![D_{j-1}]\!]$ with

$$\psi(in_r(E)/ <\sim_{D_{j-1}}>) = \psi(in_s(F)/ <\sim_{D_{j-1}}>).$$

Then $(in_r(E), in_s(F)) \in <\sim_{D_j}> \subseteq \rho$ and $r \neq r_j \neq s$ hold, so we get

$$(in_r(E), in_s(F)) \in \rho_1 = <\sim_{D_{j-1}}>$$

and $in_r(E)/ <\sim_{D_{j-1}}> = in_s(F)/ <\sim_{D_{j-1}}>$ which proves the injectivity.
Closedness of ψ:
Let $in_r(E)/ <\sim_{D_{j-1}}>, in_s(F)/ <\sim_{D_{j-1}}> \in [\![D_{j-1}]\!]$ such that

$$\psi(in_r(E)/ <\sim_{D_j}>) \oplus \psi(in_s(F)/ <\sim_{D_j}>)$$

exists. Then there exist $t \in R_j$, $G, H \in \mathcal{P}^{\text{cofin}}_{\text{fin}}(t)$ with

$$G \cap H = \emptyset, (in_r(E), in_t(G)) \in <\sim_{D_j}> \text{ and } (in_s(F), in_t(H)) \in <\sim_{D_j}> .$$

The sum $\tau_t(G) \oplus \tau_t(H)$ exists because τ_t is an homomorphism,

$$\psi(\tau_t(G)) = in_t(G)/ <\sim_{D_j}> = in_r(E)/ <\sim_{D_j}> = \psi(in_r(E)/ <\sim_{D\ j-1}>) \text{ and}$$

$$\psi(\tau_t(H)) = in_t(H)/ <\sim_{D_j}> = in_s(F)/ <\sim_{D_j}> = \psi(in_s(F)/ <\sim_{D_{j-1}}>)$$

hold, and because of the injectivity of ψ we get $\tau_t(G) = in_r(E)/ <\sim_{D_{j-1}})$ and $\tau_t(H) = in_r(E)/ <\sim_{D_{j-1}}>$ and therefore ψ is closed. This proves, that ψ and $\phi_j = \psi \circ \phi_{j-1}$ are isomorphism.
D_j is a **diagram:**
Let $a \in r \in R_j, b \in s \in R_j$ with $a \neq b$. Then

$$in_r(\{a\})/ <\sim_{D_j}> = \phi_j(in_r(\{a\}/ <\sim_D>) \neq \phi_j(in_r(\{b\})/ <\sim_D>) = in_s(\{b\})/ <\sim_{D_j}>$$

holds, so (C1) is satisfied for the hypergraph D_j. Let $r \in R_j$. Then $r = atoms(\underline{B})$ for a block $\underline{B} \leq [\![D]\!]$. $\underline{B}$ is generated by $atoms(\underline{B})$ and $\phi_j(\underline{B})$ is a block of $[\![D_j]\!]$, so with Theorem 3 we get $in_r(\mathcal{P}^{\text{cofin}}_{\text{fin}}(r))/ <\sim_{D_j}> = \phi_j(\underline{B})$ which proves condition (C3). For $a \in r \in R_j$ the element $in_r(\{a\})/ <\sim_D>$ is an atom of $in_r(\mathcal{P}^{\text{cofin}}_{\text{fin}}(r))/ <\sim_D>$, so

$in_r(\{a\})/ <\sim_{D_j}>$ is an atom of $in_r(\mathcal{P}_{\text{fin}}^{\text{cofin}}(r))/ <\sim_{D_j}>$ because of the isomorphism ϕ_j. With Lemma 10 D_j is a diagram.
So $D_T := (P, R \cup T)$ is a diagram and the map

$$\phi_T : [\![D]\!] \to [\![D_T]\!], in_r(E)/ <\sim_D> \mapsto in_r(E)/ <\sim_{D_T}>$$

is an isomorphism for all finite subsets $T \subseteq R_c$.
Now we prove that $\phi : [\![D]\!] \to [\![Comp(D)]\!]$ is an isomorphism. ϕ is compatible with $\oplus, '$ and 0 because of $R_j \subseteq R_c$ and $\sim_D \subseteq \sim_{Comp(D)}$.
Surjectivity of ϕ:
Let $x \in [\![Comp(D)]\!]$. Then there exist $r \in R_c, E \in \mathcal{P}_{\text{fin}}^{\text{cofin}}(r)$ with

$$in_r(E)/ <\sim_{Comp(D)}> = x.$$

For $T := \{r\}$ there exists an element $y = in_s(F)/ <\sim_D> \in [\![D]\!]$ with

$$\phi_T(y) = in_r(E)/ <\sim_{D_T}>$$

because ϕ_T is surjective. Therefore

$$in_s(F)/ <\sim_{D_T}> = \phi_T(y) = in_r(E)/ <\sim_{D_T}> \text{ and}$$

$$\phi(y) = in_s(F)/ <\sim_{Comp(D)}> = in_r(E)/ <\sim_{Comp(D)}> = x$$

hold, which proves the surjectivity.
Injectivity of ϕ:
Let $in_r(E)/ <\sim_D>, in_s(F)/ <\sim_D> \in [\![D]\!]$ with

$$\phi(in_r(E)/ <\sim_D>) = \phi(in_s(F)/ <\sim_D>).$$

Then $(in_r(E), in_s(F)) \in <\sim_{Comp(D)}>$ and because of Lemma 13 there is a finite set $T \subseteq R_c$ with $(in_r(E), in_s(F)) \in <\sim_{D_T}>$ and because of the injectivity of ϕ_T we get $in_r(E)/ <\sim_D> = in_s(F)/ <\sim_D>$ which proves the injectivity of ϕ.
Closedness of ϕ:
This proof is analogously to the proof of the injecivity.
Therefore ϕ is an isomorphism.
$Comp(D)$ is a **diagram:**
This prove is the same as the prove for D_j (see above).
Completeness of Comp(D):
Now assume that $atoms(\underline{B}) \subseteq P$ holds for each block $\underline{B} \leq [\![D]\!]$. Let $\underline{B}$ be a block of $[\![Comp(D)]\!]$. Then $\underline{C} := \phi^{-1}(\underline{B})$ is a block of $[\![D]\!]$ and $\underline{C}$ is generated by $r := atoms(\underline{C}) \in R_c$, so $in_r(\mathcal{P}_{\text{fin}}^{\text{cofin}}(r))/ <\sim_{Comp(D)}> = B$ and therefore $Comp(D)$ is complete. ∎

For a diagram D in which each block is generated by its atoms this theorem states, that we can compute the completion without changing the interpretation. The completion is again a diagram and the interpretation of the completion is isomorphic to the interpretation of D. If $atoms(\underline{B}) \subseteq P$ holds for each block $\underline{B} \leq [\![D]\!]$ then every block of $Comp(D)$ is induced by a line $r \in R_c$.

Let D be a diagram such that each block of $[\![D]\!]$ is generated by its atoms. Then $Comp(D)$ is an OMA-diagram iff D is an OMA-diagram. If these diagrams are OMA-diagrams then $Comp(D)$ is a complete diagram with $[\![D]\!] \cong [\![Comp(D)]\!]$ and the lines of $Comp(D)$ are exactly the cliques of D (see Theorems 16 and 22 and Corollary 21).

Definition 23 *Let $\underline{A}$ be a nontrivial OMA ($|\underline{A}| > 1$) such that each block of $\underline{A}$ is generated by its atoms. Define $Diag(\underline{A}) := (P, R)$ with $P = atoms(\underline{A})$ and $R = \{atoms(\underline{B}) | \underline{B}$ block of $\underline{A}\}$.*

Note that for $Diag(\underline{A}) = (P, R)$ we get $P = \bigcup R$ because of Theorem 5. We have $\emptyset \notin R$ because the Boolean subalgebra $\{0, 1\}$ is contained in a block which is generated by its atoms. Therefore $Diag(\underline{A})$ is a hypergraph.

Lemma 24 *Let $D = (P, R)$ be a hypergraph in which each line is finite. Let $E \subseteq r \in R$ and $F \subseteq s \in R$. Let $t \in R$ such that t is the disjoint union of E and $s \setminus F$. Then $(in_r(E), in_s(F)) \in <\!\sim_D\!>$ holds.*

Proof. Because t is finite we get $E \in \mathcal{P}_{\text{fin}}^{\text{cofin}}(t)$ and $s \setminus F \in \mathcal{P}_{\text{fin}}^{\text{cofin}}(t)$ and therefore

$$in_s(F) = (in_s(s \setminus F))' <\!\sim_D\!> (in_t(s \setminus F))' = in_t(E) <\!\sim_D\!> in_r(E)$$

■

In the following theorem we use this lemma to prove that every nontrivial OMA, in which each block is finite, is induced by a complete OMA-diagram.

Theorem 25 *Let $\underline{A}$ be an nontrivial OMA in which each block is finite. Then $D := (P, R) := Diag(\underline{A})$ is a complete OMA-diagram with $\underline{A} \cong [\![D]\!]$.*

Proof. Every line $r \in R$ is finite because of the finiteness of the blocks. Let

$$\psi : \coprod_{r \in R} \underline{\mathcal{P}_{\text{fin}}^{\text{cofin}}(r)} \to \underline{A},$$

$$in_r(E) \mapsto \bigoplus E$$

for $E \subseteq r \in R$. Then ψ is a welldefined homomorphism because of Theorem 3. For $r \in R$ the restriction

$$\tau_r := \psi|_{in_r(\mathcal{P}^{\text{cofin}}_{\text{fin}}(r))} : \underline{in_r(\mathcal{P}^{\text{cofin}}_{\text{fin}}(r))} \to < r >$$

is an isomorphism because of Theorem 3.
For $(in_r(\{a\}), in_s(\{a\})) \in \sim_D$ we have $\psi(in_r(\{a\})) = a = \psi(in_s(\{a\}))$ so we get

$$\sim_D \subseteq kern(\psi)$$

and therefore $<\sim_D> \subseteq kern(\psi)$. Let $\phi : [\![D]\!] \to \underline{A}$ be the induced homomorphism with $\phi(in_r(E)/ <\sim_D>) = \psi(in_r(E))$ for $E \subseteq r \in R$.
Now we prove that $e \leq \phi(in_r(E)/ <\sim_D>)$ holds in $\underline{A}$ for all $e \in E \subseteq r \in R$. For $e \in E$ we have $in_r(\{e\}) \leq in_r(E)$ in the Boolean OMA $in_r(\mathcal{P}^{\text{cofin}}_{\text{fin}}(r))$, and because of the isomorphism τ_r we get

$$e = \tau_r(in_r(\{e\})) \leq \tau_r(in_r(E)) = \phi(in_r(E)/ <\sim_D>).$$

Surjectivity of ϕ:
Let $a \in A$. Then a is in a block $\underline{B} \leq \underline{A}$, so a is generated by $atoms(\underline{B}) =: r$ and with Theorem 3 we get $a \in \phi([\![r]\!])$. Therefore ϕ is surjective.
Injectivity of ϕ:
Let $in_r(E)/ <\sim_D>, in_s(F)/ <\sim_D> \in [\![D]\!]$ with

$$\phi(in_r(E)/ <\sim_D>) = \phi(in_s(F)/ <\sim_D>).$$

Let $e \in E$ and $g \in s \setminus F$. Then $g \leq \phi(in_s(s \setminus F)/ <\sim_D>) = \phi(in_s(F)/ <\sim_D>)'$ holds, so we get the existence of

$$\begin{aligned}\phi(in_s(F)/ <\sim_D>) \oplus g &= \phi(in_r(E)/ <\sim_D>) \oplus g \\ &= \phi(in_r(E \setminus \{e\})/ <\sim_D> \oplus in_r(\{e\})/ <\sim_D>) \oplus g \\ &= (\phi(in_r(E \setminus \{e\})/ <\sim_D>) \oplus e) \oplus g\end{aligned}$$

and with axiom (A5) we get the existence of $e \oplus g$. With Theorem 3 the set $E \cup (s \setminus F)$ generates a Boolean algebra which is contained in a block $\underline{B}$. Let $t := atoms(\underline{B}) \in R$. Each element $a \in E \cup (s \setminus F) \subseteq P$ is an atom of a block, so with theorem 5 we have $a \in atoms(\underline{A})$ and therefore $E \cup (s \setminus F) \subseteq atoms(\underline{B}) = t$. The union $E \cup (s \setminus F)$ is disjoint because of the existence of $e \oplus g$ for all $e \in E$ and $g \in s \setminus F$.

We have

$$\bigoplus(E \cup (s \setminus F)) = \bigoplus E \oplus \bigoplus(s \setminus F) =$$
$$\phi(in_t(E)/ <\sim_D>) \oplus \bigoplus(s \setminus F) =$$
$$\phi(in_r(E)/ <\sim_D>) \oplus \bigoplus(s \setminus F) =$$
$$\phi(in_s(F)/ <\sim_D>) \bigoplus(s \setminus F) =$$
$$\bigoplus s = 1$$

Therefore $E \dot\cup (s \setminus F) = atoms(\underline{B}) = t$. With Lemma 24 we get $in_r(E)/ <\sim_D>= in_s(F)/ <\sim_D>$, which proves the injectivity.

Closedness of ϕ:

Let $E \subseteq r \in R$ and $F \subseteq s \in R$, such that

$$\phi(in_r(E)/ <\sim_D>) \oplus \phi(in_s(F)/ <\sim_D>)$$

exists. With Theorem 3 the set $H := \{\phi(in_r(E)/ <\sim_D>), \phi(in_s(F)/ <\sim_D>)\}$ generates a Boolean subalgebra of $\underline{A}$ which is contained in a block $\underline{B} \leq \underline{A}$.

Let $t := atoms(\underline{B}) \in R$. The function τ_t is surjective, so there exist $E_2, F_2 \subseteq t$ with $\tau_t(in_t(E_2)) = \phi(in_r(E)/ <\sim_D>)$ and $\tau_t(in_t(F_2)) = \phi(in_s(F)/ <\sim_D>)$. The function τ_t is closed, so we get the existence of $in_t(E_2) \oplus in_t(F_2)$ and because of the homomorphism $nat_{<\sim_D>}$ we get the existence $in_t(E_2)/ <\sim_D> \oplus in_t(F_2)/ <\sim_D>$. We have

$$\phi(in_t(E_2)/ <\sim_D>) = \tau_t(in_t(E_2)) = \phi(in_r(E)/ <\sim_D>) \text{ and}$$
$$\phi(in_t(F_2)/ <\sim_D>) = \tau_t(in_t(F_2)) = \phi(in_s(F)/ <\sim_D>),$$

and because of the injectivity of ϕ we get $in_t(E_2)/ <\sim_D>= in_r(E)/ <\sim_D>$ and $in_t(F_2)/ <\sim_D>= in_s(F)/ <\sim_D>$, so ϕ is closed.

Therefore ϕ is an isomorphism.

D is a **diagram:**

For $a \in r \in R$ and $b \in s \in R$ with $a \neq b$ we get

$$\phi(in_r(\{a\})/ <\sim_D>) = a \neq b = \phi(in_s(\{b\})/ <\sim_D>),$$

so $in_r(\{a\})/ <\sim_D>\neq in_s(\{b\})/ <\sim_D>$ which proves (C1). For $r \in R$ the function $nat_{<\sim_D>} \circ in_r = \phi^{-1} \circ \tau_r \circ in_r$ is closed and injective, so (C2) holds.

Let $r \in R$ and $\underline{B} \leq \underline{A}$ be the block generated by $r = atoms(\underline{B})$.

Then $\phi^{-1}(B) = [\![r]\!]$ is a block of $[\![D]\!]$, which proves (C3).

Completeness of D:

Each block $\underline{B} \leq [\![D]\!]$ is induced by the line $atoms(\phi(B)) =: r \in R$. ■

Corollary 26 *The mappings Diag and* $[\![\cdot]\!]$ *are bijective functions (up to isomorphy) between the class of all nontrivial OMAs with finite blocks and all complete OMA-diagrams with finite lines. For every nontrivial OMA* $\underline{A}$ *with finite blocks* $[\![Diag(\underline{A})]\!] \cong \underline{A}$ *holds. For every complete OMA-diagrams D with finite lines* $Diag([\![D]\!]) \cong D$ *holds.*

Proof. $[\![Diag(\underline{A})]\!] \cong \underline{A}$ was proved in Theorem 25 and

$$Diag([\![D]\!]) = (atoms([\![D]\!]), \{atoms(\underline{B}) | \underline{B} \text{ block of } [\![D]\!]\}) \cong (P, R) = D$$

holds because of Theorem 16 and completeness of D. ∎

So every nontrivial OMA $\underline{A}$ in which each block is finite is induced by a complete OMA-diagram

$$Diag(\underline{A}) = (atoms(\underline{A}), \{atoms(\underline{B}) | \underline{B} \text{ block of } \underline{A}\})$$

with finite lines. For every complete OMA-diagram D with finite lines each block of $[\![D]\!]$ is finite because the block is induced by a line. For complete diagrams D_1 and D_2 with finite lines with $[\![D_1]\!] \cong [\![D_2]\!]$ we have

$$D_1 \cong Diag([\![D_1]\!]) \cong Diag([\![D_2]\!]) \cong D_2$$

and for two OMAs $\underline{A}_1$ and $\underline{A}_2$ in which each block is finite with $Diag(\underline{A}_1) \cong Diag(\underline{A}_2)$ we have

$$\underline{A}_1 \cong [\![Diag(\underline{A}_1)]\!] \cong [\![Diag(\underline{A}_2)]\!] \cong \underline{A}_2.$$

So these operators are bijections between the isomorphy classes of all complete OMA-diagrams with finite lines and the isomorphy classes of all nontrivial OMAs in which each block is finite.

The following theorem shows, that the congruence relation $<\sim_D>$ can easily be computed, if the diagram satisfies some conditions. Later it will be shown that these conditions allways hold for complete OMA-diagrams with finite lines.

Theorem 27 *Let* $D = (P, R)$ *be a diagram in which each line is finite such that the following two conditions hold:*

1. *For every* $r, s, t \in R$ *there exists a line* $u \in R$ *with* $(r \cap s) \cup (s \cap t) \cup (r \cap t) \subseteq u$ *and*

2. *for every* $r, s, t \in R$ *with* $s \subseteq r \cup t$ *there exists a line* $u \in R$ *with* $(r \setminus s) \cup (t \setminus s) \cup (r \cap t) \subseteq u$.

Let $\rho := \{(in_r(E), in_s(F)) |$ *there exist* $t, u \in R$ *with* $t = E\dot{\cup}(s \setminus F)$ *and* $u = F\dot{\cup}(r \setminus E)\}$. *Then* $\rho = <\sim_D>$ *holds.*

Proof.
Because of Lemma 24 ρ is a subset of $<\sim_D>$. Now we prove, that ρ is a congruence relation on $\coprod_{r\in R} \underline{\mathcal{P}_{\text{fin}}^{\text{cofin}}(r)}$:
Reflexivity of ρ:
For $r \in R$ and $E \in \mathcal{P}_{\text{fin}}^{\text{cofin}}(r)$ take $t := r =: u$, so $(in_r(E), in_r(E)) \in \rho$ holds.
Symmetry of ρ:
For $(in_r(E), in_s(F)) \in \rho$ we have the existence of $t, u \in R$ with $t = E\dot{\cup}(s \setminus F)$ and $u = F\dot{\cup}(r \setminus E)$, so $(in_s(F), in_r(E)) \in \rho$ holds.
Transitivity of ρ:
Let $(in_r(E), in_s(F)) \in \rho$ and $(in_s(F), in_t(G)) \in \rho$. Then we get the existence of $v, q \in R$ with $v = F\dot{\cup}(r \setminus E)$ and $q = G\dot{\cup}(s \setminus F)$. So we have $s \subseteq q \cup v$ and because of condition 2 we get a line $w \in R$ with $(q \setminus s) \cup (v \setminus s) \cup (q \cap v) \subseteq w$. Now we show $r \setminus E \subseteq w$ and $G \subseteq w$. Let $a \in r \setminus E$. If $a \notin s$ then $a \in v \setminus s \subseteq w$ and if $a \in s$ then $a \in s \setminus F$ because of $(r \setminus E) \cap F = \emptyset$, so we have $a \in (s \setminus F) \cap (r \setminus E) \subseteq q \cap v \subseteq w$. Therefore $r \setminus E \subseteq w$ holds. Let $a \in G$. If $a \notin s$ then $a \in q \setminus s \subseteq w$ and if $a \in s$ then $a \in F$ because of $G \cap (s \setminus F) = \emptyset$, so we have $a \in G \cap F \subseteq q \cap v \subseteq w$. Therefore $G \subseteq w$ holds. Because of $\rho \subseteq <\sim_D>$ we have $(in_r(E), in_t(G)) \in <\sim_D>$ because $<\sim_D>$ is transitive, so we get

$$\begin{aligned} in_w(G)/ <\sim_D> &= in_t(G)/ <\sim_D> \\ &= in_r(E)/ <\sim_D> \\ &= (in_r(r \setminus E)/ <\sim_D>)' \\ &= (in_w(r \setminus E)/ <\sim_D>)' \\ &= in_w(w \setminus (r \setminus E))/ <\sim_D> \end{aligned}$$

and because of condition (C2) we have $G = w \setminus (r \setminus E)$, so $w = G\dot{\cup}(r \setminus E)$. Analogously we get a line $u = E\dot{\cup}(t \setminus G) \in R$, and therefore $(in_r(E), in_t(G)) \in \rho$. This proves the transitivity.
Compatibility with $'$:
Let $(in_r(E), in_s(F)) \in \rho$. Then there exist $t, u \in R$ with

$$t = E\dot{\cup}(s \setminus F) \text{ and } u = F\dot{\cup}(r \setminus E).$$

So $(in_r(E)', in_s(F)') = (in_r(r \setminus E), in_s(s \setminus F)) \in \rho$ and this proves that ρ is compatible with $'$.
Compatibility with $\oplus$:
Let $(in_{r_1}(E_1), in_{s_1}(F_1)) \in \rho$ and $(in_{r_2}(E_2), in_{s_2}(F_2)) \in \rho$ such that $in_{r_1}(E_1) \oplus in_{r_2}(E_2)$ and $in_{s_1}(F_1) \oplus in_{s_2}(F_2)$ exist. Then we have $r_1 = r_2$ (or $E_1 = \emptyset$ or $E_2 = \emptyset$, but in that case we can redefine r_1 or r_2 to get the same situation because $in_{r_1}(\emptyset) = 0 = in_{r_2}(\emptyset)$ holds) and $s_1 = s_2$. Because of the existence of the sums we have

$$E_1 \cap E_2 = \emptyset = F_1 \cap F_2.$$

From the definition of ρ we get the existence of elements $t, u \in R$ with

$$t = E_1 \dot{\cup} (s_1 \setminus F_1) \text{ and } u = E_2 \dot{\cup} (s_2 \setminus F_2) = E_2 \dot{\cup} (s_1 \setminus F_2).$$

With condition 1 we get a line $v \in R$ with

$$E_1 \cup E_2 \cup (s_1 \setminus (F_1 \cup F_2)) \subseteq (r_1 \cap t) \cup (r_1 \cap u) \cup (t \cap u) \subseteq v.$$

Because of $(in_{r_1}(E_1), in_{s_1}(F_1)) \in \rho \subseteq <\sim_D>$ and $(in_{r_2}(E_2), in_{s_2}(F_2)) \in \rho \subseteq <\sim_D>$ we have

$$\begin{aligned} in_v(s_1 \setminus (F_1 \oplus F_2))/ <\sim_D> &= in_{s_1}(s_1 \setminus (F_1 \oplus F_2))/ <\sim_D> \\ &= (in_{s_1}(F_1 \oplus F_2)/ <\sim_D>)' \\ &= (in_{r_1}(E_1 \oplus E_2)/ <\sim_D>)' \\ &= (in_v(E_1 \oplus E_2)/ <\sim_D>)' \\ &= in_v(v \setminus (E_1 \oplus E_2))/ <\sim_D> \end{aligned}$$

and because of condition (C2) we get

$$s_1 \setminus (F_1 \oplus F_2) = v \setminus (E_1 \oplus E_2)$$

and therefore $v = (E_1 \oplus E_2) \dot{\cup} (s_1 \setminus (F_1 \oplus F_2))$. Analogously we get a line

$$w = (F_1 \oplus F_2) \dot{\cup} (r_1 \setminus (E_1 \oplus E_2)) \in R$$

so $(in_{r_1}(E_1) \oplus in_{r_2}(E_2), in_{s_1}(F_1) \oplus in_{s_2}(F_2)) \in \rho$.
Therefore ρ is a congruence relation.
For $(in_r(\{a\}), in_s(\{a\})) \in \sim_D$ we have $(in_r(\{a\}), in_s(\{a\})) \in \rho$ with $t := s$ and $u := r$. So we have $\sim_D \subseteq \rho$ and $<\sim_D> \subseteq \rho$. Therefore $<\sim_D> = \rho$ holds.

■

In this theorem condition 1 says, that for every triangle of D there exists a line which contains the corners of the triangle, where the corner of two lines is defined as their intersection. Condition 2 says, that for every line which is covered by two other lines there exists a line containing the rest of the other lines and their intersection. These conditions are sufficient for the property, that the condition of Lemma 24 characterises the whole congruence relation $<\sim_D>$.

The following lemma gives a sufficient condition for the property that a nontrivial diagram is an OMA-diagram.

Lemma 28 *Let $D = (P, R)$ be a nontrivial diagram such that the following three conditions hold:*

1. *For every $r, s, t \in R$ there exists a line $u \in R$ with $(r \cap s) \cup (s \cap t) \cup (r \cap t) \subseteq u$ and*
2. *for every $r, s \in R$ and $E \in \mathcal{P}^{cofin}_{fin}(r)$ and $F \in \mathcal{P}^{cofin}_{fin}(s)$ with $(in_r(E), in_s(F)) \in <\sim_D>$ there exists $t \in R$ with $E \cup (s \setminus F) \subseteq t$,*
3. *for every $r, s \in R$ and $E \in \mathcal{P}^{cofin}_{fin}(r)$ with $E \subseteq s \neq r$ the set E is finite.*

Then D is an OMA-diagram.

Proof.
The axioms (A0)-(A4), (A7) and (A9) hold because of Theorem 8.
Proof of (A5):
Let $x, y, z \in [\![D]\!]$ such that $(x \oplus y) \oplus z$ exists. Then there exist $G, H \subseteq r \in R$ and $E, F \subseteq s \in R$ with $x = in_r(G)/<\sim_D>, y = in_r(H)/<\sim_D>, z = in_s(E)/<\sim_D>$ and $x \oplus y = in_s(F)/<\sim_D>$ such that $G \oplus H$ and $F \oplus E$ exist. Then we have $(in_r(G \oplus H), in_s(F)) \in <\sim_D>$ and with condition 2 we get a line $t \in R$ with $(G \oplus H) \cup (s \setminus F) \subseteq t$, so $E, G, H \subseteq t$ because of the existence of $F \oplus E$. With condition 3 we get $x = in_t(G)/<\sim_D>, y = in_t(H)/<\sim_D>, z = in_t(E)/<\sim_D>$. Because of condition (C2) and the existence of

$$(x \oplus y) \oplus z = (in_t(G)/<\sim_D> \oplus in_t(H)/<\sim_D>) \oplus in_t(E)/<\sim_D>$$

we get the existence of $(G \oplus H) \oplus E$ in $\underline{\mathcal{P}^{\text{cofin}}_{\text{fin}}(t)}$ and therefore $G \oplus (H \oplus E)$ exists and

$$(G \oplus H) \oplus E = G \oplus (H \oplus E)$$

holds because $\underline{\mathcal{P}^{\text{cofin}}_{\text{fin}}(t)}$ is an OMA. So

$$(x \oplus y) \oplus z = (in_t(G)/<\sim_D> \oplus in_t(H)/<\sim_D>) \oplus in_t(E)/<\sim_D>$$

$$= in_t(G \oplus H \oplus E)/<\sim_D> = x \oplus (y \oplus z)$$

which proves (A5).

Proof of (A8):

Let $x, y, z \in [\![D]\!]$ such that $x \oplus y, y \oplus z$ and $x \oplus z$ exist.
Then there exist $E_x, E_y \subseteq r \in R$ and $F_y, F_z \subseteq s \in R$ and $G_x, G_z \subseteq t \in R$ with

$$in_r(E_x)/ <\sim_D> = x = in_t(G_x)/ <\sim_D>,$$

$$in_r(E_y)/ <\sim_D> = y = in_s(F_y)/ <\sim_D>,$$

$$in_s(F_z)/ <\sim_D> = z = in_t(G_z)/ <\sim_D>$$

such that $E_x \oplus E_y, F_y \oplus F_z$ and $G_x \oplus G_z$ exist. Because of $(in_r(E_x), in_t(G_x)) \in <\sim_D>$ condition 2 implies the existence of $u \in R$ with $E_x \cup (t \setminus G_x) \subseteq u$. Because of the existence of $G_x \oplus G_z$ we have $E_x, G_z \subseteq u$. Analogously we get $v, w \in R$ with $F_y, G_z \subseteq v$ and $E_x, F_y \subseteq w$. Condition 1 implies the existence of a line $q \in R$ with $E_x \cup F_y \cup G_z \subseteq (u \cap w) \cup (v \cap w) \cup (u \cap v) \subseteq q$. With condition 3 we get $x = in_q(E_x)/ <\sim_D>, y = in_q(F_y)/ <\sim_D>, z = in_q(G_z)/ <\sim_D>$. Because of condition (C2) $E_x \oplus F_y$ and $F_y \oplus G_z$ and $E_x \oplus G_z$ exist in $\underline{\mathcal{P}^{\text{cofin}}_{\text{fin}}(q)}$, so $E_x \oplus (F_y \oplus G_z)$ exists because $\underline{\mathcal{P}^{\text{cofin}}_{\text{fin}}(q)}$ satisfies axiom (A8). So we have the existence of

$$in_q(E_x)/ <\sim_D> \oplus (in_q(F_y)/ <\sim_D> \oplus in_q(G_z)/ <\sim_D>) = x \oplus (y \oplus z)$$

which proves (A8).
Axiom (A6) follows from the other axioms, so $[\![D]\!]$ is an OMA. ■

In the following characterisation Theorem 29 we use Theorem 27 and Lemma 28 to get some conditions which are equivalent to the property, that a complete diagram with finite lines is an OMA-diagram.

Theorem 29 *Let $D = (P, R)$ be a nontrivial complete diagram in which each line is finite. The following conditions are equivalent:*

1. *D is an OMA-diagram.*
2. (a) *For every $r, s, t \in R$ there exists a line $u \in R$ with $(r \cap s) \cup (s \cap t) \cup (r \cap t) \subseteq u$ and*

   (b) *for every $r, s, t \in R$ with $s \subseteq r \cup t$ there exists a line $u \in R$ with $(r \setminus s) \cup (t \setminus s) \cup (r \cap t) \subseteq u$.*
3. (a) *For every $r, s, t \in R$ there exists a line $u \in R$ with $(r \cap s) \cup (s \cap t) \cup (r \cap t) \subseteq u$ and*

   (b) *for every $r, s \in R$ and $E \in \mathcal{P}^{cofin}_{fin}(r)$ and $F \in \mathcal{P}^{cofin}_{fin}(s)$ with $(in_r(E), in_s(F)) \in <\sim_D>$ there exists $t \in R$ with $E \cup (s \setminus F) \subseteq t$.*

Proof.
$1 \to 2$:
Because of the completeness each block of $[\![D]\!]$ is generated by its atoms.
Let $r, s, t \in R$ and $E := (r \cap s) \cup (s \cap t) \cup (r \cap t)$. For all $a, b \in E$ there exists $v \in \{r, s, t\} \subseteq R$ with $a, b \in v$. So because of Theorem 19 E generates a Boolean subalgebra, which is contained in a block. D is complete, so there exists a line $u \in R$ with $E \subseteq [\![u]\!]$ and therefore $E \subseteq u$ because of Theorem 17 which proves condition 2a.
Now let $r, s, t \in R$ with $s \subseteq r \cup t$. For $a \in r \setminus s$ and $b \in t \setminus s$ we have $a \in r \setminus (s \setminus t)$ and $b \notin s \cap t$ so $\bigoplus(s \cap t) \oplus b$ exists, $s \setminus t \subseteq r$ holds and

$$\begin{aligned}\bigoplus(s \cap t) \oplus b &= (\bigoplus(s \setminus t))' \oplus b \\ &= \bigoplus(r \setminus (s \setminus t)) \oplus b \\ &= (\bigoplus(r \setminus (\{a\} \cup (s \setminus t))) \oplus a) \oplus b\end{aligned}$$

With axiom (A5) we get the existence of $a \oplus b$. Define $E := (r \setminus s) \cup (t \setminus s) \cup (r \cap t)$. Then for all $a, b \in E$ with $a \neq b$ the sum $a \oplus b$ exists, and therefore E generates a Boolean subalgebra which is contained in a block. D is complete, so there exists a line $u \in R$ with $E \subseteq u$ which proves condition 2b.
$2 \to 3$:
See Theorem 27.
$3 \to 1$:
See Lemma 28. ■

The implication $1 \to 2$ of Theorem 29 may not hold if D is contains infinite lines. This will be proved in the next chapter. The characterisation of the congruence relation in Theorem 27 also holds for complete OMA-diagrams with finite lines:

Corollary 30 *Let $D = (P, R)$ be a complete OMA-diagram in which each line is finite, $r, s \in R$ and $E \in \mathcal{P}_{fin}^{cofin}(r)$ and $F \in \mathcal{P}_{fin}^{cofin}(s)$. Then*

$$(in_r(E), in_s(F)) \in <\sim_D> \quad \textit{iff } E\dot{\cup}(s \setminus F) \in R \textit{ and } F\dot{\cup}(r \setminus E) \in R$$

holds.

Proof. See Theorems 27 and 29. ■

In Theorem 33 we will give a characterisation of complete OMA-diagrams with a weaker precondition than in Theorem 29. First we need the following lemma:

Lemma 31 *Let $D = (P, R)$ be a hypergraph in which each clique is a line. Then condition (2a) of Theorem 29 is satisfied.*

Proof. Let $r, s, t \in R$. In $E := (r \cap s) \cup (s \cap t) \cup (r \cap t) \subseteq u$ each pair of points is connected by a line (for example $a \in r \cap s$ is connected to $b \in s \cap t$ by the line s), so E is contained in a clique, and there exists a line $u \in R$ with $E \subseteq u$. ■

Note that the property that each clique is a line is equivalent to the property that each clique is contained in a line because a line can not be a proper superset of a clique because of the maximality.

For nontrivial finite hypergraphs the two conditions of this lemma are equivalent:

Lemma 32 *Let $D = (P, R)$ be a nontrivial finite hypergraph. Each clique is a line iff condition (2a) of Theorem 29 is satisfied.*

Proof. Assume that condition (2a) holds and that there exists a clique which is not a line. Because of the finiteness of P there exists a minimal subset $E \subseteq P$, in which each pair of points is connected by a line, but E is not contained in a line. We have $|E| > 1$ because of $P \neq \emptyset$, so there exist $a, b \in E$ with $a \neq b$. Because of the minimality of E the set $E \setminus \{a\}$ is contained in a line $r \in R$ and $E \setminus \{b\}$ is contained in a line $s \in R$. The set $\{a, b\}$ is also contained in a line $t \in R$ because a is connected to b. With condition (2a) we get a line $u \in R$ with $E = (E \setminus \{a, b\}) \cup \{b\} \cup \{a\} \subseteq (r \cap s) \cup (r \cap t) \cup (s \cap t) \subseteq u$, which is a contradiction. ■

For infinite hypergraphs this lemma may not hold. In chapter 3 we show that there exists an OMA-diagram with finite lines, such condition (2a) of Theorem 29 is satisfied, but there exists a clique which is not contained in a line (see example 4).

Theorem 33 *Let $D = (P, R)$ be a diagram in which each line is finite such that every block of $[\![D]\!]$ is generated by its atoms. The following conditions are equivalent:*

1. *D is a complete OMA-diagram.*
2. *(a) Every clique is a line, and*
   (b) for every $r, s, t \in R$ with $s \subseteq r \cup t$ there exists a line $u \in R$ with $(r \setminus s) \cup (t \setminus s) \cup (r \cap t) \subseteq u$.
3. *(a) Every clique is a line, and*
   (b) for every $r, s \in R$ and $E \in \mathcal{P}^{cofin}_{fin}(r)$ and $F \in \mathcal{P}^{cofin}_{fin}(s)$ with $(in_r(E), in_s(F)) \in <\sim_D>$ there exists $t \in R$ with $E \cup (s \setminus F) \subseteq t$.

Proof.
$1 \to 2$:
See Theorem 20 and Theorem 29.
$2 \to 3$:
See Lemma 31 and Theorem 27.
$3 \to 1$:
We have $R \neq \emptyset$ because otherwise the empty set would be a clique which is not a line. Lemma 31 and Lemma 28 imply that D is an OMA-diagram. Let $\underline{B}$ be a block of $[\![D]\!]$. With Theorem 20 $\underline{B}$ is induced by a clique $E \subseteq P$, so E is a line $E = r \in R$. The Boolean subalgebra $\underline{B}$ is maximal, so we get $\underline{B} = [\![r]\!]$ and D is complete. ■

This theorem characterises complete OMA-diagrams under the assumption that each block of $[\![D]\!]$ for a diagram D with finite lines is generated by its atoms: D is a complete OMA-diagram iff every clique is a line and for every line which is covered by two other lines there exists a line containing the rest of the other lines and their intersection. In the following characterisation theorem the assumption that D is a diagram is not needed, so the theorem holds for every hypergraph with finite lines, such that each block is generated by its atoms.

Theorem 34 *Let $D = (P, R)$ be a hypergraph in which each line is finite such that each block of $[\![D]\!]$ is generated by its atoms. D is a complete OMA-diagram iff the following conditions are satisfied:*

1. $|r \setminus s| > 1$ *holds for all $r, s \in R$ with $r \neq s$,*
2. *the relation $\rho := \{(in_r(E), in_s(F))|$ there exist $t, u \in R$ with $t = E\dot{\cup}(s \setminus F)$ and $u = F\dot{\cup}(r \setminus E)\}$ is transitive and compatible with $\oplus$ (in the coproduct $\coprod_{r\in R} \underline{\mathcal{P}^{cofin}_{fin}(r)}$),*
3. *every clique is a line,*
4. *for every $r, s, t \in R$ with $s \subseteq r \cup t$ there exists a line $u \in R$ with $(r \setminus s) \cup (t \setminus s) \cup (r \cap t) \subseteq u$.*

Proof.
$\to$:
If D is a complete OMA-diagram then the conditions 1-4 follow from Lemma 9, Theorem 33 and Theorem 27.
$\leftarrow$:
Let the four conditions be satisfied. Then $\rho = <\sim_D>$ holds (see proof of Theorem 27). Now we use Theorem 12 to show that D is a diagram.
Let $(in_r(\{a\}), in_s(E)) \in <\sim_D> = \rho$. Then there exists a line $u \in R$ with $u =$

$\{a\}\dot{\cup}(s \setminus E)$, so with condition 1 we get $u = s$. Therefore $E = \{a\}$ holds.
Now we proof (C2):
Let $(in_r(E), in_r(F)) \in <\sim_D> = \rho$. Then there exists a line $u \in R$ with $u = E\dot{\cup}(r \setminus F)$. With condition 1 we get $u = r$ and $E = F$, so $nat_{<\sim_D>} \circ in_r$ is injective. Let $E, F \subseteq r \in R$ such that $in_r(E)/ <\sim_D> \oplus in_r(F)/ <\sim_D>$ exists. Then there exist $G, H \subseteq s \in R$ with $(in_r(E), in_s(G)) \in <\sim_D> = \rho$, $(in_r(F), in_s(H)) \in <\sim_D> = \rho$ and $G \cap H = \emptyset$. Then there exist $t, u \in R$ with $t = E\dot{\cup}(s \setminus G)$ and $u = F\dot{\cup}(s \setminus H)$. With condition 3 and Lemma 31 we get the existence of $v \in R$ with $(s \cap t) \cup (t \cap u) \cup (s \cap u) \subseteq v$. We have $s = G \cup H \cup (s \setminus (G \cup H)) \subseteq (u \cap s) \cup (t \cap s) \cup (t \cap s) \subseteq v$, so with condition 1 we get $s = v$. We have $E \cap F \subseteq t \cap u \subseteq v = s$. The union $E\dot{\cup}(s \setminus G)$ is disjoint, so we get $(E \cap F) \cap (s \setminus G) = \emptyset$ and analogously $(E \cap F) \cap (s \setminus H) = \emptyset$. Therefore $E \cap F = (E \cap F) \cap s = (E \cap F) \cap ((s \setminus G) \cup (s \setminus H)) = \emptyset$ holds and $nat_{<\sim_D>} \circ in_r$ is closed. So D is a diagram because of Theorem 12. D is a complete OMA-diagram because of Theorem 33. ∎

If D is a hypergraph with finite lines and each block of $[\![D]\!]$ is generated by its atoms, the four conditions of this theorem are usefull to check wether a D is a complete OMA-diagram. If D is finite then all lines of D are finite and each block is generated by its atoms, so for finite diagrams it can be easily checked, whether D is a complete OMA-diagram: We need not to compute the interpretation $[\![D]\!]$, we just have to check the conditions of Theorem 34.

3 Examples

The first example is a counterexample for Theorem 29 with infinite lines.

Example 1:
Let $R = \{r, s, t\}$ with

$$r = A_1 \cup A_2, s = A_2 \cup A_3, t = A_3 \cup A_4 \text{ and}$$

$$A_1 := \mathbb{N} \times \{1\}, A_2 := \mathbb{N} \times \{2\}, A_3 := \mathbb{N} \times \{3\}, A_4 := \mathbb{N} \times \{4\}.$$

Let $P = \bigcup R$ and $D = (P, R)$. Now we show that D is a complete OMA-diagram, but condition 2b of Theorem 29 is not satisfied. Let

$$\rho_1 := \{(in_r(E), in_s(E) | E \subseteq A_2, E \text{ finite }\},$$

$$\rho_2 := \{(in_r(r \setminus E), in_s(s \setminus E) | E \subseteq A_2, E \text{ finite }\},$$

$$\rho_3 := \{(in_s(E), in_t(E) | E \subseteq A_3, E \text{ finite }\},$$

$$\rho_4 := \{(in_s(s \setminus E), in_t(t \setminus E) | E \subseteq A_3, E \text{ finite }\} \text{ and}$$

$$\rho := \rho_1 \cup \rho_2 \cup \rho_3 \cup \rho_4$$

It is not difficult to see that the congruence relation generated by ρ is the reflexive and symmetrical closure:

$$< \rho >= ref(sym(\rho))$$

We have $\sim_D \subseteq ref(sym(\rho))$ and therefore $<\sim_D> \subseteq ref(sym(\rho))$.
The relation $ref(sym(\rho))$ is generated by $\sim_D$, so we get

$$<\sim_D>= ref(sym(\rho)) =< \rho > .$$

D is a **diagram:**
(C1) holds because of $<\sim_D>= ref(sym(\rho))$.
Let $u \in R$. The mapping $nat_{<\sim_D>} \circ in_u$ is injective because of $<\sim_D>= ref(sym(\rho))$.
Now let $in_u(E)/ <\sim_D>, in_u(F)/ <\sim_D> \in [\![D]\!]$ such that

$$in_u(E)/ <\sim_D> \oplus in_u(F)/ <\sim_D>$$

exists. Then there exist $w \in R$ and $G, H \in \mathcal{P}^{\text{cofin}}_{\text{fin}}(w)$ with $G \cap H = \emptyset$ and

$$(in_u(E), in_w(G)) \in <\sim_D> = ref(sym(\rho)) \text{ and}$$

$$(in_u(F), in_w(H)) \in <\sim_D> = ref(sym(\rho)).$$

Then G or H must be finite (because of $G \cap H = \emptyset$), so we assume now that G is finite. Therefore $E = G$ holds. If $F = H$ holds then we get $E \cap F = \emptyset$ otherwise we get $E = G \subseteq w \setminus H = u \setminus F$, so $E \cap F = \emptyset$. Therefore $E \oplus F$ exists in $\mathcal{P}^{\text{cofin}}_{\text{fin}}(u)$ and the mapping $nat_{<\sim_D>} \circ in_u$ is closed which proves (C2).
Let $u \in R$ and $in_v(E)/ <\sim_D> \in [\![D]\!]$ such that the set $[\![u]\!] \cup \{in_v(E)/ <\sim_D>\}$ is contained in a Boolean subalgebra $\underline{B} \leq [\![D]\!]$. Now we show that $in_v(E)/ <\sim_D> \in [\![u]\!]$ holds. For $a \in u$ we have $in_u(\{a\})/ <\sim_D> \in atoms(\underline{B})$ because of Lemma 11. We only have to consider the case $u \neq v$ because if $u = v$ then $in_v(E)/ <\sim_D> \in [\![u]\!]$ holds.
Case 1: u = s
Then we can assume that $v = r$ holds because $v = t$ works analogously.
For $a := (3, 3) \in s \setminus r$ we have

$$in_s(\{a\})/ <\sim_D> \leq in_v(F)/ <\sim_D> \text{ or } in_s(\{a\})/ <\sim_D> \leq in_v(v \setminus F)/ <\sim_D>$$

because of Lemma 2. If

$$in_s(\{a\})/ <\sim_D> \leq in_v(v \setminus F)/ <\sim_D>$$

holds then

$$in_s(\{a\})/ <\sim_D> \oplus in_v(F)/ <\sim_D>$$

exists and because of $<\sim_D>= ref(sym(\rho))$ we have $[\![t]\!] \cap [\![r]\!] = \{0,1\}$, so we get a set $G \in \mathcal{P}^{\text{cofin}}_{\text{fin}}(s)$ with $(in_v(F), in_s(G)) \in <\sim_D>$ and therefore

$$in_v(F)/ <\sim_D> \in [\![s]\!].$$

If $in_s(\{a\})/ <\sim_D> \leq in_v(F)/ <\sim_D>$ holds then we get analogously

$$in_v(v \setminus F)/ <\sim_D> \in [\![s]\!].$$

Case 2: u = r
Analogiosly case 1 with $a := (1,1)$.
Therefore we have $\underline{B} = [\![u]\!]$ and (C3) holds, so D is a diagram.
Completeness of D:
Assume that there exists a block $\underline{B} \leq [\![D]\!]$ with $[\![u]\!] \neq \underline{B}$ for all $u \in R$. If $\underline{B} \subseteq [\![s]\!]$ holds then we get $\underline{B} = [\![s]\!]$ because $\underline{B}$ is a maximal Boolean subalgebra. Therefore $\underline{B}$ cannot be a subset of $[\![s]\!]$ and there exists an element $x \in \underline{B}$ with $x \notin [\![s]\!]$. In the following we assume that $x \in [\![r]\!]$ holds because the proof with $x \in [\![t]\!]$ works analogously. The block $\underline{B}$ cannot be a subset of $[\![r]\!]$, so there exists an element $y \in \underline{B}$ with $y \notin [\![r]\!]$. With [BM98] we get the existence of $a, b, c \in \underline{B}$ with $a \oplus b = x, b \oplus c = y$ such that $a \oplus c$ exists. Because of $x \notin [\![s]\!]$ and $[\![t]\!] \cap [\![r]\!] = \{0,1\}$, we we get the existence of $E, F \in \mathcal{P}^{\text{cofin}}_{\text{fin}}(r)$ with $E \cap F = \emptyset$ and

$$in_r(E)/ <\sim_D>= a \text{ and } in_r(F)/ <\sim_D>= b.$$

There exists an $u \in R$ and $G, H \in \mathcal{P}^{\text{cofin}}_{\text{fin}}(u)$ with $G \cap H = \emptyset$ and

$$in_u(G)/ <\sim_D>= b \text{ and } in_u(H)/ <\sim_D>= c.$$

We have $b \oplus c = y \notin [\![r]\!]$, so $u \neq r$ holds. Because of

$$in_r(F)/ <\sim_D>= b = in_u(G)/ <\sim_D> \text{ and}$$

$$[\![t]\!] \cap [\![r]\!] = \{0,1\},$$

we get $u = s$ or $b = 0$.
Case 1: $b = 0$ or $(in_r(F), in_s(G)) \in \rho_1$
Then F and G are finite subsets of A_2 with $F = G$. Because of the existence of $a \oplus c$ at least one of the sets E and H must be a finite subset of A_2. If E is a finite subset of A_2 then $x = in_r(E \cup F)/ <\sim_D>= in_s(E \cup F)/ <\sim_D>$ holds, which is a contradiction to

$x \notin [\![s]\!]$. If H is a finite subset of A_2 then $y = in_s(G \cup H)/ <\sim_D>= in_r(G \cup H)/ <\sim_D>$ holds, which is a contradiction to $y \notin [\![r]\!]$.
Case 2: $u = s$ and $(in_r(F), in_s(G)) \notin \rho_1$
Then $(in_r(F), in_s(G)) \in \rho_2$, so there is a finite subset $J \subseteq A_2$ with $F = r \setminus J$ and $G = s \setminus J$, therefore E is a finite subset of A_2 because of $E \cap F = \emptyset$. So $x = in_r(E \cup F)/ <\sim_D>= in_s(E \cup G)/ <\sim_D>$ holds which is a contradiction to $x \notin [\![s]\!]$.
Therefore D is complete. D is an OMA-diagram because of Lemma 28. Condition 2b of Theorem 29 is not satisfied.

The next example shows that there exist OMAs, which are not induced by diagrams, so Theorem 25 does not hold for OMAs with infinite blocks. We also show that there exist two different OMAs (with infinite blocks) for that the operator $Diag$ of definition 23 produces isomorphic diagrams.

Example 2:
Let $D = (P, R)$ with $P = \bigcup R$ and $R = \{r, s\}$ with

$$r = \mathbb{N} \cup \{-1, -2\},$$

$$s = \mathbb{N} \cup \{-3, -4\}.$$

Now we show that D is an OMA-diagram. Let

$$\rho := \{(in_r(E), in_s(E)) | E \subseteq \mathbb{N}, E \text{ finite } \} \cup$$

$$\{(in_r(\{-1, -2\} \cup (\mathbb{N} \setminus E)), in_s(\{-3, -4\} \cup (\mathbb{N} \setminus E))) | E \subseteq \mathbb{N}, E \text{ finite } \}$$

Then we have $< \rho >= ref(sym(\rho)) =<\sim_D>$ like in example 1.
D is a **diagram:**
(C1) holds because of $<\sim_D>= ref(sym(\rho))$. We show (C2) only for the line r, because for s the proof works analogously. The mapping $nat_{<\sim_D>} \circ in_r$ is injective because of $<\sim_D>= ref(sym(\rho))$. Now let $in_r(E)/ <\sim_D>, in_r(F)/ <\sim_D> \in [\![D]\!]$ such that $in_r(E)/ <\sim_D> \oplus in_r(F)/ <\sim_D>$ exists. Then there exist $w \in R$ and $G, H \in \mathcal{P}_{\text{fin}}^{\text{cofin}}(w)$ with $G \cap H = \emptyset$ and $(in_r(E), in_w(G)) \in <\sim_D>= ref(sym(\rho))$ and $(in_r(F), in_w(H)) \in <\sim_D>= ref(sym(\rho))$. Then we get $E \cap F = \emptyset$, so $E \oplus F$ exists in $\mathcal{P}_{\text{fin}}^{\text{cofin}}(r)$ and the mapping $nat_{<\sim_D>} \circ in_r$ is closed which proves (C2). The prove of (C3) is the same like in example 1 (with $a := -3$ in case 1 and $a := -1$ in case 2). Therefore D is a diagram.
$[\![D]\!]$ is an **OMA:**
The axioms (A0)-(A4), (A7) and (A9) hold because of Theorem 8.
Proof of (A5):
Let $x, y, z \in [\![D]\!]$ such that $(x \oplus y) \oplus z$ exists. Then there exist $u, v \in R$ and

$G, H \in \mathcal{P}^{\text{cofin}}_{\text{fin}}(u)$ and $E, F \in \mathcal{P}^{\text{cofin}}_{\text{fin}}(v)$ with

$$x = in_u(G)/ <\sim_D>,$$

$$y = in_u(H)/ <\sim_D>,$$

$$z = in_v(E)/ <\sim_D>,$$

$$x \oplus y = in_v(F)/ <\sim_D>$$

such that $G \cap H = \emptyset = F \oplus E$ hold. We can assume $u \neq v$ because axiom (A5) holds in the Boolean OMAs $[\![r]\!]$ and $[\![s]\!]$. Then we have $(in_u(G \oplus H), in_v(F)) \in <\sim_D> = ref(sym(\rho))$ so we only have to consider the following two cases:
Case 1: $G \cup H = F \subseteq \mathbb{N}$ and F is finite.
Then $x, y, z \in [\![v]\!]$ and $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ holds.
Case 2: $u \setminus (G \cup H) = v \setminus F \subseteq \mathbb{N}$ and $v \setminus F$ is finite.
Then $E \subseteq v \setminus F$ is finite and $x, y, z \in [\![u]\!]$ and $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ holds.
Proof of (A8):
Let $x, y, z \in [\![D]\!]$ such that $x \oplus y, y \oplus z$ and $x \oplus z$ exist.
Then there exist $E_x, E_y \subseteq u \in R$ and $F_y, F_z \subseteq v \in R$ and $G_x, G_z \subseteq w \in R$ with

$$in_u(E_x)/ <\sim_D> = x = in_w(G_x)/ <\sim_D>,$$

$$in_u(E_y)/ <\sim_D> = y = in_v(F_y)/ <\sim_D>,$$

$$in_v(F_z)/ <\sim_D> = z = in_w(G_z)/ <\sim_D>$$

such that $E_x \oplus E_y, F_y \oplus F_z$ and $G_x \oplus G_z$ exist. We have $|R| = 2$, so we can assume $u = v$ (the cases $u = w$ and $v = w$ work analogously). Then we get $x, y, z \in [\![u]\!]$ and $x \oplus (y \oplus z)$ exists because $[\![u]\!]$ is an OMA.
Axiom (A6) follows from the other axioms, so $[\![D]\!]$ is an OMA.
Let $\underline{C} := \coprod_{r \in R} \underline{\mathcal{P}^{\text{cofin}}_{\text{fin}}(r)}$ and

$$\begin{aligned} \sigma :=& \{(in_r(E), in_s(E)) | E \in \mathcal{P}^{\text{cofin}}_{\text{fin}}(\mathbb{N})\} \cup \\ & \{(in_r(\{-1, -2\} \cup E), in_s(\{-3, -4\} \cup E)) | E \in \mathcal{P}^{\text{cofin}}_{\text{fin}}(\mathbb{N})\} \end{aligned}$$

Then the congruence relation $< \sigma >$ generated by σ is the reflexive symmetrical closure of σ:

$$< \sigma >= ref(sym(\sigma)).$$

Let $\underline{A} := \underline{C}/ < \sigma >$. To show that $\underline{A}$ is an OMA, just use an analogue proof like above with $< \sigma >$ instead of $<\sim_D>$.

Note that $in_r(\mathcal{P}_{\text{fin}}^{\text{cofin}}(r))/ < \sigma >$ and $in_s(\mathcal{P}_{\text{fin}}^{\text{cofin}}(s))/ < \sigma >$ are blocks of $\underline{A}$. This can be shown like the proof of (C2) and (C3) for D with $< \sigma >$ instead of $<\sim_D>$.

Proof of $Diag(\underline{A}) \cong D \cong Diag([\![D]\!])$**:**

Let $\gamma \in \{\sim_D, \sigma\}$. Now assume that there exists a block $\underline{B} \leq \underline{C}/ < \gamma >$ with $in_r(\mathcal{P}_{\text{fin}}^{\text{cofin}}(r))/ < \gamma > \neq \underline{B} \neq in_s(\mathcal{P}_{\text{fin}}^{\text{cofin}}(s))/ < \gamma >$. If $\underline{B} \subseteq in_s(\mathcal{P}_{\text{fin}}^{\text{cofin}}(s))/ < \gamma >$ holds then we get $\underline{B} = in_s(\mathcal{P}_{\text{fin}}^{\text{cofin}}(s))/ < \gamma >$ because $\underline{B}$ is a maximal Boolean subalgebra. So $\underline{B}$ cannot be a subset of $in_s(\mathcal{P}_{\text{fin}}^{\text{cofin}}(s))/ < \gamma >$ and there exists an element $x \in \underline{B}$ with $x \notin in_s(\mathcal{P}_{\text{fin}}^{\text{cofin}}(s))/ < \gamma >$ and analogously we get the existence of an element $y \in \underline{B}$ with $y \notin in_r(\mathcal{P}_{\text{fin}}^{\text{cofin}}(r))/ < \gamma >$. With [BM98] we get the existence of $a, b, c \in \underline{B}$ with $a \oplus b = x, b \oplus c = y$ such that $a \oplus c$ exists. Because of $x \notin in_s(\mathcal{P}_{\text{fin}}^{\text{cofin}}(s))/ < \gamma >$ and $y \notin in_r(\mathcal{P}_{\text{fin}}^{\text{cofin}}(r))/ < \gamma >$ we get the existence of $E, F \in \mathcal{P}_{\text{fin}}^{\text{cofin}}(r)$ and $G, H \in \mathcal{P}_{\text{fin}}^{\text{cofin}}(s)$ with

$$in_r(E)/ < \gamma > = a,$$

$$in_r(F)/ < \gamma > = b = in_s(G)/ < \gamma > \text{ and}$$

$$in_s(H)/ < \gamma > = c$$

such that $E \cap F = \emptyset = G \cap H$ holds. Then we have $(in_r(F), in_s(G)) \in < \gamma >$.

Case 1: $\gamma = \sigma$

Because of

$$in_r(E \cup F)/ < \sigma > = a \oplus b = x \notin in_s(\mathcal{P}_{\text{fin}}^{\text{cofin}}(s))/ < \sigma >$$

we get $|(E \cup F) \cap \{-1, -2\}| = 1$. Let $\{e\} := (E \cup F) \cap \{-1, -2\}$. Because of $(in_r(F), in_s(G)) \in < \sigma >$ we have $e \notin F$, so $e \in E$ and $|E \cap \{-1, -2\}| = 1$. Analogously we get $|H \cap \{-3, -4\}| = 1$, which is a contradiction to the existence of $a \oplus c = in_r(E)/ < \sigma > \oplus in_s(H)/ < \sigma >$.

Case 2: $\gamma = \sim_D$

If there is a finite subset $J \subseteq \mathbb{N}$ with $F = \{-1, -2\} \cup \mathbb{N} \setminus J$ and $G = \{-3, -4\} \cup \mathbb{N} \setminus J$ then E is a finite subset of $\mathbb{N}$ because of $E \cap F = \emptyset$, so

$$x = in_r(E \cup F)/ <\sim_D> = in_s(E \cup G)/ <\sim_D>$$

holds which is a contradiction to $x \notin [\![s]\!]$. Therefore F and G are finite subsets of $\mathbb{N}$ with $F = G$. Because of the existence of $a \oplus c$ at least one of the sets E and H must be a finite subset of $\mathbb{N}$. If E is a finite subset of $\mathbb{N}$ then $x = in_r(E \cup F)/ <\sim_D> = in_s(E \cup F)/ <\sim_D>$ holds, which is a contradiction to $x \notin [\![s]\!]$. If H is a finite subset of $\mathbb{N}$ then

$$y = in_s(G \cup H)/ <\sim_D> = in_r(G \cup H)/ <\sim_D>$$

holds, which is a contradiction to $y \notin [\![r]\!]$.

So $in_r(\mathcal{P}_{\text{fin}}^{\text{cofin}}(r))/ < \gamma >$ and $in_s(\mathcal{P}_{\text{fin}}^{\text{cofin}}(s))/ < \gamma >$ are the only blocks of $\underline{C}/ < \gamma >$.

We get $R = \{atoms(\underline{B}) | \underline{B}$ block of $\underline{C}/ < \gamma >\}$ and $P = atoms(\underline{C}/ < \gamma >)$ with Theorem 5. Therefore $Diag(\underline{A}) \cong D \cong Diag([\![D]\!])$ holds and D is complete.
Now we show that $\underline{A}$ is not the interpretation of a diagram. Assume that there exists a diagram D_2 with $[\![D_2]\!] \cong \underline{A}$. Each block of $[\![D_2]\!]$ is generated by its atoms, because this property holds in $\underline{A}$. Because of the Theorems 16 and 22 the diagram $Comp(D_2)$ is complete and $[\![Comp(D_2)]\!] \cong [\![D_2]\!] \cong \underline{A}$ holds. So we get

$$Comp(D_2) \cong (atoms([\![Comp(D_2)]\!]), \{atoms(\underline{B}) | \underline{B} \text{ block of } [\![Comp(D_2)]\!]\})$$
$$\cong Diag(\underline{A}) \cong D.$$

Therefore we get $[\![D]\!] \cong [\![Comp(D_2)]\!] \cong \underline{A}$.
The OMA $\underline{A}$ contains two blocks: $in_r(\mathcal{P}_{\text{fin}}^{\text{cofin}}(r))/ < \sigma >$ and $in_s(\mathcal{P}_{\text{fin}}^{\text{cofin}}(s))/ < \sigma >$. The corresponding blocks in $[\![D]\!]$ are $[\![r]\!] = in_r(\mathcal{P}_{\text{fin}}^{\text{cofin}}(r))/ <\sim_D>$ and $[\![s]\!] = in_r(\mathcal{P}_{\text{fin}}^{\text{cofin}}(s))/ <\sim_D>$. In the block $in_r(\mathcal{P}_{\text{fin}}^{\text{cofin}}(r))/ < \sigma >$ there exist two elements $x := in_r(\{-1\})/ < \sigma >$ and $y := in_r(\{-2\})/ < \sigma >$ with

$$x, y \notin in_s(\mathcal{P}_{\text{fin}}^{\text{cofin}}(s))/ < \sigma > \text{ and}$$
$$x \oplus y = in_r(\{-1,-2\})/ < \sigma >= in_s(\{-3,-4\})/ < \sigma > \in in_s(\mathcal{P}_{\text{fin}}^{\text{cofin}}(s))/ < \sigma > .$$

But in the two blocks of $[\![D]\!]$ such elements with these properties do not exist, so $[\![D]\!] \not\cong \underline{A}$, which is a contradiction. Therefore $\underline{A}$ is not induced by a diagram and $[\![Diag(\underline{A})]\!] \not\cong \underline{A}$ holds.

The following example shows that there exists a diagram D in which each block is generated by its atoms such that the atoms of a block $\underline{B} \leq [\![D]\!]$ are not contained in P. So Theorem 16 does not hold for diagrams which are not OMA-diagrams. This example also shows that there may exist blocks in the interpretation of a diagram which are not generated by its atoms.

Example 3:
Let M be a set with $|M| > 3$ and $N := \mathcal{P}(M) \setminus \{\emptyset, M\}$. Let $D = (P, R)$ with

$$P = \{a_i^T | 1 \leq i \leq 4, T \in N\} \cup$$
$$\cup \{b_i^{T,U} | 1 \leq i \leq 6, T, U \in N, T \cap U = \emptyset, T \cup U \neq M\} \cup$$
$$\cup \{b_i^{T,U} | 1 \leq i \leq 4, T, U \in N, T \cap U = \emptyset, T \cup U = M\}$$

with $a_i^T = (i, T)$ and $b_i^{T,U} = (i, T, U)$ and

$$\begin{aligned} R = \{r^T | T \in N\} \cup \\ \cup \{s^{T,U} | T, U \in N, T \cap U = \emptyset\} \cup \\ \cup \{u^{T,U} | T, U \in N, T \cap U = \emptyset\} \cup \\ \cup \{v^{T,U} | T, U \in N, T \cap U = \emptyset\} \cup \\ \cup \{w^{T,U} | T, U \in N, T \cap U = \emptyset, T \cup U \neq M\} \end{aligned}$$

with

$$\begin{aligned} r^T &= \{a_1^T, a_2^T, a_3^T, a_4^T\}, \\ u^{T,U} &= \{a_3^T, a_4^T, b_1^{T,U}, b_2^{T,U}\}, \\ v^{T,U} &= \{a_3^U, a_4^U, b_3^{T,U}, b_4^{T,U}\}, \\ w^{T,U} &= \{b_5^{T,U}, b_6^{T,U}, a_1^{T \cup U}, a_2^{T \cup U}\} \\ s^{T,U} &= \{b_1^{T,U}, b_2^{T,U}, b_3^{T,U}, b_4^{T,U}, b_5^{T,U}, b_6^{T,U}\}, \text{ if } T \cup U \neq M \text{ and} \\ s^{T,U} &= \{b_1^{T,U}, b_2^{T,U}, b_3^{T,U}, b_4^{T,U}\}, \text{ if } T \cup U = M. \end{aligned}$$

For $T \in N$ let

$$
\begin{aligned}
x^T := & \{in_{r^T}(\{a_1^T, a_2^T\}), in_{r^{M\setminus T}}(\{a_3^{M\setminus T}, a_4^{M\setminus T}\})\} \cup \\
& \{in_{u^{M\setminus T,U}}(\{a_3^{M\setminus T}, a_4^{M\setminus T}\}) | U \in N, (M \setminus T) \cap U = \emptyset\} \cup \\
& \{in_{u^{T,U}}(\{b_1^{T,U}, b_2^{T,U}\}) | U \in N, T \cap U = \emptyset\} \cup \\
& \{in_{v^{U,M\setminus T}}(\{a_3^{M\setminus T}, a_4^{M\setminus T}\}) | U \in N, (M \setminus T) \cap U = \emptyset\} \cup \\
& \{in_{v^{U,T}}(\{b_3^{U,T}, b_4^{U,T}\}) | U \in N, T \cap U = \emptyset\} \cup \\
& \{in_{s^{T,U}}(\{b_1^{T,U}, b_2^{T,U}\}) | U \in N, T \cap U = \emptyset\} \cup \\
& \{in_{s^{U,T}}(\{b_3^{U,T}, b_4^{U,T}\}) | U \in N, T \cap U = \emptyset\} \cup \\
& \{in_{s^{U,V}}(\{b_5^{U,V}, b_6^{U,V}\}) | U, V \in N, U \cap V = \emptyset, U \cup V = M \setminus T\} \cup \\
& \{in_{s^{U,V}}(\{b_1^{U,V}, b_2^{U,V}, b_3^{U,V}, b_4^{U,V}\}) | U, V \in N, U \cap V = \emptyset, U \cup V = T\} \cup \\
& \{in_{s^{U,M\setminus T}}(\{b_1^{U,M\setminus T}, b_2^{U,M\setminus T}, b_5^{U,M\setminus T}, b_6^{U,M\setminus T}\}) | U \in N, U \cap (M \setminus T) = \emptyset, U \cup (M \setminus T) \neq M\} \cup \\
& \{in_{s^{M\setminus T,U}}(\{b_3^{M\setminus T,U}, b_4^{M\setminus T,U}, b_5^{M\setminus T,U}, b_6^{M\setminus T,U}\}) | U \in N, U \cap (M \setminus T) = \emptyset, U \cup (M \setminus T) \neq M\} \cup \\
& \{in_{w^{U,V}}(\{b_5^{U,V}, b_6^{U,V}\}) | U, V \in N, U \cap V = \emptyset, U \cup V = M \setminus T\} \cup \\
& \{in_{w^{U,V}}(\{a_1^T, a_2^T\}) | U, V \in N, U \cap V = \emptyset, U \cup V = T\}
\end{aligned}
$$

Define

$$\rho_1 :=\{(in_{r^{T\cup U}}(\{a\}), in_{w^{T,U}}(\{a\}))|$$
$$T, U \in N, T \cap U = \emptyset, T \cup U \neq M, a \in \{a_1^{T\cup U}, a_2^{T\cup U}\}\}$$
$$\rho_2 :=\{(in_{w^{T,U}}(\{a\}), in_{w^{V,W}}(\{a\}))|$$
$$T, U \in N, T \cap U = \emptyset = V \cap W, T \cup U = V \cup W \neq M, a \in \{a_1^{T\cup U}, a_2^{T\cup U}\}\}$$
$$\rho_3 :=\{(in_{r^T}(\{a\}), in_{u^{T,U}}(\{a\}))|$$
$$T, U \in N, T \cap U = \emptyset, a \in \{a_3^T, a_4^T\}\}$$
$$\rho_4 :=\{(in_{r^T}(\{a\}), (in_{v^{U,T}}(\{a\}))|$$
$$T, U \in N, T \cap U = \emptyset, a \in \{a_3^T, a_4^T\}\}$$
$$\rho_5 :=\{(in_{u^{T,U}}(\{a\}), in_{u^{T,V}}(\{a\}))|$$
$$T, U, V \in N, T \cap U = \emptyset = T \cap V, a \in \{a_3^T, a_4^T\}\}$$
$$\rho_6 :=\{(in_{u^{T,U}}(\{a\}), in_{v^{V,T}}(\{a\}))|$$
$$T, U, V \in N, T \cap U = \emptyset = T \cap V, a \in \{a_3^T, a_4^T\}\}$$
$$\rho_7 :=\{(in_{v^{U,T}}(\{a\}), (in_{v^{V,T}}(\{a\}))|$$
$$T, U, V \in N, T \cap U = \emptyset = T \cap V, a \in \{a_3^T, a_4^T\}\}$$
$$\rho_8 :=\{(in_{u^{T,U}}(\{b\}), (in_{s^{T,U}}(\{b\}))|$$
$$T, U \in N, T \cap U = \emptyset, b \in \{b_1^{T,U}, b_2^{T,U}\}\}$$
$$\rho_9 :=\{(in_{v^{T,U}}(\{b\}), (in_{s^{T,U}}(\{b\}))|$$
$$T, U \in N, T \cap U = \emptyset, b \in \{b_3^{T,U}, b_4^{T,U}\}\}$$
$$\rho_{10} :=\{(in_{w^{T,U}}(\{b\}), (in_{s^{T,U}}(\{b\}))|$$
$$T, U \in N, T \cap U = \emptyset, T \cup U \neq M, b \in \{b_5^{T,U}, b_6^{T,U}\}\}$$
$$\rho_{j+10} :=\rho'_j := \{(y', z')|(y, z) \in \rho_j\} \text{ for } 1 \leq j \leq 10$$
$$\rho_{21} :=\bigcup\{x^T \times x^T | T \in N\}$$

Let $\rho := \bigcup_{1\leq i\leq 21} \rho_i$. Now we show that $ref(sym(\rho))$ is a congruence relation. $ref(sym(\rho))$ is reflexive, symmetrical and compatible with $'$.

Transitivity of $ref(sym(\rho))$:
For $i, j \leq 21$ let

$$\rho_i^{-1} := \{(x,y)|(y,x) \in \rho_i\} \text{ and}$$

$$\rho_i \circ \rho_j := \{(x,z)| \text{ there exists an } y \text{ with } (x,y) \in \rho_i \text{ and } (y,z) \in \rho_j\}.$$

We just have to consider those relations ρ_i, ρ_j which have a common component.
We only consider pairs $(x,y) \in \rho_i \cup \rho_i^{-1}$ and $(y,z) \in \rho_j \cup \rho_j^{-1}$ with $i \leq j$ because if $i > j$ holds then we have $(z,y) \in \rho_j \cup \rho_j^{-1}$ and $(y,x) \in \rho_i \cup \rho_i^{-1}$ with $j \leq i$. We have

$$\rho_1^{-1} \circ \rho_1 \subseteq \rho_2 \subseteq ref(sym(\rho))$$

$$\rho_1 \circ \rho_2 \subseteq \rho_1 \subseteq ref(sym(\rho))$$

$$\rho_2 \circ \rho_2 \subseteq \rho_2 \subseteq ref(sym(\rho))$$

$$\rho_3^{-1} \circ \rho_3 \subseteq \rho_5 \subseteq ref(sym(\rho))$$

$$\rho_3^{-1} \circ \rho_4 \subseteq \rho_6 \subseteq ref(sym(\rho))$$

$$\rho_3 \circ \rho_5 \subseteq \rho_3 \subseteq ref(sym(\rho))$$

$$\rho_3 \circ \rho_6 \subseteq \rho_4 \subseteq ref(sym(\rho))$$

$$\rho_4^{-1} \circ \rho_4 \subseteq \rho_7 \subseteq ref(sym(\rho))$$

$$\rho_4 \circ \rho_6^{-1} \subseteq \rho_3 \subseteq ref(sym(\rho))$$

$$\rho_4 \circ \rho_7 \subseteq \rho_4 \subseteq ref(sym(\rho))$$

$$\rho_5 \circ \rho_5 \subseteq \rho_5 \subseteq ref(sym(\rho))$$

$$\rho_5 \circ \rho_6 \subseteq \rho_6 \subseteq ref(sym(\rho))$$

$$\rho_6 \circ \rho_6^{-1} \subseteq \rho_5 \subseteq ref(sym(\rho))$$

$$\rho_6^{-1} \circ \rho_6 \subseteq \rho_7 \subseteq ref(sym(\rho))$$

$$\rho_6 \circ \rho_7 \subseteq \rho_6 \subseteq ref(sym(\rho))$$

$$\rho_7 \circ \rho_7 \subseteq \rho_7 \subseteq ref(sym(\rho))$$

For $(x,y) \in \rho_i \cup \rho_i^{-1}$ and $(y,z) \in \rho_j \cup \rho_j^{-1}$ with $11 \leq i \leq j \leq 20$ we have

$$(x',y') \in \rho_{i-10} \cup \rho_{i-10}^{-1} \text{ and } (y',z') \in \rho_{j-10} \cup \rho_{j-10}^{-1}$$

so $(x',z') \in ref(sym(\rho))$ and $(x,z) \in ref(sym(\rho))' = ref(sym(\rho))$.
We have $\rho_{21} \circ \rho_{21} = \rho_{21}$, therefore ρ is transitive.
Compatibility with $\oplus$:
Let $(x_1,y_1) \in ref(sym(\rho_i))$ and $(x_2,y_2) \in ref(sym(\rho_j))$ with $1 \leq i \leq j \leq 21$ such that $x_1 \oplus x_2$ and $y_1 \oplus y_2$ exist. We can assume $0 \notin \{x_1,x_2,y_1,y_2\}$ because if for example $x_1 = 0$ then we get $y_1 = 0$ and $(x_1 \oplus x_2, y_1 \oplus y_2) = (x_2,y_2) \in ref(sym(\rho))$.
If $x_1 = y_1$ holds then we get $x_2 = y_2$ because of the existence of the sums, so $(x_1 \oplus x_2, y_1 \oplus y_2) \in ref(sym(\rho))$.
If $i,j \leq 10$ then $i = j$ and $(x_1 \oplus x_2, y_1 \oplus y_2) \in \rho_{21} \subseteq ref(sym(\rho))$.
If $i \leq 10 < j \leq 20$ then $j = i+10$ and $(x_1 \oplus x_2, y_1 \oplus y_2) = (1,1) \in ref(sym(\rho))$.
If $i = 1$ and $j = 21$ then there exist $T,U \in N$ and $a \in \{a_1^{T\cup U}, a_2^{T\cup U}\}$ with

$$x_1 = in_{r^{T\cup U}}(\{a\}), y_1 = in_{w^{T,U}}(\{a\}) \text{ and}$$

$$x_2 = in_{r^{T\cup U}}(\{a_3^{T\cup U}, a_4^{T\cup U}\}), y_2 = in_{w^{T,U}}(\{b_5^{T,U}, b_6^{T,U}\}),$$

so $(x_1 \oplus x_2, y_1 \oplus y_2) \in \rho_{11} \subseteq ref(sym(\rho))$.
Analogously for $i \in \{2,3,4,5,6,7\}, j = 21$.
If $i = 8$ and $j = 21$ then there exist $T,U \in N$ and $b \in \{b_1^{T,U}, b_2^{T,U}\}$ with

$$x_1 = in_{u^{T,U}}(\{b\}), y_1 = in_{s^{T,U}}(\{b\}) \text{ and}$$

$$x_2 = in_{u^{T,U}}(\{a_3^T, a_4^T\}), y_2 = in_{s^{T,U}}(s^{T,U} \setminus \{b_1^{T,U}, b_2^{T,U}\}),$$

so $(x_1 \oplus x_2, y_1 \oplus y_2) \in \rho_{18} \subseteq ref(sym(\rho))$.
Analogously for $i \in \{9,10\}, j = 21$.
If $11 \leq i \leq 20$ then the sums $x_1 \oplus x_2$ and $y_1 \oplus y_2$ do not exist because of $j \geq i$.
If $i = 21 = j$ then $(x_1 \oplus x_2, y_1 \oplus y_2) \in \rho_{21} \cup \{(1,1)\} \subseteq ref(sym(\rho))$.
Therefore $ref(sym(\rho))$ is a congruence relation. We have $\sim_D \subseteq ref(sym(\rho))$ and therefore $<\sim_D> \subseteq ref(sym(\rho))$.
Proof of $ref(sym(\rho)) \subseteq <\sim_D>$:
For $i \leq 20$ we have $\rho_i \subseteq <\sim_D>$. Now we show $\rho_{21} \subseteq <\sim_D>$.
We have $\rho_1 \subseteq <\sim_D>$ and with the operation $\oplus$ we get

$$(in_{r^T}(\{a_1^T, a_2^T\}), in_{w^{U,V}}(\{a_1^T, a_2^T\})) \in <\sim_D>$$

for all $T,U,V \in N$ with $U \dot\cup V = T$ and with the operation $'$ we get

$$(in_{r^{M\setminus T}}(\{a_3^{M\setminus T}, a_4^{M\setminus T}\}), in_{w^{U,V}}(\{b_5^{U,V}, b_6^{U,V}\})) \in <\sim_D>$$

for all $T, U, V \in N$ with $U \dot{\cup} V = M \setminus T$.
We have $\rho_3 \subseteq <\sim_D>$ and with the operation $\oplus$ we get

$$(in_{r^{M\setminus T}}(\{a_3^{M\setminus T}, a_4^{M\setminus T}\}), in_{u^{M\setminus T,U}}(\{a_3^{M\setminus T}, a_4^{M\setminus T}\})) \in <\sim_D>$$

and with the operation $'$ we get

$$(in_{r^T}(\{a_1^T, a_2^T), in_{u^{T,U}}(\{b_1^{T,U}, b_2^{T,U}\})) \in <\sim_D> .$$

We have $\rho_4 \subseteq <\sim_D>$ and with the operation $\oplus$ we get

$$(in_{r^{M\setminus T}}(\{a_3^{M\setminus T}, a_4^{M\setminus T}\}), in_{v^{U,M\setminus T}}(\{a_3^{M\setminus T}, a_4^{M\setminus T}\})) \in <\sim_D>$$

and with the operation $'$ we get

$$(in_{r^T}(\{a_1^T, a_2^T\}), in_{v^{U,T}}(\{b_3^{U,T}, b_4^{U,T}\})) \in <\sim_D> .$$

We have $\rho_8 \subseteq <\sim_D>$ and with the operation $\oplus$ we get

$$(in_{u^{T,U}}(\{b_1^{T,U}, b_2^{T,U}\}), in_{s^{T,U}}(\{b_1^{T,U}, b_2^{T,U}\})) \in <\sim_D>$$

and with the operation $'$ we get

$$(in_{u^{M\setminus T,U}}(\{a_3^{M\setminus T}, a_4^{M\setminus T}\}), in_{s^{M\setminus T,U}}(\{b_3^{M\setminus T,U}, b_4^{M\setminus T,U}, b_5^{M\setminus T,U}, b_6^{M\setminus T,U}\})) \in <\sim_D> .$$

We have $\rho_9 \subseteq <\sim_D>$ and with the operation $\oplus$ we get

$$(in_{v^{U,T}}(\{b_3^{U,T}, b_4^{U,T}\}), in_{s^{U,T}}(\{b_3^{U,T}, b_4^{U,T}\})) \in <\sim_D>$$

and with the operation $'$ we get

$$(in_{v^{U,M\setminus T}}(\{a_3^{M\setminus T}, a_4^{M\setminus T}\}), in_{s^{U,M\setminus T}}(\{b_1^{U,M\setminus T}, b_2^{U,M\setminus T}, b_5^{U,M\setminus T}, b_6^{U,M\setminus T}\})) \in <\sim_D> .$$

We have $\rho_{10} \subseteq <\sim_D>$ and with the operation $\oplus$ we get

$$(in_{w^{U,V}}(\{b_5^{U,V}, b_6^{U,V}\}), in_{s^{U,V}}(\{b_5^{U,V}, b_6^{U,V}\})) \in <\sim_D>$$

and with the operation $'$ we get

$$(in_{w^{U,V}}(\{a_1^{U\cup V}, a_2^{U\cup V}\}), in_{s^{U,V}}(\{b_1^{U,V}, b_2^{U,V}, b_3^{U,V}, b_4^{U,V}\})) \in <\sim_D> .$$

We have $\rho_8 \subseteq <\sim_D>$ and with the operation $\oplus$ we get

$$(in_{u^{T,M\setminus T}}(\{b_1^{T,M\setminus T}, b_2^{T,M\setminus T}\}), in_{s^{T,M\setminus T}}(\{b_1^{T,M\setminus T}, b_2^{T,M\setminus T}\})) \in <\sim_D>$$

and with the operation $'$ we get

$$(in_{u^{T,M\setminus T}}(\{a_3^T, a_4^T\}), in_{s^{T,M\setminus T}}(\{b_3^{T,M\setminus T}, b_4^{T,M\setminus T}\}) \in <\sim_D>,$$

and with the transitivity of $<\sim_D>$ we get

$$(in_{r^T}(\{a_3^T, a_4^T\}), in_{v^{T,M\setminus T}}(\{b_3^{T,M\setminus T}, b_4^{T,M\setminus T}\})) \in <\sim_D>,$$

and again with the operation $'$ and transitivity we get

$$(in_{r^T}(\{a_1^T, a_2^T\}), in_{r^{M\setminus T}}(\{a_3^{M\setminus T}, a_4^{M\setminus T}\})) \in <\sim_D> .$$

With the transitivity of $<\sim_D>$ we get $x^T \times x^T \subseteq <\sim_D>$ for all $T \in N$. Therefore $<\rho> = ref(sym(\rho)) = <\sim_D>$ holds.
D is a **diagram:**
For $a \in p \in R$ and $E \subseteq q \in R$ with $(in_p(\{a\}), in_q(E)) \in <\sim_D> = ref(sym(\rho))$ we get $E = \{a\}$, so with Theorem 12 we only have to proof (C2). Let $p \in R$. The mapping $nat_{<\sim_D>} \circ in_p$ is injective because of $<\sim_D> = ref(sym(\rho))$.
Now let $in_p(E)/ <\sim_D>, in_p(F)/ <\sim_D> \in [\![D]\!]$ such that

$$in_p(E)/ <\sim_D> \oplus in_p(F)/ <\sim_D>$$

exists. Then there exist $q \in R$ and $G, H \in \mathcal{P}_{\text{fin}}^{\text{cofin}}(q)$ with $G \cap H = \emptyset$ and

$$(in_p(E), in_q(G)) \in <\sim_D> = ref(sym(\rho)) \text{ and}$$

$$(in_p(F), in_q(H)) \in <\sim_D> = ref(sym(\rho)).$$

If $E = \emptyset$ or $F = \emptyset$ holds then we get $E \cap F = \emptyset$ and $E \oplus F$ exists, so in the following we can assume that $E \neq \emptyset \neq F$ holds. We get $G \neq \emptyset \neq H$. If $p = q$ holds then we get $E = G$ and $F = H$ because of the definition of ρ, so $E \cap F = \emptyset$. In the following we can assume $p \neq q$. Let $i, j \leq 21$ with

$$(in_p(E), in_q(G)) \in \rho_i \cup \rho_i^{-1} \text{ and}$$

$$(in_p(F), in_q(H)) \in \rho_j \cup \rho_j^{-1}.$$

We can assume that $i \leq j$ holds, because otherwise we exchange E and F. If $j \leq 10$ holds then we get $i \leq 10$ and $E \cap F = G \cap H = \emptyset$. If $11 \leq j \leq 20$ holds then we get $|q \setminus H| = 1$ and $G = q \setminus H$, therefore

$$(in_p(p \setminus E), in_q(H)) = (in_p(E)', in_q(G)') \in ref(sym(\rho)),$$

so $(in_p(p \setminus E), in_p(F)) \in ref(sym(\rho))$ and $p \setminus E = F$ which implies $E \cap F = \emptyset$. Analogously for $11 \leq i \leq 20$.
Now let $j = 21$. If $i \leq 10$ holds, then we get $E = G$ and $|E| = 1$. Let $E = \{e\}$. Assume that $E \cap F \neq \emptyset$ holds. Note that for $T, U \in N$ we have $T = U$ iff $x^T = x^U$. Because of $e \in p \cap q$ and $e \in F$ and $(in_p(F), in_q(H)) \in \rho_{21} \cup \rho_{21}^{-1}$ we just have to consider the following four cases:
Case 1: $F \subseteq p \cap q$
Then we get $(in_q(F), in_q(H)) \in ref(sym(\rho))$ and $F = H$, therefore

$$E \cap F = G \cap H = \emptyset$$

which is a contradiction.
Case 2: There exist $U, V \in N$ with $p = s^{U,V}$ and $F = \{b_1^{U,V}, b_2^{U,V}, b_3^{U,V}, b_4^{U,V}\}$
Then we have $i \in \{8, 9\}$ because of $G = E \subseteq F$. If $i = 8$ holds then we get $q = u^{U,V}$ and $H = \{a_3^U, a_4^U\}$, which is a contradiction to $(in_p(F), in_q(H)) \in \rho_{21} \cup \rho_{21}^{-1}$. If $i = 9$ holds then we get $q = v^{U,V}$ and $H = \{a_3^V, a_4^V\}$, again a contradiction.
Case 3: There exist $U, V \in N$ with $p = s^{U,V}$ and $F = \{b_1^{U,V}, b_2^{U,V}, b_5^{U,V}, b_6^{U,V}\}$
Then we have $i \in \{8, 10\}$. If $i = 8$ holds then we get $q = u^{U,V}$ and $H = \{a_3^U, a_4^U\}$, which is a contradiction to $(in_p(F), in_q(H)) \in \rho_{21} \cup \rho_{21}^{-1}$. If $i = 10$ holds then we get $q = w^{U,V}$ and $H = \{a_1^{U \cup V}, a_2^{U \cup V}\}$, again a contradiction to $(in_p(F), in_q(H)) \in \rho_{21} \cup \rho_{21}^{-1}$.
Case 4: There exist $U, V \in N$ with $p = s^{U,V}$ and $F = \{b_3^{U,V}, b_4^{U,V}, b_5^{U,V}, b_6^{U,V}\}$
Then we have $i \in \{9, 10\}$. If $i = 9$ holds then we get $q = v^{U,V}$ and $H = \{a_3^V, a_4^V\}$, again a contradiction to $(in_p(F), in_q(H)) \in \rho_{21} \cup \rho_{21}^{-1}$.
If $i = 10$ holds then we get $q = w^{U,V}$ and $H = \{a_1^{U \cup V}, a_2^{U \cup V}\}$, again a contradiction to $(in_p(F), in_q(H)) \in \rho_{21} \cup \rho_{21}^{-1}$.
Now let $i = 21 = j$. If $G = q \setminus H$ then we get $E \cap F = \emptyset$ like in the case $11 \leq j \leq 20$, so we assume $G \neq q \setminus H$. Therefore there exist $T, U \in N$ with $T \cup U \neq M$ and $q = s^{T,U}$ and $|G| = 2 = |H|$. Assume that $E \cap F \neq \emptyset$ holds. We have $E \neq F$ because otherwise we get $(in_q(G), in_q(H)) \in ref(sym(\rho))$ because of the transitivity. Therefore there exist $V, W \in N$ with $V \cup W \neq M, p = s^{V,W}, |E| = 4 = |F|$ and $|E \cap F| = 2$. Then we just have to consider three cases:
Case 1: $G = \{b_1^{T,U}, b_2^{T,U}\}$
Then $in_q(G) \in x^T$ holds and we get $in_p(E) \in x^T$ because of $(in_p(E), in_q(G)) \in \rho_{21}$. We have $H = \{b_3^{T,U}, b_4^{T,U}\}$ or $H = \{b_5^{T,U}, b_6^{T,U}\}$, so we get $in_q(H) \in x^U$ or $in_q(H) \in x^{M \setminus (T \cup U)}$.
Case 1.1: $E = \{b_1^{V,W}, b_2^{V,W}, b_3^{V,W}, b_4^{V,W}\}$ then we get $V \dot{\cup} W = T$ because of the definition of x^T. We have $F = \{b_1^{V,W}, b_2^{V,W}, b_5^{V,W}, b_6^{V,W}\}$ or $F = \{b_3^{V,W}, b_4^{V,W}, b_5^{V,W}, b_6^{V,W}\}$, so we get $in_p(F) \in x^{M \setminus W}$ or $in_p(F) \in x^{M \setminus V}$. We have $T \cap (M \setminus W) \neq \emptyset \neq T \cap (M \setminus V)$ but $T \cap U = \emptyset = T \cap (M \setminus (T \cup U))$, so $U \neq M \setminus W \neq M \setminus (T \cup U)$ and $U \neq M \setminus V \neq M \setminus (T \cup U)$, so $in_p(F) \notin x^U$ and $in_p(F) \notin x^{M \setminus (T \cup U)}$ which is a contradiction to $(in_p(F), in_q(H)) \in \rho_{21}$.
Case 1.2: $E = \{b_1^{V,W}, b_2^{V,W}, b_5^{V,W}, b_6^{V,W}\}$

Then we get $M \setminus W = T$, so $in_p(F) \in x^{M \setminus V}$ or $in_p(F) \in x^{V \cup W}$. We have $U \subseteq M \setminus T = W$ but $M \setminus V \not\subseteq W$ because of $V \cup W \neq M$. We have $U \subseteq M \setminus T = W \subseteq M \setminus V$ but $U \not\subseteq M \setminus (T \cup U)$, therefore $U \neq M \setminus V \neq M \setminus (T \cup U)$ holds. We have $V \subseteq M \setminus W = T$, so $V \not\subseteq U$ and $V \not\subseteq M \setminus (T \cup U)$, therefore $U \neq V \cup W \neq M \setminus (T \cup U)$ holds. This is a contradiction to $(in_p(F), in_q(H)) \in \rho_{21}$.
Case 1.3: $E = \{b_3^{V,W}, b_4^{V,W}, b_5^{V,W}, b_6^{V,W}\}$
Then we get $M \setminus V = T$, so $in_p(F) \in x^{M \setminus W}$ or $in_p(F) \in x^{V \cup W}$ which is a contradiction to $(in_p(F), in_q(H)) \in \rho_{21} \cup \rho_{21}^{-1}$ like in Case 1.2 (just exchange V and W).
Case 2: $G = \{b_3^{T,U}, b_4^{T,U}\}$
Then $in_q(G) \in x^U$ holds and we get $in_q(H) \in x^T$ or $in_q(H) \in x^{M \setminus (T \cup U)}$, so we get the contradiction like in case 1 (just exchange T and U).
Case 3: $G = \{b_5^{T,U}, b_6^{T,U}\}$
Then $H = \{b_1^{T,U}, b_2^{T,U}\}$ or $H = \{b_3^{T,U}, b_4^{T,U}\}$ holds, so we can exchange E with F and G with H to get the same situation like in case 1 or case 2.
In all cases we get a contradiction, so $E \cap F = \emptyset$ holds and $E \oplus F$ exists in $\mathcal{P}_{\text{fin}}^{\text{cofin}}(p)$ and the mapping $nat_{<\sim_D>} \circ in_p$ is closed which proves (C2). So D is a diagram because of Theorem 12.
For $T \in N$ the set x^T is an equivalence class: $x^T = in_{r^T}(\{a_1^T, a_2^T\}) / <\sim_D> \in [\![D]\!]$
Define $x^{\emptyset} := 0, x^M := 1$ and $B := \{x^T | T \subseteq M\} \subseteq [\![D]\!]$.
Proof of $\underline{B} \cong \mathcal{P}(M)$:
We have $x^T \neq x^U$ iff $T \neq U$ for $T, U \subseteq M$. Therefore the mapping

$$\phi : \underline{\mathcal{P}(M)} \to \underline{B}, T \mapsto x^T$$

is bijective. For $T, U \in N$ with $U = M \setminus T$ we have

$$\begin{aligned} \phi(T)' = (x^T)' &= (in_{s^{T,U}}(\{b_1^{T,U}, b_2^{T,U}\}) / <\sim_D>)' \\ &= in_{s^{T,U}}(\{b_3^{T,U}, b_4^{T,U}\}) / <\sim_D> = x^U = \phi(T'), \end{aligned}$$

so ϕ is compatible with $'$.
For $T = \emptyset$ the sum $x^T \oplus x^U$ exists and equals to x^U for all $U \subseteq M$. Now let $T, U \in N$ such that $T \oplus U$ exists in $\mathcal{P}(M)$. Then we get $x^T = in_{s^{T,U}}(\{b_1^{T,U}, b_2^{T,U}\}) / <\sim_D>$ and $x^U = in_{s^{T,U}}(\{b_3^{T,U}, b_4^{T,U}\}) / <\sim_D>$, so $x^T \oplus x^U$ exists. If $T \cup U = M$ then we get

$$\phi(T \oplus U) = x^M = 1 =$$

$$in_{s^{T,U}}(\{b_1^{T,U}, b_2^{T,U}, b_3^{T,U}, b_4^{T,U}\}) / <\sim_D> = x^T \oplus x^U = \phi(T) \oplus \phi(U)$$

and if $T \cup U \neq M$ then we get

$$\phi(T \oplus U) = x^{T \cup U} = in_{s^{T,U}}(\{b_1^{T,U}, b_2^{T,U}, b_3^{T,U}, b_4^{T,U}\}) / <\sim_D> = x^T \oplus x^U = \phi(T) \oplus \phi(U),$$

so ϕ is compatible with $\oplus$.
Now let $T, U \in N$ such that $x^T \oplus x^U$ exists. Then there exist $E, F \subseteq q \in R$ with

$$E \cap F = \emptyset \text{ and } x^T = in_q(E)/ <\sim_D> \text{ and } x^U = in_q(F)/ <\sim_D> .$$

Because of the definition of x^T we have to consider the following cases:
Case 1: There exists a set $V \in N$ with

$$q \in \{r^T, r^{M\setminus T}, u^{M\setminus T,V}, u^{T,V}, v^{V,M\setminus T}, v^{V,T}, w^{V,M\setminus(T\cup V)}, w^{V,T\setminus V}, s^{V,T\setminus V}, s^{V,M\setminus T}, s^{M\setminus T,V}\}$$

Then we get $T = M \setminus U$ because of $E \cap F = \emptyset$, so $T \oplus U$ exists in $\mathcal{P}(M)$.
Case 2: There exists a set $V \in N$ with $T \cap V = \emptyset$ and $q = s^{T,V}$
Then we have

$$E = \{b_1^{T,V}, b_2^{T,V}\} \text{ and } F \in \{\{b_3^{T,V}, b_4^{T,V}\}, \{b_5^{T,V}, b_6^{T,V}\}, \{b_3^{T,V}, b_4^{T,V}, b_5^{T,V}, b_6^{T,V}\}\}.$$

If $F = \{b_3^{T,V}, b_4^{T,V}\}$ then we get $U = V$ and $T \oplus U$ exists.
If $F = \{b_5^{T,V}, b_6^{T,V}\}$ then we get $U = M \setminus (T \cup V)$ and $T \oplus U$ exists.
If $F = \{b_3^{T,V}, b_4^{T,V}, b_5^{T,V}, b_6^{T,V}\}$ then we get $U = M \setminus T$ and $T \oplus U$ exists.
Case 3: There exists a set $V \in N$ with $T \cap V = \emptyset$ and $q = s^{V,T}$
Then we have

$$E = \{b_3^{V,T}, b_4^{V,T}\} \text{ and } F \in \{\{b_1^{V,T}, b_2^{V,T}\}, \{b_5^{V,T}, b_6^{V,T}\}, \{b_1^{V,T}, b_2^{V,T}, b_5^{V,T}, b_6^{V,T}\}\}.$$

If $F = \{b_1^{V,T}, b_2^{V,T}\}$ then we get $U = V$ and $T \oplus U$ exists.
If $F = \{b_5^{V,T}, b_6^{V,T}\}$ then we get $U = M \setminus (T \cup V)$ and $T \oplus U$ exists.
If $F = \{b_1^{V,T}, b_2^{V,T}, b_5^{V,T}, b_6^{V,T}\}$ then we get $U = M \setminus T$ and $T \oplus U$ exists.
Case 4: There exists a set $V \in N$ with $T \cap V = \emptyset$ and $q = s^{V,M\setminus(T\cup V)}$
Then we have

$$\begin{aligned} E &= \{b_5^{V,M\setminus(T\cup V)}, b_6^{V,M\setminus(T\cup V)}\} \text{ and} \\ F &\in \{\{b_1^{V,M\setminus(T\cup V)}, b_2^{V,M\setminus(T\cup V)}\}, \{b_3^{V,M\setminus(T\cup V)}, b_4^{V,M\setminus(T\cup V)}\}, \\ &\quad \{b_1^{V,M\setminus(T\cup V)}, b_2^{V,M\setminus(T\cup V)}, b_3^{V,M\setminus(T\cup V)}, b_4^{V,M\setminus(T\cup V)}\}\}. \end{aligned}$$

If $F = \{b_1^{V,M\setminus(T\cup V)}, b_2^{V,M\setminus(T\cup V)}\}$ then we get $U = V$ and $T \oplus U$ exists.
If $F = \{b_3^{V,M\setminus(T\cup V)}, b_4^{V,M\setminus(T\cup V)}\}$ then we get $U = M \setminus (T \cup V)$ and $T \oplus U$ exists.
If $F = \{b_1^{V,M\setminus(T\cup V)}, b_2^{V,M\setminus(T\cup V)}, b_3^{V,M\setminus(T\cup V)}, b_4^{V,M\setminus(T\cup V)}\}$ then we get $U = M \setminus T$ and $T \oplus U$ exists.
Therefore ϕ is closed. We have $\phi(0) = 0$, therefore ϕ is an isomorphism.
Now we show that $\underline{B}$ is a block of $[\![D]\!]$. Let $\underline{C} \leq [\![D]\!]$ a Boolean subalgebra which contains $\underline{B}$. Assume that there exists an element $y = in_q(E)/ <\sim_D> \in C \setminus B$. Here

we choose E with minimal cardinality. Then $y' = in_q(q \setminus E)/ <\sim_D> \in C \setminus B$ also holds. Let $T_0, U_0 \in N$ with

$$q \in \{r^{T_0}, s^{T_0,U_0}, u^{T_0,U_0}, v^{T_0,U_0}, w^{T_0,U_0}\}.$$

Because of $|M| > 3$ there exists a set $Z \subseteq M$ with

$$Z \notin \{\emptyset, M, T_0, U_0, M \setminus T_0, M \setminus U_0, T_0 \cup U_0, M \setminus (T_0 \cup U_0)\}.$$

We have the existence (see [BM98]) of $a, b, c \in \underline{C}$ with $a \oplus b = x^Z, b \oplus c = y$ such that $a \oplus c$ exists. There exist $F, G \subseteq p \in R$ with

$$b = in_p(F)/ <\sim_D>, c = in_p(G)/ <\sim_D> \text{ and } in_p(F \dot{\cup} G)/ <\sim_D> = in_q(E)/ <\sim_D> .$$

Because of $y \notin B$ we have

$$\begin{aligned}
&E \neq \emptyset,\\
&E \neq \{a_1^T, a_2^T\},\\
&E \neq \{a_3^T, a_4^T\},\\
&E \neq \{b_1^{T,U}, b_2^{T,U}\},\\
&E \neq \{b_3^{T,U}, b_4^{T,U}\},\\
&E \neq \{b_5^{T,U}, b_6^{T,U}\},\\
&E \neq \{b_1^{T,U}, b_2^{T,U}, b_3^{T,U}, b_4^{T,U}\},\\
&E \neq \{b_1^{T,U}, b_2^{T,U}, b_5^{T,U}, b_6^{T,U}\},\\
&E \neq \{b_3^{T,U}, b_4^{T,U}, b_5^{T,U}, b_6^{T,U}\},\\
&E \neq \{b_1^{T,M\setminus T}, b_2^{T,M\setminus T}\},\\
&E \neq \{b_3^{T,M\setminus T}, b_4^{T,M\setminus T}\}
\end{aligned}$$

for all $T, U \in N$ with $T \cap U = \emptyset, T \cup U \neq M$. Therefore we have $(in_p(F \dot{\cup} G), in_q(E)) \notin \rho_{21}$. Because of the minimality of $|E|$ we have $|q \setminus E| > 1$, so $(in_p(F \dot{\cup} G), in_q(E)) \notin \rho_j \cup \rho_j^{-1}$ for $j \geq 10$ and we get $F \dot{\cup} G = E \subseteq q$. At least one of the elements b and c is not contained in B because we have $b \oplus c = y \notin B$. Because of the minimality of $|E|$ we get $E \in \{F, G\}$ and therefore $b = 0$ or $c = 0$.
Case 1: b = 0
We get the existence of $a \oplus c = x^Z \oplus in_q(E)/ <\sim_D>$ and therefore there exists

$H, I \subseteq l \in R$ with $x^Z = in_l(H)/ <\sim_D>, in_q(E)/ <\sim_D> = in_l(I)/ <\sim_D>$ and $H \cap I = \emptyset$. We have $l \neq q$ because of

$$Z \notin \{\emptyset, M, T_0, U_0, M \setminus T_0, M \setminus U_0, T_0 \cup U_0, M \setminus (T_0 \cup U_0)\}.$$

We have $|H| \geq 2$ because of the definition of x^Z, therefore $|l \setminus I| > 1$ holds. We have $(in_l(I), in_q(E)) \notin \rho_{21}$. Therefore there exists $1 \leq i \leq 10$ with

$$(in_l(I), in_q(E)) \in \rho_i \cup \rho_i^{-1}.$$

If $(in_l(I), in_q(E)) \in \rho_1$ then we get $q = w^{T_0, U_0}$ and

$$in_l(H) = in_{r^{T_0 \cup U_0}}(a_3^{T_0 \cup U_0}, a_4^{T_0 \cup U_0}) \in x^{M \setminus (T_0 \cup U_0)} \neq x^Z$$

which is a contradiction to $in_l(H)/ <\sim_D> = x^Z$.
If $(in_l(I), in_q(E)) \in \rho_1^{-1}$ then we get $q = r^{T_0}$ and

$$in_l(H) = in_{w^{U \cup V}}(b_5^{U,V}, b_6^{U,V}) \in x^{M \setminus T_0} \neq x^Z$$

which is again a contradiction.
Analogously we get a contradiction if $(in_l(H), in_q(E)) \in \rho_j \cup \rho_j^{-1}$ holds with $2 \leq j \leq 10$.
Case 2: c = 0
Then we get $a \oplus in_q(E)/ <\sim_D> = a \oplus b = x^Z$, so $in_q(E)/ <\sim_D> \leq x^Z$ and

$$in_q(E)/ <\sim_D> \oplus (x^Z)' = in_q(E)/ <\sim_D> \oplus x^{M \setminus Z}$$

exists. We have

$$M \setminus Z \notin \{\emptyset, M, T_0, U_0, M \setminus T_0, M \setminus U_0, T_0 \cup U_0, M \setminus (T_0 \cup U_0)\},$$

so we get the contradiction like in case 1 (with $M \setminus Z$ instead of Z). Therefore $\underline{B}$ is a block. The atoms of $\underline{B}$ are not contained in P. If M is finite then each block of $[\![D]\!]$ is finite and therefore each block is generated by its atoms. So theorem 16 does not hold for diagrams which are not OMA-diagrams. If M is infinite then the block $\underline{B}$ is not generated by $atoms(\underline{B})$.

The following example shows that there exists an OMA-diagram with finite lines, such that condition (2a) of Theorem 29 is satisfied, but there exists a clique which is not contained in a line.

Example 4:
Let $D = (P, R)$ with $R = \{r_n | n \geq 3\}$ with $r_n = \{a_0, a_1, a_2, \ldots, a_n, b_n, c_n\}$ for $n \geq 3$

and $a_n = (0,n)$ for $n \in \mathbb{N}$ and $b_n = (1,n)$ and $c_n = (2,n)$ for $n \geq 3$ and $P = \bigcup R$. Let $\rho = \rho_1 \cup \rho_2$ with

$$\rho_1 = \{(in_r(E), in_s(E)) | E \subseteq r \cap s \text{ and } r,s \in R\} \text{ and}$$

$$\rho_2 = \{(in_r(r \setminus E), in_s(s \setminus E)) | E \subseteq r \cap s \text{ and } r,s \in R\}.$$

Then ρ is reflexive and symmetrical. We have $\rho_1' = \rho_2$ and $\rho_2' = \rho_1$, so ρ is compatible with $'$. It is not difficult to see that ρ is transitive and compatible with $\oplus$, so ρ is a congruence relation. ρ is generated by $\sim_D$, so we get $<\sim_D>= \rho$.
D is a **diagram:**
For $a \in r \in R$ and $E \subseteq s \in R$ with $(in_r(\{a\}), in_s(E)) \in <\sim_D>= \rho$ we get $E = \{a\}$, so with Theorem 12 we only have to proof (C2). Let $r \in R$. The mapping $nat_{<\sim_D>} \circ in_r$ is injective because of $<\sim_D>= \rho$.
Now let $in_r(E)/ <\sim_D>, in_r(F)/ <\sim_D> \in [\![D]\!]$ such that

$$in_r(E)/ <\sim_D> \oplus in_r(F)/ <\sim_D>$$

exists. Then there exist $s \in R$ and $G, H \in \mathcal{P}^{\text{cofin}}_{\text{fin}}(s)$ with $G \cap H = \emptyset$ and

$$in_r(E)/ <\sim_D>= in_s(G)/ <\sim_D> \text{ and } in_r(F)/ <\sim_D>= in_s(H)/ <\sim_D> .$$

Because of $<\sim_D>= \rho$ we get $E \cap F = \emptyset$, so $nat_{<\sim_D>} \circ in_r$ is closed. Therefore D is a diagram. D is an OMA-diagram because of Theorem 27 and Lemma 28. The set $A = \{a_n | n \in \mathbb{N}\}$ is a clique which is not contained in a line. Condition (2a) of theorem 29 is satisfied. The diagram D is not complete because with Theorem 3 the set $A = \{[\![a_n]\!] | n \in \mathbb{N}\}$ generates an infinite Boolean subalgebra, but D contains only finite lines, so every block would be finite, if D is complete.

4 Conclusion

With the theorems of chapter 2 we get an algorithm how to check whether the interpretation of a finite hypergraph is a complete OMA-diagram:
Input: finite hypergraph $D = (P, R)$
Output: "yes" if D is a complete OMA-diagram, "no" otherwise
Algorithm: If there exist $r, s \in R$ with $r \neq s$ and $|r \setminus s| \leq 1$ then the algorithm ends with output "no". If the relation $\rho := \{(in_r(E), in_s(F)) |$ there exist $t, u \in R$ with $t = E \dot\cup (s \setminus F)$ and $u = F \dot\cup (r \setminus E)\}$ is not transitive or not compatible with $\oplus$ then output "no". If there exists a triangle but no line containing the corners of the triangle then output "no". If there exists a line $s \in R$ which is covered by two other lines but there does not exist a line containing the rest of the two other lines and their intersection, then output "no". Otherwise output "yes".

This algorithm has been implemented by the author in the program "omacheck". The correctness of this algorithm follows from Theorem 34 and from Lemma 32.

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