$\mathbf{L_q}$-$\mathbf{L_r}$-Estimates for the Non-Stationary Stokes Equations in an Aperture Domain

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Abstract

This article deals with asymptotic estimates of strong solutions of Stokes equations in aperture domain. An aperture domain is a domain, which outside a bounded set is identical to two half spaces separated by a wall and connected inside the bounded set by one or more holes in the wall. It is known that the corresponding Stokes operator generates a bounded analytic semigroup in the closed subspace $J_q(\Omega)$ of divergence free vector fields of $L_q(\Omega)^n$. We deal with $L_q - L_r$-estimates for the semigroup, which are known for $\mathbb{R}^n$, the half space and exterior domains.

Key words: Stokes equations, aperture domain, asymptotic behaviour, asymptotic expansions
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1 Introduction and Main Results

Suppose that $\Omega \subset \mathbb{R}^n$, $n \geq 3$ is an aperture domain with smooth boundary, i.e.

$$\Omega \cup B_r(0) = \mathbb{R}^n_+ \cup \mathbb{R}^n_- \cup B_r(0)$$

with $\mathbb{R}^n_+ = \{x = (x_1, \dots, x_n) : x_n > 0\}$ and $\mathbb{R}^n_- = \{x = (x_1, \dots, x_n) : x_n < -d\}$, $d, r > 0$. We consider the homogeneous non-stationary Stokes equations in $(0, \infty) \times \Omega$

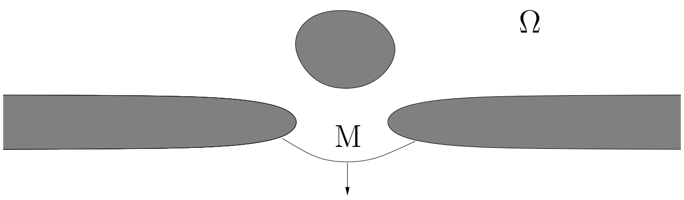

Figure 1: An aperture domain

concerning the velocity field $u(t, x)$ and the scalar pressure $p(t, x)$:

$$\begin{aligned} \partial_t u - \Delta u + \nabla p &= f && \text{in } (0,\infty) \times \Omega, && (1)\\ \operatorname{div} u &= 0 && \text{in } (0,\infty) \times \Omega, && (2)\\ u|_{\partial\Omega} &= 0 && \text{in } (0,\infty) \times \Omega, && (3)\\ \Phi(u) &= \alpha && \text{in } (0,\infty), && (4)\\ u|_{t=0} &= u_0 && \text{in } \Omega, && (5) \end{aligned}$$

where

$$\Phi(u(t)) = \int_M N \cdot u(t,x) d\sigma(x) = \alpha(t)$$

is the flux through a smooth, bounded $(n-1)$-dimensional manifold M with normal vector N directed downwards dividing Ω into two connected components. This flux has to be prescibed in order to get a unique solution with $u(t) \in L_q(\Omega)$ with $\frac{n}{n-1} < q < \infty$. In the case $1 < q \leq \frac{n}{n-1}$ the flux has to vanish, i.e. $\Phi(u) = 0$. (See [8] for the corresponding resolvent problem.)
In this paper we only deal with the case $f = 0$ and $\Phi(u) = 0$. We consider the asymptotic behaviour of the solutions $u(t)$. The general case can be derived from this case depending on the asymptotic behaviour of $f(t)$ and $\alpha(t)$. Since the Stokes operator A_q generates a bounded semigroup in $J_q(\Omega) = \overline{\{u \in \mathbb{C}_0^\infty(\Omega)^n, \operatorname{div} u = 0\}}^{\|\cdot\|_q}$ the estimate $\|u(t)\|_q \leq C\|u_0\|_q$ holds. The goal of this paper is to prove the following decay rate measuring $u(t)$ and u_0 in the norm of L_q for different $1 < q < \infty$.

Theorem 1.1 Let $1 < q \leq r < \infty$. Then there is a constant $C = C(\Omega, q, r)$ such that

$$\|u(t)\|_{L_r(\Omega)} \leq Ct^{-\sigma}\|u_0\|_{L_q(\Omega)} \tag{6}$$

with $\sigma = \frac{n}{2}\left(\frac{1}{q} - \frac{1}{r}\right)$ for all $t > 0$ and $u_0 \in J_q(\Omega)$.

Theorem 1.2 Let $1 < q \leq r < n$. Then there is a constant $C = C(\Omega, q, r)$ such that

$$\|\nabla u(t)\|_{L_r(\Omega)} \leq Ct^{-\sigma-\frac{1}{2}}\|u_0\|_{L_q(\Omega)} \tag{7}$$

with $\sigma = \frac{n}{2}\left(\frac{1}{q} - \frac{1}{r}\right)$ for all $t > 0$ and $u_0 \in J_q(\Omega)$.

These inequalities are known for other unbounded domains. In [11] Ukai showed these estimates for $1 < q < \infty$ if the domain is the half-space $\mathbb{R}^n_+$. This is done by using an explicit solution formula in terms of Riesz operators and the heat kernel in $\mathbb{R}^n_+$. In the case of an exterior domain, Iwashita [5] showed the validity of (6) for $1 < q \leq r < \infty$ and (7) for $1 < q \leq r \leq n$.
The proof of Theorem 1.1 and Theorem 1.2 uses a similar technique as in [5]. It consists of first showing a local estimate of the L_q-norm of $u(t)$ and then comparing the full L_q-norm with suitable solutions of the non-stationary Stokes equations in $\mathbb{R}^n_+$. The local estimate is derived from an asymptotic expansion of the resolvent of the Stokes operator in the aperture domain around 0 in special weighted L_q-spaces. The resolvent expansion is constructed by using a similar resolvent expansion of the Stokes operator in the half-space $\mathbb{R}^n_+$. For the latter expansion we combine Ukai's solution formula [11] with an resolvent expansion of the Laplace operator Δ in $\mathbb{R}^n$, based on the results of Murata [7].

Remark 1.3 With the methods of this article we can't prove Theorem 1.2 for the case $r = n$, which is done by Iwashita in the case of the exterior domain. This is due to a slightly weaker estimate of the local part of the L_q-norm. (See Corollary 6.2 and [5, Theorem 1.2 (i)].) We get this condition because we have to deal with weighted L_q-spaces of the kind $L_q(\Omega; \omega^{sq})$ such that ω^{sq} is a Muckenhoupt weight (see preliminaries); this condition on the weights is not needed in [5].

2 Preliminaries and Notation

We will consider the resolvent expansion in a scale of weighted L_q-spaces

$$\begin{aligned} L_q(\Omega; \omega^{sq}) &:= \left\{ f : \Omega \to \mathbb{R} \text{ measurable} : \|f\|_{L_q(\Omega;\omega^{sq})} < \infty \right\}, \qquad s \in \mathbb{R}, \\ \|f\|_{L_q(\Omega;\omega^{sq})} &:= \left(\int_\Omega |f(x)|^q \omega^{sq}(x) dx \right)^{\frac{1}{q}}. \end{aligned}$$

Analogously we define the weighted Sobolev spaces as

$$W_q^m(\Omega; \omega^{sq}) := \left\{ f \in L_{1,loc}(\overline{\Omega}) : D^\alpha f \in L_q(\Omega; \omega^{sq}), \forall |\alpha| \le m \right\}$$

and $W_{0,q}^m(\Omega; \omega^{sq}) := \overline{C_0^\infty(\Omega)}^{W_q^m(\Omega;\omega^{sq})}$. Recall that $f \in L_{1,loc}(\overline{\Omega})$ means that $f \in L_1(\Omega \cap B)$ for all balls B with $\Omega \cap B \neq \emptyset$. Moreover $D^\alpha f(x) = \partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n} f(x)$ for $\alpha \in \mathbb{N}_0^n$. By $\dot{W}_q^m(\Omega; \omega^{sq})$ we denote the corresponding homogeneous Sobolev space of $L_{1,loc}$-functions f with $D^\alpha f \in L_q(\Omega; \omega^{sq})$ for all $|\alpha| = m$. Finally

$$J_q(\Omega; \omega_n^{sq}) := \overline{\{u \in C_0^\infty(\Omega)^n : \operatorname{div} u = 0\}}^{L_q(\Omega;\omega_n^{sq})}.$$

For simplicity we often will skip the exponent n if we deal with spaces of vector fields; e.g. we write $f \in L_q(\Omega)$ instead of $f \in L_q(\Omega)^n$. If X, Y are two Banach spaces, we denote by $\mathcal{L}(X, Y)$ the space of all bounded linear maps $T : X \to Y$; furthermore $\mathcal{L}(X) := \mathcal{L}(X, X)$.

In [5, 7] the simple weight $\omega(x) = \langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}$ is used. For $-\frac{n}{q} < s < \frac{n}{q'}$ the weight $\langle x \rangle^{sq}$ is an element of the Muckenhoupt class $\mathcal{A}_q$. This is the class of all measurable functions $\omega : \mathbb{R}^n \to [0, \infty)$ with

$$\frac{1}{|B|} \int_B \omega(x) dx \left(\frac{1}{|B|} \int_B \omega(x)^{-\frac{q'}{q}} dx \right)^{\frac{q}{q'}} \le A < \infty,$$

where B is an arbitrary ball in $\mathbb{R}^n$ and A is independent of B. The weights $\omega \in \mathcal{A}_q$ have the important property that singular integral operators like the Riesz transforms

$$R_j f(x) := \mathcal{F}^{-1} \left[\frac{i\xi_j}{|\xi|} \hat{f}(\xi) \right] = c_n \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \frac{x_j - y_j}{|x - y|^{n+1}} f(y) dy,$$

$j = 1, \dots, n$, are continuous on $L_q(\mathbb{R}^n; \omega)$ into itself. Here $\mathcal{F}[u](\xi) = \hat{u}(\xi)$ denotes the Fourier transform with respect to x. (See for example [10, Chapter V: §4.2, Theorem 2] for the continuity and [9, Chapter III, Section 1] for Riesz transforms.)
We will also use the partial Riesz transforms

$$S_j f(x) := \mathcal{F}^{-1}_{\xi' \mapsto x'} \left[\frac{i\xi_j}{|\xi'|} \tilde{f}(\xi', x_n) \right] = c_{n-1} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^{n-1} \setminus B_\varepsilon(x')} \frac{x'_j - y'_j}{|x' - y'|^n} f(y', x_n) dy,$$

$j = 1, \dots, n-1$, $x = (x', x_n), \xi = (\xi', \xi_n)$, for functions f defined on $\mathbb{R}^n_+$ or $\mathbb{R}^n$. These partial Riesz transforms are used in Ukai's solution formula.
Unfortunately the weight $\langle x \rangle^{sq}$ considered for fixed x_n as weight in $\mathbb{R}^{n-1}$ is in the class $\mathcal{A}_q$ only if $-\frac{n-1}{q} < s < \frac{n-1}{q'}$. Therefore we will use the slightly weaker weight $\omega_n(x) := \prod_{i=1}^n \langle x_i \rangle^{\frac{1}{n}}$. For this weight $\omega_n(x)^{sq}$ considered for fixed x_n is in $\mathcal{A}_q$ on $\mathbb{R}^n$ for $-\frac{n}{q} < s < \frac{n}{q'}$. This is easily derived from the special product structure and the fact that $\langle x_i \rangle^{\frac{s}{n}}$ is a one-dimensional weight in $\mathcal{A}_q$. Therefore we get:

Lemma 2.1 *Let $\Omega = \mathbb{R}^n$ or $\Omega = \mathbb{R}^n_+$, $1 < q < \infty$, $-\frac{n}{q} < s < \frac{n}{q'}$ and $\omega_n(x) = \prod_{i=1}^n \langle x_i \rangle^{\frac{1}{n}}$. Then the (partial) Riesz transforms are continuous from $L_q(\Omega; \omega_n^{sq})$ into itself.*

Moreover we introduce $\Sigma_\delta = \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \delta\}$ and $\Sigma_{\delta,\varepsilon} = \Sigma_\delta \cap B_\varepsilon(0)$.
Recall the Helmholtz decomposition of a vector field $f \in L_q(\Omega; \omega_n^{sq})^n$, i.e. the unique decomposition $f = f_0 + \nabla p$ with $f_0 \in J_q(\Omega; \omega_n^{sq}), p \in \dot{W}^1_q(\Omega; \omega_n^{sq})$. The existence and continuity of the corresponding Helmholtz projection $P_q : L_q(\Omega; \omega_n^{sq})^n \to J_q(\Omega; \omega_n^{sq}), f \mapsto P_q f = f_0$ is proved in [3, Theorem 5] for the case that $\Omega = \mathbb{R}^n, \mathbb{R}^n_+$ or Ω is a bounded domain. For the case of an aperture domain and $s = 0$ the result is proved in [8, Theorem 2.6].
Furthermore we define the Stokes operator $A_q = -P_q \Delta$ in $J_q(\Omega)$ with $\mathcal{D}(A_q) = W^2_q(\Omega) \cap W^1_{0,q}(\Omega) \cap J_q(\Omega)$. Note that the resolvent of A_q satisfies the estimate

$$\|(z + A_q)^{-1} f\|_{L_q(\Omega)} \leq C_\delta |z|^{-1} \|f\|_{L_q(\Omega)} \tag{8}$$

for $z \in \Sigma_\delta, \delta \in (0, \pi)$, if Ω is an aperture domain (see [8, Theorem 2.5]). Therefore $-A_q$ generates an analytic semigroup.

3 The Resolvent Expansion in R^n_+

We consider the resolvent equations

$$(z - \Delta)u + \nabla p = f \qquad \text{in } \mathbb{R}^n_+, \tag{9}$$
$$\operatorname{div} u = 0 \qquad \text{in } \mathbb{R}^n_+, \tag{10}$$
$$u|_{\partial\mathbb{R}^n_+} = 0 \qquad \text{on } \partial\mathbb{R}^n_+. \tag{11}$$

Let $R_0(z) = (z - \Delta)^{-1}$ denote the resolvent of the Laplace operator in $\mathbb{R}^n$.

Lemma 3.1 *Let* $1 \le p \le \infty$, $0 < \delta < \pi$, $\alpha \in \mathbb{N}_0^n$, $|\alpha| \le 2$, $\frac{|\alpha|}{2} < \sigma < \frac{n+|\alpha|}{2}$, $-\frac{n}{p} < s' < s < \frac{n}{p'}$, $s' = s - 2\sigma + |\alpha|$. *Then*

$$D^\alpha R_0(z) = \sum_{j=0}^{[\sigma]-1} z^j D^\alpha G_{0j} + G_{0r}(z)$$

where $G_{0r}(z) = O(z^{\sigma-1})$ *in* $\mathcal{L}(W_p^m(\mathbb{R}^n;\omega_n^{sp}), W_p^{m+2-|\alpha|}(\mathbb{R}^n;\omega_n^{s'p}))$ *for* $z \to 0$ *with* $z \in \Sigma_\delta$.

Proof: The proof is the same as [7, Lemma 2.3.(i)]. It is based on the estimate for the convolution operator with the heat kernel $E_0(t)$:

$$\|D^\alpha E_0(t)\|_{\mathcal{L}(L_p(\mathbb{R}^n;\omega^{sp}),L_p(\mathbb{R}^n;\omega^{s'p}))} \le |t|^{-\frac{|\alpha|}{2}} \langle t\rangle^{-\sigma} \tag{12}$$

for $\omega(x) = \omega_n(x)$, $t \in \Sigma_{\delta_0}$, $0 < \delta_0 < \frac{\pi}{2}$, $\alpha \in \mathbb{N}_0^n$, $0 \le \sigma < \frac{n}{2}$ and $-\frac{n}{p} < s' < s < \frac{n}{p'}$, $s' = s - 2\sigma$.
The estimate (12) is proved in [7, Lemma 2.2] for the case $\omega(x) = \langle x\rangle$. But this case implies the estimate for $\omega(x) = \omega_n(x)$ since

$$\begin{aligned}
&\|D^\alpha E_0(t) f\|_{L_p(\mathbb{R}^n;\omega_n^{s'p})} \\
&\le \left\| \int_{\mathbb{R}^{n-1}} \left| D^{\alpha'} \frac{e^{-\frac{|x'-y'|^2}{4t}}}{(4\pi t)^{\frac{n-1}{2}}} \right| \left\| \int_{\mathbb{R}} \partial_{x_n}^{\alpha_n} \frac{e^{-\frac{|x_n-y_n|^2}{4t}}}{\sqrt{4\pi t}} f(y',y_n)dy_n \right\|_{L_p\left(\mathbb{R};\langle x_n\rangle^{\frac{s'p}{n}}\right)} dy' \right\|_{L_p\left(\mathbb{R}^{n-1};\omega_{n-1}^{s'p\frac{n-1}{n}}(x')\right)} \\
&\le C|t|^{-\frac{\alpha_n}{2}} \langle t\rangle^{-\frac{\sigma}{n}} \left\| \int_{\mathbb{R}^{n-1}} \left| D^{\alpha'} \frac{e^{-\frac{|x'-y'|^2}{4t}}}{(4\pi t)^{\frac{n-1}{2}}} \right| \|f(y',.)\|_{L_p\left(\mathbb{R};\langle x_n\rangle^{\frac{sp}{n}}\right)} dy' \right\|_{L_p\left(\mathbb{R}^{n-1};\omega_{n-1}^{s'p\frac{n-1}{n}}(x')\right)} \\
&\le C\left(\prod_{i=1}^n |t|^{-\frac{\alpha_i}{2}} \langle t\rangle^{-\frac{\sigma}{n}}\right) \|f\|_{L_p\left(\mathbb{R}^n;\omega_n^{sp}\right)} = C|t|^{-\frac{|\alpha|}{2}} \langle t\rangle^{-\sigma} \|f\|_{L_p\left(\mathbb{R}^n;\omega_n^{sp}\right)}.
\end{aligned}$$

with $\alpha = (\alpha', \alpha_n)$. ■

Remark 3.2 The operators G_{0j} and $G_{0r}(z)$ are given by

$$G_{0j} = \int_0^\infty E_0(t) \frac{(-t)^j}{j!} dt, \tag{13}$$

$$G_{0r}(z) = \int_0^\infty E_0(t) f_{[\sigma]}(zt) dt \qquad \text{with} \tag{14}$$

$$f_{[\sigma]}(zt) = e^{-zt} - \sum_{j=0}^{[\sigma]-1} \frac{(-zt)^j}{j!}.$$

We recall Ukai's solution formula for the homogeneous non-stationary Stokes equations in $\mathbb{R}^n_+$ (see [11]), i.e. (1)-(3), (5) for $\Omega = \mathbb{R}^n_+$, $f = 0$ with compatibility condition $\operatorname{div} u_0 = 0$ in $\mathbb{R}^n_+$ and $u_0^n = 0$, $u_0 = (u_0', u_0^n)$, on $\partial\mathbb{R}^n_+$. Let R_j, S_j be as above. Moreover let $rf = f|_{\mathbb{R}^n_+}$, $\gamma f = f|_{\partial\mathbb{R}^n_+}$ and e be the extension operator from $\mathbb{R}^n_+$ to $\mathbb{R}^n$ with value 0. Finally let $E(t)$ be the solution operator for the heat equation in $\mathbb{R}^n_+$, which is derived from $E_0(t)$ by odd extension from $\mathbb{R}^n_+$ to $\mathbb{R}^n$.
Then the solution $(u(t), p(t))$ of the non-stationary Stokes equations in $\mathbb{R}^n_+$ is $u(t) = WE(t)Vu_0$ and $p(t) = -D\gamma\partial_n E(t)V_1 u_0$ where

$$W = \begin{pmatrix} I & -SU \\ 0 & U \end{pmatrix}, \qquad V = \begin{pmatrix} V_2 \\ V_1 \end{pmatrix}, \qquad U = rR' \cdot S(R' \cdot S + R_n)e,$$
$$V_1 u_0 = -S \cdot u_0' + u_0^n, \qquad V_2 u = u_0' + S u_0^n,$$
$$R' = (R_1, \dots, R_{n-1})^T, \quad S = (S_1, \dots, S_{n-1})^T$$

and D is the Poisson operator for the Dirichlet problem of the Laplace equation in $\mathbb{R}^n_+$.
Using this result, we get:

Theorem 3.3 Let $1 < q < \infty, 0 < \delta < \pi$, $n \geq 3$, $\frac{|\alpha|}{2} < \sigma < \frac{n+|\alpha|}{2}$, $\alpha \in \mathbb{N}_0^n$, $|\alpha| \leq 2, -\frac{n}{q} < s' \leq 0 \leq s < \frac{n}{q'}$ and $s' = s - 2\sigma + |\alpha|$. Then there exist operators $R_+(z)$, $P_+(z)$ with $D^\alpha R_+(z) \in \mathcal{L}(L_q(\mathbb{R}^n_+; \omega_n^{sq}), W_q^{2-|\alpha|}(\mathbb{R}^n_+; \omega_n^{s'q}))$ and $P_+(z) \in \mathcal{L}(L_q(\mathbb{R}^n_+; \omega_n^{sq}), \dot{W}_q^1(\mathbb{R}^n_+; \omega_n^{s'q}))$ depending continuously on $z \in \Sigma_\delta \cup \{0\}$ with:

1. $u = R_+(z)f$ and $p = P_+(z)f, f \in L_q(\mathbb{R}^n_+; \omega_n^{sq})$, is a solution of (9) - (11) for $z \in \Sigma_\delta$.

2. $R_+(z) \in \mathcal{L}(L_q(\mathbb{R}^n_+; \omega_n^{sq}), W_q^2(\mathbb{R}^n_+))$ and $P_+(z) \in \mathcal{L}(L_q(\mathbb{R}^n_+; \omega_n^{sq}), \dot{W}_q^1(\mathbb{R}^n_+))$ for every $z \in \Sigma_\delta$.

3. The asymptotic expansions

$$\begin{aligned} D^\alpha R_+(z) &= \sum_{j=0}^{[\sigma]-1} z^j D^\alpha G_j + O(z^{\sigma-1}) \ \text{ in } \mathcal{L}(L_q(\mathbb{R}^n_+; \omega_n^{sq}), W_q^{2-|\alpha|}(\mathbb{R}^n_+; \omega_n^{s'q})), \\ P_+(z) &= \sum_{j=0}^{[\sigma]-1} z^j P_{+,j} + O(z^{\sigma-1}) \ \text{ in } \mathcal{L}(L_q(\mathbb{R}^n_+; \omega_n^{sq}), \dot{W}_q^1(\mathbb{R}^n_+; \omega_n^{s'q})) \end{aligned}$$

   hold for $z \to 0, z \in \Sigma_\delta$.

Proof: Because of the Helmholtz decomposition in weighted L_q-Spaces (see [3, Theorem 5]), we can assume w.l.o.g. that $f \in J_q(\Omega; \omega^{sq})$.

Therefore the asymptotic expansion for $R_+(z)$ simply follows from the expansion of $R_0(z)$, the equations (13)-(14),the continuity of the Riesz-transforms S_j and R_j in $L_q(\mathbb{R}^n;\omega_n^{sq})$ and $L_q(\mathbb{R}^n_+;\omega_n^{sq})$ if $-\frac{n}{q} < s < \frac{n}{q'}$ and the fact

$$R_+(z)f = \int_0^\infty e^{-tz} W E(t) V f \, dt.$$

In order to get the result for $D^\alpha R_+(z)$, $|\alpha| \leq 2$ we use the relations

$$\begin{aligned} \partial_n U &= (I-U)|\nabla'| = -(I-U)\sum_{i=1}^{n-1} S_i \partial_i, \\ \partial_i S &= S\partial_i \qquad i = 1, \dots, n, \\ \partial_i U &= U\partial_i \qquad i = 1, \dots, n-1 \end{aligned}$$

and prove the expansion in the same way as in the case $\alpha = 0$. We note that the first equation is a consequence of

$$\mathcal{F}_{x'\mapsto\xi'}\left[Uf\right](\xi', x_n) = |\xi'| \int_0^{x_n} e^{-|\xi|(x_n - y_n)} \tilde{f}(\xi', x_n) dy_n \tag{15}$$

(see the proof of [11, Theorem 1.1]); the other equations are obvious.
Finally we get the expansion of $\nabla P_+(z)$ in the same way using $|\nabla'|D\gamma = \partial_n U - U\partial_n$. ■

Because of the estimate (12) and Ukai's formula we also easily get

Lemma 3.4 *Let $u(t) = WE(t)Vu_0$, $u_0 \in J_q(\mathbb{R}^n_+;\omega_n^{sq})$, denote the solution of the homogeneous non-stationary Stokes equations (1)-(3), (5) for $\Omega = \mathbb{R}^n_+$, $f = 0$. Then*

$$\|u(t)\|_{L_q(\mathbb{R}^n_+;\omega_n^{s'q})} \leq C(1+t)^{-\sigma} \|u_0\|_{L_q(\mathbb{R}^n_+;\omega_n^{sq})}$$

with $1 < q < \infty$, $-\frac{n}{q} < s' \leq 0 \leq s < \frac{n}{q'}$, $s' = s - 2\sigma$, $t \geq 0$.

4 Resolvent Expansions in Aperture Domains

We consider the resolvent equation

$$\begin{aligned} (z-\Delta)u + \nabla p &= f \qquad \text{in } \Omega, &(16)\\ \operatorname{div} u &= 0 \qquad \text{in } \Omega, &(17)\\ u|_{\partial\Omega} &= 0 \qquad \text{on } \partial\Omega, &(18)\\ \Phi(u) &= 0 &(19) \end{aligned}$$

for an aperture domain Ω.

Theorem 4.1 Let $1 < q < \infty, 0 < \delta < \pi$, $n \geq 3$, $1 < \sigma < \frac{n}{2}$, $-\frac{n}{q} < s' \leq 0 \leq s < \frac{n}{q'}$ and $s' := s - 2\sigma$. Then there are an $\varepsilon > 0$ and operators $R(z) \in \mathcal{L}(L_q(\Omega;\omega_n^{sq}), W_q^2(\Omega;\omega_n^{s'q}))$ and $P(z) \in \mathcal{L}(L_q(\Omega;\omega_n^{sq}), \dot{W}_q^1(\Omega;\omega_n^{s'q}))$, depending continuously on $z \in \Sigma_{\delta,\varepsilon} \cup \{0\}$ with the following properties:

1. The pair $u = R(z)f$ and $p = P(z)f$ is a solution of (16)-(19).
2. $R(z) \in \mathcal{L}(L_q(\Omega;\omega_n^{sq}), W_q^2(\Omega))$ for every $z \in \Sigma_{\delta,\varepsilon}$.
3. The operator-valued function $R(z)$, $z \in \Sigma_{\delta,\varepsilon_0}$ has an expansion

$$R(z) = \sum_{j=0}^{[\sigma]-1} z^j G_j + G_r(z)$$

in $\mathcal{L}(L_q(\Omega;\omega_n^{sq}), W_q^2(\Omega;\omega_n^{s'q}))$ where $G_r(z) = O(z^{\sigma-1})$ for $z \to 0$.

Proof: We use the technique used in the proof of Theorem 3.1 in [5]. Let $\Omega \cup B_r(0) = \mathbb{R}^n_+ \cup \mathbb{R}^n_- \cup B_r(0)$. We choose $b, R \in \mathbb{R}$ such that $b > R > r + 3$ and denote $\mathbb{R}^n_\pm := \mathbb{R}^n_+ \cup \mathbb{R}^n_-, \Omega_\pm := \Omega \cap \mathbb{R}^n_\pm, \Omega_b := \Omega \cap B_b(0)$. Let $\varphi, \psi \in C^\infty(\Omega)$ be cut-off functions with $\varphi(x) = 1$ for $|x| > R$, $\varphi(x) = 0$ for $|x| < R-1$, $\psi(x) = 1$ for $|x| > R-2$ and $\psi(x) = 0$ for $|x| < R-3$. We identify ψf with its extension by 0 to $\mathbb{R}^n_\pm$. Moreover we define $R_\pm(z) : L_q(\mathbb{R}^n_\pm;\omega_n^{sq}) \to W_q^2(\mathbb{R}^n_\pm;\omega_n^{s'q})$ by

$$R_\pm(z)g(x) = \begin{cases} R_+(z)(g|_{\mathbb{R}^n_+})(x) & \text{if } x \in \mathbb{R}^n_+ \\ R_-(z)(g|_{\mathbb{R}^n_-})(x) & \text{if } x \in \mathbb{R}^n_-. \end{cases}$$

The operator $P_\pm(z) : L_q(\mathbb{R}^n_\pm;\omega_n^{sq}) \to \dot{W}_q^1(\mathbb{R}^n_\pm;\omega_n^{s'q})$ is defined analogously. Let $f_b := f|_{\Omega_b}$ and $(L,P) : L_q(\Omega_b)^n \to W_q^2(\Omega_b)^n \times \dot{W}_q^1(\Omega_b)$ be the solution operator of the Stokes equation in the bounded domain Ω_b. Set $R_1(z) \in \mathcal{L}(L_q(\Omega;\omega_n^{sq}), W_q^2(\Omega;\omega_n^{s'q}))$ by

$$R_1(z)f := \varphi R_\pm(z)(\psi f) + (1-\varphi)Lf_b.$$

Similarly define $\Pi(z) \in \mathcal{L}(L_q(\Omega;\omega_n^{sq}), \dot{W}_q^1(\Omega;\omega_n^{s'q}))$ by

$$\Pi(z)f := \varphi P_\pm(z)(\psi f) + (1-\varphi)Pf_b.$$

Obviously the operator $R_1(z)$ has the same type of expansion as $R_\pm(z)$. Let

$$P_\pm(z) = \sum_{j=0}^{[\sigma]-1} z^j P_{\pm,j} + P_{\pm,r}(z)$$

with $P_{\pm,r}(z) = O(z^{\sigma-1})$ in $\mathcal{L}(L_q(\mathbb{R}^n_\pm;\omega_n^{sq}), \dot{W}^1_q(\mathbb{R}^n_\pm;\omega_n^{s'q}))$ be the expansion for $P_\pm(z)$. We choose $P_{\pm,j}f, P_{\pm,r}f \in \dot{W}^1_q(\mathbb{R}^n_\pm)$ such that

$$\begin{aligned}\int_{D_R\cap\Omega} P_{\pm,0}f\,dx &= \int_{D_R\cap\Omega} Pf_b dx, \qquad \int_{D_R\cap\Omega} P_{\pm,r}(z)f\,dx = 0,\\ \int_{D_R\cap\Omega} P_{\pm,j}f\,dx &= 0 \qquad\qquad \text{for } j = 1,\cdots,[\sigma]-1\end{aligned}$$

where $D_R := \{x \in \Omega : R-1 < |x| < R\}$. Applying Poincaré's inequality

$$\|f\|_q \le C\left(\|\nabla f\|_q + \left|\int_D f(x)dx\right|\right)$$

for a bounded domain D with C^0-boundary (see [1, V. Theorem 4.19]) it follows that

$$\begin{aligned}\|P_{\pm,0}f - Pf_b\|_{L_q(D_R\cap\Omega)} &\le C\left(\|\nabla P_{\pm,0}f\|_{L_q(D_R\cap\Omega)} + \|\nabla Pf_b\|_{L_q(\Omega_b)}\right) \le C\|f\|_{L_q(\Omega;\omega_n^{sq})},\\ \|P_{\pm,j}f\|_{L_q(D_R\cap\Omega)} &\le C\,\|\nabla P_{\pm,j}f\|_{L_q(D_R\cap\Omega)} \le C\|f\|_{L_q(\Omega;\omega_n^{sq})},\\ \|P_{\pm,r}(z)f\|_{L_q(D_R\cap\Omega)} &\le C\,\|\nabla P_{\pm,r}(z)f\|_{L_q(D_R\cap\Omega)} \le C|z|^{\sigma-1}\|f\|_{L_q(\Omega;\omega_n^{sq})}.\end{aligned}$$

Because of these inequalities and the identity $\nabla\Pi(z)f = \varphi\nabla P_\pm(z)(\psi f) + (1-\varphi)\nabla Pf_b + (\nabla\varphi)(P_\pm(z)(\psi f) - Pf)$ the operator $\Pi(z)$ has the same type of expansion as $P_\pm(z)$.
It remains to correct the divergence of $R_1(z)f$. For this we apply Bogovskii's Theorem (see e.g. [4, Theorem 3.2]) to $\operatorname{div}(R_1(z)f) = \nabla\varphi\cdot\{R_\pm(z)(\psi f) - Lf_b\}$, which has compact support in D_R. We note that

$$\begin{aligned}\int_{D_R}\operatorname{div}(R_1(z)f) &= -\int_{B_R\cap\mathbb{R}^n_\pm}\operatorname{div}((1-\varphi)R_\pm(z)(\psi f))dx - \int_{\Omega_b}\operatorname{div}(\varphi Lf_b)\,dx\\ &= -\int_{\partial(B_R\cap\mathbb{R}^n_\pm)} N\cdot(1-\varphi)R_\pm(z)(\psi f)d\sigma - \int_{\partial\Omega_b} N\cdot\varphi Lf_b d\sigma = 0.\end{aligned}$$

Since $\operatorname{div} R_1(z)f \in W^2_q(D_R)\cap W^1_{0,q}(D_R)$, we get a compact operator $Q(z) : L_q(\Omega;\omega_n^{sq}) \to W^2_{0,q}(D_R)$ with $\operatorname{div} Q(z)f = \operatorname{div} R_1(z)f$. The operator $Q(z)$ depends continuously on $z \in \Sigma_\delta \cup \{0\}$.
We identify $Q(z)f$ with its extension by zero to a function $Q(z)f \in W^2_{0,q}(\Omega;\omega_n^{s'q})$. Now let $R_2(z) := R_1(z) - Q(z) \in \mathcal{L}(L_q(\Omega;\omega_n^{sq}), W^2_q(\Omega;\omega_n^{s'q}))$; then

$$\begin{aligned}(z-\Delta)R_2(z)f + \nabla\Pi(z)f &= f + S(z)f \quad \text{in } \Omega,\\ \operatorname{div} R_2(z)f &= 0 \qquad\qquad \text{in } \Omega,\\ R_2(z)f &= 0 \qquad\qquad \text{on } \partial\Omega\end{aligned}$$

for all $f \in L_q(\Omega;\omega_n^{sq})$, where

$$\begin{aligned}S(z)f &= -\{2(\nabla\varphi)\cdot\nabla + (\Delta\varphi)\}\{R_\pm(z)(\psi f) - Lf_b\}\\ &\quad + z(1-\varphi)Lf_b + (\Delta - z)Q(z)f + \nabla\varphi(P_\pm(z)(\psi f) - Pf_b).\end{aligned}$$

Since $\operatorname{supp} S(z)f \subseteq \overline{D_R}$, it holds $S(z) \in \mathcal{L}(L_q(\Omega;\omega_n^{sq}))$. The term $(\Delta - z)Q(z) \in \mathcal{L}(L_q(\Omega;\omega_n^{sq}))$ is a compact operator since $Q(z) : L_q(\Omega;\omega_n^{sq}) \to W^2_{0,q}(D_R)$ is compact. Furthermore $S(z) - (\Delta - z)Q(z) : L_q(\Omega;\omega_n^{sq}) \to W^1_q(D_R)$ is continuous, so $S(z) \in \mathcal{L}(L_q(\Omega;\omega_n^{sq}))$ is a compact operator. Moreover $S(z)$ is continuous in $z \in \Sigma_\delta \cup \{0\}$ and has the same type of expansion in $\mathcal{L}(L_q(\Omega;\omega_n^{sq}))$ as $R_\pm(z)$ in $\mathcal{L}(L_q(\Omega;\omega_n^{sq}), W^2_q(\Omega;\omega_n^{s'q}))$.
In the following Lemma 4.2 we show that $I+S(0)$ is injective. Since $S(0)$ is compact, the Fredholm alternative yields that $(I+S(0))^{-1} \in \mathcal{L}(L_q(\Omega;\omega_n^{sq}))$ exists. Therefore $(I+S(z))^{-1}$ exists for all $z \in \Sigma_{\delta,\varepsilon}$ for some $\varepsilon > 0$ More precisely

$$(I+S(z))^{-1} = (I+S(0))^{-1} \sum_{k=0}^{\infty} \left[(S(0)-S(z))(I+S(0))^{-1}\right]^k$$

for all $z \in \Sigma_{\delta,\varepsilon_0}$, where $\varepsilon_0 > 0$ is chosen so small that

$$\|S(z) - S(0)\| \le \frac{1}{2\|(I+S(0))^{-1}\|} \qquad \text{for all } z \in \Sigma_{\delta,\varepsilon_0}.$$

Since $S(z)$ and therefore all powers $(S(0) - S(z)^k$ have an expansion in $\mathcal{L}(L_q(\Omega;\omega_n^{sq}))$ of the same type as $R_\pm(z)$, the inverse $(I+S(z))^{-1}$ has the same.
If we now set $R(z) := R_2(z)(I+S(z))^{-1}$ and $P(z) = \Pi(z)(I+S(z))^{-1}$, we get the solution operators of the resolvent problem with the desired expansion. ■

Lemma 4.2 *Let $S(z)$ denote the same operator as in the proof of Theorem 4.1. Then $I+S(0) \in \mathcal{L}(L_q(\Omega;\omega_n^{sq}))$ is injective.*

Proof: It is known [2, 8] that the Stokes equations in an aperture domain have a unique solution $(u,\tilde{p}) \in \left[\dot{W}^2_p(\Omega) \cap \dot{W}^1_{p^*}(\Omega)\right]^n \times \dot{W}^1_p(\Omega)$, $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$ for $1 < p < n$, for given force $f \in L_p(\Omega)$ and prescribed flux $\Phi(u) = \alpha \in \mathbb{R}$.
We calculate the flux of $R_2(0)$. Since $M \subset B_r$, the identity $R_2(0)f(x) = Lf_b(x)$ holds for all $x \in M$. Denote by B_+ the connected component of $B_r(0) \setminus M$ "above" M. Then we conclude that

$$0 \;=\; \int_{B_+} \operatorname{div} Lf_b dx = \int_{\partial B_+} Lf_b \cdot N d\sigma = \int_M Lf_b \cdot N d\sigma = \int_M R_2(0)f \cdot N d\sigma.$$

Therefore we get $R_2(0)f = 0$, $\Pi(0) = const.$ if we show that $R_2(0)f \in \left[\dot{W}^2_p(\Omega) \cap \dot{W}^1_{p^*}(\Omega)\right]^n$ and $\Pi(0)f \in \dot{W}^1_p(\Omega)$.
Let $(I+S(0))f = 0$. That means $f = -S(0)f$; and therefore the support of f is contained in $\overline{\Omega}_b$. This implies $f \in L_p(\Omega;\omega_n^{sp})$ for all $s \in \mathbb{R}$ and $1 \le p \le q$.
Claim: $\nabla^2 R_2(0)f, \nabla\Pi(0)f \in L_p(\Omega)$ for all $1 < p \le q$ and $\nabla R_2(0)f \in L_{p^*}(\Omega)$ with $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$ and $1 < p < \min\{q,n\}$.

Proof of claim: It holds for $i, j \in \{1, \ldots, n\}$:

$$\begin{aligned}\partial_i\partial_j R_2(0)f &= \varphi\partial_i\partial_j R_\pm(0)(\psi f) + \partial_i\partial_j\left[(1-\varphi)Lf_b\right] + (\partial_i\varphi)\partial_j R_\pm(0)(\psi f) \\ &\quad + (\partial_j\varphi)\partial_i R_\pm(0)(\psi f) + (\partial_i\partial_j\varphi)R_\pm(0)(\psi f) - \partial_i\partial_j Q(0)f.\end{aligned}$$

The support of every term except the first one is contained in $\overline{\Omega}_b$. Therefore each of these function is an element of $L_p(\Omega)$ for every $1 \le p \le q$.
Considering the first term Theorem 3.3 tells us that $\partial_i\partial_j R_\pm(0) \in \mathcal{L}(L_p(\mathbb{R}^n_\pm; \omega_n^{sp}), L_p(\Omega, \omega_n^{s'p}))$ for all $-\frac{n}{p} < s' \le 0 \le s < \frac{n}{p'}$, $s' = s - 2\sigma + 2$, $1 < \sigma < \frac{n}{2}$. Since $f \in L_p^s(\Omega)$ for arbitrary $s \in \mathbb{R}$ and $1 \le p \le q$, we can apply this Theorem for $s' = 0$, $s = 2\sigma - 2$. Therefore we choose $1 < \sigma < \frac{n}{2}$ such that $\frac{n}{n-2\sigma+2} < p \Leftrightarrow 2\sigma - 2 < \frac{n}{p'}$. Thus we get $\partial_i\partial_j R_\pm(0)(\psi f) \in L_p(\Omega)$ for every $1 < p \le q$. With the same choice of s and s' we see $\nabla\Pi(0)f \in L_p(\Omega)$ for all $1 < p \le q$.
The same argumentation can be applied to

$$\partial_i R_2(0)f = \varphi\partial_i R_\pm(0)(\psi f) + \partial_i\left[(1-\varphi)Lf_b\right] + (\partial_i\varphi)R_\pm(0)(\psi f) - \partial_i Q(0)f.$$

In this case $\partial_i R_\pm(0) \in \mathcal{L}(L_r(\Omega; \omega_n^{sr}), L_r(\Omega; \omega_n^{s'r}))$ holds for all $-\frac{n}{r} < s' \le 0 \le s < \frac{n}{r'}$, $s' := s - 2\sigma + 1$, $1 < \sigma < \frac{n}{2}$. The choice of $s' = 0$, $s = 2\sigma - 1$ yields the condition $2\sigma - 1 < \frac{n}{r'}$. Since $\frac{1}{r} + \frac{1}{n} = \frac{1}{p}$, this condition is equivalent to $2\sigma - 2 < \frac{n}{p'} \Leftrightarrow p > \frac{n}{n-2\sigma+2}$. This proves the claim.

Thus $R_2(0)f = 0$ and $\nabla\Pi(0)f = 0$. Since $\operatorname{supp} Q(0) \subseteq \{R - 1 \le |x| \le R\}$, it is obvious that

$$R_2(0)f(x) = R_\pm(0)(\psi f)(x) = 0, \quad \nabla\Pi(0)f(x) = \nabla P_\pm(0)(\psi f)(x) = 0$$

for $x \in \Omega, |x| \ge R$, and

$$R_2(0)f(x) = Lf_b(x) = 0, \quad \nabla\Pi(0)f(x) = \nabla Pf_b(x) = 0$$

for $x \in \Omega$, $|x| \le R - 1$. This implies $f = 0$ for $|x| \ge R$ since

$$-\Delta R_\pm(0)(\psi f) + \nabla P_\pm(0)(\psi f) = \psi f \qquad \text{in } \mathbb{R}^n_\pm.$$

Similarly we get $f = 0$ for $x \in \Omega, |x| \le R - 1$, since

$$-\Delta Lf_b + \nabla Pf_b = f_b \qquad \text{in } \Omega_b.$$

The support of $(R_\pm(0)(\psi f), P_\pm(0)(\psi f))$ and of (Lf_b, Pf_b) is contained in $\widetilde{D} := \{x \in \Omega : R - 1 < |x| < b\}$. Therefore both terms solve the Stokes equations

$$\begin{aligned}-\Delta u + \nabla p &= f \qquad \text{in } \widetilde{D}, \\ \operatorname{div} u &= 0 \qquad \text{in } \widetilde{D}, \\ u &= 0 \qquad \text{on } \partial\widetilde{D}.\end{aligned}$$

This implies that $R_\pm(0)(\psi f) = Lf_b$ and $\nabla P_\pm(0)(\psi f) = \nabla Pf_b$ in $\widetilde{D}$ because of the unique solvability of the Stokes equations in a bounded domain. Hence $Q(z)f = 0$, $Lf_b = R_2(0)f = 0$, $\nabla Pf_b = \nabla\Pi(0)f = 0$ in $\widetilde{D}$ and finally $f = 0$ in the whole domain Ω. ■

5 Decay of the Semigroup in Weighted Spaces

Let $A_q = -P_q\Delta$ denote the Stokes operator for an aperture domain Ω.

Theorem 5.1 Let $n \geq 3$, $1 < \sigma < \frac{n}{2}$, $1 < q < \infty$, $-\frac{n}{q} < s' \leq 0 \leq s < \frac{n}{q'}$, $s' = s - 2\sigma$. Then there exists a constant $C = C(q, s, s')$ such that

$$\left\|e^{-tA_q} f\right\|_{L_q(\Omega;\omega_n^{s'q})} \leq C(1+t)^{-\sigma}\|f\|_{L_q(\Omega;\omega_n^{sq})}, \qquad t \geq 0,$$

for all $f \in J_q(\Omega) \cap L_q(\Omega;\omega_n^{sq})$. Furthermore

$$\left\|e^{-tA_q} f\right\|_{W_q^2(\Omega;\omega_n^{s'q})} \leq C(1+t)^{-\sigma} \max\left\{\|f\|_{W_q^2(\Omega)}, \|f\|_{L_q(\Omega;\omega_n^{sq})}\right\}, \qquad t \geq 0,$$

for all $f \in \mathcal{D}(A_q) \cap L_q(\Omega;\omega_n^{sq})$.

Proof: The proof of the inequalities is nearly the same as the proof of Theorem 1.1 in [5]. So we give only a sketch.
Since the semigroup e^{-tA_q} is bounded in $J_q(\Omega)$, the first estimate is satisfied for $0 < t < 1$. The second estimate holds for $0 < t < 1$ because of the estimates

$$\|f\|_{W_q^2(\Omega)} \leq c\|(I + A_q)f\|_{L_q(\Omega)} \leq C\|f\|_{W_q^2(\Omega)} \tag{20}$$

for all $f \in \mathcal{D}(A_q)$. (The first inequality is a consequence of [8, Theorem 2.1]. The second inequality is obvious.)
For $t \geq 1$ consider the representation of the semigroup

$$e^{-tA_q} = \frac{1}{2\pi i}\int_\Gamma e^{tz}(z + A_q)^{-1} dz$$

where the curve Γ coincides outside a ball $B_\varepsilon(0)$, $0 < \varepsilon < \varepsilon_0$,with the rays $e^{\pm\phi i}\tilde{t}, \tilde{t} > 0$ with $\frac{\pi}{2} < \phi < \delta$. ($\delta, \varepsilon_0$ are the same numbers as in Theorem 4.1.)
We split the curve Γ into two parts $\Gamma_1 = \{z \in \Gamma : 0 < |z| < \varepsilon\}$ and $\Gamma_2 = \{z \in \Gamma : \varepsilon \leq |z|\}$. So we get

$$e^{-tA_q} f = \frac{1}{2\pi i}\int_{\Gamma_1} e^{tz} R(z) f dz + \frac{1}{2\pi i}\int_{\Gamma_2} e^{tz}(z + A_q)^{-1} f dz$$

for all $f \in J_q(\Omega) \cap L_q(\Omega;\omega_n^{sq})$ since $R(z)f = (z + A_q)^{-1}f$ for $z \in \Sigma_{\delta,\varepsilon}$.
Using the resolvent estimate $\|(z + A_q)^{-1}f\|_q \leq C|z|^{-1}\|f\|_q$ we easily get

$$\left\|\frac{1}{2\pi i}\int_{\Gamma_2} e^{tz}(z + A_q)^{-1} dz f\right\|_{L_q(\Omega;\omega_n^{s'q})} \leq C\int_\varepsilon^\infty \frac{e^{ts\cos\phi}}{s} ds \|f\|_{L_q(\Omega)} \leq C(\varepsilon, \phi)\frac{e^{-ct}}{t}\|f\|_{L_q(\Omega;\omega_n^{sq})}$$

with some constant $c > 0$. Analogously we get

$$\left\|\frac{1}{2\pi i}\int_{\Gamma_2} e^{tz}(z + A_q)^{-1} dz f\right\|_{W_q^2(\Omega;\omega_n^{s'q})} \leq C\int_\varepsilon^\infty \frac{e^{ts\cos\phi}}{s} ds \|f\|_{W_q^2(\Omega)} \leq C(\varepsilon, \phi)\frac{e^{-ct}}{t}\|f\|_{W_q^2(\Omega)}$$

if we use (20) for $f \in \mathcal{D}(A_q)$.
We use the resolvent expansion of Theorem 4.1 to estimate the first integral. Since $\sum_{j=0}^{[\sigma]-1} z^j G_j$ is holomorphic in $\mathbb{C}$, it holds that

$$\left\| \sum_{j=0}^{[\sigma]-1} \int_{\Gamma_1} e^{tz} z^j G_j dz \right\|_{\mathcal{L}(L_q(\omega_n^{sq}), W_q^2(\omega_n^{s'q}))} \leq C e^{\varepsilon t \cos(\phi)} = C e^{-ct}$$

with $c > 0$. In order to estimate the remainder term we deform the curve Γ_1 to a curve Γ^* which coincides with $z = e^{\pm\phi i}\tilde{t}$, $\tilde{t} \in [0, \varepsilon]$. Therefore

$$\left\| \frac{1}{2\pi i} \int_{\Gamma_1} e^{tz} G_r(z) dz \right\|_{\mathcal{L}(L_q(\omega_n^{sq}), W_q^2(\omega_n^{s'q}))} \leq C \int_0^\infty e^{\lambda t \cos(\phi)} \lambda^{\sigma-1} d\lambda = C' t^{-\sigma}.$$

Collecting all estimates we proved the Theorem. ■

6 The L_q-L_r-Estimate

In order to get an estimate of $\|e^{-tA_q} f\|_{L_q(\Omega_b)}$, $\Omega_b = \Omega \cap B_b(0)$, we need:

Lemma 6.1 *Let $1 < q < \infty$ and $-\frac{n}{q} < s' < 0$. Then it holds that*

$$\|e^{-tA_q} f\|_{L_q(\Omega; \omega_n^{s'q})} \leq C(1+t)^{\frac{s'}{2}} \|f\|_{L_q(\Omega)}$$

for all $f \in J_q(\Omega)$ and

$$\|e^{-tA_q} f\|_{W_q^2(\Omega; \omega_n^{s'q})} \leq C(1+t)^{\frac{s'}{2}} \|f\|_{W_q^2(\Omega)}$$

for all $f \in \mathcal{D}(A_q)$.

Corollary 6.2 *Let $1 < q < \infty$. Then for every $0 \leq s < \frac{n}{2q}$ there is a constant $C = C(s, q, \Omega)$ with*

$$\|e^{-tA_q} f\|_{L_q(\Omega_b)} \leq C(1+t)^{-s} \|f\|_{L_q(\Omega)}$$

for all $f \in J_q(\Omega)$ and

$$\|e^{-tA_q} f\|_{W_q^2(\Omega_b)} \leq C(1+t)^{-s} \|f\|_{W_q^2(\Omega)}$$

for all $f \in \mathcal{D}(A_q)$.

Proof of Lemma 6.1: If $1 < p < \frac{n}{2}$ then $\frac{n}{p} > 2$; so we can we apply Theorem 5.1 with $s = 0$. Therefore we get

$$\|e^{-tA_p} f\|_{W_p^m(\Omega;\omega_n^{\tilde{s}' p})} \leq C(1+t)^{\frac{\tilde{s}'}{2}} \|f\|_{W_p^m(\Omega)} \tag{21}$$

for $m = 0, 2$, $f \in J_p(\Omega)$ resp. $f \in \mathcal{D}(A_p)$ and $-\frac{n}{p} < \tilde{s}' < -2$. In order to get the statement of the theorem we interpolate the estimates (21) and

$$\|e^{-tA_r} f\|_{W_r^m(\Omega)} \leq C\|f\|_{W_r^m(\Omega)}, \qquad m = 0, 2, f \in J_r(\Omega) \text{ resp. } \mathcal{D}(A_r) \tag{22}$$

for suitable p close to 1 and large r. For this we need the following statement about complex interpolation:

$$\Big(L_p(\Omega;\omega_n^{\tilde{s}' p}), L_r(\Omega)\Big)_{[\theta]} \quad = \quad L_q(\Omega;\omega_n^{\tilde{s}' p(1-\theta)})$$

with $0 < \theta < 1$, $\frac{1}{q} = \frac{1-\theta}{p} + \frac{\theta}{r}$ (see for example [6, Theorem 5.5.3]).
Now let $1 < q < \infty$, $-\frac{n}{q} < s' < 0$ be given as in the assumptions. We set for $0 < \theta < 1$

$$\tilde{s}' = \frac{s'}{1-\theta} \quad \text{and} \quad \frac{1}{q} = \frac{1-\theta}{p} + \frac{\theta}{r}.$$

Then we choose $0 < \theta < 1$ such that

$$-\frac{n}{p}(1-\theta) < s' < -2(1-\theta) \Leftrightarrow -\frac{n}{p} < \tilde{s}' < -2,$$

which exists if $1 < p < \min\{\frac{n}{2}, q\}$. If we furthermore use that

$$(J_p(\Omega), J_r(\Omega))_{[\theta]} \quad = \quad J_q(\Omega)$$

(see appendix), we get with the chosen θ, p and the corresponding r that

$$\|e^{-tA_q} f\|_{L_q(\Omega;\omega_n^{s' q})} \quad \leq \quad C\left[(1+t)^{\frac{\tilde{s}'}{2}}\right]^{1-\theta} \|f\|_{L_q(\Omega)} = C(1+t)^{\frac{s'}{2}} \|f\|_{L_q(\Omega)}$$

for $f \in J_q(\Omega)$. Complex interpolation with the same parameters yields the estimate for $f \in \mathcal{D}(A_q)$. For this we use the second estimate of Theorem 5.1 and

$$(\mathcal{D}(A_p), \mathcal{D}(A_r))_{[\theta]} \quad = \quad \mathcal{D}(A_q).$$

The latter equation will be proved in the appendix. ■

Proof of Theorem 1.1: The proof is similar to that of Theorem 1.2 in [5] but a little bit shorter.
It is sufficient to show the statement for $0 < \sigma < \frac{1}{2}$ since we can reduce the general case to this statement. (Choose $q = q_0 < q_1 < \ldots < q_k = r$ such that $\sigma_i := \frac{n}{2}\left(\frac{1}{q_i} - \frac{1}{q_{i+1}}\right) < \frac{1}{2}$ and apply the statement to q_i and q_{i+1}.)

1st step: The inequality holds for $t \geq 2$.
Let $\tilde{u}_0 := e^{-A_q} u_0$. Then $\tilde{u}_0 \in \mathcal{D}(A_q)$ and $\|\tilde{u}_0\|_{W_q^2(\Omega)} \leq C\|u_0\|_{L_q(\Omega)}$. Moreover let $\tilde{u}(t) := e^{-tA_q}\tilde{u}_0$ and $\tilde{p}(t) \in \dot{W}_q^1(\Omega)$ be the pressure corresponding to $\tilde{u}(t)$.
Let $\Omega \cup B_r(0) = \mathbb{R}^n_+ \cup \mathbb{R}^n_- \cup B_r(0)$ and $b > r+1$. We choose a cut-off-function $\psi \in C^\infty(\Omega)$ with $\psi(x) = 1$ for $|x| \geq b$ and $\psi(x) = 0$ for $|x| \leq b-1$. Then $\operatorname{div}(\psi\tilde{u}(t)) = \nabla\psi \cdot \tilde{u}(t) \in W^1_{0,q}(D_b)$ with $D_b := \{x \in \Omega : b-1 < |x| < b\}$ and $\int_{D_b} \nabla\psi \cdot \tilde{u}(t)dx = 0$. Applying Bogovskii's Theorem [4, Theorem 3.2] we know that there exists a $v_0(t) \in W^2_{0,q}(D_b)$ with $\operatorname{div} v_0(t) = \operatorname{div}(\psi\tilde{u}(t))$ and

$$\|v_0(t)\|_{W_q^2(D_b)} \leq C\|\tilde{u}(t)\|_{W_q^1(D_b)}. \tag{23}$$

Therefore we have

$$\|\partial_t v_0(t)\|_{W_q^1(D_b)} \;\leq\; C\|e^{-tA_q}A_q\tilde{u}_0\|_{L_q(D_b)} \leq C(1+t)^{-\tilde{s}}\|\tilde{u}_0\|_{W_q^2(\Omega)} \tag{24}$$

with an arbitrary $0 \leq \tilde{s} < \frac{n}{2q}$.
If we define $v_1(t) := \psi\tilde{u}(t) - v_0(t)$, it solves the differential equation

$$\begin{aligned} \partial_t v_1(t) - \Delta v_1(t) + \nabla(\psi\tilde{p}(t)) &= h(t) && \text{in } (0,\infty)\times\mathbb{R}^n_\pm, && (25)\\ \operatorname{div} v_1(t) &= 0 && \text{in } (0,\infty)\times\mathbb{R}^n_\pm, && (26)\\ v_1(t)|_{\partial\mathbb{R}^n_\pm} &= 0 && \text{in } (0,\infty), && (27)\\ v_1(0) &= v_1 && && (28) \end{aligned}$$

with $v_1 = \psi\tilde{u}_0 - v_0(0)$ and

$$h(t) = -\left\{2(\nabla\psi)\cdot\nabla + (\Delta\psi)\right\}\tilde{u}(t) - (\partial_t - \Delta)v_0(t) + (\nabla\psi)\tilde{p}(t).$$

Moreover $\operatorname{supp} h(t) \subseteq \overline{D}_b$. We choose the pressure $\tilde{p}(t)$ such that $\int_{D_b}\tilde{p}(t)dx = 0$. If we now apply (23), (24), Poincaré's inequality [1, Theorem 4.19] and Corollary 6.2, we get

$$\begin{aligned} \|h(t)\|_{L_q(D_b)} &\leq C\left(\|\tilde{u}(t)\|_{W_q^1(D_b)} + \|v_0(t)\|_{W_q^2(D_b)} + \|\partial_t v_0(t)\|_{L_q(D_b)} + \|\tilde{p}(t)\|_{L_q(D_b)}\right)\\ &\leq C\left((1+t)^{-\frac{\tilde{s}}{2}}\|\tilde{u}_0\|_{W_q^2(\Omega)} + \|\nabla\tilde{p}(t)\|_{L_q(\Omega_b)}\right)\\ &\leq C\left((1+t)^{-\frac{\tilde{s}}{2}}\|\tilde{u}_0\|_{W_q^2(\Omega)} + \|\partial_t\tilde{u}(t)\|_{L_q(D_b)} + \|\tilde{u}(t)\|_{W_q^2(D_b)}\right)\\ &\leq C(1+t)^{-\frac{\tilde{s}}{2}}\|\tilde{u}_0\|_{W_q^2(\Omega)} \end{aligned}$$

with an arbitrary $\tilde{s}$ such that $0 \leq \tilde{s} < \frac{n}{q}$.
Let $E_\pm(t)$ denote the semigroup of the Stokes operator in $\mathbb{R}^n_\pm$ and $P_\pm$ denote the Helmholtz projection in $L_q(\mathbb{R}^n_\pm; \omega_n^{sq})$. Since $v_1(t)$ solves the equations (25)-(28), the identity

$$v_1(t) = E_\pm(t)v_1 + \int_0^t E_\pm(t-\tau)P_\pm h(\tau)d\tau$$

holds. Because of Corollary 3.4 and the $L_q - L_r$-estimate in the half space [11, Theorem 3.1] the semigroup $E_\pm(t)$ satisfies

$$\begin{aligned}\|E_\pm(t)f\|_{L_r(\mathbb{R}^n_\pm)} &\leq Ct^{-\sigma}\|f\|_{L_q(\mathbb{R}^n_\pm)}\\ \|E_\pm(t)f\|_{L_q(\mathbb{R}^n_\pm)} &\leq C(1+t)^{-\frac{s}{2}}\|f\|_{L_q(\mathbb{R}^n_\pm;\omega_n^{sq})}\end{aligned}$$

with $1 < q \leq r < \infty$, $0 \leq s < \frac{n}{q'}$ and $\sigma = \frac{n}{2}\left(\frac{1}{q} - \frac{1}{r}\right)$ for all $t > 0$, $f \in J_q(\mathbb{R}^n_\pm)$ resp. $f \in J_q(\mathbb{R}^n_\pm;\omega_n^{sq})$. Using both inequalities we get

$$\|E_\pm(t)f\|_{L_r(\mathbb{R}^n_\pm)} \leq Ct^{-\sigma}\left\|E_\pm\left(\frac{t}{2}\right)f\right\|_{L_q(\mathbb{R}^n_\pm)} \leq Ct^{-\sigma}(1+t)^{-\frac{s}{2}}\|f\|_{L_q(\mathbb{R}^n_\pm;\omega_n^{sq})}$$

for $f \in J_q(\mathbb{R}^n_\pm;\omega_n^{sq})$, $t > 0$. Therefore we conclude

$$\|E_\pm(t)v_1\|_{L_r(\mathbb{R}^n_\pm)} \leq Ct^{-\sigma}\|v_1\|_{L_q(\mathbb{R}^n_\pm)} \leq Ct^{-\sigma}\|\tilde{u}_0\|_{L_q(\Omega)}$$

and

$$\begin{aligned}\left\|\int_0^t E_\pm(t-\tau)P_\pm h(\tau)d\tau\right\|_{L_r(\mathbb{R}^n_\pm)} &\leq C\int_0^t (t-\tau)^{-\sigma}(1+t-\tau)^{-\frac{s}{2}}\underbrace{\|P_\pm h(\tau)\|_{L_q(\mathbb{R}^n_\pm;\omega_n^{sq})}}_{\leq C\|h(\tau)\|_{L_q(\mathbb{R}^n_\pm;\omega_n^{sq})}}d\tau\\ &\leq C\int_0^t (t-\tau)^{-\sigma}(1+t-\tau)^{-\frac{s}{2}}\|h(\tau)\|_{L_q(D_b)}d\tau\\ &\leq C\int_0^t (t-\tau)^{-\sigma}(1+t-\tau)^{-\frac{s}{2}}(1+\tau)^{-\frac{\tilde{s}}{2}}d\tau\|\tilde{u}_0\|_{W^2_q(\Omega)}.\end{aligned}$$

We now choose $0 \leq s < \frac{n}{q'}$ and $\sigma \leq \frac{\tilde{s}}{2} < \frac{n}{2q}$ such that $\frac{s}{2}+\frac{\tilde{s}}{2} > 1$, $\frac{s}{2}+\sigma \neq 1$ and $\frac{\tilde{s}}{2} \neq 1$. (This is possible since $\frac{n}{2q}+\frac{n}{2q'} = \frac{n}{2} > 1$.) If we apply Lemma A.2 with this choice of s and $\tilde{s}$, we get

$$\left\|\int_0^t E_\pm(t-\tau)P_\pm h(\tau)d\tau\right\|_{L_r(\mathbb{R}^n_\pm)} \leq Ct^{-\sigma}\|\tilde{u}_0\|_{W^2_q(\Omega)}$$

and therefore

$$\|v_1(t)\|_{L_r(\mathbb{R}^n_\pm)} \leq Ct^{-\sigma}\|\tilde{u}_0\|_{W^2_q(\Omega)}.$$

Since $u(t,x) = v_1(t,x)$ for all $x \in \Omega \setminus \Omega_b$, the previous estimates, Corollary 6.2 and Sobolev's embedding theorem imply that

$$\begin{aligned}\|\tilde{u}(t)\|_{L_r(\Omega)} &\leq \|\tilde{u}(t)\|_{L_r(\Omega_b)} + \|v_1(t)\|_{L_r(\Omega\setminus\Omega_b)} \leq C\left(\|\tilde{u}(t)\|_{W^2_q(\Omega_b)} + \|v_1(t)\|_{L_r(\Omega\setminus\Omega_b)}\right)\\ &\leq Ct^{-\sigma}\|\tilde{u}_0\|_{W^2_q(\Omega)} \leq Ct^{-\sigma}\|f\|_{L_q(\Omega)}.\end{aligned}$$

Since $\tilde{u}(t) = e^{-(t+1)A_q}u_0$ we have proved the theorem for $t \geq 2$.

2nd step: The inequality holds for $\mathbf{t < 2}$.
The case $t < 2$ is proved in the same way as in the proof of [5, Theorem 1.2] using Sobolev's embedding theorem and an interpolation method. ∎

Proof of Theorem 1.2: Because of the semigroup property of e^{-tA_q} and Theorem 1.1 it suffices to prove the statement for $\sigma = 0$, i.e. $1 < q = r < n$.
The proof for the case $t < 2$ uses the same interpolation method as in the proof of Theorem 1.2 [5].
So let $t \geq 2$ and $v_1(t)$, $v_0(t)$, $h(t)$ be the functions used in the proof of Theorem 1.1. Then it holds that

$$\nabla v_1(t) = \nabla E_\pm(t) v_1 + \int_0^t \nabla E_\pm(t-\tau) P_\pm h(\tau) d\tau.$$

The estimate for the Stokes semigroup in $\mathbb{R}^n_\pm$ yields

$$\|\nabla E_\pm(t) v_1\|_{L_q(\mathbb{R}^n_\pm)} \leq C t^{-\frac{1}{2}} \|v_1\|_{L_q(\mathbb{R}^n_\mp)}.$$

Now we choose $0 \leq s < \frac{n}{q'}$ and $1 \leq \tilde{s} < \frac{n}{q}$ with $\frac{s}{2} + \frac{\tilde{s}}{2} > 1$, $\frac{\tilde{s}}{2} \neq 1$ and $\frac{1}{2} + \frac{s}{2} \neq 1$. So we get because of Corollary 6.2 and Lemma A.2

$$\begin{aligned}
&\left\| \int_0^t \nabla E_\pm(t-\tau) P_\pm h(\tau) d\tau \right\|_{L_q(\mathbb{R}^n_\pm)} \\
&\quad \leq C \int_0^t (t-\tau)^{-\frac{1}{2}} (1+t-\tau)^{-\frac{s}{2}} \|P_\pm h(\tau)\|_{L_q(\mathbb{R}^n_\pm;\omega^{sq})} d\tau \\
&\quad \leq C \int_0^t (t-\tau)^{-\frac{1}{2}} (1+t-\tau)^{-\frac{s}{2}} \|h(\tau)\|_{L_q(\Omega_b)} d\tau \\
&\quad \leq C \int_0^t (t-\tau)^{-\frac{1}{2}} (1+t-\tau)^{-\frac{s}{2}} (1+\tau)^{-\frac{\tilde{s}}{2}} d\tau \|\tilde{u}_0\|_{W^2_q(\Omega)} \\
&\quad \leq C t^{-\frac{1}{2}} \|\tilde{u}_0\|_{W^2_q(\Omega)}.
\end{aligned}$$

Moreover let $\tilde{s} = 1 < \frac{n}{q}$. Therefore we get for $t \geq 1$

$$\begin{aligned}
\|\nabla e^{-(t+1)A_q} f\|_{L_q(\Omega)} &\leq C \left(\|\nabla \tilde{u}(t)\|_{L_q(\Omega_b)} + \|\nabla v_1(t)\|_{L_q(\mathbb{R}^n_\pm)} \right) \\
&\leq C \left((1+t)^{-\frac{\tilde{s}}{2}} + t^{-\frac{1}{2}} \right) \|\tilde{u}_0\|_{W^2_q(\Omega)} \leq C t^{-\frac{1}{2}} \|f\|_{L_q(\Omega)}.
\end{aligned}$$

Thus the theorem is also true for $t \geq 2$. ∎

A Appendix

Lemma A.1 *Let $1 < p, q, r < \infty$, $\theta \in (0,1)$ with $\frac{1}{q} = \frac{1-\theta}{r} + \frac{\theta}{p}$ and Ω be an aperture domain. Then*

$$\begin{aligned} (\mathcal{D}(A_r), \mathcal{D}(A_p))_{[\theta]} &= \mathcal{D}(A_q), \\ (J_r(\Omega), J_p(\Omega))_{[\theta]} &= J_q(\Omega). \end{aligned}$$

Proof: To prove the first equality we define a continuous projection $P_q : W_q^2(\Omega)^n \to \mathcal{D}(A_q)$ for arbitrary $1 < q < \infty$. For a function $u \in W_q^2(\Omega)^n$ let $(v, p) \in W_q^2(\Omega)^n \times \dot{W}_q^1(\Omega)$ denote the unique solution of the resolvent equations (16)-(19) with right-hand side $f = (z - \Delta)u$ for some fixed $z \in \Sigma_\delta$ (see [8, Theorem 2.1]). We set $P_q u = v$. Then it holds that

$$\|v\|_{W_q^2(\Omega)} \le C\|(z-\Delta)u\|_{L_q(\Omega)} \le C\|u\|_{W_q^2(\Omega)}.$$

If $u \in \mathcal{D}(A_q)$, $(u, 0)$ is the unique solution of these equations. Therefore P_q is a continuous projection on $\mathcal{D}(A_q)$.
If $u \in W_r^2(\Omega)^n \cap W_q^2(\Omega)^n$ the corresponding solutions in $W_r^2(\Omega)^n$ and $W_q^2(\Omega)^n$ coincide (see [2, Lemma 3.2]). Therefore we can extend P_q and P_r to a well-defined projection $P(u_r + u_q) = P_r u_r + P_q u_q$ on $W_r^2(\Omega)^n + W_p^2(\Omega)^n$ with $P|_{W_r^2(\Omega)^n} = P_r$ and $P|_{W_p^2(\Omega)^n} = P_p$. Therefore we conclude

$$\mathcal{D}(A_q) \;=\; P\left(W_r^2(\Omega)^n, W_p^2(\Omega)^n\right)_{[\theta]} = \left(PW_r^2(\Omega)^n, PW_p^2(\Omega)^n\right)_{[\theta]} = (\mathcal{D}(A_r), \mathcal{D}(A_p))_{[\theta]}.$$

The second equality immediately follows from the fact that $P_q = P_r$ on $J_q(\Omega) \cap J_r(\Omega)$ (see [8, Lemma 3.2]). ■

Lemma A.2 *Let $0 \le \alpha < 1$,$\beta \ge 0$, $\alpha \le \gamma$, $\beta + \gamma > 1$, $\alpha + \beta \ne 1$ and $\gamma \ne 1$. Then*

$$\int_0^t (t-s)^{-\alpha}(1+t-s)^{-\beta}(1+s)^{-\gamma} ds. \;\le\; Ct^{-\alpha}.$$

Proof: The case $t \in (0,1)$ is trivial. For $t > 1$ we simply estimate

$$\begin{aligned} \int_0^{\frac{t}{2}} (t-s)^{-\alpha}(1+t-s)^{-\beta}(1+s)^{-\gamma} ds &\le Ct^{-\alpha-\beta} \int_0^{\frac{t}{2}} (1+s)^{-\gamma} ds \\ &\le Ct^{-\alpha-\beta} \begin{cases} t^{1-\gamma}, & \text{if } \gamma < 1, \\ 1, & \text{if } \gamma > 1, \end{cases} \\ &\le Ct^{-\alpha}. \end{aligned}$$

Similarly we get

$$\begin{aligned} \int_{\frac{t}{2}}^t (t-s)^{-\alpha}(1+t-s)^{-\beta}(1+s)^{-\gamma} ds &\le Ct^{-\gamma} \begin{cases} t^{1-\alpha-\beta}, & \text{if } \alpha+\beta < 1, \\ 1, & \text{if } \alpha+\beta > 1, \end{cases} \\ &\le Ct^{-\alpha}. \end{aligned}$$

■

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