Banach-Lie Quotients, Enlargibility, and Universal Complexifications

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Abstract. We characterize those real Banach-Lie groups which admit universal complexifications, and present examples of Banach-Lie groups which have none. To achieve these goals, we prove new results concerning the enlargibility of Banach-Lie algebras, and derive a necessary and sufficient condition for the existence of Lie group structures on quotients of Banach-Lie groups.

Introduction

In this article, we address several interrelated problems in the theory of Banach-Lie groups, namely: (a) the existence of Lie group structures on quotient groups; (b) enlargibility of Banach-Lie algebras; (c) the existence of universal complexifications of Banach-Lie groups.

A classical fact in the theory of Banach-Lie groups asserts that the topological quotient group G/N of a real Banach-Lie group G by a normal Lie subgroup N can be made a real Banach-Lie group if N is a *split* Lie subgroup, *i.e.*, provided $\mathbf{L}(N)$ is complemented in $\mathbf{L}(G)$ as a topological vector space ([Ms62], [Bo89]; see Section 1 below for the terminology). As our first main result, we show that the assumption that N be split is superfluous (Corollary II.4):

1. Quotient Theorem. *If G is a real Banach-Lie group and N a closed normal subgroup of G, then the topological quotient group G/N can be given a real Banach-Lie group structure if and only if N is a Lie subgroup of G.*

Equipped with the Quotient Theorem, we turn to enlargibility questions of Banach-Lie algebras. Since the fundamental work of van Est and Korthagen [EK64], it is known that there are Banach-Lie algebras which are not enlargible, *i.e.*, which are not the Lie algebra of any Banach-Lie group. Van Est and Korthagen also proved the following Enlargibility Criterion: *a Banach-Lie algebra $\mathfrak{g}$ is enlargible if and only if its period group $\Pi(\mathfrak{g}) \subseteq \mathfrak{z}(\mathfrak{g})$ is discrete* [EK64, p. 24]. The Quotient Theorem allows us to approach this important classical fact more directly (Theorem III.7). Furthermore, making use of the functoriality of $\Pi(\bullet)$ (Remark III.5), we prove necessary and sufficient conditions for enlargibility of ℓ^∞-direct sums of Banach-Lie algebras (Theorem III.9), as well as a characterization of the existence of universal enlargible envelopes (Theorem III.19):

2. Existence of Universal Enlargible Envelopes. *A Banach-Lie algebra $\mathfrak{g}$ has a universal enlargible envelope if and only if there is a smallest closed vector subspace $\mathfrak{a}$ of $\mathfrak{z}(\mathfrak{g})$ such that $\mathfrak{a}$ is open in $\mathfrak{a} + \Pi(\mathfrak{g})$.*

See also [DL66], [Sw71], [Pe92], and [Pe93] for discussions related to enlargibility.

The remaining sections of this article are devoted to the study of universal complexifications of Banach-Lie groups. Although it is a classical fact that every finite-dimensional Lie group has a universal complexification ([Bo89], cf. [Ho65], [Ho66]), according to the authors' best knowledge, the existence question of universal complexifications of Banach-Lie groups has never been addressed in the literature until the recent investigations in [Gl00], where an explicit existence criterion for universal complexifications was formulated. We strengthen this existence criterion in Corollary IV.8 below. More importantly, making use of the Quotient Theorem, we derive a complete characterization of the existence of universal complexifications (Theorem IV.6):

3. Complexification Theorem. *Given a real Banach-Lie group G, let N_G be the intersection of all kernels of smooth homomorphisms from G into complex Banach-Lie groups. Then G has a universal complexification if and only if N_G is a Lie subgroup of G and the complexification of $\mathbf{L}(G)/\mathbf{L}(N_G)$ is enlargible.*

We provide an example of a Banach-Lie group for which N_G fails to be a Lie subgroup (Section V), and also examples where $N_G = \{\mathbf{1}\}$ but $\mathbf{L}(G)_{\mathbb{C}}$ is not enlargible (Section VI). Cf. [Le97] for a Fréchet-Lie group whose Lie algebra has a non-enlargible complexification.

For simply connected Banach-Lie groups, we also give an alternative characterization of the existence of universal complexifications in terms of properties of the Lie algebra (Theorem IV.11):

4. Complexifications of Simply Connected Banach-Lie Groups. *A simply connected Banach-Lie group G has a universal complexification if and only if the complexification of its Lie algebra has a universal enlargible envelope in the category of complex Banach-Lie algebras.*

Part of the results and techniques developed here carry over to more general classes of infinite-dimensional Lie groups, including all smooth mapping groups, test function groups,and classical direct limit Lie groups. We refer to [Gl01] for these generalizations.

I. Preliminaries, Notation and Terminology

In this section, we describe our terminology concerning enlargibility, Lie subgroups, and universal complexifications. We also assemble various basic facts.

Recall that a real (resp., complex) Banach-Lie group is a group, equipped with a smooth (resp., complex analytic) Banach manifold structure, such that the group operations are smooth (resp., complex analytic). Since every continuous homomorphism between real Banach-Lie groups is smooth, there is at most one real Banach-Lie group structure on a given topological group, whence a real Banach-Lie group can be identified with its underlying topological group. Furthermore, every real Banach-Lie group can be given a unique real analytic structure. For standard results, notation and terminology concerning Banach-Lie groups, the reader is referred to [Bo89, Chapter 3] and [Ms62].

Definition I.1. A Banach-Lie algebra $\mathfrak{g}$ is called *enlargible* if there exists a Banach-Lie group G with Lie algebra $\mathfrak{g}$. ■

In [EK64] one finds several results on enlargibility of Banach-Lie algebras, containing in particular the construction of examples of non-enlargible Lie algebras.

Lemma I.2. *If $\mathfrak{g}$ is enlargible and $\varphi\colon \mathfrak{h} \to \mathfrak{g}$ is an injective morphism of Banach-Lie algebras, then $\mathfrak{h}$ is enlargible.*

Proof. This follows from [EK64, $(***)$ in §3]. ■

Lemma I.3. *If $\varphi : G \to H$ is a morphism of Banach-Lie groups, then $\mathbf{L}(G)/\ker \mathbf{L}(\varphi)$ is enlargible.*

Proof. The map $\mathbf{L}(\varphi)\colon \mathbf{L}(G) \to \mathbf{L}(H)$ factors through an injection $\mathbf{L}(G)/\ker \mathbf{L}(\varphi) \to \mathbf{L}(H)$, so that Lemma I.2 applies. ■

It is useful to distinguish various types of subgroups of Banach-Lie groups. Since the terminology is not uniform in the literature, we need to explain ours.

Definition I.4. Let G be a Banach-Lie group over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

(a) An *analytic subgroup of G* is a Banach-Lie group H over $\mathbb{K}$ whose underlying abstract group is a subgroup of G, such that the inclusion map $\varepsilon : H \to G$ is smooth and $\mathbf{L}(\varepsilon)$:

$\mathbf{L}(H) \to \mathbf{L}(G)$ is an embedding of topological $\mathbb{K}$-Lie algebras. We identify $\mathbf{L}(H)$ with its image $\mathfrak{h} \subseteq \mathbf{L}(G)$ under $\mathbf{L}(\varepsilon)$. Thus, the exponential function of H is $\exp_G|_{\mathfrak{h}}$.

(b) An analytic subgroup H of G is called a *Lie subgroup* of G if the analytic subgroup topology on H coincides with the topology induced by G, *i.e.*, if the above mapping ε is a topological embedding. If, in addition, $\mathbf{L}(H)$ is complemented in $\mathbf{L}(G)$ as a topological $\mathbb{K}$-vector space, we call H a *split* Lie subgroup of G. ∎

Remark I.5. Note that $\mathbf{L}(H) = \{X \in \mathbf{L}(G) : \exp_G(\mathbb{R}X) \subseteq H\}$ whenever H is a Lie subgroup of G in the preceding situation, and note that any Lie subgroup is closed, being locally closed. Conversely, let H be any closed subgroup of G. Then $\mathfrak{h} := \{X \in \mathbf{L}(G) : \exp_G(\mathbb{R}X) \subseteq H\}$ is a closed real Lie subalgebra of $\mathbf{L}(G)$, and a closed real Lie algebra ideal if H is a closed *normal* subgroup (see [Ms62, Satz 12.4, Satz 12.6]). The closed subgroup H can be given a (necessarily unique) Banach-Lie group structure over $\mathbb{K}$ making it a Lie subgroup of G if and only if there exists a zero-neighbourhood U in $\mathbf{L}(G)$ such that $\exp_G|_U$ is injective and $\exp_G(U) \cap H = \exp_G(U \cap \mathfrak{h})$, and if furthermore $\mathfrak{h}$ is a complex Lie subalgebra of $\mathbf{L}(G)$ if $\mathbb{K} = \mathbb{C}$. In this case, we shall call the closed subgroup H a *Lie subgroup* of G, by abuse of language.

Remark I.6. To prevent confusion, let us point out that "split Lie subgroups" in the our sense are called "Lie subgroups" in [Bo89] and "differentiable subgroups" in [Ms62], whereas "Lie subgroups" in the our sense are called "Lie quasi-subgroups" by Bourbaki. Analytic subgroups in our sense are Maissen's "Lie subgroups."

Definition I.7. Let G be a real Banach-Lie group. A complex Banach-Lie group $G_{\mathbb{C}}$, together with a smooth homomorphism $\eta_G : G \to G_{\mathbb{C}}$, is called a *universal complexification* of G if for every smooth homomorphism $f : G \to H$ from G into a complex Banach-Lie group H, there exists a unique complex analytic homomorphism $\widetilde{f} : G_{\mathbb{C}} \to H$ such that $\widetilde{f} \circ \eta_G = f$. ∎

II. Lie group structures on quotient groups

A classical fact in the theory of Banach-Lie groups asserts that the topological quotient group G/N of a real Banach-Lie group G by a split normal Lie subgroup N can be made a real Banach-Lie group ([Ms62, Satz 13.1]; [Bo89, Chapter 3, §1.6, Proposition 11]). In this section, we show that the hypothesis that N be split is superfluous.

First, we recall a useful lemma from [Ne00a].

Lemma II.1. *If $f : G \to H$ is a smooth homomorphism between real Banach-Lie groups, then $S := f^{-1}(T)$ is a Lie subgroup of G, for every Lie subgroup T of H.*

Proof. Set $\mathfrak{g} := \mathbf{L}(G)$. The naturality of exp entails that $\mathfrak{s} = \mathbf{L}(f)^{-1}(\mathfrak{t})$, where $\mathfrak{t} := \mathbf{L}(T)$ and $\mathfrak{s} := \mathbf{L}(S) := \{X \in \mathfrak{g} : \exp_G(\mathbb{R}X) \subseteq S\}$. If S fails to be a Lie subgroup of G, there exists a sequence $(X_n)_{n\in\mathbb{N}}$ in $\mathfrak{g} \setminus \mathfrak{s}$ such that $\exp_G(X_n) \in S$ for all n, and $X_n \to 0$ in $\mathfrak{g}$ as $n \to \infty$. Let V be a zero-neighbourhood in $\mathbf{L}(H)$ such that $\exp_H$ is injective on V, and $T \cap \exp_H(V) = \exp_H(\mathfrak{t} \cap V)$. Since $U := \mathbf{L}(f)^{-1}(V)$ is a zero-neighbourhood in $\mathfrak{g}$, there exists $n_0 \in \mathbb{N}$ such that $X_n \in U$ for all $n \geq n_0$. Then $\exp_H(\mathbf{L}(f).X_n) = f(\exp_G(X_n)) \in T$ forces $\mathbf{L}(f).X_n \in \mathfrak{t}$ for all $n \geq n_0$. Thus $X_n \in \mathbf{L}(f)^{-1}(\mathfrak{t}) = \mathfrak{s}$, which is a contradiction. Therefore S is a Lie subgroup. ∎

Theorem II.2 (Quotient Theorem). *Let G be a Banach-Lie group over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, with Lie algebra $\mathbf{L}(G) = \mathfrak{g}$, and suppose that N is a closed normal subgroup of G. Define $\mathfrak{n} := \{X \in \mathfrak{g} : \exp_G(\mathbb{R}X) \subseteq N\}$, and let $q : G \to G/N$, $Q : \mathfrak{g} \to \mathfrak{g}/\mathfrak{n}$ be the canonical quotient maps. If $\mathbb{K} = \mathbb{C}$, assume in addition that $\mathfrak{n}$ is a complex Lie subalgebra of $\mathfrak{g}$. Then the following conditions are equivalent:*

(a) *There exists a smooth (resp., complex analytic) homomorphism* $\varphi\colon G \to H$ *into a Banach-Lie group* H *over* $\mathbb{K}$ *such that* $\ker(\varphi) = N$.
(b) G/N *can be made a Banach-Lie group over* $\mathbb{K}$ *with Lie algebra* $\mathfrak{g}/\mathfrak{n}$, *such that* $q \circ \exp_G = \exp_{G/N} \circ Q$.
(c) N *is a Lie subgroup of* G.

Proof. We give the proof in the real case only; the case $\mathbb{K} = \mathbb{C}$ follows the same lines.

The implication (b) $\Rightarrow$ (a) is trivial.

(a) $\Rightarrow$ (c): This is Lemma II.1.

(c) $\Rightarrow$ (b): We choose norms on $\mathfrak{g}$ and $\mathfrak{g}/\mathfrak{n}$ compatible with the topologies which make $\mathfrak{g}$, resp., $\mathfrak{g}/\mathfrak{n}$ normed Lie algebras. Then the Campbell-Hausdorff series converges absolutely on $V \times V$ for a sufficiently small open ball V with center 0 in $\mathfrak{g}/\mathfrak{n}$. There is an open ball $W \subseteq V$ centered at 0 such that $W * W \subseteq V$; thus $X * Y * Z$ is defined for all $X, Y, Z \in W$. Furthermore, there is an open ball U centered at 0 in $\mathfrak{g}$ such that the Campbell-Hausdorff series converges absolutely on U. Shrinking U if necessary, we may assume that $\exp_G|_U$ is a diffeomorphism onto an open subset of G, and that $\exp_G(U) \cap N = \exp_G(U \cap \mathfrak{n})$. There is an open, connected, symmetric zero-neighbourhood $A \subseteq U$ in $\mathfrak{g}$ such that $A * A \subseteq U$ and $Q(A) \subseteq W$. Then $\exp_G(X * Y) = \exp_G(X)\exp_G(Y)$ for all $X, Y \in A$.

Claim 1: *If* $X, Y \in A$ *and* $Q(X) = Q(Y)$, *then* $q(\exp_G(X)) = q(\exp_G(Y))$. In fact, from $Q(X) = Q(Y)$ we deduce that $Q(X * (-Y)) = Q(X) * (-Q(Y)) = 0$, *i.e.*, $X * (-Y) \in \mathfrak{n}$. Thus $1 = q(\exp_G(X * (-Y))) = q(\exp_G(X)\exp_G(Y)^{-1})$, which implies the claim.

Claim 2: *If* $X, Y \in A$ *and* $q(\exp_G(X)) = q(\exp_G(Y))$, *then* $X - Y \in \mathfrak{n}$. In fact, we have $\exp_G(X * (-Y)) = \exp_G(X)\exp_G(Y)^{-1} \in N$ in this case, where $X, -Y \in A$ and thus $X * (-Y) \in U$. From the choice of U, we deduce that $X * (-Y) \in \mathfrak{n}$. Thus $0 = Q(X * (-Y)) = Q(X) * (-Q(Y))$. Since $Q(X), Q(Y) \in W$, multiplication with $Q(Y)$ on the right yields $Q(Y) = Q(X)$, as required.

Let $B := Q(A)$ now. By Claim 1, a mapping $E\colon B \to G/N$ can be defined via $E(Q(X)) := q(\exp_G(X))$ for $X \in A$. The mapping $Q|_A^B\colon A \to B$ being an open surjection, we deduce from the continuity and openness of $q \circ \exp_G|_A$ that E is continuous and open. Furthermore, E is injective by Claim 2. Let $C_1 \subseteq A$ be an open zero-neighbourhood in $\mathfrak{g}$ such that $C_1 * C_1 \subseteq A$, and define $C := Q(C_1)$. Then for every $X, Y \in C$, say $X = Q(X_1)$, $Y = Q(Y_1)$ with $X_1, Y_1 \in C_1$, we have $E(X * Y) = q(\exp_G(X_1 * Y_1)) = q(\exp_G(X_1)\exp_G(Y_1)) = E(X)E(Y)$. We deduce from [Bo89, Chapter 3, §1.9, Proposition 18] that there is a unique Banach-Lie group structure on $\langle E(C)\rangle = \langle E(B)\rangle = (G/N)_0$ which makes $E|_C^{E(C)}$ a diffeomorphism onto the open submanifold $E(C)$. Since $E(C)$ is open in G/N and $E|_C^{E(C)}$ a homeomorphism with respect to the topology on $E(C)$ induced by G/N, clearly the topology underlying the Banach-Lie group $(G/N)_0$ is the topology induced by G/N. The automorphisms $(G/N)_0 \to (G/N)_0$, $g \mapsto xgx^{-1}$ being continuous and hence analytic on the open normal subgroup $(G/N)_0$ of G/N for all $x \in G/N$, we deduce from [Bo89, Chapter 3, §1.9, Proposition 18] that G/N is a Banach-Lie group. We extend E to a function $\exp_{G/N} : \mathfrak{g}/\mathfrak{n} \to G/N$ via $\exp_{G/N}(X) := E(\frac{1}{n}X)^n$, where $X \in \mathfrak{g}/\mathfrak{n}$ and $n \in \mathbb{N}$ is chosen such that $\frac{1}{n}X \in C$. Then $\exp_{G/N}$ is well-defined, is analytic, and is an exponential function for G/N (cf. [Bo89, Chapter 3, §6.4]). By construction of E, we have $\exp_{G/N} \circ Q = q \circ \exp_G$. ∎

Remark II.3. Our construction of a Banach-Lie group structure on G/N closely resembles Maissen's in the case where N is a split Lie subgroup [Ms62, Satz 13.1]. In fact, Maissen already noted that the definition of our mapping E (which he called $\overline{\exp}$) does not require that $\mathfrak{n}$ be complemented in $\mathfrak{g}$. However, he didn't realize that a certain mapping $\overline{\pi}$ he defined is simply the Campbell-Hausdorff multiplication on $\mathfrak{g}/\mathfrak{n}$ (and thus analytic), and believed that nothing could be said about the differentiability of $\overline{\pi}$ in the absence of a vector complement.

Corollary II.4. *Suppose that* G *is a real Banach-Lie group and* N *a closed normal subgroup of* G. *Then the topological quotient group* G/N *can be given a real Banach-Lie group structure compatible with the quotient topology if and only if* N *is a Lie subgroup of* G. ∎

Remark II.5. Let N be a normal Lie subgroup of the real Banach-Lie group G. According to Michael's Theorem ([Mi59]), the quotient map $q\colon \mathbf{L}(G) \to \mathbf{L}(G)/\mathbf{L}(N) = \mathbf{L}(G/N)$ has a continuous section $\sigma\colon \mathbf{L}(G/N) \to \mathbf{L}(G)$. Since the exponential function of G/N is a local homeomorphism, it follows that the quotient map $G \to G/N$ has continuous local sections, hence is a locally trivial principal bundle. ∎

III. Period groups and enlargibility of Banach-Lie algebras

The period group $\Pi(\mathfrak{g})$ of a Banach-Lie algebra $\mathfrak{g}$ is an additive subgroup of its center. Using a result of van Est on the existence of certain central extensions ([Es62]) and the Quotient Theorem we refine the results on the period group given in [EK64] and thus obtain a quite direct proof of the classical result that $\mathfrak{g}$ is enlargible if and only if its period group $\Pi(\mathfrak{g})$ is discrete. With this characterization, we study the enlargibility of ℓ^∞-direct sums of Banach-Lie algebras and derive a characterization which also provides a method to construct non-enlargible Banach–Lie algebras as ℓ^∞-direct sums of enlargible ones (Theorem III.9). Finally we characterize in Theorem III.19 those Banach–Lie algebras which have a universal enlargible envelope, which means that there exists an enlargible quotient $q\colon \mathfrak{g} \to \mathfrak{g}/\mathfrak{a}$ such that all continuous homomorphisms of $\mathfrak{g}$ to enlargible Banach–Lie algebras factor through q.

The period group of a Banach-Lie algebra

In this subsection we give a direct definition of the period group $\Pi(\mathfrak{g})$ of a Banach-Lie algebra $\mathfrak{g}$. This group has been defined in [EK64], but we need some refinements, so that we have to go through part of the process leading to this group. It will be an additive subgroup of the center $\mathfrak{z}(\mathfrak{g})$ of $\mathfrak{g}$.

Definition III.1. Let G be a connected Banach-Lie group. We write

$$P(G) := \{\gamma \in C([0,1], G) :\ \gamma(0) = \mathbf{1}\}$$

for the *path group of G*, where the multiplication on $P(G)$ is pointwise. This group is a Banach-Lie group, and if $\mathfrak{g} = \mathbf{L}(G)$ is the Lie algebra of G, then

$$P(\mathfrak{g}) := \{\gamma \in C([0,1], \mathfrak{g}) :\ \gamma(0) = 0\}$$

is the Lie algebra of $P(G)$. The evaluation map

$$\mathrm{ev}_1 : P(G) \to G, \quad \gamma \mapsto \gamma(1)$$

being a morphism of Lie groups, its kernel $\Omega(G)$ is a Lie subgroup of $P(G)$ (Lemma II.1), called the *loop group of G*. Clearly $G \cong P(G)/\Omega(G)$.

It is easy to see that $P(G)$ is contractible, hence simply connected, so that the universal covering group $\widetilde{G}$ can be identified with $P(G)/\Omega(G)_0$, in accordance with $\pi_0(\Omega(G)) \cong \pi_1(G)$.

On the Lie algebra level we have the Banach-Lie algebra $P(\mathfrak{g})$ and its Lie subalgebra $\Omega(\mathfrak{g})$. Although $\{\alpha \in P(\mathfrak{g}) : (\forall t)\ \alpha(t) = t\,\alpha(1)\}$ is a natural vector space complement to $\Omega(\mathfrak{g})$, this subspace is not a Lie subalgebra unless $[\mathfrak{g}, \mathfrak{g}] = \{0\}$. ∎

Definition III.2. Let $\mathfrak{g}$ be a Banach-Lie algebra, $\mathfrak{z}$ its center and $\mathfrak{g}_{\mathrm{ad}} := \mathfrak{g}/\mathfrak{z}$, endowed with its natural Banach space topology. Then

$$\mathfrak{z} \hookrightarrow \mathfrak{g} \twoheadrightarrow \mathfrak{g}_{\mathrm{ad}} \tag{3.1}$$

is a central extension, but it is not clear whether it has a continuous linear section, so that we cannot in general describe it by a continuous Lie algebra cocycle. Lemma I.2 implies that $\mathfrak{g}_{\rm ad}$ is enlargible to a simply connected Banach-Lie group $G_{\rm ad}$.

The central extension (3.1) can be pulled back via the evaluation map $\mathrm{ev}_1\colon P(\mathfrak{g}_{\rm ad}) \to \mathfrak{g}_{\rm ad}$ to a central extension

$$\mathfrak{z} \hookrightarrow \widehat{P}(\mathfrak{g}) \twoheadrightarrow P(\mathfrak{g}_{\rm ad}) \quad \text{with} \quad \widehat{P}(\mathfrak{g}) := \{(\alpha, x) \in P(\mathfrak{g}_{\rm ad}) \times \mathfrak{g} : \alpha(1) = x + \mathfrak{z}\}.$$

The restriction of this extension to $\Omega(\mathfrak{g}_{\rm ad})$ splits by the continuous section

$$\sigma\colon \Omega(\mathfrak{g}_{\rm ad}) \to \widehat{P}(\mathfrak{g}), \quad \alpha \mapsto (\alpha, 0),$$

so that the inverse image $\widehat{\Omega}(\mathfrak{g})$ of $\Omega(\mathfrak{g}_{\rm ad})$ in $\widehat{P}(\mathfrak{g})$ is isomorphic to the direct product $\Omega(\mathfrak{g}_{\rm ad}) \times \mathfrak{z}$.

Since the group $P(G_{\rm ad})$ is contractible, we derive from [Es62, Theorem 7.1] that there exists a central group extension

$$\mathfrak{z} \hookrightarrow \widehat{P}(G) \xrightarrow{\;q\;} P(G_{\rm ad}),$$

where the group $\widehat{P}(G)$ is simply connected (we can always pass to the simply connected covering group). Here we need that the singular cohomology $H^2_{\rm sing}(P(G_{\rm ad}), \mathfrak{z})$ vanishes, which follows from the contractibility of $P(G_{\rm ad})$. [1]

Consider the homomorphism $\gamma\colon \widehat{P}(G) \to G_{\rm ad}$, $\gamma(g) = q(g)(1)$. On the Lie algebra level we have $\mathbf{L}(\gamma)(\alpha, x) = \alpha(1) = x + \mathfrak{z}$ with $\ker \mathbf{L}(\gamma) = \widehat{\Omega}(\mathfrak{g}) \cong \Omega(\mathfrak{g}_{\rm ad}) \times \mathfrak{z}$. Moreover, $\widehat{\Omega}(G) := \ker \gamma$ is a Lie subgroup of $\widehat{P}(G)$, with $\widehat{P}(G)/\widehat{\Omega}(G) \cong G_{\rm ad}$ (Theorem II.2).

Since the group $G_{\rm ad}$ is simply connected, the group $\widehat{\Omega}(G)$ is connected, and its universal covering group is isomorphic to $\widetilde{\Omega}(G_{\rm ad}) \times \mathfrak{z}$, because its Lie algebra is $\Omega(\mathfrak{g}_{\rm ad}) \times \mathfrak{z}$. In view of [Ne00b, Prop. II.8], the group $\widehat{\Omega}(G)$ is isomorphic to a quotient

$$\big(\widetilde{\Omega}(G_{\rm ad}) \times \mathfrak{z}\big) / \Gamma(-\mathrm{per}_{\mathfrak{g}}),$$

where $\mathrm{per}_{\mathfrak{g}} : \pi_1(\Omega(G_{\rm ad})) \cong \pi_2(G_{\rm ad}) \to \mathfrak{z}$ is a homomorphism and $\Gamma(-\mathrm{per}_{\mathfrak{g}})$ is the graph of $-\mathrm{per}_{\mathfrak{g}}$. We call $\mathrm{per}_{\mathfrak{g}}$ the *period homomorphism of* $\mathfrak{g}$ and its image $\Pi(\mathfrak{g}) := \mathrm{im}(\mathrm{per}_{\mathfrak{g}}) \subseteq \mathfrak{z}$ the *period group.* ∎

Lemma III.3. *Let $\varphi\colon \mathfrak{g} \to \mathfrak{h}$ be a homomorphism of Banach-Lie algebras with $\varphi(\mathfrak{z}(\mathfrak{g})) \subseteq \mathfrak{z}(\mathfrak{h})$ and $\varphi_{\rm ad}^G\colon G_{\rm ad} \to H_{\rm ad}$ the group homomorphism induced by φ. Then $\varphi(\Pi(\mathfrak{g})) \subseteq \Pi(\mathfrak{h})$ and, moreover, the following diagram is commutative:*

$$\begin{array}{ccc} \pi_2(G_{\rm ad}) & \xrightarrow{\pi_2(\varphi_{\rm ad}^G)} & \pi_2(H_{\rm ad}) \\ \big\downarrow {\scriptstyle \mathrm{per}_{\mathfrak{g}}} & & \big\downarrow {\scriptstyle \mathrm{per}_{\mathfrak{h}}} \\ \mathfrak{z}(\mathfrak{g}) & \xrightarrow{\quad\varphi\quad} & \mathfrak{z}(\mathfrak{h}). \end{array}$$

Proof. Since φ maps $\mathfrak{z}(\mathfrak{g})$ to $\mathfrak{z}(\mathfrak{h})$, it induces a homomorphism $\varphi_{\rm ad} : \mathfrak{g}_{\rm ad} \to \mathfrak{h}_{\rm ad}$ and hence a homomorphism $\widehat{P}(\varphi) : \widehat{P}(\mathfrak{g}) \to \widehat{P}(\mathfrak{h})$ with $\widehat{P}(\varphi)\big(\widehat{\Omega}(\mathfrak{g})\big) = \widehat{\Omega}(\mathfrak{h})$. Integration to the simply connected group $\widehat{P}(G)$ further leads to a group homomorphism

$$\widehat{P}(\varphi)^G\colon \widehat{P}(G) \to \widehat{P}(H) \quad \text{with} \quad \mathbf{L}\left(\widehat{P}(\varphi)^G\right) = \widehat{P}(\varphi).$$

[1] Another possibility to obtain the group $\widehat{P}(G)$ is to use the results in [Sw71]. There it is shown that $\mathfrak{g} \mapsto P(\mathfrak{g})$ is an exact functor, so that $P(\mathfrak{g}_{\rm ad}) \cong P(\mathfrak{g})/P(\mathfrak{z})$, and we obtain a central extension as $\mathfrak{z} \cong P(\mathfrak{z})/\Omega(\mathfrak{z}) \hookrightarrow \widehat{P}(\mathfrak{g}) \cong P(\mathfrak{g})/\Omega(\mathfrak{z}) \twoheadrightarrow P(\mathfrak{g}_{\rm ad})$. Using the existence of a simply connected group H with Lie algebra $P(\mathfrak{g})$ ([Sw71]), we obtain a description $P(G_{\rm ad}) \cong H/N$, where $N \subseteq H$ is a normal subgroup with Lie algebra $P(\mathfrak{z})$. Since H is a locally trivial N-bundle (Remark II.5), the contractibility of the group $P(G_{\rm ad})$ and the exact homotopy sequence of the locally trivial principal bundle $N \hookrightarrow H \twoheadrightarrow H/N$ implies that $N \hookrightarrow H$ is a weak homotopy equivalence, and in particular that N is simply connected, hence isomorphic to $P(\mathfrak{z})$. From that it follows that $\Omega(\mathfrak{z}) \subseteq P(\mathfrak{z})$ is a normal Lie subgroup of H, so that $\widehat{P}(G) := H/\Omega(\mathfrak{z})$ is a Banach–Lie group.

It is clear that this homomorphism maps the subgroup $\widehat{\Omega}(G)$ to $\widehat{\Omega}(H)$, hence induces a homomorphism

$$\pi_1(\widehat{\Omega}(G)) \to \pi_1(\widehat{\Omega}(H)).$$

This means that the induced map

$$\widetilde{\Omega}(G_{\rm ad}) \times \mathfrak{z}(\mathfrak{g}) \to \widetilde{\Omega}(H_{\rm ad}) \times \mathfrak{z}(\mathfrak{h})$$

of the simply connected covering groups maps the graph of $\mathrm{per}_{\mathfrak{g}}$ into the graph of $\mathrm{per}_{\mathfrak{h}}$. We conclude that

$$\varphi|_{\mathfrak{z}(\mathfrak{g})} \circ \mathrm{per}_{\mathfrak{g}} = \mathrm{per}_{\mathfrak{h}} \circ \pi_1\big(\Omega(\varphi_{\rm ad}^G)\big), \tag{3.2}$$

where $\varphi_{\rm ad}^G : G_{\rm ad} \to H_{\rm ad}$ is the homomorphism induced by φ with $\mathbf{L}(\varphi_{\rm ad}^G) = \varphi_{\rm ad}$, and $\Omega(\varphi_{\rm ad}^G)$ is the corresponding map $\Omega(G_{\rm ad}) \to \Omega(H_{\rm ad})$. From the isomorphism of functors $\pi_1 \circ \Omega \cong \pi_2$ from topological groups to abelian groups, it follows that $\pi_1(\Omega(\varphi_{\rm ad}^G))$ corresponds to the map $\pi_2(\varphi_{\rm ad}^G)$ if we identify $\pi_2(G_{\rm ad})$ with $\pi_1(\Omega(G_{\rm ad}))$. Therefore (3.2) implies the commutativity of the diagram and hence in particular that $\varphi(\Pi(\mathfrak{g})) \subseteq \Pi(\mathfrak{h})$. ■

Corollary III.4. (a) *If $\varphi\colon \mathfrak{g} \to \mathfrak{h}$ is a homomorphism of Lie algebras with $\varphi(\mathfrak{z}(\mathfrak{g})) \subseteq \mathfrak{z}(\mathfrak{h})$ for which the induced map $\pi_2(G_{\rm ad}) \to \pi_2(H_{\rm ad})$ is surjective, then $\varphi(\Pi(\mathfrak{g})) = \Pi(\mathfrak{h})$.*
(b) *If $\varphi\colon \mathfrak{g} \to \mathfrak{h}$ is a quotient homomorphism of Lie algebras with $\varphi(\mathfrak{z}(\mathfrak{g})) = \mathfrak{z}(\mathfrak{h})$ and $\ker\varphi \subseteq \mathfrak{z}(\mathfrak{g})$, then $\varphi(\Pi(\mathfrak{g})) = \Pi(\mathfrak{h})$.*

Proof. (a) This is an immediate consequence of Lemma III.3.
(b) Our assumption implies that

$$\mathfrak{g}_{\rm ad} = \mathfrak{g}/\mathfrak{z}(\mathfrak{g}) \cong \mathfrak{h}/\varphi(\mathfrak{z}(\mathfrak{g})) \cong \mathfrak{h}_{\rm ad}.$$

Therefore the induced map $\varphi_{\rm ad}^G\colon G_{\rm ad} \to H_{\rm ad}$ is an isomorphism, and the assertion follows from (a). ■

Remark III.5. Let **LZ** denote the category whose objects are Banach-Lie algebras and whose morphisms are continuous Lie algebra homomorphisms mapping center to center. Then Lemma III.3 means that $\Pi\colon \mathfrak{g} \mapsto \Pi(\mathfrak{g})$ can be viewed as a functor from **LZ** to the category of abelian topological groups. ■

Lemma III.6. *Let $A \subseteq \widehat{P}(G)$ be the connected analytic subgroup corresponding to the closed Lie subalgebra $\Omega(\mathfrak{g}_{\rm ad}) \subseteq \widehat{P}(\mathfrak{g})$. Then $A \cap \mathfrak{z} = \Pi(\mathfrak{g})$, and A is a Lie subgroup if and only if $\Pi(\mathfrak{g})$ is a discrete subgroup of $\mathfrak{z}$.*

Proof. The description of $\widehat{\Omega}(G)$ as the quotient $(\widetilde{\Omega}(G_{\rm ad}) \times \mathfrak{z})/\,\Gamma(-\mathrm{per}_{\mathfrak{g}})$ (cf. Definition III.1) shows that

$$A \cap \mathfrak{z} \cong \mathrm{im}(\mathrm{per}_{\mathfrak{g}}) = \Pi(\mathfrak{g})$$

because A is the image of $\widetilde{\Omega}(G_{\rm ad})$ in $\widehat{\Omega}(G)$.

That the normal subgroup $A \subseteq \widehat{P}(G)$ is a Lie subgroup is equivalent to A being a Lie subgroup of $\widehat{\Omega}(G)$. The Lie algebra $\widehat{\Omega}(\mathfrak{g})$ is a direct product $\Omega(\mathfrak{g}_{\rm ad}) \times \mathfrak{z}$. Therefore A is a Lie subgroup if and only if there exists a 0-neighborhood U in $\mathfrak{z}$ with $A \cap U = \{0\}$, which is equivalent to $\Pi(\mathfrak{g})$ being discrete. ■

The following theorem is also contained in [EK64]. As our proof shows, it can be obtained as a rather direct consequence of the existence of the group $\widehat{P}(G)$.

Theorem III.7 (Characterization Theorem for enlargible Lie algebras). *The Banach-Lie algebra $\mathfrak{g}$ is enlargible if and only if $\Pi(\mathfrak{g})$ is discrete.*

Proof. We have seen in the construction of $\Pi(\mathfrak{g})$ that there exists a group extension

$$\widehat{\Omega}(G) \hookrightarrow \widehat{P}(G) \twoheadrightarrow G_{\rm ad},$$

where $\widehat{P}(G)$ is a simply connected group with Lie algebra $\widehat{P}(\mathfrak{g})$.

If $\mathfrak{g}$ is enlargible and G is a corresponding simply connected group, then the simple connectedness of $\widehat{P}(G)$ permits us to integrate the natural homomorphism $\widehat{P}(\mathfrak{g}) \to \mathfrak{g}$ to a Lie group homomorphism $p\colon \widehat{P}(G) \to G$ with

$$\ker \mathbf{L}(p) = \Omega(\mathfrak{g}_{\rm ad}).$$

In view of Theorem II.2, we then have $G \cong \widehat{P}(G)/\ker p$, where $\ker p$ is connected because G is simply connected. Thus $\ker p$ coincides with the connected analytic subgroup A corresponding to the Lie subalgebra $\Omega(\mathfrak{g}_{\rm ad})$ of $\widehat{P}(\mathfrak{g})$. In view of Lemma III.6, this implies that $\Pi(\mathfrak{g})$ is discrete.

If, conversely, $\Pi(\mathfrak{g})$ is discrete, then A is a Lie subgroup, and Theorem II.2 implies that $\widehat{P}(G)/A$ is a Lie group with Lie algebra $\widehat{P}(\mathfrak{g})/\,\Omega(\mathfrak{g}_{\rm ad}) \cong \mathfrak{g}$. ■

Proposition III.8. *If G is a simply connected Lie group with Lie algebra $\mathfrak{g}$, then*

$$Z(G)_0 \cong \mathfrak{z}/\Pi(\mathfrak{g}) \quad \text{and} \quad \pi_1(Z(G)) \cong \Pi(\mathfrak{g}) = \ker(\exp_G|_{\mathfrak{z}}).$$

Proof. As in the proof of Theorem III.7, we write G as $\widehat{P}(G)/\ker p$. Since G is simply connected, the group $\ker p$ is connected. Moreover, $\mathfrak{z}(\mathfrak{g}) \cong \widehat{\Omega}(\mathfrak{g})/\Omega(\mathfrak{g}_{\rm ad})$ implies that

$$Z(G)_0 \cong \widehat{\Omega}(G)/\ker p,$$

so that $\widehat{\Omega}(G) \cong (\widetilde{\Omega}(G_{\rm ad}) \times \mathfrak{z})/\,\Gamma(-\operatorname{per}_{\mathfrak{g}})$ implies that $Z(G)_0 \cong \mathfrak{z}/\operatorname{im}(\operatorname{per}_{\mathfrak{g}}) = \mathfrak{z}/\Pi(\mathfrak{g})$. ■

Enlargibility of products

In the present subsection, we study enlargibility of ℓ^∞-direct sums of Banach-Lie algebras. It is important for these considerations to endow each Banach-Lie algebra with a fixed norm (rather than considering it as a completely normable topological Lie algebra).

Theorem III.9. *Let $(\mathfrak{g}_j)_{j\in J}$ be a family of Banach-Lie algebras whose norms satisfy $\|[x,y]\| \leq \|x\|\,\|y\|$ for $x, y \in \mathfrak{g}_j$, $\delta_j := \inf\{\|\gamma\| : 0 \neq \gamma \in \Pi(\mathfrak{g}_j)\} \in [0,\infty]$,*

$$\mathfrak{g} := \Big\{(x_j)_{j\in J} \in \prod_{j\in J} \mathfrak{g}_j : \sup_{j\in J} \|x_j\| < \infty\Big\}$$

their ℓ^∞-direct sum, and $\mathfrak{g}_0 \subseteq \mathfrak{g}$ their c_0-direct sum, i.e., the closure of $\sum_j \mathfrak{g}_j$. Then

$$\bigoplus_{j\in J} \Pi(\mathfrak{g}_j) \subseteq \Pi(\mathfrak{g}_0) \subseteq \Pi(\mathfrak{g}) \subseteq \prod_{j\in J} \Pi(\mathfrak{g}_j), \tag{3.3}$$

and the following assertions are equivalent

(1) $\mathfrak{g}$ *is enlargible.*

(2) $\mathfrak{g}_0$ *is enlargible.*

(3) $\inf_{j\in J} \delta_j > 0$.

Proof. We consider the inclusion maps $\alpha_j : \mathfrak{g}_j \to \mathfrak{g}_0$ and the projection maps $\beta_j : \mathfrak{g} \to \mathfrak{g}_j$. Both map centers into centers, so that Lemma III.3 implies that

$$\alpha_j(\Pi(\mathfrak{g}_j)) \subseteq \Pi(\mathfrak{g}_0) \subseteq \Pi(\mathfrak{g}) \quad \text{and} \quad \beta_j(\Pi(\mathfrak{g})) \subseteq \Pi(\mathfrak{g}_j).$$

This entails (3.3).

Let $\delta := \inf\{\|\gamma\| : 0 \neq \gamma \in \Pi(\mathfrak{g})\}$ and $\delta_0 := \inf\{\|\gamma\| : 0 \neq \gamma \in \Pi(\mathfrak{g}_0)\}$. In view of Theorem III.7, $\mathfrak{g}$, resp., $\mathfrak{g}_0$ is enlargible if and only if $\delta > 0$, resp., $\delta_0 > 0$. By (3.3), we have $\delta \leq \delta_0 \leq \delta_j$ for each j because $\Pi(\mathfrak{g}_0)$ contains each $\Pi(\mathfrak{g}_j)$. Thus $\delta \leq \delta_0 \leq \inf_{j\in J} \delta_j$.

If $0 \neq \gamma \in \Pi(\mathfrak{g})$, then there exists some $j \in J$ with $\gamma_j := \beta_j(\gamma) \neq 0$. Then $\|\gamma\| \geq \|\gamma_j\| \geq \delta_j$. This implies the converse inequality $\delta \geq \inf_{j\in J} \delta_j$. Thus $\delta = \delta_0 = \inf_{j\in J} \delta_j$. ■

Corollary III.10. *If $\mathfrak{h}$ is an enlargible Banach–Lie algebra and J is a set, then $\mathfrak{g} := \ell^\infty(J,\mathfrak{h})$ is enlargible.*

Proof. On $\mathfrak{h}$ we choose a norm compatible with the topology such that $\|[x,y]\| \leq \|x\| \cdot \|y\|$ holds for $x, y \in \mathfrak{h}$. Then we apply Theorem III.9 with $\mathfrak{g}_j := \mathfrak{h}$ for each $j \in J$. Now all δ_j are equal and positive because $\mathfrak{h}$ is enlargible, and therefore $\mathfrak{g}$ is enlargible. ■

The following lemma illuminates the meaning of δ.

Lemma III.11. *Let G be a simply connected Lie group with Lie algebra $\mathfrak{g}$. Suppose that $\|[x,y]\| \leq \|x\| \cdot \|y\|$ holds for $x, y \in \mathfrak{g}$ and put $\delta := \inf\{\|\gamma\| : 0 \neq \gamma \in \Pi(\mathfrak{g})\}$. Then for $R = \min(\pi, \frac{\delta}{2})$ the exponential function $\exp|_{B_R(0)} : B_R(0) \to G$ is injective.*

Proof. Let $x, y \in B_R(0)$, i.e., $\|x\|, \|y\| < R$, and assume that $\exp x = \exp y$. Then $\|\operatorname{ad} x\| \leq \|x\| < \pi$ implies that $\operatorname{Spec}(\operatorname{ad} x) \cap 2\pi i\mathbb{Z} \subseteq \{0\}$, so that the exponential function is regular in x. Therefore [Ne01c, Lemma V.3] implies that $[x,y] = 0$ and $\exp(x-y) = \mathbf{1}$. For $z := x - y$ we then have $\mathbf{1} = \operatorname{Ad}(\exp z) = e^{\operatorname{ad} z}$, so that $\operatorname{ad} z$ is diagonalizable with $\operatorname{Spec}(\operatorname{ad} z) \subseteq 2\pi i\mathbb{Z}$ ([Ne01c, Lemma III.13]). On the other hand $\|\operatorname{ad} z\| \leq \|z\| < 2\pi$, so that $\operatorname{ad} z = 0$, and we get $z \in \mathfrak{z}(\mathfrak{g})$. Now Proposition III.8 yields $z \in \ker \exp|_{\mathfrak{z}(\mathfrak{g})} = \Pi(\mathfrak{g})$, so that $\|z\| < \delta$ eventually leads to $z = 0$. ■

The preceding lemma is sharp in the sense that for each $z \in \Pi(\mathfrak{g})$ we have $\exp\left(\frac{z}{2}\right) = \exp\left(-\frac{z}{2}\right)$ and $\frac{\|z\|}{2}$ may be arbitrarily close to $\frac{\delta}{2}$.

Proposition III.12. *If, under the assumptions of* Theorem III.9, *$\mathfrak{g}$ is enlargible, $\widetilde{G}$ is a simply connected Lie group with Lie algebra $\mathfrak{g}$, and G_j, $j \in J$, are groups with Lie algebra $\mathfrak{g}_j$, then the following assertions hold:*

(i) *There exists a continuous homomorphism $\varphi : \widetilde{G} \to \prod_{j\in J} G_j$ with $\varphi(\exp_{\widetilde{G}} x) = (\exp_{G_j} x_j)_{j\in J}$ for $x \in \mathfrak{g}$.*

(ii) *If $\ker\varphi$ is discrete, then $G := \widetilde{G}/\ker\varphi$ is a Banach–Lie group and φ factors through an injective homomorphism $G \hookrightarrow \prod_{j\in J} G_j$.*

(iii) *$\ker\varphi$ is discrete if and only if*

$$\inf_j r_j > 0 \quad \text{holds for} \quad r_j := \inf\{\|z\| : 0 \neq z \in \mathfrak{z}(\mathfrak{g}_j), \exp_{G_j} z = \mathbf{1}\}.$$

(iv) *The following conditions are sufficient for $\ker\varphi$ to be discrete:*
   (1) *The groups G_j, $j \in J$, are simply connected.*
   (2) *We have $\mathfrak{g}_j = \mathfrak{h}$ for each $j \in J$ and $G_j = H$.*

Proof. (i) First the enlargibility of $\mathfrak{g}$ implies the existence of a simply connected Lie group $\widetilde{G}$ with Lie algebra $\mathfrak{g}$. Let $p_k : \prod_{j\in J} G_j \to G_k$ denote the projection homomorphisms. In view of the simple connectedness of $\widetilde{G}$, there exists for each $k \in J$ a Banach–Lie group homomorphism $\varphi_k : \widetilde{G} \to G_k$ for which $\mathbf{L}(\varphi_k) : \mathfrak{g} \to \mathfrak{g}_k$ is the projection map. Then $\varphi := (\varphi_j)_{j\in J} : \widetilde{G} \to \prod_{j\in J} G_j$ is a continuous group homomorphism with $p_k \circ \varphi = \varphi_k$ for $k \in J$. Let $N := \ker\varphi \subseteq \widetilde{G}$.

(ii) Since the Lie algebra homomorphisms $\mathbf{L}(\varphi_k) : \mathfrak{g} \to \mathfrak{g}_k$ separate the points of $\mathfrak{g}$, we have $\mathbf{L}(N) = \{0\}$, and each $n \in N$ acts via the adjoint representation trivially on each $\mathfrak{g}_j$, hence on $\mathfrak{g}$. This implies that $N \subseteq Z(\widetilde{G})$. Moreover, N is a Lie subgroup of $\widetilde{G}$ if and only if it is discrete. If this is the case, then we put $G := \widetilde{G}/N$ and obtain the required injection $G \hookrightarrow \prod_{j\in J} G_j$.

(iii) If N is not discrete, then there exists a sequence $g_n \in N$ with $\mathbf{1} \neq g_n \to \mathbf{1}$. Let $U \subseteq \mathfrak{g}$ be a 0-neighborhood on which $\exp_{\widetilde{G}}$ is a diffeomorphism onto $\exp_{\widetilde{G}}(U)$. We may w.l.o.g. assume that $g_n = \exp_{\widetilde{G}} x_n$ with $x_n \in U$. Then $x_n \to 0$, and since $Z(\widetilde{G}) = \ker \operatorname{Ad}$ is a Lie subgroup of $\widetilde{G}$, we may further assume that $x_n \in \mathfrak{z}(\mathfrak{g})$. Pick $j \in J$ with $\mathbf{L}(p_j).x_n \neq 0$. Then

$$\exp_{G_j} \mathbf{L}(p_j).x_n = p_j(\exp_{\widetilde{G}} x_n) = p_j(g_n) = \mathbf{1}$$

implies that

$$r_j \leq \|\mathbf{L}(p_j).x_n\| \leq \|x_n\|.$$

Therefore $\inf_j r_j = 0$.

Suppose, conversely, that $\inf_j r_j = 0$. Let $\varepsilon_j\colon \widetilde{G}_j \to \widetilde{G}$ denote the homomorphism for which $\mathbf{L}(\varepsilon_j)\colon \mathfrak{g}_j \to \mathfrak{g}$ is the inclusion map.

For $z_j \in \mathfrak{g}_j$ with $\exp_{G_j} z_j = \mathbf{1}$ we then have $\varphi(\exp_{\widetilde{G}} z_j) = \mathbf{1}$, which means that $\exp_{\widetilde{G}} z_j \in N$. Since $\inf_j r_j = 0$, there exist sequences $j_n \in J$ and $0 \neq z_{j_n} \in \mathfrak{z}(\mathfrak{g}_j)$ with $\exp_{G_j} z_{j_n} = \mathbf{1}$ and $z_{j_n} \to 0$ in $\mathfrak{g}$. Then $\exp_{\widetilde{G}} z_{j_n} \in N$ converges to $\mathbf{1}$, and for sufficiently large n we have $\exp_{\widetilde{G}} z_{j_n} \neq \mathbf{1}$, so that N is not discrete.

(iv) (1) If all the groups G_j are simply connected and $R := \min(\pi, \frac{\delta}{2})$ as in Lemma III.11, then $\inf_j r_j \geq R > 0$, and (iii) implies that N is discrete.

(2) If $H = G_j$ for each $j \in J$, then we choose R such that $\exp_H$ is injective on $\{y \in \mathfrak{h}\colon \|y\| < R\}$. Then $\inf_j r_j \geq R > 0$, and again (iii) shows that N is discrete. ■

Remark III.13. (a) An interesting consequence of Theorem III.9 is that if $J = \mathbb{N}$, $\delta_n > 0$ for each $n \in \mathbb{N}$, and $\delta_n \to 0$, then $\mathfrak{g}$ is a non-enlargible Lie algebra whose homomorphisms to enlargible Lie algebras separate points.

(b) If $\varphi\colon \widetilde{G} \to \prod_{j\in J} G_j$ is the homomorphism from Proposition III.12 and all the groups G_j are simply connected, then one might expect that φ is injective, i.e., the "analytic subgroup" of $\prod_{j\in J} G_j$ corresponding to the subspace $\mathfrak{g} \subseteq \prod_{j\in J} \mathfrak{g}_j$ is simply connected. We think that this is probably true, but we do not have any proof.

This would imply in particular that the exponential function of G is just the componentwise exponential function, so that $\Pi(\mathfrak{g}_j) = \ker(\exp_{G_j}|_{\mathfrak{z}(\mathfrak{g}_j)})$ for each j leads to

$$\Pi(\mathfrak{g}) = \ker(\exp_G|_{\mathfrak{z}(\mathfrak{g})}) = \Big\{(x_j)_{j\in J} \in \prod_{j\in J} \Pi(\mathfrak{g}_j) : \sup_{j\in J} \|x_j\| < \infty\Big\}.$$

(c) Let $\mathbf{BLa_c}$ denote the category whose objects are Banach–Lie algebras $(\mathfrak{g}, \|\cdot\|)$, where $\|[x,y]\| \leq \|x\|\,\|y\|$ for $x, y \in \mathfrak{g}$ and whose morphisms are contractive Lie algebra homomorphisms. Then it is easy to see that the ℓ^∞-direct sum $\mathfrak{g} := \oplus_{j\in J}^\infty \mathfrak{g}_j$ is a categorical direct product of $(\mathfrak{g}_j)_{j\in J}$ in $\mathbf{BLa_c}$.

On the group level we consider the category $\mathbf{CBLg_c}$ whose objects are pairs $(G, \|\cdot\|)$, where G is a connected Banach–Lie group and $(\mathbf{L}(G), \|\cdot\|)$ is an object of $\mathbf{BLa_c}$. The morphisms in $\mathbf{CBLg_c}$ are those Lie group morphisms φ for which $\mathbf{L}(\varphi)$ is a morphism in $\mathbf{BLa_c}$.

Let $(G_j)_{j\in J}$ be a family of objects of $\mathbf{CBLg_c}$. We claim that their direct product exists in $\mathbf{CBLg_c}$ if and only if there exists an $r > 0$ such that for each $j \in J$ the exponential function $\exp_{G_j}$ is injective on the open ball $B_r(0)$ of radius r in $\mathbf{L}(G_j)$.

Suppose first that the injectivity condition is satisfied. For $0 \neq z \in \Pi(\mathfrak{g}_j)$ we have $\exp z = \mathbf{1}$ by Proposition III.8, so that the injectivity of $\exp_{G_j}$ on $B_r(0)$ implies that $\delta_j \geq r_j \geq r$ and hence that $\mathfrak{g} := \oplus_{j\in J}^\infty \mathbf{L}(G_j)$ is enlargible (Theorem III.9) and the kernel of the homomorphism $\varphi\colon \widetilde{G} \to \prod_{j\in J} G_j$ is discrete by Proposition III.12(iii). Now $G := \widetilde{G}/\ker\varphi$ is a direct product of $(G_j)_{j\in J}$ in $\mathbf{CBLg_c}$. In fact, let $\varphi_j\colon H \to G_j$ be a collection of morphisms in $\mathbf{CBLg_c}$. Then the $\mathbf{L}(\varphi_j)$ yield a continuous Lie algebra homomorphism $\alpha\colon \mathbf{L}(H) \to \mathfrak{g}$ which integrates to a unique continuous group homomorphism $\widetilde{\alpha}_H\colon \widetilde{H} \to G$. Let $p_j\colon G \to G_j$ denote the projection maps. Then all homomorphisms $p_j \circ \widetilde{\alpha}_H$ factor through H, and therefore $\pi_1(H) \subseteq \ker\widetilde{\alpha}_H$ implies that $\widetilde{\alpha}_H$ factors through a homomorphism $\alpha_H\colon H \to G$ with $p_j \circ \alpha_H = \varphi_j$ for each $j \in J$.

Suppose, conversely, that G is a direct product of the system $(G_j)_{j\in J}$ in $\mathbf{CBLg_c}$ and write $p_j\colon G \to G_j$ for the corresponding projection morphisms and $\varepsilon_j\colon G_j \to G$ for the inclusion maps which are uniquely determined by $p_k \circ \varepsilon_j = \mathbf{1}$ for $j \neq k$ and $p_j \circ \varepsilon_j = \mathrm{id}_{G_j}$. If there exists no $r > 0$ such that the restriction of the map $\exp_{G_j}$ to the ball $B_r(0)$ in $\mathbf{L}(G_j)$ is injective for each $j \in J$, then there exist $j_n \in J$ and $x_n, y_n \in \mathfrak{g}_{j_n}$ with $x_n \neq y_n$, $\|x_n\|, \|y_n\| < \frac{1}{n}$, and $\exp_{G_{j_n}} x_n = \exp_{G_{j_n}} y_n$.

Then $a_n := \mathbf{L}(\varepsilon_{j_n}).x_n$ and $b_n := \mathbf{L}(\varepsilon_{j_n}).y_n$ are null sequences in $\mathbf{L}(G)$ with $\mathbf{L}(p_{j_n}).a_n = x_n \neq y_n = \mathbf{L}(p_{j_n}).b_n$ and

$$\exp_G a_n = \varepsilon_{j_n}(\exp_{G_{j_n}} x_n) = \varepsilon_{j_n}(\exp_{G_{j_n}} y_n) = \exp_G b_n.$$

This contradicts the injectivity of $\exp_G$ in a neighborhood of 0.

As in the proof of Proposition III.12, we see that the injectivity condition is satisfied if all groups G_j are equal or simply connected. In this case we obtain a direct product in $\mathbf{CBLg_c}$. ■

Universal enlargible envelopes

Definition III.14. Let $\mathfrak g$ be a Banach-Lie algebra over $\mathbb K \in \{\mathbb R, \mathbb C\}$. A continuous homomorphism $\zeta_{\mathfrak g} \colon \mathfrak g \to e(\mathfrak g)$ is called a $\mathbb K$-*universal enlargible envelope of* $\mathfrak g$ if $e(\mathfrak g)$ is an enlargible $\mathbb K$-Banach-Lie algebra and for every continuous homomorphism $\varphi \colon \mathfrak g \to \mathfrak h$, where $\mathfrak h$ is an enlargible $\mathbb K$-Banach-Lie algebra, there exists a unique continuous homomorphism $\overline{\varphi} \colon e(\mathfrak g) \to \mathfrak h$ with $\overline{\varphi} \circ \zeta_{\mathfrak g} = \varphi$. ■

Remark III.15. (a) Whenever universal enlargible envelopes exist, they are unique up to isomorphism.

(b) Let $\zeta_{\mathfrak g} : \mathfrak g \to e(\mathfrak g)$ be a universal enlargible envelope. Then Lemma I.2 implies that $\mathfrak g/\ker \zeta_{\mathfrak g}$ is enlargible, and it further follows that this Banach-Lie algebra has the universal property of an enlargible envelope. Therefore $e(\mathfrak g) \cong \mathfrak g/\ker \zeta_{\mathfrak g}$ and $\zeta_{\mathfrak g} : \mathfrak g \to e(\mathfrak g)$ is a quotient homomorphism. Moreover, $\ker \zeta_{\mathfrak g} \subseteq \ker \mathrm{ad} = \mathfrak z(\mathfrak g)$ (Lemma I.2). ■

Lemma III.16. *Let Z be a Banach space, $\Gamma \subseteq Z$ an additive subgroup, and $X \subseteq Z$ a closed vector subspace. Then the following conditions are equivalent:*

(1) *X is an open subgroup of $X + \Gamma$.*

(2) *$X + \Gamma$ is a Lie subgroup of Z with Lie algebra X.*

(3) *The image of Γ in Z/X is discrete.*

The set of all subspaces X satisfying these conditions is closed under finite intersections.

Proof. The equivalence of (1)-(3) is a trivial consequence of the definitions.

Suppose that $X_1, \ldots, X_n$ satisfy this condition and let $U_j \subseteq Z$ be an open 0-neighborhood with $U_j \cap (X_j + \Gamma) \subseteq X_j$. Then $U := \bigcap_{j=1}^n U_j$ satisfies

$$U \cap \big((\cap_{j=1}^n X_j) + \Gamma\big) \subseteq X_k$$

for each k, and therefore $U \cap ((\cap_{j=1}^n X_j) + \Gamma) \subseteq \bigcap_{j=1}^n X_j$. This completes the proof. ■

Lemma III.17. *Let G be a connected Banach-Lie group, $N \trianglelefteq G$ a normal Lie subgroup, and $H \supseteq N$ a subgroup. Then the following conditions are equivalent:*

(1) *H/N is a Lie subgroup of G/N.*

(2) *H is a Lie subgroup of G.*

Proof. That the quotient G/N carries the structure of a Banach-Lie group follows from Theorem II.2. Let $q \colon G \to G/N$ denote the quotient map.

(1) $\Rightarrow$ (2) In view of $H = q^{-1}(H/N)$, the subgroup H is the inverse image of a Lie subgroup of G/N, hence a Lie subgroup of G by Lemma II.1.

(2) $\Rightarrow$ (1): As N is a Lie subgroup of G, it is a Lie subgroup of H, whence H/N is a Banach-Lie group by Theorem II.2. The topology on the Banach-Lie group H/N being the one induced by G/N, we easily deduce that H/N is a Lie subgroup of G/N. ■

Lemma III.18. *Let $\mathfrak a \subseteq \mathfrak z(\mathfrak g)$ be a closed vector subspace. Then the following are equivalent:*

(1) *$\mathfrak a$ is an open subgroup of $\Pi(\mathfrak g) + \mathfrak a$.*

(2) *$\mathfrak a + \Pi(\mathfrak g)$ is a Lie subgroup of $\mathfrak z(\mathfrak g)$ with Lie algebra $\mathfrak a$.*

(3) *The quotient Lie algebra $\mathfrak g/\mathfrak a$ is enlargible.*

Proof. (1) $\Leftrightarrow$ (2) follows from Lemma III.16.

(2) $\Rightarrow$ (3): We consider the central extension

$$\mathfrak{z}(\mathfrak{g}) \hookrightarrow \widehat{\Omega}(\mathfrak{g}) \to \Omega(G_{\mathrm{ad}})$$

and write $B := \langle \exp \Omega(\mathfrak{g}_{\mathrm{ad}})\rangle$ for the connected analytic subgroup of $\widehat{\Omega}(G)$ corresponding to the closed Lie subalgebra $\Omega(\mathfrak{g}_{\mathrm{ad}})$ (cf. Definition III.2). Then

$$\Pi(\mathfrak{g}) = \mathfrak{z} \cap B$$

(Lemma III.6) and on the Lie algebra level we have a trivial central extension $\widehat{\Omega}(\mathfrak{g}) \cong \mathfrak{z} \times \Omega(\mathfrak{g}_{\mathrm{ad}})$. Therefore $\mathfrak{a} \times \Omega(\mathfrak{g}_{\mathrm{ad}})$ is a closed ideal of $\widehat{\Omega}(\mathfrak{g})$, and for $A := \exp_{\widehat{\Omega}(G)} \mathfrak{a}$ the product $AB \subseteq \widehat{\Omega}(G)$ is the normal analytic subgroup corresponding to the ideal $\mathfrak{a} \times \Omega(\mathfrak{g}_{\mathrm{ad}})$.

We claim that AB is a Lie subgroup of $\widehat{\Omega}(G)$. Let $p\colon \widetilde{\Omega}(G_{\mathrm{ad}}) \times \mathfrak{z} \to \widehat{\Omega}(G)$ denote the universal covering homomorphism (cf. Definition III.2). Then

$$p^{-1}(AB) = (\widetilde{\Omega}(G) \times \mathfrak{a})\Gamma(-\operatorname{per}_{\mathfrak{g}}) = \widetilde{\Omega}(G) \times (\mathfrak{a} + \Pi(\mathfrak{g})).$$

The assumption that $\mathfrak{a}$ is open in $\mathfrak{a} + \Pi(\mathfrak{g})$ implies that $\widetilde{\Omega}(G) \times \mathfrak{a}$ is open in $p^{-1}(AB)$, hence that $p^{-1}(AB)$ is a normal Lie subgroup in $\widetilde{\Omega}(G) \times \mathfrak{z}$ with Lie algebra $\Omega(\mathfrak{g}) \times \mathfrak{a}$. We conclude with Lemma III.17 that $AB = p(p^{-1}(AB))$ is a normal Lie subgroup of $\widehat{\Omega}(G)$ with Lie algebra $\Omega(\mathfrak{g}) \times \mathfrak{a}$, hence a normal Lie subgroup of $\widehat{P}(G)$. In view of Theorem II.2, the quotient group $\widehat{P}(G)/AB$ is a Banach–Lie group and its Lie algebra coincides with

$$\widehat{P}(\mathfrak{g})/(\Omega(\mathfrak{g}) \times \mathfrak{a}) \cong (\widehat{P}(\mathfrak{g})/\Omega(\mathfrak{g}))/(\mathfrak{a} + \Omega(\mathfrak{g})/\Omega(\mathfrak{g})) \cong \mathfrak{g}/\mathfrak{a}.$$

Therefore $\mathfrak{g}/\mathfrak{a}$ is enlargible.

(3) $\Rightarrow$ (1): Suppose that the Lie algebra $\mathfrak{g}_{\mathfrak{a}} := \mathfrak{g}/\mathfrak{a}$ with the quotient map $q_{\mathfrak{a}} : \mathfrak{g} \to \mathfrak{g}_{\mathfrak{a}}$ is enlargible. Then $q_{\mathfrak{a}}(\mathfrak{z}(\mathfrak{g})) \subseteq \mathfrak{z}(\mathfrak{g}_{\mathfrak{a}})$, so that Lemma III.3 implies that $q_{\mathfrak{a}}(\Pi(\mathfrak{g})) \subseteq \Pi(\mathfrak{g}_{\mathfrak{a}})$. Since $\Pi(\mathfrak{g}_{\mathfrak{a}})$ is discrete by Theorem III.7, $\{0\}$ is an open subgroup of $\Pi(\mathfrak{g}_{\mathfrak{a}})$, and therefore the inverse image $\mathfrak{a} = q_{\mathfrak{a}}^{-1}(0)$ is an open subgroup of $q_{\mathfrak{a}}^{-1}(\Pi(\mathfrak{g}_{\mathfrak{a}})) \supseteq \mathfrak{a} + \Pi(\mathfrak{g})$. ■

Theorem III.19. *Let $\mathfrak{g}$ be a Banach-Lie algebra and $\Pi(\mathfrak{g}) \subseteq \mathfrak{z}(\mathfrak{g})$ its period group. A universal enlargible envelope of $\mathfrak{g}$ exists if and only if there exists a minimal closed vector subspace $\mathfrak{a} \subseteq \mathfrak{z}(\mathfrak{g})$ for which $\mathfrak{a}$ is an open subgroup of $\mathfrak{a} + \Pi(\mathfrak{g})$.*

Proof. If $\zeta_{\mathfrak{g}} \colon \mathfrak{g} \to e(\mathfrak{g})$ exists, then Remark III.15(b) implies that $\zeta_{\mathfrak{g}}$ is a quotient map with $\mathfrak{a} := \ker \zeta_{\mathfrak{g}} \subseteq \mathfrak{z}(\mathfrak{g})$. Now Lemma III.18 entails that $\mathfrak{a}$ is open in $\mathfrak{a} + \Pi(\mathfrak{g})$. In view of Lemma III.18 and the universality of $e(\mathfrak{g}) \cong \mathfrak{g}/\mathfrak{a}$, the subspace $\mathfrak{a} \subseteq \mathfrak{z}(\mathfrak{g})$ is contained in all other closed subspaces $\mathfrak{b} \subseteq \mathfrak{z}(\mathfrak{g})$ for which $\mathfrak{b}$ is open in $\mathfrak{b} + \Pi(\mathfrak{g})$ because this property is equivalent to $\mathfrak{g}/\mathfrak{b}$ being enlargible.

Suppose, conversely, that $\mathfrak{a} \subseteq \mathfrak{z}(\mathfrak{g})$ is a minimal closed subspace with the property that $\mathfrak{a}$ is open in $\mathfrak{a} + \Pi(\mathfrak{g})$. Since the set of all these subspaces is closed under finite intersections, it follows that $\mathfrak{a}$ is unique and contained in all other closed subspaces with this property (Lemma III.16). Then Lemma III.18 implies that $e(\mathfrak{g}) := \mathfrak{g}/\mathfrak{a}$ is enlargible. Let $\zeta_{\mathfrak{g}} : \mathfrak{g} \to e(\mathfrak{g})$ denote the quotient homomorphism. We show that this map has the required universal property. If $\varphi \colon \mathfrak{g} \to \mathfrak{h}$ is a homomorphism into an enlargible Lie algebra $\mathfrak{h}$, then Lemma I.2 implies that $\mathfrak{g}/\ker\varphi$ is enlargible, and since φ factors through $\mathfrak{g}/\ker\varphi$, we may assume that φ is a quotient homomorphism, so that it remains to show that $\mathfrak{b} := \ker\varphi \supseteq \mathfrak{a}$.

Since $\varphi(\mathfrak{z}(\mathfrak{g})) \subseteq \mathfrak{z}(\mathfrak{h})$, it follows from Lemma III.3 that $\varphi(\Pi(\mathfrak{g})) \subseteq \Pi(\mathfrak{g}/\mathfrak{b})$, and since $\Pi(\mathfrak{g}/\mathfrak{b})$ is discrete (Theorem III.7), the subgroup $\mathfrak{b}_{\mathfrak{z}} := \mathfrak{b} \cap \mathfrak{z}(\mathfrak{g})$ is open in $\varphi^{-1}(\varphi(\Pi(\mathfrak{g}))) \cap \mathfrak{z}(\mathfrak{g}) = \Pi(\mathfrak{g}) + \mathfrak{b}_{\mathfrak{z}}$. Now the minimality of $\mathfrak{a}$ implies that $\mathfrak{a} \subseteq \mathfrak{b}_{\mathfrak{z}} \subseteq \mathfrak{b}$, and hence that φ factors through $\mathfrak{g}/\mathfrak{a}$. This proves the universal property of $\mathfrak{g}/\mathfrak{a}$. ■

IV. Enlargibility and universal complexifications

In this section, we characterize those real Banach-Lie groups which have universal complexifications (Theorem IV.6), and give examples of such groups. In the case of simply connected Banach-Lie groups, the existence of universal complexifications can be characterized on the Lie-algebra level (Theorem IV.11).

Lemma IV.1. *If $\eta_G : G \to G_{\mathbb{C}}$ is a universal complexification of the real Banach-Lie group G, then the following assertions hold:*

(i) *There exists a unique antiholomorphic involutive automorphism σ of $G_{\mathbb{C}}$ with $\sigma \circ \eta_G = \eta_G$.*

(ii) *The complexification of $\mathbf{L}(G)/\ker \mathbf{L}(\eta_G)$ is enlargible and $\ker \widetilde{\mathbf{L}}(\eta_G) = (\ker \mathbf{L}(\eta_G))_{\mathbb{C}}$, where $\widetilde{\mathbf{L}}(\eta_G)$ is the complex linear extension of $\mathbf{L}(\eta_G)$ to $\mathbf{L}(G)_{\mathbb{C}}$.*

(iii) *$\mathbf{L}(G)/\ker \mathbf{L}(\eta_G)$ is enlargible.*

(iv) *$G/\ker \eta_G$ is a Banach-Lie group, and $G_{\mathbb{C}}$ is also universal for this group.*

Proof. (i) First we prove the uniqueness. If $\sigma_1, \sigma_2 : G_{\mathbb{C}} \to G_{\mathbb{C}}$ are antiholomorphic morphisms with $\sigma_j \circ \eta_G = \eta_G$, then $\sigma_1 \circ \sigma_2$ is holomorphic, so that $\sigma_1 \circ \sigma_2 \circ \eta_G = \eta_G$ implies that $\sigma_1 \circ \sigma_2 = \mathrm{id}_{G_{\mathbb{C}}}$. We likewise obtain $\sigma_2 \circ \sigma_1 = \mathrm{id}_{G_{\mathbb{C}}}$. This implies in particular that $\sigma_1^2 = \mathrm{id}_{G_{\mathbb{C}}}$, so that $\sigma_2 = \sigma_1^{-1} = \sigma_1$.

Let $\overline{G}_{\mathbb{C}}$ denote the real Banach-Lie group $G_{\mathbb{C}}$ endowed with the opposite complex structure. Then $\eta_G : G \to \overline{G}_{\mathbb{C}}$ yields a holomorphic morphism $\sigma : G_{\mathbb{C}} \to \overline{G}_{\mathbb{C}}$ with $\sigma \circ \eta_G = \eta_G$. This means that we can view σ as an antiholomorphic endomorphism of $G_{\mathbb{C}}$ with

$$\mathrm{im}(\eta_G) \subseteq G_{\mathbb{C}}^{\sigma} := \{g \in G_{\mathbb{C}} : \sigma(g) = g\}.$$

As we have seen above, σ is uniquely determined by this property, and it is an involution.

(ii) Since $\sigma \circ \eta_G = \eta_G$ and σ is antiholomorphic, we obtain for $x, y \in \mathbf{L}(G)$:

$$\mathbf{L}(\sigma)\widetilde{\mathbf{L}}(\eta_G)(x+iy) = \mathbf{L}(\sigma)\big(\mathbf{L}(\eta_G)(x) + i\,\mathbf{L}(\eta_G)(y)\big) = \mathbf{L}(\eta_G)(x) - i\,\mathbf{L}(\eta_G)(y) = \widetilde{\mathbf{L}}(\eta_G)(x-iy).$$

Therefore $\ker \widetilde{\mathbf{L}}(\eta_G) \subseteq \mathbf{L}(G)_{\mathbb{C}}$ is a conjugation invariant closed subalgebra of $\mathbf{L}(G)_{\mathbb{C}}$. Hence it coincides with $(\ker \mathbf{L}(\eta_G))_{\mathbb{C}}$. This means that $\widetilde{\mathbf{L}}(\eta_G) : \mathbf{L}(G)_{\mathbb{C}} \to \mathbf{L}(G_{\mathbb{C}})$ induces an injective map

$$(\mathbf{L}(G)/\ker \mathbf{L}(\eta_G))_{\mathbb{C}} \cong \mathbf{L}(G)_{\mathbb{C}}/(\ker \mathbf{L}(\eta_G))_{\mathbb{C}} \hookrightarrow \mathbf{L}(G_{\mathbb{C}}).$$

In view of Lemma I.2, this implies that the complexification of $\mathbf{L}(G)/\ker \mathbf{L}(\eta_G)$ is enlargible.

(iii) follows from Lemma I.3.

(iv) follows from Theorem II.2. ∎

In the following, $N_G \subseteq G$ denotes the intersection of the kernels of all continuous homomorphisms of G to complex Banach-Lie groups.

Corollary IV.2. *If $\mathbf{L}(G)_{\mathbb{C}}$ is not enlargible and $\mathbf{L}(N_G) = \{0\}$, then G has no universal complexification.*

Proof. Suppose that $\eta_G : G \to G_{\mathbb{C}}$ is a universal complexification. Then $\ker \eta_G = N_G$ implies that $\ker \mathbf{L}(\eta_G) = \mathbf{L}(N_G) = \{0\}$. Therefore Lemma IV.1(ii) implies that $\mathbf{L}(G)_{\mathbb{C}}$ is enlargible. ∎

Lemma IV.3. *If $\mathbf{L}(G)_{\mathbb{C}}$ is enlargible and G is simply connected, then there exists a universal complexification $\eta_G : G \to G_{\mathbb{C}}$, where $G_{\mathbb{C}}$ is simply connected and $\mathbf{L}(G_{\mathbb{C}}) \cong \mathbf{L}(G)_{\mathbb{C}}$.*

Proof. Let $G_{\mathbb{C}}$ be a simply connected Lie group with Lie algebra $\mathbf{L}(G)_{\mathbb{C}}$. Since G is simply connected, the inclusion map $\mathbf{L}(G) \hookrightarrow \mathbf{L}(G)_{\mathbb{C}}$ integrates to a smooth homomorphism $\eta_G : G \to G_{\mathbb{C}}$. If $\alpha : G \to H$ is a smooth homomorphism into a complex Lie group H, then $\mathbf{L}(\alpha) : \mathbf{L}(G) \to \mathbf{L}(H)$ extends to a complex linear map $\widetilde{\mathbf{L}}(\alpha) : \mathbf{L}(G)_{\mathbb{C}} \to \mathbf{L}(H)$, which integrates to a complex analytic homomorphism $\beta : G_{\mathbb{C}} \to H$ with $\beta \circ \eta_G = \alpha$. Clearly β is uniquely determined by the latter property. Therefore η_G is a universal complexification. ∎

The next lemma allows us to focus on connected Banach-Lie groups in the following proofs.

Lemma IV.4. *Let G be a real Banach-Lie group whose identity component G_0 has a universal complexification $(G_0)_{\mathbb{C}}$. Then G has a universal complexification $(G_{\mathbb{C}}, \gamma_G)$. Furthermore, $((G_{\mathbb{C}})_0, \gamma_G|_{G_0}^{(G_{\mathbb{C}})_0})$ is a universal complexification of G_0, and $G_{\mathbb{C}}/(G_{\mathbb{C}})_0 \cong G/G_0$.*

Proof. Part (b) of the proof of [Bo89, Chapter 3, §6.10, Proposition 20 (a)] and Remark (1) following that proposition can be copied line by line. ∎

Lemma IV.5. *Let G be a real Banach-Lie group. If $N_G = \{\mathbf{1}\}$ and $\mathbf{L}(G)_{\mathbb{C}}$ is enlargible, then G has a universal complexification with $\mathbf{L}(G_{\mathbb{C}}) \cong \mathbf{L}(G)_{\mathbb{C}}$.*

Proof. By the preceding lemma, we may assume that G is connected. In view of Lemma IV.3, the covering group $\widetilde{G}$ has a universal complexification $\eta_{\widetilde{G}}: \widetilde{G} \to (\widetilde{G})_{\mathbb{C}}$ with $\mathbf{L}((\widetilde{G})_{\mathbb{C}}) = \mathbf{L}(G)_{\mathbb{C}}$ and $(\widetilde{G})_{\mathbb{C}}$ simply connected.

Lemma IV.1(i) provides a unique antiholomorphic involution σ of $(\widetilde{G})_{\mathbb{C}}$ with $\sigma \circ \eta_{\widetilde{G}} = \eta_{\widetilde{G}}$. Then $\mathbf{L}(\sigma)$ is the complex conjugation of $\mathbf{L}(G)_{\mathbb{C}}$ with respect to the real form $\mathbf{L}(G)$.

If $\varphi: G \to H$ is a Lie group morphism to a complex Lie group H, then $\varphi \circ q_G: \widetilde{G} \to H$ induces a unique holomorphic morphism $\varphi_{\mathbb{C}}: (\widetilde{G})_{\mathbb{C}} \to H$ with $\varphi_{\mathbb{C}} \circ \eta_{\widetilde{G}} = \varphi \circ q_G$. Hence $q_G(\ker \eta_{\widetilde{G}}) \subseteq \ker \varphi$, so that $N_G = \{\mathbf{1}\}$ implies that $\ker \eta_{\widetilde{G}} \subseteq \pi_1(G)$, and hence that

$$G_1 := \big((\widetilde{G})_{\mathbb{C}}^{\sigma}\big)_0 = \eta_{\widetilde{G}}(\widetilde{G}) \cong \widetilde{G}/\ker \eta_{\widetilde{G}}$$

is a covering group of G. Since

$$\eta_{\widetilde{G}}(\pi_1(G)) \cong \pi_1(G)/\ker \eta_{\widetilde{G}} \subseteq \widetilde{G}/\ker \eta_{\widetilde{G}} \cong G_1$$

is a discrete subgroup, it follows that $\eta_{\widetilde{G}}(\pi_1(G))$ is a discrete subgroup of $(\widetilde{G})_{\mathbb{C}}$. It is central because it is contained in $\ker \mathrm{Ad}_{\mathbf{L}(G)_{\mathbb{C}}}$.

Let $G_{\mathbb{C}} := (\widetilde{G})_{\mathbb{C}}/\eta_{\widetilde{G}}(\pi_1(G))$, and observe that the map $\widetilde{G} \to G_{\mathbb{C}}$ factors through a map $\eta_G: G \to G_{\mathbb{C}}$. It is easy to verify that we thus obtain a universal complexification. ∎

Theorem IV.6 (Complexification Theorem). *Let G be a real Banach-Lie group. Then G has a universal complexification if and only if the following two conditions are satisfied:*

(i) *The intersection $N_G \subseteq G$ of all kernels of smooth homomorphisms to complex Banach-Lie groups (which always is a closed normal subgroup of G) is a Lie subgroup.*

(ii) *The Banach-Lie algebra $\big(\mathbf{L}(G)/\mathbf{L}(N_G)\big)_{\mathbb{C}}$ is enlargible.*

Proof. "$\Rightarrow$" Suppose that $\eta_G: G \to G_{\mathbb{C}}$ is a universal complexification. Then $N_G = \ker(\eta_G)$ follows from the universal property of η_G, and we conclude that N_G is a kernel, hence a Lie subgroup. The remaining assertion is Lemma IV.1(ii).

"$\Leftarrow$" If N_G is a Lie subgroup, then we use Theorem II.2 to see that $H := G/N_G$ has a natural structure of a Banach-Lie group with Lie algebra $\mathbf{L}(H) = \mathbf{L}(G)/\mathbf{L}(N_G)$. Obviously $N_H = \{\mathbf{1}\}$ because the homomorphisms from H into complex Lie groups separate points, and (ii) means that $\mathbf{L}(H)_{\mathbb{C}}$ is enlargible, so that the assertion follows from Lemma IV.5. ∎

Corollary IV.7. *If $N_G = \{\mathbf{1}\}$, then G has a universal complexification if and only if $\mathbf{L}(G)_{\mathbb{C}}$ is enlargible.* ∎

Corollary IV.8. *Suppose that G admits a smooth homomorphism $f: G \to H$ into a complex Banach-Lie group H such that $\widetilde{\mathbf{L}}(f): \mathbf{L}(G)_{\mathbb{C}} \to \mathbf{L}(H)$, $X + iY \mapsto \mathbf{L}(f).X + i\,\mathbf{L}(f).Y$ is injective. Then G has a universal complexification $G_{\mathbb{C}}$, and $\mathbf{L}(G_{\mathbb{C}}) \cong \mathbf{L}(G)_{\mathbb{C}}$.*

Proof. The hypothesis entails that N_G is discrete. Replacing G by G/N_G if necessary, we may assume that $N_G = \{\mathbf{1}\}$. In view of Lemma I.2 and the hypothesis, $\mathbf{L}(G)_{\mathbb{C}}$ is enlargible. Thus Corollary IV.7 applies. ∎

Examples IV.9. We give simple examples of Banach-Lie groups with universal complexifications.

(a) Let A be a real Banach algebra. Then every analytic subgroup G of the group of units $A^\times$ has a universal complexification, as Corollary IV.8 applies to the inclusion map $f\colon G \to (A_{\mathbb{C}})^\times$ (see also [Gl00, Corollary 24.21], and [Gl01] for the C^∞-analogue).

(b) Let K be a compact topological space and F be a real Banach-Lie group such that $\eta_F : F \to F_{\mathbb{C}}$ has discrete kernel. Then the Banach-Lie group $C(K,F)$ of continuous F-valued mappings on M has a universal complexification, as Corollary IV.8 applies to the homomorphism $f := C(K,\gamma_F)\colon C(K,F) \to C(K,F_{\mathbb{C}})$ (see also [Gl00, Proposition 25.5]). ■

Remark IV.10. The preceding results suggest the following algorithm to decide whether G has a universal complexification:

1. First check if the intersection N_G of all kernels of smooth homomorphisms $G \to H$, H a complex Banach-Lie group, is a Lie subgroup. If this is not the case, then G has no universal complexification.

2. If N_G is a Lie subgroup, then G/N_G is a Banach-Lie group by Theorem II.2. Replacing G by G/N_G, we may assume that $N_G = \{\mathbf{1}\}$, i.e., that the morphisms to complex Banach-Lie groups separate points. Then G has a universal complexification if and only if $\mathbf{L}(G)_{\mathbb{C}}$ is enlargible (Corollary IV.7).

This means that we have *two levels*, where the existence of $G_{\mathbb{C}}$ can fail. An example of a Banach-Lie group which fails to satisfy Condition (i) of Theorem IV.6 will be given in Section V. Banach-Lie groups which do not satisfy Condition (ii) are described in Example VI.4 below. ■

Complexifications of simply connected groups

If G is a simply connected Lie group, then the general philosophy of Lie theory says that every group theoretic property of G is somehow encoded in the Lie algebra $\mathfrak{g}$. Therefore one would expect a characterization of those simply connected groups having a universal complexification in terms of their Lie algebra. The following theorem is a criterion of this type.

Theorem IV.11. *Let G be a simply connected Banach-Lie group. Then G has a universal complexification if and only if the complexification $\mathbf{L}(G)_{\mathbb{C}}$ of its Lie algebra has a universal enlargible envelope in the category of complex Banach-Lie algebras.*

Proof. Let $\eta_G : G \to G_{\mathbb{C}}$ be a universal complexification and $\varphi : \mathbf{L}(G)_{\mathbb{C}} \to \mathfrak{h}$ a complex linear homomorphism into an enlargible complex Lie algebra. Since G is simply connected, we can integrate $\varphi|_{\mathbf{L}(G)}$ to a group homomorphism of $\varphi_G\colon G \to H$, where H is a simply connected complex group H with Lie algebra $\mathfrak{h}$. Since this homomorphism factors through η_G, we obtain a Lie algebra homomorphism $\mathbf{L}(\varphi_G)^\sharp\colon \mathbf{L}(G_{\mathbb{C}}) \to \mathfrak{h}$ with $\mathbf{L}(\varphi_G)^\sharp \circ \mathbf{L}(\eta_G) = \varphi|_{\mathbf{L}(G)}$. This implies that the homomorphism $\widetilde{\mathbf{L}}(\eta_G) : \mathbf{L}(G)_{\mathbb{C}} \to \mathbf{L}(G_{\mathbb{C}})$ has the universal property of the universal enlargible complex envelope of $\mathbf{L}(G)_{\mathbb{C}}$.

Suppose, conversely, that $\zeta\colon \mathbf{L}(G)_{\mathbb{C}} \to \mathfrak{e}$ is a complex universal enlargible envelope and that E is a simply connected Lie group with Lie algebra $\mathfrak{e}$. Since G is simply connected, there exists a continuous homomorphism $\eta_G\colon G \to E$ with $\mathbf{L}(\eta_G) = \zeta$. We claim that η_G is a universal complexification.

Let $\varphi : G \to H$ be a continuous homomorphism into a complex Banach-Lie group H. Then $\widetilde{\mathbf{L}}(\varphi)\colon \mathbf{L}(G)_{\mathbb{C}} \to \mathfrak{h}$ is a continuous Lie algebra homomorphism which then factors through ζ. In view of the simple connectedness of E, the homomorphism φ factors through η_G. This proves the universality of η_G. ■

Complexifications of elliptic groups

Elliptic Banach-Lie algebras defined below are natural generalizations of finite-dimensional compact Lie algebras. In this section we extend the result that a finite-dimensional connected Lie group G with a compact Lie algebra has a faithful universal complexification with a polar decomposition to elliptic Lie algebras. Here the remarkable part is that the existence of a faithful homomorphism into a complex Lie group is obtained from general geometric results on polar decompositions.

Definition IV.12. (a) We call a Banach-Lie algebra $\mathfrak{g}$ *elliptic* if there exists a norm on $\mathfrak{g}$ defining the topology which is invariant under the operators $e^{\mathrm{ad}\, x}$, $x \in \mathfrak{g}$. We say that a connected Banach-Lie group G is *elliptic* if its Lie algebra $\mathfrak{g}$ is elliptic, i.e., there exists a norm on $\mathfrak{g}$ defining the topology which is invariant under the group $\mathrm{Ad}(G)$.
(b) Let G be a Banach-Lie group endowed with an involutive automorphism τ. Then the eigenspace decomposition of $\mathfrak{g} = \mathbf{L}(G)$ with respect to $\mathbf{L}(\tau)$ yields a direct sum decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k} = \ker(\mathbf{L}(\tau) - 1)$ and $\mathfrak{p} = \ker(\mathbf{L}(\tau) + 1)$. We say that the group G *has a polar decomposition* if for $K := \{g \in G : \tau(g) = g\}$ the *polar map*

$$p: K \times \mathfrak{p} \to G, \quad (k, x) \mapsto k \exp x$$

is a diffeomorphism. This implies in particular that the inclusion map $K \hookrightarrow G$ is a homotopy equivalence, hence induces an isomorphism $\pi_2(K) \to \pi_2(G)$. ∎

Lemma IV.13. *If $\mathfrak{g}$ is an elliptic Banach-Lie algebra, then $\mathfrak{g}_{\mathbb{C}}$ is enlargible.*

Proof. Let $\|\cdot\|$ be an $e^{\mathrm{ad}\,\mathfrak{g}}$-invariant norm on $\mathfrak{g}$ compatible with the topology on $\mathfrak{g}$. Then the quotient norm on $\mathfrak{g}_{\mathrm{ad}}$ is also invariant, showing that $\mathfrak{g}_{\mathrm{ad}}$ is elliptic. Moreover, the complexification $\mathfrak{g}_{\mathrm{ad},\mathbb{C}}$ of $\mathfrak{g}_{\mathrm{ad}}$ in enlargible because it is contained in $\mathrm{der}(\mathfrak{g}_{\mathbb{C}})$ (Lemma I.2). Let $G_{\mathrm{ad},\mathbb{C}}$ be a corresponding simply connected group. Now [Ne01c, Cor. IV.9 and Th. V.1] imply that the group $G_{\mathrm{ad},\mathbb{C}}$ has a polar decomposition $G_{\mathrm{ad},\mathbb{C}} = G_{\mathrm{ad}} \exp(i\mathfrak{g}_{\mathrm{ad}})$, where $G_{\mathrm{ad}} \subseteq G_{\mathrm{ad},\mathbb{C}}$ is the fixed point group for the antiholomorphic automorphism τ of $G_{\mathrm{ad},\mathbb{C}}$ for which $\mathbf{L}(\tau)$ is the conjugation of $\mathfrak{g}_{\mathrm{ad},\mathbb{C}}$ with respect to the real form $\mathfrak{g}_{\mathrm{ad}}$. Since the inclusion $G_{\mathrm{ad}} \hookrightarrow G_{\mathrm{ad},\mathbb{C}}$ is a homotopy equivalence, the group G_{ad} is simply connected, so that our notation here is compatible with the definition of G_{ad} in Section III.

Next we observe that Corollary III.4(a) applies to the inclusion map $\mathfrak{g} \hookrightarrow \mathfrak{g}_{\mathbb{C}}$ because this maps $\mathfrak{z}(\mathfrak{g})$ into $\mathfrak{z}(\mathfrak{g}_{\mathbb{C}}) \cong \mathfrak{z}(\mathfrak{g})_{\mathbb{C}}$ and the induced map $G_{\mathrm{ad}} \hookrightarrow (G_{\mathbb{C}})_{\mathrm{ad}} \cong G_{\mathrm{ad},\mathbb{C}}$ induces an isomorphism of the second homotopy groups. Corollary III.4(a) implies that $\Pi(\mathfrak{g}_{\mathbb{C}}) = \Pi(\mathfrak{g})$ and hence that $\mathfrak{g}_{\mathbb{C}}$ is enlargible by Theorem III.7. ∎

The following proposition generalizes the standard result on the polar decomposition and the existence of a universal complexification of compact Lie groups.

Proposition IV.14. *If G is an elliptic Banach-Lie group, then G has an injective universal complexification $\eta_G: G \to G_{\mathbb{C}}$ such that $G_{\mathbb{C}}$ has a polar decomposition $G_{\mathbb{C}} = G \exp(i\mathfrak{g})$.*

Proof. Let $\widetilde{G}_{\mathbb{C}}$ be a simply connected Lie group with Lie algebra $\mathfrak{g}_{\mathbb{C}}$ (Lemma IV.13). From [Ne01c, Cor. IV.9 and Th. V.1] we conclude that $\widetilde{G}_{\mathbb{C}}$ has a polar decomposition $\widetilde{G}_{\mathbb{C}} = \widetilde{G} \exp(i\mathfrak{g})$, where the subgroup $\widetilde{G} \subseteq G_{\mathbb{C}}$ can be identified with the simply connected covering group of G. Identifying $\pi_1(G)$ with a discrete central subgroup of $\widetilde{G}$, we observe that it is also central in $\widetilde{G}_{\mathbb{C}}$ because it acts trivially by the adjoint representation on $\mathfrak{g}_{\mathbb{C}}$. Therefore $G_{\mathbb{C}} := \widetilde{G}_{\mathbb{C}}/\pi_1(G)$ is a complex Lie group containing $\widetilde{G}/\pi_1(G) \cong G$ as a real subgroup corresponding to the Lie subalgebra $\mathfrak{g} \subseteq \mathfrak{g}_{\mathbb{C}}$. Moreover, the polar decomposition of $\widetilde{G}_{\mathbb{C}}$ induces a polar decomposition $G_{\mathbb{C}} = G \exp(i\mathfrak{g})$ of $G_{\mathbb{C}}$. From that one easily derives that the inclusion map $\eta_G: G \hookrightarrow G_{\mathbb{C}}$ is a universal complexification. ∎

V. An example of the first kind

In this section and the next, we construct real Banach-Lie groups without universal complexifications. We begin with a Banach-Lie group which does not satisfy condition (i) of Theorem IV.6.

Step 1. Recall that the universal covering group $S := \widetilde{\mathrm{SL}}(2,\mathbb{R})$ of the special linear group $\mathrm{SL}(2,\mathbb{R})$ has discrete center $Z(S) \cong \mathbb{Z}$, and recall that $N_S = \ker \eta_S \cong \mathbb{Z}$ is a subgroup of index 2 in $Z(S)$, where $\eta_S : S \to S_{\mathbb{C}} \cong \mathrm{SL}(2,\mathbb{C})$. Let z_0 be a generator for N_S. For every $n \in \mathbb{N}$, there is a unique homomorphism $\varphi_n : N_S \to \mathbb{R}$ such that $\varphi_n(z_0) = \frac{1}{n}$. Then the graph Γ_n of φ_n is a discrete normal subgroup of $S \times \mathbb{R}$, and $H_n := (S \times \mathbb{R})/\Gamma_n$ is a Lie group with Lie algebra $\mathfrak{h} := \mathfrak{sl}(2,\mathbb{R}) \times \mathbb{R}$ and exponential function $\exp_{H_n} = q_n \circ (\exp_S \times \mathrm{id}_{\mathbb{R}})$, where $\exp_S$ is the exponential function of S and $q_n : S \times \mathbb{R} \to H_n$ the canonical quotient morphism. The mapping $i_n : S \to H_n$, $g \mapsto q_n(g,0)$ is an embedding of topological groups.

Step 2. It is apparent from the definition of N_S that $i_n(N_S) \subseteq \ker \eta_{H_n} =: N_{H_n}$, where $\eta_{H_n} : H_n \to (H_n)_{\mathbb{C}}$ is the universal homomorphism. Note that

$$i_n(z_0) = (z_0, 0)\,\Gamma_n = (\mathbf{1}, -\tfrac{1}{n})\,\Gamma_n \in N_{H_n} \tag{5.1}$$

in particular. On the other hand, $N_{H_n} \subseteq q_n(N_S \times \frac{1}{n}\mathbb{Z}) \cong \mathbb{Z}$, whence N_{H_n} is discrete. In fact, since $\Gamma_n \subseteq N_S \times \frac{1}{n}\mathbb{Z}$, the composition

$$\widetilde{\mathrm{SL}}(2,\mathbb{R}) \times \mathbb{R} \to \frac{\widetilde{\mathrm{SL}}(2,\mathbb{R}) \times \mathbb{R}}{N_S \times \frac{1}{n}\mathbb{Z}} \overset{\cong}{\to} \mathrm{SL}(2,\mathbb{R}) \times \frac{\mathbb{R}}{\frac{1}{n}\mathbb{Z}} \hookrightarrow \mathrm{SL}(2,\mathbb{C}) \times \frac{\mathbb{C}}{\frac{1}{n}\mathbb{Z}}$$

factors through H_n, giving rise to a continuous homomorphism from H_n into a complex Lie group with kernel $q_n(N_S \times \frac{1}{n}\mathbb{Z})$.

Step 3. Having chosen any norm $\|.\|'$ on $\mathfrak{sl}(2,\mathbb{R})$ making it a normed Lie algebra, we make $\mathfrak{h}$ a normed Lie algebra via $\|(X,t)\| := \max\{\|X\|', |t|\}$ for $(X,t) \in \mathfrak{h}$. Then the ℓ^∞-direct sum $\mathfrak{g} := \ell^\infty(\mathbb{N}, \mathfrak{h})$ is a Banach-Lie algebra with respect to pointwise operations and the supremum-norm (cf. Theorem III.9).

Step 4. In view of Theorem III.9 and $\delta_n = \infty$ for $n \in \mathbb{N}$, there exists a simply connected Lie group $\widetilde{G}$ with Lie algebra $\mathfrak{g}$, and with Proposition III.12 we obtain a continuous homomorphism $\widetilde{\psi}: \widetilde{G} \to \prod_{n\in\mathbb{N}} H_n$ with $\widetilde{\psi}(\exp X) = (\exp_{H_n}(X_n))_{n\in\mathbb{N}}$ for $X \in \mathfrak{g}$.

Step 5. Since N_S is discrete in S, there exists an identity neighbourhood W in S such that $W^{-1}W \cap N_S = \{\mathbf{1}\}$. For suitable $R > 0$, we may assume that $\exp_S(B_R(0)) \subseteq W$, and that $\exp_S$ is injective on $B_R(0)$.

Step 6. Then $\exp_{H_n}$ is injective on $B_R(0) \times \mathbb{R}$, for every $n \in \mathbb{N}$. In fact, suppose that $X_1, X_2 \in B_R(0) \subseteq \mathfrak{sl}(2,\mathbb{R})$ and $t_1, t_2 \in \mathbb{R}$ such that $\exp_{H_n}(X_1,t_1) = \exp_{H_n}(X_2,t_2)$. Then there is $z \in N_S$ such that $\exp_S(X_1)z = \exp_S(X_2)$ and $t_1 + \varphi_n(z) = t_2$. Thus $\exp_S(X_1)^{-1}\exp_S(X_2) = z \in N_S \cap W^{-1}W = \{\mathbf{1}\}$ and therefore $\exp_S(X_1) = \exp_S(X_2)$ whence $X_1 = X_2$ by injectivity. Since $z = \mathbf{1}$, we also have $t_1 = t_2$.

Step 7. We deduce from Step 6 that $\widetilde{\psi} \circ \exp_{\widetilde{G}}$ is injective on the open ball $B_R(0) \subseteq \mathfrak{g}$ and hence that $\ker \widetilde{\psi}$ is discrete. We define $G := \widetilde{G}/\ker\widetilde{\psi}$ and note that $\widetilde{\psi}$ factors to a continuous injection $\psi: G \to \prod_{n\in\mathbb{N}} H_n$ such that for all projections $p_n: \prod_{m\in\mathbb{N}} H_m \to H_n$ the composition $\pi_n := p_n \circ \psi: G \to H_n$ is a Lie group homomorphism for which $\mathbf{L}(p_n \circ \psi)$ is the point evaluation $e_n: \mathfrak{g} = l^\infty(\mathbb{N}, \mathfrak{h}) \to \mathfrak{h}, (X_m)_{m\in\mathbb{N}} \mapsto X_n$ at n.

Step 8. Let $\gamma_n: \mathfrak{h} \hookrightarrow \mathfrak{g}$ denote the inclusion map with $\mathfrak{e}_m \circ \gamma_n = \delta_{nm}\,\mathrm{id}_{\mathfrak{h}}$. Then the fact that ψ is injective implies that the Lie algebra homomorphism γ_n integrates to a group homomorphism $\varepsilon_n: H_n \to G$ with $p_n \circ \psi \circ \varepsilon_n = \mathrm{id}_{H_n}$ because the corresponding homomorphism $\widetilde{H}_n \to G \to \prod_{m\in\mathbb{N}} H_m$ factors through H_n. We then have $\psi(\varepsilon_n(h))_m = \mathbf{1}$ for $m \neq n$ and $\psi(\varepsilon_n(h))_n = h$.

Step 9. Define $N_G \subseteq G$ as in Section IV. As $\eta_{H_n} \circ \pi_n$ is a smooth homomorphism into a complex Lie group for each n, with kernel $\pi_n^{-1}(N_{H_n})$, we have $N_G \subseteq \big(G \cap \prod_{n\in\mathbb{N}} N_{H_n}\big) =: P$, identifying G with $\operatorname{im}\psi$ now. Here P is totally disconnected, since the continuous homomorphisms $\pi_n|_P^{N_{H_n}}$ into discrete groups separate points on P. Hence N_G is totally disconnected as well. On the other hand, by necessity $\varepsilon_n(N_{H_n}) \subseteq N_G$ for each n, whence

$$\Big\{(h_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} N_{H_n} : h_n = 1 \text{ for almost all } n\Big\} \subseteq N_G. \tag{5.2}$$

Step 10. Note that $\{X \in \mathfrak{g} : \exp_G(\mathbb{R}X) \subseteq N_G\} = \{0\}$, since N_G is totally disconnected. However, N_G is not discrete: for every $0 < \delta < R$, we have $D := \exp_G(B_\delta(0)) \cap N_G \neq \{\mathbf{1}\}$, as $\mathbf{1} \neq (\mathbf{1}, -\frac{1}{n})\Gamma_n \in \pi_n(D)$ by (5.1) and (5.2), where $n \in \mathbb{N}$ is chosen such that $\frac{1}{n} < \delta$. Hence N_G is not a Lie subgroup of G, and we have reached our goal: condition (i) of Theorem IV.6 is not satisfied by G. In particular, G does not have a universal complexification.

Remark V.1. It is interesting to take a closer look at the topology of the groups in the construction above, to understand them in the context of Proposition III.12. Here we can get a quite explicit picture of the simply connected group $\widetilde{G}$. We recall that the group S is homeomorphic to $\mathbb{R}^3$ which can most easily be seen from its polar decomposition $S = K_S \exp \mathfrak{p}_\mathfrak{s}$, where $K_S \cong \widetilde{\mathrm{SO}}(2,\mathbb{R}) \cong \mathbb{R}$ and $\mathfrak{p}_\mathfrak{s} \cong \mathbb{R}^2$. From that it is not hard to derive that the subgroup of $(S \times \mathbb{R})^{\mathbb{N}}$ corresponding to $\ell^\infty(\mathbb{N},\mathfrak{h})$ is homeomorphic to

$$\ell^\infty(\mathbb{N},\mathbb{R}) \times \ell^\infty(\mathbb{N},\mathbb{R}^2) \times \ell^\infty(\mathbb{N},\mathbb{R}),$$

hence in particular simply connected, whence isomorphic to $\widetilde{G}$.

This implies that for the natural map $\widetilde{\psi}\colon \widetilde{G} \to \prod_{n\in\mathbb{N}} H_n$ the kernel is $\widetilde{G} \cap \prod_{n\in\mathbb{N}} \Gamma_n$, and this group is the graph Γ of the homomorphism

$$\varphi\colon \ell^\infty(\mathbb{N},\mathbb{Z}) \to \ell^\infty(\mathbb{N},\mathbb{R}), \quad \varphi(x_n)_{n\in\mathbb{N}} = \Big(\frac{1}{n}x_n\Big)_{n\in\mathbb{N}}.$$

This subgroup is discrete and $G \cong \widetilde{G}/\Gamma$, so that $\pi_1(G) \cong \Gamma$. ∎

VI. Examples of the second kind

In this section, we construct examples of real Banach-Lie groups G which satisfy condition (i) of Theorem IV.6 (as $N_G = \{\mathbf{1}\}$) but not condition (ii).

Lemma VI.1. *Let $\mathfrak{g}_1, \ldots, \mathfrak{g}_n$ be Banach-Lie algebras with centers $\mathfrak{z}_1, \ldots, \mathfrak{z}_n$. Let*

$$\mathfrak{g} := \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_n, \qquad \mathfrak{b} \subseteq \mathfrak{z}(\mathfrak{g}) = \mathfrak{z}_1 \oplus \ldots \oplus \mathfrak{z}_n$$

be a subspace intersecting each $\mathfrak{z}_j$ trivially, and $q\colon \mathfrak{g} \to \mathfrak{g}/\mathfrak{b}$ denote the quotient map. Then

$$\mathfrak{z}(\mathfrak{g}/\mathfrak{b}) = \mathfrak{z}(\mathfrak{g})/\mathfrak{b}, \quad q(\Pi(\mathfrak{g})) = \Pi(\mathfrak{g}/\mathfrak{b}), \quad \text{and} \quad \Pi(\mathfrak{g}) \cong \bigoplus_{j=1}^n \Pi(\mathfrak{g}_j).$$

Proof. It is clear that $\mathfrak{z}(\mathfrak{g})/\mathfrak{b}$ is central in $\mathfrak{g}/\mathfrak{b}$. If, conversely, $q(x)$ is central in $\mathfrak{g}/\mathfrak{b}$, then for each j we have $[x, \mathfrak{g}_j] \subseteq \mathfrak{b} \cap \mathfrak{z}_j = \{0\}$, so that x is central in $\mathfrak{g}$. Therefore $\mathfrak{z}(\mathfrak{g}/\mathfrak{b}) = \mathfrak{z}(\mathfrak{g})/\mathfrak{b} = q(\mathfrak{z}(\mathfrak{g}))$. Now Corollary III.4(b) shows that $q(\Pi(\mathfrak{g})) = \Pi(\mathfrak{g}/\mathfrak{b})$. The relation $\Pi(\mathfrak{g}) \cong \bigoplus_{j=1}^n \Pi(\mathfrak{g}_j)$ follows directly from Theorem III.9. ∎

Lemma VI.2. *Let $\mathfrak{a}$ be a Banach-Lie algebra over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ with $\dim \mathfrak{z}(\mathfrak{a}) = 1$ and period group $\Pi(\mathfrak{a}) = \mathbb{Z}\gamma_0 \cong \mathbb{Z}$. Then each of the Lie algebras*

$$\mathfrak{g}_n(\mathfrak{a}) := (\mathfrak{a} \oplus \mathfrak{a})/\mathbb{K}(\gamma_0, n\gamma_0)$$

is enlargible, but their ℓ^∞-direct sum

$$\mathfrak{g}(\mathfrak{a}) := \bigoplus_{n\in\mathbb{N}}^{\infty} \mathfrak{g}_n(\mathfrak{a}) := \Big\{(x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} \mathfrak{g}_n(\mathfrak{a}) \colon \sup_{n\in\mathbb{N}} \|x_n\| < \infty\Big\}$$

is not enlargible, whereas the homomorphisms to the enlargible Lie algebras $\mathfrak{g}_n$ separate points.

Proof. First we note that (3.3) implies that

$$\Pi(\mathfrak{a} \oplus \mathfrak{a}) = \Pi(\mathfrak{a}) \oplus \Pi(\mathfrak{a}) \cong \mathbb{Z}^2.$$

Now $\mathfrak{b}_n := \mathbb{K}(\gamma_0, n\gamma_0)$ is a one-dimensional central subspace of $\mathfrak{a} \oplus \mathfrak{a}$, so that $\mathfrak{g}_n := \mathfrak{g}_n(\mathfrak{a})$ is a Banach-Lie algebra with $\mathfrak{z}(\mathfrak{g}_n) = \mathfrak{z}(\mathfrak{a} \oplus \mathfrak{a})/\mathfrak{b}_n$ (Lemma VI.1). Since the period group $\Pi(\mathfrak{a})$ is discrete, the Lie algebra $\mathfrak{a}$ is enlargible (Theorem III.7). Moreover,

$$\big(\Pi(\mathfrak{a} \oplus \mathfrak{a}) + \mathfrak{b}_n\big)/\mathfrak{b}_n \cong \Pi(\mathfrak{a} \oplus \mathfrak{a})/\big(\Pi(\mathfrak{a} \oplus \mathfrak{a}) \cap \mathfrak{b}_n\big)$$

is isomorphic to $\mathbb{Z}^2/\mathbb{Z}(1,n) \cong \mathbb{Z}$ and therefore discrete in $\mathfrak{z}(\mathfrak{g}_n) \cong \mathfrak{z}(\mathfrak{a} \oplus \mathfrak{a})/\mathfrak{b}_n$. Hence Lemma VI.1 implies that

$$\Pi(\mathfrak{g}_n) \cong \Pi(\mathfrak{a} \oplus \mathfrak{a})/\mathbb{Z}(\gamma_0, n\gamma_0) \cong \mathbb{Z}^2/\mathbb{Z}(1,n) \cong \mathbb{Z}.$$

We endow the Lie algebra $\mathfrak{a} \oplus \mathfrak{a}$ with the l^∞-norm $\|(x_1, x_2)\| = \max(\|x_1\|, \|x_2\|)$. Then $\mathfrak{z}(\mathfrak{a}\oplus\mathfrak{a}) \cong (\mathbb{K}^2, \|\cdot\|_\infty)$ as a normed space and $\Pi(\mathfrak{a}\oplus\mathfrak{a}) \cong \delta\mathbb{Z}^2$, where $\delta = \min\{\|\gamma\| \colon 0 \neq \gamma \in \Pi(\mathfrak{a})\}$. In

$$\mathfrak{z}(\mathfrak{g}_n) = \mathfrak{z}(\mathfrak{a} \oplus \mathfrak{a})/\mathfrak{b}_n \cong \mathbb{K}^2/\mathbb{K}(1,n)$$

we have with $\overline{x} := q(x)$, $q \colon \mathfrak{a} \oplus \mathfrak{a} \to \mathfrak{g}_n$:

$$\begin{aligned}\|\overline{(1,0)}\| &= \inf\nolimits_{\lambda\in\mathbb{K}} \|(1+\lambda, n\lambda)\|_\infty = \inf\nolimits_{\lambda\in\mathbb{K}} \max(|1+\lambda|, n|\lambda|) \\ &= \inf\nolimits_{\lambda\in[-2,0]} \max(|1+\lambda|, n|\lambda|) = \frac{n}{n+1}.\end{aligned}$$

The elements of the group $\frac{1}{\delta}\Pi(\mathfrak{g}_n)$ correspond to

$$\mathbb{Z}^2/\mathbb{Z}(1,n) = \mathbb{Z}\overline{(1,0)} + \mathbb{Z}\overline{(0,1)} = \mathbb{Z}\overline{(1,0)} + \mathbb{Z}\overline{(\tfrac{1}{n}, 0)} = \mathbb{Z}\overline{(\tfrac{1}{n}, 0)}.$$

This means that

$$\delta_n := \inf\{\|\gamma\| \colon 0 \neq \gamma \in \Pi(\mathfrak{g}_n)\} = \frac{n}{n+1}\delta\tfrac{1}{n} = \frac{\delta}{n+1}.$$

Therefore the Lie algebras $\mathfrak{g}_n$ do not satisfy the assumptions of Theorem III.9 because $\delta_n \to 0$. This means that their l^∞-sum $\mathfrak{g}(\mathfrak{a}) := \bigoplus_{n\in\mathbb{N}}^\infty \mathfrak{g}_n(\mathfrak{a})$ is not enlargible, whereas the continuous Lie algebra homomorphisms to the enlargible Lie algebras $\mathfrak{g}_n$ separate points. ■

Next we construct examples of Banach-Lie algebras $\mathfrak{a}$ with one-dimensional center and period group isomorphic to $\mathbb{Z}$ because these are needed as input for the construction in Lemma VI.2.

Example VI.3. In this remark we discuss Lie algebras $\mathfrak{a}$ satisfying the assumptions of Lemma VI.2. This turns out to be of particular interest for $\Pi(\mathfrak{a}) = \{0\}$ and $\Pi(\mathfrak{a}_\mathbb{C}) \cong \mathbb{Z}$ which is satisfied for the algebras in (b) and (c).

(a) The most prominent example of a Lie algebra $\mathfrak{a}$ with these properties is $\mathfrak{a} = \mathfrak{u}(H)$, where H is an infinite-dimensional complex Hilbert space. In view of Kuiper's Theorem, the unitary group $\mathrm{U}(H)$ of H is contractible, hence in particular a simply connected Banach-Lie group. Its center is isomorphic to $\mathbb{T}$, so that Proposition III.8 implies that $\Pi(\mathfrak{a}) \cong \pi_1(\mathbb{T}) \cong \mathbb{Z}$.

If we replace $\mathfrak{a}$ by its complexification $B(H)$, the Lie algebra of all bounded operators on H, then the polar decomposition of $\mathrm{GL}(H)$ implies that it is also contractible, so that

$$\Pi((\mathfrak{a})_{\mathbb{C}}) \cong \pi_1(Z(\mathrm{GL}(H))) \cong \pi_1(\mathbb{C}^\times) \cong \mathbb{Z}.$$

(b) Next we construct an example of a real Banach-Lie algebra $\mathfrak{a}$ with $\Pi(\mathfrak{a}) = \{0\}$ and $\Pi(\mathfrak{a}_{\mathbb{C}}) \cong \mathbb{Z}$.

Let

$$\mathfrak{a}_0 := \Omega^1(\mathfrak{sl}(2,\mathbb{K})) := \{f \in C^1(\mathbb{T},\mathfrak{sl}(2,\mathbb{K})) \colon f(1) = 0\}.$$

Then $\mathfrak{a}_0$ is a $\mathbb{K}$-Banach-Lie algebra which has a central extension $\mathfrak{a}$ given by the cocycle

$$\omega(f,g) := \int_{\mathbb{T}} \kappa(f(t), g'(t))\, dt,$$

where κ is the Killing form of $\mathfrak{sl}(2,\mathbb{K})$. More precisely, $\mathfrak{a} = \mathfrak{a}_0 \times \mathbb{K}$ with the bracket

$$[(x,z),(x',z')] = ([x,x'],\omega(x,x')).$$

Since $\mathfrak{z}(\mathfrak{a}_0) = \{0\}$, we see that $\mathfrak{z}(\mathfrak{a}) = \{0\} \times \mathbb{K}$ is one-dimensional.

Now

$$A_0 := \Omega^1(\mathrm{SL}(2,\mathbb{K}))_0 \cong \Omega^1(\mathrm{SL}(2,\mathbb{K})^{\sim})$$

is a Banach-Lie group with Lie algebra $\mathfrak{a}_0$. Smoothing of loops easily implies that A_0 is homotopy equivalent to the continuous loop group $\Omega(\mathrm{SL}(2,\mathbb{K})^{\sim})$, so that

$$\pi_2(\Omega^1(\mathrm{SL}(2,\mathbb{K})^{\sim}) \cong \pi_2(\Omega(\mathrm{SL}(2,\mathbb{K})^{\sim}) \cong \pi_2(\Omega(\mathrm{SL}(2,\mathbb{K}))) \cong \pi_3(\mathrm{SL}(2,\mathbb{K})) \cong \begin{cases} \mathbf{0} & \text{for } \mathbb{K} = \mathbb{R} \\ \mathbb{Z} & \text{for } \mathbb{K} = \mathbb{C} \end{cases}$$

because $\mathrm{SL}(2,\mathbb{R})$ is homeomorphic to $\mathbb{T} \times \mathbb{R}^2$, so that $\mathrm{SL}(2,\mathbb{R})^{\sim}$ is homeomorphic to $\mathbb{R}^3$, hence has trivial third homotopy, and

$$\pi_3(\mathrm{SL}(2,\mathbb{C})) \cong \pi_3(\mathrm{SU}(2,\mathbb{C})) \cong \pi_3(\mathbb{S}^3) \cong \mathbb{Z}.$$

Since $\Pi(\mathfrak{a})$ is a homomorphic image of $\pi_2(A_0)$ (Definition III.1), this group vanishes for $\mathbb{K} = \mathbb{R}$. For $\mathbb{K} = \mathbb{C}$ we have $\Pi(\mathfrak{a}) \cong \mathbb{Z}$, as is shown in [EK64, p. 26] and [Ne01a, Th. II.5], because the simply connected group corresponding to the Lie algebra $\mathfrak{a}$ has a center which is not simply connected.

Note that we cannot take C^0 instead of C^1 in the above construction since the Lie algebra $C^0(\mathbb{T},\mathfrak{sl}(2,\mathbb{C}))$ has no non-trivial central extensions.[2]

(c) The following example is simpler and still satisfies $\Pi(\mathfrak{a}) = \{0\}$ and $\Pi(\mathfrak{a}_{\mathbb{C}}) \cong \mathbb{Z}$.

Let H be an infinite-dimensional complex Hilbert space and $H^{\mathbb{R}}$ the underlying real Hilbert space. Let $J \colon H \to H, v \mapsto iv$, denote the complex structure on H and define the symplectic form $\Omega(v,w) := \mathrm{Im}\langle v,w\rangle$. We write $\mathrm{Sp}(H,\Omega)$ for the group of all real linear continuous automorphisms of H preserving the form Ω and consider the subgroup

$$\mathrm{Sp}_{\mathrm{res}}(H,\Omega) := \{g \in \mathrm{Sp}(H,\Omega) \colon \|[g,J]\|_2 < \infty\},$$

called the *restricted symplectic group*. This group has a polar decomposition $K \exp \mathfrak{p}$ with $K \cong \mathrm{U}(H)$ and $\mathfrak{p}$ is the space of antilinear symmetric operators on $H^{\mathbb{R}}$. Kuiper's Theorem implies that $\mathrm{U}(H)$ and hence $\mathrm{Sp}_{\mathrm{res}}(H,\Omega)$ is contractible, hence in particular simply connected. As we have seen in [Ne01b, Sect. IV], the group $\mathrm{Sp}_{\mathrm{res}}(H,\Omega)$ has a universal complexification $\mathrm{Sp}_{\mathrm{res}}(H,\Omega)_{\mathbb{C}} \subseteq \mathrm{GL}(H_{\mathbb{C}})$ which is also simply connected but has a second homotopy group isomorphic to $\mathbb{Z}$.

Let $A := \mathrm{Mp}(H,\Omega)$ denote the *metaplectic group* which is a central $\mathbb{T}$-extension of $\mathrm{Sp}_{\mathrm{res}}(H,\Omega)$ ([Ne01b, Sect. IV]) and the center of its Lie algebra $\mathfrak{a}$ is the Lie algebra of $\mathbb{T}$. Therefore the fact that $\mathrm{Sp}_{\mathrm{res}}(H,\Omega)$ is simply connected implies that $A_{\mathrm{ad}} \cong \mathrm{Sp}_{\mathrm{res}}(H,\Omega)$, and the contractibility of this group further implies that $\Pi(\mathfrak{a}) = \{0\}$. The simply connected group corresponding to the complexification $\mathfrak{a}_{\mathbb{C}}$ is a central $\mathbb{C}^\times$-extension of $\mathrm{Sp}_{\mathrm{res}}(H,\Omega)_{\mathbb{C}}$ ([Ne01b, Sect. IV]). Hence Proposition III.8 implies that $\Pi(\mathfrak{a}_{\mathbb{C}}) \cong \pi_1(\mathbb{C}^\times) \cong \mathbb{Z}$. ■

Now we construct an example of a connected Banach-Lie group G for which the homomorphisms to complex Lie groups separate points, but G has no universal complexification.

[2] This follows from [Ma01 Corollary 12] and the C^0-analogue of [Ma01, Theorem 15].

Example VI.4. In Example VI.3(b),(c) we have seen that there exist real Banach-Lie algebras $\mathfrak{a}$ with $\mathfrak{z}(\mathfrak{a}) \cong \mathbb{R}$, $\Pi(\mathfrak{a}) = \{0\}$ and $\Pi(\mathfrak{a}_{\mathbb{C}}) \cong \mathbb{Z}$. For each Lie algebra $\mathfrak{g}_n(\mathfrak{a})$ we then have

$$\Pi(\mathfrak{g}_n(\mathfrak{a})) = \{0\} \quad \text{and} \quad \Pi(\mathfrak{g}_n(\mathfrak{a}_{\mathbb{C}})) \cong \mathbb{Z},$$

as follows from Lemma VI.1 and the arguments in Lemma VI.2. Therefore Theorem III.9 implies that $\Pi(\mathfrak{g}(\mathfrak{a})) = \{0\}$ and that $\Pi(\mathfrak{g}(\mathfrak{a}_{\mathbb{C}}))$ is not discrete.

For each $n \in \mathbb{N}$ let $G_n(\mathfrak{a}_{\mathbb{C}})$ denote a simply connected Lie group with Lie algebra $\mathfrak{g}_n(\mathfrak{a}_{\mathbb{C}})$ and $G_n(\mathfrak{a})$ the Lie subgroup corresponding to the real form $\mathfrak{g}_n(\mathfrak{a})$ of $\mathfrak{g}_n(\mathfrak{a}_{\mathbb{C}})$. That $G_n(\mathfrak{a})$ is a Lie subgroup follows from the fact that it is the fixed point set of the antiholomorphic involution on $G_n(\mathfrak{a})$ whose derivative is the complex conjugation of $\mathfrak{g}_n(\mathfrak{a}_{\mathbb{C}})$. Now let $G(\mathfrak{a}) \subseteq \prod_{n\in\mathbb{N}} G_n(\mathfrak{a})$ be the analytic subgroup with Lie algebra $\mathfrak{g}(\mathfrak{a})$. Since the $G_n(\mathfrak{a})$ are subgroups of complex groups, the homomorphism of $G(\mathfrak{a})$ to complex groups separate points, i.e., $N = \{\mathbf{1}\}$. Moreover, we have $\mathbf{L}(G(\mathfrak{a}))_{\mathbb{C}} = \mathfrak{g}(\mathfrak{a})_{\mathbb{C}} \cong \mathfrak{g}(\mathfrak{a}_{\mathbb{C}})$, and this Lie algebra is not enlargible. ■

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