Classical Hilbert–Lie groups, their extensions and their homotopy groups

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Abstract. Let H be a complex Hilbert space and D a hermitian operator on H with finite spectrum. Then the operators for which the commutator with D is of Schatten class p form a Banach algebra $B_p(H,D)$. In the present paper we study groups $\mathrm{GL}_p(H,D)$ associated to this kind of Lie algebra and also groups $\mathrm{GL}_p(H,I,D)$ associated to the sub Lie algebras $B_p(H,I,D):=\{x\in B_p(H,D): Ix^*I^{-1}=-x\}$, where I is an antilinear isometry with $I^2\in\{\pm\mathbf{1}\}$. For $p=2$ we determine the full second continuous cohomology for these Lie algebras and for the groups we compute all homotopy groups. These results then lead to a direct description of universal central extensions of the groups $\mathrm{GL}_2(H,D)$,$\mathrm{GL}_2(H,I,D)$ and some of their real forms. In particular we obtain the infinite-dimensional metaplectic and the metagonal group as special examples. In a last section we discuss associated complex flag manifolds and show that the unitary forms of the complex groups act transitively.

Introduction

An important feature of finite-dimensional reductive Lie algebras $\mathfrak{g}$ is that they always possess a positive definite bilinear form $\langle\cdot,\cdot\rangle$ and a so called *Cartan involution* θ such that

$$\langle [x,y],z\rangle = -\langle y,[\theta(x),z]\rangle \quad \text{for} \quad x,y,z\in\mathfrak{g}. \tag{0.1}$$

Here the case $\theta=\mathrm{id}_{\mathfrak{g}}$ corresponds to the case of a compact Lie algebra.

An interesting class of infinite-dimensional Lie algebras generalizing finite-dimensional real reductive Lie algebras, in the sense that they still have the structure provided by (0.1), are the *L^*-algebras.* More precisely, these are Lie algebras $\mathfrak{g}$ which are real Hilbert spaces endowed with an isometric Lie algebra involution $x\mapsto x^*$, i.e.,

$$x^{**}=x \quad \text{and} \quad [x,y]^*=[y^*,x^*] \quad \text{for} \quad x,y\in\mathfrak{g},$$

such that

$$\langle [x,y],z\rangle = \langle y,[x^*,z]\rangle \quad \text{for} \quad x,y,z\in\mathfrak{g}. \tag{0.2}$$

Using the Closed Graph Theorem, one can derive the continuity of the Lie bracket on $\mathfrak{g}$ from (0.2), so that this requirement does not have to be put into the axioms of an L^*-algebra. If $\mathfrak{g}$ is finite-dimensional real reductive, we may define $x^*:=-\theta(x)$ for a Cartan involution θ. A *complex L^*-algebra* is a real L^*-algebra which is a complex Lie algebra for which the involution $*$ is antilinear. This easily implies that $\mathfrak{g}$ can be turned into a complex Hilbert space by

$$\langle x,y\rangle_{\mathbb{C}} := \langle x,y\rangle - i\langle ix,y\rangle, \quad x,y\in\mathfrak{g},$$

such that (0.2) is satisfied for the hermitian scalar product $\langle\cdot,\cdot\rangle_{\mathbb{C}}$. The structure theory of real and complex L^*-algebras has mostly been developed by Schue, Balachandran, de la Harpe and Unsain, where certain key references are [Sch60], [Sch61], [Ba69], [dlH72] and [Un72]. One finds further references in de la Harpe's book.

The L^*-algebras of *compact type*, where $x^* = -x$ for all $x \in \mathfrak{g}$, are natural generalizations of compact Lie algebras, so that the corresponding groups G are generalizations of compact groups. This point of view suggests that their representation theory lies at the heart of the representation theory of infinite-dimensional Lie groups. Although we won't deal with representations in the present paper, it has very much been motivated by [Ne01a], where we approach the representation theory via coadjoint orbits of central extensions, or equivalently affine coadjoint orbits of G. At many points it turns out to be important to have specific geometric and topological information on groups corresponding to simple L^*-algebras of compact type and their complexifications. It is the goal of this paper to provide such information in the concrete setting of groups of operators on Hilbert spaces, without using the structure theory of L^*-algebras.

The simple infinite-dimensional complex L^*-algebras arise in three series which can be described as follows. If H is a complex Hilbert space, we write

$$\mathfrak{gl}_2(H) := \{x \in B(H): \mathrm{tr}(xx^*) < \infty\}$$

for the Lie algebra of Hilbert-Schmidt operators on H (cf. Definition I.8). If $I: H \to H$ is an antilinear isometry with $I^2 \in \{\pm \mathbf{1}\}$, we define

$$\mathfrak{gl}_2(H, I) := \mathfrak{g}(H, I) \cap \mathfrak{gl}_2(H) \quad \text{and} \quad \mathfrak{gl}(H, I) := \{X \in \mathfrak{gl}(H): X + IX^*I^{-1} = 0\}.$$

For $I^2 = -\mathbf{1}$ we also write $\mathfrak{sp}_2(H, I) := \mathfrak{gl}_2(H, I)$ and for $I^2 = \mathbf{1}$ we write $\mathfrak{o}_2(H, I) := \mathfrak{gl}_2(H, I)$. This notation is motivated by the observation that $\beta(x, y) := \langle x, I.y\rangle$ defines a complex bilinear form on H which is symmetric for $I^2 = \mathbf{1}$ and skew-symmetric for $I^2 = -\mathbf{1}$ and which satisfies

$$\mathfrak{gl}_2(H, I) = \{x \in \mathfrak{gl}_2(H): (\forall v, w \in H)\, \beta(x.v, w) + \beta(v, x.w) = 0\}.$$

Each simple infinite-dimensional L^*-algebra $\mathfrak{g}$ is isomorphic to $\mathfrak{gl}_2(H)$, $\mathfrak{sp}_2(H, I)$ or $\mathfrak{o}_2(H, I)$ for some infinite-dimensional Hilbert space H, and all these algebras are pairwise non-isomorphic (see [Sch60] for the separable case and [CGM90], [Neh93] and [St99] for different proofs for the general case). Since we want to treat $\mathfrak{sp}_2(H, I)$ and $\mathfrak{o}_2(H, I)$ in a uniform way, this leaves us with the two types $\mathfrak{gl}_2(H)$ and $\mathfrak{gl}_2(H, I)$. Separable real simple L^*-algebras have been classified independently by Balachandran ([Ba69]), de la Harpe ([dlH70, 71a]) and Unsain ([Un71, 72]).

Passing from finite-dimensional semisimple complex Lie algebras to infinite-dimensional L^*-algebras, an important new feature is that these algebras have many non-trivial central extensions and outer derivations which lead to a large family of related Banach–Lie algebras, where each one of them has its own specific merits.

It is a general phenomenon in infinite-dimensional Lie theory that non-trivial central extensions play an important role because geometric actions of groups on certain manifolds M do not lift to actions on line bundles over M. One first has to enlarge the group by a central extension. On the representation theoretic side this means that the original symmetry groups only have projective representations in natural function spaces, and that central extensions are required to obtain genuine representations. For a general approach to central extensions of infinite-dimensional Lie groups and criteria for their existences in terms of topological data we refer to [Ne00b].

It is related, but in general of a different nature, that infinite-dimensional Lie algebras have many outer derivations, leading to a different kind of extension of the original Lie algebra. For the classical L^*-algebras, this kind of extension process leads in particular to the corresponding *restricted Lie algebras and groups*. For $\mathfrak{g} = \mathfrak{gl}_2(H)$, resp., $\mathfrak{gl}_2(H, I)$ we put $\mathfrak{g}_b := \mathfrak{gl}(H) := B(H)$, resp., $\mathfrak{gl}(H, I)$. Let $D \in \mathfrak{g}_b$ be a hermitian element with finite spectrum. Then the Lie algebra $\mathfrak{g}(D) := \mathfrak{g} + \mathfrak{z}_{\mathfrak{g}_b}(D)$ is called the *restricted Lie algebra* associated to $\mathfrak{g}$ and D. For $\mathfrak{g} = \mathfrak{gl}_2(H)$ the restricted Lie algebras $\mathfrak{gl}_2(H, D) := \mathfrak{g}(D)$ has the form

$$\mathfrak{gl}_2(H, D) = \mathfrak{gl}_2(H) + \sum_{j=1}^{k} \mathfrak{gl}(H_j) = \mathfrak{g} + \mathfrak{z}_{\mathfrak{g}_b}(D),$$

where $H = H_1 \oplus \cdots \oplus H_k$ is the orthogonal eigenspace decomposition for D, and $\mathfrak{z}_{\mathfrak{g}_b}(D)$ corresponds to the Lie algebra of all operators preserving all the spaces H_j. The terminology "restricted" comes from the fact that operator $x = (x_{ij}) \in \mathfrak{gl}(H)$, viewed as a $k \times k$-block matrix with entries $x_{ij} \in B(H_j, H_i)$, is contained in $\mathfrak{gl}(H, D)$ if and only if all its off-diagonal blocks $x_{ij} \in B(H_j, H_i)$, $i \neq j$, are Hilbert–Schmidt, which we view as a restriction on x.

Now we describe the contents of this paper in some more detail. In Section I we deal with the central extensions of the Lie algebras related to $\mathfrak{gl}_2(H)$ and $\mathfrak{gl}_2(H, I)$. In particular we show that $\mathfrak{gl}(H)$ and $\mathfrak{gl}(H, I)$ have no non-trivial central extension if H is infinite-dimensional. Using this information, we calculate the continuous second Lie algebra cohomology group $H^2_c(\mathfrak{g}(D), \mathbb{C})$ for the restricted Lie algebras $\mathfrak{g}(D)$ which turns out to be finite-dimensional. This in turn implies the existence of a universal central extension which we describe explicitly in Section IV. For universality of central extensions we use results from [Ne01b], where universal central extensions of infinite-dimensional Lie groups are studied in detail.

Sections II and III are devoted to the homotopy groups of the corresponding groups. Here the groups π_k, $k = 0, 1, 2$, are of particular importance. For $k = 0$ this is obvious, the group $\pi_1(G)$ is an obstruction to integrate Lie algebra homomorphisms to group homomorphisms, and the groups $\pi_1(G)$ and $\pi_2(G)$ are closely related to obstructions for the existence of central extensions ([Ne00b]).

For the full operator groups $\mathrm{GL}(H)$ and

$$\mathrm{GL}(H, I) := \{g \in \mathrm{GL}(H) \colon g^{-1} = I g^* I^{-1}\}$$

one can use Kuiper's Theorem saying that the group $\mathrm{GL}(H)$ is contractible for a separable Hilbert space H over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. If H is inseparable, then there seems to be no immediate reference for this result for real and quaternionic Hilbert spaces. As we will see in Section II, the inseparable case has a quite elementary proof which does not depend on $\mathbb{K}$. In Section II we also compute the homotopy groups of the congruence groups $\mathrm{GL}_p(H)$ of the Schatten ideals $B_p(H) \subseteq B(H)$, of $\mathrm{GL}_p(H, I) := \mathrm{GL}(H, I) \cap \mathrm{GL}_p(H)$ for $1 \leq p \leq \infty$, and of the corresponding direct limit groups without the restriction that H is separable. Since the homotopy groups of the restricted groups $\mathrm{GL}_p(H, D)$ and $\mathrm{GL}_p(H, I, D)$ are somewhat more involved, we deal with them separately in Section III.

Throughout this paper we never have to assume that the Hilbert spaces H under consideration are separable. All statements hold for general Hilbert spaces. To obtain this generality, we frequently have to extend results on homotopy groups of groups of operators which are known for the separable case.

Combining the information on central extensions from Section I and on the homotopy groups from Section III, we show in Section IV that the identity components $\mathrm{GL}_2(H, D)_e$ and $\mathrm{GL}_2(H, I, D)_e$ of the restricted groups have a universal central extension in the category of complex Banach–Lie groups whose central fiber is of the type $(\mathbb{C}^\times)^k$. We extend these results to certain real forms of these groups. If D has only two eigenvalues, this construction leads, for $\mathrm{GL}_2(H, D)$, to the central extension of the restricted general linear group of a polarized Hilbert space, which plays a crucial role in the theory of loop groups (cf. [PS86]). For the real forms $\mathrm{Sp}_{\mathrm{res}}(H, \Omega)$ of $\mathrm{Sp}_2(H_\mathbb{C}, I, D)$ and $\mathrm{O}_{\mathrm{res}}(H^\mathbb{R})$ of $\mathrm{O}_2(H_\mathbb{C}, I, D)$ (see Section IV for the notation), we obtain as universal central extensions the metagonal and the metaplectic groups discussed systematically by Vershik in [Ve90]. The metaplectic group has been introduced by I. Segal and Shale ([Se59], [Sh62]) and the metagonal group by Shale/Stinespring in [ShSt65] (see also [dlH71b] for a construction of the spin group as a two fold cover of $\mathrm{O}_1(H^\mathbb{R})^+$, a real form of the identity component $\mathrm{O}_1(H, I)^+$ of $\mathrm{O}_1(H, I)$). These two groups are relatives of the finite-dimensional spin and metaplectic group, which are twofold covers of $\mathrm{SO}(2n, \mathbb{R})$, resp., $\mathrm{Sp}(2n, \mathbb{R})$. The results of Section IV on the universality of certain central extensions form the heart of the paper. It requires essentially all the information on the homotopy groups collected in Sections II and III and also the information on Lie algebra cohomology from Section I. Our structure theoretic approach to central extensions, as opposed to the representation theoretic one, has the advantage that it immediately provides a good deal of structural and topological information on the groups: their Banach–Lie group structure, their Lie algebra cocycles, and their topology.

The representation theoretic approach usually has to face the problem to deal with, a priori, unbounded operators on a Hilbert spaces. A general motivation is to understand the full set of central extensions of groups like $\mathrm{O}_{\mathrm{res}}(H^{\mathbb{R}})$ and $\mathrm{Sp}_{\mathrm{res}}(H,\Omega)$ is that they act naturally as symmetry groups of geometric objects (mostly symmetric spaces), so that the knowledge of their central extensions is important to understand the implementation of the these symmetry groups in natural Hilbert spaces attached to the geometric objects.

In Section V we finally discuss flag manifolds for the groups $\mathrm{GL}_2(H)$ and $\mathrm{GL}_2(H,I)$. For $\mathrm{GL}_2(H)$ we consider a flag $\mathcal{F} = (F_0, F_1, \ldots, F_k)$, where

$$\{0\} = F_0 \subseteq F_1 \subseteq F_2 \subseteq \ldots \subseteq F_k = H$$

are closed subspaces of H. For $\mathrm{GL}_2(H,I)$ we consider flags $\mathcal{F}$ of the type

$$\{0\} = F_0 \subseteq F_1 \subseteq F_2 \subseteq \ldots \subseteq F_k \subseteq F_k^{\perp_\beta} \subseteq \ldots \subseteq F_1^{\perp_\beta} \subseteq F_0^{\perp_\beta} = H,$$

which means that the spaces F_j, $j = 1, \ldots, k$, are isotropic for the bilinear form $\beta(x,y) = \langle x, I.y\rangle$. Let $P(\mathcal{F}) \subseteq G$ denote the stabilizer of the flag $\mathcal{F}$. Then the homogeneous space $G/P(\mathcal{F})$ has a natural manifold structure and its elements can be viewed as flags $g.\mathcal{F}$ of closed subspaces of H. We will see that in all cases the action of G on $G/P(\mathcal{F})$ extends naturally to an action of the corresponding restricted group and that the unitary real form $U = G \cap \mathrm{U}(H)$ acts transitively on $G/P(\mathcal{F})$. For $G = \mathrm{GL}_2(H)$ and $F_2 = H$ this construction leads to the restricted Graßmannians $\mathrm{Gr}_{\mathrm{res}}(F_1)$. For $G = \mathrm{GL}_2(H,I)$ and $k = 2$ we obtain for $F_1 \subseteq H$ maximal isotropic the restricted Graßmannian of maximal isotropic subspaces and for $\dim F_1 = 1$ the space of isotropic lines in H. Both are hermitian symmetric spaces modeled over Hilbert spaces. Using some structural results from Section III, it is not hard to get basic information on the homotopy groups and the corresponding period maps for the flag manifolds. This kind of information is important for the quantizability of symplectic structures on these manifolds, viewed as affine coadjoint orbits of the unitary real forms $U = G \cap \mathrm{U}(H)$. For more details on this interpretation and on affine coadjoint actions of these groups we refer to [Ne01a].

Several special classes of these flag manifolds show up at various places in the literature. The flag manifolds for $\mathrm{GL}_2(H,D)$ for separable H have been introduced by A. and G. Helminck in [HH94a] and [HH94b]. They apply the representations of central extensions of the complex group $\mathrm{GL}_2(H,D)$ in Hilbert spaces of holomorphic sections of line bundles on the flag manifolds to integrable systems. Moreover, they study cell decompositions of the flag manifolds and use them to obtain a Birkhoff decomposition of the group $\mathrm{GL}_2(H,D)$ ([HH94b, Prop. 2.4.16]).

The restricted Graßmannian $\mathrm{Gr}_{\mathrm{res}}(F)$ of a polarized Hilbert space plays a central role for the structure of loop groups ([PS86]). The Graßmannians are particular cases of hermitian symmetric spaces, which are dual to symmetric Hilbert domains. These manifolds and their automorphism have been studied in [Ka75] and [DNS89], [DNS90]. A classification of hermitian symmetric Hilbert manifolds has been obtained by W. Kaup in [Ka83].

For separable Hilbert spaces the groups $\mathrm{GL}_p(H)$ and $\mathrm{GL}_p(H,I)$, $1 \leq p \leq \infty$, and their real forms have been studied in detail by de la Harpe in [dlH72], where one finds all kinds of information such as the cohomology, the automorphisms, and the derivations of their Lie algebras, which we use in Section I. De la Harpe's book also contains a discussion of Riemannian symmetric spaces of the real forms of these groups, where the aforementioned Graßmannians and several other related manifolds show up.

In this paper we do not deal with representations of the groups under consideration, although this paper was motivated by and provides important information useful for the theory of unitary representations of real L^*-groups and their realization in Hilbert spaces of holomorphic sections of holomorphic line bundles over coadjoint orbits which are Kähler manifolds. The geometry of the "elliptic" coadjoint orbits and the corresponding unitary representations will be studied in forthcoming papers (cf. [Ne01a]). This theory includes in particular the spin representation of the metagonal group (fermionic second quantization) and the metaplectic representation (Segal–Shale–Weil representation) of the metaplectic group (bosonic second quantization). For a nice exposition of the construction of these representations in an ad hoc fashion, we refer to Ottesen's book [Ot95], where it is also explained how embeddings of diffeomorphism groups and loop groups into $\mathrm{Sp}_{\mathrm{res}}(H,\Omega)$ and $\mathrm{U}_2(H,D)$ lead to interesting unitary representations of their central extensions (see also [PS86], [CR87] and [Mi89]). The mixed cases correspond to the infinite wedge representations of the restricted unitary group $U_{\mathrm{res}}(H_+,H_-)$, which in our terminology is $U_2(H,D) := U(H) \cap \mathrm{GL}_2(H,D)$, where D has only two eigenvalues (cf. [PS86] and also [Wu98] which contains a lot of information on the physical background). The general L^*-approach to these representations provides in particular direct geometric explanations for their intricate analytic properties such as boundedness properties of the corresponding operators (cf. [Ot95]).

Throughout this paper the letter I will always denote an antilinear isometry on a complex Hilbert space H with $I^2 \in \{\pm \mathbf{1}\}$. We call I a *conjugation* if $I^2 = \mathbf{1}$ and an *anticonjugation* for $I^2 = -\mathbf{1}$.

I. The second cohomology of classical Lie algebras

In this first section we discuss the second continuous Lie algebra cohomology of the Lie algebras $\mathfrak{gl}(H)$ and $\mathfrak{gl}(H,I)$ and also for the corresponding restricted Lie algebras $\mathfrak{gl}_2(H,D)$ and $\mathfrak{gl}_2(H,I,D)$. In particular we will see that $H^2_c(\mathfrak{g}(D),\mathbb{C})$ is always finite-dimensional and that it has a universal central extension ([Ne01b]).

The second cohomology groups of full classical Lie algebras

In this subsection we will show that for every Banach space $\mathfrak{z}$, considered as a trivial module of $\mathfrak{g} \in \{\mathfrak{gl}(H), \mathfrak{gl}(H,I)\}$, the second continuous cohomology group $H^2_c(\mathfrak{g},\mathfrak{z})$ (cf. Definition I.4) vanishes.

If J is a set we define the set $J^{\pm}$ as the disjoint union $J \dot{\cup} -J$, where $-J$ is a copy of the set J whose elements are denoted $-j$ with the convention that $-(-j) = j$.

Lemma I.1. *Let H be a complex Hilbert space and $I\colon H \to H$ an antilinear isometry with $I^2 = \pm\mathbf{1}$. If H is infinite-dimensional or of finite even-dimension, then there exists an orthonormal basis $(e_j)_{j\in J^{\pm}}$ with*

$$I.e_j = \begin{cases} e_{-j} & \text{for } j \in J \\ \pm e_{-j} & \text{for } j \in -J. \end{cases}$$

Proof. (cf. [dlH72, App. I]) First we consider the case $I^2 = -\mathbf{1}$. Since I and the complex structure on H generate a finite group, H is an orthogonal direct sum of complex subspaces on which I acts irreducibly. Let E be one of these subspaces and $v \in E$ a unit vector. The complex bilinear form $\beta(x,y) := \langle x, I.y\rangle$ is skew-symmetric because

$$\beta(y,x) = \langle y, I.x\rangle = \langle x, I^{-1}.y\rangle = -\beta(x,y).$$

Therefore $\{v, I.v\}$ is an orthonormal basis of E with the required properties. Since H is an orthogonal direct sum of copies of E, the assertion follows.

Next we consider the case $I^2 = \mathbf{1}$. Then our assumptions imply that the real Hilbert space $H_{\mathbb{R}} := \{v \in H\colon I.v = v\}$ has an orthonormal basis of the form $(f_j)_{j\in J^{\pm}}$. We define

$$e_{\pm j} := \frac{1}{\sqrt{2}}(f_j \pm i f_{-j})$$

and obtain a basis with the required properties. ∎

Remark I.2. (a) To obtain a more explicit description of the Lie algebra $\mathfrak{sp}(H,I)$, we use Lemma I.1 to obtain an orthonormal basis $(e_j)_{j\in J^\pm}$ of H with $I.e_j = e_{-j}$ for $j \in J$. Then the closed subspace H_0 generated by the elements e_j, $j \in J$, satisfies $H_0 \cong l^2(J,\mathbb{C})$, and we obtain a conjugation σ_0 on this space by $\sigma_0((x_j)_{j\in J}) = (\overline{x_j})_{j\in J}$. If we identify $H = H_0 \oplus I.H_0$ with the space $H_0 \oplus H_0$, the anticonjugation I is given by $I.(a,b) = (-\sigma_0(b), \sigma_0(a))$.

For $x \in B(H_0)$ we define $x^\top = \sigma_0 x^* \sigma_0$. Then the Lie algebra $\mathfrak{sp}(H,I) \subseteq B(H_0 \oplus H_0)$ can be described in terms of (2×2)-block matrices as

$$\mathfrak{sp}(H,I) \cong \Big\{ \begin{pmatrix} a & b \\ c & -a^\top \end{pmatrix} \in \mathfrak{gl}(H) \colon b = b^\top, c = c^\top \Big\}.$$

To get a similar description on the group level, we write I as a composition of (σ_0, σ_0) and the operator with the matrix

$$\widetilde{I} = \begin{pmatrix} \mathbf{0} & -\mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix}.$$

Then we have for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ the relation

$$Ig^*I^{-1} = \widetilde{I} g^\top \widetilde{I}^{-1} = \begin{pmatrix} \mathbf{0} & -\mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix} \begin{pmatrix} a^\top & c^\top \\ b^\top & d^\top \end{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix} = \begin{pmatrix} d^\top & -b^\top \\ -c^\top & a^\top \end{pmatrix},$$

which shows that

$$\mathrm{Sp}(H,I) \cong \Big\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(H) \colon ad^\top - bc^\top = \mathbf{1}, ab^\top = ba^\top, cd^\top = dc^\top \Big\}.$$

In particular we see that

$$\mathrm{GL}(H_0) \cong \Big\{ \begin{pmatrix} a & \mathbf{0} \\ \mathbf{0} & (a^\top)^{-1} \end{pmatrix} \colon a \in \mathrm{GL}(H_0) \Big\} \subseteq \mathrm{Sp}(H,I).$$

(b) For $\mathfrak{o}(H,I)$ and $\dim H = \infty$ or $\dim H$ even we write $H \cong H_0 \oplus H_0$ with $I.(a,b) = (\sigma_0(b), \sigma_0(a))$, and keep the other notations from above. Then the Lie algebra $\mathfrak{o}(H,I)$ can be described in terms of (2×2)-block matrices as

$$\mathfrak{o}(H,I) \cong \Big\{ \begin{pmatrix} a & b \\ c & -a^\top \end{pmatrix} \in \mathfrak{gl}(H_0 \oplus H_0) \colon b = -b^\top, c = -c^\top \}.$$

(c) If $\dim H = \infty$ or $\dim H$ is odd, then we have an orthogonal decomposition $H \cong H_0 \oplus \mathbb{C} \oplus H_0$ with $I.(a,z,b) = (\sigma_0(b), \overline{z}, \sigma_0(a))$ and obtain a similar explicit description as above by (3×3)-block matrices. ■

Lemma I.3. *Let H be a complex Hilbert space. The Lie algebra $\mathfrak{gl}(H)$ is perfect if and only if $\dim H = \infty$, and $\mathfrak{gl}(H,I)$ is perfect if and only if not* ($\dim H = 2$ *and* $I^2 = \mathbf{1}$).

Proof. If H is of finite dimension n, then $\mathfrak{gl}(H)$ is not perfect because $\mathrm{tr}\colon \mathfrak{gl}(H) \to \mathbb{C}$ is a non-trivial Lie algebra homomorphism. If H is infinite-dimensional, then we use [Ha67, Cor. 2 to Probl. 186] to see that every element in $\mathfrak{gl}(H)$ is the sum of two commutators, so that $\mathfrak{gl}(H)$ is in particular perfect.

Now we consider $\mathfrak{g} := \mathfrak{gl}(H,I)$. If H is finite-dimensional, then this Lie algebra is perfect unless $I^2 = \mathbf{1}$ and $\dim H = 2$. Suppose that H is infinite-dimensional. Then there exists a closed subspace $H_0 \subseteq H$ such that $H = H_0 \oplus I.H_0$ is an orthogonal direct sum (Lemma I.1). We consider the element

$$X := \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix} \in \mathfrak{gl}(H,I)$$

and define $\mathfrak{g}^0 := \mathfrak{z}_{\mathfrak{g}}(X)$. Then $\mathfrak{g} = \mathfrak{g}^0 \oplus [X, \mathfrak{g}]$, and $\mathfrak{g}^0 \cong \mathfrak{gl}(H_0)$ (Remark I.2). Since the first part of the proof implies that $\mathfrak{g}^0$ is perfect, we conclude that $\mathfrak{gl}(H,I)$ is perfect. ■

Definition I.4. Let $\mathfrak{g}$ a *topological Lie algebra*, i.e., a Lie algebra which is a topological vector space with a continuous Lie bracket, and $\mathfrak{z}$ be a topological vector space, considered as a trivial $\mathfrak{g}$-module. A *continuous* $\mathfrak{z}$*-valued* 2*-cocycle* is a continuous skew-symmetric function $\omega\colon \mathfrak{g}\times\mathfrak{g}\to\mathfrak{z}$ with

$$\omega([x,y],z)+\omega([y,z],x)+\omega([z,x],y)=0.$$

It is called a *coboundary* if there exists a continuous linear map $\alpha\colon\mathfrak{g}\to\mathfrak{z}$ with $\omega(x,y)=\alpha([x,y])$ for all $x,y\in\mathfrak{g}$. We write $Z^2_c(\mathfrak{g},\mathfrak{z})$ for the space of continuous $\mathfrak{z}$-valued 2-cocycles and $B^2_c(\mathfrak{g},\mathfrak{z})$ for the subspace of coboundaries. We define the *second continuous Lie algebra cohomology space*

$$H^2_c(\mathfrak{g},\mathfrak{z}):=Z^2_c(\mathfrak{g},\mathfrak{z})/B^2_c(\mathfrak{g},\mathfrak{z}).$$

See [Ja62, Sect. III.10] for the basic concepts related to Lie algebra cohomology. ■

In [dlH79] it is shown that the second homology space $H_2(\mathfrak{gl}(H))$ vanishes (on the algebraic level) and it is also shown that this implies that all Banach Lie algebra extensions of $\mathfrak{gl}(H)$ with finite-dimensional centers are trivial (cf. [dlH79, Cor. 4]). The following proposition sharpens this result.

Proposition I.5. *If H is a complex Hilbert space, then*

$$H^2_c(\mathfrak{gl}(H),\mathfrak{z})=\mathbf{0}$$

holds for all trivial Banach $\mathfrak{g}$-modules $\mathfrak{z}$.

Proof. Let $\mathfrak{g}:=\mathfrak{gl}(H)$. If $\dim H=n$ is finite, then $\mathfrak{g}\cong\mathfrak{gl}(n,\mathbb{C})$ and therefore $H^2_c(\mathfrak{g},\mathfrak{z})=H^2(\mathfrak{g},\mathfrak{z})=\mathbf{0}$. In fact, in view of Levi's Theorem, each element $[\omega]\in H^2(\mathfrak{g},\mathfrak{z})$ can be represented by an $\mathfrak{sl}(n,\mathbb{C})$-invariant $\mathfrak{z}$-valued cocycle. Since there is no non-zero skew-symmetric invariant bilinear form on $\mathfrak{sl}(n,\mathbb{C})$, and $\mathfrak{z}(\mathfrak{gl}(n,\mathbb{C}))\cong\mathbb{C}$ is one-dimensional, it follows that $H^2(\mathfrak{g},\mathfrak{z})$ is trivial.

Now we assume that H is infinite-dimensional. We consider $\mathfrak{z}$ as a trivial $\mathfrak{g}$-module. Let $\omega\in Z^2_c(\mathfrak{g},\mathfrak{z})$ be a continuous $\mathfrak{z}$-valued cocycle. In view of $H^2(\mathfrak{g},\mathfrak{z})\cong\mathrm{Lin}(H_2(\mathfrak{g}),\mathfrak{z})=\mathbf{0}$ ([dlH79]), there exists a linear map $\lambda\colon\mathfrak{g}\to\mathfrak{z}$ with $\omega(x,y)=\lambda([x,y])$ for all $x,y\in\mathfrak{g}$. It remains to show that λ is continuous.

Let $x\in\mathfrak{g}$. According to [Ha67, Cor. 2 to Probl. 186], there exist operators $a,b,c,d\in\mathfrak{g}$ with $x=[a,b]+[c,d]$, where $\|a\|,\|c\|\leq 2\|x\|$ and $\|b\|,\|d\|\leq 1$. We obtain

$$\|\lambda(x)\|=\|\lambda([a,b]+[c,d])\|\leq\|\omega(a,b)\|+\|\omega(c,d)\|\leq\|\omega\|\,\|a\|\,\|b\|+\|\omega\|\,\|c\|\,\|d\|\leq 4\|\omega\|\,\|x\|.$$

This proves that λ is continuous and therefore that $H^2_c(\mathfrak{g},\mathfrak{z})=\mathbf{0}$. ■

Next we show that the second cohomology of the Lie algebras $\mathfrak{gl}(H,I)$ vanishes. The proof is based on a modification of the strategy used in [dlH79].

Proposition I.6. *If H is a complex Hilbert space, then the Lie algebra $\mathfrak{sp}(H,I)$ satisfies*

$$H^2_c(\mathfrak{sp}(H,I),\mathfrak{z})=\mathbf{0}$$

for every Banach space $\mathfrak{z}$, considered as a trivial module.

Proof. If H is finite-dimensional with $\dim H=2n$, then $\mathfrak{g}:=\mathfrak{sp}(H,I)\cong\mathfrak{sp}(2n,\mathbb{C})$ is a simple complex Lie algebra, and the Whitehead Lemmas ([Ja62, Lemma III.9.6]) imply that $H^2_c(\mathfrak{g},\mathfrak{z})=H^2(\mathfrak{g},\mathfrak{z})=\mathbf{0}$ for every Banach space $\mathfrak{z}$.

Now we assume that H is infinite-dimensional. We consider the element

$$X:=\begin{pmatrix}\mathbf{1}&\mathbf{0}\\ \mathbf{0}&-\mathbf{1}\end{pmatrix}\in\mathfrak{g}$$

which defines the 3-grading

$$\mathfrak{g}=\mathfrak{g}^-\oplus\mathfrak{g}^0\oplus\mathfrak{g}^+,$$

where $\mathfrak{g}^0 = \ker \operatorname{ad} X \cong \mathfrak{gl}(H_0)$ (Remark I.2) and $\mathfrak{g}^\pm = \ker(\operatorname{ad} X \mp 2)$. We know from Proposition I.5 that $H^2_c(\mathfrak{g}^0, \mathfrak{z}) = \mathbf{0}$. This means that all the assumptions of Corollary III.9 in [Ne01a] are satisfied with $\mathfrak{d} = \mathfrak{g}^0$ and $D_0 = X$, so that it suffices to consider a $\mathfrak{g}^0$-invariant cocycle $\varphi \in Z^2_c(\mathfrak{g}, \mathfrak{z})$ satisfying $\varphi(\mathfrak{g}^0, \mathfrak{g}) = \mathbf{0}$. Therefore it suffices to show that every $\mathfrak{g}^0$-invariant bilinear form $\varphi\colon \mathfrak{g}^+ \times \mathfrak{g}^- \to \mathfrak{z}$ vanishes.

For the sake of simpler notation, we identify $\mathfrak{g}^0$ with $\mathfrak{gl}(H_0)$ by the map

$$\mathfrak{gl}(H_0) \to \mathfrak{g}^0, \quad x \mapsto \begin{pmatrix} x & 0 \\ 0 & -x^\top \end{pmatrix}$$

(Remark I.2). We further identify $\mathfrak{g}^\pm$ in the canonical way with $\operatorname{Sym}(H_0) := \{a \in B(H_0)\colon a^\top = a\}$ and consider the natural action of $\mathfrak{gl}(H_0)$ on $\operatorname{Sym}(H_0)$ given by $x.a := xa + ax^\top$. This corresponds to the action of $\mathfrak{g}^0$ on $\mathfrak{g}^+$, and on $\mathfrak{g}^-$ the action of $\mathfrak{g}^0$ corresponds to $[x, d] = -x^\top.d = -x^\top d - dx$. Therefore φ corresponds to a bilinear form

$$\varphi\colon \operatorname{Sym}(H_0) \times \operatorname{Sym}(H_0) \to \mathfrak{z}$$

satisfying

$$\varphi(x.a, d) - \varphi(a, x^\top.d) = 0 \quad \text{for all} \quad x \in \mathfrak{gl}(H_0), a, d \in \operatorname{Sym}(H_0).$$

For $x = x^\top \in \mathfrak{gl}(H_0)$ we have $x.\mathbf{1} = x\mathbf{1} + \mathbf{1}x^\top = 2x$, and therefore

$$\varphi(a, d) = \frac{1}{2}\varphi(a.\mathbf{1}, d) = \frac{1}{2}\varphi(\mathbf{1}, a.d) = \frac{1}{2}\varphi(\mathbf{1}, da + ad).$$

It follows in particular that φ is symmetric and that it suffices to show that $\varphi(\mathbf{1}, \cdot) = 0$. The $\mathfrak{g}^0$-invariance of φ leads for $x = -x^\top \in \mathfrak{gl}(H_0)$ to

$$0 = \varphi(x.a, \mathbf{1}) - \varphi(a, x^\top.\mathbf{1}) = \varphi(xa + ax^\top, \mathbf{1}) + \varphi(a, \underbrace{x.\mathbf{1}}_{=0}) = \varphi(xa - ax, \mathbf{1}) = \varphi([x, a], \mathbf{1}).$$

To see that φ vanishes, it therefore suffices to show that

$$[\operatorname{Skew}(H_0), \operatorname{Sym}(H_0)] = \operatorname{Sym}(H_0).$$

We know already that the Lie algebra $\mathfrak{gl}(H_0) = \operatorname{Skew}(H_0) \oplus \operatorname{Sym}(H_0)$ is perfect (Lemma I.3), which implies that

$$\begin{aligned} \mathfrak{gl}(H_0) &= [\mathfrak{gl}(H_0), \mathfrak{gl}(H_0)] \\ &= \big([\operatorname{Skew}(H_0), \operatorname{Skew}(H_0)] + [\operatorname{Sym}(H_0), \operatorname{Sym}(H_0)]\big) + [\operatorname{Skew}(H_0), \operatorname{Sym}(H_0)], \end{aligned}$$

and this implies in particular that $\operatorname{Sym}(H_0) = [\operatorname{Skew}(H_0), \operatorname{Sym}(H_0)]$. ■

Proposition I.7. *If* $\dim H > 2$, *then*

$$H^2_c(\mathfrak{o}(H, I), \mathfrak{z}) = \mathbf{0}$$

for every Banach space $\mathfrak{z}$, *considered as a trivial module.*

Proof. Assume first that $n := \dim H$ is finite. Then $n > 2$ implies that $\mathfrak{o}(H, I) \cong \mathfrak{o}(n, \mathbb{C})$ is a semisimple complex Lie algebra, and the assertion follows from the Whitehead Lemmas.

Now we assume that H is infinite-dimensional. Let $\mathfrak{g} := \mathfrak{o}(H, I)$. We consider the element

$$X := \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix} \in \mathfrak{g}$$

which defines the 3-grading $\mathfrak{g} = \mathfrak{g}^- \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^+$, where $\mathfrak{g}^j = \ker(\operatorname{ad} X - 2j)$.

With the same argument as in the proof of Proposition I.6, we see that we may w.l.o.g. assume that φ is $\mathfrak{g}^0$-invariant and satisfies $\varphi(\mathfrak{g}^0, \mathfrak{g}) = \mathbf{0}$. As in Proposition I.6, this leads to a bilinear form

$$\varphi\colon \mathrm{Skew}(H_0) \times \mathrm{Skew}(H_0) \to \mathfrak{z}$$

satisfying

$$\varphi(x.a, d) - \varphi(a, x^\top.d) = 0 \quad \text{for all} \quad x \in \mathfrak{gl}(H_0), a, d \in \mathrm{Skew}(H_0),$$

where $x.a = xa + ax^\top$ for $x \in \mathfrak{gl}(H_0)$ and $a \in \mathrm{Skew}(H_0)$. We have to show that this implies that $\varphi = 0$.

We write H_0 as $H_1 \oplus H_1$, where H_1 is endowed with an antilinear isometric involution σ_1 such that $\sigma_0(v, w) = (\sigma_1.v, \sigma_1.w)$ for $v, w \in H_1$. From now on we write operators in $\mathfrak{gl}(H_0)$ as (2×2)-block matrices according to the decomposition of H_0 into $H_1 \oplus H_1$. Let

$$S := \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix} \in \mathrm{Skew}(H_0).$$

Then

$$x.S = \begin{pmatrix} x_{12}^\top - x_{12} & x_{11} + x_{22}^\top \\ -x_{11}^\top - x_{22} & x_{21} - x_{21}^\top \end{pmatrix}$$

shows that for each $a \in \mathrm{Skew}(H_0)$ we have

$$a = \widetilde{a}.S \quad \text{for} \quad \widetilde{a} := \begin{pmatrix} a_{12} & -\frac{1}{2}a_{11} \\ \frac{1}{2}a_{22} & 0 \end{pmatrix}.$$

Therefore

$$\varphi(a, d) = \varphi(\widetilde{a}.S, d) = \varphi(S, \widetilde{a}^\top.d),$$

and it suffices to prove that $\varphi(S, \cdot) = \mathbf{0}$.

For $x \in \mathfrak{gl}(H_0)$ with $x^\top.S = 0$ we have

$$0 = \varphi(S, x.a) - \varphi(x^\top.S, a) = \varphi(S, x.a).$$

For $x = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}$, which satisfies $x^\top.S = x^\top S + Sx = 0$, we obtain in particular

$$x.a = xa + ax = \begin{pmatrix} a_{11} & -a_{12} \\ -a_{21} & a_{22} \end{pmatrix},$$

showing that $x.\mathrm{Skew}(H_0) = \mathrm{Skew}(H_0)$. This implies that $\varphi(S, \cdot) = 0$, and hence that $\varphi = 0$. ■

The second cohomology groups of restricted Lie algebras

In this subsection we will use the results of the preceding subsection and the general tools developed in [Ne01a] to compute the second cohomology of the restricted versions of the Lie algebras $\mathfrak{gl}_2(H)$ and $\mathfrak{gl}_2(H, I)$.

Definition I.8. (a) Let H be a Hilbert space. For $1 \leq p < \infty$ we define

$$B_p(H) := \{X \in B(H)\colon \mathrm{tr}\left((XX^*)^{\frac{p}{2}}\right) < \infty\}.$$

For $p = \infty$ we define $B_\infty(H) := K(H)$ (cf. [RS78]) as the ideal of compact operators on H. More generally we define for two Hilbert spaces H_1, H_2:

$$B_p(H_1, H_2) := \{X \in B(H_1, H_2)\colon \mathrm{tr}\left((XX^*)^{\frac{p}{2}}\right) < \infty\} \quad \text{and} \quad B_\infty(H_1, H_2) := K(H_1, H_2).$$

These sets are are invariant under left and right multiplication with bounded operators and they are Banach spaces with respect to the norms

$$\|X\|_p := \operatorname{tr}\big((XX^*)^{\frac{p}{2}}\big)^{\frac{1}{p}} \quad \text{and} \quad \|X\|_\infty := \|X\|,$$

satisfying

$$\|XY\|_p \le \|X\| \cdot \|Y\|_p \quad \text{and} \quad \|XY\|_p \le \|X\|_p \cdot \|Y\|.$$

The spaces $B_p(H)$ are called the *Schatten ideals* of $B(H)$.

The *congruence subgroups*

$$\mathrm{GL}_p(H) := \mathrm{GL}(H) \cap (\mathbf{1} + B_p(H))$$

with respect to the Schatten ideals are Banach–Lie groups with Lie algebra $\mathfrak{gl}_p(H) := B_p(H)$ (cf. [Ne00a, Def. IV.20] and [Mi89]). The group $\mathrm{GL}_\infty(H)$ is called the *Fredholm group.* It is contained in the monoid

$$\mathrm{Fred}(H) := \{A \in B(H)\colon \dim\ker A, \dim\operatorname{coker} A < \infty\}$$

of *Fredholm operators* on H. The group

$$\mathrm{U}_p(H) := \mathrm{U}(H) \cap (\mathbf{1} + B_p(H))$$

is a Lie group with Lie algebra

$$\mathfrak{u}_p(H) := \mathfrak{u}(H) \cap B_p(H) = \{X \in B_p(H)\colon X^* = -X\}.$$

With $\mathrm{Herm}_p(H) := \mathrm{Herm}(H) \cap B_p(H) = i\mathfrak{u}_p(H)$ we then have

$$\mathfrak{gl}_p(H) = \mathfrak{u}_p(H) \oplus \mathrm{Herm}_p(H) = \mathfrak{u}_p(H) \oplus i\mathfrak{u}_p(H)$$

and the polar map

$$\mathrm{U}_p(H) \times \mathrm{Herm}_p(H) \to \mathrm{GL}_p(H), \quad (u, X) \mapsto ue^X$$

is a diffeomorphism ([Ne00a, Prop. A.4]).

(b) The restricted classical Lie algebras are defined as follows. For $\mathfrak{g} = \mathfrak{gl}_2(H)$ we put $\mathfrak{g}_b := \mathfrak{gl}(H)$, and for $\mathfrak{g} = \mathfrak{gl}_2(H, I)$ we put $\mathfrak{g}_b := \mathfrak{gl}(H, I)$. Let $D \in \mathfrak{g}_b$ be a hermitian element with finite spectrum, $\mathfrak{g}_b^0 := \mathfrak{z}_{\mathfrak{g}_b}(D)$ and $\mathfrak{g}^0 := \mathfrak{z}_{\mathfrak{g}}(D)$. Then the Lie algebra

$$\mathfrak{g}_r := \mathfrak{g}(D) := \mathfrak{g} + \mathfrak{g}_b^0$$

is called the *restricted Lie algebra* associated to $\mathfrak{g}$ and D. For $\mathfrak{g} = \mathfrak{gl}_2(H)$ we also write $\mathfrak{gl}_2(H, D) := \mathfrak{g}(D)$ and for $\mathfrak{g} = \mathfrak{gl}_2(H, I)$ we likewise write $\mathfrak{gl}_2(H, I, D) := \mathfrak{g}(D)$. ■

Examples I.9. (a) Let $\mathfrak{g} = \mathfrak{gl}_2(H)$ for an infinite-dimensional complex Hilbert space H and $D = D^* \in \mathfrak{gl}(H)$ diagonalizable with the eigenvalues $d_1, \ldots, d_k$ and the corresponding eigenspaces $H_j := \ker(D - d_j\mathbf{1})$. Then $H = H_1 \oplus \ldots \oplus H_k$ is an orthogonal decomposition, $\mathfrak{g}^0$ consists of all elements in $\mathfrak{g}$ preserving this decomposition, and therefore

$$\mathfrak{g}^0 \cong \bigoplus_{j=1}^k \mathfrak{gl}_2(H_j) \quad \text{and} \quad \mathfrak{g}_b^0 \cong \bigoplus_{j=1}^k \mathfrak{gl}(H_j)$$

lead to

$$\mathfrak{gl}_2(H, D) = \{X = (x_{ij})_{i,j=1,\ldots,k}\colon (\forall i \ne j)\, x_{ij} \in B_2(H_j, H_i)\}.$$

(b) For $\mathfrak{g} = \mathfrak{gl}_2(H, I)$ and $D = D^* \in \mathfrak{gl}(H, I)$, we write $d_1, \ldots, d_k$ for the positive eigenvalues of D, $d_{-j} := -d_j$, and $d_0 := 0$. Then $\mathrm{Spec}(D) \cup \{0\} = \{d_j\colon j = -k, \ldots, k\}$ (cf. [Ne01a, Lemma III.12]) and for $H_j := \ker(D - d_j\mathbf{1})$ we obtain an orthogonal decomposition

$$H = H_k \oplus \ldots \oplus H_0 \oplus \ldots \oplus H_{-k}$$

with $I.H_j = H_{-j}$, so that $H_0 = \ker D$ is I-invariant, but this space might be trivial. With $I_0 := I|_{H_0}$ and Remark I.2 we now obtain

$$\mathfrak{g}^0 \cong \mathfrak{gl}_2(H_0, I_0) \oplus \bigoplus_{j=1}^k \mathfrak{gl}_2(H_j) \quad \text{and} \quad \mathfrak{g}_b^0 \cong \mathfrak{gl}(H_0, I_0) \oplus \bigoplus_{j=1}^k \mathfrak{gl}(H_j). \qquad ■$$

In the following we will keep the notation of Examples I.9 whenever we discuss specific properties of the Lie algebras $\mathfrak{g}(D)$ and the corresponding groups.

Proposition I.10. (a) $\mathfrak{gl}_2(H,D)$ *is perfect if and only if* $\dim H = \infty$.
(b) $\mathfrak{gl}_2(H,I,D)$ *is perfect if not* $(\dim H = 2$ *and* $I^2 = \mathbf{1})$.

Proof. (a) Let $\mathfrak g := \mathfrak{gl}_2(H)$. If If $\dim H < \infty$, then $\mathfrak g = \mathfrak{gl}(H)$ is not perfect, and $[\mathfrak g,\mathfrak g] = \mathfrak{sl}(H)$. Suppose that $\dim H = \infty$. Then there exists at least on j for which H_j is infinite-dimensional. We consider the direct sum decomposition

$$\mathfrak g(D) = \mathfrak g_b^0 \oplus \bigoplus_{i\neq j} B_2(H_j,H_i).$$

Since $\mathfrak g(D) = \mathfrak g_b^0 + [D,\mathfrak g]$, it suffices to show that $\mathfrak g_b^0 \subseteq [\mathfrak g(D),\mathfrak g(D)]$. We recall that $\mathfrak g_b^0 \cong \bigoplus_{j=1}^k \mathfrak{gl}(H_j)$ (Examples I.9) and view each $\mathfrak{gl}(H_j)$ as a subalgebra of $\mathfrak g_b^0$.

The commutator algebra of $\mathfrak g_b^0$ contains the full algebra $\mathfrak{gl}(H_i) \subseteq \mathfrak g_b^0$ whenever H_i is infinite-dimensional ([Ha67, Cor. 2 to Probl. 186]). If H_i is finite-dimensional, then we choose j with H_j infinite-dimensional and consider elements of $B(H_i \oplus H_j)$ as (2×2)-block matrices. For such matrices we have

$$\Big[\begin{pmatrix} 0 & B \\ 0 & 0\end{pmatrix}, \begin{pmatrix} 0 & 0 \\ C & 0\end{pmatrix}\Big] = \begin{pmatrix} BC & 0 \\ 0 & -CB\end{pmatrix}.$$

Then $[\mathfrak g(D),\mathfrak g(D)]$ contains $\mathfrak{gl}(H_j)$ and therefore also $B_2(H_i,H_j)B_2(H_j,H_i) = \mathfrak{gl}(H_i)$. Hence $[\mathfrak g(D),\mathfrak g(D)]$ contains $\mathfrak g_b^0$, which shows that $\mathfrak g(D)$ is perfect.
(b) Let $\mathfrak g := \mathfrak{gl}_2(H,I)$. If H is of finite dimension n, then $\mathfrak g(D) = \mathfrak g \cong \mathfrak{sp}(n,\mathbb C)$ or $\mathfrak o(n,\mathbb C)$ is semisimple and therefore perfect unless $n=2$ and $I^2 = \mathbf 1$ (Lemma I.3).

Suppose that H is infinite-dimensional. We have $\mathfrak g(D) = \mathfrak g_b^0 + [D,\mathfrak g]$, so that it suffices to show that $\mathfrak g_b^0 \subseteq [\mathfrak g(D),\mathfrak g(D)]$. Using the 3×3-block description of $\mathfrak{gl}(H,I)$ according to $H = H_+ \oplus H_0 \oplus H_-$ with $H_\pm := \sum_{j=1}^k H_{\pm j}$, we see that $\mathfrak g(D)$ is adapted to this decomposition, and we get with $D_+ := D|_{H_+}$:

$$\mathfrak g(D) \supseteq \mathfrak{gl}_2(H_+,D_+) \cong \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -a^\top\end{pmatrix} : a \in \mathfrak{gl}_2(H_+,D_+)\right\} \supseteq \sum_{j=1}^k \mathfrak{gl}(H_j).$$

If H_+ is finite-dimensional, then the same holds for $H_- = I.H_+$, so that $\mathfrak g(D) = \mathfrak{gl}(H,I)$, and the perfectness of $\mathfrak g(D)$ follows from Lemma I.3.

If H_+ is infinite-dimensional, then $\mathfrak{gl}_2(H_+,D_+)$ is perfect by (a), and each $\mathfrak{gl}(H_j), j = 1,\ldots,k$, is contained in this algebra, so that it remains to see that $\mathfrak{gl}(H_0,I_0) \subseteq [\mathfrak g(D),\mathfrak g(D)]$. If $\mathfrak{gl}(H_0,I_0)$ itself is perfect, this is trivial. If this is not the case, then $\dim H_0 = 2$ and $I^2 = \mathbf 1$ (Lemma I.3). Then we extend H_0 to a four-dimensional I-invariant subspace $\widetilde H_0$ of $H_1 + H_0 + H_{-1}$, set $\widetilde I_0 := I|_{\widetilde H_0}$, and obtain

$$[\mathfrak g(D),\mathfrak g(D)] \supseteq [\mathfrak o(\widetilde H_0,\widetilde I_0),\mathfrak o(\widetilde H_0,\widetilde I_0)] = \mathfrak o(\widetilde H_0,\widetilde I_0) \supseteq \mathfrak o(H_0,I_0) \cong \mathfrak{gl}(H_0,I_0).$$

This implies that $\mathfrak g(D)$ is perfect. ∎

Proposition I.11. *For* $k_\infty := |\{j \in \{1,\ldots,k\} \colon \dim H_j = \infty\}|$ *we have*

$$H^2_c(\mathfrak{gl}_2(H,D),\mathbb C) \cong \mathbb C^{k_\infty - 1} \quad \textit{and} \quad H^2_c(\mathfrak{gl}_2(H,I,D),\mathbb C) \cong \mathbb C^{k_\infty}.$$

Each cohomology class contains a cocycle of the form

$$\varphi(z)(x+d,x'+d') := \mathrm{tr}([z,x]x') = \mathrm{tr}(z[x,x']) \quad \textit{for} \quad d,d' \in \mathfrak g(D)^0, x,x' \in \mathfrak g,$$

where $z \in \mathfrak z(\mathfrak g(D)^0)$.

Proof. In [Ne01a, Ex. III.13] we have seen how to describe the space $H^2_c(\mathfrak{g}(D),\mathbb{C})$ in all cases. Each continuous 2-cocycle on $\mathfrak{g}(D)$ is equivalent to a cocycle $\varphi(z)$ given as follows. For $z \in \mathfrak{z}(\mathfrak{g}_b^0)$ we define

$$\varphi(z)(x+d,x'+d') := \mathrm{tr}([z,x]x') = \mathrm{tr}(z[x,x']), \quad d,d' \in \mathfrak{g}_b^0, x,x' \in \mathfrak{g}.$$

This cocycle is trivial if and only if $z \in \mathfrak{z}(\mathfrak{g}^0) + \mathbb{C}\mathbf{1}$, which for $\mathfrak{g} = \mathfrak{gl}_2(H,I)$ is equivalent to $z \in \mathfrak{z}(\mathfrak{g}^0)$. We always have $\mathfrak{z}(\mathfrak{gl}(H_0,I_0)) = \mathfrak{z}(\mathfrak{gl}_2(H_0,I_0))$, and for $j>0$ we have $\mathfrak{z}(\mathfrak{gl}(H_j,I_j)) = \mathbb{C}\,\mathrm{id}_{H_j}$ and $\mathfrak{z}(\mathfrak{gl}_2(H_j,I_j)) = \mathbf{0}$ if H_j is infinite-dimensional. We conclude that each cohomology class can be represented by $\varphi(z)$ with $z = \sum_{\dim H_j=\infty} z_j\,\mathrm{id}_{H_j}$, and that such a cocycle is trivial if and only if $z=0$, for $\mathfrak{g} = \mathfrak{gl}_2(H,I)$, or $z_1 = \ldots = z_k$, for $\mathfrak{g} = \mathfrak{gl}_2(H)$. This implies the assertion. ∎

Universal central extensions

Definition I.12. (a) Let $\mathfrak{g}$ be a topological Lie algebra over $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$, $\mathfrak{z}$ a topological vector space, and $\omega \in Z^2_c(\mathfrak{g},\mathfrak{z})$ a continuous $\mathfrak{z}$-valued 2-cocycle. Then we write $\mathfrak{g}\oplus_\omega\mathfrak{z}$ for the topological Lie algebra whose underlying topological vector space is the product space $\mathfrak{g}\times\mathfrak{z}$ and whose Lie bracket is defined by

$$[(x,z),(x',z')] = \big([x,x'],\omega(x,x')\big).$$

Then $q\colon \mathfrak{g}\oplus_\omega\mathfrak{z} \to \mathfrak{g}, (x,z)\mapsto x$ is a central extension and $\sigma\colon \mathfrak{g}\to\mathfrak{g}\oplus_\omega\mathfrak{z}, x\mapsto (x,0)$ is a continuous linear section of q.
(b) Let $\mathfrak{a}$ be a topological vector space considered as a trivial $\mathfrak{g}$-module. We call a central extension $q\colon \widehat{\mathfrak{g}} = \mathfrak{g}\oplus_\omega\mathfrak{z} \to \mathfrak{g}$ with $\mathfrak{z} = \ker q$ *weakly* $\mathfrak{a}$*-universal* if the map

$$\delta_\mathfrak{a}\colon \mathrm{Lin}(\mathfrak{z},\mathfrak{a}) \to H^2_c(\mathfrak{g},\mathfrak{a}), \quad \gamma\mapsto[\gamma\circ\omega]$$

is bijective.

We call $q\colon\widehat{\mathfrak{g}}\to\mathfrak{g}$ *universal for* $\mathfrak{a}$ if for each central $\mathfrak{a}$-extension $q_1\colon\widehat{\mathfrak{g}}_1 := \mathfrak{g}\oplus_f\mathfrak{a}\to\mathfrak{g}$ of $\mathfrak{g}$ there exists a unique continuous homomorphism $\varphi\colon\widehat{\mathfrak{g}}\to\widehat{\mathfrak{g}}_1$ with $q_1\circ\varphi = q$. In view of [Ne01b, Remark I.10(b)], the $\mathfrak{a}$-universality is equivalent to the weak $\mathfrak{a}$-universality plus $\mathrm{Hom}(\mathfrak{g},\mathfrak{a}) = \mathbf{0}$.

For $\mathfrak{a}\neq\{0\}$ this implies in particular that $\mathrm{Hom}(\mathfrak{g},\mathbb{K}) = \{0\}$ which for a Banach–Lie algebra $\mathfrak{g}$ implies that $\mathfrak{g}$ is topologically perfect. ∎

Proposition I.13. *Let $\mathfrak{g}$ be a perfect $\mathbb{K}$-Banach–Lie algebra for which $H^2_c(\mathfrak{g},\mathbb{K})$ is finite-dimensional. Then $\mathfrak{g}$ has up to isomorphism a unique $\mathbb{K}$-universal central extension*

$$\mathfrak{z}\hookrightarrow\widehat{\mathfrak{g}} := \mathfrak{g}\oplus_\omega\mathfrak{z}\to\mathfrak{g}$$

which, in addition, is universal for all Fréchet spaces.

Proof. The uniqueness follows from [Ne01b, Lemma I.13] and the existence from [Ne01b, Cor. II.12]. ∎

Remark I.14. The background for Definition I.12(b) is that the central extension $q\colon\widehat{\mathfrak{g}} = \mathfrak{g}\oplus_\omega\mathfrak{z}\to\mathfrak{g}$ defines for each topological vector space $\mathfrak{a}$ an exact sequence containing $\delta_\mathfrak{a}$. To describe this exact sequence, let $Z^2_c(\widehat{\mathfrak{g}},\mathfrak{z},\mathfrak{a})$ denotes the set of all continuous $\mathfrak{a}$-valued 2-cocycles $\omega\in Z^2_c(\widehat{\mathfrak{g}},\mathfrak{a})$ with $\omega(\mathfrak{z},\widehat{\mathfrak{g}}) = \{0\}$. Then $B^2_c(\widehat{\mathfrak{g}},\mathfrak{a})\subseteq Z^2_c(\widehat{\mathfrak{g}},\mathfrak{z},\mathfrak{a})$ because $\beta([\widehat{\mathfrak{g}},\mathfrak{z}]) = \{0\}$ for $\beta\in\mathrm{Lin}(\widehat{\mathfrak{g}},\mathfrak{z})$, and we define

$$H^2_c(\widehat{\mathfrak{g}},\mathfrak{z},\mathfrak{a}) := Z^2_c(\widehat{\mathfrak{g}},\mathfrak{z},\mathfrak{a})/B^2_c(\widehat{\mathfrak{g}},\mathfrak{z}).$$

According to [Ne01a, Th. I.4], we now have the exact sequence

$$(1.1) \qquad \mathbf{0}\to\mathrm{Hom}(\mathfrak{g},\mathfrak{a})\xrightarrow{q^*}\mathrm{Hom}(\widehat{\mathfrak{g}},\mathfrak{a})\xrightarrow{\mathrm{res}}\mathrm{Lin}(\mathfrak{z},\mathfrak{a})\xrightarrow{\delta_\mathfrak{a}}H^2_c(\mathfrak{g},\mathfrak{a})\xrightarrow{q^*}H^2_c(\widehat{\mathfrak{g}},\mathfrak{z},\mathfrak{a})\to\mathbf{0}.$$

∎

We will see in Section IV below how to realize the universal central extensions of the restricted Lie algebras $\mathfrak{gl}_2(H,D)$ and $\mathfrak{gl}_2(H,I,D)$ explicitly.

II. Homotopy groups of classical groups

In this section we first discuss a quite elementary proof of Kuiper's Theorem for inseparable Hilbert spaces H over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, and then we use this result to prove that various classical groups of operators on Hilbert spaces such as $\mathrm{GL}(H, I)$ are contractible. Then we turn to the direct limit groups $\mathrm{GL}(J, \mathbb{K})$ of those invertible $J \times J$-matrices g for which $g - \mathbf{1}$ has only finitely many non-zero entries. We will see that for an infinite set J this group is weakly homotopy equivalent to $\mathrm{GL}(\mathbb{N}, \mathbb{K})$, i.e., to the direct limit of the groups $\mathrm{GL}(n, \mathbb{K})$. Combining these insights with general results of Palais, we compute the homotopy groups of the congruence groups $\mathrm{GL}_p(H)$ of the Schatten ideals $B_p(H) \subseteq B(H)$ and $\mathrm{GL}_p(H, I) := \mathrm{GL}(H, I) \cap \mathrm{GL}_p(H)$ for $1 \leq p \leq \infty$. In the next section we deal with groups corresponding to the restricted Lie algebras $\mathfrak{g}(D)$.

Kuiper's Theorem

In this subsection we explain how Kuiper's Theorem that the group $\mathrm{GL}(H, \mathbb{K})$ of $\mathbb{K}$-linear continuous operators on an infinite-dimensional separable $\mathbb{K}$-Hilbert space H is contractible ([Ku65]) can be obtained in a quite elementary way for inseparable Hilbert spaces. [1]

The observation is based on the following lemma, which is a refinement of [vNeu50, Th. 14.10].

Lemma II.1. *Let H be a Hilbert space over $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ and $\mathcal{M} \subseteq B(H, \mathbb{K})$ a separable set of operators. Then there exists an orthogonal decomposition $H \cong \widehat{\bigoplus}_{j \in J} H_j$ into $\mathcal{M}$-invariant subspaces such that each H_j is separable.*

If, in addition, H is infinite-dimensional, then the spaces H_j can be chosen in such a way that they are all infinite-dimensional, hence isomorphic to $l^2(\mathbb{N}, \mathbb{K})$.

Proof. Since the closed $*$-subalgebra of $B(H)$ generated by $\mathcal{M}$ is separable, we may assume that $\mathcal{M}$ is $*$-invariant with $\mathbf{1} \in \mathcal{M}$, otherwise we replace it by $\mathcal{M} \cup \mathcal{M}^* \cup \{\mathbf{1}\}$. Now the assertion follows by a standard application of Zorn's Lemma. Let H_j, $j \in J$, be a maximal set of non-zero closed $\mathcal{M}$-invariant separable subspaces of H such that the sum $\sum_{j \in J} H_j$ is orthogonal. Set $H_0 := \overline{\sum_{j \in J} H_j}$. Then $H_0^\perp$ is $\mathcal{M}$-invariant because $\mathcal{M}$ is $*$-invariant. Assume that $H_0 \neq H$. For $0 \neq v \in H_0^\perp$ the subspace $H_v := \overline{\mathrm{span}\, \mathcal{M}.v}$ is a cyclic hence separable subspace orthogonal to all the spaces H_j, contradicting the maximality of the family $(H_j)_{j \in J}$. This proves the first assertion.

To prove the second part, let us assume that H is infinite-dimensional and consider a decomposition $H \cong \widehat{\bigoplus}_{j \in J} H_j$ as above. Let

$$I := \{j \in J \colon \dim H_j < \infty\}.$$

Case 1: If I is finite, then there exists a $j_0 \in J \setminus I$. Replacing H_{j_0} by $H_{j_0} + \sum_{i \in I} H_i$, we obtain the desired decomposition.

Case 2: If I is infinite, then $|I \times \mathbb{N}| = |I|$ ([La93, App. 2]) implies that I can be partitioned into infinite countable subsets I_i, $i \in I$. Then all the subspaces $K_i := \overline{\sum_{j \in I_i} H_j}$ are infinite-dimensional and separable, and we have the derived orthogonal decomposition of H:

$$H = \widehat{\bigoplus}_{j \in J \setminus I} H_j \oplus \widehat{\bigoplus}_{i \in I} K_i. \qquad \blacksquare$$

[1] The proof is based on a hint in a footnote in Kuiper's paper but we don't know of any direct reference in the literature which provides the result also for real and quaternionic inseparable Hilbert spaces. For complex Hilbert spaces it follows from results of Brüning and Willgerodt on the contractibility of unit groups of von Neumann algebras of infinite type ([BW76]).

Theorem II.2. (Palais) *For a metrizable topological manifold modeled over a sequentially complete locally convex space the following are equivalent:*

(1) $\pi_n(X) = 0$ *for all* $n \in \mathbb{N}_0$.

(2) X *is contractible.*

Proof. [Pa66, Cor. to Th. 15] ■

The proof of the following proposition is inspired by the setting in Mityagin's paper [Mit70].

Proposition II.3. *Let Y be a separable topological space and H an inseparable Hilbert space over $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. Then each continuous map $f\colon Y \to \mathrm{GL}(H, \mathbb{K})$ is homotopic to a constant map.*

Proof. Since $f(Y)$ is a separable set of operators, Lemma II.1 implies that there exists a set J and an isomorphism $H \to l^2(J, H_s)$ with $H_s := l^2(\mathbb{N}, \mathbb{K})$ such that the operators in $f(Y)$ are diagonal operators on $l^2(J, H_s)$.

Step 1: Since H is not separable, the set J is (uncountably) infinite. First we consider a decomposition $J = J_1 \dot\cup J_2$ with $|J_1| = |J_2| = |J|$. This leads to an orthogonal decomposition $H \cong H \oplus H$, and we consider operators on H accordingly as block 2×2-matrices. Let $f\colon Y \to \mathrm{GL}(H, \mathbb{K})$ be as above. Then

$$f(y) = \begin{pmatrix} g_1(y) & 0 \\ 0 & g_2(y) \end{pmatrix},$$

where $g_j\colon Y \to \mathrm{GL}(H, \mathbb{K})$ are continuous maps. We claim that f is homotopic to the map

$$f_1(y) = \begin{pmatrix} g_1(y)g_2(y) & 0 \\ 0 & \mathbf{1} \end{pmatrix}. \tag{2.1}$$

It suffices to show that

$$f(y)^{-1} f_1(y) = \begin{pmatrix} g_2(y) & 0 \\ 0 & g_2(y)^{-1} \end{pmatrix}$$

is homotopic to a constant map. This is implemented by the homotopy

$$H(t, a) := \begin{pmatrix} 1 & 0 \\ t(a^{-1} - 1) & 1 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t(a - 1) & 1 \end{pmatrix} \begin{pmatrix} 1 & -ta^{-1} \\ 0 & 1 \end{pmatrix}$$

which satisfies

$$H(1, a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \quad \text{and} \quad H(0, a) = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix}.$$

Step 2: In view of Step 1, we may assume that $f_1\colon Y \to \mathrm{GL}(H, \mathbb{K})$ has the form (2.1). Next we observe that $H \cong l^2(\mathbb{N}, H)$ because $|J| = |\mathbb{N} \times J|$ ([La93, App. 2]). Therefore

$$H \cong H \oplus l^2(\mathbb{N}, H),$$

and we may assume that

$$f_1(y) = \mathrm{diag}(g(y), \mathbf{1}, \mathbf{1}, \ldots).$$

Partitioning $\mathbb{N}$ into odd numbers $\mathbb{N}_{\mathrm{odd}}$ and even numbers $\mathbb{N}_{\mathrm{even}}$, and writing accordingly

$$l^2(\mathbb{N}, H) \cong l^2(\mathbb{N}_{\mathrm{odd}}, H) \oplus l^2(\mathbb{N}_{\mathrm{even}}, H),$$

it follows from Step 1 that the constant map $Y \to \mathrm{GL}(l^2(\mathbb{N}, H), \mathbb{K})$ is equivalent to the map

$$y \mapsto \mathrm{diag}(g(y)^{-1}, g(y), g(y)^{-1}, g(y), \ldots).$$

Therefore f_1 is homotopic to

$$f_2(y) = \mathrm{diag}(g(y), g(y)^{-1}, g(y), g(y)^{-1}, \ldots).$$

Applying the same argument again to the decomposition

$$H \cong H \oplus l^2(\mathbb{N}, H) \cong l^2(\{0\} \cup \mathbb{N}_{\mathrm{even}}, H) \oplus l^2(\mathbb{N}_{\mathrm{odd}}, H),$$

we see that f_2 is homotopic to a constant map. ■

Theorem II.4. (Kuiper's Theorem for general Hilbert spaces) *If H is an infinite-dimensional Hilbert space over $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, then the group $\mathrm{GL}(H, \mathbb{K})$ is contractible.*

Proof. In view of Theorem II.2, it suffices to show that all homotopy groups of $\mathrm{GL}(H,\mathbb{K})$ vanish. In [Ku65], this is proved for infinite-dimensional separable Hilbert spaces, and for inseparable Hilbert spaces, this follows from Proposition II.3 because the spheres $\mathbb{S}^k$, $k \in \mathbb{N}_0$, are separable. ■

Consequences of Kuiper's Theorem

Definition II.5. (a) If H is a Hilbert space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, then we define

$$\mathrm{U}(H,\mathbb{K}) := \{g \in \mathrm{GL}(H,\mathbb{K}) \colon g^*g = gg^* = \mathbf{1}\}$$

as the unitary part of this group. We also write

$$\mathrm{O}(H) := \mathrm{U}(H,\mathbb{R}), \quad \mathrm{U}(H) := \mathrm{U}(H,\mathbb{C}) \quad \text{and} \quad \mathrm{Sp}(H) := \mathrm{U}(H,\mathbb{H}).$$

(b) Let H be a complex Hilbert space and I be an antilinear isometry with $I^2 \in \{\pm\mathbf{1}\}$. For $I^2 = \mathbf{1}$ we then have

$$\mathrm{U}(H,I) := \mathrm{U}(H) \cap \mathrm{GL}(H,I) \cong \mathrm{O}(H_\mathbb{R}) \quad \text{with} \quad H_\mathbb{R} := \{x \in H \colon I.x = x\},$$

and for $I^2 = -\mathbf{1}$ we have

$$\mathrm{U}(H,I) \cong \mathrm{U}(H,\mathbb{H}) \cong \mathrm{Sp}(H),$$

where the quaternionic structure on H is given by the subalgebra $\mathbb{C}\mathbf{1} + \mathbb{C}I \cong \mathbb{H}$ of $B(H,\mathbb{R})$, the real linear endomorphisms of H.

(c) (Hermitian groups) Let H be a complex Hilbert space and $H = H_+ \oplus H_-$ be an orthogonal decomposition. Further let $T = T^* \in B(H)$ with $H_\pm = \ker(T \mp \mathbf{1})$. We define the corresponding *pseudo-unitary group*

$$\mathrm{U}(H_+, H_-) := \{g \in \mathrm{GL}(H) \colon Tg^*T^{-1} = g^{-1}\}.$$

We define $\Omega(x,y) := \mathrm{Im}\langle x, y\rangle$ and write $H^\mathbb{R}$ for the real Hilbert space underlying H. Then

$$\mathrm{Sp}(H,\Omega) := \{g \in \mathrm{GL}(H^\mathbb{R},\mathbb{R}) \colon (\forall v,w \in H^\mathbb{R})\, \Omega(g.v, g.w) = \Omega(v,w)\}$$

is called the *symplectic group of* H. If we start with the real Hilbert space $H^\mathbb{R}$ and consider an isometric complex structure I on $H^\mathbb{R}$, then we can define

$$\Omega(x,y) := -\langle I.x, y\rangle = \langle x, I.y\rangle$$

and put

$$\mathrm{Sp}(H^\mathbb{R}, I) := \{g \in \mathrm{GL}(H^\mathbb{R},\mathbb{R}) \colon (\forall v,w \in H^\mathbb{R})\, \Omega(g.v, g.w) = \Omega(v,w)\}.$$

It is easy to see that both constructions lead to isomorphic groups $\mathrm{Sp}(H^\mathbb{R}, I) \cong \mathrm{Sp}(H,\Omega)$.

Now let I be a conjugation on the complex Hilbert space H and $H_+ \subseteq H$ a subspace for which we get an orthogonal decomposition $H = H_+ \oplus H_-$ with $H_- := I.H_+$. Then we define

$$\mathrm{O}^*(H,I) := \mathrm{U}(H,I) \cap \mathrm{U}(H_+,H_-).$$ ■

Theorem II.6. *If H is an infinite-dimensional Hilbert space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, then the following groups are contractible:*

(i) *the group of $\mathbb{K}$-linear automorphisms $\mathrm{GL}(H,\mathbb{K})$.*

(ii) *the group of isometric* $\mathbb{K}$*-linear automorphisms* $\mathrm{U}(H,\mathbb{K})$*, and in particular the groups* $\mathrm{O}(H) = \mathrm{U}(H,\mathbb{R})$*,* $\mathrm{U}(H) = \mathrm{U}(H,\mathbb{C})$ *and* $\mathrm{Sp}(H) = \mathrm{U}(H,\mathbb{H})$*.*
(iii) *the group* $\mathrm{GL}(H,I)$ *if* H *is complex and* I *an antilinear isometry with* $I^2 \in \{\pm\mathbf{1}\}$*. Moreover,* $\mathrm{GL}(H,I)$ *has a smooth polar decomposition.*
(iv) *the hermitian groups* $\mathrm{U}(H_+,H_-)$*,* $H = H_+ \oplus H_-$ *an orthogonal decomposition with two infinite-dimensional summands,* $\mathrm{Sp}(H,\Omega)$*, and* $\mathrm{O}^*(H,I)$*.*

Proof. (i) is Theorem II.4.
(ii) follows from (i) and the polar decomposition $\mathrm{GL}(H,\mathbb{K}) \cong \mathrm{U}(H,\mathbb{K}) \times \mathrm{Herm}(H,\mathbb{K})$ of the group $\mathrm{GL}(H,\mathbb{K})$ with the unitary part $\mathrm{U}(H,\mathbb{K})$.
(iii) In view of Definition II.5(b), the group $\mathrm{U}(H,I)$ is contractible, because it is one of the groups in (ii). Hence the assertion follows from the polar decomposition of $\mathrm{GL}(H,I)$ which can be obtained as follows. We consider the automorphism $\tau(g) := I(g^*)^{-1}I^{-1}$ of $\mathrm{GL}(H)$ and write $\tau_{\mathfrak{g}}(x) := -Ix^*I^{-1}$ for the corresponding antilinear automorphism of its Lie algebra $\mathfrak{gl}(H)$. Then

$$\mathrm{GL}(H,I) = \mathrm{GL}(H)^\tau := \{g \in \mathrm{GL}(H) : \tau(g) = g\}.$$

Let $g = ue^x$ be the polar decomposition of $g \in \mathrm{GL}(H)$. Then $\tau(g) = \tau(u)e^{\tau_{\mathfrak{g}}(x)}$ is the polar decomposition of $\tau(g)$, so that the uniqueness of this decomposition implies that $\tau(g) = g$, is equivalent to $\tau(u) = u$ and $\tau_{\mathfrak{g}}(x) = x$, i.e., $u \in \mathrm{U}(H,I)$ and $x \in \mathrm{Herm}(H,I)$.
(iv) For the hermitian groups we will see below that they have polar decompositions with

$$\mathrm{U}(H_+,H_-) \cap \mathrm{U}(H) \cong \mathrm{U}(H_+) \times \mathrm{U}(H_-), \quad \mathrm{Sp}(H,\Omega) \cap \mathrm{O}(H^{\mathbb{R}}) \cong \mathrm{U}(H)$$

and

$$\mathrm{O}^*(H,I) \cap \mathrm{U}(H) \cong \mathrm{U}(H_+),$$

where $H \cong H_+ \oplus I.H_+$ as in Definition II.5(c). Therefore (ii) implies that all these groups are contractible.

To prove the polar decomposition of $\mathrm{U}(H_+,H_-)$, let $g \in \mathrm{GL}(H)$ with polar decomposition $g = ue^x$, $u \in \mathrm{U}(H)$ and $x = x^*$. For T as in Definition II.5(c) we consider the automorphism $\tau(g) := T(g^*)^{-1}T^{-1}$ of $\mathrm{GL}(H)$ and write $\tau_{\mathfrak{g}}(x) := -Tx^*T^{-1}$ for the corresponding antilinear automorphism of its Lie algebra $\mathfrak{gl}(H)$. Then $\tau(g) = \tau(u)e^{\tau_{\mathfrak{g}}(x)}$ is the polar decomposition of $\tau(g)$, so that the uniqueness of this decomposition implies that $\tau(g) = g$ is equivalent to $\tau(u) = u$ and $\tau_{\mathfrak{g}}(x) = x$. Therefore $g \in \mathrm{U}(H_+,H_-)$ if and only if

$$u \in \mathrm{U}(H_+,H_-) \cap \mathrm{U}(H) \cong \mathrm{U}(H_+) \times \mathrm{U}(H_-) \quad \text{and} \quad x \in \mathfrak{u}(H_+,H_-).$$

To see that $\mathrm{Sp}(H,\Omega)$ is adapted to the polar decomposition, we observe that

$$\Omega(x,y) = \mathrm{Im}\langle x,y\rangle = \mathrm{Re}\langle x,iy\rangle = (x,Jy),$$

where $(\cdot,\cdot) := \mathrm{Re}\langle\cdot,\cdot\rangle$ denotes the real scalar product on $H^{\mathbb{R}}$. Therefore $g \in \mathrm{Sp}(H,\Omega)$ is equivalent to $g^\top Jg = J$, i.e., $g = \tau(g) := J(g^\top)^{-1}J^{-1}$. Then τ is an involutive automorphism of $\mathrm{GL}(H^{\mathbb{R}})$ and $\tau_{\mathfrak{g}}(x) := -Jx^\top J^{-1}$ is the corresponding Lie algebra automorphism. Let $g = ue^x$ be the polar decomposition of $g \in \mathrm{GL}(H^{\mathbb{R}})$, where $u \in \mathrm{O}(H^{\mathbb{R}})$ and $x^\top = x$. Then $\tau(g) = \tau(u)e^{\tau_{\mathfrak{g}}(x)}$ is the polar decomposition of $\tau(g)$ because ue^{-x} is the polar decomposition of $g^{-\top}$. Therefore $g \in \mathrm{Sp}(H,\Omega)$ is equivalent to $\tau(u) = u$, i.e., $u \in \mathrm{U}(H)$, and to $Jx = -xJ$, i.e., x is antilinear.

The argument for the group $\mathrm{O}^*(H,I)$ is similar. ■

Homotopy groups of direct limit groups

Definition II.7. Let (X,x_0) and (Y,y_0) be pointed topological spaces. A map $f \in C_*(X,Y) := \{h \in C(X,Y) : h(x_0) = y_0\}$ is called a *weak homotopy equivalence* if all induced maps $\pi_k(f) : \pi_k(X,x_0) \to \pi_k(Y,y_0)$ are bijections.

A map $f \in C_*(X,Y)$ is called a *homotopy equivalence* if there exists $g \in C_*(Y,X)$ such that fg, resp., gf are homotopic to id_Y, resp., id_X in $C_*(Y,Y)$, resp., $C_*(X,X)$. ■

Theorem II.8. (a) *Let X be a locally convex topological vector space and $E \subseteq X$ a dense subspace. We endow E with the direct limit topology with respect to the finite-dimensional subspaces. For each open subset $U \subseteq X$ the continuous map $U \cap E \to U$ is a weak homotopy equivalence if $U \cap E$ is considered as a topological subspace of E.*
(b) *Let X and Y be metrizable locally convex topological vector spaces and $f: X \to Y$ a continuous linear map with dense range. For each open subset $U \subseteq Y$ let $V := f^{-1}(U)$ and $f_V := f|_V : V \to U$. Then f_V is a homotopy equivalence.*

Proof. These are Theorems 12 and 16 in [Pa66]. ■

Lemma II.9. *Let E be a real vector space endowed with the direct limit topology with respect to its finite-dimensional subspaces. Then the following assertions hold:*
(i) *Each linearly independent subset of E is closed and discrete.*
(ii) *Each compact subset of E is contained a finite-dimensional subspace.*
(iii) *For each subset $U \subseteq E$ and $u_0 \in U$ we have*

$$\pi_k(U, u_0) \cong \lim_{F \in \mathcal{F}} \pi_k(U \cap F, u_0),$$

where $\mathcal{F}$ denotes the directed set of all finite-dimensional subspaces $F \subseteq E$ containing u_0.
(iv) *If $U \subseteq E$ is a subset for which the intersection with all finite-dimensional subspaces are open, then the subspace topology on U coincides with the direct limit topology with respect to the sets $U \cap F$, $F \subseteq E$ a finite-dimensional subspace.*

Proof. (i) (cf. [Pa66, Lemma 5.2]) Let $S \subseteq E$ be a linearly independent subset. Then for each finite-dimensional subspace $F \subseteq E$ the intersection $S \cap F$ is closed, and therefore S is closed in E. The same argument implies that each subset of S is also closed in E. It follows in particular that S is a discrete topological space.
(ii) (cf. [Pa66, Lemma 5.3]) Let $K \subseteq E$ be a compact subset and $S \subseteq K$ a maximal linearly independent subset. Then $K \subseteq \operatorname{span} S$. In view of (i), S is closed, hence compact. On the other hand, S is discrete and therefore finite.
(iii) Let Y be a compact space with base point y_0 and $f: Y \to U$ a continuous map with $f(y_0) = u_0$. Then $f(Y)$ is a compact subset of E, hence contained in a finite-dimensional subspace F, and we clearly have $u_0 = f(y_0) \in F$.

For $Y = \mathbb{S}^k$, it follows that the natural homomorphism

$$\eta : \lim_{F \in \mathcal{F}} \pi_k(U \cap F, u_0) \to \pi_k(U, u_0)$$

is surjective. To see that it is also injective, suppose that $f: \mathbb{S}^k \to U \cap F$ is a continuous map which in U is homotopic to the constant map $\mathbb{S}^k \to \{u_0\}$. Let $H: [0,1] \times \mathbb{S}^k \to U$ be a homotopy with $H(0,x) = f(x)$ and $H(1,x) = u_0$. Then $\operatorname{im}(H)$ is contained in a finite-dimensional subspace $F \subseteq E$, and therefore the homotopy class of f in $\pi_k(U \cap F, u_0)$ is trivial. This implies that η is injective.
(iv) This follows from the observation that a subset $V \subseteq U$ is open in the subspace topology if and only if all intersections $V \cap F$, $F \subseteq E$ a finite-dimensional subspace, are open, because this already implies that V is open in E. ■

Definition II.10. Let J be an infinite set. We view a function $m: J \times J \to \mathbb{K}$ as a matrix with entries $m(i,j)$. In this sense we write $\mathbb{M}(J, \mathbb{K})$ for the set of all $J \times J$-matrices with at most finitely many non-zero entries in $\mathbb{K}$. Then $\mathbb{M}(J, \mathbb{K})$ is a real algebra with respect to matrix multiplication. It has a unit if and only if J is finite. We write $\mathbf{1} = (\delta_{ij})_{i,j \in J}$ for the identity matrix. Then $\mathbf{1} + \mathbb{M}(J, \mathbb{K})$ is a multiplicative monoid, and we define $\mathrm{GL}(J, \mathbb{K})$ to be its group of units. We endow $\mathrm{GL}(J, \mathbb{K})$ with the direct limit topology with respect to the subgroups $\mathrm{GL}(F, \mathbb{K}) := \mathrm{GL}(J, \mathbb{K}) \cap (\mathbf{1} + \mathbb{M}(F, \mathbb{K}))$, where $F \subseteq J$ is a finite subset. It follows directly from the constructions that the left and right multiplications in the group $\mathrm{GL}(J, \mathbb{K})$ are continuous, but if J is uncountable, then the multiplication is not jointly continuous ([Gl99, Th. 7.1]). Here we identify $\mathbb{M}(F, \mathbb{K})$ in a natural way with a subset of $\mathbb{M}(J, \mathbb{K})$ and likewise $\mathrm{GL}(F, \mathbb{K})$ with a subset of $\mathrm{GL}(J, \mathbb{K})$. ■

Proposition II.11. *Let J be an infinite set. Then for each injective map $\mathbb{N} \hookrightarrow J$ the corresponding map $\mathrm{GL}(\mathbb{N}, \mathbb{K}) \to \mathrm{GL}(J, \mathbb{K})$ is a weak homotopy equivalence.*

Proof. We may w.l.o.g. assume that $\mathbb{N} \subseteq J$. Let $\eta\colon \mathrm{GL}(\mathbb{N}, \mathbb{K}) \hookrightarrow \mathrm{GL}(J, \mathbb{K})$ be the corresponding embedding of groups.

Let Y be a compact space and $f\colon Y \to \mathrm{GL}(J, \mathbb{K})$ be a continuous map. Then there exists a finite subset $F \subseteq J$ with $f(Y) \subseteq \mathrm{GL}(F, \mathbb{K})$ (Lemma II.9(ii)). If $F' \subseteq J$ is finite with $F \cap F' = \emptyset$ and $|F| = |F'| = n$, then $\mathrm{GL}(F \cup F', \mathbb{K}) \cong \mathrm{GL}(2n, \mathbb{K})$, where we identify F with $\{1, \ldots, n\}$ and F' with $\{n+1, \ldots, 2n\}$. Then f is a map of the form

$$f(y) = \begin{pmatrix} g(y) & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{pmatrix},$$

where we write the elements of $\mathrm{GL}(2n, \mathbb{K})$ as block 2×2-matrices with entries in $\mathbb{M}(n, \mathbb{K})$, and $g\colon Y \to \mathrm{GL}(n, \mathbb{K})$ is a continuous map.

We consider the map $H\colon [0,1] \times \mathrm{GL}(n, \mathbb{K}) \to \mathrm{GL}(2n, \mathbb{K})$ given by

$$H(t, a) := \begin{pmatrix} 1 & 0 \\ t(a^{-1} - 1) & 1 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t(a-1) & 1 \end{pmatrix} \begin{pmatrix} 1 & -ta^{-1} \\ 0 & 1 \end{pmatrix}$$

which satisfies

$$H(1, a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \quad \text{and} \quad H(0, a) = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix}.$$

Then

$$\widetilde{H}\colon [0,1] \times Y \to \mathrm{GL}(2n, \mathbb{K}), \quad (t, y) \mapsto f(y) H(t, g(y))^{-1}$$

is continuous with $\widetilde{H}(0, y) = f(y)$ and

$$\widetilde{H}(1, y) = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & g(y)^{-1} \end{pmatrix}.$$

This construction shows that every continuous map $f\colon Y \to \mathrm{GL}(F, \mathbb{K})$ is homotopic in $\mathrm{GL}(J, \mathbb{K})$ to a continuous map $f'\colon Y \to \mathrm{GL}(F', \mathbb{K})$.

In particular we see that for each continuous map $f\colon Y \to \mathrm{GL}(J, \mathbb{K})$ there exists a finite subset $E \subseteq \mathbb{N}$ such that f is homotopic to a continuous map $\widetilde{f}\colon Y \to \mathrm{GL}(E, \mathbb{K})$. In fact, with F as above, we simply choose $E \subseteq \mathbb{N}$ such that $|E| = |F|$ and $E \cap F = \emptyset$. This argument shows that the natural homomorphism $\pi_k(\eta)\colon \pi_k(\mathrm{GL}(\mathbb{N}, \mathbb{K})) \to \pi_k(\mathrm{GL}(J, \mathbb{K}))$ is surjective.

To see that $\pi_k(\eta)$ is injective, suppose that $f\colon Y \to \mathrm{GL}(n, \mathbb{K}) \subseteq \mathrm{GL}(\mathbb{N}, \mathbb{K})$ is in $\mathrm{GL}(J, \mathbb{K})$ homotopic to a constant map. Let $H\colon [0,1] \times Y \to \mathrm{GL}(J, \mathbb{K})$ be a homotopy with $H(0, y) = \mathbf{1}$ and $H(1, y) = f(y)$ for all $y \in Y$. Then there exists a finite subset $F \subseteq J$ with $\mathrm{im}(H) \subseteq \mathrm{GL}(F, \mathbb{K})$. Then we may assume that $F \supseteq \{1, \ldots, n\}$, and since $\mathrm{GL}(|F|, \mathbb{K}) \cong \mathrm{GL}(F, \mathbb{K})$, we see that the homotopy class of f vanishes in $\mathrm{GL}(|F|, \mathbb{K}) \subseteq \mathrm{GL}(\mathbb{N}, \mathbb{K})$. In particular f is homotopic to a constant map in $\mathrm{GL}(\mathbb{N}, \mathbb{K})$. ■

The homotopy groups for $\mathrm{GL}(\mathbb{N}, \mathbb{K})$ and hence for all groups $\mathrm{GL}(J, \mathbb{K})$, where J is an infinite set, are given by the following theorem.

Theorem II.12. (Bott Periodicity Theorem) *Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ and $d := \dim_{\mathbb{R}} \mathbb{K}$. Then for $k \leq d(n+1) - 3$ and $q \in \mathbb{N}$ the maps*

$$\pi_k(\mathrm{GL}(n, \mathbb{K})) \to \pi_k(\mathrm{GL}(n+q, \mathbb{K}))$$

are isomorphisms, so that

$$\pi_k(\mathrm{GL}(\mathbb{N}, \mathbb{K})) \cong \pi_k(\mathrm{GL}(n, \mathbb{K})).$$

Moreover, we have the periodicity relations

$$\pi_{n+2}(\mathrm{GL}(\mathbb{N}, \mathbb{C})) \cong \pi_n(\mathrm{GL}(\mathbb{N}, \mathbb{C})), \quad \pi_{n+4}(\mathrm{GL}(\mathbb{N}, \mathbb{R})) \cong \pi_n(\mathrm{GL}(\mathbb{N}, \mathbb{H})),$$

$$\pi_{n+4}(\mathrm{GL}(\mathbb{N},\mathbb{H})) \cong \pi_n(\mathrm{GL}(\mathbb{N},\mathbb{R})), \quad \pi_n(\mathrm{GL}(\mathbb{N},\mathbb{H})/\mathrm{GL}(\mathbb{N},\mathbb{C})) \cong \pi_{n+1}(\mathrm{GL}(\mathbb{N},\mathbb{H})),$$
$$\pi_n(\mathrm{GL}(\mathbb{N},\mathbb{C})/\mathrm{GL}(\mathbb{N},\mathbb{R})) := \lim_{m\to\infty} \pi_n(\mathrm{GL}(m,\mathbb{C})/\mathrm{GL}(m,\mathbb{R})) \cong \pi_{n+2}(\mathrm{GL}(\mathbb{N},\mathbb{H})),$$
$$\pi_n(\mathrm{GL}(\mathbb{N},\mathbb{R})/\mathrm{GL}(\mathbb{N},\mathbb{C})) := \lim_{m\to\infty} \pi_n(\mathrm{GL}(2m,\mathbb{R})/\mathrm{GL}(m,\mathbb{C})) \cong \pi_{n+1}(\mathrm{GL}(\mathbb{N},\mathbb{R})),$$
$$\pi_n(\mathrm{GL}(\mathbb{N},\mathbb{C})/\mathrm{GL}(\mathbb{N},\mathbb{H})) := \lim_{m\to\infty} \pi_n(\mathrm{GL}(2m,\mathbb{C})/\mathrm{GL}(m,\mathbb{H})) \cong \pi_{n+2}(\mathrm{GL}(\mathbb{N},\mathbb{R})).$$

In particular the homotopy groups of $\mathrm{GL}(\mathbb{N},\mathbb{K})$ *are determined by the following table:*

	$\mathrm{GL}(\mathbb{N},\mathbb{R})$	$\mathrm{GL}(\mathbb{N},\mathbb{C})$	$\mathrm{GL}(\mathbb{N},\mathbb{H})$
π_0	$\mathbb{Z}_2$	$\mathbf{0}$	$\mathbf{0}$
π_1	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbf{0}$
π_2	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
π_3	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$

Proof. The first easy part is [Hu94, Th. 8.4.1] and the remaining assertions can be found in [Bo59, pp.314ff] (cf. also [Hu94, Cor. 9.5.2]).

For the sake of completeness, we include a proof of the first part. Using the polar decomposition, we may consider the corresponding maps of the unitary groups $\mathrm{U}(n,\mathbb{K})$. To understand the effect of the inclusion maps $\mathrm{U}(n,\mathbb{K}) \to \mathrm{U}(n+1,\mathbb{K})$ for the homotopy groups, we consider the transitive action of $\mathrm{U}(n+1,\mathbb{K})$ on the sphere $\mathbb{S}^{d(n+1)-1}$ which leads to a locally trivial principal bundle

$$\mathrm{U}(n,\mathbb{K}) \hookrightarrow \mathrm{U}(n+1,\mathbb{K}) \to \mathbb{S}^{d(n+1)-1}.$$

The exact homotopy sequence of this bundles contains the piece

$$\ldots \to \pi_{k+1}(\mathbb{S}^{d(n+1)-1}) \to \pi_k(\mathrm{U}(n,\mathbb{K})) \xrightarrow{\pi_k(\eta_n)} \pi_k(\mathrm{U}(n+1,\mathbb{K})) \to \pi_k(\mathbb{S}^{d(n+1)-1}) \to \ldots.$$

For $k < d(n+1)-1$, i.e., $k \leq d(n+1)-2$ the group $\pi_k(\mathbb{S}^{d(n+1)-1})$ vanishes (this follows by smoothing and Sard's Theorem), so that $\pi_k(\eta_n)$ is surjective. If, in addition, $k \leq d(n+1)-3$, then $k+1 < d(n+1)-1$ implies that also $\pi_{k+1}(\mathbb{S}^{d(n+1)-1})$ vanishes, so that the injectivity of $\pi_k(\eta_n)$ follows. ■

Homotopy groups of congruence subgroups for Schatten ideals

Let H be a Hilbert space over $\mathbb{K} \in \{\mathbb{R},\mathbb{C},\mathbb{H}\}$. For $x,y \in H$ we define $P_{x,y}(v) := \langle v,y\rangle x$ and put $P_x := P_{x,x}$. Note that $P_{x,y} \in B_1(H)$.

Lemma II.13. *Let* H *be a* $\mathbb{K}$*-Hilbert space and* $(e_j)_{j\in J}$ *an orthonormal basis. Then* $B_0(H) := \mathrm{span}\{P_{e_j,e_k} \colon j,k \in J\}$ *is dense in each of the spaces* $B_p(H)$, $1 \leq p \leq \infty$.

Proof. For each $p \in [1,\infty]$ we have $\|x\|_\infty \leq \|x\|_p \leq \|x\|_1$ and accordingly $B_1(H) \subseteq B_p(H) \subseteq K(H) = B_\infty(H)$.

Since $B_0(H)$ and $B_p(H)$ are $*$-invariant, it suffices to see that each hermitian operator in $B_p(H)$ is contained in the closure of $B_0(H)$. The Spectral Theorem for Compact Hermitian Operators directly implies that the ideal $B_{\mathrm{fin}}(H)$ of continuous maps with finite-dimensional image is dense in $B_p(H)$, and hence that $B_1(H)$ is dense in $B_p(H)$. Therefore it suffices to see that $B_0(H)$ is dense in $B_1(H)$ because $\|x\|_p \leq \|x\|_1$.

In view of the Hahn–Banach Theorem, we have to show that each continuous linear functional $f \in B_1(H)'$ vanishing on $B_0(H)$ is zero. As $B_1(H)' \cong B(H)$ ([Ne99, Prop. A.I.10(vi)]), the functional f can be written as $f(X) = \mathrm{tr}(AX)$ for some $A \in B(H)$. Hence

$$f(P_{e_j,e_k}) = \mathrm{tr}(AP_{e_j,e_k}) = \mathrm{tr}(P_{A.e_j,e_k}) = \langle A.e_j, e_k\rangle.$$

If f vanishes on $B_0(H)$, then the matrix of A with respect to the orthonormal basis $(e_j)_{j\in J}$ vanishes, and this means that $A = 0$. ■

The following theorem is well known for the case of separable Hilbert spaces (cf. [Pa65] and [dlH72, p.II.29]). The results on direct limit groups obtained in the preceding subsection easily permit us to extend it to general Hilbert spaces.

Theorem II.14. *Let H be an infinite-dimensional Hilbert space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ and $p \in [1, \infty]$. Then the following assertions hold:*

(i) *For every $k \in \mathbb{N}_0$ we have $\pi_k(\mathrm{GL}_p(H)) \cong \pi_k(\mathrm{GL}(\mathbb{N}, \mathbb{K})) \cong \varinjlim \pi_k(\mathrm{GL}(n, \mathbb{K}))$.*

(ii) *If $H_s \subseteq H$ is an infinite-dimensional separable subspace, then the inclusion map $\mathrm{GL}_p(H_s) \hookrightarrow \mathrm{GL}_p(H)$ is a weak homotopy equivalence.*

(iii) *For $1 \leq p \leq q \leq \infty$ the inclusion map $\mathrm{GL}_p(H) \hookrightarrow \mathrm{GL}_q(H)$ is a homotopy equivalence.*

Proof. (i) Let e_j, $j \in J$, be an orthonormal basis of H. Then Lemma II.13 above shows that $B_0(H) = \mathrm{span}\{P_{e_j, e_k} : j, k \in J\}$ is dense $B_p(H)$. We endow $B_0(H)$ with the direct limit topology with respect to the directed set of finite-dimensional subspaces of $B_0(H)$.

For the open subset $U := \mathrm{GL}_p(H) - \mathbf{1} \subseteq B_p(H)$ we have

$$U \cap B_0(H) = \mathrm{GL}(J, \mathbb{K}) - \mathbf{1},$$

where $\mathrm{GL}(J, \mathbb{K})$ is the set of those elements $g \in \mathrm{GL}(H)$ for which the matrix of $g - \mathbf{1}$ with respect to $(e_j)_{j \in J}$ has only finitely many entries, i.e., g and g^* fix all but finitely many e_j. It easily follows from $(g^*)^{-1} = (g^{-1})^*$ that g^{-1} has the same property. Therefore the natural identification of $B_0(H)$ with the matrix algebra $\mathbb{M}(J, \mathbb{K})$ leads to an identification of the group $\mathbf{1} + (U \cap B_0(H))$ with $\mathrm{GL}(J, \mathbb{K})$ as in Definition II.10.

Theorem II.8 implies that if we endow $\mathrm{GL}(J, \mathbb{K})$ with the final topology with respect to the subgroups $\mathrm{GL}(F, \mathbb{K})$, $F \subseteq J$ a finite subset, the inclusion map $\mathrm{GL}(J, \mathbb{K}) \to \mathrm{GL}_p(H)$ is a weak homotopy equivalence. Further Proposition II.11 shows that we have a weak homotopy equivalence $\mathrm{GL}(\mathbb{N}, \mathbb{K}) \hookrightarrow \mathrm{GL}(J, \mathbb{K})$. Composition of these two weak homotopy equivalences yields a weak homotopy equivalence, and, in view of Lemma II.9(iii), this proves (i).

(ii) We first choose an orthonormal basis $(e_n)_{n \in \mathbb{N}}$ of H_s and then complete it to an orthonormal basis $(e_j)_{j \in J}$ of H. We consider the corresponding maps

$$\varphi_1 : \mathrm{GL}(\mathbb{N}, \mathbb{K}) \to \mathrm{GL}_p(H_s), \quad \varphi_2 : \mathrm{GL}(J, \mathbb{K}) \to \mathrm{GL}_p(H), \quad \varphi_3 : \mathrm{GL}(\mathbb{N}, \mathbb{K}) \to \mathrm{GL}(J, \mathbb{K})$$

and

$$\varphi_4 : \mathrm{GL}_p(H_s) \to \mathrm{GL}_p(H)$$

with $\varphi_4 \circ \varphi_1 = \varphi_2 \circ \varphi_3$. Since φ_1 and φ_2 are weak homotopy equivalences by the first part of the proof, and φ_3 is a weak homotopy equivalence by Proposition II.11, it follows that φ_4 also is a weak homotopy equivalence.

(iii) From the elementary inclusion $l^p(\mathbb{N}, \mathbb{R}) \subseteq l^q(\mathbb{N}, \mathbb{R})$ we derive that $B_p(H) \subseteq B_q(H)$, and Lemma II.13 implies that $B_p(H)$ is a dense subspace. Therefore (iii) follows by applying Theorem II.8(b) to the open subset $U := \mathrm{GL}_q(H) - \mathbf{1} \subseteq B_q(H)$ which satisfies $U \cap B_p(H) = \mathrm{GL}_p(H) - \mathbf{1}$. ■

Corollary II.15. *If H is an infinite-dimensional complex Hilbert space, $1 \leq p \leq \infty$, and*

$$\mathrm{GL}_p(H, I) := \mathrm{GL}(H, I) \cap \mathrm{GL}_p(H),$$

then the following assertions hold:

(i) $\pi_k(\mathrm{GL}_p(H, I)) \cong \begin{cases} \pi_k(\mathrm{GL}(\mathbb{N}, \mathbb{R})) & \text{for } I^2 = \mathbf{1} \\ \pi_k(\mathrm{GL}(\mathbb{N}, \mathbb{H})) & \text{for } I^2 = -\mathbf{1}. \end{cases}$

(ii) *If $H_s \subseteq H$ is an infinite-dimensional separable I-invariant subspace, then the inclusion map $\mathrm{GL}_p(H_s, I|_{H_s}) \hookrightarrow \mathrm{GL}_p(H, I)$ is a weak homotopy equivalence.*

(iii) *For $1 \leq p \leq q \leq \infty$ the inclusion map $\mathrm{GL}_p(H, I) \hookrightarrow \mathrm{GL}_q(H, I)$ is a homotopy equivalence.*

Proof. We first observe that the polar decomposition of $\mathrm{GL}_p(H)$ ([Ne00a, Prop. A.4]) implies that its intersection with $\mathrm{GL}_p(H, I)$ also has a polar decomposition (see the proof of Theorem II.6), hence is homotopy equivalent to $\mathrm{U}_p(H, I) := \mathrm{U}_p(H) \cap \mathrm{GL}(H, I)$. For $I^2 = -\mathbf{1}$ we have $\mathrm{U}_p(H, I) \cong \mathrm{U}_p(H, \mathbb{H})$, and for $I^2 = \mathbf{1}$ we get $\mathrm{U}_p(H, I) \cong \mathrm{U}_p(H_{\mathbb{R}}, \mathbb{R})$, where $H_{\mathbb{R}} = \{x \in H : I.x = x\}$. Since the group $\mathrm{U}_p(H, \mathbb{K})$ is homotopy equivalent to $\mathrm{GL}_p(H, \mathbb{K})$ (Theorem II.6), the assertions on the groups $\mathrm{GL}_p(H, I)$ follow Theorem II.14 and the existence of polar decompositions. ■

III. Homotopy groups of restricted groups

In this section we turn to the homotopy groups of the restricted groups $\mathrm{GL}_p(H,D)$ and $\mathrm{GL}_p(H,I,D)$ defined below for a complex Hilbert space H, an antilinear isometry I with $I^2 = \pm\mathbf{1}$, and a hermitian operator D with finite spectrum.

The main motivation for the study of the restricted groups defined by a hermitian element $D = D^* \in \mathfrak{gl}(H)$ comes from the results in [Ne01a]. There it is shown that each continuous Lie algebra cocycle on $\mathfrak g \in \{\mathfrak{gl}_2(H), \mathfrak{gl}_2(H,I)\}$ can be written in the form $\omega_D(x,y) := \mathrm{tr}([D,x]y)$ for an element $D \in \mathfrak g_b \in \{\mathfrak{gl}(H), \mathfrak{gl}(H,I)\}$. If one considers the unitary real form $\mathfrak g_{\mathbb R} := \{x \in \mathfrak g \colon x^* = -x\}$, then the corresponding real cocycles are of the form ω_D for $D^* = -D$. Each of these cocycles determines an affine action of $G_{\mathbb R} := G \cap \mathrm U(H)$ on $\mathfrak g_{\mathbb R} \cong \mathfrak g_{\mathbb R}'$ (a twisted coadjoint action) given by

$$g.x = gxg^{-1} + D - gDg^{-1} = g(x-D)g^{-1} + D$$

whose orbits carry natural weak symplectic structures generalizing the Kirillov–Kostant–Souriau structure on coadjoint orbits of compact groups. The orbit $\mathcal O_D$ of $0 \in \mathfrak g_{\mathbb R}$ is most naturally attached to D, and we show in [Ne01a] that it is a submanifold of $\mathfrak g_{\mathbb R}$ if and only if its symplectic structure is strongly symplectic if and only if D has finite spectrum, which of course is equivalent to the hermitian operator iD having finite spectrum. In this sense the condition of having finite spectrum shows up as a natural condition in regard of the geometry of coadjoint orbits. Now the restricted real group $G_{\mathbb R}(D) = G_{\mathbb R} Z_{G_{\mathbb R,b}}(D)$ acts naturally on these orbits and one obtains unitary representations of a central $\mathbb T$-extension on Hilbert spaces of holomorphic sections of holomorphic line bundles on the orbits. For more details we refer to [Ne01a] and [Ne00a].

Restricted classical groups

Definition III.1. (a) Let H be a $\mathbb K$-Hilbert space with $\mathbb K \in \{\mathbb R, \mathbb C, \mathbb H\}$. If $H = H_1 \oplus \ldots \oplus H_k$ is the eigenspace decomposition of $D = D^* \in B(H)$, then we write operators on H accordingly as $(k\times k)$-block matrices and consider for $1 \le p \le \infty$ the space

$$B_p(H,D) := \{A \in B(H) \colon \|[D,A]\|_p < \infty\} = \{A = (a_{ij}) \in B(H) \colon (\forall i \neq j) a_{ij} \in B_p(H_j, H_i)\}.$$

This space carries the structure of a Banach algebra given by the natural composition of operators and the norm

$$\|X\| := \max\{\|a_{jj}\|, j = 1,\ldots,k; \|a_{jl}\|_p, j \neq l\}.$$

It is clear that A is complete with respect to this norm. That the norm on A satisfies $\|XY\| \le k\|X\|\,\|Y\|$ can be seen as follows. We have

$$\|(XY)_{jj}\| = \|\sum_l X_{jl}Y_{lj}\| \le \|X_{jj}\|\,\|Y_{jj}\| + \sum_{l\neq j} \|X_{jl}\|_p \|Y_{lj}\|_p \le k\|X\|\,\|Y\|$$

because $\|A\| \le \|A\|_p$. We likewise obtain for $j \neq l$ the estimate

$$\begin{aligned}\|(XY)_{jl}\|_p &\le \|X_{jj}Y_{jl}\|_p + \|X_{jl}Y_{ll}\|_p + \sum_{i\neq j,l} \|X_{ji}Y_{il}\|_p \\ &\le \|X_{jj}\|\,\|Y_{jl}\|_p + \|X_{jl}\|_p\|Y_{ll}\| + \sum_{i\neq j,l} \|X_{ji}\|_p\,\|Y_{il}\|_p \le k\|X\|\,\|Y\|.\end{aligned}$$

Since $B_p(H_i,H_j) = B(H_i,H_j)$ if and only if one of the spaces H_i and H_j is finite-dimensional, the algebra $B_p(H,D)$ coincides with $B(H)$ if and only if at most one of the spaces H_j is infinite-dimensional.

(b) Next we modify the construction in (a) slightly. The space

$$B_{1,2}(H,D) := \{A = (a_{ij}) \in B_2(H,D)\colon (\forall j)\, a_{jj} \in B_1(H_j)\}$$

is a Banach algebra with respect to operator composition and the norm

$$\|X\| := \max\{\|a_{jj}\|_1; \|a_{jl}\|_2, j \neq l\}.$$

It is clear that A is complete with respect to this norm. We recall that

$$\|XY\|_1 \leq \|X\|_2 \|Y\|_2 \leq \|X\|_1 \|Y\|_1 \quad \text{for} \quad X, Y \in B_2(H).$$

In view of this fact, we can show that the norm on $B_{1,2}(H)$ satisfies $\|XY\| \leq k\|X\|\,\|Y\|$: For each j we have

$$\|(XY)_{jj}\|_1 = \|\sum_l X_{jl}Y_{lj}\|_1 \leq \|X_{jj}\|_1\,\|Y_{jj}\|_1 + \sum_{l\neq j} \|X_{jl}\|_2 \|Y_{lj}\|_2 \leq k\|X\|\,\|Y\|.$$

We likewise obtain for $j \neq l$ the estimate

$$\begin{aligned}\|(XY)_{jl}\|_2 &= \|X_{jj}Y_{jl}\|_2 + \|X_{jl}Y_{ll}\|_2 + \sum_{i\neq j,l} \|X_{ji}Y_{il}\|_2 \\ &\leq \|X_{jj}\|\,\|Y_{jl}\|_2 + \|X_{jl}\|_2\|Y_{ll}\| + \sum_{i\neq j,l} \|X_{ji}\|_2\,\|Y_{il}\|_2 \leq k\|X\|\,\|Y\|\end{aligned}$$

because $\|A\| \leq \|A\|_2 \leq \|A\|_1$. ■

Proposition III.2. *In the setting of* Definition III.1 *we have:*
(a) *For each* $p \in [1,\infty]$ *the set* $\mathrm{GL}_p(H,D) := \mathrm{GL}(H) \cap B_p(H,D)$ *is a group.*
(b) $\mathrm{GL}_{1,2}(H,D) := \mathrm{GL}(H) \cap \big(\mathbf{1} + B_{1,2}(H,D)\big)$ *is a group.*
(c) *The inclusion maps* $\mathrm{GL}_1(H) \hookrightarrow \mathrm{GL}_{1,2}(H,D) \hookrightarrow \mathrm{GL}_2(H)$ *are homotopy equivalences.*
(d) *For each* $p \in [1,\infty]$ *the inclusion* $\mathrm{GL}_p(H,D) \hookrightarrow \mathrm{GL}_\infty(H,D)$ *is a homotopy equivalence.*
(e) *For each* $p \in [1,\infty]$ *the the polar map*

$$\mathrm{U}_p(H,D) \times \mathrm{Herm}_p(H,D) \to \mathrm{GL}_p(H,D), \quad (u,x) \mapsto ue^x$$

is a diffeomorphism and the inclusion $\mathrm{U}_p(H,D) \hookrightarrow \mathrm{GL}_p(H,D)$ *is a homotopy equivalence.*

Proof. (a) Let $g \in \mathrm{GL}_p(H,D)$. We only have to show that $(g^{-1})_{il} \in B_p(H_l, H_i)$ holds for $i \neq l$. First we observe that

$$\mathbf{1} = (g^{-1})_{ii}g_{ii} + \sum_{j\neq i}(g^{-1})_{ij}g_{ji} \in (g^{-1})_{ii}g_{ii} + B_p(H_i).$$

We also have

$$g_{ii}(g^{-1})_{il} = -\sum_{j\neq i} g_{ij}(g^{-1})_{jl} \in B_p(H_l, H_i).$$

Multiplying this equation with $(g^{-1})_{ii}$, we obtain

$$(g^{-1})_{ii}g_{ii}(g^{-1})_{il} \in B_p(H_l,H_i) \cap \big((g^{-1})_{il} + B_p(H_l,H_i)\big).$$

The fact that the intersection of these two sets is not empty shows that $(g^{-1})_{il} \in B_p(H_l,H_i)$.
(b) Let $g \in \mathrm{GL}(H) \cap \big(\mathbf{1} + B_{1,2}(H,D)\big) \subseteq B_2(H,D)$. Then (a) implies that for $j \neq l$ we have $(g^{-1})_{jl} \in B_2(H_l,H_j)$. We further have for $i =, 1 \ldots, k$:

$$\mathbf{1} = g_{ii}(g^{-1})_{ii} + \sum_{j\neq i} g_{ij}(g^{-1})_{ji} \in (\mathbf{1} + B_1(H_i))(g^{-1})_{ii} + B_1(H_i) \subseteq (g^{-1})_{ii} + B_1(H_i),$$

so that $g^{-1} \in \mathbf{1} + B_{1,2}(H,D)$.
(c) First we observe that the space $B_1(H)$ is dense in $B_{1,2}(H,D)$, so that

$$\mathrm{GL}_1(H) = \mathrm{GL}(H) \cap (\mathbf{1} + B_1(H))$$

further yields

$$\mathrm{GL}_1(H) = \mathrm{GL}_{1,2}(H,D) \cap (\mathbf{1} + B_1(H)),$$

and now Theorem II.8(b) applies.

Next we note that $B_{1,2}(H,D) \subseteq B_2(H)$ is a dense subspace and that (b) implies that $\mathrm{GL}_{1,2}(H,D) = \mathrm{GL}_2(H) \cap \big(\mathbf{1} + B_{1,2}(H,D)\big)$. Therefore Theorem II.8(b) applies again and shows that the inclusion $\mathrm{GL}_{1,2}(H,D) \to \mathrm{GL}_2(H)$ is a homotopy equivalence.
(d) follows as in (c) from Theorem II.8(b) because $B_p(H,D)$ is dense in $B_\infty(H,D)$ with

$$\mathrm{GL}_p(H,D) = \mathrm{GL}(H) \cap B_p(H,D) = \mathrm{GL}(H) \cap B_\infty(H,D) \cap B_p(H,D) = \mathrm{GL}_\infty(H,D) \cap B_p(H,D).$$

(e) (See also [HH94b, Prop. 2.1.14] for the existence of a polar decomposition). In view of (a), $\mathrm{GL}_p(H,D) = \mathrm{GL}(H) \cap B_p(H,D)$ is the unit group of the Banach algebra $B_p(H,D)$. Hence the spectrum of an element of the Banach algebra $B_p(H,D)$ is the same as the spectrum as an element of $B(H)$.

Let $g \in \mathrm{GL}_p(H,D)$. Then $g^*g \in B_p(H,D)$, and $\mathrm{Spec}_{B_p(H,D)}(g^*g) = \mathrm{Spec}(g^*g)$ is contained in $]0,\infty[$. Therefore [Ne00a, Lemma A.1] implies that $\log(g^*g) \in B_p(H,D)$ and that the map $\mathrm{GL}_p(H,D) \to B_p(H,D), g \mapsto \log(g^*g)$, is smooth. If $g = ue^x$ is the polar decomposition of g, then we conclude that $x = \log(g^*g) \in B_p(H,D)$, hence that $e^x \in \mathrm{GL}_p(H,D)$ and therefore $u = ge^{-x} \in \mathrm{U}_p(H,D)$. Moreover, the polar map

$$\mathrm{U}_p(H,D) \times \mathrm{Herm}_p(H,D) \to \mathrm{GL}_p(H,D), \quad (u,x) \mapsto ue^x$$

is a diffeomorphism since its inverse is also smooth. This means that $\mathrm{GL}_p(H,D)$ has a smooth polar decomposition, and (e) follows. ■

Definition III.3. (Restricted groups) Let $\mathfrak{g} \in \{\mathfrak{gl}_2(H), \mathfrak{gl}_2(H,I)\}$ and accordingly $\mathfrak{g}_b \in \{\mathfrak{gl}(H), \mathfrak{gl}(H,I)\}$. We fix a hermitian element $D \in \mathfrak{g}_b$ with finite spectrum.

For $\mathfrak{g} = \mathfrak{gl}_2(H)$ we define

$$G_b := \mathrm{GL}(H) \quad \text{and} \quad G := \mathrm{GL}_2(H).$$

For $\mathfrak{g} = \mathfrak{gl}_2(H,I)$ we likewise put $G_b := \mathrm{GL}(H,I)$ and $G := \mathrm{GL}_2(H,I)$.

In both cases we define the *restricted groups* associated to $\mathfrak{g}$ and $D = D^* \in \mathfrak{g}_b$ by $\mathrm{GL}_p(H,D)$ (cf. Proposition III.2) for $\mathfrak{g} = \mathfrak{gl}_2(H)$ and

$$\mathrm{GL}_p(H,I,D) := \mathrm{GL}(H,I) \cap \mathrm{GL}_p(H,D), \quad 1 \leq p \leq \infty,$$

for $\mathfrak{g} = \mathfrak{gl}_2(H,I)$. We likewise define

$$\mathrm{GL}_{1,2}(H,I,D) := \mathrm{GL}(H,I) \cap \mathrm{GL}_{1,2}(H,D).$$

We also put

$$\mathrm{GL}(H)^0 := \mathrm{GL}(H,D)^0 := Z_{\mathrm{GL}(H)}(D) \cong \prod_{j=1}^k \mathrm{GL}(H_j)$$

and

$$\mathrm{GL}(H,I)^0 := \mathrm{GL}(H,I,D)^0 := Z_{\mathrm{GL}(H,I)}(D) \cong \mathrm{GL}(H_0,I_0) \times \prod_{j=1}^k \mathrm{GL}(H_j),$$

where the last isomorphism follows easily from Remark I.2. ■

Remark III.4. Suppose that H is infinite-dimensional. The following remark shows that we may often reduce considerations about the groups $\mathrm{GL}_p(H,D)$ or $\mathrm{GL}_p(H,I,D)$ to the case where all spaces H_j are infinite-dimensional.

Let $\mathfrak{g}_b \in \{\mathfrak{gl}(H), \mathfrak{gl}(H,I)\}$ for $\mathfrak{g} \in \{\mathfrak{gl}_2(H), \mathfrak{gl}_2(H,I)\}$. Let D, D' be hermitian elements of $\mathfrak{g}_b$ with finite spectrum for which $D - D'$ has finite rank. Then for $x \in B(H)$ the conditions $[D,x] \in B_p(H)$ and $[D',x] \in B_p(H)$ are equivalent because their difference has finite rank, whence $B_p(H,D) = B_p(H,D')$ and therefore $B_p(H,I,D) = B_p(H,I,D')$.

We explain the construction for $\mathfrak{gl}_p(H,I,D)$, the other case is even simpler. If D is given, then we construct D' as follows. If H_j is infinite-dimensional, then we define $D'|_{H_j}$ as $D|_{H_j}$. If H_0 is finite-dimensional, then it is even-dimensional and there exists a subspace H_0^+ for which $H_0 = H_0^+ \oplus I.H_0^+$ is an orthogonal direct sum (cf. Remark I.2). Pick $j_0 > 0$ such that H_{j_0} is infinite-dimensional. Then we define D' on all finite-dimensional spaces H_j, $j > 0$, and H_0^+ in such a way that it has the same eigenvalue d_{j_0} as D on H_{j_0}. Likewise we define it by $-d_{j_0}$ on H_0^- and the finite-dimensional spaces H_{-j}. Then $D' \in \mathfrak{g}_b$ and $D - D'$ has finite rank, so that $\mathrm{GL}_p(H,I,D) = \mathrm{GL}_p(H,I,D')$. ■

In the remainder of this section we discuss the homotopy groups of the two types of restricted groups $\mathrm{GL}_p(H,D)$ and $\mathrm{GL}_p(H,I,D)$, where $D = D^* \in \mathfrak{g}_b$ is a hermitian element with finite spectrum.

The homotopy groups of $\mathrm{GL}_p(H,D)$

In Proposition III.2(d) we have seen that for $p \in [1,\infty]$ the natural inclusion maps $\mathrm{GL}_p(H,D) \hookrightarrow \mathrm{GL}_\infty(H,D)$ are homotopy equivalences. Therefore it suffices to determine the homotopy groups of $\mathrm{GL}_2(H,D)$ to know them for all the groups $\mathrm{GL}_p(H,D)$.

In the following we keep the setting of Definition III.1, resp., Examples I.9 and set

$$k_\infty := |\{j \in \{1,\ldots,k\} \colon \dim H_j = \infty\}|.$$

The determination of the connected components of $\mathrm{GL}_2(H,D)$ described in the following proposition can be found in [HH94b, Prop. 2.3.1].

Proposition III.5. *For each $g = (g_{ij}) \in \mathrm{GL}_2(H,D)$ the diagonal operators g_{jj} are Fredholm operators, and the connected components of the group $\mathrm{GL}_2(H,D)$ coincide with the fibers of the continuous homomorphism*

$$\mathrm{Ind}\colon \mathrm{GL}_2(H,D) \to \mathbb{Z}^{k_\infty}, \quad g \mapsto (\mathrm{ind}(g_{jj}))_{\dim H_j = \infty}$$

whose image is the set of those tuples $(n_j) \in \mathbb{Z}^{k_\infty}$ with $\sum_j n_j = 0$. Moreover, the identity component $\mathrm{GL}_2(H,D)_e$ of $\mathrm{GL}_2(H,D)$ is given by

$$\ker(\mathrm{Ind}) = \mathrm{GL}_2(H,D)_e = \mathrm{GL}_2(H)\,\mathrm{GL}(H,D)^0 = \mathrm{GL}_{1,2}(H,D)\,\mathrm{GL}(H,D)^0_\infty,$$

where $\mathrm{GL}(H,D)^0_\infty \subseteq \mathrm{GL}(H,D)^0$ is the subgroup corresponding to the infinite-dimensional ones among the spaces H_j, $j = 1,\ldots,k$, and

$$\pi_0(\mathrm{GL}_2(H,D)) \cong \begin{cases} \mathbb{Z}^{k_\infty - 1} & \text{for } k_\infty \geq 1 \\ \mathbf{0} & \text{for } k_\infty = 0. \end{cases}$$

Proof. For each $p \in [1,\infty]$ we have $\mathrm{GL}_p(H,D) \subseteq \mathrm{GL}_\infty(H,D)$, and each element $g = (g_{ij})$ of this group, written as a $k \times k$-matrix with $g_{ij} \in B(H_j, H_i)$, is a diagonal matrix modulo compact operators and invertible as such. Therefore all diagonal entries g_{jj} are invertible modulo compact operators, hence contained in the monoid $\mathrm{Fred}(H_j)$ of Fredholm operators on H_j. This means that $\mathrm{ind}(g_{jj}) := \dim\ker g_{jj} - \dim\mathrm{coker}\, g_{jj}$ is well defined. That Ind is a group homomorphism follows from the observation that modulo compact operators we can view elements of $\mathrm{GL}_2(H,D)$

as diagonal operators, so that the assertion follows from $\mathrm{ind}(ab) = \mathrm{ind}(a) + \mathrm{ind}(b)$ for Fredholm operators on each space H_j.

The inclusions

$$\mathrm{GL}_{1,2}(H,D)\,\mathrm{GL}(H,D)^0_\infty \subseteq \mathrm{GL}_2(H)\,\mathrm{GL}(H,D)^0 \subseteq \mathrm{GL}_2(H,D)_e \subseteq \ker(\mathrm{Ind}) \tag{3.1}$$

follow from the connectedness of the groups $\mathrm{GL}_{1,2}(H)$ and $\mathrm{GL}_2(H)$ (Theorem II.14, Proposition III.2(c)) and $\mathrm{GL}(H_j)$ and the continuity of the index function.

For the converse, let $g \in \ker(\mathrm{Ind})$. Since each g_{jj} is a Fredholm operator of index 0, we conclude that whenever $\dim H_j = \infty$, there exists a finite rank operator $b_j \in B(H_j)$ with $\ker b_j = (\ker g_{jj})^\perp$ mapping $\ker(g_{jj})$ bijectively onto $\mathrm{im}(g_{jj})^\perp$. Then $d_j := g_{jj} + b_j \in \mathrm{GL}(H_j)$ satisfies

$$g_{jj} = g_{jj} + b_j - b_j \in (g_{jj} + b_j)(\mathbf{1} + B_1(H_j)).$$

For $\dim H_j < \infty$ we put $d_j := \mathbf{1}$. Then $d := \mathrm{diag}(d_1, \ldots, d_k) \in \mathrm{GL}(H,D)^0_\infty$ satisfies $d^{-1}g \in \mathrm{GL}_{1,2}(H,D)$. We thus obtain $\ker(\mathrm{Ind}) \subseteq \mathrm{GL}_{1,2}(H,D)\,\mathrm{GL}(H,D)^0_\infty$ and hence equality in (3.1).

Since the off-diagonal entries of g are compact, the invertibility of g implies that $0 = \mathrm{ind}(g) = \sum_j \mathrm{ind}(g_{jj})$, hence the corresponding restriction on the image of Ind. If, conversely, $(n_j) \in \mathbb{Z}^{k_\infty}$ satisfies $\sum_j n_j = 0$, then we can write H_j as $l^2(J_j, \mathbb{C})$ and accordingly H as $l^2(J, \mathbb{C})$ with $J = \dot\cup_j J_j$. Now there exists a permutation σ of J for which

$$n_j = |J_j \cap \sigma^{-1}(J \setminus J_j)| - |(J_j \cap \sigma(J \setminus J_j))|.$$

We leave the easy proof to the reader. Then the isometry $\widetilde{\sigma}$ of H defined by σ is contained in $\mathrm{GL}_2(H,D)$ and satisfies $\mathrm{Ind}(\widetilde{\sigma}) = (n_j)_{\dim H_j = \infty}$. ∎

Definition III.6. Let G be a Banach–Lie group with Lie algebra $\mathfrak{g} = \mathbf{L}(G)$ and $\mathfrak{h} \subseteq \mathfrak{g}$ a closed subalgebra. We call the subgroup $H := \langle \exp \mathfrak{h} \rangle$ generated by the exponential image of $\mathfrak{h}$ the corresponding *analytic subgroup* of G. According to [Ma62], this group has a natural Lie group structure such that the map $H \hookrightarrow G$ is a morphism of Lie groups.

For a closed subgroup $H \subseteq G$ we consider the closed Lie subalgebra

$$\mathfrak{h} := \mathbf{L}(H) := \{X \in \mathfrak{g} \colon \exp(\mathbb{R}X) \subseteq H\}$$

of $\mathfrak{g}$ ([Ne00a, Cor. IV.3]) and say that H is a *Lie subgroup* if there exists an open 0-neighborhood $V \subseteq \mathfrak{g}$ such that $\exp|_V$ is a diffeomorphism onto an open subset $\exp(V)$ and $\exp(V \cap \mathfrak{h}) = (\exp V) \cap H$. Then H carries a natural Lie group structure such that the map $H \hookrightarrow G$ is a homomorphism of Lie groups which is a homeomorphism onto its image (cf. [Ne00a, Prop. IV.5]).

We call a Lie subgroup $H \subseteq G$ *complemented* or *split* if $\mathfrak{g}$ contains a closed subspace E complementing the closed subalgebra $\mathfrak{h}$. If this condition is satisfied, then H is a submanifold in the sense of Bourbaki, and in particular the homogeneous space G/H carries a natural manifold structure such that the canonical map $\pi \colon G \to G/H$ is a submersion (cf. [Ne00a, Prop. IV.5]; see also [Bou90, Ch. 3, §1.6, Prop. 11]). ∎

The next step is the determination of all homotopy groups of the restricted group $\mathrm{GL}_p(H,D)$.

Theorem III.7. *If H is infinite-dimensional, then*

$$\pi_m(\mathrm{GL}_2(H,D)) \cong \pi_{m-1}(\mathrm{GL}(\mathbb{N},\mathbb{C}))^{k_\infty - 1} \cong \begin{cases} \mathbb{Z}^{k_\infty - 1} & \text{for } m \text{ even} \\ \mathbf{0} & \text{for } m \text{ odd.} \end{cases}$$

Proof. We consider the short exact sequence of groups

$$A := \mathrm{GL}_2(H) \cap \mathrm{GL}(H,D)^0_\infty \hookrightarrow B := \mathrm{GL}_2(H) \rtimes \mathrm{GL}(H,D)^0_\infty \twoheadrightarrow C := \mathrm{GL}_2(H,D)_e, \tag{3.2}$$

where the surjectivity of the multiplication map $B \to C$ follows from Proposition III.5. Moreover, the assumptions of Proposition A.6 are satisfied because $\mathrm{GL}_2(H)$ is connected and a normal subgroup of $\mathrm{GL}_2(H,D)$, since $B_2(H)$ is an algebra ideal of $B(H)$. Furthermore, it is clear that

$$A \cong \prod_{\dim H_j = \infty} \mathrm{GL}_2(H_j)$$

is a complemented Lie subgroup of $\mathrm{GL}_2(H)$. It follows that B has a natural Banach–Lie group structure and that the map $B \to C$ is a locally trivial A-principal bundle.

The homotopy groups of A are given by

$$\pi_m(A) \cong \pi_m(\mathrm{GL}_2(H)^{k_\infty}) \cong \pi_m(\mathrm{GL}_2(H))^{k_\infty}$$

(Theorem II.14). Since the group

$$\mathrm{GL}(H,D)^0_\infty \cong \prod_{\dim H_j = \infty} \mathrm{GL}(H_j)$$

is contractible (Theorem II.4), the homomorphisms

$$\chi_m : \pi_m(A) \cong \pi_m(\mathrm{GL}_2(H))^{k_\infty} \to \pi_m(\mathrm{GL}_2(H)) \cong \pi_m(B)$$

can be viewed as the k_∞-fold summation maps in the abelian group $\pi_m(\mathrm{GL}_2(H))$. This map is surjective with

$$\ker \chi_m \cong \pi_m(\mathrm{GL}_2(H))^{k_\infty - 1}.$$

Therefore the exact homotopy sequence of (3.2) yields for each $m \in \mathbb{N}_0$ a short exact sequence

$$\cdots \xrightarrow{0} \pi_{m+1}(C) \hookrightarrow \pi_m(A) \xrightarrow{\chi_m} \pi_m(B) \xrightarrow{0} \pi_m(C) \cdots,$$

whence

$$\pi_{m+1}(C) \cong \ker \chi_m \cong \pi_m(\mathrm{GL}_2(H))^{k_\infty - 1}.$$

The remaining assertions follow from Theorem II.12 and II.14. ∎

The groups $\mathrm{GL}_p(H,I,D)$

Our next step it to determine the homotopy groups of the groups $\mathrm{GL}_p(H,I,D)$. For that we need some preparation because the case $H_0 \neq \mathbf{0}$ is more complicated than the case of $\mathrm{GL}_p(H,D)$ discussed above.

Up to a discussion of the connected components, Propositions III.9 below reduces the general case $p \in [1,\infty]$ to the special case $p = \infty$, which for $\mathrm{GL}_p(H,I,D)$ seems to be better accessible than the case $p = 2$ to determine the connected components.

Proposition III.8. *For each $1 \leq p \leq \infty$ the groups $\mathrm{GL}_p(H,D)$ and $\mathrm{GL}_p(H,I,D)$ have smooth polar decompositions. In particular, the inclusion maps*

$$\mathrm{U}_p(H,I,D) \to \mathrm{GL}_p(H,I,D)$$

are homotopy equivalences.

Proof. In Proposition II.2(e) we have seen that $\mathrm{GL}_p(H,D)$ has a smooth polar decomposition. If $g = ue^x$ is the polar decomposition of $g \in \mathrm{GL}_p(H,I,D)$, then it follows from the polar decompositions of $\mathrm{GL}(H,I)$ (Theorem II.6) and of $\mathrm{GL}_p(H,D)$ that $u \in \mathrm{U}_p(H,D)$ and $x \in B_p(H,D)$. That the polar map of $\mathrm{GL}_p(H,I,D)$ is a diffeomorphism follows by restriction from the corresponding result for $\mathrm{GL}_p(H,D)$. ∎

Proposition III.9. *For $1 \leq p \leq q \leq \infty$ the inclusion map of the identity components*

$$\mathrm{GL}_p(H,I,D)_e \to \mathrm{GL}_q(H,I,D)_e$$

is a weak homotopy equivalence.

Proof. In view of Remark III.4, we may assume that all spaces H_j are infinite-dimensional. Then the group

$$\mathrm{GL}(H,I,D)^0 \cong \mathrm{GL}(H_0,I_0) \times \prod_{j=1}^k \mathrm{GL}(H_j)$$

is contractible (Theorem II.6). Since $\mathrm{GL}(H)$ acts smoothly by conjugation on the normal subgroup $\mathrm{GL}_p(H)$, the group $\mathrm{GL}(H,I,D)^0$ acts smoothly on $\mathrm{GL}_p(H,I)$, so that we can form the connected Banach–Lie group $G_p := \mathrm{GL}_p(H,I)_e \rtimes \mathrm{GL}(H,I,D)^0$. From $\mathfrak{gl}_p(H,I,D) = \mathfrak{gl}_p(H,I) + \mathfrak{gl}(H,I,D)^0$ and Lemma A.5 we derive that the multiplication map $m\colon G_p \to M_p := \mathrm{GL}_p(H,I,D)_e$ is surjective with kernel

$$H_p := \mathrm{GL}_p(H,I,D)^0 \cong \mathrm{GL}_p(H_0,I_0) \times \prod_{j=1}^k \mathrm{GL}_p(H_j).$$

The assumptions of Proposition A.6 are satisfied, so that the map $G_p \to M_p$ defines a locally trivial H_p-bundle. Since all the groups $\mathrm{GL}(H_j)$ and $\mathrm{GL}(H_0,I_0)$ are contractible, Corollary II.15(iii) implies that the inclusion maps $H_p \hookrightarrow H_q$ and $G_p \hookrightarrow G_q$ are homotopy equivalences. Now Proposition A.8 implies that the inclusion map $M_p \to M_q$ is a weak homotopy equivalence.■

Now we prepare the discussion of the case $p = \infty$.

Lemma III.10. *Let A be a unital C^*-algebra and τ a linear antiautomorphism of A commuting with the $*$-map. Then*

$$G := \{g \in \mathrm{GL}(A) \colon \tau(g) = g^{-1}\}$$

is adapted to the polar decomposition $G(A) = \mathrm{U}(A)\exp(\mathrm{Herm}(A))$ of $G(A)$.

Proof. First we note that G is an algebraic subgroup of $G(A)$, hence a Lie group with Lie algebra $\mathfrak{g} = \{x \in A \colon \tau(x) = -x\}$ ([Ne00a, Prop. IV.14]). Since τ commutes with $*$, the group G is $*$-invariant.

Now we consider the automorphism of $G(A)$ given by $\sigma(g) := \tau(g)^{-1}$. The fact that τ commutes with $*$ implies that this automorphism preserves the subgroup $\mathrm{U}(A)$ of unitary elements and the subset $\exp(\mathrm{Herm}(A))$. Let $g \in G(A)$, and let $g = ue^x$ denote its polar decomposition. Then

$$\sigma(g) = \sigma(u)\sigma(e^x) = \tau(u)^{-1}e^{-\tau(x)}$$

is the polar decomposition of $\sigma(g)$. Therefore g is fixed by σ if and only if σ fixes u and e^x separately. This means that $u \in G$ and $\tau(x) = -x$, i.e., $x \in \mathfrak{g} \cap \mathrm{Herm}(A)$. We conclude that $G = (G \cap \mathrm{U}(A))\exp(\mathfrak{g} \cap \mathrm{Herm}(A))$. ■

Proposition III.11. *Let $I\colon H \to H$ be an antilinear isometry with $I^2 \in \{\pm \mathbf{1}\}$. We define*

$$\mathrm{Fred}(H,I) := \{g \in \mathrm{Fred}(H) \colon gIg^*I^{-1} \in \mathbf{1} + K(H)\}.$$

Then

$$\{g \in \mathrm{Fred}(H,I) \colon \mathrm{ind}(g) = 0\} = \mathrm{GL}(H,I)(\mathbf{1} + K(H)) = \mathrm{GL}(H,I) + K(H).$$

Proof. It is obvious that $\mathrm{GL}(H,I) + K(H) \subseteq \{g \in \mathrm{Fred}(H,I) \colon \mathrm{ind}(g) = 0\}$. The proof of the converse is more involved.

Let $A := \mathrm{Cal}(H) := B(H)/K(H)$, write $q\colon B(H) \to A$ for the quotient map, and observe that the antiautomorphism $a \mapsto Ia^*I^{-1}$ of $B(H)$ preserves $K(H)$, hence induces an antiautomorphism τ on A with

$$\tau(q(a)) := q(Ia^*I^{-1}).$$

We consider the group

$$G(A)_\tau := \{g \in G(A)\colon \tau(g) = g^{-1}\}.$$

Since $\mathrm{Fred}(H) = q^{-1}(G(A))$, $G(A)_e = \{q(g)\colon \mathrm{ind}(g) = 0\}$, and $q(g) \in G(A)_\tau$ is equivalent to $q(g) \in G(A)$ and $q(gIg^*I^{-1}) = q(g)\tau(q(g)) = \mathbf{1}$, we see that

$$\mathrm{Fred}(H, I) = q^{-1}(G(A)_\tau) \quad \text{and} \quad q(\mathrm{Fred}(H, I)) = G(A)_\tau.$$

The assertion of the theorem means that $G(A)_\tau \cap G(A)_e = q(\mathrm{GL}(H, I))$.

Since $G(A)_\tau$ is adapted to the polar decomposition of $G(A)$ (Lemma III.10), we have $G(A)_\tau = (G(A)_\tau \cap \mathrm{U}(A)) \exp(\mathbf{L}(G(A)_\tau) \cap \mathrm{Herm}(A))$ with

$$\exp(\mathbf{L}(G(A)_\tau) \cap \mathrm{Herm}(A)) = q(\exp(\mathfrak{gl}(H, I) \cap \mathrm{Herm}(H)) \subseteq q(\mathrm{GL}(H, I)).$$

Therefore it suffices to show that $G(A)_\tau \cap \mathrm{U}(A)_e \subseteq q(\mathrm{GL}(H, I))$ which in turn will follow from

$$\mathrm{Fred}(H, I) \cap \mathrm{U}(H) \subseteq \mathrm{GL}(H, I) + K(H)$$

because $q(\mathrm{U}(H)) = \mathrm{U}(A)_e$ follows from the connectedness of $\mathrm{U}(H)$ (Theorem II.4) and $q(\mathfrak{u}(H)) = \{a \in A\colon a^* = -a\} = \mathfrak{u}(A)$.

First we consider an element $u \in U_\infty(H) = \mathrm{U}(H) \cap \mathrm{GL}_\infty(H)$ with $Iu^{-1}I^{-1} = u$ and the eigenspaces $H_\mu = \ker(u - \mu\mathbf{1})$. Then we have for $v \in H_\mu$ the relation

$$uI.v = Iu^{-1}.v = I.\overline{\mu}v = \mu I.v,$$

showing that $I.H_\mu = H_\mu$. This implies that $u = \exp x$ for some $x \in \mathfrak{u}_\infty(H) = \mathfrak{u}(H) \cap B_\infty(H)$ and $IxI^{-1} = -x$ because we can choose x in such a way that on H_μ it is given by $i\lambda\,\mathrm{id}_{H_\mu}$, where $e^{i\lambda} = \mu$ for some $\lambda \in [-\pi, \pi]$, and $u \in U_\infty(H)$ implies $x \in \mathfrak{u}_\infty(H)$. Let $\mathfrak{u}_\infty(H)_- := \{x \in \mathfrak{u}_\infty(H)\colon IxI^{-1} = -x\}$. Then

$$\{u \in U_\infty(H)\colon Iu^{-1}I^{-1} = u\} = \exp(\mathfrak{u}_\infty(H)_-).$$

Now let $g \in \mathrm{Fred}(H, I) \cap \mathrm{U}(H)$ and define $u := gIg^{-1}I^{-1}$. Then

$$Iu^{-1}I^{-1} = I^2gI^{-1}g^{-1}I^{-1} = gI^2I^{-1}g^{-1}I^{-1} = gIg^{-1}I^{-1} = u,$$

so that the preceding paragraph shows that $u = \exp x$ with $x \in \mathfrak{u}_\infty(H)_-$. We put

$$\widetilde{g} := \exp(-\tfrac{1}{2}x)g$$

and obtain

$$\begin{aligned}\widetilde{g}I\widetilde{g}^{-1}I^{-1} &= \exp(-\tfrac{1}{2}x)gIg^{-1}\exp(\tfrac{1}{2}x)I^{-1} \\ &= \exp(-\tfrac{1}{2}x)uI\exp(\tfrac{1}{2}x)I^{-1} = \exp(\tfrac{1}{2}x)\exp(-\tfrac{1}{2}x) = \mathbf{1}.\end{aligned}$$

This means that $\widetilde{g} \in \mathrm{U}(H, I)$. We conclude that

$$g = \exp\big(\tfrac{1}{2}x\big)\widetilde{g} \in \widetilde{g} + K(H) \subseteq \mathrm{GL}(H, I) + K(H).$$

This completes the proof. ■

Lemma III.12. *For $g \in \mathrm{GL}_\infty(H,I,D)$ and $\dim H = \infty$ the following assertions hold:*
(i) *For each j we have $g_{jj} \in \mathrm{Fred}(H_j)$ with $\mathrm{ind}(g_{jj}) = \mathrm{ind}(g_{-j,-j})$.*
(ii) *We consider the map*

$$\mathrm{Ind}\colon \mathrm{GL}_\infty(H,I,D) \to \mathbb{Z} \times \mathbb{Z}^{k_\infty}, \quad \mathrm{Ind}(g) = \big(\mathrm{ind}(g_{00}), \mathrm{ind}(g_{jj})_{1 \leq j \leq k, \dim H_j = \infty}\big).$$

Then Ind *is a continuous group homomorphism,*

$$\mathrm{GL}_\infty(H,I,D)_{\mathrm{Ind}} := \ker(\mathrm{Ind}) = \mathrm{GL}_\infty(H,I)\,\mathrm{GL}(H,I,D)^0 = \mathrm{GL}_\infty(H,I)\,\mathrm{GL}(H,I,D)^0_\infty,$$

where $\mathrm{GL}(H,I,D)^0_\infty \subseteq \mathrm{GL}(H,I,D)^0$ is the subgroup corresponding to the infinite-dimensional ones among the spaces H_j, $j = 0,\ldots,k$. Moreover

$$\mathrm{im}(\mathrm{Ind}) = \begin{cases} \{(n_j)\colon n_0 + 2\sum_{0<j} n_j = 0\} \cong \mathbb{Z}^{k_\infty} & \text{for } \dim H_0 = \infty \\ \{(n_j)\colon n_0 = 0 = \sum_{0<j} n_j\} \cong \mathbb{Z}^{k_\infty - 1} & \text{for } \dim H_0 < \infty. \end{cases}$$

Proof. (i) The operator g has an inverse g^{-1} in $B_\infty(H,I,D)$, which means that all off-diagonal blocks of g^{-1} are compact. Therefore the diagonal block $g_{jj} \in B(H_j)$ is invertible modulo $K(H_j)$, and this means that $g_{jj} \in \mathrm{Fred}(H_j)$.

In view of $Ig^*I^{-1} = g^{-1}$ and $I.H_j = H_{-j}$, we have $(g^{-1})_{jj} = Ig^*_{-j,-j}I^{-1}$, and therefore

$$-\mathrm{ind}(g_{jj}) = \mathrm{ind}(g^*_{-j,-j}) = -\mathrm{ind}(g_{-j,-j}).$$

(ii) That Ind is a group homomorphism follows from Proposition III.5. We may w.l.o.g. assume that the spaces H_j are infinite-dimensional for $1 \leq j \leq k_\infty$ and finite-dimensional for $j > k_\infty$.

In view of (i), for each $g \in \mathrm{GL}_\infty(H,I,D)$ we have

$$0 = \mathrm{ind}\, g = \mathrm{ind}(g_{00}) + 2\sum_{j=1}^k \mathrm{ind}(g_{jj}).$$

Therefore a necessary condition for $n = (n_j) \in \mathrm{im}(\mathrm{Ind})$ is $n_0 + 2\sum_{j=1}^{k_\infty} n_j = 0$.

For tuples with $n_0 = 0$ this leads to the requirement $\sum_j n_j = 0$, and Proposition III.5 shows that all these tuples can be obtained from the subgroup

$$\mathrm{GL}_\infty(H_+ \oplus I.H_+, I, D) \cong \mathrm{GL}_\infty(H_+, D_+),$$

where $H_+ = \sum_{j=1}^k H_j$ (Remark I.2). Therefore we may assume that $n_2 = \ldots = n_{k_\infty} = 0$. Considering only those operators which act non-trivially on the subspaces $H_{\pm 1}$ and H_0, we may even assume that $k = k_\infty = 1$, and that H_1 and H_0 are infinite-dimensional and separable. Then we identify H with $l^2(\mathbb{Z}^\pm, \mathbb{C})$ (in the notation of Lemma I.1), where

$$H_{\pm 1} = l^2(\pm\mathbb{Z}_{>0}, \mathbb{C}) \quad \text{and} \quad H_0 = l^2(\pm\mathbb{Z}_{\leq 0}, \mathbb{C})$$

and $I.e_j = e_{-j}$ for $j \in \mathbb{Z}$. We consider the operator $S \in \mathrm{U}(H)$ given by $S.e_{\pm j} := e_{\pm(j+1)}$. Since this is a unitary operator commuting with I, it is an element of $\mathrm{U}(H,I)$. Moreover, its off-diagonal terms in the (3×3)-block decomposition are of finite rank. This implies that $S \in \mathrm{GL}_p(H,I,D)$ for $p \in [1,\infty]$. The operator S_{11} is a unilateral right shift operator, so that $\mathrm{ind}(S_{11}) = -1$, and S_{00} is a (2×2)-block diagonal operator, where both components are unilateral left shift operators, so that $\mathrm{ind}(S_{00}) = 2$. Therefore $(2,-1) \in \mathrm{im}(\mathrm{Ind})$, and from that the description of $\mathrm{im}(\mathrm{Ind})$ follows.

It is clear that $\mathrm{GL}_\infty(H,I)\,\mathrm{GL}(H,I,D)^0 \subseteq \ker(\mathrm{Ind})$. For the converse, assume that $\mathrm{Ind}(g) = 0$. Then each g_{jj} is a Fredholm operator of index 0, so that we find $d_j \in \mathrm{GL}(H_j)$, $j = 1,\ldots,k$, such that $d_j^{-1}g_{jj} \in \mathbf{1} + B_1(H_j) \subseteq \mathbf{1} + K(H_j)$. For $j = 0$ the relation $gIg^*I^{-1} = \mathbf{1}$

implies that $g_{00} \in \mathrm{Fred}(H_0, I_0)$, so that Proposition III.11 yields an element $d_0 \in \mathrm{GL}(H_0, I_0)$ with $d_0^{-1} g_{00} \in \mathbf{1} + K(H_0)$. Then

$$\mathrm{diag}(d_k^{-1}, \ldots, d_1^{-1}, d_0, I d_1^* I^{-1}, \ldots, I d_k^* I^{-1}) \in \mathrm{GL}(H, I, D)^0,$$

so that we may w.l.o.g. assume that $g_{jj} \in \mathbf{1} + B_1(H_j)$ holds for $j = 1, \ldots, k$ and $g_{00} \in \mathbf{1} + K(H_0)$. Then for $j > 0$ the relation

$$(g^{-1})_{jj} g_{jj} \in \mathbf{1} + B_1(H_j)$$

implies that $(g^{-1})_{jj} \in \mathbf{1} + B_1(H_j)$, and therefore $g^{-1} = I g^* I^{-1}$ leads to $g_{-j,-j} \in \mathbf{1} + B_1(H_{-j})$. Therefore $g \in \mathrm{GL}_\infty(H, I)$.

If $\dim H_j < \infty$, we may put $d_j = \mathbf{1}$, so that we get the sharper assertion that

$$\ker(\mathrm{Ind}) = \mathrm{GL}_\infty(H, I)\, \mathrm{GL}(H, I, D)^0_\infty. \qquad \blacksquare$$

We will see below that the group $\mathrm{GL}_\infty(H, I, D)_{\mathrm{Ind}}$ is not always connected.

Corollary III.13. *For $1 \leq p \leq q \leq \infty$ the inclusion map $\mathrm{GL}_p(H, I, D) \to \mathrm{GL}_q(H, I, D)$ is a weak homotopy equivalence.*

Proof. In view of Proposition III.9, it remains to show that the induced homomorphism

$$\alpha_{p,q} \colon \pi_0\big(\mathrm{GL}_p(H, I, D)\big) \to \pi_0\big(\mathrm{GL}_q(H, I, D)\big)$$

is bijective. For this we may assume that $q = \infty$ because if we can show the assertion in this case, the corollary follows from $\alpha_{p,\infty} = \alpha_{q,\infty} \alpha_{p,q}$.

The proof of Lemma III.12 shows in particular that $\mathrm{Ind}|_{\mathrm{GL}_p(H,I,D)}$ has the same range for each $p \in [1, \infty]$. Moreover, $\mathrm{GL}(H, I, D)^0 \subseteq \mathrm{GL}_p(H, I, D)$ implies that

$$\mathrm{GL}_p(H, I, D)_{\mathrm{Ind}} := \ker \mathrm{Ind} \cap \mathrm{GL}_p(H, I, D) = \mathrm{GL}_p(H, I)\, \mathrm{GL}(H, I, D)^0_\infty.$$

Modulo connected components, the inclusion

$$\mathrm{GL}_p(H, I, D)_{\mathrm{Ind}} \hookrightarrow \mathrm{GL}_\infty(H, I, D)_{\mathrm{Ind}}$$

therefore corresponds to the inclusion map $\mathrm{GL}_p(H, I) \hookrightarrow \mathrm{GL}_\infty(H, I)$ which is a homotopy equivalence (Corollary II.15). This completes the proof. $\blacksquare$

Theorem III.14. (Homotopy groups of $\mathrm{GL}_p(H, I, D)$) *If H is an infinite-dimensional complex Hilbert space and $k_\infty := |\{j \in \{1, \ldots, k\} : \dim H_j = \infty\}|$, then*

$$\pi_0(\mathrm{GL}_p(H, I, D)_{\mathrm{Ind}}) \cong \begin{cases} \mathbb{Z}_2 & \text{for } \dim H_0 < \infty \text{ and } I^2 = \mathbf{1} \\ \mathbf{0} & \text{otherwise,} \end{cases}$$

$$\pi_0(\mathrm{GL}_p(H, I, D)) \cong \begin{cases} \mathbb{Z}_2 \times \mathbb{Z}^{k_\infty - 1} & \text{for } \dim H_0 < \infty \text{ and } I^2 = \mathbf{1} \\ \mathbb{Z}^{k_\infty - 1} & \text{for } \dim H_0 < \infty \text{ and } I^2 = -\mathbf{1} \\ \mathbb{Z}^{k_\infty} & \text{for } \dim H_0 = \infty, \end{cases}$$

$$\pi_1(\mathrm{GL}_p(H, I, D)) = \mathbf{0}, \quad \text{and} \quad \pi_2(\mathrm{GL}_p(H, I, D)) \cong \mathbb{Z}^{k_\infty}.$$

The higher homotopy groups are 8-periodic, i.e., $\pi_{n+8}(\mathrm{GL}_p(H, I, D)) \cong \pi_n(\mathrm{GL}_p(H, I, D))$, $n \in \mathbb{N}_0$, and therefore determined by the following table.

	$\dim H_0 = \infty$	$\dim H_0 < \infty, I^2 = \mathbf{1}$	$\dim H_0 < \infty, I^2 = -\mathbf{1}$
π_1	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
π_2	$\mathbb{Z}^{k_\infty}$	$\mathbb{Z}^{k_\infty}$	$\mathbb{Z}^{k_\infty}$
π_3	$\mathbf{0}$	$\mathbf{0}$	$\mathbb{Z}_2$
π_4	$\mathbb{Z}^{k_\infty}$	$\mathbb{Z}^{k_\infty - 1}$	$\mathbb{Z}_2 \oplus \mathbb{Z}^{k_\infty}$
π_5	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
π_6	$\mathbb{Z}^{k_\infty}$	$\mathbb{Z}^{k_\infty}$	$\mathbb{Z}^{k_\infty}$
π_7	$\mathbf{0}$	$\mathbb{Z}_2$	$\mathbf{0}$
π_8	$\mathbb{Z}^{k_\infty}$	$\mathbb{Z}_2 \oplus \mathbb{Z}^{k_\infty - 1}$	$\mathbb{Z}^{k_\infty - 1}$

Proof. First we use Remark III.4 to see that we may assume that all spaces H_j are infinite-dimensional and Corollary III.13 to see that we may assume that $p = \infty$.

In view of Lemma III.12(b), we have a surjective homomorphism

$$\varphi\colon \mathrm{GL}_\infty(H,I) \rtimes \mathrm{GL}(H,I,D)^0 \to \mathrm{GL}_\infty(H,I,D)_{\mathrm{Ind}}$$

whose kernel is isomorphic to

$$A := \mathrm{GL}_\infty(H,I) \cap \mathrm{GL}(H,I,D)^0 \cong \begin{cases} \prod_{j=1}^k \mathrm{GL}_\infty(H_j) & \text{for } H_0 = \mathbf{0} \\ \mathrm{GL}_\infty(H_0,I_0) \times \prod_{j=1}^k \mathrm{GL}_\infty(H_j) & \text{for } H_0 \neq \mathbf{0}. \end{cases}$$

Let $B := \mathrm{GL}_\infty(H,I) \rtimes \mathrm{GL}(H,I,D)^0$ and $C := \mathrm{GL}_\infty(H,I,D)_{\mathrm{Ind}}$. To see that the map $B \twoheadrightarrow C$ defines a locally trivial A-principal bundle, we first observe that, although the group $\mathrm{GL}_\infty(H,I)$ need not be connected, the group $\mathrm{GL}(H,I < D)^0$ acts smoothly on it since it acts smoothly on the Banach algebra $K(H)$, hence on $\mathrm{GL}_\infty(H)$, and $\mathrm{GL}_\infty(H,I)$ is a complemented Lie subgroup invariant under this action. In view of Remark A.7, and since all assumptions of Proposition A.6 are easily verified, the multiplication map $B \to C$ is a locally trivial A-principal bundle.

The exact homotopy sequence of the principal bundle $A \hookrightarrow B \twoheadrightarrow C$ yields a long exact sequence of homotopy groups

$$\cdots \to \pi_{k+1}(C) \to \pi_k(A) \to \pi_k(B) \to \pi_k(C) \to \pi_{k-1}(A) \to \ldots \tag{3.3}$$

ending as an exact sequence in

$$\cdots \to \pi_1(C) \to \pi_0(A) \to \pi_0(B) \to \pi_0(C) \to \mathbf{0}.$$

Since the groups $\mathrm{GL}_\infty(H_j)$ are all connected, we have

$$\pi_0(A) \cong \begin{cases} \pi_0(\mathrm{GL}_\infty(H_0,I_0)) \cong \mathbb{Z}_2 & \text{for } H_0 \neq \mathbf{0} \text{ and } I^2 = \mathbf{1} \\ \mathbf{0} & \text{otherwise} \end{cases}$$

(Theorem II.14). The contractibility of the group $\mathrm{GL}(H,I,D)^0$ (Theorem II.6) further leads to

$$\pi_0(B) \cong \pi_0(\mathrm{GL}_\infty(H,I)) \cong \begin{cases} \mathbb{Z}_2 & \text{for } I^2 = \mathbf{1} \\ \mathbf{0} & \text{for } I^2 = -\mathbf{1}. \end{cases}$$

Here the homomorphism $\pi_0(A) \to \pi_0(B)$ is the identity if $\pi_0(A)$ is non-trivial because the inclusions

$$\mathrm{O}(\mathbb{N},\mathbb{C}) \hookrightarrow \mathrm{GL}_\infty(H_0,I_0) \hookrightarrow \mathrm{GL}_\infty(H,I)$$

are weak homotopy equivalences (cf. Corollary II.15). This leads directly to

$$\pi_0(C) \cong \begin{cases} \mathbb{Z}_2 & \text{for } H_0 = \mathbf{0} \text{ and } I^2 = \mathbf{1} \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

Now we turn to the higher homotopy groups π_m, $m \geq 1$. We observe that in all cases the contractibility of $\mathrm{GL}(H,I,D)^0$ shows that B is homotopy equivalent to $\mathrm{GL}_\infty(H,I)$. First we deal with the simpler case $H_0 \neq \mathbf{0}$. Then the inclusion map $\mathrm{GL}_\infty(H_0,I_0) \to \mathrm{GL}_\infty(H,I)$ is a weak homotopy equivalence (cf. Corollary II.15). Hence for each $m \in \mathbb{N}_0$ the homomorphism $\chi_m\colon \pi_m(A) \to \pi_m(B)$ is surjective. Therefore the exact homotopy sequence of $A \hookrightarrow B \twoheadrightarrow C$ leads to for each $m \in \mathbb{N}$ to

$$\begin{aligned}\pi_m(C) &\cong \ker \chi_{m-1} \cong \prod_{j=1}^k \pi_{m-1}(\mathrm{GL}_\infty(H_j)) \\ &\cong \pi_{m-1}(\mathrm{GL}(\mathbb{N},\mathbb{C}))^{k_\infty} \cong \begin{cases} \mathbb{Z}^{k_\infty} & \text{for } m \text{ even} \\ \mathbf{0} & \text{for } m \text{ odd.} \end{cases}\end{aligned}$$

Now we turn to the case where $H_0 = \mathbf{0}$. Here we will need results from Bott's paper [Bo59]. Since the first two homotopy groups will be particularly important in the following, it is instructive to determine them directly. For that we note that in all cases the homomorphism $\pi_0(A) \to \pi_0(B)$ is injective, so that the homomorphism $\pi_1(B) \to \pi_1(C)$ is surjective by the exactness of (3.3). Furthermore $\pi_2(A) = \mathbf{0}$ and $\pi_2(B) = \mathbf{0}$ (Theorems II.6, II.14, Corollary II.15), so that $\pi_2(C) = \ker \chi_1$ and $\pi_1(C) \cong \operatorname{coker} \chi_1$. To determine these groups, we observe that

$$\pi_1(A) \cong \begin{cases} \mathbb{Z}_2 \times \mathbb{Z}^k & \text{for } \dim H_0 = \infty \text{ and } I^2 = \mathbf{1} \\ \mathbb{Z}^{k_\infty} & \text{otherwise} \end{cases}$$

and

$$\pi_1(B) \cong \pi_1\big(\mathrm{GL}_\infty(H,I)\big) \cong \left\{ \begin{matrix} \mathbb{Z}_2 & \text{for } I^2 = \mathbf{1} \\ \mathbf{0} & \text{for } I^2 = -\mathbf{1} \end{matrix} \right\}.$$

For $I^2 = -\mathbf{1}$ the homomorphism χ_1 is trivial, so that $\pi_2(C) \cong \pi_1(A) \cong \mathbb{Z}^k$ and $\pi_1(C) \cong \pi_1(B) = \mathbf{0}$.

For the case $I^2 = \mathbf{1}$ we first observe that the block diagonal map

$$\mathrm{GL}(n,\mathbb{C}) \to \mathrm{O}(2n,\mathbb{C}), \quad g \mapsto \begin{pmatrix} g & 0 \\ 0 & (g^\top)^{-1} \end{pmatrix}$$

induces a surjective map

$$\pi_1(\mathrm{GL}(n,\mathbb{C})) \cong \mathbb{Z} \to \pi_1(\mathrm{O}(2n,\mathbb{C})) \cong \begin{cases} \mathbb{Z} & \text{for } n = 1 \\ \mathbb{Z}_2 & \text{for } n > 1, \end{cases}$$

because the generator of $\pi_1(\mathrm{O}(2n,\mathbb{C}))$ can be obtained with the natural embedding of $\mathrm{SO}(2,\mathbb{R}) \cong \mathbb{T}$. The homomorphism χ_1 is therefore given by

$$\chi_1 \colon \pi_1(A) \cong \mathbb{Z}^k \to \pi_1(B) \cong \mathbb{Z}_2, \quad \chi_1((n_j)) = \sum_j [n_j],$$

where $[n] \in \mathbb{Z}_2 \cong \mathbb{Z}/2\mathbb{Z}$ denotes the congruence class modulo 2 of $n \in \mathbb{Z}$. Since H is infinite-dimensional, we have $k > 0$, showing that χ_1 is surjective, so that $\pi_1(C) = \mathbf{0}$. Since $\ker \chi_1$ is a subgroup of $\mathbb{Z}^k$ of index 2, we obtain $\pi_2(C) \cong \ker \chi_1 \cong \mathbb{Z}^k$.

We finally turn to the higher homotopy groups. The group $A \cong \prod_{j=1}^k \mathrm{GL}_\infty(H_j)$ is homotopy equivalent to $\mathrm{GL}(\mathbb{N},\mathbb{C})^k$. In particular we have $\pi_{2n-1}(A) \cong \mathbb{Z}^k$ and $\pi_{2n}(A) = \mathbf{0}$ for all $n \in \mathbb{N}$ (Theorem II.12). Therefore the exact homotopy sequence of $A \hookrightarrow B \twoheadrightarrow C$ contains the exact pieces

$$\pi_{2n}(B) \hookrightarrow \pi_{2n}(C) \to \pi_{2n-1}(A) \xrightarrow{\chi_{2n-1}} \pi_{2n-1}(B) \twoheadrightarrow \pi_{2n-1}(C).$$

Since every subgroup of $\pi_{2n-1}(A) \cong \mathbb{Z}^k$ is free, hence projective, we can apply this to $\ker \chi_{2n-1} \subseteq \pi_{2n-1}(A)$ to obtain

$$\pi_{2n}(C) \cong \pi_{2n}(B) \oplus \ker \chi_{2n-1} \quad \text{and} \quad \pi_{2n-1}(C) \cong \operatorname{coker} \chi_{2n-1}. \tag{3.4}$$

We are thus left with the determination of kernel and cokernel of χ_{2n-1}. Let

$$\eta_m \colon \mathrm{GL}(\mathbb{C}^m) \to \mathrm{GL}(\mathbb{C}^{2m}, I), \quad g \mapsto \begin{pmatrix} g & 0 \\ 0 & (g^\top)^{-1} \end{pmatrix}$$

and $\eta \colon \mathrm{GL}(\mathbb{N},\mathbb{C}) \to \mathrm{GL}(2\mathbb{N},\mathbb{C},I) := \varinjlim \mathrm{GL}(\mathbb{C}^{2m}, I)$ be the corresponding limit map. Then the homomorphism χ_{2n-1} is equivalent to

$$\begin{aligned} &\pi_{2n-1}(A) \cong \pi_{2n-1}(\mathrm{GL}(\mathbb{N},\mathbb{C}))^k \to \pi_{2n-1}(B) \cong \pi_{2n-1}(\mathrm{GL}(2\mathbb{N},\mathbb{C},I)), \\ &(x_1,\ldots,x_k) \mapsto \sum_{j=1}^k \pi_{2n-1}(\eta)(x_j). \end{aligned} \tag{3.5}$$

We conclude in particular that

$$\operatorname{coker} \chi_{2n-1} = \operatorname{coker} \pi_{2n-1}(\eta).$$

Next we use polar decompositions to see that for $I^2 = \mathbf{1}$ we have a homotopy equivalence

$$\mathrm{GL}(2\mathbb{N}, \mathbb{C}, I) \sim \varinjlim \mathrm{O}(2m, \mathbb{C}) \sim \varinjlim \mathrm{O}(2m, \mathbb{R}) \sim \mathrm{GL}(\mathbb{N}, \mathbb{R})$$

and for $I^2 = -\mathbf{1}$ we get

$$\mathrm{GL}(2\mathbb{N}, \mathbb{C}, I) \sim \varinjlim \mathrm{Sp}(2m, \mathbb{C}) \sim \varinjlim \mathrm{U}(\mathbb{H}^m, \mathbb{H}) \sim \mathrm{GL}(\mathbb{N}, \mathbb{H}).$$

The natural embedding $\mathrm{GL}(\mathbb{N}, \mathbb{C}) \hookrightarrow \mathrm{GL}(2\mathbb{N}, \mathbb{C}, I)$ correspond then to the natural inclusions

$$\mathrm{GL}(\mathbb{N}, \mathbb{C}) \hookrightarrow \mathrm{GL}(2\mathbb{N}, \mathbb{R}) \cong \mathrm{GL}(\mathbb{N}, \mathbb{R}) \quad \text{and} \quad \mathrm{GL}(\mathbb{N}, \mathbb{C}) \hookrightarrow \mathrm{GL}(\mathbb{N}, \mathbb{H}).$$

Information on the effect of these maps on the level of the homotopy groups comes from Theorem II.12, where we find
(3.6)

$$\pi_n(\mathrm{GL}(\mathbb{N}, \mathbb{R})/\mathrm{GL}(\mathbb{N}, \mathbb{C})) \cong \pi_{n+1}(\mathrm{GL}(\mathbb{N}, \mathbb{R})), \quad \pi_n(\mathrm{GL}(\mathbb{N}, \mathbb{H})/\mathrm{GL}(\mathbb{N}, \mathbb{C})) \cong \pi_{n+1}(\mathrm{GL}(\mathbb{N}, \mathbb{H})).$$

We first consider the case $I^2 = \mathbf{1}$ in detail. For each $m \in \mathbb{N}$ we obtain with (3.6) an exact sequence

$$\begin{aligned} &\pi_{2m}(\mathrm{GL}(\mathbb{N}, \mathbb{C})) = \mathbf{0} \to \pi_{2m}(\mathrm{GL}(\mathbb{N}, \mathbb{R})) \hookrightarrow \pi_{2m+1}(\mathrm{GL}(\mathbb{N}, \mathbb{R})) \\ &\to \pi_{2m-1}(\mathrm{GL}(\mathbb{N}, \mathbb{C})) \xrightarrow{\pi_{2m-1}(\eta)} \pi_{2n-1}(\mathrm{GL}(\mathbb{N}, \mathbb{R})) \twoheadrightarrow \pi_{2m}(\mathrm{GL}(\mathbb{N}, \mathbb{R})) \\ &\to \pi_{2m-2}(\mathrm{GL}(\mathbb{N}, \mathbb{C})) = \mathbf{0}. \end{aligned}$$

In view of the Bott periodicity, it suffices to consider $m = 1, 2, 3, 4$, which lead to

$$\begin{aligned} m = 1: &\quad \mathbf{0} \hookrightarrow \mathbb{Z} \to \mathbb{Z} \xrightarrow{\pi_1(\eta)} \mathbb{Z}_2 \twoheadrightarrow \mathbf{0} \\ m = 2: &\quad \mathbf{0} \hookrightarrow \mathbf{0} \to \mathbb{Z} \xrightarrow{\pi_3(\eta)} \mathbb{Z} \twoheadrightarrow \mathbf{0} \\ m = 3: &\quad \mathbf{0} \hookrightarrow \mathbb{Z} \to \mathbb{Z} \xrightarrow{\pi_5(\eta)} \mathbf{0} \twoheadrightarrow \mathbf{0} \\ m = 4: &\quad \mathbb{Z}_2 \hookrightarrow \mathbb{Z}_2 \to \mathbb{Z} \xrightarrow{\pi_7(\eta)} \mathbb{Z} \twoheadrightarrow \mathbb{Z}_2. \end{aligned}$$

Using (3.4) and (3.5), this information leads to the following table:

m	$\pi_n(B)$	$\ker \chi_{n-1}$	$\operatorname{coker} \chi_n$	$\pi_n(C)$
1	$\mathbb{Z}_2$	—	$\mathbf{0}$	$\mathbf{0}$
2	$\mathbf{0}$	$\mathbb{Z}^k$	—	$\mathbb{Z}^k$
3	$\mathbb{Z}$	—	$\mathbf{0}$	$\mathbf{0}$
4	$\mathbf{0}$	$\mathbb{Z}^{k-1}$	—	$\mathbb{Z}^{k-1}$
5	$\mathbf{0}$	—	$\mathbf{0}$	$\mathbf{0}$
6	$\mathbf{0}$	$\mathbb{Z}^k$	—	$\mathbb{Z}^k$
7	$\mathbb{Z}$	—	$\mathbb{Z}_2$	$\mathbb{Z}_2$
8	$\mathbb{Z}_2$	$\mathbb{Z}^{k-1}$	—	$\mathbb{Z}_2 \oplus \mathbb{Z}^{k-1}$

From (3.4) and Bott periodicity we further derive that the homotopy groups of C are 8-periodic, so that the table above contains all the information.

For $I^2 = -\mathbf{1}$, Bott periodicity implies that we obtain a similar picture shifted by 4, therefore the entries for $I^2 = -\mathbf{1}$ in the table can simply be obtained from them for $I^2 = \mathbf{1}$ by a 4-shift. ■

Proposition III.15. *The inclusion maps*

$$\mathrm{GL}_1(H,I) \hookrightarrow \mathrm{GL}_{1,2}(H,I,D) \hookrightarrow \mathrm{GL}_2(H,I)$$

are weak homotopy equivalences.

Proof. Since the subgroup $G^0 := \mathrm{GL}_2(H,I,D)^0 \subseteq G := \mathrm{GL}_2(H,I)$ acts smoothly on $G_{1,2}; = \mathrm{GL}_{1,2}(H,I,D)$ by conjugation, the semidirect product group $B := G_{1,2} \rtimes G^0$ has a natural Banach–Lie group structure. Moreover, the fact hat $\mathfrak{gl}_2(H,I,D) = \mathfrak{gl}_{1,2}(H,I,D) + \mathfrak{gl}_2(H,I)^0$ implies that the multiplication map $m \colon B \to G$ has an open image which therefore is a union of connected components. Since the inclusion map $G_1 := \mathrm{GL}_1(H,I) \hookrightarrow G$ is a weak homotopy equivalence (Corollary III.13), the map m is surjective.

Using Proposition A.6, we see that $G \cong B/A$, where

$$A \cong G_{1,2} \cap G^0 = G_1^0 \cong \mathrm{GL}_1(H_0, I_0) \times \prod_{j=1}^k \mathrm{GL}_1(H_j).$$

We consider the exact homotopy sequence of the A-principal bundle B:

$$\ldots \to \pi_k(A) \to \pi_k(B) \cong \pi_k(G_{1,2}) \times \pi_k(G_2^0) \to \pi_k(G) \to \pi_{k-1}(A) \to \ldots$$

Since the inclusion map $A \cong G_1^0 \hookrightarrow G_2^0$ is a weak homotopy equivalence, the homomorphism $\pi_k(A) \to \pi_k(G_{1,2}) \times \pi_k(G_2^0)$ corresponds to the inclusion of the second factor. In particular, it is injective, so that the exactness of the sequence implies that the maps

$$\pi_k(G_{1,2}) \to \pi_k(G), \quad k \in \mathbb{N},$$

are isomorphisms. ■

Problems III. It is natural to ask for the range of the index map ind on the monoid $\mathrm{Fred}(H,I)$. If $I^2 = \mathbf{1}$, then $\mathrm{Fred}(H,I)$ contains in particular all operators g on the real space $H_{\mathbb{R}} := \{x \in H \colon I.x = x\}$ for which $gg^* - \mathbf{1}$ is compact. Since this includes unilateral shift operators on separable subspaces, it follows that $\mathrm{im(ind)} = \mathbb{Z}$ in this case.

For $I^2 = -\mathbf{1}$, we may consider H as a quaternionic Hilbert space and obtain, with similar arguments as above, that im(ind) contains all even numbers, because an unilateral shift operator on $l^2(\mathbb{N}, \mathbb{H})$ has index 2. It is an interesting question if for $I^2 = -\mathbf{1}$ there exists an element $g \in \mathrm{Fred}(H,I)$ with $\mathrm{ind}(g) = 1$. It is clear that this cannot be realized as an $\mathbb{H}$-linear operator, because all these operator have even index.

In both cases we see that $\mathrm{im(ind)} \cong \mathbb{Z}$, so that we get $\pi_0(G(A)_\tau) \cong \mathbb{Z}$ with the notation from the proof of Proposition III.11. ■

IV. Universal central extensions of restricted groups

In this section we will draw the results from the preceding section together to describe universal central extensions $\widehat{G}_r$ of the identity components G_r of the restricted groups $\mathrm{GL}_2(H,D)$ and $\mathrm{GL}_2(H,I,D)$ in the category of complex Banach–Lie groups. Extending the results to real forms of these groups leads in particular to the metagonal and the metaplectic groups, of which we show that they are universal central extensions of restricted versions of the real groups $\mathrm{O}(H^{\mathbb{R}})$ and $\mathrm{Sp}(H,\Omega)$ in the category of real Banach–Lie groups. This section is the heart of the paper because the proof of the universality of the central extensions requires the results on Lie algebra cohomology from Section I and also the detailed knowledge on the homotopy groups from Section III, which in turn uses Section II. All statements concerning the universality of the considered group extensions are new.

Throughout this section H is assumed to be an infinite-dimensional complex Hilbert space.

Central extensions of complex restricted groups

First we deal with the groups $G_r := \mathrm{GL}_2(H,D)_e$. We start with a few preparations.

Lemma IV.1. *For $z \in \mathfrak{z}(\mathfrak{gl}(H,D)^0)$ the functional $x \mapsto \mathrm{tr}(zx)$ on $\mathfrak{g}_{1,2} := \mathfrak{gl}_{1,2}(H,D)$ vanishes on the commutator algebra if and only if all z_j are equal.*

Proof. We have $[\mathfrak{g}_{1,2}, \mathfrak{g}_{1,2}] \subseteq [\mathfrak{gl}_2(H), \mathfrak{gl}_2(H)] = \mathfrak{sl}(H)$, so that the condition that all z_j are equal is sufficient.

On the other hand we have for $i \neq j$ and $A \in B_2(H_i, H_j)$, $B \in B_2(H_j, H_i)$:

$$[A, B] = ABE_{ii} - BAE_{jj}$$

in terms of $(k \times k)$-block matrices. Therefore

$$\mathrm{tr}(z[A,B]) = z_i \,\mathrm{tr}(AB) - z_j \,\mathrm{tr}(BA) = (z_i - z_j)\,\mathrm{tr}(AB).$$

Now $B_2(H_i, H_j)B_2(H_j, H_i) = B_1(H_i)$ implies that $z_i = z_j$ is also necessary for $\mathrm{tr}(z[\mathfrak{g}_{1,2}, \mathfrak{g}_{1,2}])$ to vanish. ■

Lemma IV.2. *The continuous Lie algebra homomorphism $\mathrm{tr}\colon \mathfrak{gl}_{1,2}(H,D) \to \mathbb{C}$ integrates to a continuous group homomorphism $\det\colon \mathrm{GL}_{1,2}(H,D) \to \mathbb{C}^\times$.*

Proof. (cf. [HH94b] for a similar construction) Since the inclusion map

$$\mathrm{GL}_1(H) \to \mathrm{GL}_{1,2}(H,D)$$

is a homotopy equivalence (Proposition III.2(c)), we obtain a homomorphism $\iota\colon \widetilde{\mathrm{GL}}_1(H) \to \widetilde{\mathrm{GL}}_{1,2}(D)$ which induces a surjective homomorphism

$$\pi_1(\mathrm{GL}_1(H)) \cong \mathbb{Z} \to \pi_1(\mathrm{GL}_{1,2}(H,D)) \cong \mathbb{Z}.$$

If we compose the unique homomorphism $\widetilde{\det}\colon \widetilde{\mathrm{GL}}_{1,2}(D) \to \mathbb{C}^\times$ satisfying $\mathbf{L}(\widetilde{\det}) = \mathrm{tr}$ with ι, then we obtain a lift of the determinant map $\det\colon \mathrm{GL}_1(H) \to \mathbb{C}^\times$. We conclude that

$$\pi_1(\mathrm{GL}_{1,2}(H,D)) \subseteq \ker \widetilde{\det},$$

and this implies the assertion. ■

Definition IV.3. We define the group

$$\mathrm{SL}(H,D) := \ker(\det\colon \mathrm{GL}_{1,2}(H,D) \to \mathbb{C}^\times).$$

Let $v \in H_j$ for some infinite-dimensional space H_j be a unit vector and define the holomorphic homomorphism $\gamma\colon \mathbb{C}^\times \to \mathrm{GL}_1(H_1) \subseteq \mathrm{GL}_{1,2}(H,D)$ by $\gamma(z).v = zv$ and $\gamma(z).w = w$ for $w \bot v$. Then $\det \circ \gamma = \mathrm{id}_{\mathbb{C}^\times}$, and we conclude that the map

$$\mathrm{GL}_{1,2}(H,D) \mapsto \mathrm{SL}(H,D) \rtimes \mathbb{C}^\times, \quad g \mapsto (g\gamma(\det g)^{-1}, \det(g))$$

is a diffeomorphism. On the Lie algebra level we have a corresponding semidirect decomposition

$$\mathfrak{gl}_{1,2}(H,D) \cong \mathfrak{sl}(H,D) \rtimes \mathbb{C}. \qquad ■$$

The idea for the direct construction of the central extension in Definition IV.4 is a slight modification of the construction used in [HH94a,b]. Different constructions for the special case $k = 2$ can be found in [PS86] and [Mi89]. These central extensions could also be obtained more indirectly with the general methods described in [Ne00b], which requires to calculate the corresponding period maps (cf. Proposition IV.9), but in any case it is more convenient to have a concrete realization of the central extension.

Definition IV.4. Let $G = \mathrm{GL}_2(H)$ and $G_r := \mathrm{GL}_2(H,D)_e$. Since $G^0_{b,\infty} := \mathrm{GL}(H,D)^0_\infty$ (cf. Lemma III.1) acts smoothly on the Banach algebra $B_{1,2}(H,D)$, it acts smoothly by conjugation on $G_{1,2} := \mathrm{GL}_{1,2}(H,D)$, so that we can form the semidirect product Banach–Lie group $G_{1,2} \rtimes G^0_{b,\infty}$, and the multiplication map $G_{1,2} \rtimes G^0_{b,\infty} \twoheadrightarrow G_r$ induces an isomorphism

$$(G_{1,2} \rtimes G^0_{b,\infty})/N \to G_r, \quad (a,d)N \mapsto ad, \quad \text{where} \quad N \cong G_{1,2} \cap G^0_{b,\infty} \cong \prod_{\dim H_j = \infty} \mathrm{GL}_1(H_j)$$

(cf. Proposition A.6 and Remark A.7). Here we use that $G_{1,2}$ and $G^0_{b,\infty}$ are connected (Proposition III.2(c) and Theorem III.6).

In view of Definition IV.3, $S := \mathrm{SL}(H,D) \subseteq G_{1,2}$ is a Lie subgroup which also satisfies $G_r = SG^0_{b,\infty}$, so that

$$G_r \cong (S \rtimes G^0_{b,\infty})/N_S \quad \text{with} \quad N_S := N \cap (S \rtimes G^0_{b,\infty}).$$

The group N_S has a natural holomorphic homomorphism

$$\Delta\colon N_S \to (\mathbb{C}^\times)^{k_\infty}, \quad (g, g^{-1}) \mapsto (\det(g_j))_{\dim H_j = \infty}.$$

With

$$Z := \Delta(N_S) = \Big\{(z_j) \in (\mathbb{C}^\times)^{k_\infty} \colon \prod_j z_j = 1\Big\}$$

we then have $N_S \cong \ker\Delta \rtimes Z$. Since Δ is invariant under conjugation with elements of $G^0_{b,\infty}$, the subgroup $\ker\Delta$ is a normal Lie subgroup in $S \rtimes G^0_{b,\infty}$, and it is complemented because N_S is complemented in S. Therefore we can form the quotient group

$$\widehat{G}_r := (S \rtimes G^0_{b,\infty})/\ker\Delta$$

whose elements we write as $[(a,d)] := (a,d)\ker\Delta$ (cf. Definition III.6). This group has a natural homomorphism

$$q\colon \widehat{G}_r \to G_r, \quad q([(a,d)]) := ad \quad \text{with} \quad \ker q \cong N_S/\ker\Delta \cong Z.$$

We thus obtain a central extension $Z \hookrightarrow \widehat{G}_r \xrightarrow{q} G_r$. On the subgroup $G_{1,2} = S\gamma(\mathbb{C}^\times) \subseteq G_r$ (in the notation of Definition IV.3), this central extension has a natural splitting given by the homomorphism

$$\sigma_1\colon G_{1,2} = S\gamma(\mathbb{C}^\times) \to G_r^\sharp, \quad \sigma_1(g\gamma(z)) := [(g,\gamma(z))]. \qquad \blacksquare$$

Remark IV.5. On the Lie algebra level the construction in Definition IV.4 leads, for $\mathfrak{g} = \mathfrak{gl}_2(H)$ and $\mathfrak{g}_r := \mathfrak{gl}_2(H,D)$, to a surjective homomorphism

$$\mathfrak{s} \rtimes \mathfrak{g}^0_{b,\infty} := \mathfrak{sl}(H,D) \rtimes \mathfrak{gl}(H,D)^0_\infty \twoheadrightarrow \mathfrak{g}_r := \mathfrak{gl}_2(H,D), \quad (x,y) \mapsto x+y,$$

(cf. Definition III.1(b)), and $\mathfrak{g}^0_{b,\infty} = \mathfrak{gl}(H,D)^0_\infty$ is a closed Lie subalgebra of $\mathfrak{g}_r$. We have the central extension

$$\mathfrak{z} \cong \mathbb{C}^{k_\infty - 1} \hookrightarrow \widehat{\mathfrak{g}}_r \twoheadrightarrow \mathfrak{g}_r.$$

To describe this central extension by a continuous cocycle, we need a continuous splitting map $\sigma\colon \mathfrak{g}_r \to \widehat{\mathfrak{g}}_r$. This can be obtained by from the decomposition $\mathfrak{g}_r = \mathfrak{g}_r^0 \oplus [D,\mathfrak{g}]$, where $[D,\mathfrak{g}] \subseteq \mathfrak{s}$ denotes the closed subspace corresponding to the off-diagonal blocks. Writing elements $x \in \mathfrak{g}_r$ as $x = x_0 + x_1$ according to this decomposition, we define

$$\sigma(x_0 + x_1) := [(x_1, x_0, 0)].$$

It is clear that this is a continuous linear splitting map. The corresponding cocycle is given by

$$\begin{aligned}\omega(x_0+x_1,y_0+y_1) &= [\sigma(x_0+x_1),\sigma(y_0+y_1)] - \sigma([x_0+x_1,y_0+y_1]) \\ &= [([x_0,y_1]+[x_1,y_0]+[x_1,y_1],[x_0,y_0],0)] \\ &\quad - \sigma([x_0,y_0]+[x_1,y_1]_0,[x_1,y_1]_1+[x_0,y_1]+[x_1,y_0]) \\ &= [([x_0,y_1]+[x_1,y_0]+[x_1,y_1],[x_0,y_0],0)] \\ &\quad - [([x_1,y_1]_1+[x_0,y_1]+[x_1,y_0],[x_0,y_0]+[x_1,y_1]_0,0)] \\ &= [([x_1,y_1]_0,-[x_1,y_1]_0,0)] \\ &= [\big(0,0,d\Delta([x_1,y_1]_0)\big)] \cong d\Delta([x_1,y_1]_0).\end{aligned}$$

If $E := [x_1,y_1]_0$ denotes the block diagonal part of $[x_1,y_1]$, then E is of trace class with trace 0 because $[B_2(H),B_2(H)] \subseteq B_1(H)$ consists of matrices with vanishing trace. Now

$$d\Delta(E) = (\mathrm{tr}(E_j))_{\dim H_j=\infty} \in \mathfrak{z} = \Big\{(z_j) \in \mathbb{C}^{k_\infty} \colon \sum_j z_j = 0\Big\}. \qquad \blacksquare$$

Remark IV.6. It is interesting to compare the group $\widehat{G}_r$ with the group constructed in [HH94b]. As in Definition IV.4, we write G_r as a quotient $(G_{1,2} \rtimes G^0_{b,\infty})/N$, and consider the homomorphism $\Delta_N \colon N \to (\mathbb{C}^\times)^{k_\infty}$ given by the same formula as in Definition IV.4. We now obtain a central extension

$$G_r^\sharp := (G_{1,2} \rtimes G^0_{b,\infty})/\ker \Delta_N$$

by the same arguments. It is clear that we may view $\widehat{G}_r$ as a subgroup of $G_r^\sharp$ which we now describe as a kernel of a homomorphism to $\mathbb{C}^\times$.

Since the homomorphism $\det\colon G_{1,2} := \mathrm{GL}_{1,2}(H,D) \to \mathbb{C}^\times$ is invariant under the action of the group $G^0_{b,\infty} = \mathrm{GL}(H,D)^0_\infty$, it extends to a holomorphic homomorphism

$$G_{1,2} \rtimes G^0_{b,\infty} \to \mathbb{C}^\times$$

which obviously vanishes on the normal subgroup $\ker \Delta_N$, so that we obtain a holomorphic homomorphism

$$D\colon G_r^\sharp = (G_{1,2} \rtimes G^0_{b,\infty})/\ker\Delta_N \to \mathbb{C}^\times$$

which on $Z^\sharp := N/\ker\Delta_N \cong (\mathbb{C}^\times)^{k_\infty}$ restricts to the multiplication map $(z_j) \mapsto \prod_j z_j$. From that we conclude that

$$\ker D = \widehat{G}_r \quad \text{and} \quad G_r^\sharp \cong \widehat{G}_r \times \mathbb{C}^\times,$$

where the complementary factor can be chosen as the first factor in $Z \cong (\mathbb{C}^\times)^{k_\infty}$. Our construction further implies that the section $\sigma\colon G_{1,2} \to G_r^\sharp, g \mapsto [(g,\mathbf{1})]$ satisfies $D \circ \sigma = \det$. $\blacksquare$

Definition IV.7. For $\mathfrak{g} = \mathfrak{gl}_2(H,I)$ and $\mathfrak{g}_r = \mathfrak{gl}_2(H,I,D)$ we have a surjective homomorphism

$$\mathfrak{gl}_{1,2}(H,I,D) \rtimes \mathfrak{gl}(H,I,D)^0_\infty \twoheadrightarrow \mathfrak{g}_r := \mathfrak{gl}_2(H,I,D),$$

where $\mathfrak{gl}_{1,2}(H,I,D) := \mathfrak{gl}_2(H,I) \cap B_{1,2}(H,D)$ (cf. Definition III.1(b)). On the group level, we obtain for $G = \mathrm{GL}_2(H,I)$ as in Definition IV.4 an isomorphism

$$(G_{1,2} \rtimes G^0_{b,\infty})/N \to G_r, \quad (a,d)N \mapsto ad,$$

where $G_r := \mathrm{GL}_2(H,I,D)_e$, $G^0_{b,\infty} := \mathrm{GL}(H,I,D)^0_\infty$, $G_{1,2} := \mathrm{GL}_{1,2}(H,I,D)_e$, and

$$N \cong \prod_{\dim H_j=\infty, j>0} \mathrm{GL}_1(H_j) \times \left\{\begin{matrix} \mathrm{GL}_1(H_0,I_0) & \text{for } \dim H_0 = \infty \\ \mathbf{0} & \text{for } \dim H_0 < \infty \end{matrix}\right\}$$

(cf. Definition III.3). The group N has a natural holomorphic homomorphism

$$\Delta\colon N \to Z^\sharp := (\mathbb{C}^\times)^{k_\infty}, \quad g \mapsto (\det(g_j))_{\dim H_j=\infty, j>0}$$

with $N \cong \ker\Delta \rtimes Z^\sharp$. As in Definition IV.4, we now obtain a central extension

$$Z^\sharp \hookrightarrow G_r^\sharp := (G_{1,2} \rtimes G^0_{b,\infty})/\ker\Delta \twoheadrightarrow G_r.$$

which splits on the subgroup $G_{1,2}$ (cf. Remark IV.6). If $\widehat{\mathfrak{g}}_r$ denotes the Lie algebra of $G_r^\sharp$ and $\mathfrak{z}$ the Lie algebra of $Z^\sharp$, then we obtain a central Lie algebra extension $\mathfrak{z} \hookrightarrow \widehat{\mathfrak{g}}_r \twoheadrightarrow \mathfrak{g}_r$. $\blacksquare$

Proposition IV.8. *For $\mathfrak{g}_r = \mathfrak{gl}_2(H,D)$ let $\widehat{\mathfrak{g}}_r$ be as in* Definition IV.4 *and for $\mathfrak{g}_r = \mathfrak{gl}_2(H,I,D)$ as in* Definition IV.7. *Then the following assertions hold:*

(i) *The central extension $\mathfrak{z} \hookrightarrow \widehat{\mathfrak{g}}_r \twoheadrightarrow \mathfrak{g}_r$ is Banach universal over $\mathbb{C}$.*

(ii) *Every real form $\mathfrak{z}_{\mathbb{R}} \hookrightarrow (\widehat{\mathfrak{g}}_r)_{\mathbb{R}} \twoheadrightarrow \mathfrak{g}_{r,\mathbb{R}}$ of this central extension is a Banach universal central extension of the real form $\mathfrak{g}_{r,\mathbb{R}}$ of $\mathfrak{g}_r$.*

Proof. (i) First we assume that $\mathfrak{g}_r = \mathfrak{gl}_2(H,D)$. Let ω_j, $\dim H_j = \infty$, denote the components of the cocycle ω from Remark IV.5. Then $\sum_j \omega_j = 0$, and Proposition I.11 shows that this is the only non-trivial relation between the cohomology classes $[\omega_j] \in H^2_c(\mathfrak{g}_r, \mathbb{C})$. Hence we obtain an isomorphism

$$\delta_{\mathbb{C}} \colon \mathfrak{z}' = \mathrm{Lin}(\mathfrak{z}, \mathbb{C}) \to H^2_c(\mathfrak{g}_r, \mathbb{C}), \quad \alpha \mapsto [\alpha \circ \omega]$$

so that the central extension $\widehat{\mathfrak{g}}_r \to \mathfrak{g}_r$ with kernel $\mathfrak{z}$ is $\mathbb{C}$-universal by Proposition I.13 because $\mathfrak{g}_r$ is perfect (Proposition I.10). Moreover, Proposition I.13 implies that $\widehat{\mathfrak{g}}_r$ is a Banach universal central extension of $\mathfrak{g}_r$.

For $\mathfrak{g}_r = \mathfrak{gl}_2(H,I,D)$ Proposition I.11 shows that the components ω_j with $\dim H_j = \infty$ of the corresponding Lie algebra cocycle ω (Remark IV.5) yield a basis $[\omega_j]$ of $H^2_c(\mathfrak{g}_r, \mathbb{C})$, so that $\widehat{\mathfrak{g}}_r$ is a universal central extension of $\mathfrak{g}_r$ by Proposition I.13.

(ii) is an immediate consequence of (i) and Remark I.10(c) in [Ne01b]. ∎

Before we can turn to the universality assertions on the group level, we have to compute some homotopy groups of $\widehat{G}_r$ and $G_r^\sharp$. For any central extension $Z \hookrightarrow \widehat{G} \twoheadrightarrow G$ of a connected Banach–Lie group G with an abelian Banach–Lie group Z, which is a locally trivial Z-bundle, the vanishing of $\pi_m(Z)$, $m \geq 2$, in view of the exact homotopy sequence of the Z-bundle implies that the maps $\pi_m(G) \to \pi_m(\widehat{G})$ are isomorphisms for $m \geq 3$, and we have an exact sequence

$$\mathbf{0} \to \pi_2(\widehat{G}) \hookrightarrow \pi_2(G) \xrightarrow{\delta} \pi_1(Z) \to \pi_1(\widehat{G}) \to \pi_1(G) \to \pi_0(Z) \to \mathbf{0}$$

which describes the relations between π_1 and π_2 of G and $\widehat{G}$. Here the *period map* $\delta \colon \pi_2(G) \to \pi_1(Z)$ plays a key role because it determines $\pi_2(\widehat{G}) \cong \ker \delta$ and $\mathrm{im}\, \delta$ is the kernel of the map $\pi_1(\widehat{G}) \to \pi_1(G)$. If G is simply connected, we obtain in particular $\pi_1(\widehat{G}) \cong \mathrm{coker}\, \delta$.

Next we will analyze the period map for the central extensions $\widehat{G}_r \to G_r$ and $G_r^\sharp \to G_r$ explicitly in terms of the description of the group $\pi_2(G_r)$ given in Theorems III.6 and III.14. We will see that in both cases δ is injective, but that it is not surjective for $G_r^\sharp$. Since G_r is simply connected, this leads to

$$\pi_1(G_r^\sharp) \cong \mathrm{coker}\, \delta.$$

Proposition IV.9.

(i) *For $G_r = \mathrm{GL}_2(H,D)_e$ the period map $\delta \colon \pi_2(G_r) \to \pi_1(Z)$ is an isomorphism. The group $\widehat{G}_r$ is simply connected, and $\pi_2(\widehat{G}_r) = \mathbf{0}$.*

(ii) *For $G_r = \mathrm{GL}_2(H,I,D)_e$ the period map δ is always injective. If $I^2 = -\mathbf{1}$, then it is also surjective, and for $I^2 = \mathbf{1}$ we have $\mathrm{coker}\, \delta \cong \mathbb{Z}_2$. The group $G_r^\sharp$ satisfies $\pi_1(G_r^\sharp) \cong \mathbb{Z}_2$ and $\pi_2(G_r^\sharp) = \mathbf{0}$.*

Proof. (i) In our direct construction of $\widehat{G}_r$ in Definition IV.4, we used the description of G_r as $(S \rtimes G^0_{b,\infty})/N_S$ and a homomorphism $\Delta \colon N_S \to Z$ with

$$\widehat{G}_r \cong (S \rtimes G^0_{b,\infty})/\ker \Delta \quad \text{and} \quad N_S \cong \ker \Delta \rtimes Z.$$

The exact homotopy sequence of the N_S-principal bundle $S \rtimes G^0_{b,\infty}$ contains a piece

$$\cdots \pi_2(G_{1,2} \rtimes G^0_{b,\infty}) \to \pi_2(G_r) \xrightarrow{\delta_r} \pi_1(N_S) \to \pi_1(G_{1,2} \rtimes G^0_{b,\infty}) \to \cdots,$$

and since $\pi_2(G_{1,2} \rtimes G^0_{b,\infty}) = \mathbf{0}$ (Theorems II.6, II.14, and Proposition III.2(c)), the connecting map $\delta_r \colon \pi_2(G_r) \to \pi_1(N_S)$ is injective. The Z-bundle $\widehat{G}_r \to G_r$ is associated to the N_S-bundle $G_{1,2} \rtimes G^0_{b,\infty} \to G_r$ via the homomorphism $\Delta \colon N_S \to Z$, so that the corresponding connecting map δ satisfies $\delta = \pi_1(\Delta) \circ \delta_r$. Therefore we simply have to identify the image of δ_r in $\pi_1(N_S)$ and the homomorphism $\pi_1(\Delta)\colon \pi_1(N_S) \to \pi_1(Z)$, restricted to this subgroup.

We have

$$N_S = \Big\{ d = \mathrm{diag}(d_j) \in \prod_{\dim H_j = \infty} \mathrm{GL}_1(H_j) \colon \det d = \prod_j \det(d_j) = 1 \Big\},$$

$\ker\Delta \cong \prod_{\dim H_j=\infty} \mathrm{SL}(H_j)$, and $Z \cong (\mathbb{C}^\times)^{k_\infty - 1}$. From the proof of Theorem III.7 we derive that

$$\mathrm{im}(\delta_r) \cong \Big\{ (n_j) \in \mathbb{Z}^{k_\infty} \colon \sum n_j = 0 \Big\} = \pi_1(N) \cong \mathbb{Z}^{k_\infty - 1},$$

and that the map $\pi_1(\Delta)\colon \pi_1(N) \to \pi_1(Z) \cong \mathbb{Z}^{k_\infty - 1}$ is an isomorphism. Therefore $\delta\colon \pi_2(G_r) \to \pi_1(Z)$ is an isomorphism. The remaining assertions follows from the exact homotopy sequence of the Z-principal bundle $Z \hookrightarrow \widehat{G}_r \twoheadrightarrow G_r$ because $\pi_1(G_r) = \mathbf{0}$ (Theorem III.7) implies that $\pi_1(\widehat{G}_r) \cong \mathrm{coker}\,\delta = \mathbf{0}$ and $\pi_2(\widehat{G}_r) \cong \ker\delta = \mathbf{0}$.

(ii) For $q\colon G_r^\sharp \to G_r = \mathrm{GL}_2(H, I, D)_e$ the situation is slightly different. Here we have $N \cong \prod_{\dim H_j=\infty} \mathrm{GL}_1(H_j)$ with $N \cong \ker\Delta \rtimes Z$, where $\ker\Delta \cong \prod_{\dim H_j=\infty} \mathrm{SL}(H_j)$ is simply connected. Therefore the map $\pi_1(\Delta)\colon \pi_1(N) \to \pi_1(Z)$ is an isomorphism, and again we have a factorization $\delta = \pi_1(\Delta) \circ \delta_r$ with $\delta_r\colon \pi_2(G_r) \to \pi_1(N)$. Here the vanishing of $\pi_2(G_{1,2} \rtimes G^0_{b,\infty})$ follows from the contractibility of $G^0_{b,\infty}$ (Theorem II.6), Proposition III.15, and Corollary II.15. We conclude that δ_r is injective.

In view of $\pi_1(G_r) = \mathbf{0}$ (Theorem II.14), the exactness of

$$\cdots \pi_2(G_r) \xrightarrow{\delta_r} \pi_1(N) \to \pi_1(G_{1,2} \rtimes G^0_{b,\infty}) \to \pi_1(G_r) = \mathbf{0}$$

implies that

$$\mathrm{coker}\,\delta_r \cong \pi_1(G_{1,2} \rtimes G^0_{b,\infty}) \cong \pi_1(G_{1,2}).$$

Using again Proposition III.15 and Corollary II.15, we obtain

$$\pi_1(G_{1,2}) \cong \begin{cases} \mathbb{Z}_2 & \text{for } I^2 = \mathbf{1} \\ \mathbf{0} & \text{for } I^2 = -\mathbf{1}. \end{cases}$$

As in (i), we now obtain $\pi_2(G_r^\sharp) = \mathbf{0}$ and $\pi_1(G_r^\sharp) \cong \mathrm{coker}\,\delta \cong \mathrm{coker}\,\delta_r$. ■

From the topological information contained in Proposition IV.9 and the universality of the Lie algebra extensions $\widehat{\mathfrak{g}}_r \to \mathfrak{g}_r$, we will now derive the description of a universal central group extension $\widehat{G}_r \to G_r$ in the category of complex Banach–Lie groups. For $I^2 = \mathbf{1}$ it turns out that we will have to a twofold covering group $\widehat{G}_r$ of $G_r^\sharp$, which corresponds to the usual passage from orthogonal groups to spin groups.

Theorem IV.10. *If H is infinite-dimensional, then for $G_r = \mathrm{GL}_2(H, D)_e$ the central extension*

$$Z \cong (\mathbb{C}^\times)^{k_\infty - 1} \hookrightarrow \widehat{G}_r \xrightarrow{q} G_r$$

and for $G_r = \mathrm{GL}_2(H, I, D)_e$ the universal covering group

$$Z \hookrightarrow \widehat{G}_r := \widetilde{G}_r^\sharp \xrightarrow{q} G_r$$

is universal for all abelian complex Banach–Lie groups in the following sense: For each central extension $q_H\colon H \to G_r$ which is a locally trivial A-principal bundle for an abelian Banach–Lie group A, there exists a unique morphism $\varphi\colon \widehat{G}_r \to H$ with $q_H \circ \varphi = q$.

Proof. We have seen in Proposition IV.9 that for $G_r = \mathrm{GL}_2(H, I, D)_e$ the period map is not always surjective onto $\pi_1(Z)$. Let $q\colon \widehat{G}_r \to G_r^\sharp$ denote the universal covering map for $G_r = \mathrm{O}_2(H_{\mathbb{C}}, I, D)^+ := \mathrm{GL}_2(H_{\mathbb{C}}, I, D)_e$. For $Z := q^{-1}(Z^\sharp) \subseteq \widehat{G}_r$ we then have $G_r \cong \widehat{G}_r/Z$, and since G_r is simply connected, the group Z is connected; otherwise $\widehat{G}_r/Z_e$ would be a nontrivial connected covering group of G_r. Therefore $Z = \exp_{\widehat{G}_r} \mathfrak{z}$ is central in $\widehat{G}_r$, and we see that $q: \widehat{G}_r \to G_r$ is in fact a central extension of $\mathfrak{g}_r$. Furthermore, Proposition IV.9 implies that $\pi_2(\widehat{G}_r) \cong \pi_2(G_r^\sharp) = \mathbf{0}$.

For $G_r = \mathrm{GL}_2(H,D)_e$ we directly get from Proposition IV.9 that $\pi_2(\widehat{G}_r) = \pi_1(\widehat{G}_r) = \mathbf{0}$.

Therefore in both cases G_r is simply connected (Theorem III.7, Theorem III.14), $\widehat{G}_r$ is simply connected, and $\widehat{\mathfrak{g}}_r$ is Banach universal (Proposition IV.8). Now the assertion follows from [Ne01a, Th. IV.14]. ∎

Remark IV.11. If $\omega \in Z^2_c(\mathfrak{g}_r, \mathbb{C})$ is represented by $\mathrm{diag}(\lambda_j) \in \bigoplus_{\dim H_j = \infty} \mathbb{C}\, \mathrm{id}_{H_j}$ (Proposition I.11), then Proposition IV.9 implies that the corresponding period map $\mathrm{per}_\omega: \pi_2(G_r) \to \mathbb{C}$ factors through

$$\pi_2(G_r) \to \pi_1(Z) \hookrightarrow \mathfrak{z} \xrightarrow{\alpha} \mathbb{C},$$

where the linear functional $\alpha: \mathfrak{z} \to \mathbb{C}$ is given by $\alpha((z_j)) = \sum_j \lambda_j z_j$. Then

$$\mathrm{per}_\omega((n_j)) = \sum_j \lambda_j n_j.$$

For $\mathfrak{g} = \mathfrak{gl}_2(H, I)$ this leads to

$$\mathrm{im}(\mathrm{per}_\omega) = \sum_j \mathbb{Z}\lambda_j \subseteq \mathbb{C}$$

and for $\mathfrak{g} = \mathfrak{gl}_2(H)$ to

$$\mathrm{im}(\mathrm{per}_\omega) = \Big\{ \sum_j n_j \lambda_j : \sum_j n_j = 0 \Big\} = \sum_{j \neq k} \mathbb{Z}(\lambda_j - \lambda_k) \subseteq \mathbb{C}.$$ ∎

The metaplectic and the metagonal group

In this subsection H denotes a complex *infinite-dimensional* Hilbert space and J its complex structure given by $J.v = iv$, $v \in H$. We write $H^{\mathbb{R}}$ for the underlying real Hilbert space. The complexification $H_{\mathbb{C}} := (H^{\mathbb{R}})_{\mathbb{C}}$ decomposes into the $\pm i$-eigenspaces $H_{\mathbb{C}}^{\pm}$ for the complex extension of J which we also denote by J. In the following we will consider $H_{\mathbb{C}}$ as a space with this decomposition, so that $J.(x,y) = (ix, -iy)$ in these "coordinates." We write I for the complex conjugation on $H_{\mathbb{C}}$ with $H^{\mathbb{R}} = \{x \in H_{\mathbb{C}}: I.x = x\}$ and note that $IJ = JI$ on $H_{\mathbb{C}}$ because J preserves the subspace $H^{\mathbb{R}}$. The $\pm i$-eigenvectors of J can be written $v \mp iJv$, $v \in H^{\mathbb{R}}$, and the antilinearity of I implies that $I(v \mp iJv) = v \pm iJv$. To obtain a convenient setting, we identify the i-eigenspace $H_{\mathbb{C}}^+$ of J with H via the mapping $v \mapsto \frac{1}{\sqrt{2}}(v - iJv)$ which is a complex linear isometry. From $IJ = IJ$ and the antilinearity of I, we get $I.H_{\mathbb{C}}^+ = H_{\mathbb{C}}^-$. Since each orthonormal basis of $H_{\mathbb{C}}^+$ is mapped by I into an orthonormal basis of $H_{\mathbb{C}}^-$, we see that we may identify $H_{\mathbb{C}}^-$ with H in such a way that there exists an antilinear involution σ of H (fixing the elements of a given orthonormal basis) such that I is given by product coordinates on $H_{\mathbb{C}} \cong H \oplus H$ by the formula $I.(x,y) = (\sigma(y), \sigma(x))$ (cf. Remark I.2).

Definition IV.12. (a) We define the *restricted real linear group of* H by

$$\mathrm{GL}_{\mathrm{res}}(H^{\mathbb{R}}) := \{ g \in \mathrm{GL}(H^{\mathbb{R}}): gJg^{-1} - J \in B_2(H^{\mathbb{R}}) \}.$$

The condition $gJg^{-1} - J \in B_2(H^{\mathbb{R}})$ is equivalent to $[g, J] \in B_2(H^{\mathbb{R}})$. The elements of the group $\mathrm{GL}_{\mathrm{res}}(H^{\mathbb{R}})$ are called *almost linear automorphisms of the complex Hilbert space* H.
(b) It is clear that $\mathrm{GL}(H)$ is a subgroup of $\mathrm{GL}_{\mathrm{res}}(H^{\mathbb{R}})$. We also define the corresponding *restricted orthogonal and symplectic group*

$$\mathrm{O}_{\mathrm{res}}(H^{\mathbb{R}}) := \mathrm{O}(H^{\mathbb{R}}) \cap \mathrm{GL}_{\mathrm{res}}(H^{\mathbb{R}}) \quad \text{and} \quad \mathrm{Sp}_{\mathrm{res}}(H, \Omega) := \mathrm{Sp}(H, \Omega) \cap \mathrm{GL}_{\mathrm{res}}(H^{\mathbb{R}}).$$ ∎

Lemma IV.13. *The operator* $D := -iJ$ *is a hermitian involution on* $H_{\mathbb{C}}$*, and the following assertions hold:*

(i) $\mathrm{GL}_{\mathrm{res}}(H^{\mathbb{R}}) = \{g \in \mathrm{GL}_2(H_{\mathbb{C}}, D)\colon Ig = gI\} = \mathrm{GL}_2(H_{\mathbb{C}}, D) \cap \mathrm{GL}(H^{\mathbb{R}})$ *and the group* $\mathrm{GL}_{\mathrm{res}}(H^{\mathbb{R}})$ *has a smooth polar decomposition with* $\mathrm{GL}_{\mathrm{res}}(H_{\mathbb{R}}) \cap \mathrm{U}(H_{\mathbb{C}}) = \mathrm{O}_{\mathrm{res}}(H^{\mathbb{R}})$.

(ii) $\mathrm{O}(H_{\mathbb{C}}, I) \cap \mathrm{U}(H_{\mathbb{C}}) = \mathrm{O}(H_{\mathbb{C}}, I) \cap \mathrm{GL}(H^{\mathbb{R}}) = \mathrm{O}(H^{\mathbb{R}})$ *and* $\mathrm{O}(H_{\mathbb{C}}, I) \cap \mathrm{U}_2(H_{\mathbb{C}}, D) = \mathrm{O}_{\mathrm{res}}(H^{\mathbb{R}})$.

(iii) *The operator* $\widetilde{I} := iJI$ *is an antilinear isometry with* $\widetilde{I}^2 = -\mathbf{1}$ *and*

$$\mathrm{Sp}(H, \Omega) = \{g \in \mathrm{Sp}(H_{\mathbb{C}}, \widetilde{I})\colon Ig = gI\} \quad \text{and} \quad \mathrm{Sp}_{\mathrm{res}}(H, \Omega) = \{g \in \mathrm{Sp}_2(H_{\mathbb{C}}, \widetilde{I}, D)\colon Ig = gI\}.$$

In particular $\mathrm{Sp}(H, \Omega)$ *is a real form of* $\mathrm{Sp}_2(H_{\mathbb{C}}, I)$ *and* $\mathrm{Sp}_{\mathrm{res}}(H, \Omega)$ *is a real form of* $\mathrm{Sp}(H_{\mathbb{C}}, I, D)$*. The group* $\mathrm{Sp}_{\mathrm{res}}(H, \Omega)$ *has a polar decomposition with* $\mathrm{Sp}_{\mathrm{res}}(H^{\mathbb{R}}) \cap \mathrm{U}(H^{\mathbb{R}}) = \mathrm{U}(H)$*, and it is contractible.*

Proof. First we note that D is a hermitian operator on $H_{\mathbb{C}}$ with spectrum $\{\pm 1\}$, because in product coordinates on $H_{\mathbb{C}}$ it is given by $D(x, y) = (-x, y)$.

(i) The condition $g \in \mathrm{GL}_{\mathrm{res}}(H^{\mathbb{R}})$ means that if we write its complex linear extension, also denoted g, to $H_{\mathbb{C}}$ according to the decomposition of $H_{\mathbb{C}}$ as a matrix

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then

$$[J, g] = \left[\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right] = i \begin{pmatrix} 0 & 2b \\ -2c & 0 \end{pmatrix},$$

so that $g \in \mathrm{GL}_{\mathrm{res}}(H^{\mathbb{R}})$ is equivalent to $g \in \mathrm{GL}_2(H_{\mathbb{C}}, D)$ and $gI = Ig$.

That the group $\mathrm{GL}_2(H_{\mathbb{C}}, D)$ has a polar decomposition has been shown in Proposition III.7. Let $g \in \mathrm{GL}_2(H_{\mathbb{C}}, D)$ and $g = ue^x$ be its polar decomposition, where $u \in \mathrm{U}(H_{\mathbb{C}})$ and $x = x^* \in \mathrm{Herm}(H_{\mathbb{C}})$. We then have $IgI^{-1} = IuI^{-1}e^{IxI^{-1}}$, which is the polar decomposition of IgI^{-1}. Therefore the uniqueness of the polar decomposition and the first part of the proof imply that $g \in \mathrm{GL}_{\mathrm{res}}(H_{\mathbb{R}})$ if and only if $u \in \mathrm{GL}_{\mathrm{res}}(H_{\mathbb{R}}) \cap \mathrm{U}(H_{\mathbb{C}}) \cong \mathrm{O}_{\mathrm{res}}(H^{\mathbb{R}})$ and $IxI^{-1} = x$, i.e., $x \in \mathrm{Herm}(H_{\mathbb{C}}) \cap \mathfrak{gl}_{\mathrm{res}}(H^{\mathbb{R}})$.

(ii) It is clear that the group $\mathrm{O}(H^{\mathbb{R}})$ acts unitarily on $H_{\mathbb{C}}$. For $g \in \mathrm{O}(H^{\mathbb{R}})$ we therefore have $Ig^*I^{-1} = g^*II^{-1} = g^* = g^{-1}$ which implies that $g \in \mathrm{O}(H_{\mathbb{C}}, I)$. If, conversely, $g \in \mathrm{O}(H_{\mathbb{C}}, I)$, then $gIg^* = I$, so that $g \in \mathrm{O}(H^{\mathbb{R}})$ holds if and only if g commutes with I if and only if g is unitary. This is the first assertion. With (i) and the first part we get

$$\begin{aligned} \mathrm{O}(H_{\mathbb{C}}, I) \cap \mathrm{U}_2(H_{\mathbb{C}}, D) &= \mathrm{O}(H_{\mathbb{C}}, I) \cap \mathrm{U}(H_{\mathbb{C}}) \cap \mathrm{GL}_2(H_{\mathbb{C}}, D) = \mathrm{O}(H^{\mathbb{R}}) \cap \mathrm{GL}_2(H_{\mathbb{C}}, D) \\ &= \mathrm{O}(H^{\mathbb{R}}) \cap \mathrm{GL}(H^{\mathbb{R}}) \cap \mathrm{GL}_2(H_{\mathbb{C}}, D) = \mathrm{O}(H^{\mathbb{R}}) \cap \mathrm{GL}_{\mathrm{res}}(H^{\mathbb{R}}) = \mathrm{O}_{\mathrm{res}}(H^{\mathbb{R}}). \end{aligned}$$

(iii) Its definition implies that $\widetilde{I} = iJI$ is isometric and antilinear. Further $\widetilde{I}^2 = iJIiJI = -i^2JIJI = J^2I^2 = -\mathbf{1}$. For $g \in \mathrm{Sp}(H, \Omega) \subseteq \mathrm{GL}(H^{\mathbb{R}})$ we have $\widetilde{I}g^*\widetilde{I}^{-1} = Jg^*J^{-1} = g^{-1}$, showing that $g \in \mathrm{Sp}(H_{\mathbb{C}}, \widetilde{I})$. If, conversely, $g \in \mathrm{Sp}(H_{\mathbb{C}}, \widetilde{I})$, then $IgI^{-1} = J(g^*)^{-1}J^{-1}$, so that $Ig = gI$ is equivalent to $g = J(g^*)^{-1}J^{-1}$. We further get with (i):

$$\begin{aligned} \mathrm{Sp}_{\mathrm{res}}(H, \Omega) &= \mathrm{Sp}(H, \Omega) \cap \mathrm{GL}_{\mathrm{res}}(H^{\mathbb{R}}) = \mathrm{Sp}(H, \Omega) \cap \mathrm{GL}_2(H_{\mathbb{C}}, D) \\ &= \mathrm{GL}(H^{\mathbb{R}}) \cap \mathrm{Sp}_2(H_{\mathbb{C}}, \widetilde{I}, D) = \{g \in \mathrm{Sp}_2(H_{\mathbb{C}}, \widetilde{I}, D)\colon Ig = gI\}. \end{aligned}$$

To obtain the polar decomposition of $\mathrm{Sp}_{\mathrm{res}}(H, \Omega)$, we can argue as in (i), and we get

$$\mathrm{Sp}_{\mathrm{res}}(H, \Omega) \cap \mathrm{O}(H^{\mathbb{R}}) = \{g \in \mathrm{O}(H^{\mathbb{R}})\colon Jg^{-1}J = Jg^*J^{-1} = g^{-1}\} = \mathrm{U}(H).$$

The contractibility of $\mathrm{U}(H)$ (Theorem II.6) implies that $\mathrm{Sp}_{\mathrm{res}}(H, \Omega)$ is contractible. ∎

Remark IV.14. From Lemma IV.13(ii) we get $\mathrm{O}(H^{\mathbb{R}}) = \mathrm{O}(H_{\mathbb{C}}, I) \cap \mathrm{GL}(H^{\mathbb{R}})$ and therefore

$$\mathrm{O}_{\mathrm{res}}(H^{\mathbb{R}}) = \mathrm{O}(H_{\mathbb{C}}, I) \cap \mathrm{GL}_{\mathrm{res}}(H^{\mathbb{R}}) = \mathrm{O}_2(H_{\mathbb{C}}, I, D) \cap \mathrm{GL}(H^{\mathbb{R}}),$$

showing that $\mathrm{O}_{\mathrm{res}}(H^{\mathbb{R}})$ is a real form of $\mathrm{O}_2(H_{\mathbb{C}}, I, D)$. Moreover

$$\mathrm{O}_{\mathrm{res}}(H^{\mathbb{R}}) = \mathrm{O}_2(H_{\mathbb{C}}, I, D) \cap \mathrm{U}(H_{\mathbb{C}})$$

by Lemma IV.13(ii), so that the polar decomposition of $\mathrm{O}_2(H_{\mathbb{C}}, I, D)$ implies that the inclusion map

$$\mathrm{O}_{\mathrm{res}}(H^{\mathbb{R}}) \hookrightarrow \mathrm{O}_2(H_{\mathbb{C}}, I, D)$$

is a homotopy equivalence. Therefore Theorem III.14 leads to

$$\pi_k(\mathrm{O}_{\mathrm{res}}(H^{\mathbb{R}})) \cong \begin{cases} \mathbb{Z}_2 & \text{for } k = 0 \\ \mathbf{0} & \text{for } k = 1 \\ \mathbb{Z} & \text{for } k = 2. \end{cases}$$

To see elements of $\mathrm{O}_{\mathrm{res}}(H^{\mathbb{R}}) \setminus \mathrm{O}_{\mathrm{res}}(H^{\mathbb{R}})^+$, we recall from Lemma III.12 that

$$\mathrm{O}_\infty(H_{\mathbb{C}}, I, D) = \mathrm{O}_\infty(H_{\mathbb{C}}, I)\, \mathrm{O}(H_{\mathbb{C}}, I, D)^0_\infty,$$

where the group $\mathrm{O}(H_{\mathbb{C}}, I, D)^0_\infty \cong \mathrm{O}(H)$ is contractible. Moreover, the inclusion map $\mathrm{O}_1(H_{\mathbb{C}}, I) \hookrightarrow \mathrm{O}_\infty(H_{\mathbb{C}}, I)$ is a homotopy equivalence by Corollary III.9. Therefore $\mathrm{O}_1(H_{\mathbb{C}}, I)^- := \{g \in \mathrm{O}_1(H_{\mathbb{C}}, I) \colon \det g = -1\}$ is not contained in the identity component of $\mathrm{O}_\infty(H_{\mathbb{C}}, I, D)$, and hence

$$\mathrm{O}_{\mathrm{res}}(H^{\mathbb{R}})^- \supseteq \mathrm{O}_1(H_{\mathbb{C}}, I)^-. \qquad \blacksquare$$

The terminology for the groups defined below is taken from Vershik [Ve90], where these groups are also discussed. Here the main new point is that we can show their universality as central extensions of the corresponding restricted groups.

Definition IV.15. (Metaplectic and metagonal group) (a) Let $D := -iJ$ as above. Then $\omega(x+d, x'+d') := \mathrm{tr}(D[x,x'])$ is the universal cocycle for $\mathfrak{g}_r := \mathfrak{sp}_2(H_{\mathbb{C}}, \widetilde{I}, D)$ (Proposition I.11, Remark IV.5). For the antilinear involution $\theta(x) = IxI^{-1}$ we have $\theta(D) = -IiJI = iIJI = iJ = -D$. Therefore

$$\begin{aligned}(\theta.\omega)(x+d, x'+d') &= \overline{\mathrm{tr}(D[\theta.x, \theta.x'])} = \overline{\mathrm{tr}(D\theta.[x,x'])} \\ &= \mathrm{tr}((\theta.D)[x,x']) = -\mathrm{tr}(D[x,x']) = -\omega(x+d, x'+d').\end{aligned}$$

We conclude that

$$\widehat{\theta}(x,z) := (\theta(x), -\overline{z})$$

defines a complex conjugation of $\widehat{\mathfrak{sp}}_2(H_{\mathbb{C}}, \widetilde{I}, D) := \widehat{\mathfrak{g}}_r$ whose real form is the *metaplectic Lie algebra*

$$\mathfrak{mp}(H, \Omega) := \{(x,z) \in \widehat{\mathfrak{sp}}_2(H_{\mathbb{C}}, \widetilde{I}, D) \colon x \in \mathfrak{sp}_{\mathrm{res}}(H, \Omega), z \in i\mathbb{R}\}.$$

It is a universal central extension of the real Banach–Lie algebra $\mathfrak{sp}_{\mathrm{res}}(H, \Omega)$ (Proposition IV.3(iii)).

The involution $\widehat{\theta}$ integrates to an antiholomorphic involution $\widehat{\theta}_G$ of the simply connected complex group $\widehat{\mathrm{Sp}}_2(H_{\mathbb{C}}, \widetilde{I}, D) := \widehat{G}_r$ (Proposition IV.9), and we define the *metaplectic group*

$$\mathrm{Mp}(H, \Omega) := \{g \in \widehat{\mathrm{Sp}}_2(H_{\mathbb{C}}, \widetilde{I}, D) \colon \widehat{\theta}_G(g) = g\}.$$

Then $\mathrm{Mp}(H, \Omega)$ is a real Lie subgroup of $\widehat{\mathrm{Sp}}_2(H_{\mathbb{C}}, \widetilde{I}, D)$ with Lie algebra $\mathfrak{mp}(H, \Omega)$. This group is a central extension of $\mathrm{Sp}_{\mathrm{res}}(H, \Omega)$ by $\mathbb{T} = \{z \in \mathbb{C}^\times \colon \overline{z} = z^{-1}\}$.

(b) The universal cocycle $\omega(x+d, x'+d') := \mathrm{tr}(D[x,x'])$ of $\mathfrak{g}_r := \mathfrak{o}_2(H_{\mathbb{C}}, I, D)$ satisfies $\theta.\omega = -\omega$ for $\theta(x) = \sigma x \sigma^{-1}$. Therefore

$$\widehat{\theta}(x,z) := (\theta(x), -\overline{z})$$

defines a complex conjugation of $\widehat{\mathfrak{o}}_2(H_{\mathbb{C}}, D) := \widehat{\mathfrak{g}}_r$ whose real form is the *metagonal Lie algebra*

$$\mathfrak{mo}(H^{\mathbb{R}}) := \{(x,z) \in \widehat{\mathfrak{o}}_2(H_{\mathbb{C}}, D) \colon x \in \mathfrak{o}_{\mathrm{res}}(H^{\mathbb{R}}), z \in i\mathbb{R}\}$$

which is a universal central extension of the real Banach–Lie algebra $\mathfrak{o}_{\mathrm{res}}(H^{\mathbb{R}})$ (Proposition IV.8(iii)).

The involution $\widehat{\theta}$ integrates to an antiholomorphic involution $\widehat{\theta}_G$ of the simply connected complex group $\widehat{\mathrm{O}}_2(H_{\mathbb{C}}, I, D)^+ := \widehat{G}_r$ (Theorem IV.10), and we define the *connected metagonal group*

$$\mathrm{MO}(H^{\mathbb{R}})^+ := \{g \in \widehat{\mathrm{O}}_2(H_{\mathbb{C}}, I, D)^+ \colon \widehat{\theta}_G(g) = g\}.$$

Then $\mathrm{MO}(H^{\mathbb{R}})^+$ is a real Lie subgroup of $\widehat{\mathrm{O}}_2(H_{\mathbb{C}}, I, D)^+$ with Lie algebra $\mathfrak{mo}(H^{\mathbb{R}})$ which is a central extension of the identity component $\mathrm{O}_{\mathrm{res}}(H^{\mathbb{R}})^+$ by $Z_{\mathbb{R}} \cong \mathbb{T}$. Its connectedness now follows from the connectedness of $Z_{\mathbb{R}}$ and of $\mathrm{O}_{\mathrm{res}}(H^{\mathbb{R}})^+$.

Let $r \in \mathrm{O}(H^{\mathbb{R}})^-$ be a *simple reflection*, i.e., $r - \mathbf{1}$ has one-dimensional range, which is J-antilinear (cf. Remark IV.14). Then

$$\mathrm{O}_{\mathrm{res}}(H^{\mathbb{R}}) \cong \mathrm{O}_{\mathrm{res}}(H^{\mathbb{R}})^+ \rtimes \{\mathbf{1}, r\}.$$

The relation $rDr^{-1} = -D$ implies that $\omega(rxr^{-1}, ryr^{-1}) = -\omega(x,y)$, so that r acts as an involutive automorphism on the Lie algebra $\mathfrak{mo}(H^{\mathbb{R}})$ by $\tau_{\mathfrak{g}}.(x,z) := (rxr^{-1}, -z)$.

Anticipating the result that $\mathrm{MO}(H^{\mathbb{R}})^+$ is simply connected (Theorem IV.18), it follows that $\tau_{\mathfrak{g}}$ integrates to an involution τ_G on this group. We let the group $\mathbb{Z}_4 := \mathbb{Z}/4\mathbb{Z}$ act on $\mathrm{MO}(H^{\mathbb{R}})^+$ in such a way that $[n] := n + 4\mathbb{Z}$ acts as τ_G^n. Then we define the *full metagonal group*

$$\mathrm{MO}(H^{\mathbb{R}}) := (\mathrm{MO}(H^{\mathbb{R}})^+ \rtimes \mathbb{Z}_4)/\{\mathbf{1}, (-1, [2])\},$$

where -1 denotes the unique non-trivial involution in the central circle $Z_{\mathbb{R}} \subseteq \mathrm{MO}(H^{\mathbb{R}})^+$. Using reflections s in $H_{\mathbb{R}}$ with $\dim(\mathrm{im}(s - \mathbf{1})) = 2$, we obtain, as in the finite-dimensional case, elements $\widetilde{s} \in \mathrm{MO}(H^{\mathbb{R}})^+$ with $\widetilde{s}^2 = -1$, and we may even assume that r and s commute. Then $(\widetilde{s}, [1])^2 = (-1, [2])$ implies that the group $\mathrm{MO}(H^{\mathbb{R}})$ can also be written as a semidirect product $\mathrm{MO}(H^{\mathbb{R}})^+ \rtimes \mathbb{Z}_2$. The full metagonal group is a natural analog of the groups $\mathrm{Pin}(2n, \mathbb{R})$. ■

Theorem IV.16. *The metaplectic group* $\mathrm{Mp}(H,\Omega)$ *satisfies*

$$\pi_m(\mathrm{Mp}(H,\Omega)) \cong \pi_m(\mathbb{T}) \cong \begin{cases} \mathbf{0} & \text{for } m \neq 1 \\ \mathbb{Z} & \text{for } m = 1. \end{cases}$$

Its universal covering group $\widetilde{\mathrm{Mp}}(H,\Omega)$ *is contractible, the identity component of its center is isomorphic to* $\mathbb{R}$*, and it is a universal central extension of the real Banach–Lie group* $\mathrm{Sp}_{\mathrm{res}}(H,\Omega)$*.*

Proof. Since $\mathrm{Mp}(H,\Omega)$ is a central $\mathbb{T}$-extension of the contractible group $\mathrm{Sp}_{\mathrm{res}}(H,\Omega)$, it is connected, and the exact homotopy sequence of the locally trivial principal bundle

$$\mathbb{T} \hookrightarrow \mathrm{Mp}(H,\Omega) \twoheadrightarrow \mathrm{Sp}_{\mathrm{res}}(H,\Omega)$$

yields the assertion on the homotopy groups of $\mathrm{Mp}(H,\Omega)$. Therefore all homotopy groups of the universal covering group $\widetilde{\mathrm{Mp}}(H,\Omega)$ are trivial, which implies that it is contractible (Theorem II.2). It is easy to see that the center of $\mathrm{Sp}_{\mathrm{res}}(H,\Omega)$ consists of $\{\pm\mathbf{1}\}$, so that the identity component of the center of $\widetilde{\mathrm{Mp}}(H,\Omega)$ is the universal covering group of $\mathbb{T} = Z(\mathrm{Mp}(H,\Omega))_e$, hence isomorphic to $\mathbb{R}$. Finally the fact that $\mathfrak{mp}(H,\Omega)$ is a universal central extension of the Banach–Lie algebra $\mathfrak{sp}_{\mathrm{res}}(H,\Omega)$ (Proposition IV.8(ii)), and the simple connectedness of $\widetilde{\mathrm{Mp}}(H,\Omega)$ and $\mathrm{Sp}_{\mathrm{res}}(H,\Omega)$, imply the universality of $\widetilde{\mathrm{Mp}}(H,\Omega)$ as a central extension of $\mathrm{Sp}_{\mathrm{res}}(H,\Omega)$ ([Ne01a, Th. IV.14]). ■

Remark IV.17. (a) Since the group $\mathrm{Sp}_{\mathrm{res}}(H,\Omega)$ is contractible, the central extension $\mathrm{Mp}(H,\Omega)$ is a trivial principal bundle, hence has a continuous global section. There is no general reason for this to imply that there is a smooth global section because we do not know whether there exists a contraction which is a smooth map. For a discussion of the existence of smooth global sections of central extensions we refer to Section V in [Ne00b]. The polar decomposition of $\mathrm{Sp}_{\mathrm{res}}(H,\Omega)$ implies that it is diffeomorphic to a product of $\mathrm{U}(H)$ and a Hilbert space. Since the space $C([0,1],\mathbb{R})$ embeds isometrically into $\mathfrak{u}(H)$ and has no smooth functions with arbitrarily small support (cf. [KM97]), the same holds for $\mathrm{U}(H)$, showing that $\mathrm{U}(H)$ and therefore also $\mathrm{Sp}_{\mathrm{res}}(H,\Omega)$ is not smoothly paracompact.

Nevertheless, one can also show directly as follows that $\mathrm{Mp}(H,\Omega)$ has a smooth global section. First we observe that the the invariance of $\mathfrak{sp}_{\mathrm{res}}(H,\Omega)$ under the involution $x\mapsto x^*$ leads to the decomposition

$$\mathfrak{g}:=\mathfrak{sp}_{\mathrm{res}}(H,\Omega)=\mathfrak{k}\oplus\mathfrak{p}\quad\text{with}\quad \mathfrak{k}=\mathfrak{u}(H)=\{x\in\mathfrak{g}\colon x^*=-x\},\quad \mathfrak{p}=\{x\in\mathfrak{g}\colon x^*=x\}.$$

Writing elements $x\in\mathfrak{g}$ accordingly as $x=x_{\mathfrak{k}}+x_{\mathfrak{p}}$, the universal cocycle satisfies

$$\omega(x,x')=\mathrm{tr}(D[x,x'])=\mathrm{tr}(D[x_{\mathfrak{p}},x'_{\mathfrak{p}}])$$

because $[D,x_{\mathfrak{k}}]=0$. Therefore the inverse image

$$\widehat{\mathfrak{k}}=\mathfrak{k}\oplus i\mathbb{R}\subseteq\widehat{\mathfrak{g}}=\mathfrak{g}\oplus_\omega i\mathbb{R}$$

of $\mathfrak{k}$ is a direct Lie algebra sum $\mathfrak{k}\oplus i\mathbb{R}$. Moreover, $\widehat{\mathfrak{g}}=\widehat{\mathfrak{k}}\oplus\mathfrak{p}$ has the structure of a symmetric Lie algebra because $[\widehat{\mathfrak{k}},\mathfrak{p}]\subseteq\mathfrak{p}$ and $[\mathfrak{p},\mathfrak{p}]\subseteq\widehat{\mathfrak{k}}$. Now the use the methods from [Ne00c] to see that $\mathrm{Mp}(H,\Omega)$ has a polar decomposition $\widehat{K}\exp(\mathfrak{p})$. That the assumptions of [Ne00c, Th. IV.1] are satisfied follows from the fact that for each $x\in\mathfrak{p}$ the operator $(\mathrm{ad}\,x)^2\mid_{\mathfrak{p}}\colon\mathfrak{p}\to\mathfrak{p}$ is positive hermitian ([Ne00c, Prop. III.16]). We then conclude that that simply connected covering group $\widetilde{\mathrm{Mp}}(H,\Omega)$ has a polar decomposition $\big(\mathrm{U}(H)\times i\mathbb{R}\big)\exp\mathfrak{p}$ because $\mathrm{U}(H)\times i\mathbb{R}$ is the simply connected group with Lie algebra $\widehat{\mathfrak{k}}$. From that it follows directly that $\mathrm{Mp}(H,\Omega)$ has a polar decomposition $\widehat{K}\exp\mathfrak{p}$ with $\widehat{K}\cong\mathrm{U}(H)\times\mathbb{T}$. Now we get a smooth section

$$\sigma\colon\mathrm{Sp}_{\mathrm{res}}(H,\Omega)=\mathrm{U}(H)\exp\mathfrak{p}\to\mathrm{Mp}(H,\Omega)=(\mathrm{U}(H)\times\mathbb{T})\exp\mathfrak{p},\quad u\exp x\mapsto(u,\mathbf{1})\exp x$$

for $u\in\mathrm{U}(H)$ and $x\in\mathfrak{p}$.

(b) In the finite-dimensional case $H\cong\mathbb{C}^n$ we have $\pi_1(\mathrm{Sp}(H,\Omega))\cong\pi_1(\mathrm{Sp}(2n,\mathbb{R}))\cong\mathbb{Z}$, and the Lie algebra $\mathfrak{sp}(H,\Omega)$ is centrally closed. Hence the central $\mathbb{T}$-extensions of $\mathrm{Sp}(H,\Omega)$ are classified by

$$\mathrm{Hom}(\pi_1(\mathrm{Sp}(\Omega,H)),\mathbb{T})\cong\mathrm{Hom}(\mathbb{Z},\mathbb{T})\cong\mathbb{T},$$

and the metaplectic group (which in this case is also called $\mathrm{Mp}_c(H,\Omega)$) is defined by the homomorphism $\mathbb{Z}\to\mathbb{T}$ mapping 1 to -1. The corresponding Lie algebra extension is trivial, but the commutator group of $\mathrm{Mp}_c(H,\Omega)$ is a twofold covering of $\mathrm{Sp}(H,\Omega)$ (which is also frequently called metaplectic group). ■

Theorem IV.18. *The inclusion map*

$$\mathrm{MO}(H^{\mathbb{R}})^+\hookrightarrow\widehat{\mathrm{O}}_2(H_{\mathbb{C}},I,D)^+$$

is a weak homotopy equivalence. In particular the group $\mathrm{MO}(H^{\mathbb{R}})^+$ *is simply connected. Moreover, it is a universal central extension of the real Banach–Lie group* $\mathrm{O}_{\mathrm{res}}(H^{\mathbb{R}})^+$.

Proof. Since $\mathrm{O}_2(H_{\mathbb{C}},I,D)^+$ has a polar decomposition with unitary part $\mathrm{O}_{\mathrm{res}}(H_{\mathbb{R}})^+$ (Remark IV.14), the inclusion map $\mathrm{O}_{\mathrm{res}}(H_{\mathbb{R}})^+\to\mathrm{O}_2(H_{\mathbb{C}},I,D)^+$ is a homotopy equivalence. Further the inclusion map $Z_{\mathbb{R}}\cong\mathbb{T}\hookrightarrow Z\cong\mathbb{C}^\times$ is a homotopy equivalence. Therefore Proposition A.8 implies that the inclusion

$$\mathrm{MO}(H^{\mathbb{R}})^+\hookrightarrow\widehat{\mathrm{O}}_2(H_{\mathbb{C}},I,D)^+$$

is a weak homotopy equivalence. We conclude in particular that $\mathrm{MO}(H^{\mathbb{R}})^+$ is simply connected. Therefore the fact that $\mathfrak{mo}(H^{\mathbb{R}})$ is a universal central extension of the real Banach–Lie algebra $\mathfrak{o}_{\mathrm{res}}(H^{\mathbb{R}})$ implies the universality of $\mathrm{MO}(H^{\mathbb{R}})^+$ as a central extension of the simply connected group $\mathrm{O}_{\mathrm{res}}(H^{\mathbb{R}})^+$ ([Ne01a, Th. IV.14]). ■

Remark IV.19. If $H \cong \mathbb{C}^n$ is finite-dimensional and $n > 1$, then $\pi_1(\mathrm{O}(2n,\mathbb{R})) \cong \mathbb{Z}_2$, and there exists a natural $\mathbb{T}$-extension of $\mathrm{O}(2n,\mathbb{R})^+ = \mathrm{SO}(2n,\mathbb{R})$ corresponding to the inclusion homomorphism $\pi_1(\mathrm{O}(2n,\mathbb{R})) \cong \mathbb{Z}_2 \hookrightarrow \mathbb{T}$. This central extension would be a natural analog of $\mathrm{MO}(H^{\mathbb{R}})$. Its commutator subgroup is the universal covering group $\mathrm{Spin}(2n,\mathbb{R})$ of $\mathrm{SO}(2n,\mathbb{R})$.■

V. Infinite-dimensional flag manifolds

In this section we discuss analogs of complex flag manifolds for the groups

$$G \in \{\mathrm{GL}_2(H), \mathrm{GL}_2(H,I)\}.$$

We define these manifolds as the orbits of certain flags $\mathcal{F} = (F_0, F_1, \ldots, F_k)$ in H under G. Let $P(\mathcal{F}) \subseteq G$ denote the stabilizer of such a flag. Then the homogeneous space $G/P(\mathcal{F})$ is a complex manifold, called a *flag manifold*. We also show that the unitary real form $U := G \cap \mathrm{U}(H)$ acts transitively on $G/P(\mathcal{F})$. Similar results hold for the restricted groups G_r.

Definition V.1. (a) We consider a *flag* $\mathcal{F} = (F_0, F_1, \ldots, F_k)$, where

$$\{0\} = F_0 \subseteq F_1 \subseteq F_2 \subseteq \ldots \subseteq F_k = H$$

are closed subspaces of H. Let

$$P_b(\mathcal{F}) := \{g \in \mathrm{GL}(H) \colon (\forall j) g.F_j = F_j\}$$

denote the stabilizer of this flag. To get a better description of this group, we define closed subspaces $H_j := F_j \cap F_{j-1}^\perp$ for $j = 1, \ldots, k$ and thus obtain an orthogonal decomposition $H = H_1 \oplus \ldots \oplus H_k$. Accordingly we view operators on H as matrices $(x_{ij})_{i,j=1,\ldots,k}$ with $x_{ij} \in B(H_j, H_i)$. Then

$$P_b(\mathcal{F}) = \{g \in \mathrm{GL}(H) \colon (\forall i > j)\, g_{ij} = 0\} \cong N(\mathcal{F}) \rtimes M(\mathcal{F}),$$

where

$$M(\mathcal{F}) = \{g \in \mathrm{GL}(H) \colon (\forall j) g.H_j = H_j\} = \{g \in \mathrm{GL}(H) \colon (\forall i \neq j) g_{ij} = 0\} \cong \prod_{j=1}^k \mathrm{GL}(H_j)$$

and

$$N(\mathcal{F}) = \{g \in \mathrm{GL}(H) \colon (\forall j)(g - \mathbf{1}).H_j \subseteq H_{j-1}\} = \{g \in P_b(\mathcal{F}) \colon (\forall j)\, g_{jj} = \mathbf{1}\}.$$

(b) For $G = \mathrm{GL}_2(H)$ we now define $P := P(\mathcal{F}) := P_b(\mathcal{F}) \cap G$. In this case the description in (a) implies immediately that $P(\mathcal{F})$ is a complemented Lie subgroup of G, so that the homogeneous space $G/P(\mathcal{F})$ has a natural structure of a complex Banach manifold (cf. Definition III.6) which we call a *flag manifold* associated to G.
(c) For $G = \mathrm{GL}_2(H,I)$ we now consider a chain of closed subspaces

$$\{0\} = F_0 \subseteq F_1 \subseteq \ldots \subseteq F_k$$

which are *isotropic*, i.e., that all spaces F_j are isotropic for the bilinear form $\beta(x,y) = \langle x, I.y\rangle$, which in turn is equivalent to $I.F_j \perp F_j$. We extend this chain of subspaces to the flag $\mathcal{F}$ defined by

$$\{0\} = F_0 \subseteq F_1 \subseteq F_2 \subseteq \ldots \subseteq F_k \subseteq F_k^{\perp_\beta} \subseteq \ldots \subseteq F_1^{\perp_\beta} \subseteq F_0^{\perp_\beta} = H.$$

For $g \in \mathrm{GL}(H)$ and $\tau(g) := I(g^*)^{-1}I^{-1}$ the condition $g \in \mathrm{GL}(H,I)$ is equivalent to $g = \tau(g)$. Moreover, the condition $g.E = E$ for a closed subspace $E \subseteq H$ is equivalent to $g^*.E^\perp = E^\perp$, hence to $\tau(g).E^{\perp_\beta} = E^{\perp_\beta}$ because $E^{\perp_\beta} = I.E^\perp$. Therefore the subgroup

$P_b(\mathcal{F}) \subseteq \mathrm{GL}(H)$ is invariant under the involution τ, and its intersection with $\mathrm{GL}(H,I)$ coincides with the set of all elements preserving the subspaces $F_1, \ldots, F_k$.

To fix the notation in such a way that it is compatible with Examples I.9, we define for the flag $\mathcal{F}$ the spaces $H_1, \ldots, H_k$ as above, $H_0 := F_k^\perp \cap F_k^{\perp_\beta}$, and $H_{-j} := I.H_j$ for $j = 1, \ldots, k$. Then

$$F_j^{\perp_\beta} = H_1 \oplus \ldots \oplus H_k \oplus H_0 \oplus H_{-k} \oplus \ldots \oplus H_{-j-1}.$$

Note that H_0 is zero if and only if F_k is maximal isotropic for β.

It is clear that the group $M(\mathcal{F})$ is invariant under τ. For $g \in P_b(\mathcal{F})$ we have $\tau(g)_{jj} = I(g^*_{-j,-j})^{-1}I^{-1}$, showing that also $N(\mathcal{F})$ is τ-invariant. Therefore the semidirect decomposition of $P_b(\mathcal{F})$ leads with

$$M_b := M(\mathcal{F})^\tau \cong \mathrm{GL}(H_0, I_0) \times \prod_{j=1}^k \mathrm{GL}(H_j)$$

(cf. Remark I.2) and $N_b := N(\mathcal{F})^\tau$ to the semidirect decomposition $P_b = N_b \rtimes M_b$. On the Lie algebra level the strictly lower triangular matrices in $\mathfrak{gl}(H)$ provide a complement invariant under $\tau_{\mathfrak{g}}(x) := -Ix^*I^{-1}$, so that passing to $\tau_{\mathfrak{g}}$-fixed points yields a closed complement to the Lie algebra $\mathbf{L}(P_b)$ of P_b in $\mathfrak{gl}(H,I)$. Therefore P_b is a complemented Lie subgroup of $G_b = \mathrm{GL}(H,I)$.

Similar results hold for the group $P := P_b \cap \mathrm{GL}_2(H,I)$, so that we obtain a complex manifold structure on the homogeneous spaces G/P and G_b/P_b (cf. Definition III.6). ∎

Remark V.2. (a) The equation

$$N(\mathcal{F}) - \mathbf{1} = \{x \in B(H) \colon (\forall i \leq j)\ x_{ij} = 0\}$$

shows that this is a closed subspace of $B(H)$, hence that $N(\mathcal{F})$ is contractible. Similar assertions hold for the intersection with $\mathrm{GL}_2(H)$.

(b) To obtain the corresponding result for $N(\mathcal{F}) \cap \mathrm{GL}(H,I)$, we note that the exponential function $\exp\colon N(\mathcal{F}) - \mathbf{1} \to N(\mathcal{F})$ is a polynomial diffeomorphism inverted by the logarithm function given by

$$\log(\mathbf{1} + x) = \sum_{j=1}^\infty \frac{(-1)^{n+1}}{n} x^n.$$

This property is inherited by the subgroup $N(\mathcal{F}) \cap \mathrm{GL}(H,I)$, so that it is also contractible, and the same holds for $N(\mathcal{F}) \cap \mathrm{GL}_2(H,I)$. ∎

Now we turn to the homotopy groups of the flag manifolds $G/P(\mathcal{F})$.

Proposition V.3. (a) *For $G = \mathrm{GL}_2(H)$ and $P = P_b(\mathcal{F}) \cap G$ the manifold G/P satisfies*

$$\pi_0(G/P) = \mathbf{0}, \quad \pi_1(G/P) = \mathbf{0} \quad \textit{and} \quad \pi_2(G/P) \cong \mathbb{Z}^{k-1}.$$

(b) *For $G = \mathrm{GL}_2(H,I)$ and $P = P_b(\mathcal{F}) \cap G$ the flag manifold G/P satisfies*

$$\pi_0(G/P) \cong \left\{ \begin{matrix} \mathbf{0} & \textit{for } I^2 = -\mathbf{1} \\ \mathbb{Z}_2 & \textit{for } I^2 = \mathbf{1} \textit{ and } H_0 \neq \mathbf{0} \end{matrix} \right\}, \quad \pi_1(G/P) = \mathbf{0} \quad \textit{and} \quad \pi_2(G/P) \cong \mathbb{Z}^k.$$

Proof. (a) We have $P \cong N \rtimes M$, where N is diffeomorphic to a Banach space. Therefore P is homotopy equivalent to M. We conclude that

$$\pi_0(P) = \pi_0(G) = \mathbf{0}, \quad \pi_1(P) \cong \mathbb{Z}^k, \quad \pi_1(G) \cong \mathbb{Z} \quad \text{and} \quad \pi_2(P) \cong \pi_2(G) = \mathbf{0}.$$

Hence G/P is connected, and if $\chi\colon \pi_1(P) \to \pi_1(G)$ is the homomorphism induced by the inclusion map, the exact homotopy sequence of the principal P-bundle $G \to G/P$ implies that

$$\pi_1(G/P) \cong \operatorname{coker} \chi \quad \text{and} \quad \pi_2(G/P) \cong \ker \chi.$$

In view of $\chi(n_1, \ldots, n_k) = \sum_j n_j$, we get

$$\pi_1(G/P) \cong \mathbf{0} \quad \text{and} \quad \pi_2(G/P) \cong \mathbb{Z}^{k-1}.$$

(b) For $G = \mathrm{GL}_2(H, I)$ we also have $P \cong N \rtimes M$, which is homotopy equivalent to M (Remark V.2(b)). For $H_0 = \mathbf{0}$ we have

$$\pi_0(P) = \mathbf{0}, \quad \pi_1(P) \cong \mathbb{Z}^k \quad \text{and} \quad \pi_2(P) = \mathbf{0},$$

and

$$\pi_0(G) \cong \pi_1(G) \cong \begin{Bmatrix} \mathbf{0} & \text{for } I^2 = -\mathbf{1} \\ \mathbb{Z}_2 & \text{for } I^2 = \mathbf{1} \end{Bmatrix} \quad \text{and} \quad \pi_2(G) = \mathbf{0}.$$

Therefore the exact homotopy sequence of the bundle $P \hookrightarrow G \to G/P$ yields

$$\pi_0(G/P) \cong \begin{Bmatrix} \mathbf{0} & \text{for } I^2 = -\mathbf{1} \\ \mathbb{Z}_2 & \text{for } I^2 = \mathbf{1} \end{Bmatrix}, \quad \pi_1(G/P) = \mathbf{0} \quad \text{and} \quad \pi_2(G/P) \cong \mathbb{Z}^k$$

because for $I^2 = \mathbf{1}$ the homomorphism $\pi_1(P) \to \pi_1(G)$ is given by

$$\mathbb{Z}^k \to \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}, \quad (n_j) \mapsto \sum_j [n_j].$$

For $H_0 \neq \mathbf{0}$ we have

$$\pi_0(P) \cong \begin{Bmatrix} \mathbf{0} & \text{for } I^2 = -\mathbf{1} \\ \mathbb{Z}_2 & \text{for } I^2 = \mathbf{1} \end{Bmatrix}, \quad \pi_1(P) \cong \begin{Bmatrix} \mathbb{Z}^k & \text{for } I^2 = -\mathbf{1} \\ \mathbb{Z}_2 \times \mathbb{Z}^k & \text{for } I^2 = \mathbf{1} \end{Bmatrix}, \quad \text{and} \quad \pi_2(P) = \mathbf{0}$$

and

$$\pi_0(G) \cong \pi_1(G) \cong \begin{Bmatrix} \mathbf{0} & \text{for } I^2 = -\mathbf{1} \\ \mathbb{Z}_2 & \text{for } I^2 = \mathbf{1} \end{Bmatrix} \quad \text{and} \quad \pi_2(G) = \mathbf{0}.$$

Therefore the exact homotopy sequence yields

$$\pi_0(G/P) = \mathbf{0}, \quad \pi_1(G/P) = \mathbf{0} \quad \text{and} \quad \pi_2(G/P) \cong \mathbb{Z}^k$$

because for $I^2 = \mathbf{1}$ the homomorphism $\pi_0(P) \to \pi_0(G)$ is surjective. ■

Remark V.4. (a) In Proposition III.5 we have seen that for $G = \mathrm{GL}_2(H)$ we have $G_r = GG_b^0$ which implies that with $P_r := G_r \cap P(\mathcal{F})$ we have $G_r = GP_r$ with $P_r \cap G = P$, so that G acts transitively on G_r/P_r, and we obtain $G/P \cong G_r/P_r$.

For $G = \mathrm{GL}_2(H, I)$ we have on the Lie algebra level $\mathfrak{g}_r = \mathfrak{g} + \mathfrak{g}_b^0$, which implies that $G_r = G_e(G_b^0)_e$ (Lemma A.5), and hence that the identity component G_e of G acts transitively on the connected manifold G_r/P_r for $P_r := G_r \cap P(\mathcal{F})$. From $P_r \cap G = P$ we derive that $G_r/P_r \cong G_e/(G_e \cap P)$ is the connected component of G/P containing the base point $\mathbf{1}P$.

(b) Suppose that all spaces H_j in Definition V.1 are infinite-dimensional and that $G_r = \mathrm{GL}(H, I, D)_e$, where D is compatible with the flag $\mathcal{F}$ in the sense of Examples I.9. Then the group $P_r \cong N_r \rtimes M_r$ is contractible because N_r is contractible and all factors in M_r are contractible (Theorem II.6, Remark V.2). Therefore the exact homotopy sequence of the P_r-principal bundle $G_r \to G_r/P_r \cong G_e/(P \cap G_e)$ implies that the quotient map is a weak homotopy equivalence. In particular we obtain for each $m \in \mathbb{N}_0$ the relation

$$\pi_m(G/P) \cong \pi_m(G_r).$$

■

Transitivity of the action of the unitary real form on the flag manifolds

In this subsection we show that the unitary real forms of the restricted groups G_r also act transitively on the corresponding flag manifolds G_r/P_r.

Lemma V.5. *If $\varphi: H_1 \to H_2$ is a topological isomorphism of Hilbert spaces, then there exists a unitary isomorphism $\psi: H_1 \to H_2$.*

Proof. The map $\varphi^*\varphi \in \mathrm{GL}(H_1)$ is an invertible positive operator, so that $\gamma := \sqrt{\varphi^*\varphi} \in \mathrm{GL}(H_1)$ is uniquely defined. Now $\psi := \varphi \circ \gamma^{-1}: H_1 \to H_2$ is unitary because it is invertible and

$$\psi^*\psi = \gamma^{-1}\varphi^*\varphi\gamma^{-1} = \gamma^{-1}\gamma^2\gamma^{-1} = \mathbf{1}. \qquad \blacksquare$$

Lemma V.6. *Let H_1 and H_2 be complex Hilbert spaces and $I_j: H_j \to H_j$ antilinear isometries with $I_j^2 = \pm\mathbf{1}$ (same signs). If $\varphi: (H_1, I_1) \to (H_2, I_2)$ is a topological isomorphism with $\varphi I_1 \varphi^* = I_2$, then there exists a unitary isomorphism $\psi: H_1 \to H_2$ with $\psi I_1 \psi^* = I_2$.*

Proof. First we observe that the condition $\varphi I_1 \varphi^* = I_2$ means that φ is an isometry between the spaces (H_j, β_j), where $\beta_j(x,y) = \langle x, I_j.y\rangle$ is a non-degenerate complex bilinear form. Indeed,

$$\beta_2(\varphi(x), \varphi(y)) := \langle \varphi(x), I_2.\varphi(y)\rangle = \langle x, I_1.y\rangle =: \beta_1(x,y), \quad x, y \in H_1,$$

is equivalent to $\varphi^* I_2 \varphi = I_1$, i.e., to $\varphi^{-1} I_2 (\varphi^*)^{-1} = I_1$ which in turn means that $\varphi I_1 \varphi^* = I_2$.

We define ψ as in the proof of Lemma V.5. The remark above implies that $\varphi^*\varphi$ is a β_1-isometry, hence in $\mathrm{GL}(H_1, I_1)$. The polar decomposition of this group (Theorem II.6(iii)) implies that $\gamma \in \mathrm{GL}(H_1, I_1)$, so that $\psi = \varphi\gamma^{-1}: H_1 \to H_2$ satisfies

$$\psi I_1 \psi^* = \varphi\gamma^{-1} I_1 (\gamma^*)^{-1} \varphi^* = \varphi I_1 \varphi^* = I_2. \qquad \blacksquare$$

Proposition V.7. *Let $H = H_1 \oplus \ldots \oplus H_k$ be the orthogonal eigenspace decomposition of $D = D^* \in B(H)$, $F_j := H_1 + \ldots + H_j$, $P_b := P_b(\mathcal{F})$, $P_r := P_b \cap \mathrm{GL}_2(H, D)$ and $P := P_b \cap \mathrm{GL}_2(H)$. Then*

$$\mathrm{GL}(H) = \mathrm{U}(H)P_b, \quad \mathrm{GL}_2(H,D) = \mathrm{U}_2(H,D)P_r = \mathrm{U}_2(H)P_r, \quad \textit{and} \quad \mathrm{GL}_2(H) = \mathrm{U}_2(H)P,$$

i.e., $\mathrm{U}(H)$ acts transitively on $\mathrm{GL}(H)/P_b$ and $\mathrm{U}_2(H)$ acts transitively on $\mathrm{GL}_2(H,D)/P_r$ and $\mathrm{GL}_2(H)/P$. Moreover, if $u \in \mathrm{U}(H)$ and $g \in \mathrm{GL}_2(H,D)$ satisfy $u^{-1}g \in P_b$, then $u \in \mathrm{U}_2(H,D)$.

Proof. (see [PS86, Prop. 7.13] for the case $k = 2$) Let $F_j' := g.F_j$. Then g maps F_1 isomorphically onto the Hilbert space F_1'. Hence Lemma V.5 implies that there exists a unitary isomorphism $u_1: F_1 \to F_1'$. Moreover, g induces a topological isomorphism

$$H_2 \cong F_2/F_1 \to F_2'/F_1' \cong H_2' := (F_1')^\perp \cap F_2'.$$

Applying Lemma V.5 again, we find a unitary isomorphism $u_2: H_2 \to H_2'$. Continuing this way, we obtain unitary isomorphisms

$$u_j: H_j \to H_j' := (F_{j-1}')^\perp \cap F_j', \quad j = 1, \ldots, k.$$

Putting these maps together, we obtain a unitary map $u \in \mathrm{U}(H)$ with $u(H_j) = H_j'$ for all j and therefore in particular with

$$u(F_j) = \sum_{m=1}^{j} H_m' = F_j'.$$

This means that $u^{-1}g$ preserves all spaces F_j. We conclude that $\mathrm{GL}(H) = \mathrm{U}(H)P_b$.

Suppose that $g \in \mathrm{GL}_2(H,D)$. Then for each j the orthogonal projection

$$p_j : F'_j \to F_j^\perp$$

is Hilbert–Schmidt because its composition with g is Hilbert–Schmidt as an operator $F_j \to F_j^\perp$. We conclude that $p_j \circ u : F_j \to F_j^\perp$ is Hilbert–Schmidt, which implies that for $i > j$ the operator $u_{ij} \in B(H_j, H_i)$ is Hilbert–Schmidt. Moreover, for $j > 1$ the orthogonal projection

$$q_j : H'_j \to F_{j-1}$$

is Hilbert–Schmidt because its composition with g corresponds to the operators $g_{1j}, \ldots, g_{j-1,j}$, hence is Hilbert–Schmidt. Therefore $q_j \circ u\,|_{H_j} : H_j \to F_{j-1}$ is Hilbert–Schmidt, which means that $u_{1j}, \ldots, u_{j-1,j}$ are Hilbert–Schmidt. We conclude that also for $i < j$ we have $u_{ij} \in B_2(H_j, H_i)$, and hence that $u \in U_2(H,D)$. Thus $u^{-1}g \in P_b \cap \mathrm{GL}_2(H,D) = P_r$, and this proves that $\mathrm{GL}_2(H,D) = \mathrm{U}_2(H,D)P_r$. From the connectedness of the groups $\mathrm{U}_2(H,D)$ we derive $\mathrm{U}_2(H,D) = \mathrm{U}_2(H)\,\mathrm{U}(H)^0 \subseteq \mathrm{U}_2(H)P_r$ (Lemma A.5), whence $\mathrm{GL}_2(H,D) = \mathrm{U}_2(H)P_r$.

Finally $\mathrm{GL}_2(H) \subseteq \mathrm{GL}_2(H,D) = \mathrm{U}_2(H)P_r$ leads to $\mathrm{GL}_2(H) = \mathrm{U}_2(H)(P_r \cap \mathrm{GL}_2(H)) = \mathrm{U}_2(H)P$. ∎

Proposition V.8. *Let $H = H_{-k} \oplus \ldots \oplus H_k$ be the orthogonal eigenspace decomposition of $D = D^* \in B(H,I)$ with $IH_j = H_{-j}$, $F_j := H_1 + \ldots + H_j$ for $j = 1, \ldots, k$, and define $P_b := P_b(\mathcal{F})$ and $P_e := P_b \cap \mathrm{GL}_2(H,I)_e$. Then*

$$\mathrm{GL}(H,I) = \mathrm{U}(H,I)P_b \quad \text{and} \quad \mathrm{GL}_2(H,I,D) \subseteq \mathrm{U}_2(H,I,D)P_b.$$

Moreover, with $G_r := \mathrm{GL}_2(H,I,D)_e$, $U_r := G_r \cap \mathrm{U}(H)$ and $P_r := P_b \cap G_r$ we get

$$G_r = U_rP_r = \mathrm{U}_2(H,I)P_r \text{ and } \quad \mathrm{GL}_2(H,I)_e = \mathrm{U}_2(H,I)_eP_e.$$

In particular $\mathrm{U}(H,I)$ acts transitively on $\mathrm{GL}(H,I)/P_b$ and $\mathrm{U}_2(H,I)$ acts transitively on G_r/P_r.

Proof. Let $g \in \mathrm{GL}(H,I)$. As in the proof of Proposition V.7, we put $F'_j := g.F_j$ and obtain unitary operators

$$u_j : H_j \to H'_j := (F'_{j-1})^\perp \cap F'_j, \quad j = 1, \ldots, k.$$

Putting these maps together, we obtain a unitary map $u_+ : H_+ := F_k \to H'_+ := F'_k$ mapping each H_j, $j = 1, \ldots, k$, unitarily onto H'_j. From $g \in \mathrm{GL}(H,I)$ we derive that F'_k is isotropic for the bilinear form $\beta(x,y) := \langle x, I.y\rangle$, which means that $H'_- := I.H'_+ \subseteq H_+^\perp$. Therefore $H'_+ + I.H'_+ \subseteq H$ is an orthogonal direct sum, hence a closed subspace of H. We define a unitary map

$$u_- : H_- := I.H_+ \to H'_-, \quad v \mapsto I(u_+^*)^{-1}I^{-1}.v.$$

Let $H'_0 := g.(H_0 + H_+) \cap (H'_+)^\perp$. Then

$$(H'_0 + H'_+)^\perp = \big(g.(H_0 + H_+)\big)^\perp = (g^*)^{-1}.\big((H_0 + H_+)^\perp\big) = I^{-1}gI.H_- = I.(g.H_+) = I.H'_+ = H'_-$$

implies that $H = H'_+ \oplus H'_0 \oplus H'_-$ is an orthogonal direct sum.

Since $(H_0, \beta|_{H_0 \times H_0})$ is, as a space with bilinear form, isomorphic to

$$(H_+ + H_0)/(H_+ + H_0)^{\perp_\beta} = (H_+ + H_0)/H_+ \cong H_0,$$

the map $g \in \mathrm{GL}(H,I)$ induces an isomorphism

$$H_0 \to g(H_0 + H_+)/\big(g(H_0 + H_+)\big)^{\perp_\beta} \cong (H'_0 + H'_+)/H'_+ \cong H'_0$$

with respect to the restriction of β to both spaces. Therefore we obtain with with Lemma V.6 a unitary β-isometric map $u_0 : H_0 \to H'_0$.

Combining u_0 with $u_\pm$, we now obtain with $H = H_+ \oplus H_0 \oplus H_-$ a unitary map $u\colon H \to H$. To see that $u \in \mathrm{U}(H,I)$, we first recall that u_0 is β-isometric. Moreover, u maps $(H_0)^{\perp_\beta} = H_+ + H_-$ to the closed subspace $H'_+ + H'_- = H'_+ + I.H'_+$ with

$$(H'_+ + H'_-)^{\perp_\beta} = (H'_+)^{\perp_\beta} \cap (H'_-)^{\perp_\beta} = (H'_+ + H'_0) \cap (I.H'_-)^\perp = (H'_+ + H'_0) \cap (H'_+)^\perp = H'_0.$$

Therefore it remains to show that $u\,|_{H_+ + H_-}$ is β-isometric. The subspaces $H_\pm$ and $H'_\pm$ are β-isotropic, so that the assertion follows from

$$\beta(u.v_+, u.v_-) = \beta(u_+.v_+, I(u_+^*)^{-1}I^{-1}.v_-) = -\langle u_+.v_+, (u_+^*)^{-1}I^{-1}.v_-\rangle = \langle v_+, I.v_-\rangle = \beta(v_+, v_-)$$

for $v_\pm \in H_\pm$. We conclude that $u \in \mathrm{U}(H,I)$, and that $u^{-1}g$ preserves the spaces $F_1, \ldots, F_k$, hence is contained in P_b.

Suppose that $g \in \mathrm{GL}_2(H,I,D)$. Then g maps the flag

$$F_1 \subseteq F_2 \subseteq \ldots \subseteq F_k \subseteq F_k^{\perp_\beta} \subseteq \ldots \subseteq F_1^{\perp_\beta}$$

to

$$F'_1 \subseteq F'_2 \subseteq \ldots \subseteq F'_k \subseteq (F'_k)^{\perp_\beta} \subseteq \ldots \subseteq (F'_1)^{\perp_\beta}$$

and u does the same. Therefore the last assertion in Proposition V.7 entails that $u \in \mathrm{U}_2(H,D)$ and hence that $u \in \mathrm{U}_2(H,I,D)$. Finally $u^{-1}g \in P_b$ implies that $\mathrm{GL}_2(H,I,D) \subseteq \mathrm{U}_2(H,I,D)P_b$.

We conclude in particular that the group $U_r = \mathrm{U}_2(H,I,D)_e \subseteq G_r$ acts on G_r/P_r with open orbits, and therefore transitively because G_r/P_r is connected, whence $G_r = U_rP_r$. With Lemma A.5 we now obtain from $\mathfrak{u}_r = \mathfrak{u}_2(H,I) + \mathfrak{u}(H,I,D)^0$ that $U_r = \mathrm{U}_2(H,I)_e\,\mathrm{U}(H,I,D)^0 \subseteq \mathrm{U}_2(H,I)_eP_r$, so that $G_r = \mathrm{U}_2(H,I)_eP_r$, and therefore $\mathrm{GL}_2(H,I)_e \subseteq G_r = \mathrm{U}_2(H,I)_eP_r$ yields $\mathrm{GL}_2(H,I)_e = \mathrm{U}_2(H,I)_eP_e$. ■

Remark V.9. (a) The decompositions of type $G = UP$ obtained in Propositions V.7/8 are analogs of the Iwasawa decomposition of finite-dimensional complex reductive Lie groups. It is an interesting question whether such decompositions could be obtained for infinite flags.
(b) It follows from the proof of Theorem II.14 that the manifolds G/P contain a dense subset which is the directed union of orbits of finite-dimensional groups G_F which are compact complex flag manifolds. Since the orbits $G_FP/P \subseteq G/P$ have the property that all holomorphic functions on them are constant, it follows easily that all holomorphic functions on G/P are constant (cf. [HH94b, Cor. 3.2.2] for the case $G = \mathrm{GL}_2(H)$).

For refined information on holomorphic sections of complex line bundles on the manifolds G/P we refer to [HH94a,b], [Ne00a] and [Ne01a]. For an extension of the Bott–Borel–Weil Theorem to direct limit groups, which is closely related to our setting, we refer to [NRW00]. ■

Remark V.10. (a) First let $G = \mathrm{GL}_2(H)$. For $k = 2$ and $n := \dim F_1 < \infty$ the orbit $G.F_1$ consists of all n-dimensional subspaces of H. Therefore $\mathrm{Gr}_n(H) := G/P$ is the *Graßmannian* of all n-dimensional subspaces of H. For $n = 1$ we obtain in particular the *projective space* $\mathbb{P}(H) = \mathrm{Gr}_1(H)$.

For $k = 2$, H separable, and $F := F_1$ of infinite dimension and codimension, the manifold $\mathrm{Gr}_{\mathrm{res}}(F) := G/P$ is the *restricted Graßmannian* of the separable Hilbert space H based in F. This manifold plays a crucial role in the structure theory of loop groups and in theoretical physics (cf. [PS86], [Wu98]).
(b) The manifolds G/P for arbitrary length of the flag and $G = \mathrm{GL}_2(H)$ have been introduced in two papers of A. and G. Helminck (cf. [HH94a] and [HH94b]).
(c) For $G = \mathrm{GL}_2(H)$ and $k = 2$ the manifolds $G/P \cong U/(U \cap P)$ (Proposition V.7) are symmetric spaces because the group $U \cap P$ can be written as the fixed point set of an involution on U defined as $\tau(u) = pup^{-1}$, where $p^2 = \mathbf{1}$ and $\ker(p - \mathbf{1}) = F_1$.

Other symmetric spaces are obtained for $G = \mathrm{GL}_2(H,I)$ and $k = 1$ and either $\dim F_1 = 1$ or $F_1 \subseteq H$ maximal isotropic. In the first case we obtain the space of all isotropic lines in $\mathbb{P}(H)$ for the bilinear form $\beta(x,y) = \langle x, I.y\rangle$, and in the second case we obtain a subset of the restricted

Graßmannian associated to F_1, consisting of all those subspaces which are isotropic for β. In the first case the involution on $\mathrm{U}_2(H,I)$ can be obtained from $p^2 = p$ with $\ker(p+\mathbf{1}) = H_0$ and $\ker(p-\mathbf{1}) = F_1 + I.F_1$. Then each element in $\mathrm{U}_2(H,I)$ commuting with p either preserves F_1 or maps it to $I.F_1$, showing that $\mathrm{U}_2(H,I) \cap P$ has index 2 in the fixed point group of $\tau(u) = pup^{-1}$, hence is an open subgroup. In the second case, where F_1 is maximal isotropic, we define $p \in \mathrm{GL}(H,I)$ by i on F_1 and $-i$ on $F_1^\perp$. Then $\tau(u) := pup^{-1}$ defines an involution on $\mathrm{U}_2(H,I)$ with the required properties. ∎

The important role of the flag manifolds $G/P \cong U/(U \cap P)$ stems from the fact that these are *precisely* those coadjoint orbits of central extensions of the real group U which are strong Kähler manifolds, hence have the "best" geometric structure. We refer to [Ne01a] for this characterization which was one of the main motivations for writing the present paper.

Appendix

Lemma A.1. *Let X and Y be Banach spaces and $A\colon X \to Y$ a continuous linear map. Suppose that X_1, Y_1 are Banach spaces with continuous injective linear maps $\eta_X\colon X_1 \to X$ and $\eta_Y\colon Y_1 \to Y$. If $A(\eta_X(X_1)) \subseteq \eta_Y(Y_1)$, then the induced map*

$$A_1\colon X_1 \to Y_1 \quad \text{with} \quad \eta_Y \circ A_1 = A \circ \eta_X$$

is continuous.

Proof. We argue with the Closed Graph Theorem. Assume that $(x_n, A_1.x_n) \to (x,y) \in X_1 \times Y_1$. Then $\eta_Y(A_1.x_n) = A.\eta_X(x_n) \to A.\eta_X(x)$ implies that $\eta_Y(y) = A.\eta_X(x) = \eta_Y(A_1.x)$, and therefore $A_1.x = y$. Therefore A_1 is continuous. ∎

Lemma A.2. *Let X, Y, Z be Banach spaces and $A\colon X \times Y \to Z$ a continuous bilinear map. Suppose that X_1, Y_1, Z_1 are Banach spaces with continuous injective linear maps $\eta_X\colon X_1 \to X$, $\eta_Y\colon Y_1 \to Y$ and $\eta_Z\colon Z_1 \to Z$. If $A(\eta_X(X_1) \times \eta_Y(Y_1)) \subseteq \eta_Z(Z_1)$, then the induced bilinear map*

$$A_1\colon X_1 \times Y_1 \to Z_1 \quad \text{with} \quad \eta_Z \circ A_1 = A \circ (\eta_X \times \eta_Y)$$

is continuous.

Proof. In view of [Ru73, Th. 2.17], it suffices to show that A_1 is separately continuous. Fix $y_1 \in Y_1$. Then the map $A(\cdot, \eta_Y(y_1))$ maps $\eta_X(X_1)$ to $\eta_Z(Z_1)$, so that the continuity of the map $A_1(\cdot, y_1)\colon X_1 \to Z_1$ follows from Lemma A.1. We likewise obtain the continuity of the maps $A_1(x_1, \cdot)$. Therefore A_1 is separately continuous and therefore continuous. ∎

Lemma A.3. *If $\mathfrak{g}$ is a Banach–Lie algebra and $\mathfrak{a}, \mathfrak{b}$ are Banach-Lie algebras with continuous injective homomorphisms $\eta_{\mathfrak{a}}\colon \mathfrak{a} \to \mathfrak{g}$ and $\eta_{\mathfrak{b}}\colon \mathfrak{b} \to \mathfrak{g}$ such that $\eta_{\mathfrak{b}}(\mathfrak{b})$ normalizes $\eta_{\mathfrak{a}}(\mathfrak{a})$, then the induced action of $\mathfrak{b}$ on $\mathfrak{a}$ is continuous.*

Proof. We apply Lemma A.2 with the continuous bilinear map $[\cdot,\cdot]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ which maps $\eta_{\mathfrak{a}}(\mathfrak{a}) \times \eta_{\mathfrak{b}}(\mathfrak{b})$ to $\eta_{\mathfrak{a}}(\mathfrak{a})$. ∎

Lemma A.4. *If M and N are Banach manifolds, $M_1 \subseteq M$ is a submanifold, and $f\colon M_1 \to N$ is a smooth map, then the graph $\Gamma(f) := \{(x, f(x))\colon x \in M_1\}$ is a submanifold of $M \times N$.*

Proof. Passing to local charts, we may assume that $M = M_0 \times M_1$ holds for a Banach manifold M_0. Let $x = (x_0, x_1) \in M$. Then we may further assume that N is an open subset of a Banach space X and that $f(x_1) = 0$. We then consider the smooth function

$$F\colon M_0 \times M_1 \times N \to M_0 \times M_1 \times X, \quad F(y_0, y_1, y_2) := (y_0, y_1, y_2 - f(y_1))$$

which is a local diffeomorphism around $(x_0, x_1, 0) \in \Gamma(f)$ with

$$F^{-1}(M \times \{0\}) = M_0 \times \Gamma(f).$$

This implies that $\Gamma(f)$ is a submanifold of $M \times N$. ∎

Lemma A.5. *Let A, B, C be Banach–Lie groups with morphisms $\eta_A: A \to C$ and $\eta_B: B \to C$. Assume that*

(1) *C is connected,*

(2) $\operatorname{im} \mathbf{L}(\eta_A) + \operatorname{im} \mathbf{L}(\eta_B) = \mathbf{L}(C)$, *and*

(3) *$\eta_B(B)$ normalizes $\eta_A(A)$.*

Then $C = \eta_A(A)\eta_B(B)$.

Proof. The multiplication map $m: A \times B \to C, (a,b) \mapsto \eta_A(a)\eta_B(b)$ is a smooth map whose differential in (e,e) is given by $(x,y) \mapsto \mathbf{L}(\eta_A)(x) + \mathbf{L}(\eta_B)(y)$, and hence is surjective. Therefore the Implicit Function Theorem implies that $\operatorname{im}(m)$ has inner points. Further (3) implies that $\operatorname{im}(m)$ is a subgroup of C and therefore an open subgroup. Now (1) shows that m is surjective. ∎

Proposition A.6. *Let A, B, C be Banach–Lie groups and assume that A is connected and that there exist injective morphisms $\eta_A: A \to C$ and $\eta_B: B \to C$. Let $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$ denote the corresponding Lie algebras and identify $\mathfrak{a}$ and $\mathfrak{b}$ with their image under $\mathbf{L}(\eta_A)$, resp., $\mathbf{L}(\eta_B)$. We assume that*

(1) *$\mathfrak{b}$ is a closed subalgebra of $\mathfrak{c}$,*

(2) $\mathfrak{c} = \mathfrak{a} + \mathfrak{b}$,

(3) *B normalizes A, and*

(4) *$\mathfrak{a} \cap \mathfrak{b}$ is complemented in $\mathfrak{a}$.*

Then the conjugation action of B on A is smooth, AB is an open subgroup of C, and the multiplication map $m: A \rtimes B \to AB \subseteq C$ is a locally trivial $A \cap B$-principal bundle.

Proof. Since $\mathfrak{b}$ normalizes $\mathfrak{a}$, Lemma A.3 implies that the bracket map $\mathfrak{a} \times \mathfrak{b} \to \mathfrak{a}$ is continuous with respect to the Banach space structure on $\mathfrak{a}$ which might be finer than that inherited from $\mathfrak{c}$. Therefore $\mathfrak{a} \rtimes \mathfrak{b}$ carries the structure of a Banach–Lie algebra. Lemma A.1 also implies that for each $b \in B$ the map $\operatorname{Ad}_{\mathfrak{a}}(b) := \operatorname{Ad}(b)|_{\mathfrak{a}}: \mathfrak{a} \to \mathfrak{a}$ is a continuous automorphism. The action of B on $\mathfrak{a}$ is obtained by integrating a continuous representation of its Lie algebra $\mathfrak{b}$, hence is a continuous homomorphism $\operatorname{Ad}_{\mathfrak{a}}: B \to \operatorname{Aut}(\mathfrak{a})$.

For each $b \in B$ the conjugation map $c_A(b): A \to A, a \mapsto bab^{-1}$ is a group automorphism with $c_A(b)(\exp_A(x)) = \exp_A(\operatorname{Ad}_{\mathfrak{a}}(x))$ for $x \in \mathfrak{a}$, because both sides have the same image in C. Therefore each $c_A(b)$ is smooth in an identity neighborhood, thus a smooth automorphism of A. Moreover, the smoothness of the action of B on $\mathfrak{a}$ implies that the B-orbit maps of elements of A are smooth for regular values of the exponential map $\exp_A: \mathfrak{a} \to A$, hence for all points in an identity neighborhood of A. Since B acts by automorphisms of A, the set of all points with smooth orbit map is a subgroup, so that the connectedness of A entails that all orbit maps of elements in A are smooth. Now let $b_0 \in B$ and $a_0 \in A_e$. For $b \in B$ and $a \in A$ we then have

$$b_0 b a a_0 (b_0 b)^{-1} = c_A(b_0)\big((bab^{-1} c_A(b)(a_0)\big).$$

Since B acts by smooth automorphisms and with smooth orbits maps, the smoothness of the action of B on A follows from the smoothness close to the identity which in turn follows from the fact that the exponential function is a local diffeomorphism and the action of B on $\mathfrak{a}$ is smooth. Thus the group $A \rtimes B$ carries a natural structure of a Banach–Lie group.

As in the proof of Lemma A.5, we see that the multiplication map $m: A \rtimes B \to C, (a,b) \mapsto ab$ and an open map. Therefore AB is an open subgroup of C. The kernel of m is the Lie subgroup

$$N = \{(a, a^{-1}) \in A \rtimes B : a \in A \cap B\}$$

([Ne00a, Lemma IV.11]). The Lie algebra of N is the subalgebra $\{(x, -x) \in \mathfrak{a} \rtimes \mathfrak{b} : x \in \mathfrak{a} \cap \mathfrak{b}\}$ which is complemented in $\mathfrak{a} \rtimes \mathfrak{b}$ because, in view of (4), $\mathfrak{a} \cap \mathfrak{b}$ is complemented as a subspace of $\mathfrak{a}$. Therefore we see with Definition III.6 that the quotient group $(A \rtimes B)/N$ is a Lie group, which then is isomorphic to AB. Moreover, the quotient map $A \rtimes B \to (A \rtimes B)/N$ is a submersion, hence defines a locally trivial $A \cap B$-principal bundle. ∎

Remark A.7. If, under the assumption of Proposition A.6, A is not connected, then we first obtain with Proposition A.6 that $A_e \rtimes B \to A_e B$ is an $A_e \cap B$-principal bundle over the open subgroup $A_e B \subseteq C$. Moreover, $A_e B_e$ is an open connected subgroup of C, hence the identity component C_e of C.

Assume that $AB = C$. Then the multiplication map $A \rtimes B \to C \to C/C_e = \pi_0(C)$ factors through a surjective homomorphism

$$(A/A \cap C_e) \rtimes (B/B \cap C_e) \to \pi_0(C).$$

If we know, in addition, that B acts continuously on A, then $A \rtimes B$ is a topological group and the open subgroup $A_e \rtimes B$ is a Banach–Lie group. This implies that $A \rtimes B$ is a Banach–Lie group because because conjugating by elements of A induces continuous, hence smooth, isomorphisms of $A_e \rtimes B$. ∎

Proposition A.8. *For $j = 1, 2$ let G_j be a topological group and $H_j \subseteq G_j$ a closed subgroup. We further assume that $q_j : G_j \to M_j := G_j/H_j$ defines a locally trivial principal bundle and that we have a continuous homomorphism $\varphi: G_1 \to G_2$ with $\varphi(H_1) \subseteq H_2$. Let $\varphi_M : M_1 \to M_2, g_1H_1 \mapsto g_2H_2$ denote the map induced by φ. If two of the three maps φ, φ_M and $\varphi_H := \varphi|_{H_1} : H_1 \to H_2$ are weak homotopy equivalences, then the same holds for the third one.*

Proof. Since the map φ is compatible with the subgroups $H_j \subseteq G_j$, it induces maps between the exact homotopy sequences

$$\cdots \to \pi_k(H_j) \to \pi_k(G_j) \to \pi_k(M_j) \to \pi_{k-1}(H_j) \to \pi_{k-1}(G_j) \to \pi_{k-1}(M_j)$$

of $G_j \twoheadrightarrow M_j$:

$$\begin{array}{ccccccccccccc}
\cdots & \to & \pi_k(H_1) & \to & \pi_k(G_1) & \to & \pi_k(M_1) & \to & \pi_{k-1}(H_1) & \to & \pi_{k-1}(G_1) & \to & \cdots \\
 & & \downarrow{\scriptstyle \pi_k(\varphi_H)} & & \downarrow{\scriptstyle \pi_k(\varphi)} & & \downarrow{\scriptstyle \pi_k(\varphi_M)} & & \downarrow{\scriptstyle \pi_{k-1}(\varphi_H)} & & \downarrow{\scriptstyle \pi_{k-1}(\varphi)} & & \\
\cdots & \to & \pi_k(H_2) & \to & \pi_k(G_2) & \to & \pi_k(M_2) & \to & \pi_{k-1}(H_2) & \to & \pi_{k-1}(G_2) & \to & \cdots .
\end{array}$$

We assume that φ and φ_H are weak homotopy equivalences; the other cases are similar. Then the maps $\pi_k(\varphi)$ and $\varphi_k(\varphi_H)$ are isomorphisms, and the rows in the above diagram are exact, so that the 5-Lemma ([CE56, Prop. I.1.1]) implies that all homomorphisms $\pi_k(\varphi_M)$, $k \in \mathbb{N}$, are isomorphisms. To obtain this also for $k = 0$ we may extend the exact homotopy sequence by zeros on the right hand side because the maps $\pi_0(G_j) \to \pi_0(M_j)$ are trivially surjective. ∎

References

[Ba69] Balachandran, V. K., *Simple L^*-algebras of classical type*, Math. Ann. **180** (1969), 205–219.

[Bo59] Bott, R., *The stable homotopy groups of the classical groups*, Ann. of Math. **70:2**(1959), 313–337.

[Bou90] Bourbaki, N., *Groupes et algèbres de Lie, Chapitres 1–9*, Masson, Paris, 1990.

[BW76] Brüning, J., and W. Willgerodt, *Eine Verallgemeinerung eines Satzes von N. Kuiper*, Math. Ann. **220** (1976), 47–58.

[Ca89] Carey, A. and J. Palmer, *Infinite complex spin groups*, J. Funct. Anal. **83:1** (1989), 1–43.

[CR87] Carey, A., and S. N. M. Ruijsenaars, *On fermion gauge groups, current algebras, and Kac–Moody algebras*, Acta Appl. Math. **10** (1987), 1–86.

[CE56] Cartan, H., and S. Eilenberg, "Homological Algebra," Princeton Univ. Press, 1956.

[CGM90] Cuenca Mira, J. A., A. Garcia Martin, and C. Martin Gonzalez, *Structure theory of L^*-algebras*, Math. Proc. Camb. Phil. Soc. **107** (1990), 361–365.

[DNS89] Dorfmeiser, J., E. Neher, and J. Szmigielski, *Automorphisms of Banach manifolds associated with the KP-equation*, Quart. J. Math. Oxford (2) **40** (1989), 161–195.

[DNS90] —, *Banach manifolds and their automorphisms associated with groups of type C_∞ and D_∞*, in "Lie Algebras and Related Topics," Proc. Res. Conf., Madison/WI (USA) 1988, Contemp. Math. **110** (1990), 43–65.

[EK64] van Est, W. T., and Th. J. Korthagen, *Non-enlargible Lie algebras*, Proc. Kon. Ned. Acad. v. Wet. A **67** (1964), 15–31.

[Gl99] Glöckner, H., *Direct limit Lie groups and manifolds*, Preprint **2025**, TU Darmstadt, 1999.

[Ha67] Halmos, P. R., "A Hilbert Space Problem Book," Graduate Texts in Math. **19**, Springer-Verlag, 1967.

[dlH70] de la Harpe, P., "Classification of simple real L^*-algebras," Univ. of Warwick, July 1990.

[dlH71a] —, *Classification des L^*-algèbres semi-simples réelles séparable*, C. R. Acad. Sci. Paris, Ser. A **272** (1971), 1559–1561.

[dlH71b] —, *Groupe spinoriel associé à un espace de Hilbert*, C. R. Acad. Sci. Paris, Ser. A **273** (1971), 305–307.

[dlH72] de la Harpe, P., "Classical Banach–Lie Algebras and Banach–Lie Groups of Operators in Hilbert Space," Lecture Notes in Math. **285**, Springer-Verlag, Berlin, 1972.

[dlH79] —, *Les extensions de $\mathfrak{gl}(H)$ par un noyau de dimension finie sont triviales*, J. Funct. Anal. **33** (1979), 362–373.

[dlH82] —, *Classical groups and classical Lie algebras of operators*, Proceedings of Symp. in Pure Math. **38:1** (1982), 477–513.

[HH94a] Helminck, A. G., and Helminck, G. F., *Holomorphic line bundles over Hilbert flag varieties*, in "Algebraic Groups and their Generalizations: Quantum and Infinite-dimensional Methods," W. J. Haboush ed., Proc. Symp. Pure Math. **56:2** (1994), 349–375.

[HH94b] —, *The structure of Hilbert flag varieties*, Publ. Res. Inst. Math. Sci. **30:3** (1994), 401–441.

[Hu94] Husemoller, D., "Fibre Bundles," Graduate Texts in Math., Springer-Verlag, New York, 1994.

[Ja62] Jacobsen, N., "Lie Algebras," Dover Publ., Inc., New York, 1962.

[Ka75] Kaup, W., *Über die Automorphismen Graßmannscher Mannigfaltigkeiten unendlicher Dimension*, Math. Z. **144** (1975), 75–96.

[Ka83] —, *Über die Klassifikation der symmetrischen hermiteschen Mannigfaltigkeiten unendlicher Dimension I, II*, Math. Annalen **257**(1981), 463–486; **262**(1983), 57–75.

[KM97] Kriegl, A., and P. Michor, "The Convenient Setting of Global Analysis," Math. Surveys and Monographs **53**, Amer. Math. Soc., 1997.

[La93] Lang, S., "Algebra," Addison Wesley Publ. Comp., London, 1993.

[Ma62] Maissen, B., *Lie-Gruppen mit Banachräumen als Parameterräumen*, Acta Math. **108** (1962), 229–269.

[Mi89] Mickelsson, J., "Current Algebras and Groups," Plenum Press, New York, 1989.

[Mit70] Mityagin, B. S., *The homotopy structure of the linear group of a Banach space*, Russian Math. Surveys **25** (1970), 59–103.

[NRW00] Natarajan, L., E. Rodriguez-Carrington, and J. A. Wolf, *The Bott-Borel Weil Theorem for direct limit groups*, Trans. Amer. Math. Soc., to appear.

[Ne99] Neeb, K.-H., "Holomorphy and Convexity in Lie Theory," Expositions in Mathematics **28**, de Gruyter Verlag, Berlin, 1999.

[Ne00a] —, *Infinite-dimensional Lie groups and their representations*, "Lectures at the European School in Group Theory," SDU-Odense Univ., August 2000 (available as Preprint).

[Ne00b] —, *Central extensions of infinite-dimensional Lie groups*, Preprint **2084**, TU Darmstadt, April 2000.

[Ne00c] —, *A Cartan–Hadamard Theorem for Banach–Finsler manifolds*, Geometriae Dedicata, to appear.

[Ne01a] Neeb, K.-H., *Geometry and structure of L^*-groups*, in preparation.

[Ne01b] Neeb, K.-H., *Universal central extensions of Lie groups*, in preparation.

[NØ98] Neeb, K.–H., and B. Ørsted, *Unitary highest weight representations in Hilbert spaces of holomorphic functions on infinite dimensional domains*, J. Funct. Anal. **156** (1998), 263–300.

[Neh92] Neher, E., *Lie groups, hermitian symmetric spaces and Jordan pairs*, in "Hadronic Mechanics and Nonpotential Interactions, Part 1," Cedar Falls, IA, 1990, 243–258, Nova Sci. Publ., Commack, NY, 1992.

[Neh93] —, *Generators and relations for 3-graded Lie algebras*, Journal of Algebra **155** (1993), 1–35.

[Ot95] Ottesen, J. T., "Infinite Dimensional Groups and Algebras in Quantum Physics," Springer-Verlag, Lecture Notes in Physics **m 27**, 1995.

[Pa65] Palais, R. S., *On the homotopy type of certain groups of operators*, Topology **3** (1965), 271–279.

[Pa66] —, *Homotopy theory of infinite dimensional manifolds*, Topology **5** (1966), 1–16.

[PS86] Pressley, A., and G. Segal, "Loop Groups," Oxford University Press, Oxford, 1986.

[RS78] Reed, S., and B. Simon, "Methods of Modern Mathematical Physics IV: Analysis of Operators", Academic Press, New York, 1978.

[Ru73] Rudin, W., "Functional Analysis," McGraw Hill, 1973.

[Sch60] Schue, J. R., *Hilbert space methods in the theory of Lie algebras*, Trans. Amer. Math. Soc. **95** (1960), 69–80.

[Sch61] —, *Cartan decompositions for L^*-algebras*, Trans. Amer. Math. Soc. **98** (1961), 334–349.

[Se59] Segal, I. E., *Foundations of the theory of dynamical systems of infinitely many degrees of freedom. I*, Mat. Fys. Medd. Danske Vid. Selsk. **31:12** (1959), 1–39.

[Sh62] Shale, D., *Linear symmetries of free boson fields*, Trans. Amer. Math. Soc. **103** (1962), 146–169.

[ShSt65] Shale, D., and W. F. Stinespring, *Spinor representations of the infinite orthogonal groups*, J. Math. Mech. **14** (1965), 315–322.

[St99] Stumme, N., "The Structure of Locally Finite Split Lie algebras," Ph.D. thesis, Darmstadt University of Technology, 1999.

[Un71] Unsain, I., "Classification of the simple real separable L^*-algebras," Bull. of the Amer. Math. Soc. **72** (1971), 462–466.

[Un72] —, *Classification of the simple real separable L^*-algebras*, J. Diff. Geom. **7** (1972), 423–451.

[Up85] Upmeier, H., "Symmetric Banach Manifolds and Jordan C^*-algebras," North Holland Mathematics Studies, 1985.

[Ve90] Vershik, A. M., *Metagonal groups of finite and infinite dimension*, in "Representations of Lie Groups and Related Topics," A. M. Vershik and D. P. Zhelobenko eds., Gordon and Breach Science Publishers, 1990; 1–37.

[Wu98] Wurzbacher, T., *Fermionic second quantization and the geometry of the restricted Grassmannian*, DMV-Seminar "Infinite Dimensional Kähler Manifolds," Oberwolfach, November, 1995, Birkhäuser, to appear.