An Existence Result for a Model of Granular Material with Non-constant Density

Piotr Gwiazda *

University of Technology Darmstadt (Germany)

e-mali: gwiazda@mathematik.tu-darmstadt.de

Abstract

We consider a model developed by Savage and Hutter [12], [13] which describes the flow of granular avalanches down a smoothly varying slope. The system consists of two nonstrictly hyperbolic equations for height and momentum.
The existence of entropy solutions to this model is proved using the vanishing viscosity method, where we make extensive use of a generalised version of the invariant region theorem in order to prove a priori estimates. Since the model has a discontinuous source term, a new definition of entropy solution must be introduced.

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1 Introduction

The problem which we study can be written in the following form: For an avalanche down an inclined slope find the height $h : \mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{R}$, the density $\rho : \mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{R}$ and the velocity $v : \mathbb{R}_+ \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfying the following system of equations in the form of conservation laws

$$\begin{aligned} \frac{\partial}{\partial t}(\rho h) + \frac{\partial}{\partial x}(\rho h v) &= 0 \\ \frac{\partial}{\partial t}(\rho h v) + \frac{\partial}{\partial x}\left(\rho h v^2 + \frac{1}{2}\beta\rho h^2\right) &= \rho h g, \end{aligned} \tag{SH}$$

where $\beta := \beta(x)$ and $g := g(v, x)$ are given functions. Since these two conservation laws are not enough for an investigation of the evolution containing three variables (ρ, h, v), we need an additional constitutive relation. One of the methods is to assume

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that ρ is a function of h and v. In this case we obtain the system of two differential equations for two independent variables:

$$\begin{aligned}\frac{\partial}{\partial t}(\rho(h,v)h) + \frac{\partial}{\partial x}(\rho(h,v)hv) &= 0\\ \frac{\partial}{\partial t}(\rho(h,v)hv) + \frac{\partial}{\partial x}\left(\rho(h,v)hv^2 + \frac{1}{2}\beta\rho(h,v)h^2\right) &= \rho(h,v)hg.\end{aligned}$$

Since it is very difficult to examine the last system with the general constitutive relation $\rho := \rho(h,v)$, we will start with the very special (from the mathematical point of view) relation $\rho := h^\alpha$ with $0 \leq \alpha$, which will simplify our considerations very much[1]. Moreover, we assume that the functions $\beta(x)$, $g(v,x)$ are defined by

$$\beta(x) = k\cos(\gamma(x)),$$

$$g(v,x) = \sin(\gamma(x)) - \mathrm{sig}_0(v)\cos(\gamma(x))\tan\delta_F(x),$$

where $\tan\delta_F(x)$ is the dynamic friction angle at the position x and

$$\mathrm{sig}_0(v) = \begin{cases} 0 & \text{for } v = 0 \\ \frac{v}{|v|} & \text{for } v \neq 0. \end{cases}$$

Here $-\frac{\pi}{2} < \gamma(x) < \frac{\pi}{2}$ is an angle between the horizontal direction and the tangent to the base curve. In general the Savage-Hutter model defined in [12], [13] allows for a function β with a constant k switching between two positive states depending on whether $\frac{\partial}{\partial x}v$ is positive or negative.
The first problem is, the discontinuity of the function g at $v = 0$. In fact we do not need the continuity of g, but the maximal monotonicity of $-g$ with respect to the variable v. Thus we have to extend the function g to a multifunction

$$\tilde{g}(v,x) = \sin(\gamma(x)) - \mathrm{sig}(v)\cos(\gamma(x))\delta_F(x),$$

where

$$\mathrm{sig}(v) = \begin{cases} [-1,1] & \text{for } v = 0 \\ \frac{v}{|v|} & \text{for } v \neq 0. \end{cases}$$

Now we introduce a new variable $H = \rho h = h^{1+\alpha}$ and rewrite the system (SH) as

$$\begin{aligned}\frac{\partial}{\partial t}(H) + \frac{\partial}{\partial x}(Hv) &= 0\\ \frac{\partial}{\partial t}(Hv) + \frac{\partial}{\partial x}\left(Hv^2 + \frac{1}{2}\beta H^{\frac{2+\alpha}{1+\alpha}}\right) &\in H\tilde{g}(v,x).\end{aligned}$$

A further problem is the flux term

$$F(x,H,v) = \begin{pmatrix} Hv \\ \\ Hv^2 + \frac{1}{2}\beta(x)H^{\frac{2+\alpha}{1+\alpha}} \end{pmatrix}$$

[1]Up to now this system has been examined only with the very restrictive constitutive relation $\rho := \rho_0$ where ρ_0 is constant [1], [2].

containing β which is a function of x. For future considerations we need a flux term in homogeneous form, i.e., without x-dependence. Therefore we choose the new variable u defined by $u = (u_1, u_2) = ((\frac{\beta}{2\kappa})^{1+\alpha} H, (\frac{\beta}{2\kappa})^{1+\alpha} Hv)$. Here the constant $\kappa = \frac{1}{4}\frac{1}{(1+\alpha)(2+\alpha)}$ is introduced in consistency with the standard notation from [6], [7]. We obtain:

$$\frac{\partial}{\partial t} u + \frac{\partial}{\partial x} F(u) \in \tilde{G}(u, x), \qquad (CL)$$

where

$$F(u_1, u_2) = \begin{pmatrix} u_2 \\ \dfrac{(u_2)^2}{u_1} + \kappa (u_1)^{\frac{2+\alpha}{1+\alpha}} \end{pmatrix}$$

and

$$\tilde{G}(u_1, u_2, x) = \begin{pmatrix} \dfrac{\beta'(2\kappa)^{\alpha-1}}{\beta^{\alpha-1}} u_2 \\ \dfrac{\beta'(2\kappa)^{\alpha-1}}{\beta^{\alpha-1}} \left(\dfrac{(u_2)^2}{u_1} + \kappa (u_1)^{\frac{2+\alpha}{1+\alpha}} \right) + u_1 \tilde{g} \left(\dfrac{u_2}{u_1}, x \right) \end{pmatrix}.$$

It is well known that some classes of "very weak" solutions for conservation laws can not have physical meaning, for example the class of "measure valued solutions" [4]. First we must define a proper class of solutions we are looking for. This class contains "the weak entropy solutions" defined below.

Definition 1.1
Suppose that $\eta = \eta(u_1, u_2)$, $q = q(u_1, u_2)$ are scalar $\mathbb{C}^1$-functions satisfying

$$\nabla_{(u_1,u_2)} \eta(u_1, u_2) \cdot \nabla_{(u_1,u_2)} F(u_1, u_2) = \nabla_{(u_1,u_2)} q(u_1, u_2).$$

Such functions (η, q) are called Entropy-Flux Pairs. If η is convex, then (η, q) is called a Convex Entropy-Flux Pair.

Definition 1.2
We call $u \in \mathbb{L}^\infty([0, T) \times \mathbb{R}; \mathbb{R}_+ \times \mathbb{R})$ a Weak Entropy Solution to the system (CL) with the initial data $u^0 \in \mathbb{C}^0(\mathbb{R}; \mathbb{R}_+ \times \mathbb{R})$ iff:
1. there exists $G(t, x) \in \tilde{G}(u(t, x), x)$ for a.a. $(t, x) \in [0, T) \times \mathbb{R}$ with $G \in \mathbb{L}^\infty_{loc}([0, T) \times \mathbb{R}; \mathbb{R}^2)$,
2. u is a weak solution, i.e.;

$$\int\limits_{\mathbb{R}\times[0,T)} [u(t, x) \cdot \frac{\partial}{\partial t} \psi(t, x) + F(u(t, x)) \cdot \frac{\partial}{\partial x} \psi(t, x) + G(t, x) \cdot \psi(t, x)] \, dt dx = \int\limits_{\mathbb{R}} u^0(x) \cdot \psi(0, x) dx,$$

for all test functions $\psi \in \mathbb{C}^1_0([0, T) \times \mathbb{R}; \mathbb{R}^2)$,
3. the entropy inequality

$$\int\limits_{\mathbb{R}\times[0,T)} [\eta(u(t, x)) \frac{\partial}{\partial t} \phi(t, x) + q(u(t, x)) \frac{\partial}{\partial x} \phi(t, x) + \nabla_u \eta(u(t, x)) \cdot G(t, x) \phi(t, x)] \, dt dx \geq \int\limits_{\mathbb{R}} \eta(u^0(x)) \phi(0, x) dx$$

holds for all non-negative test functions $\phi \in \mathbb{C}_0^1([0,T) \times \mathbb{R}; \mathbb{R})$ and all convex weak entropy-flux pairs (η, q).
Here the family "convex weak entropy" is the class where $\nabla_u \eta(u) \in \mathbb{L}^\infty_{loc}([0,T) \times \mathbb{R}; \mathbb{R})$, $\eta(\cdot, \cdot)$ is $\mathbb{C}^2$-function for $(u_1, u_2) \in (0, \infty) \times \mathbb{R}$ and $\eta(0, \cdot) = 0$.

Remark
The last definition is not standard because of the differential inclusion appearing in the problem. More about the definition of the proper class of "convex weak entropies" see Chapter 4, step 2 below.

Main results:

Theorem 1.1.a (Local in time existence)
Assume an initial data $u^0 = (u_1^0, u_2^0) \in \mathbb{C}^2(\mathbb{R}; \mathbb{R}^2)$, $(u_1^0 - \overline{u}_1, u_2^0) \in \mathbb{L}^2(\mathbb{R}; \mathbb{R}^2)$ such that

1. $\inf_{x \in \mathbb{R}} u_1^0(x) > 0, \ \lim_{|x| \to \infty} u_1^0(x) = \overline{u}_1$

2. $\lim_{|x| \to \infty} u_2^0(x) = 0$

where $\overline{u}_1$ is a positive constant, and $\beta \in \mathbb{C}^3(\mathbb{R})$ with $\beta(x) \geq \beta_0 > 0$.
Then the equation (CL) possesses a local in time (i.e. $T = T_{max}$) weak entropy solution in sense of Definition 1.2.

Theorem 1.1.b (Global in time existence)
Assume an initial data $u^0 = (u_1^0, u_2^0) \in \mathbb{C}^2(\mathbb{R}; \mathbb{R}^2)$, $(u_1^0 - \overline{u}_1, u_2^0) \in \mathbb{L}^2(\mathbb{R}; \mathbb{R}^2)$ such that

1. $\inf_{x \in \mathbb{R}} u_1^0(x) > 0, \ \lim_{|x| \to \infty} u_1^0(x) = \overline{u}_1$

2. $\lim_{|x| \to \infty} u_2^0(x) = 0$

where $\overline{u}_1$ is a positive constant and $\beta = \beta_{const} > 0$.
Then the equation (CL) possesses a global in time weak entropy solution (i.e. $T = \infty$) in sense of Definition 1.2.

2 The Viscous Approximation - Existence and Estimates

Our idea is to apply the vanishing viscosity method by adding a second order elliptic term to the first order conservation law. For $\varepsilon, \lambda > 0$ let us define the viscous, regularized system

$$\frac{\partial}{\partial t} u + \frac{\partial}{\partial x} F(u) = G_\lambda(u, x) + \varepsilon \frac{\partial^2}{\partial x^2} u, \qquad (VP)$$

with initial data

$$u^0 = \left(u_1^0, u_2^0\right).$$

Here we introduced a smooth function G_λ defined by

$$G_\lambda(u_1,u_2,x) = \Lambda\left(\lambda x\right)\left(\begin{array}{c} \dfrac{\beta'(2\kappa)^{\alpha-1}}{\beta^{\alpha-1}}u_2 \\ \dfrac{\beta'(2\kappa)^{\alpha-1}}{\beta^{\alpha-1}}\left(\dfrac{(u_2)^2}{u_1}+\kappa u_1^{\frac{2+\alpha}{1+\alpha}}\right)+u_1\Lambda\left(\dfrac{u_2}{\lambda u_1}\right)g\left(\dfrac{u_2}{u_1},x\right)\end{array}\right).$$

Furthermore Λ is a symmetric, smooth, monotonically decreasing function for positive arguments with $\Lambda(0)=1$, $\Lambda(2)=0$, and $\mathrm{supp}\Lambda' \subset [-2,-1]\cup[1,2]$.

Theorem 2.1
Assume an initial data $u^0=(u_1^0,u_2^0)\in\mathbb{C}^2(\mathbb{R})$ such that

1. $\inf\limits_{x\in\mathbb{R}} u_1^0(x)>0,\ \lim\limits_{|x|\to\infty} u_1^0(x)=\overline{u}_1$

2. $\lim\limits_{|x|\to\infty} u_2^0(x)=0$

where $\overline{u}_1$ is a positive constant. Moreover let $\lambda,\varepsilon>0$ and $\beta\in\mathbb{C}^3(\mathbb{R})$ with $\beta(x)\geq\beta_0>0$.
Then the equation (VP) possesses a local in time classical solution, i.e., $u\in\mathbb{C}^0([0,T_{loc});\mathbb{C}^2(\mathbb{R}))$, $\frac{\partial}{\partial t}u\in C^0([0,T_{loc})\times\mathbb{R})$ where $T_{loc}=T_{loc}(\varepsilon,\lambda)$; moreover for all $t\in[0,T_{loc})$

1. $\inf\limits_{x\in\mathbb{R}} u_1(t,x)>0,\ \lim\limits_{|x|\to\infty} u_1(t,x)=\overline{u}_1$

2. $\lim\limits_{|x|\to\infty} u_2(t,x)=0$

holds.

Proof
Using the Gaussian kernel

$$\Theta(t,x)=\frac{1}{\sqrt{4\pi\varepsilon t}}\exp\left(\frac{-x^2}{4\varepsilon t}\right)$$

of the heat equation we rewrite (VP) in the integral form

$$\begin{aligned} u(t,x) &= \int\limits_{\mathbb{R}}\Theta(x-y,t)u^0(y)dy+\int\limits_0^t\int\limits_{\mathbb{R}}\frac{\partial}{\partial x}\Theta(x-y,t-\tau)F(u(y,\tau))dy\,d\tau \\ &\quad+\int\limits_0^t\int\limits_{\mathbb{R}}\Theta(x-y,t-\tau)G_\lambda(u(y,\tau),y)dy\,d\tau. \end{aligned} \tag{IE}$$

Note that

$$\left|\frac{\partial}{\partial x}\Theta(t,x)\right|\leq\frac{c}{\varepsilon t}\exp\left(\frac{-x^2}{2\varepsilon t}\right)$$

and

$$\int\limits_0^t\int\limits_{\mathbb{R}}\left|\frac{\partial}{\partial x}\Theta(x-y,t-\tau)\right|dyd\tau\leq c\sqrt{\frac{t}{\varepsilon}}.$$

Defining the bounded set

$$\mathcal{U} = \mathrm{conv}\{u^0(x) | x \in \mathbb{R}\},$$

and the compact set

$$\mathcal{U}_\delta = \{y \in \mathbb{R}^2 \,|\, \mathrm{dist}(y, \mathcal{U}) \le \delta\},$$

we can choose $\delta > 0$ small enough such that $\mathcal{U} \subset \mathcal{U}_\delta \subset (0, \infty) \times \mathbb{R}$. The functions F and G_λ, are locally lipschitz on the set $(0, \infty) \times \mathbb{R}$ and global lipschitz on $\mathcal{U}_\delta$.
To solve (IE) we define the sequence $u^{(1)}, u^{(2)}, ...$ by

$$u^{(1)}(t,x) = \int_{\mathbb{R}} \Theta(x-y,t) u^0(y) dy$$

and

$$\begin{aligned} u^{(n+1)}(t,x) &= \int_{\mathbb{R}} \Theta(x-y,t) u^0(y) dy + \int_0^t \int_{\mathbb{R}} \frac{\partial}{\partial x} \Theta(x-y,t-\tau) F(u^{(n)}(y,\tau)) dy\, d\tau \\ &+ \int_0^t \int_{\mathbb{R}} \Theta(x-y,t-\tau) G_\lambda(u^{(n)}(y,\tau), y) dy\, d\tau. \end{aligned}$$

Obviously $u^{(1)}(t,x) \in \mathcal{U}$ for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$. It is also easy to observe that there exists a sufficiently small time T_{loc} such that

$$\sum_{n=1}^{\infty} \|u^{(n+1)} - u^{(n)}\|_{\mathbb{C}^0([0,T_{loc}) \times \mathbb{R})} \le \delta.$$

Then $\lim_{n \to \infty} u^{(n)}(t,x) = u(t,x)$ for all $(t,x) \in [0, T_{loc}) \times \mathbb{R}$, where $u(t,x)$ is a solution to (IE), and thus a "mild solution" to (VP). Moreover $u(t,x) \in \mathcal{U}_\delta$ for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$. Note that $F, G_\lambda \in \mathbb{C}^2(\mathcal{U}_\delta)$. Then regularity of the solution (it means $u \in \mathbb{C}^0((0, T_{loc}); \mathbb{C}^2(\mathbb{R}))$) will be obtained by examining difference quotients of first and second order (see also [11]).

Remark
It is easy to observe that the norm $\sup_{x \in \mathbb{R}} |u(t,x)| + \sup_{x \in \mathbb{R}} |\frac{\partial}{\partial x} u(t,x)| + \sup_{x \in \mathbb{R}} |\frac{\partial^2}{\partial x^2} u(t,x)|$ can blow-up at time T if and only if $\lim_{t \to T} \inf_{x \in \mathbb{R}} u_1(t,x) = 0$ or the norm $\|u_1(t)\|_{\mathbb{L}^\infty(\mathbb{R})} + \|\frac{u_2}{u_1}(t)\|_{\mathbb{L}^\infty(\mathbb{R})}$ will blow-up at time T.

The last remark gives us a hint to prove an apriori estimate of the solution $u(t)$.

Definition 2.1
Let $R_i : \mathbb{R}^2 \to \mathbb{R}$, $i = 1, ..., k$, and $M_i : [0, T) \to \mathbb{R}$ be smooth functions. We define the region

$$\Sigma(t) = \bigcap_{i=1}^{k} \{u \in \mathbb{R}^2 : R_i(u) \le M_i(t)\}. \qquad (IR)$$

If $u : [0,T) \times \mathbb{R} \to \mathbb{R}^2$, we call $\Sigma(t)$ a one parameter family of invariant regions for u iff $u(x, 0) \in \Sigma(0)$ implies that $u(t,x) \in \Sigma(t)$ for $t > 0$.

Lemma 2.1
Let $u : [0,T) \times \mathbb{R} \to (0,\infty) \times \mathbb{R}$ be a classical solution of the system

$$\frac{\partial}{\partial t}u + \nabla_u F(u)\frac{\partial}{\partial x}u = G_\lambda(u,x) + \varepsilon\frac{\partial^2}{\partial x^2}u$$

and let $\Sigma(t)$ be defined by (IR). Suppose that the following conditions hold:

1. $R_i(u(0,x)) \le M_i(0)$
2. $\nabla R_i(a)$ *is a left eigenvector of* $\nabla F(a)$ *for all* $a \in \partial\Sigma_i(t)$
3. $\Sigma(t)$ *is a convex set for all* $t \in [0,T)$
4. $\sup_{x\in\mathbb{R}} \sup_{a\in\partial\Sigma_i(t)} \nabla R_i(a)\cdot G_\lambda(a,x) \le M_i'(t)$

where $\partial\Sigma_i(t) = \partial\Sigma(t) \cap \{a \in \mathbb{R}^2 \mid R_i(a) = M_i(t)\}$.

Then $\Sigma(t)$ is a one parameter family of invariant regions for u.

Proof
cf. [1], [10].

Theorem 2.2.a (Local in time estimate)
Assume that $u(t)$ is the local in time classical solution to the problem (VP) with $\|u_1^0\|_{\mathbb{L}^\infty(\mathbb{R})} + \|\frac{u_2^0}{u_1^0}\|_{\mathbb{L}^\infty(\mathbb{R})} \le C_1$, $\|\beta'\|_{\mathbb{L}^\infty(\mathbb{R})} \le C_2$ and $\beta(x) \ge \beta_0 > 0$.
Then there exists a time T_{max} and a function $C : [0,T_{max}) \to \mathbb{R}_+$ depending only on C_1, C_2 such that $\|u_1\|_{\mathbb{L}^\infty([0,T)\times\mathbb{R})} + \|\frac{u_2}{u_1}\|_{\mathbb{L}^\infty([0,T)\times\mathbb{R})} \le C(T)$ for all $T \in [0,T_{max})$.

Proof
We apply Lemma 2.1 to the so-called Riemann invariants $R_1 = \frac{u_2}{u_1} + (u_1)^{\frac{1}{2(1+\alpha)}}$, $R_2 = -\frac{u_2}{u_1} + (u_1)^{\frac{1}{2(1+\alpha)}}$; moreover let

$$M_1(0) = M_2(0) = \max\left\{\sup_{x\in\mathbb{R}}\left[\frac{u_2^0(x)}{u_1^0(x)} + (u_1^0(x))^{\frac{1}{2(1+\alpha)}}\right]; \sup_{x\in\mathbb{R}}\left[-\frac{u_2^0(x)}{u_1^0(x)} + (u_1^0(x))^{\frac{1}{2(1+\alpha)}}\right]\right\}.$$

Then

$$\frac{\partial R_1}{\partial u_1} = -\frac{u_2}{(u_1)^2} + \frac{1}{2(1+\alpha)}(u_1)^{\frac{-(1+2\alpha)}{2(1+\alpha)}}, \qquad \frac{\partial R_1}{\partial u_2} = \frac{1}{u_1}$$

and

$$\frac{\partial R_2}{\partial u_1} = \frac{u_2}{(u_1)^2} + \frac{1}{2(1+\alpha)}(u_1)^{\frac{-(1+2\alpha)}{2(1+\alpha)}}, \qquad \frac{\partial R_1}{\partial u_2} = -\frac{1}{u_1}.$$

Hence for $u_2 \ge 0$

$$\begin{aligned}\nabla R_1(u_1,u_2)\cdot G_\lambda(u_1,u_2,x) &\le \frac{\beta'(2\kappa)^{\alpha-1}}{\beta^{\alpha-1}}\left[-\frac{(u_2)^2}{(u_1)^2} + \frac{u_2}{2(1+\alpha)}(u_1)^{\frac{-(1+2\alpha)}{2(1+\alpha)}}\right] \\ &\quad + \frac{\beta'(2\kappa)^{\alpha-1}}{\beta^{\alpha-1}}\left(\frac{(u_2)^2}{(u_1)^2} + \kappa u_1^{\frac{1}{1+\alpha}}\right) + \sin\gamma((x)) \\ &\le \frac{\beta'(2\kappa)^{\alpha-1}}{\beta^{\alpha-1}}\left[\kappa u_1^{\frac{1}{1+\alpha}} + \frac{u_2}{2(1+\alpha)}(u_1)^{\frac{-(1+2\alpha)}{2(1+\alpha)}}\right] + \sin\gamma((x)).\end{aligned}$$

Similarly for $u_2 \leq 0$

$$\nabla R_1(u_1,u_2)\cdot G_\lambda(u_1,u_2,x) \;\leq\; \frac{\beta'(2\kappa)^{\alpha-1}}{\beta^{\alpha-1}}\left[-\kappa u_1^{\frac{1}{1+\alpha}} + \frac{u_2}{2(1+\alpha)}(u_1)^{\frac{-(1+2\alpha)}{2(1+\alpha)}}\right] - \sin\gamma((x)).$$

Now can observe that the fourth condition of Lemma 2.1 can be satisfied if $M(t) = M_1(t) = M_2(t)$ is a function such that

$$\frac{dM(t)}{dt} \geq C[M(t)]^2 + \sup_{x\in\mathbb{R}} |\sin(\gamma(x))|.$$

Here C is a positive constant depending only on $\|\beta'\|_{\mathbb{L}^\infty(\mathbb{R})}$, β_0 and α.

Theorem 2.2.b (Global in time estimate)
Assume that $u(t)$ is the local in time solution to the problem (VP) with $\|u_1^0\|_{\mathbb{L}^\infty(\mathbb{R})} + \|\frac{u_2^0}{u_1^0}\|_{\mathbb{L}^\infty(\mathbb{R})} \leq C_1$, $\|\sin(\gamma)\|_{\mathbb{L}^\infty(\mathbb{R})} \leq C_2$ and $\beta(x) = \beta_{const.}$. Then there exists a constant C such that $\|u_1\|_{\mathbb{L}^\infty([0,T)\times\mathbb{R})} + \|\frac{u_2}{u_1}\|_{\mathbb{L}^\infty([0,T)\times\mathbb{R})} \leq C(C_1 + C_2T)$, for all positive T.

Proof
Proof is analogues to the proof of Theorem 2.2.a.

Lemma 2.2
Assume that the equation (VP) possesses a local in time solution $u \in \mathbb{C}([0,T_{loc});\mathbb{C}^2(\mathbb{R}))$ such that

1. $\lim_{|x|\to\infty} u_1(t,x) = \overline{u}_1$,
2. $\lim_{|x|\to\infty} u_2(t,x) = 0$.

Moreover let us suppose that $u_1^0 - \overline{u}_1, u_2^0 \in \mathbb{L}^2(\mathbb{R})$ and $u_1^0 \geq \delta$, where $\delta, \overline{u}_1$ are positive constants.
Then the following estimates hold

$$\|u_1(t)-\overline{u}_1\|_{\mathbb{L}^2(\mathbb{R})} + \|u_2(t)\|_{\mathbb{L}^2(\mathbb{R})} \leq C_1(\varepsilon, T_{loc}) \quad \text{for all } t\in[0,T_{loc}),$$

$$\left\|\frac{\partial}{\partial x}u\right\|_{\mathbb{L}^2([0,T_{loc})\times\mathbb{R})} \leq C_2(\varepsilon, T_{loc}).$$

Proof
Define $w = (w_1,w_2) = (u_1-\overline{u}_1, u_2)$. Therefore

$$\frac{\partial}{\partial t}w + \frac{\partial}{\partial x}F(w_1+\overline{u}_1,w_2) = G_\lambda(w_1+\overline{u}_1,w_2,x) + \varepsilon\frac{\partial^2}{\partial x^2}w.$$

Multiplying by w and integrating on $[0,t]\times[-l,l]$ we obtain

$$\frac{1}{2}\left[\|w(t)\|^2_{\mathbb{L}^2([-l,l])}\right]_0^t - \int_0^t\int_{-l}^l \varepsilon\frac{\partial^2}{\partial x^2}w\cdot w\,dxd\tau \;=\; \int_0^t\int_{-l}^l G_\lambda(w_1+\overline{u}_1,w_2,x)\cdot w\,dxd\tau - \int_0^t\int_{-l}^l \frac{\partial}{\partial x}F(w_1+\overline{u}_1,w_2)\cdot w\,dxd\tau.$$

Integrating by parts gives

$$\begin{aligned}
\frac{1}{2}\left[\|w(t)\|^2_{\mathbb{L}^2([-l,l])}\right]_0^t \quad +\varepsilon\left\|\frac{\partial}{\partial x}w\right\|^2_{\mathbb{L}^2([0,t)\times[-l,l])} &= \int_0^t\int_{-l}^{l} G_\lambda(w_1+\overline{u}_1,w_2,x)\cdot w\,dxd\tau \\
&+ \int_0^t\int_{-l}^{l} F(w_1+\overline{u}_1,w_2)\cdot\frac{\partial}{\partial x}w\,dxd\tau + \int_0^t \left[\frac{\partial}{\partial x}w(x,\tau)\cdot w(x,\tau)\right]_{-l}^{l} d\tau \\
&- \int_0^t \left[F\left(w_1(x,\tau)+\overline{u}_1,w_2(x,\tau)\right)\cdot w(x,\tau)\right]_{-l}^{l} d\tau.
\end{aligned}$$

Considering the second term of the right hand side recall that

$$F(w_1+\overline{u}_1,w_2)\cdot\frac{\partial}{\partial x}w \quad = \quad w_2\frac{\partial}{\partial x}w_1+\frac{\partial}{\partial x}w_2\left(\frac{(w_2)^2}{w_1+\overline{u}}+\kappa(w_1+\overline{u}_1)^{\frac{2+\alpha}{1+\alpha}}\right).$$

However the term $|\kappa(w_1+\overline{u}_1)^{\frac{2+\alpha}{1+\alpha}}|$ can not directly be estimated by $C_1|w_1|$. But by the boundedness of w_1 we obtain

$$-C|w_1|\left|\frac{\partial}{\partial x}w_2\right|+(\overline{u}_1)^{\frac{2+\alpha}{1+\alpha}}\frac{\partial}{\partial x}w_2 \le (w_1+\overline{u}_1)^{\frac{2+\alpha}{1+\alpha}}\frac{\partial}{\partial x}w_2 \le C|w_1|\left|\frac{\partial}{\partial x}w_2\right|+(\overline{u}_1)^{\frac{2+\alpha}{1+\alpha}}\frac{\partial}{\partial x}w_2.$$

Hence

$$\begin{aligned}
\left|\int_0^t\int_{-l}^{l} F(w_1+\overline{u}_1,w_2)\cdot\frac{\partial}{\partial x}w\,dxd\tau\right| \quad &\le \quad C\left(\|w_1\|_{\mathbb{L}^\infty},\left\|\frac{w_2}{w_1+\overline{u}_1}\right\|_{\mathbb{L}^\infty}\right)\int_0^t\int_{-l}^{l}\left|\frac{\partial}{\partial x}w\right||w|\,dxd\tau \\
&+\left|\int_0^t\left[\kappa\overline{u}_1^{\frac{2+\alpha}{1+\alpha}}w_2(x,\tau)\right]_{-l}^{l}\right|.
\end{aligned}$$

By the properties of G_λ and the fact that $\lim_{|x|\to\infty}(w_1(x,\tau),w_2(x,\tau)) = (0,0)$ for all $\tau\in[0,T_{loc})$ we can pass to the limit $l\to\infty$ getting rid of the boundary terms.
We complete the proof using Gronwall's lemma.

Lemma 2.3
Assume that $u_1\in\mathbb{C}^0([0,T_{loc});\mathbb{C}^2(\mathbb{R}))$, $u_2\in\mathbb{C}^0([0,T_{loc});\mathbb{C}^1(\mathbb{R}))$ *is a solution to the equation*

$$\frac{\partial}{\partial t}u_1+\frac{\partial}{\partial x}u_2=\Lambda(\lambda x)\frac{\beta'(2\kappa)^{\alpha-1}}{\beta^{\alpha-1}}u_2+\varepsilon\frac{\partial^2}{\partial x^2}u_1,$$

moreover let $\inf_{x\in\mathbb{R}}u_1^0(x) > 0$, $\|\frac{u_2}{u_1}\|_{\mathbb{L}^\infty([0,T_{loc})\times\mathbb{R})}+\sup_{t\in[0,T_{loc})}\|u_1(t)-\overline{u}_1\|_{\mathbb{L}^2(\mathbb{R})}\le C_1$ *and* $\|\Lambda(\lambda x)\frac{\beta'(2\kappa)^{\alpha-1}}{\beta^{\alpha-1}}\|_{\mathbb{L}^\infty(\mathbb{R})}\le C_2$.
Then there exists a constant $b(\varepsilon,T_{loc})$ *such that*

$$u_1(t,x)>b(\varepsilon,T_{loc})>0$$

for all $t\in[0,T_{loc})$.

Proof
The proof is similar to the proof of Theorem 5.3 from [10], cf. also [1], though we include it for completeness. First let us define $r = \frac{1}{e^{at}u_1}$, where the constant a will be determined later. Then

$$\frac{\partial}{\partial t}r + u_2\frac{\partial}{\partial x}r - \frac{\partial}{\partial x}ur(a+u_2)r = \varepsilon\left(\frac{\partial^2}{\partial x^2}r - 2\left(\frac{\partial}{\partial x}r\right)^2 r^{-1}\right). \qquad (*)$$

To prove Lemma 2.3 we recall a technical lemma, cf. [10]:

Lemma 2.4
If $\phi : [c_0, \infty) \to [0, \infty)$ is a nonincreasing function, and if for positive constants K, a and b

$$\phi(c_2) \leq K c_1^a (c_2 - c_1)^{-1}\phi(c_1)^{1+b} \quad \text{for all} \quad c_2 > c_1 \geq c_0,$$

then there exists a $c^* < \infty$ such that $\phi(c^*) = 0$.

We define $\xi = \max\{0, r - c_1\}$ for some $c_1 \geq c_0$ large enough so that $\xi(0, x) = 0$, and $\phi(c)$ as the measure $|\{(t, x) \in [0, T_{loc}) \times \mathbb{R} : r(t, x) > c\}|$; observe that $\phi(c) < \infty$ for all $c < \overline{u}_1$.
Multiply (*) by ξ^3 and integrate over $[0, t) \times \mathbb{R}$ for $t \in (0, T_{loc})$ to get

$$\frac{1}{4}\int_{\mathbb{R}} \xi(t)dx + (a+\gamma u)\int_0^t\int_{\mathbb{R}} (\xi^4 + c_1\xi^3)dxd\tau + \frac{3\varepsilon}{4}\int_0^t\int_{\mathbb{R}} |(\xi^2)^2|dxd\tau =$$

$$= 2\varepsilon\int_0^t\int_{\mathbb{R}} \frac{\frac{\partial}{\partial x}\xi^2}{\xi + c_1}dxd\tau - \int_0^t\int_{\mathbb{R}} (5\xi^3 + 3c_1\xi^2)u_2\frac{\partial}{\partial x}\xi dxd\tau.$$

Let us define $C = \left\|\frac{u_2}{u_1}\right\|^2_{\mathbb{L}^\infty([0,T_{loc})\times\mathbb{R})}$. Considering the last integral Young's inequality yields

$$-\int_0^t\int_{\mathbb{R}} 5\xi^3 u_2\frac{\partial}{\partial x}\xi dxd\tau \leq \frac{\varepsilon}{4}\int_0^t\int_{\mathbb{R}} \left|\frac{\partial}{\partial x}(\xi)^2\right|^2 dxd\tau + \frac{25}{4\varepsilon}C\int_0^t\int_{\mathbb{R}} \xi^4 dxd\tau$$

and

$$\begin{aligned}-\int_0^t\int_{\mathbb{R}} 3c_1\xi^2 u_2\frac{\partial}{\partial x}\xi dxd\tau \quad &\leq \quad \frac{\varepsilon}{4}\int_0^t\int_{\mathbb{R}} \left|\frac{\partial}{\partial x}(\xi)^2\right|^2 dxd\tau \\ &\quad + \frac{9c_1^2}{4\varepsilon}C\phi(c_1)^{\frac{3}{2}}\int_0^t \|\xi\|^2_{\mathbb{L}^6(\mathbb{R})}dxd\tau.\end{aligned}$$

Now choose

$$a = \frac{\varepsilon}{4} + \frac{25}{4\varepsilon}C + \left\|\frac{\beta'(2\kappa)^{\alpha-1}}{\beta^{\alpha-1}}\frac{u_2}{u_1}\right\|_{\mathbb{L}^\infty([0,T_{loc})\times\mathbb{R})}.$$

We thus derive

$$\int_{\mathbb{R}} \xi(t)dx + \int_0^t\int_{\mathbb{R}} \left(\xi^4 + \left|\frac{\partial}{\partial x}\left(\xi^2\right)\right|^2\right)dxd\tau \leq \frac{9c_1^2}{\varepsilon}C\phi(c_1)^{\frac{3}{2}}\int_0^t \|\xi\|^2_{\mathbb{L}^6(\mathbb{R})}dxd\tau.$$

Combining Sobolev's embedding theorem $\|\xi\|^2_{\mathbb{L}^6(\mathbb{R})} \le C\|\xi\|^2_{\mathbb{H}^1(\mathbb{R})}$ with another application of Young's inequality we derive

$$\sup_{t\in[0,T_{loc}]} \int_{\mathbb{R}} \xi(t)dx \le \frac{\alpha t}{\varepsilon} c_1^4 \phi(c_1)^{\frac{4}{3}}$$

where α is a positive constant. Furthermore, by definition of ξ, for $c_2 > c_1$ we have

$$\sup_{t\in[0,T_{loc}]} \int_{\mathbb{R}} \xi(t)dx \ge (c_2 - c_1)^4 \phi(c_2).$$

The last two inequalities show that Lemma 2.4 can be applied to complete the proof of Lemma 2.3.

Theorem 2.3.a
Assume an initial data $u^0 = (u_1^0, u_2^0) \in \mathbb{C}^2(\mathbb{R};\mathbb{R}^2)$, $(u_1^0 - \overline{u}_1, u_2^0) \in \mathbb{L}^2(\mathbb{R};\mathbb{R}^2)$ *such that*

1. $\inf_{x\in\mathbb{R}} u_1^0(x) > 0$, $\lim_{|x|\to\infty} u_1^0(x) = \overline{u}_1$

2. $\lim_{|x|\to\infty} u_2^0(x) = 0$

where $\overline{u}_1$ *is a positive constant. Moreover let* $\lambda, \varepsilon > 0$, *and* $\beta \in \mathbb{C}^3(\mathbb{R})$ *with* $\beta(x) \ge \beta_0 > 0$.
Then the equation (VP) possesses a local in time classical solution, i.e., $u \in \mathbb{C}^0([0,T_{max});\mathbb{C}^2(\mathbb{R}))$, $\frac{\partial}{\partial t}u \in \mathbb{C}^0([0,T_{max}) \times \mathbb{R})$; *moreover*

1. $\inf_{x\in\mathbb{R}} u_1(t,x) > 0$, $\lim_{|x|\to\infty} u_1(t,x) = \overline{u}_1$

2. $\lim_{|x|\to\infty} u_2(t,x) = 0$

for all $t \in [0, T_{max})$.

Proof
Apply Theorem 2.1, Theorem 2.2.a, Lemma 2.2, Lemma 2.3.

Theorem 2.3.b
Assume an initial data $u^0 = (u_1^0, u_2^0) \in \mathbb{C}^2(\mathbb{R};\mathbb{R}^2)$, $(u_1^0 - \overline{u}_1, u_2^0) \in \mathbb{L}^2(\mathbb{R};\mathbb{R}^2)$ *such that*

1. $\inf_{x\in\mathbb{R}} u_1^0(x) > 0$, $\lim_{|x|\to\infty} u_1^0(x) = \overline{u}_1$

2. $\lim_{|x|\to\infty} u_2^0(x) = 0$

where $\overline{u}_1$ *is a positive constant. Moreover* $\lambda, \varepsilon > 0$, *and* $\beta = \beta_{const}$.
Then the equation (VP) possesses a global in time classical solution i.e. $u \in \mathbb{C}^0([0,T);\mathbb{C}^2(\mathbb{R}))$, $\frac{\partial}{\partial t}u \in \mathbb{C}^0([0,T) \times \mathbb{R})$ *for all* $T \in [0,\infty)$; *moreover*

1. $\inf_{x\in\mathbb{R}} u_1(T,x) > 0$, $\lim_{|x|\to\infty} u_1(T,x) = \overline{u}_1$

2. $\lim_{|x|\to\infty} u_2(T,x) = 0$

holds.

Proof
Apply Theorem 2.1, Theorem 2.2.b, Lemma 2.2, Lemma 2.3.

Lemma 2.5
Assume that $u(t,x)$ is a strong solution of the equation

$$\frac{\partial}{\partial t}u + \frac{\partial}{\partial x}F(u) = G_\lambda(u,x) + \varepsilon\frac{\partial^2}{\partial^2 x}u.$$

Moreover suppose that $u, F(u), G_\lambda(u,x) \in \mathbb{L}^\infty([0,T) \times \mathbb{R}; \mathbb{R}^2)$, $u_1 > 0$ and that (η, q) is a convex entropy-flux pair with $|\nabla\eta| \leq C$ and $\eta(\cdot,\cdot)$ is $\mathbb{C}^2$-function for $(u_1, u_2) \in (0,\infty) \times \mathbb{R}$.
Then $\varepsilon|\frac{\partial}{\partial x}u|^2$, $\varepsilon(\frac{\partial}{\partial x}u^T \cdot \nabla^2\eta(u) \cdot \frac{\partial}{\partial x}u)$ are bounded independently of $0 < \varepsilon < 1$ in the space $\mathbb{L}^1(\Omega)$ for any bounded Ω relatively open in $[0,\infty) \times \mathbb{R}$.

Proof
If (η, q) is a convex entropy-flux pair ($\frac{\partial}{\partial x}u^T \cdot \nabla^2\eta(u) \cdot \frac{\partial}{\partial x}u \geq 0$) with $|\nabla\eta| \leq C$, then multiplying (VP) by $\nabla\eta$ gives:

$$\frac{\partial}{\partial t}\eta + \frac{\partial}{\partial x}q = \varepsilon\left(\frac{\partial^2}{\partial^2 x}\eta - \frac{\partial}{\partial x}u^T \cdot \nabla^2\eta(u) \cdot \frac{\partial}{\partial x}u\right) + \nabla\eta \cdot G_\lambda.$$

Now we multiply by a non-negative test function $\theta \in \mathbb{C}_0^\infty([0,\infty) \times \mathbb{R})$ with $\theta(t,x)_{|\Omega} = 1$ and integrate over $[0,\infty) \times \mathbb{R}$. We obtain the inequality

$$\begin{aligned}\varepsilon\left\|\frac{\partial}{\partial x}u^T \cdot \nabla^2\eta(u) \cdot \frac{\partial}{\partial x}u\right\|_{\mathbb{L}^1(\Omega)} \leq\ & \int_{\mathbb{R}}\int_0^\infty \left\{\left|\eta\frac{\partial^2}{\partial^2 x}\theta\right| + q\frac{\partial}{\partial x}\theta + \eta\frac{\partial}{\partial t}\theta + \theta\nabla\eta \cdot G_\lambda\right\} dx\, dt \\ & + \int_{\mathbb{R}} \theta\, \eta\, dx,\end{aligned}$$

where the right hand side and consequently also the left hand side is bounded independently of ε.
Similarly if (η, q) is an strictly convex entropy-flux pair ($\frac{\partial}{\partial x}u^T \cdot \nabla^2\eta(u) \cdot \frac{\partial}{\partial x}u > K|\frac{\partial}{\partial x}u|^2$ $K > 0$), for example η_E defined in Chapter 4 step 2, then

$$\varepsilon K\left\|\frac{\partial}{\partial x}u\right\|^2_{\mathbb{L}^2(\Omega)} \leq \int_{\mathbb{R}}\theta\eta dx + \int_{\mathbb{R}}\int_0^\infty \left\{\left|\eta\frac{\partial^2}{\partial^2 x}\theta\right| + q\frac{\partial}{\partial x}\theta + \eta\frac{\partial}{\partial t}\theta + \theta\nabla\eta \cdot G_\lambda\right\} dx\, dt.$$

3 Young Measure Limits

First we recall two important theorems:

Theorem (convergence in the sense of Young measures)
Assume that the sequence $\{u_k\}$ is bounded in $\mathbb{L}^\infty(\Omega; K)$, where $\Omega \subset [0,T) \times \mathbb{R}$, K is a

compact, convex set in $\mathbb{R}^2$.
Then there exists a subsequence $\{u_{k_j}\} \subset \{u_k\}$ *and for a.e.* $(t,x) \in \Omega$ *a Borel probability measure* $\mu_{(t,x)}$ *on* K *such that for each* $\mathcal{F} \in \mathbb{C}(\mathbb{R} \times K; \mathbb{R}^2)$ *we have*

$$\lim_{k_j \to \infty} \int_\Omega \mathcal{F}(x, u_{k_j}(t,x))\varphi(t,x)dxdt = \int_\Omega \overline{\mathcal{F}(t,x)}\varphi(t,x)dxdt \quad \forall \varphi \in \mathbb{L}^1(\Omega, \mathbb{R}^2),$$

where

$$\overline{\mathcal{F}(t,x)} = \int_K \mathcal{F}(x,y)d\mu_{(t,x)}(y)$$

for a.e. $(t,x) \in \Omega$.

Proof
The main idea is to use the *Banach-Alaoglu Theorem* and the fact that $\mathbb{L}^\infty_w(\Omega; \mathcal{M}(K))$, is the dual of the separable space $\mathbb{L}^1(\Omega; \mathbb{C}^0(K))$ cf. [17], and for $\mathcal{F}$ without x-dependence [3],[9].

Remark
By Lemma 2.2 we have $\left|\frac{u_2}{u_1}\right| + |u_1| \leq C(T)$ and $u_1 \geq 0$. For such (u_1, u_2) the flux term F is continuous.

Theorem (Div-Curl lemma)
Assume that (q_k^1, η_k^1) *and* $(q_k^2, -\eta_k^2)$ *are bounded sequences in* $\mathbb{L}^2(\Omega; \mathbb{R}^2)$ *such that* $(\frac{\partial}{\partial t}q_k^i, \frac{\partial}{\partial x}\eta_k)$ *(for i=1,2) lies in a compact subset of* $\mathbb{W}^{-1,2}(\Omega; \mathbb{R}^2)$. *Suppose further that* $(q_k^1, \eta_k^1) \rightharpoonup (q^2, \eta^2)$, $(q_k^2, -\eta_k^2) \rightharpoonup (q^2, -\eta^2)$ *in* $\mathbb{L}^2(\Omega; \mathbb{R}^2)$.
Then

$$\int_\Omega (q_k^1\eta_k^2 - \eta_k^1 q_k^2)\varphi \, dxdt \to \int_\Omega (q^1\eta^2 - \eta^1 q^2)\varphi \, dxdt \quad \forall \varphi \in \mathbb{C}_0^\infty(\Omega, \mathbb{R}).$$

Proof
For a similar statement see [9]. Note that $(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}) \cdot (q, \eta)$ is the two-dimensional divergence operator and $(\frac{\partial}{\partial t}, -\frac{\partial}{\partial x}) \cdot (q, \eta)$ is the two-dimensional curl operator.

Now we prove the existence of a so-called "measure valued solution" defined in [4] by R. DiPerna to our problem (CL). We also show the very important property that for a.e. (t,x) the measures reduce to Dirac measures concentrated in points.
Let us divide our analysis into three steps:

Step 1
First we fix $\lambda = \varepsilon$ and write the approximate system in the weak form

$$\begin{aligned} \int\limits_{[0,T)\times\mathbb{R}} \left[u^\varepsilon \cdot \frac{\partial}{\partial t}\psi + F(u^\varepsilon) \cdot \frac{\partial}{\partial x}\psi + G_\varepsilon(u^\varepsilon, x) \cdot \psi \right] dtdx \;\; &= \;\; \varepsilon \int\limits_{[0,T)\times\mathbb{R}} \frac{\partial}{\partial x}u^\varepsilon \cdot \frac{\partial}{\partial x}\psi \, dtdx \\ &\quad + \int\limits_{\mathbb{R}} u^0(x) \cdot \psi(0,x) \, dx \end{aligned}$$

for all test functions $\psi \in \mathbb{C}_0^1([0,T) \times \mathbb{R};\mathbb{R}^2)$. Defining the new variable z^ε by $z^\varepsilon = (z_1^\varepsilon, z_2^\varepsilon) = (\frac{u_2^\varepsilon}{u_1^\varepsilon}, u_1^\varepsilon)$ we obtain

$$\int\limits_{[0,T)\times\mathbb{R}} \left[\mathcal{H}_1(z^\varepsilon)\cdot\frac{\partial}{\partial t}\psi + \mathcal{H}_2(z^\varepsilon)\cdot\frac{\partial}{\partial x}\psi + \mathcal{H}_3^\varepsilon(z^\varepsilon,x)\cdot\psi\right] dtdx = \varepsilon\int\limits_{[0,T)\times\mathbb{R}} \frac{\partial}{\partial x}u^\varepsilon\cdot\frac{\partial}{\partial x}\psi\, dtdx + \int\limits_{\mathbb{R}} u^0(x)\cdot\psi(0,x)dx \tag{VP}$$

for all test functions $\psi \in \mathbb{C}_0^1([0,T) \times \mathbb{R};\mathbb{R}^2)$, where

$$\mathcal{H}_1(z_1^\varepsilon, z_2^\varepsilon) = \begin{pmatrix} z_2^\varepsilon \\ z_1^\varepsilon z_2^\varepsilon \end{pmatrix}, \quad \mathcal{H}_2(z_1^\varepsilon, z_2^\varepsilon) = \begin{pmatrix} z_1^\varepsilon z_2^\varepsilon \\ (z_1^\varepsilon)^2 z_2^\varepsilon + \kappa(z_2^\varepsilon)^{\frac{2+\alpha}{1+\alpha}} \end{pmatrix}$$

and

$$\mathcal{H}_3^\varepsilon(z_1^\varepsilon, z_2^\varepsilon, x) = \Lambda(\varepsilon x)\begin{pmatrix} \frac{\beta'(2\kappa)^{\alpha-1}}{\beta^{\alpha-1}} z_1^\varepsilon z_2^\varepsilon \\ \frac{\beta'(2\kappa)^{\alpha-1}}{\beta^{\alpha-1}}\left((z_1^\varepsilon)^2 z_2^\varepsilon + \kappa(z_2^\varepsilon)^{\frac{2+\alpha}{1+\alpha}}\right) + \Lambda\left(\frac{1}{\varepsilon z_1^\varepsilon}\right) z_1^\varepsilon g\left(z_1^\varepsilon z_2^\varepsilon, x\right) \end{pmatrix}.$$

Note that ψ_t, ψ_x, ψ are elements from the space $\mathbb{L}^1([0,T)\times\mathbb{R};\mathbb{R}^2)$. Then for the sequence of solutions z^ε of the approximate problems (for example setting $\varepsilon = \frac{1}{k}$, $k \in \mathbb{N}$) we can pass to the limit in the sense of Young measures. Since $\frac{1}{k}\frac{\partial}{\partial x}u^k \to 0$ strongly in $\mathbb{L}^2([0,T) \times \mathbb{R};\mathbb{R}^2)$ by Lemma 2.5 and hence the first term on the right hand side of (VP) goes to zero, we obtain:

$$\int\limits_{[0,T)\times\mathbb{R}} \left[\overline{\mathcal{H}_1(t,x)}\psi_t(t,x) + \overline{\mathcal{H}_2(t,x)}\psi_x(t,x) + \overline{\mathcal{H}_3(t,x)}\psi(t,x)\right] dtdx = \int\limits_{\mathbb{R}} u^0\psi(0,x)dx; \tag{ME}$$

here by the characterisation of Young measures we know that

$$\overline{\mathcal{H}_1(t,x)} = \int_K \mathcal{H}_1(y_1,y_2)d\mu_{(t,x)}(y_1,y_2),$$

$$\overline{\mathcal{H}_2(t,x)} = \int_K \mathcal{H}_2(y_1,y_2)d\mu_{(t,x)}(y_1,y_2)$$

for a.e $(t,x) \in \Omega$ and that $\mu_{(t,x)}$ is the Young measure generated by a subsequence $\{(z_1,z_2)^{k_j}\}$. However we can not characterise the "weak-star" limits of $\mathcal{H}_3^\varepsilon$ in the same way. We know only that the limit function $\mathcal{H}_3 \in \mathbb{L}^\infty(\mathbb{R}\times\mathbb{R}_+;\mathbb{R}^2)$.
In the standard theory (ME) defines a weak solution if and only if Young measures $\mu_{(t,x)}$ are Dirac measures. Here the characterisation of the limit $\mathcal{H}_3^\varepsilon$ poses an additional difficulty.

Step 2

To prove that $\mu_{(t,x)}$ is a Dirac measure we apply the Div-Curl Lemma to the family of entropies. Multiplying the equation (VP) by $\nabla_u \eta(u^k)$ we obtain

$$\frac{\partial}{\partial t} u^k \nabla_u \eta(u^k) + \frac{\partial}{\partial x} F(u^k) \nabla_u \eta(u^k) = G_k(u^k, x) \nabla_u \eta(u^k) + \frac{1}{k} \frac{\partial^2}{\partial^2 x} u^k \nabla_u \eta(u^k),$$

and by a simple calculation

$$\frac{\partial}{\partial t}\eta(u^k) + \frac{\partial}{\partial x} q(u^k) = G_k(u^k,x)\nabla_u \eta(u^k) + \frac{1}{k}\left(\frac{\partial^2}{\partial^2 x}\eta(u^k) - (\frac{\partial}{\partial x}u^k)^T \cdot \nabla_u^2 \eta(u^k) \cdot \frac{\partial}{\partial x} u^k\right). \tag{EE}$$

We want to prove that the right hand side lies in a compact subset of $\mathbb{W}^{-1,p}(\Omega;\mathbb{R}^2)$ for all bounded, relatively open sets Ω in $[0,T)\times\mathbb{R}$ and $p \in (1,\infty)$. It is well known that any "weak entropy" defined by R. DiPerna [4] can be expressed in the form of a convolution:

$$\eta(u_1,u_2) = \int_{\mathbb{R}} f(\xi)\chi\left(u_1, \xi - \frac{u_2}{u_1}\right) d\xi$$

where

$$\chi(a,b) = \left(a^{\frac{1}{2(1+\alpha)}} - b^2\right)_+^{2+4\alpha}$$

and

$$(c)_+^\lambda = \begin{cases} c^\lambda & \text{for } c > 0 \\ 0 & \text{for } c \leq 0. \end{cases}$$

Now rewrite the convolution as

$$\eta(u_1,u_2) = u_1 \int_{-1}^{1} f\left(\frac{u_2}{u_1} + \xi u_1^{\frac{1}{2(1+\alpha)}}\right)(1+\xi^2)_+^{2+4\alpha} d\xi.$$

Differentiation yields

$$\begin{aligned}\frac{\partial}{\partial u_1}\eta(u_1,u_2) &= \int_{-1}^{1} f\left(\frac{u_2}{u_1} + \xi u_1^{\frac{1}{2(1+\alpha)}}\right)(1+\xi^2)_+^{2+4\alpha} d\xi \\ &+ \int_{-1}^{1}\left(\frac{1}{2(1+\alpha)}\xi u_1^{\frac{1}{2(1+\alpha)}} - \frac{u_2}{u_1}\right) f'\left(\frac{u_2}{u_1} + \xi u_1^{\frac{1}{2(1+\alpha)}}\right)(1+\xi^2)_+^{2+4\alpha} d\xi,\end{aligned}$$

and

$$\frac{\partial}{\partial u_2}\eta(u_1,u_2) = \int_{-1}^{1} f'\left(\frac{u_2}{u_1} + \xi u_1^{\frac{1}{2(1+\alpha)}}\right)(1+\xi^2)_+^{2+4\alpha} d\xi.$$

Obviously the assumption $\|f\|_{\mathbb{W}^{1,\infty}(\mathbb{R})} < \infty$ implies that $\|\nabla_u \eta(u^k)\|_{\mathbb{L}^\infty(\Omega;\mathbb{R}^2)} \leq C$.

We can also show (see [4],[10]) that if $\|f\|_{C^2(\mathbb{R})} < \infty$, then

$$|r^T \cdot \nabla_u^2 \eta(u^k) \cdot r| \leq C(\|f\|_{C^2(\mathbb{R})})\; r^T \cdot \nabla_u^2 \eta_E(u^k) \cdot r \;\text{ for all }\; r \in \mathbb{R}^2$$

and that $\eta(\cdot,\cdot)$ is a $\mathbb{C}^2$-function for $(u_1,u_2)\in(0,\infty)\times\mathbb{R}$. Here η_E denotes the mechanical energy obtained for $f(\xi)=\frac{\xi^2}{2}$ and is given by

$$\eta_E(a,b)=\frac{a^2}{2b}+\kappa(1+\alpha)b^{\frac{2+\alpha}{1+\alpha}}.$$

Thus for each entropy generated by $f\in\mathbb{C}_0^\infty(\mathbb{R})$:

1. $\{G_k(u^k,x)\nabla_u\eta(u^k)\}_{k=1}^\infty$ is bounded in $\mathbb{L}^\infty(\Omega;\mathbb{R}^2)$ and therefore precompact in $\mathbb{W}^{-1,2}(\Omega;\mathbb{R}^2)$,
2. $\{\frac{1}{k}(\frac{\partial}{\partial x}u^k)^T\cdot\nabla_u^2\eta(u^k)\cdot\frac{\partial}{\partial x}u^k\}_{k=1}^\infty$ is bounded in $\mathbb{L}^1(\Omega;\mathbb{R}^2)$ and consequently in $\mathcal{M}(\Omega;\mathbb{R}^2)$. Hence this set is precompact in $\mathbb{W}^{-1,p}(\Omega;\mathbb{R}^2)$ for $p<2$,
3. $\frac{1}{k}\frac{\partial}{\partial x}\eta(u^k)=\sqrt{\frac{1}{k}}\nabla_u\eta(u^k)\sqrt{\frac{1}{k}}\frac{\partial}{\partial x}u^k\to 0$ in $\mathbb{L}^2(\Omega;\mathbb{R}^2)$; thus $\{\frac{1}{k}\frac{\partial^2}{\partial^2 x}\eta(u^k)\}_{k=1}^\infty$ is a precompact set in $\mathbb{W}^{-1,2}(\Omega;\mathbb{R}^2)$,
4. the left hand side of equation (EE) is bounded $\mathbb{W}^{-1,\infty}(\Omega;\mathbb{R}^2)$.

Then by Murat's Lemma [9] the left hand side of (EE) is precompact in $\mathbb{W}^{-1,2}(\Omega;\mathbb{R}^2)$.

Step 3
We apply the Div-Curl lemma to all two entropy-flux pairs (η^1,q^1), (η^2,q^2) generated by $f^1,f^2\in C_0^\infty(\mathbb{R})$. This is the starting point to the analysis from [6], [7]. In the end this will imply that in the (z_1,z_2)-coordinates the measure $\mu_{(t,x)}$ has the form

$$\mu_{(t,x)}=\begin{cases}(\delta_{z_1(t,x)},\delta_{z_2(t,x)}) & \text{if } z_2(t,x)>0\\ (\nu_{(t,x)},\delta_{z_2(t,x)}) & \text{if } z_2(t,x)=0,\end{cases}$$

where $\operatorname{supp}\{\nu_{(t,x)}\}\subset[\liminf_{k\to\infty}(z_2^k(t,x),\limsup_{k\to\infty}(z_2^k(t,x)]$. It follows that in (u_1,u_2)-coordinates the measure is concentrated in one point.

4 Strong $\mathbb{L}^p$ Convergence and Existence of Weak Entropy Solutions

To prove that our solution is a weak entropy solution to problem (CL) we need that the limit function $G(t,x)$ satisfies $G(t,x)\in\tilde{G}(z(t,x),x)$ for a.e. $(t,x)\in\Omega$, and that the entropy inequality is satisfied for all weak entropies.

First we prove $u^{k_j}\to u$ in the strong topology of $\mathbb{L}^p(\Omega;\mathbb{R}^2)$ for $1\le p<\infty$.
Note that by the Radon-Riesz theorem the assumption $f^k\rightharpoonup f$ weakly in $\mathbb{L}^p$ and $\|f^k\|_{\mathbb{L}^p}\to\|f\|_{\mathbb{L}^p}$ imply that $f^k\to f$ strongly in $\mathbb{L}^p$ (for each $p\in(1,\infty)$).
Thus we must check that $\|u^{k_j}\|_{\mathbb{L}^p}\to\|u\|_{\mathbb{L}^p}$. By the convergence in the sense of Young measures and by the fact that these measures are Dirac measures

$$\int\limits_{\mathbb{R}\times[0,T)}|u^{k_j}|^p dtdx\to\int\limits_{\mathbb{R}\times[0,T)}\int_K|u|^p d\delta_{u(t,x)}dtdx=\int\limits_{\mathbb{R}\times[0,T)}|u|^p dtdx.$$

We can also prove the strong $\mathbb{L}^p$-convergence for the terms $(u_2^{k_j}(u_1^{k_j})^{-a})^b$, $(u_1^{k_j})^b$ and $(u_1^{k_j})^a \nabla_u \eta(u^{k_j})$ for all $0 < a < 1$ and $0 < b$. Just observe that $(u_1)^a \nabla_u \eta(u)$ is a continuous bounded function, if $\|\nabla_u \eta(u)\| \leq C$, $\eta(\cdot,\cdot)$ is a $\mathbb{C}^1$-function for $(u_1, u_2) \in (0, \infty) \times \mathbb{R}$ and $\eta(0, \cdot) = 0$.

To complete the proof of the existence of a weak entropy solution we need that there exists $G(t,x) \in \tilde{G}(u(t,x), x)$ for a.a. $(t,x) \in [0,\infty) \times \mathbb{R}$ such that

1. $G_{k_j}(u^{k_j}) \rightharpoonup G$ in $\mathbb{L}^p(\Omega; \mathbb{R}^2)$,
2. $G_{k_j}(u^{k_j}) \nabla_u \eta(u^{k_j}) \rightharpoonup G \nabla_u \eta(u)$ in $\mathbb{L}^p(\Omega; \mathbb{R}^2)$

for all $1 < p < \infty$. Now we prove the following lemma:

Lemma 4.1
Assume $u^k \to u$ in the strong topology $\mathbb{L}^p(\Omega; \mathbb{R})$ for $1 \leq p < \infty$, $\text{sig}_k \to \text{sig}$ uniformly in $\mathbb{R} - [-\frac{1}{n}, \frac{1}{n}]$ for all $n \in \mathbb{N}$ and $|\text{sig}_k| \leq 1$.
Then there exists a function $S \in \mathbb{L}^p(\Omega; \mathbb{R})$ and a subsequence k_j such that

1. $\text{sig}_{k_j}(u^{k_j}) \rightharpoonup S$ *in the weak topology of* $\mathbb{L}^p(\Omega; \mathbb{R})$
2. $S \in \text{sig}(u)$ *a.e. in* Ω

holds.

Proof
We prove only $S \in \text{sig}(u)$ a.e. in Ω. Let us observe $|S(t,x)| \leq 1$ a.e. in Ω. Then we divide Ω into three sets:
$\Omega_- = \{(t,x) \in \Omega \,|\, u_2(t,x) < 0\}$,
$\Omega_0 = \{(t,x) \in \Omega \,|\, u_2(t,x) = 0\}$,
$\Omega_+ = \{(t,x) \in \Omega \,|\, u_2(t,x) > 0\}$.
It is obvious that $S \in \text{sig}(u)$ a.e. in Ω_0. Now we can assume that k_j is a subsequence with $u^{k_j} \to u$ a.e. in Ω. Thus $\text{sig}_{k_j}(u^{k_j}) \to 1$ for a.a. $(t,x) \in \Omega_+$ and $\text{sig}_{k_j}(u^{k_j}) \to -1$ for a.a. $(t,x) \in \Omega_-$. This completes the proof of the lemma.

By applying Lemma 4.1 we end the proof of the existence of a weak entropy solution to our problem (CL).

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