Central Extensions of Topological Current Algebras

Peter Maier

Abstract

In this note we describe universal central extensions of certain Fréchet current algebras, which in our context are algebras of type $A \otimes \mathfrak{g}$, where $\mathfrak{g}$ is a finite dimensional semisimple real Lie algebra and A a commutative associative Fréchet algebra.

Introduction

Although in the algebraic setting the problem to determine all central extensions of a given current algebra, i.e., a Lie algebra of type $A \otimes_F \mathfrak{g}$, where F is any field and A is some commutative associative unital F-algebra, is satisfactorily solved for $\mathrm{char}(F) \neq 2$ (see [4] for the case $\mathrm{char}(F) = 0$ and [14] for the general case) not much is known if one deals with central extensions of topological Lie algebras. In this note we construct a universal central extension for Lie algebras of type $\mathfrak{g} \otimes A$, where $\mathfrak{g}$ is a finite dimensional semisimple real Lie algebra and A a commutative associative Fréchet algebra. If A equals the algebra $C^\infty(M)$ of smooth functions on a smooth finite dimensional manifold M, we explicitly describe this universal extension by using the A-module $\Omega^1(M)$ of smooth 1-forms on M, thereby proving that this is the universal differential module for A in the category of Fréchet A-modules. As a consequence of our results, we obtain a generalisation of a theorem due to Pressley and Segal.

Topological Tensor, Alternating, and Symmetric Products

Let E and F be locally convex topological vector spaces, and let $E \otimes F$ denote their algebraic tensor product (if nothing else is specified, tensor products are always taken over the reals). The **projective topology** on $E \otimes F$ is the finest topology for which the map

$$(x, y) \mapsto x \otimes y : E \times F \to E \otimes F$$

is continuous. It has the following universal property:

1 Theorem. *Let E, F, and G be locally convex spaces, and let $\beta : E \times F \to G$ be a continuous bilinear map. Then there exists a unique continuous linear map $\overline{\beta} : E \otimes F \to G$ such that $\beta = \overline{\beta} \circ \otimes$.*

We call the so-defined topological tensor product of two locally convex vector spaces the **projective tensor product**. In the sequel we give another description for the projective topology which shows that this topology, indeed, is a locally convex vector space topology. Moreover, the description below yields that for two metrizable spaces E and F the projective tensor product again is a metrizable space. Let p be a seminorm on E and let q be a seminorm on F. We define the tensor product $p \otimes q$ of these seminorms by the prescription

$$(p \otimes q)(z) := \inf \left\{ \sum p(x_k) q(y_k) \,\middle|\, \sum x_k \otimes y_k = z \right\}.$$

It turns out that this again is a seminorm and moreover, if $(p_i)_{i \in I}$ and $(q_j)_{j \in J}$ are two families of seminorms which define the topologies on E and F, respectively, then $(p_i \otimes q_j)_{(i,j) \in I \times J}$ is a family of seminorms which defines the projective topology on $E \otimes F$ (cf. [6], 15.1). In general the projective tensor product of complete spaces E and F fails to be complete. We write $E\widehat{\otimes}F$ for its completion. Because of the universal property of the completion of a (metrizable) locally convex vector space we immediately obtain the following consequence of Theorem 1.

2 Corollary. *Let E, F, and G be complete, resp. metrizable, resp. completely metrizable, locally convex spaces, and let $\beta : E \times F \to G$ be a continuous bilinear map. Then there exists a unique continuous linear map $\overline{\beta} : E \widehat{\otimes} F \to G$ such that $\beta = \overline{\beta} \circ \otimes$.*

In the sequel the letter $\mathbf{K}$ stands for the category of locally convex vector spaces with continuous linear maps as morphisms or for one of its full subcategories consisting of all complete, resp., metrizable, resp., completely metrizable (=Fréchet), spaces. Furthermore, given two spaces $E, F \in \mathbf{K}$ we write $E \otimes_{\mathbf{K}} F$ for tensor product having the universal property described in Theorem 1, resp., Corollary 2. Note that uniqueness of the map $\overline{\beta}$ arising in Theorem 1 and Corollary 2 implies that the algebraic tensor product $E \otimes F$ always is dense in $E \otimes_{\mathbf{K}} F$. Now let $E = F \in \mathbf{K}$, then the map

$$\sigma : E \times E \to E \otimes_{\mathbf{K}} E : (x, y) \mapsto y \otimes x$$

induces a continuous linear involution

$$\overline{\sigma} : E \otimes_{\mathbf{K}} E \to E \otimes_{\mathbf{K}} E$$

which yields a decomposition

$$E \otimes_{\mathbf{K}} E = S^2_{\mathbf{K}}(E) \oplus \Lambda_{\mathbf{K}}(E),$$

where $S^2_{\mathbf{K}}(E) := \ker(1 - \overline{\sigma})$ and $\Lambda_{\mathbf{K}}(E) := \ker(1 + \overline{\sigma})$. Putting

$$f \vee g := \frac{1}{2}(f \otimes g + g \otimes f) \qquad \text{and} \qquad f \wedge g := \frac{1}{2}(f \otimes g - g \otimes f)$$

we obtain as a consequence of Theorem 1 and Corollary 2 the following result:

3 Theorem. *Let $E, F \in \mathbf{K}$. Then for any continuous symmetric, resp., skew-symmetric bilinear map $\beta : E \times E \to F$ there exists a uniquely determined continuous linear map*

$$\overline{\beta} : S^2_{\mathbf{K}}(E) \to F, \qquad \textit{resp.,} \qquad \overline{\beta} : \Lambda^2_{\mathbf{K}}(E) \to F$$

such that $\beta = \overline{\beta} \circ \vee$, resp., $\beta = \overline{\beta} \circ \wedge$.

Now, we consider a special situation which will be of interest for us later on. Let M be a finite dimensional smooth manifold and E a Fréchet space. We topologize the space $C^\infty(M, E)$ in the following way: For any two topological spaces X and Y we denote by $C(X, Y)_c$ the space $C(X, Y)$ endowed with the topology of compact convergence. We identify the tangent bundle $\mathrm{T}E$ of E with $E \times E$, so that for any smooth map $f : M \to E$ we obtain a smooth map $df : \mathrm{T}M \to E$ by letting $df(v) := \mathrm{pr}_2(\mathrm{T}f(v))$. Inductively, this yields maps $d^n f : \mathrm{T}^n M \to E$ for any $n \in \mathbb{N}_0$ by putting $d^0 f := f$ $(T^0 M := M)$ and $d^n f := d(d^{n-1} f)$ for $n > 0$. Using these maps, we get an injection

$$C^\infty(M, E) \to \prod_{n \in \mathbb{N}_0} C(\mathrm{T}^n M, E)_c : f \mapsto (d^n f)_{n \in \mathbb{N}_0}.$$

We endow $C^\infty(M,E)$ with the topology induced by the product topology via this embedding. Since for each of the spaces $\mathrm{T}^n M$, $n \in \mathbb{N}_0$, the respective topology has a countable basis consisting of relatively compact neighborhoods, the topology of each space $C(\mathrm{T}^n M, E)_c$ can be defined by a countable separating family of seminorms and therefore is locally convex and metrizable. As a subspace of a countable product of locally convex metrizable spaces the space $C^\infty(M,E)$ is locally convex and metrizable as well. In fact, it turns out that its topology even is complete (cf. the proof of Proposition III.1 in [10]) whence $C^\infty(M,E)$ is a Fréchet space. Now we restrict our attention to the special case where $E = \mathbb{R}$ and write $C^\infty(M)$ for $C^\infty(M,\mathbb{R})$. While the isomorphism $C^\infty(M)\,\widehat{\otimes}\,C^\infty(N) \cong C^\infty(M\times N)$ is well-known if M and N are open subsets of some $\mathbb{R}^n$ a proof for the general case in which M and N are smooth finite dimensional manifolds is not easy to find in the literature.

4 Theorem. *Let M and N be smooth finite dimensional manifolds. Then the map*

$$C^\infty(M)\,\widehat{\otimes}\,C^\infty(N) \to C^\infty(M\times N) : f\otimes g \mapsto \big((p,q)\mapsto f(p)g(q)\big)$$

is an isomorphism of Fréchet spaces.

Proof. We first recall some facts. Let X, Y, and Z be Hausdorff topological spaces. For $f \in C(X\times Y, Z)$ and $x\in X$ we put $f_x := \big(y\mapsto f(x,y)\big) \in C(Y,Z)$. It is well-known that the map

$$\alpha : C(X\times Y, Z)_c \to C(X, C(Y,Z)_c)_c : f \mapsto (x\mapsto f_x)$$

is a homeomorphism if Y is locally compact, and since α, obviously, is natural in X and Z, we obtain that $C(Y,\cdot)_c$ is a right adjoint self functor of the category of Hausdorff topological spaces and thus preserves limits. For the remaining proof we note that, according to [10], Theorem III.4, the image of the map $\alpha|_{C^\infty(M\times N)}$ is contained in $C^\infty(M, C^\infty(N))$, and that for any Fréchet space E the map

$$C^\infty(M)\,\widehat{\otimes}\,E \to C^\infty(M,E) : f\otimes x \mapsto fx$$

is an isomorphism of Fréchet spaces (cf. [3], Chapter II, p. 81). Hence we are done, if we can show that the map

$$\beta : C^\infty(M\times N) \to C^\infty(M, C^\infty(N)) : f \mapsto \alpha(f)$$

is an isomorphism of Fréchet spaces. Thanks to the Open Mapping Theorem for Fréchet spaces it suffices to show that β is a continuous linear bijection. Clearly, β is injective. For the prove of its surjectivity we have to show that for $g \in C^\infty(M, C^\infty(N))$ we have $\alpha^{-1}(g) \in C^\infty(M\times N)$, i.e., that $\alpha^{-1}(g)$ is smooth at any point. Since the latter is a local property we can assume M and N to be open subsets of some $\mathbb{R}^n$; but in this case the assertion is already proved, see [13], Theorem 40.1. It remains to show continuity of β. By definition of the topology of $C^\infty(M, C^\infty(N))$ the map β is continuous exactly if for any $m \in \mathbb{N}_0$ the map

$$\beta_m : C^\infty(M\times N) \to C(\mathrm{T}^m M, C^\infty(N))_c : f \mapsto d^m\beta(f)$$

is continuous. Since $C^\infty(N)$ is embedded into the product $\prod_{n\in\mathbb{N}_0} C(\mathrm{T}^n N)_c$, and since $C(\mathrm{T}^m M,\cdot)$ preserves limits, the map β_m is continuous exactly if for any $n\in\mathbb{N}_0$ the map

$$\beta_{mn} : C^\infty(M\times N) \to C(\mathrm{T}^m M, C(\mathrm{T}^n N)_c)_c : f \mapsto d^n \circ \beta_m(f)$$

is continuous. In view of the isomorphism $C(\mathrm{T}^m M, C(\mathrm{T}^n N)_c)_c \cong C(\mathrm{T}^m M \times \mathrm{T}^n N)_c$ continuity of β_{mn} is equivalent to continuity of the map

$$C^\infty(M \times N) \to C(\mathrm{T}^m M \times \mathrm{T}^n N)_c : f \mapsto d_1^m d_2^n f$$

where d_1 and d_2 denote the respective "partial derivatives". But the latter is clearly fulfilled, since f is smooth. □

Universal Differential Modules

In this section we point out that the concept of a universal differential module for a commutative associative algebra, which is well-known in the algebraic setting, not only makes sense, but even is a very useful tool, in a categorial framework. By an algebra object in the category $\mathbf{K}$, or simply a $\mathbf{K}$-algebra, we mean an object $A \in \mathbf{K}$ together with a morphism

$$\mu : A \otimes_{\mathbf{K}} A \to A$$

called multiplication. Suppose for the rest of this note that A is a unital $\mathbf{K}$-algebra with *commutative and associative* multiplication. An A-module in the category $\mathbf{K}$ is an object M together with a morphism

$$\nu : A \otimes_{\mathbf{K}} M \to M$$

that satisfies $\nu \circ (\mathrm{id}_A \otimes \nu) = \nu \circ (\mu \otimes \mathrm{id}_M)$ and $\nu(1 \otimes m) = m$ for each $m \in M$. A **derivation** from such an algebra A into an A-module M is defined to be a linear map $D : A \to M$ satisfying

$$D(ab) = aDb + bDa$$

for all $a, b \in A$. The embedding

$$a \mapsto a \otimes 1 : A \to A \otimes_{\mathbf{K}} A,$$

turns the $\mathbf{K}$-algebra $A \otimes_{\mathbf{K}} A$ into an A-module with respect to the multiplication map on $A \otimes_{\mathbf{K}} A$, and in view of this module structure the map $\overline{\mu}$ also is a morphism of A-modules. Consequently, its kernel I is an A-submodule of $A \otimes_{\mathbf{K}} A$.

5 Lemma. *Let $J := I \cap (A \otimes A)$. Then we have $J = \mathrm{span}_A\{1 \otimes b - b \otimes 1 \mid b \in A\}$ and $I = \overline{J}$.*

Proof. Obviously, we have $J' := \mathrm{span}_A\{1 \otimes b - b \otimes 1 \mid b \in A\} \subseteq J$. In order to show the reverse inclusion consider $c = \sum a_k \otimes b_k \in J$, that is, $\sum a_k b_k = 0$. Then we have

$$c = \sum a_k \otimes b_k - \Big(\sum a_k b_k\Big) \otimes 1 = \sum a_k (1 \otimes b_k - b_k \otimes 1) \in J',$$

and the first claim follows. To prove the second claim we note that the map

$$c \mapsto \big(c - \mu(c) \otimes 1,\, \mu(c)\big) : A \otimes_{\mathbf{K}} A \to I \oplus A$$

is an isomorphism of $\mathbf{K}$-A-modules whose inverse is given by

$$(b, a) \mapsto b + a \otimes 1 : I \oplus A \to A \otimes_{\mathbf{K}} A.$$

As a consequence of this, the map

$$\lambda : A \otimes_{\mathbf{K}} A \to I : c \mapsto c - \mu(c) \otimes 1$$

is a surjective morphism of $\mathbf{K}$-A-modules satisfying $\lambda(A \otimes A) = J$, by what we have just shown. As $A \otimes A$ is dense in $A \otimes_{\mathbf{K}} A$, this implies the second claim. □

Now we put $\Omega_{\mathbf{K}}(A) := I/\overline{I^2}$ and define a continuous linear map $d_A : A \to \Omega_{\mathbf{K}}(A)$ by the prescription

$$d_A(a) := [1 \otimes a - a \otimes 1],$$

where $[c]$ denotes the class of an element $c \in I$ in $\Omega_{\mathbf{K}}(A)$. Since we have

$$\begin{aligned} d_A(ab) - a d_A(b) - b d_A(a) &= [1 \otimes ab - ab \otimes 1] - [a \otimes b - ab \otimes 1] - [b \otimes a - ba \otimes 1] \\ &= [1 \otimes ab - a \otimes b - b \otimes a + ab \otimes 1] \\ &= [(1 \otimes a - a \otimes 1)(1 \otimes b - b \otimes 1)] \\ &= 0 \end{aligned}$$

for all $a, b \in A$, we see that d_A in fact is a derivation. We call the pair $(\Omega_{\mathbf{K}}(A), d_A)$ the **K-universal differential module** of the algebra A. It has the following universal property:

6 Theorem. *Let F be a $\mathbf{K}$-A-module and let $D : A \to F$ be a continuous derivation. Then there exists a unique continuous A-linear map $\overline{D} : \Omega_{\mathbf{K}}(A) \to F$ such that $D = \overline{D} \circ d_A$.*

PROOF. In order to prove the existence of the map $\overline{D}$, we consider the continuous bilinear map

$$\Delta : A \times A \to F : (a, b) \mapsto aDb$$

which induces a continuous linear map $\overline{\Delta} : A \otimes_{\mathbf{K}} A \to F$ satisfying

$$\overline{\Delta}(a \otimes b) = aDb$$

for all $a, b \in A$. As is easy to verify, this map fulfils the identity

$$\overline{\Delta}(cc') = \mu(c)\overline{\Delta}(c') + \mu(c')\overline{\Delta}(c) \tag{1}$$

for all $c, c' \in A \otimes A$, and hence for all $c, c' \in A \otimes_{\mathbf{K}} A$, because of the density of $A \otimes A$ in $A \otimes_{\mathbf{K}} A$. Equation (1) shows that $\overline{\Delta}$ vanishes on I^2 and thus on $\overline{I^2}$. Hence, the restriction $\overline{\Delta}|_I$ factors to a map

$$\overline{D} : \Omega_{\mathbf{K}}(A) \to F$$

for which we have

$$(\overline{D} \circ d_A)(a) = \overline{D}([1 \otimes a - a \otimes 1]) = 1Da - aD1 = Da,$$

as desired.

Uniqueness of the map $\overline{D}$ follows from the fact that, according to Lemma 5, the image of d_A generates a dense A-submodule of $\Omega_{\mathbf{K}}(A)$. □

Now we consider a special situation. Let M be a finite dimensional smooth manifold, $A := C^\infty(M)$, resp., $B := C^\infty(M \times M)$ the algebra of smooth functions on M, resp., $M \times M$, and let A_c and B_c the respective subalgebras consisting of compactly supported functions. Then A and B are Fréchet algebras whereas A_c and B_c in general are just locally convex algebras. Denoting the category of Fréchet spaces by $\mathbf{F}$, we seek for a convenient description of the universal module $\Omega_{\mathbf{F}}(A)$. In the sequel we view the Fréchet algebra B as a Fréchet A-module with respect to the embedding

$$A \to B : f \mapsto \big((p,q) \mapsto f(p)\big).$$

Further, we consider the following morphisms of Fréchet A-modules:

$$\delta^* : B \to A : F \mapsto F \circ \delta,$$

where δ is the diagonal map $p \mapsto (p,p) : M \to M \times M$, and

$$\overline{\theta} : A \otimes_{\mathbf{F}} A \to B$$

which is induced by the continuous bilinear map

$$\theta : A \times A \to B : (f,g) \mapsto \big((p,q) \mapsto f(p)g(q)\big).$$

Both maps δ^* and $\overline{\theta}$ are also morphisms of the underlying Fréchet algebras and moreover, $\overline{\theta}$ is a homeomorphism according to Theorem 4. Denoting by $\mu : A \otimes_{\mathbf{F}} A \to A$ the multiplication map on A, we have

$$\mu = \delta^* \circ \overline{\theta}.$$

From this relation we immediately infer $K := \ker(\delta^*) = \overline{\theta}(I)$ and therefore obtain the following isomorphism of Fréchet A-modules:

$$\Omega_{\mathbf{F}}(A) \cong K/\overline{K^2}.$$

In the sequel we think of $\Omega_{\mathbf{F}}(A)$ as $K/\overline{K^2}$ with respect to this isomorphism. Likewise, we identify $A \otimes_{\mathbf{F}} A$ with B via $\overline{\theta}$. Now let $\mathrm{T}M$ be the tangent bundle of M. Then the space $C^\infty(\mathrm{T}M)$ is a Fréchet A-module in which the space

$$\Omega^1(M) := \{\alpha \in C^\infty(\mathrm{T}M) \mid (\forall p \in M)\, \alpha|_{\mathrm{T}_pM} \text{ is linear}\}$$

of smooth 1-forms on M is a closed A-submodule and therefore is a Fréchet A-module as well. We define the **support** of a 1-form $\alpha \in \Omega^1(M)$ to be the set

$$\mathrm{supp}'(\alpha) := \overline{\{p \in M \mid \alpha|_{\mathrm{T}_pM} \neq 0\}} \subseteq M$$

and denote the space of compactly supported 1-forms on M by $\Omega^1_c(M)$. This space is a locally convex A_c-module as well as a locally convex A-module, and it is dense in $\Omega^1(M)$ since the identity element in A is a limit of elements in A_c. We want to show that $\Omega^1(M)$ and $\Omega_{\mathbf{F}}(A)$ are isomorphic Fréchet A-modules, and in order to do this, we first collect some information on $\Omega^1_c(M)$.

We put $K_c := \ker(\delta^*|_{B_c}) = K \cap B_c$ and consider the continuous linear map $\tau : B \to \Omega^1(M)$ defined by

$$\tau(F)(x,X) := dF(x,x)(0,X).$$

7 Proposition. *The kernel of* $\tau|_{K_c}$ *equals* K_c^2 *and the kernel of* $\tau|_K$ *equals* $\overline{K^2}$.

Proof. We have

$$\tau(FG) = \delta^*(F)\tau(G) + \delta^*(G)\tau(F) \tag{2}$$

for all $F, G \in B$ (which is easily verified for $F, G \in A \otimes A$ and then follows by density). Hence, we have $K_c^2 \subseteq \ker(\tau|_{K_c})$ and $\overline{K^2} \subseteq \ker(\tau|_K)$. Moreover, equation (2) shows that $\ker(\tau|_{K_c})$ and $\ker(\tau|_K)$ are ideals in B. For the rest of this proof let $(U_k)_{k\in\mathbb{N}}$ be a locally finite open covering of $M \times M$ consisting of relatively compact neighborhoods which are diffeomorphic to open convex neighborhoods in $\mathbb{R}^n$ and let $(\varphi_k)_{k\in\mathbb{N}}$ be a partition of unity subordinate to this covering. Now let $F \in B_c$. Since $\mathrm{supp}(F)$ is compact there exists some $l \in \mathbb{N}$ such that

$$\mathrm{supp}(F) \subseteq \bigcup_{k=1}^{l} \mathrm{supp}(\varphi_k).$$

Putting

$$F_k := \frac{\varphi_k}{\sum_{k=1}^{l} \varphi_k} F$$

for $k \leq l$ we have $F = F_1 + \cdots + F_l$. Since $\ker(\tau|_{K_c})$ is an ideal in B_c it follows that

$$F \in \ker(\tau|_{K_c}) \qquad \Longleftrightarrow \qquad (\forall k \leq l)\; F_k \in \ker(\tau|_{K_c}),$$

and so the problem is reduced to the case $M = \mathbb{R}^n$.

In order to prove the desired inclusion for this case, we define for each pair $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$ functions $g_{(x,y)}, h_{(x,y)} : \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}^n$ by

$$g_{(x,y)}(t) := \big(tx + (1-t)y, y\big) \qquad \text{and} \qquad h_{(x,y)}(t) := \big(x, tx + (1-t)y\big).$$

Now we have

$$\begin{aligned} 2F(x,y) &= F(x,y) - F(y,y) + F(x,y) - F(x,x) \\ &= \textstyle\int_0^1 (F \circ g_{(x,y)})'(t)dt - \int_0^1 (F \circ h_{(x,y)})'(t)dt \end{aligned}$$

and thus

$$F(x,y) = \sum_{k=1}^{n} (x_k - y_k) F_k(x,y), \tag{3}$$

where

$$F_k(x,y) := \frac{1}{2} \int_0^1 \left(\frac{\partial F}{\partial x_k}(g_{(x,y)}(t)) - \frac{\partial F}{\partial y_k}(h_{(x,y)}(t)) \right) dt$$

and $x_1, \ldots, x_n, y_1, \ldots, y_n$ denote the coordinate functions on $M \times M$. Applying τ to equation (3) leads to the 1-form

$$\tau(F) = \left(x \mapsto -\sum_{k=1}^{n} F_k(x,x) dx_k \right)$$

on $\mathbb{R}^n$ and we see that vanishing of τ on F implies that any F_k vanishes on the diagonal and so $F \in K$. We claim that each F_k has compact support, but this easily follows from the definition, since $\mathrm{supp}(F) \subseteq [-a,a]^{2n}$ for $a \in \mathbb{R}$ implies

$$\frac{\partial F}{\partial x_k} \circ g_{(x,y)} = \frac{\partial F}{\partial y_k} \circ h_{(x,y)} = 0$$

for $(x,y) \in \mathbb{R}^{2n} \setminus [-a,a]^{2n}$. Replacing the coordinate functions by functions $\overline{x}_1, \ldots, \overline{y}_n$ on $\mathbb{R}^n$ which coincide with the coordinate functions on $[-a,a]^{2n}$ and vanish outside some compact neighborhood of $[-a,a]^{2n}$ shows that $F \in K_c^2$.

Now let $F \in B$. Similar as in the previous case we put

$$F_l := \frac{\sum_{k=1}^{l} \varphi_k}{\sum_{k=1}^{\infty} \varphi_k} F$$

and obtain a sequence $(F_l)_{l\in\mathbb{N}}$ in B_c which converges to F in B (because of the local finiteness of the covering $(U_k)_{k\in\mathbb{N}}$). Now we have

$$F \in \ker(\tau|_K) \qquad \Longleftrightarrow \qquad (\forall l \in \mathbb{N})\ F_l \in \ker(\tau|_{K_c}),$$

since $\ker(\tau|_K)$ is an ideal in B and $B_c K \subseteq K_c$. But since we already know that $\ker(\tau|_{K_c}) = K_c^2$ this implies $\ker(\tau|_K) \subseteq \overline{K_c^2} \subseteq \overline{K^2}$, as desired. □

8 Proposition. *We have $\tau(B_c) = \tau(A_c \otimes A_c) = \Omega_c^1(M)$ and $\tau(B) = \Omega^1(M)$.*

Proof. Clearly, we have $\tau(A_c \otimes A_c) \subseteq \tau(B_c)$ and so it suffices to show $\tau(A_c \otimes A_c) = \Omega_c^1(M)$. So let $\alpha \in \Omega_c^1(M)$. First we consider the case that M is diffeomorphic to an open convex subset U of $\mathbb{R}^n$, where $n := \dim M$. Then we have in local coordinates

$$\alpha(p) = \sum_{k=1}^{n} f_k(p) dx_k.$$

Choosing functions $\overline{x}_1, \ldots, \overline{x}_n \in C^\infty(\mathbb{R}^n)$ with compact support in U which coincide with the coordinate functions on $\mathrm{supp}'(\alpha)$, we obtain

$$\alpha(p) = \sum_{k=1}^{n} f_k(p) d\overline{x}_k = \tau\left(\sum_{k=1}^{n} f_k(p) \otimes \overline{x}_k\right)$$

and see that $\Omega_c^1(M) = \tau(A_c \otimes A_c)$ in this case. Now let M be any finite dimensional manifold. By choosing a suitable partition of unity we get a decomposition $\alpha = \alpha_1 + \cdots + \alpha_n$ where each of the sets $\mathrm{supp}'(\alpha_k)$ is contained in some neighborhood U_k which is diffeomorphic to an open convex neighborhood in $\mathbb{R}^n$. Now $\tau(A_c \otimes A_c) = \Omega_c^1(M)$ follows by what we have just proved.

In order to prove $\tau(B) = \Omega^1(M)$ we choose a locally finite open covering $(U_k)_{k\in\mathbb{N}}$ of M consisting of relatively compact neighborhoods which are diffeomorphic to open convex neighborhoods in $\mathbb{R}^n$. Furthermore, we choose a partition of unity $(\varphi_k)_{k\in\mathbb{N}}$ subordinate to this covering. Now let $\alpha \in \Omega^1(M)$ and put $\alpha_k := \varphi_k \alpha \in \Omega_c^1(M)$ for each k. Then we have $\alpha = \sum_{k\in\mathbb{N}} \alpha_k$ in $\Omega^1(M)$. At the beginning of the proof we have seen that for any of these 1-forms α_k we find a function $F_k \in \Omega_c^1(M)$ with $\tau(F_k) = \alpha_k$ and $\mathrm{supp}(F_k) \subseteq U_k \times U_k$. We put $G_k := F_1 + \cdots + F_k$ for $k \in \mathbb{N}$ and note that the sequence $(G_k)_{k\in\mathbb{N}}$ converges to some G in B because of the local finiteness of $(U_k)_{k\in\mathbb{N}}$. Continuity of τ now yields $\tau(G) = \alpha$, and we are done. □

9 Theorem. *The map* $\overline{d} : \Omega_{\mathbf{F}}(A) \to \Omega^1(M)$ *induced by the differential* $d : A \to \Omega^1(M)$ *is an isomorphism of Fréchet* A*-modules.*

PROOF. Thanks to the Open Mapping Theorem for Fréchet spaces it suffices to show bijectivity of the map $\overline{d}$. With respect to the identification $\Omega_{\mathbf{F}}(A) = K/\overline{K^2}$ the injectivity of τ is equivalent to the equality $\ker(\tau|_K) = \overline{K^2}$ and thus is an immediate consequence of Proposition 7. In order to show surjectivity, we have to show that $\tau(K) = \Omega^1(M)$. From Proposition 8 we know that $\tau(B) = \Omega^1(M)$. But this implies $\tau(K) = \Omega^1(M)$ since we have $\tau(F - \delta^*(F) \otimes 1) = \tau(F)$ and $F - \delta^*(F) \otimes 1 \in K$ for any $F \in B$. □

10 Remark. In fact, Theorem 9 seems to be well-known (although unproved in full strength, as far as the author knows) if M is compact (cf. [2]).

The Continuous Case

An opposite to the smooth situation we are concerned with in the preceding discussion is the continuous case. Given a compact topological space X, one may ask for a universal differential module for the Banach algebra $A := C(X)$ in the category of Banach A-modules. Indeed, such an object exists, and can be obtained by our general construction described in the previous section. Surprisingly, this construction always leads to the trivial module, as we shall see in the sequel. In order to show this, we introduce the notion of an amenable Banach algebra. Let A be a Banach algebra. For any Banach A-bimodule M the dual Banach space M' also carries the structure of an A-bimodule via

$$(af)(x) := f(xa) \qquad \text{and} \qquad (fa)(x) := f(ax)$$

for $a \in A$, $x \in M$, and $f \in M'$. We call A **amenable**, if for any A-bimodule M and any continuous derivation $\delta : A \to M'$ there exists $f \in M'$ such that

$$\delta(a) = af - fa$$

for all $a \in A$. Such derivations of A are called **inner M'-derivations**. In [1] the following is shown:

11 Theorem. *If* X *is a compact topological space, then* $C(X)$ *is an amenable Banach algebra and furthermore, for any* $C(X)$*-bimodule* M *each* M*-derivation of* $C(X)$ *is inner.*

PROOF. See [1], Theorem VI.12, and Proposition VI.14. □

As an immediate consequence of Theorem 11 we now obtain the following result:

12 Corollary. *If* X *is a compact topological space, then the universal differential modul for the Banach algebra* $C(X)$ *in category of all Banach spaces is trivial.*

PROOF. Let $A := C(X)$ and let $\Omega^1_{\mathbf{B}}(A)$ denote the universal differential modul for A in category of Banach spaces. We define on $\Omega^1_{\mathbf{B}}(A)$ an A-bimodule structure by $\omega a := a\omega$. Theorem 11 now implies that each $\Omega^1_{\mathbf{B}}(A)$-derivation is inner, but with respect to the above defined bimodule structure on $\Omega^1_{\mathbf{B}}(A)$, any such inner derivation obviously is trivial, which implies that $\Omega^1_{\mathbf{B}}(A)$ itself is trivial. □

Finally we note that the situation even changes if we consider a compact C^1-manifold M, since in this case the differential $d : M \to \Omega_0^1(M)$ (where $\Omega_0^1(M)$ denotes the Banach space of continuous 1-forms on M) is a non-trivial derivation, and thus the respective universal differential module has to be non-trivial.

Central Extensions of Fréchet Current Algebras

In this section we investigate central extensions of Fréchet current algebras, which are Lie algebras of type $A \otimes_{\mathbf{F}} \mathfrak{g}$, where $\mathbf{F}$ denotes the category of Fréchet spaces and $\mathfrak{g}$ is some Fréchet–Lie algebra. We are only interested in extensions that are described by continuous Lie algebra cocycles. Given a Fréchet–Lie algebra $\mathfrak{g}$, an abelian Fréchet–Lie algebra $\mathfrak{z}$, and a continuous 2-cocycle $\omega : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{z}$, we write $\mathfrak{g} \oplus_\omega \mathfrak{z}$ for the Lie algebra $\mathfrak{g} \times \mathfrak{z}$ with the bracket

$$[(x, a), (y, b)] := \big([x, y], \omega(x, y)\big).$$

These extensions are exactly those which are given by an exact sequence

$$0 \longrightarrow \mathfrak{z} \xrightarrow{\iota} \mathfrak{h} \xrightarrow{\pi} \mathfrak{g} \longrightarrow 0$$

of Fréchet–Lie algebras in which the map π admits a continuous linear section. Such an extension is called **weakly universal** if for any other central extension $\mathfrak{g} \oplus_{\omega'} \mathfrak{z}'$ there exists a morphism of Fréchet–Lie algebras $\varphi : \mathfrak{z} \to \mathfrak{z}'$ such that $\omega' = \varphi \circ \omega$, it is called **universal** if the morphism φ is unique. In any of these cases ω is called a **universal cocycle**. We note that a weakly universal extension $\mathfrak{g} \oplus_\omega \mathfrak{z}$ is universal if $\mathfrak{g}$ is perfect (cf. [7], 1.9, Proposition 1).

Now we consider the following situation: Given a finite-dimensional Lie algebra $\mathfrak{g}$, then the space $\mathfrak{g}_A := A \otimes \mathfrak{g}$ is a Fréchet–Lie algebra with respect to the bracket defined by

$$[a \otimes x, b \otimes y] := ab \otimes [x, y].$$

Since it is skew-symmetric this bracket induces a continuous linear map

$$\beta : \Lambda^2_{\mathbf{F}}(\mathfrak{g}_A) \to \mathfrak{g}_A$$

which, because of the Jacobian identity, factors to a map

$$\overline{\beta} : \Lambda^2_{\mathbf{F}}(\mathfrak{g}_A)/B_2^{\mathbf{F}}(\mathfrak{g}_A) \to \mathfrak{g}_A,$$

where $B_2^{\mathbf{F}}(\mathfrak{g}_A)$ denotes the closure of the span of all elements of the form

$$x \wedge [y, z] + y \wedge [z, x] + z \wedge [x, y]$$

in $\Lambda^2_{\mathbf{F}}(\mathfrak{g}_A)$. Writing $\widetilde{\mathfrak{g}}_A := \Lambda^2_{\mathbf{F}}(\mathfrak{g}_A)/B_2^{\mathbf{F}}(\mathfrak{g}_A)$ and $[x] := x + B_2^{\mathbf{F}}(\mathfrak{g}_A)$ for $x \in \Lambda^2_{\mathbf{F}}(\mathfrak{g}_A)$ the prescription

$$\big[[x], [y]\big] := [x] \wedge [y]$$

defines a continuous Lie bracket on the space $\widetilde{\mathfrak{g}}_A$. Denoting the kernel of the map $\overline{\beta}$ by $Z_2^{\mathbf{F}}(\mathfrak{g}_A)$ we have the following result (cf. [11]):

13 Theorem. *The Fréchet–Lie algebra* $\mathfrak{g}_A$ *possesses a weakly universal central extension if and only if the space* $Z_2^{\mathbf{F}}(\mathfrak{g}_A)$ *has a closed vector space complement in* $\Lambda^2_{\mathbf{F}}(\mathfrak{g}_A)$*. In this case a weakly universal extension is given by the Fréchet–Lie algebra* $\widetilde{\mathfrak{g}}_A$*, and a universal cocycle is given by*

$$\omega_A : \mathfrak{g}_A \times \mathfrak{g}_A \to H_2^{\mathbf{F}}(\mathfrak{g}_A) : (x,y) \mapsto [x \wedge y],$$

where $H_2^{\mathbf{F}}(\mathfrak{g}_A) := Z_2^{\mathbf{F}}(\mathfrak{g}_A)/B_2^{\mathbf{F}}(\mathfrak{g}_A)$*.*

In case $\mathfrak{g}$ is semisimple there is a more explicit way of describing this universal central extension of $\mathfrak{g}_A$. It goes as follows: Consider the action of $\mathfrak{g}$ on $S^2(\mathfrak{g})$ given by

$$x(y \vee z) := [x,y] \vee z + y \vee [x,z],$$

put $V(\mathfrak{g}) := S^2(\mathfrak{g})/\mathfrak{g}S^2(\mathfrak{g})$, and define a symmetric bilinear map $\kappa : \mathfrak{g} \times \mathfrak{g} \to V(\mathfrak{g})$ by

$$\kappa(x,y) := [x \vee y],$$

where $[z]$ denotes the class of an element $z \in S^2(\mathfrak{g})$ in $V(\mathfrak{g})$. Since we have

$$\kappa([x,y],z) + \kappa(y,[x,z]) = \big[[x,y] \vee z + y \vee [x,z]\big] = [x(y \vee z)] = 0.$$

for all $x,y,z \in \mathfrak{g}$, this map is invariant. Furthermore, it has the following universal property:

14 Lemma. *Let* E *be a Fréchet space and* $\beta : \mathfrak{g} \times \mathfrak{g} \to E$ *a continuous invariant symmetric bilinear map. Then there exists a unique (continuous) linear map* $\overline{\beta} : V(\mathfrak{g}) \to E$ *such that* $\beta = \overline{\beta} \circ \kappa$*.*

Proof. Uniqueness of $\overline{\beta}$ is clear. For the proof of the existence we note that, because of the symmetry of β, the universal property of $S^2(\mathfrak{g})$ yields a linear map $\widetilde{\beta} : S^2(\mathfrak{g}) \to E$ with $\widetilde{\beta}(x \vee y) = \beta(x,y)$. The invariance of β then implies that $\mathfrak{g}S^2(\mathfrak{g})$ is contained in the kernel of $\widetilde{\beta}$, whence $\widetilde{\beta}$ factors to the desired map $\overline{\beta} : V(\mathfrak{g}) \to E$. □

Now we put $\mathfrak{z}_A := V(\mathfrak{g}) \otimes (\Omega_{\mathbf{F}}(A)/\overline{d_A A})$ and define a map $\omega_A : \mathfrak{g}_A \times \mathfrak{g}_A \to \mathfrak{z}_A$ by

$$\omega_A(f \otimes x, g \otimes y) := \kappa(x,y) \otimes [f d_A(g)],$$

where $[\alpha]$ denotes the class of $\alpha \in \Omega_{\mathbf{F}}(A)$ in $\Omega_{\mathbf{F}}(A)/\overline{d_A A}$. Taking into account the invariance of κ, this map is easily verified to be a continuous 2-cocycle on $\mathfrak{g}_A$ and hence defines a central extension of $\mathfrak{g}_A$. For this central extension we have the following result:

15 Theorem. *If the Lie algebra* $\mathfrak{g}$ *is semisimple, then the Lie algebra* $\widetilde{\mathfrak{g}}_A := \mathfrak{g}_A \oplus_{\omega_A} \mathfrak{z}_A$ *is a universal central extension of* $\mathfrak{g}_A$*.*

Before we prove this result, we state two preparatory lemmas. For a finite dimensional Lie group G and a locally convex G-module E we put

$$E_{\text{fix}} := \{v \in E \,|\, Gv = \{v\}\}, \qquad E_{\text{fin}} := \{v \in E \,|\, \dim \operatorname{span}(Gv) < \infty\},$$

and

$$E_{\text{Eff}} := \overline{\operatorname{span}\{gv - v \,|\, g \in G, v \in E\}}.$$

The elements of E_{fin} are called **G-finite**. Furthermore, we write $E_{\mathbb{C}}$ for the complexification $\mathbb{C} \otimes E$ of E and $G_{\mathbb{C}}$ for the complexification of G.

16 Lemma. *Let G be a connected finite dimensional semisimple Lie group, $\mathfrak{g}$ its Lie algebra, and let E be a complete locally convex G-module whose complexification is a holomorphic $G_{\mathbb{C}}$-module. Then for any closed G-submodule $F \subseteq E$ satisfying $\mathfrak{g}v \subseteq F$ for each $v \in E_{\text{fin}}$ we have $E = E_{\text{fix}} + F$.*

Proof. Let $v \in (E_{\mathbb{C}})_{\text{fin}}$ and $x \in \mathfrak{g}_{\mathbb{C}}$. According to the assumption, there exists a finite dimensional complex subspace $F' \subseteq F_{\mathbb{C}}$ containing xv and thus containing $e^x v - v \in (E_{\mathbb{C}})_{\text{fin}}$. Connectivity of G now implies $gv - v \in (E_{\mathbb{C}})_{\text{fin}}$ for any $g \in G_{\mathbb{C}}$. Let K be a compact real form of $G_{\mathbb{C}}$. Since the $G_{\mathbb{C}}$-finite elements in $E_{\mathbb{C}}$ are exactly the K-finite elements, the Big Theorem of Peter and Weyl (cf. [5], Theorem 3.51) applies and yields that the $G_{\mathbb{C}}$-finite elements are dense in $E_{\mathbb{C}}$. Therefore we obtain $(E_{\mathbb{C}})_{\text{Eff}} \subseteq F_{\mathbb{C}}$ and thus $E_{\mathbb{C}} = (E_{\mathbb{C}})_{\text{fix}} + F_{\mathbb{C}}$ in view of [5], Theorem 3.36. Now the assertion follows, since we have $(E_{\mathbb{C}})_{\text{fix}} = (E_{\text{fix}})_{\mathbb{C}}$. □

17 Lemma. *Let G be a Lie group that acts continuously on a finite dimensional vector space V. Then the induced action of G on the Fréchet A-module $A \otimes V$ also is continuous.*

Proof. Choosing a basis $v_1, \ldots, v_n$, we obtain an isomorphism $V \cong \mathbb{R}^n$ as well as an isomorphism $A \otimes V \cong A^n$ of Fréchet A-modules. With respect to these identifications, the action of G on $\mathbb{R}^n$ is given by a continuous morphism $\pi : G \to \mathrm{GL}(n, \mathbb{R})$ and the induced A-linear action on A^n is given by a morphism $\rho : G \to \mathrm{GL}(n, A)$ which is simply the push-forward of π by the embedding $\mathrm{GL}(n, \mathbb{R}) \to \mathrm{GL}(n, A)$. Since the latter embedding obviously is continuous, ρ, and therefore the related action, is continuous as well. □

Now we are ready to prove the main result of this section.

Proof of Theorem 15. We note that the central extension $\widetilde{\mathfrak{g}}_A$ is automatically universal if it is weakly universal, since $\mathfrak{g}_A$ is perfect. So it remains to show that ω_A is a universal cocycle. So let $\omega' \in Z^2_{\mathbf{F}}(\mathfrak{g}_A, \mathfrak{z}')$ be $\mathfrak{z}'$-valued cocycle on $\mathfrak{g}_A$. We denote by $L^2_c(\mathfrak{g}_A, \mathfrak{z}')$ the space of continuous $\mathfrak{z}'$-valued bilinear maps on $\mathfrak{g}_A$. Endowing this space with the compact open topology, it becomes a complete locally convex space (cf. [13], Corollary II.32.4) and as a closed subspace, $Z^2_{\mathbf{F}}(\mathfrak{g}_A, \mathfrak{z}')$ is a complete locally convex space as well.

Let G be the simply connected group associated to $\mathfrak{g}$. According to Lemma 17, the adjoint action of G on $\mathfrak{g}$ induces a continuous action of G on $\mathfrak{g}_A$, turning $\mathfrak{g}_A$ into a Fréchet G-module. For later use we note that the complexification $(\mathfrak{g}_A)_{\mathbb{C}}$ of $\mathfrak{g}_A$ is a Fréchet $G_{\mathbb{C}}$-module, since we have $(\mathfrak{g}_A)_{\mathbb{C}} \cong (\mathfrak{g}_{\mathbb{C}})_A$. If $C(\mathfrak{g}_A \times \mathfrak{g}_A, \mathfrak{z}')_c$ denotes the space of continuous maps from $\mathfrak{g}_A \times \mathfrak{g}_A$ to $\mathfrak{z}'$ endowed with the compact-open topology, then the setting

$$(g\varphi)(x, y) := \varphi(g^{-1}x, g^{-1}y) \tag{4}$$

defines a continuous action of G on $C(\mathfrak{g}_A \times \mathfrak{g}_A, \mathfrak{z}')_c$ (cf. [10], Lemma III.2). As a closed G-invariant subspace of $C(\mathfrak{g}_A \times \mathfrak{g}_A, \mathfrak{z}')_c$, the space $Z^2_{\mathbf{F}}(\mathfrak{g}_A, \mathfrak{z}')$ therefore is a complete locally convex G-module. The so-defined action of G on $Z^2_{\mathbf{F}}(\mathfrak{g}_A, \mathfrak{z}')$ induces a continuous action of $\mathfrak{g}$ on $Z^2_{\mathbf{F}}(\mathfrak{g}_A, \mathfrak{z}')$ given by

$$(x\omega)(y, z) = -\omega([x, y], z) - \omega(y, [x, z]). \tag{5}$$

Denoting the space of continuous alternating $\mathfrak{z}'$-valued p-linear maps on $\mathfrak{g}_A$ by $C^p_{\mathbf{F}}(\mathfrak{g}_A, \mathfrak{z}')$, we have for any $x \in \mathfrak{g}_A$ the insertion map $i(x) : C^p_{\mathbf{F}}(\mathfrak{g}_A, \mathfrak{z}') \to C^{p-1}_{\mathbf{F}}(\mathfrak{g}_A, \mathfrak{z}')$ defined by $\eta \mapsto \eta(x, \cdot)$. For $\omega \in Z^2_{\mathbf{F}}(\mathfrak{g}_A, \mathfrak{z}')$ the Cartan formula now yields

$$x\omega = d(i(x)\omega) + i(x)(d\omega) = d(i(x)\omega) \in B^2_{\mathbf{F}}(\mathfrak{g}_A, \mathfrak{z}').$$

Thus, applying Lemma 16, we obtain

$$Z^2_{\mathbf{F}}(\mathfrak{g}_A, \mathfrak{z}') = Z^2_{\mathbf{F}}(\mathfrak{g}_A, \mathfrak{z}')_{\text{fix}} + B^2_{\mathbf{F}}(\mathfrak{g}_A, \mathfrak{z}').$$

Therefore we can assume that ω is invariant with respect to the actions defined by (4) and (5). The latter invariance of ω implies

$$\begin{aligned} \omega(1 \otimes x, ab \otimes [y,z]) &= -\omega(a \otimes y, b \otimes [z,x]) - \omega(b \otimes z, a \otimes [x,y]) \\ &= -\omega(a \otimes [x,y], b \otimes z) - \omega(b \otimes z, a \otimes [x,y]) \\ &= 0, \end{aligned}$$

and thus

$$\omega(1 \otimes \mathfrak{g}, \mathfrak{g}_A) = 0, \tag{6}$$

since $\mathfrak{g}_A$ is perfect. Fixing $a, b \in A$, the map

$$\omega_{(a,b)} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{z}' : (x,y) \mapsto \omega(a \otimes x, b \otimes y)$$

is a continuous $\mathfrak{g}$-invariant bilinear map and therefore has to be skew-symmetric, since $\mathfrak{g}$ does not posses any non-zero symmetric $\mathfrak{g}$-invariant bilinear from (cf. [8]). In view of Lemma 14, there exists a unique continuous linear map $\overline{\omega}_{(a,b)} : V(\mathfrak{g}) \to \mathfrak{z}'$ satisfying $\omega_{(a,b)} = \overline{\omega}_{(a,b)} \circ \kappa$. From the uniqueness of the maps $\overline{\omega}_{(a,b)}$, $(a,b) \in A \times A$, and the continuity of ω, we deduce the existence of a continuous linear map

$$\eta : A \otimes_{\mathbf{F}} A \to \operatorname{Hom}(V(\mathfrak{g}), \mathfrak{z}')$$

satisfying

$$\eta(a \otimes b)\big(\kappa(x,y)\big) = \omega(a \otimes x, b \otimes y).$$

Now the skew-symmetry of ω together with the symmetry of κ and the fact that $\operatorname{im}(\kappa)$ generates $V(\mathfrak{g})$ as a vector space imply that η is skew-symmetric. Using the invariance of κ, the fact that ω is a 2-cocycle yields

$$\eta(ab \otimes c + bc \otimes a + ca \otimes b)\big(\kappa([x,y],z)\big) = 0$$

for all $a,b,c \in A$ and all $x,y,z \in \mathfrak{g}$. Since $\mathfrak{g}$ is perfect and $\operatorname{im}(\kappa)$ is generating, we conclude from the latter equation that η vanishes on all expressions of the form

$$ab \otimes c + bc \otimes a + ca \otimes b \in A \otimes_{\mathbf{F}} A.$$

From (6) it follows that η vanishes on $1 \otimes A$ and since, in view of Lemma 5, the elements of the form

$$a(1 \otimes b - b \otimes 1)(1 \otimes c - c \otimes 1) = a \otimes bc - ab \otimes c - ac \otimes b + abc \otimes 1 \in A \otimes_{\mathbf{F}} A$$

generate a dense subset of $\overline{I^2}$ (recall that I was defined to be the kernel of the multiplication map $\mu : A \otimes_{\mathbf{F}} A \to A$), we see that η vanishes on $\overline{I^2}$ and hence induces a continuous linear map

$$\overline{\eta} : \Omega_{\mathbf{F}}(A) \to \operatorname{Hom}(V(\mathfrak{g}), \mathfrak{z}').$$

For this map we have

$$\overline{\eta}(d_A(a)) = \eta(1 \otimes a - a \otimes 1) = 2\eta(1 \otimes a) = 0,$$

whence it factors to a continuous linear map

$$\xi : \Omega_{\mathbf{F}}(A)/\overline{d_A(A)} \to \mathrm{Hom}(V(\mathfrak{g}), \mathfrak{z}').$$

In view of the canonical isomorphism

$$\mathrm{Hom}_c\big(E, \mathrm{Hom}_c(F, \mathfrak{z}')\big) \cong \mathrm{Hom}_c(E \otimes F, \mathfrak{z}'),$$

we can consider ξ as a continuous linear map $\mathfrak{z}_A \to \mathfrak{z}'$, and with respect to this identification the above calculations yield $\omega = -\xi \circ \omega_A$. □

Having Theorem 9 in mind, we consider the Fréchet–Lie algebra $\mathfrak{g}_M := C^\infty(M, \mathfrak{g}) \cong \mathfrak{g}_A$. In order to obtain a convenient description of a universal central extension of this algebra we put $\mathfrak{z}_M := \Omega^1(M)/dA$, and define a continuous 2-cocycle ω_M on $\mathfrak{g}_M$ by

$$\omega_M(f \otimes x, g \otimes y) := \kappa_{\mathfrak{g}}(x, y)[fdg] \in \mathfrak{z}_M, \qquad (7)$$

where $\kappa_{\mathfrak{g}}$ denotes the Killing form of $\mathfrak{g}$. Since in case $\mathfrak{g}$ is simple all invariant symmetric bilinear forms on $\mathfrak{g}$ are multiples of the Killing form, we get the following consequence of Theorem 15 which generalizes Proposition 4.2.8 in [12]:

18 Corollary. *If the Lie algebra $\mathfrak{g}$ is simple, then the Lie algebra $\widetilde{\mathfrak{g}}_M := \mathfrak{g}_M \oplus_{\omega_M} \mathfrak{z}_M$ is a universal central extension of $\mathfrak{g}_M$.*

References

[1] Bonsall, F. F., and Duncan, J., *Complete Normed Algebras*, Ergebnisse der Math. **80**, Springer-Verlag, Berlin, 1973.

[2] Connes, A., Noncommutative differential geometry, *IHES*, **62** (1985).

[3] Grothendieck, A., *Produits tensoriels topologiques et espaces nucléaires*, Memoirs Amer. Math. Soc. **16**, 1955.

[4] Haddi, A., Homology des algèbres de Lie étendues une algèbre commutative, *Comm. in Algebra*, **20**(4), 1145–1166 (1992).

[5] Hofmann, K. H., and Morris, S. A., *The Structure of Compact Groups*, de Gruyter, Berlin, 1998.

[6] Jarchow, H., *Locally Convex Spaces*, B. G. Teubner, Stuttgart, 1981.

[7] Moody, R. V., and Pianzola, A.., *Lie Algebras With Triangular Decomposition*, Canadian Mathematical Society, Series of Monographs and Advanced Texts, 1993.

[8] Medina, A., and Revoy, P., Algèbres de Lie orthogonales, modules orthogonaux, *Comm. in Algebra*, **27**(1), 2295–2315 (1993).

[9] Neeb, K.-H., Central extensions of infinite-dimensional Lie groups, Preprint 2084, TU Darmstadt, 2000.

[10] Neeb, K.-H., Infinite-dimensional Lie groups and their representations, Preprint 2102, TU Darmstadt, 2000.

[11] Neeb, K.-H., Universal central extensions of Lie groups, manuscript, TU Darmstadt, 2000.

[12] Pressley, A., and Segal, G., *Loop Groups*, Oxford Mathematical Monographs, Oxford, 1986.

[13] Treves, F., *Topological Vector Spaces, Distributions and Kernels*, Academic Press, 1967.

[14] Zusmanovich, P., The second homology group of current Lie algebras, *S.M.F. Asterisque* 226**(1994).

Peter Maier
Technische Universität Darmstadt
Fachbereich Mathematik
Schloßgartenstraße 7
D-64289 Darmstadt
maier@mathematik.tu-darmstadt.de