Strong L^q-Theory of the Navier-Stokes Equations in Aperture Domains

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Abstract This article deals with the Stokes equations in aperture domains. Geometrically spoken, such domains consist of two halfspaces separated by a wall, but connected by a hole within that wall. It is well known that in these domains the solution of the Stokes equations is not unique, but that an additional boundary condition has to be imposed: This can be either the flux through the hole or the pressure drop between the two halfspaces. In this paper suitable Stokes operators are constructed for both cases, which are shown to generate bounded analytic semigroups. This is used to prove the existence and uniqueness of strong solutions of the Stokes and Navier-Stokes equations subject to one of the additional boundary conditions.

Key words Navier-Stokes equations – aperture domain – Stokes operator

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1 Introduction

The flow of a viscous incompressible fluid in a region Ω with rigid walls is governed by the following Navier-Stokes equations:

$$\begin{aligned} u_t - \Delta u + u \cdot \nabla u + \nabla p &= f && \text{in } \Omega \times (0,T), \\ \operatorname{div} u &= 0 && \text{in } \Omega \times (0,T), \\ u &= 0 && \text{on } \partial\Omega \times (0,T), \\ u(0) &= u_0 && \text{in } \Omega. \end{aligned} \tag{1}$$

Here u is the velocity field, p the pressure and f, u_0 are the given external force and initial velocity respectively. It turns out that for some domains even the solution of the stationary Stokes equations is not uniquely determined by the corresponding equations, but one has to impose an auxiliary condition to single out a unique solution. This was first discovered by Heywood, considering a so called *aperture domain*, see [10]:

Definition 1 *For* $n \geq 2$ *let* $\mathbb{R}^n_\pm = \{x \in \mathbb{R}^n : \pm x_n > d/2\}$, $d \geq 0$ *and* $B = \{x \in \mathbb{R}^n : |x| < R\}$, $R > 0$. *Then* $\Omega \subset \mathbb{R}^n$ *is called an aperture domain, if* Ω *is a domain satisfying the uniform cone condition and*

$$\Omega \cup B = \mathbb{R}^n_+ \cup \mathbb{R}^n_- \cup B.$$

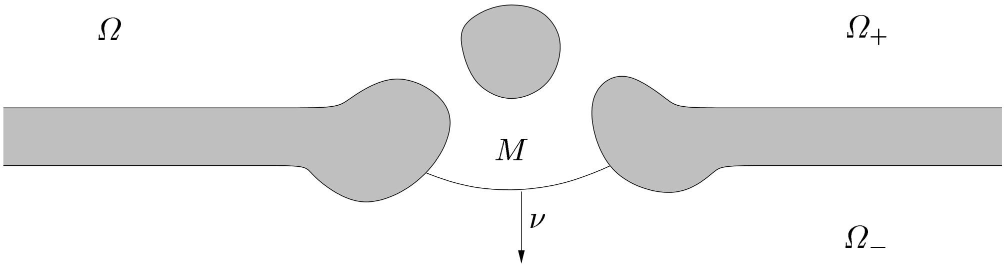

Fig. 1 An aperture domain

By means of the Galerkin method, Heywood [10], [11] has shown the local existence and uniqueness of so called generalized solutions of the Navier-Stokes equations for both of the additional boundary conditions.

Farwig and Sohr [4], [5] analysed the resolvent system for the Stokes problem. They showed that the Stokes operator, associated to a prescribed flux, is the generator of a bounded analytic semigroup.

In [7] the Stokes operator in L^2 for a prescribed pressure drop is constructed. Moreover, the local existence and uniqueness of strong solutions of the Navier-Stokes equations for both of the additional boundary conditions is shown.

In the present article the results of [7] are generalized to L^q spaces. It turns out that the behaviour of the (Navier-) Stokes equations depends on the space dimension n:

Let $n = 2$. For $1 < q \leq 2$ the solution is unique and the flux vanishes, whereas for $q > 2$ a flux has to be given. A pressure drop can never be prescribed.

Let $n \geq 3$. For $1 < q < \frac{n}{n-1} = n'$ the solution is unique, without claiming any additional boundary condition. If $n' < q < n$, either the flux or the pressure drop can be prescribed, whereas for $q \geq n$ only a flux can be given.

This paper is organized as follows: First the basic function spaces for the Navier-Stokes equations will be analysed. Then the Helmholtz decompositions appropriat to prescribe a flux or pressure drop respectively will be defined. Afterwards the associated Stokes operators will be shown to generate bounded analytic semigroups. This is used to prove the existence and uniqueness of solutions of the Stokes equations under one of the additional boundary conditions. Finally, using the technique of [9], local solutions of the corresponding Navier-Stokes equations are constructed.

2 The Basic Function Spaces

The following standard notation is used: If $\Omega \subset \mathbb{R}^n$ is a domain, then $C_0^\infty(\Omega)$ denotes the set of the smooth functions with compact support in Ω, whereas $C_0^\infty(\overline{\Omega})$ are the functions of $C_0^\infty(\mathbb{R}^n)$, restricted to Ω.

Let $1 \le q \le \infty$. Then $L^q(\Omega)$ denote the Lebesgue space and $\|\cdot\|_q$ its norm. The dual exponent q' is defined by $\frac{1}{q} + \frac{1}{q'} = 1$. If $u \in L^q(\Omega)^n$, $v \in L^{q'}(\Omega)^n$, then $\langle u, v\rangle$ denotes the duality pairing

$$\langle u, v\rangle = \int_\Omega u \cdot v \, dx \,,$$

where " $\cdot$ " stands for the scalar product in $\mathbb{R}^n$.

Let $m \in \mathbb{N}$. Then $W_q^m(\Omega)$ is the Sobolev space of all functions, whose weak derivatives up to order m are in $L^q(\Omega)$. Moreover, $\overset{\circ}{W}{}_q^1(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ with respect to the $W_q^1(\Omega)$-norm.

Beside the standard Sobolev spaces, appropriate function spaces for the velocity u and the pressure p have to be introduced. Concerning the velocity there are two possibilities:

Definition 2 *Let* $\Omega \subset \mathbb{R}^n$, $n \ge 2$ *be an aperture domain and* $1 < q < \infty$. *Then*

$$\mathcal{J}_q^1(\Omega) = \left\{ u \in \overline{C_0^\infty(\Omega)^n}^{\,\|\cdot\|_{1,q}} : \operatorname{div} u = 0 \right\}, \tag{2}$$

$$J_q^1(\Omega) = \overline{\left\{ u \in C_0^\infty(\Omega)^n : \operatorname{div} u = 0 \right\}}^{\,\|\cdot\|_{1,q}}, \tag{3}$$

where $\overline{}^{\,\|\cdot\|_{1,q}}$ *denotes the closure with respect to the* $W_q^1(\Omega)$*-norm.*

In contrast to the case of a bounded domain or an exterior domain, these two spaces do not always coincide.

To understand this phenomenon, the physical flux has to be defined precisely:

Definition 3 *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be an aperture domain and $M \subset \Omega \cap B$ a smooth $(n-1)$-dimensional manifold such that $\Omega \setminus M$ consists of two disjoint domains Ω_+ and Ω_- with $M = \partial\Omega_+ \cap \partial\Omega_-$. Furthermore let N be the normal vector on M directed into Ω_-. Then, for $u \in \mathcal{J}_q^1(\Omega)$,*

$$\Phi(u) = \int_M N \cdot u \, do$$

is called the flux through the aperture from Ω_+ to Ω_-.

By the trace theorem it is easy to see that Φ is a linear functional on $\mathcal{J}_q^1(\Omega)$. Since the vector fields are solenoidal, the flux Φ does not depend on the special shape of M; i.e. one can take $M = \mathbb{R}_+^n \cap \partial B$.

Indeed, for a special aperture domain, Heywood [10] has shown the existence of a vector field $\chi \in W_q^1(\Omega)^n$, $n' < q < \infty$ such that $\Phi(\chi) = 1$.

On the other hand it is easily seen by Gauss' theorem that $\Phi(u)$ vanishes for $u \in J_q^1(\Omega)$, hence $\mathcal{J}_q^1(\Omega) \neq J_q^1(\Omega)$. This result applies to arbitrary aperture domains:

Theorem 1 *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be an aperture domain. Then there is a vector field $\chi \in C^\infty\big(\overline{\Omega}\big)^n \cap W_q^2(\Omega)^n$, $n' < q < \infty$ such that*

$$\chi|_{\partial\Omega} = 0, \qquad \operatorname{div}\chi = 0, \qquad \Phi(\chi) = 1. \tag{4}$$

i) If $n' < q < \infty$, then

$$J_q^1(\Omega) = \Big\{ u \in \mathcal{J}_q^1(\Omega) : \Phi(u) = 0 \Big\}. \tag{5}$$

ii) If $1 < q \leq n'$, then

$$J_q^1(\Omega) = \mathcal{J}_q^1(\Omega). \tag{6}$$

Proof. Let $\{\eta_0, \eta_+, \eta_-\}$ be a partition of unity in Ω with the following property: There exists a ball $B' = \{x \in \mathbb{R}^n : |x| \leq R'\}$, $R' > R$ such that $\eta_0 = 1$ on $\Omega \cap B$, $\eta_+ = 1$ on $\Omega_+ \setminus B'$ and $\eta_- = 1$ on $\Omega_- \setminus B'$. Furthermore, let $\Omega_0 \subset \Omega \cap B'$ be a domain satisfying the uniform cone condition and $\operatorname{supp}\eta_+ \subset \overline{\Omega}_0$.

For the special aperture domain $G = \{x \in \mathbb{R}^n : x_n \neq 0 \text{ or } |x| < 1\}$ there is a vector field $\psi \in C^\infty(\overline{G})^n \cap W_q^2(G)^n$, $n' < q < \infty$, satisfying (4), see [8], chap. III.4.

Choosing B large enough guarantees that $\chi_1 = \psi(1 - \eta_0)$ vanishes on $\partial\Omega$. Furthermore, $g = -\operatorname{div}\chi_1 = \psi \cdot \nabla\eta_0 \in C_0^\infty(\Omega_0)$ with

$$\int_{\Omega_0} g \, dx = -\int_{\partial\Omega_0} N \cdot \chi_1 \, d\sigma = 0.$$

Hence by [8], Theorem III.3.2 there is a vector field $\chi_0 \in C_0^\infty(\Omega_0)^n$ with $\operatorname{div}\chi_0 = g$. Now $\chi = \chi_0 + \chi_1 \in W_q^2(\Omega)^n \cap C^\infty\big(\overline{\Omega}\big)^n$ has the desired properties.

In order to show *i)*, let $u \in \overset{\circ}{\mathcal{J}}{}_q^1(\Omega)$ with $\Phi(u) = 0$. Then $u_+ = u\eta_+ \in \overset{\circ}{W}{}_q^1(\Omega_+)^n$ and $\operatorname{div} u_+ = u \cdot \nabla\eta_+ = g_+$. Because of the compatibility condition

$$\int_{\Omega_+\cap B'} g_+\,dx = \int_{\partial(\Omega_+\cap B')} N\cdot u_+\,do = -\Phi(u) = 0,$$

[8], Theorem III.3.2 can be used once more to get $v_+ \in \overset{\circ}{W}{}_q^1(\Omega_+\cap B')^n$ such that $\operatorname{div} v_+ = g_+$. Hence $u_+ - v_+ \in \mathcal{J}_q^1(\Omega_+)$. In the same way a vector field $v_- \in \overset{\circ}{W}{}_q^1(\Omega_-\cap B')^n$ with $u_- - v_- \in \mathcal{J}_q^1(\Omega_-)$ can be found. Setting $u_0 = u\eta_0$ and $v_0 = v_+ + v_-$ yields $u_0 + v_0 \in \mathcal{J}_q^1(\Omega\cap B')$ and $u = (u_+ - v_+) + (u_- - v_-) + (u_0 + v_0)$. Now J_q^1 and $\mathcal{J}_q^1$ coincide for the perturbed halfspaces $\Omega_\pm$ and the bounded domain $\Omega\cap B'$, see [8], Chapter III.4, hence $u \in J_q^1(\Omega)$.

In order to prove *ii)*, it has to be shown that $\Phi(u) = 0$ for $u \in \mathcal{J}_q^1(\Omega)$, provided $1 < q \le n'$: For $r > R$, the Gauss theorem yields

$$\Phi(u) = \int_M N\cdot u\,d\sigma = -\int_{\Omega_+\cap\partial B_r} N\cdot u\,d\sigma\,.$$

Extending u by zero outside of Ω and using Hölder's inequality shows

$$|\Phi(u)| = C_n\big(r^{n-1}\big)^{1-1/q}\left(\int_{\partial B_r}|u|^q\,d\sigma\right)^{1/q}.$$

Integration yields

$$\int_R^\infty \frac{|\Phi(u)|^q}{r^{(n-1)(q-1)}}\,dr \le C\int_R^\infty\int_{\partial B_r}|u|^q\,d\sigma\,dr \le C\|u\|_q^q < \infty.$$

Now because of $(n-1)(q-1) \le 1$ the flux $\Phi(u)$ has to vanish. □

Similar to the case of the velocity field u, for the pressure p there are also two possibilities for the appropriate function space:

Definition 4 *Let $\Omega \subset \mathbb{R}^n$, $n \ge 2$ be an aperture domain. Then*

$$\dot{W}_q^1(\Omega) = \Big\{p \in L^q_{\mathrm{loc}}(\overline{\Omega}) : \nabla p \in L^q(\Omega)\Big\}, \tag{7}$$

$$\widehat{W}_q^1(\Omega) = \overline{C_0^\infty\big(\overline{\Omega}\big)}^{\|\nabla\cdot\|_q}. \tag{8}$$

Here $L^q_{\mathrm{loc}}(\overline{\Omega})$ denotes those functions u such that $u \in L^q(\Omega\cap B)$ for all balls $B \subset \mathbb{R}^n$.

If Ω is a bounded domain or an exterior domain, these two spaces coincide. For aperture domains however, the situation is as follows:

Theorem 2 *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be an aperture domain.*

i) If $1 < q < n$, then there are constants $p_\pm \in \mathbb{R}$ such that for $\frac{1}{r} = \frac{1}{q} - \frac{1}{n}$

$$\|p - p_\pm\|_{L^r(\Omega_\pm)} \leq C\|\nabla p\|_{L^q(\Omega_\pm)}.$$

The pressure drop $[p] = p_+ - p_-$ can be estimated by

$$|[p]| \leq C\|\nabla p\|_{L^q(\Omega)}.$$

Moreover,

$$\widehat{W}^1_q(\Omega) = \Big\{ p \in \dot{W}^1_q(\Omega) : [p] = 0 \Big\}. \tag{9}$$

ii) If $n \leq q < \infty$, then $\widehat{W}^1_q(\Omega) = \dot{W}^1_q(\Omega)$.

Proof. See [5]. It is clear from the proof that the theorem applies to domains fulfilling the uniform cone condition only. □

Theorem 3 *Let Ω be an aperture domain, $u \in \mathcal{J}^1_q(\Omega)$ and $p \in \dot{W}^1_q(\Omega)$. Then*

$$\int_\Omega \nabla p \cdot u \, dx \; = -[p]\Phi(u). \tag{10}$$

Proof. Let $u \in \mathcal{J}^1_q(\Omega)$. Then, for $p_0 \in C^\infty_0\big(\overline{\Omega}\big)$, by Gauss' theorem

$$\int_\Omega \nabla p_0 \cdot u \, dx \; = 0.$$

This applies to $p_0 \in \widehat{W}^1_q(\Omega)$ by density. Taking η_+ instead of p_0 yields

$$\int_\Omega \nabla \eta_+ \cdot u \, dx \; = \int_{\Omega_+ \cap B'} \nabla\big(\eta_+ - 1\big) \cdot u \, dx \; = -\int_M N \cdot u \, do \; = -\Phi(u).$$

Now the assertion follows, using the decomposition $\nabla p = \nabla p_0 + [p]\nabla \eta_+$, see (9). □

3 The Helmholtz Decomposition

After the introduction of suitable function spaces for the velocity u and the pressure p, the next step is to find Helmholtz decompositions appropriate for a given flux or a given pressure drop.

It will be shown, under certain restrictions on $1 < q < \infty$, that

$$L^q(\Omega)^n = J_q(\Omega) \oplus G_q(\Omega) = \mathcal{J}_q(\Omega) \oplus \mathcal{G}_q(\Omega),$$

where the underlying function spaces are defined as follows:

Definition 5 *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be an aperture domain with $\partial\Omega \in C^1$ and $1 < q < \infty$. Then*

$$\mathcal{J}_q(\Omega) = \overline{\mathcal{J}_q^1(\Omega)}^{\|\cdot\|_q}, \tag{11}$$

$$J_q(\Omega) = \overline{J_q^1(\Omega)}^{\|\cdot\|_q}, \tag{12}$$

$$\mathcal{G}_q(\Omega) = \left\{\nabla p : p \in \widehat{W}_q^1(\Omega)\right\}, \tag{13}$$

$$G_q(\Omega) = \left\{\nabla p : p \in \dot{W}_q^1(\Omega)\right\}. \tag{14}$$

The construction of both of the Helmholtz decompositions is based on the following weak Neumann problem for the Laplace equation:

Theorem 4 *Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$ be an aperture domain with $\partial\Omega \in C^1$ and $n' < q < n$.*

i) The operator $-\Delta_q : \widehat{W}_q^1(\Omega) \to \widehat{W}_q^{-1}(\Omega) = \left[\widehat{W}_{q'}^1(\Omega)\right]'$ defined by

$$-\Delta_q p = \langle \nabla p, \nabla \cdot \rangle$$

is an isomorphism, in particular, for $p \in \widehat{W}_q^1(\Omega)$

$$\|\nabla p\|_q \leq C \sup_{v \in \widehat{W}_{q'}^1(\Omega)} \frac{\langle \nabla p, \nabla v \rangle}{\|\nabla v\|_{q'}}. \tag{15}$$

ii) If $\nabla p \in \widehat{W}_q^1(\Omega)$ and

$$\sup_{v \in \widehat{W}_{r'}^1(\Omega)} \frac{\langle \nabla p, \nabla v \rangle}{\|\nabla v\|_{r'}} < \infty, \tag{16}$$

for some $n' < r < n$, then $p \in \widehat{W}_r^1(\Omega)$.

iii) If $n \geq 2$ and $1 < q$, $r < \infty$, then i), ii) apply to the corresponding $\dot{W}$-spaces as well.

Proof. If Ω is a bounded domain or a halfspace, a proof of *iii)* can be found in [12]. If Ω is an aperture domain, then the first part of *iii)* is proved in [5], whereas the second part is proved similarly to *ii)*.

To show *i)*, let $\pi_0 \in \dot{W}^1_q(\Omega)$ be the unique solution (independent of $n' < q < n$) of

$$\langle \nabla \pi_0, \nabla v \rangle = [v]$$

for $v \in \dot{W}^1_q(\Omega)$. Because of $0 < \langle \nabla \pi_0, \nabla \pi_0 \rangle = [\pi_0]$, one can normalize

$$\pi = \pi_0/[\pi_0]. \tag{17}$$

Let $v \in \dot{W}^1_{q'}(\Omega)$. Then $w = v - [v]\pi \in \widehat{W}^1_{q'}(\Omega)$ with $\|\nabla w\|_{q'} \leq C\|\nabla v\|_{q'}$. Moreover, for $p \in \widehat{W}^1_q(\Omega)$,

$$\langle \nabla p, \nabla v \rangle = \langle \nabla p, \nabla w \rangle.$$

Hence,

$$\|\nabla p\|_q \leq C \sup_{v \in \dot{W}^1_{q'}(\Omega)} \frac{\langle \nabla p, \nabla v \rangle}{\|\nabla v\|_{q'}} \leq C \sup_{w \in \widehat{W}^1_{q'}(\Omega)} \frac{\langle \nabla p, \nabla w \rangle}{\|\nabla w\|_{q'}}.$$

In particular, $-\Delta_q : \widehat{W}^1_q(\Omega) \to \widehat{W}^{-1}_q(\Omega)$ is injective and has a closed range. Obviously $(-\Delta_q)' = -\Delta_{q'}$, therefore by the closed range theorem $-\Delta_q$ is surjective.

To prove *ii)*, let $p \in \widehat{W}^1_q(\Omega)$ fulfil (16), i.e. $p \in \widehat{W}^{-1}_r(\Omega)$. Then by *i)* there exists a function $p_1 \in \widehat{W}^1_r(\Omega)$ such that

$$\int_\Omega \nabla p_1 \cdot \nabla v \, dx = \int_\Omega \nabla p \cdot \nabla v \, dx \, .$$

In the following, by interchanging p and p_1 if necessary, it can be assumed that $q > r$. Let $U = \Omega_0, \mathbb{R}^n_\pm$ and $\eta = \eta_0, \eta_\pm$ respectively. Because v and p are determined up to constants only, they can be chosen to have a vanishing mean value over Ω_0. Hence, by the Poincaré inequality $\eta p \in \widehat{W}^1_q(U)$. Furthermore,

$$\int_U \nabla(\eta p) \cdot \nabla v \, dx = \int_\Omega \Big(p \nabla \eta \cdot \nabla v - v \nabla \eta \cdot \nabla p + \nabla p_1 \cdot \nabla(\eta v) \Big) \, dx \, .$$

Since $\operatorname{supp} \eta \subset \overline{\Omega}_0$, the Poincaré and the Hölder inequality yield

$$\left| \int_U \nabla(\eta p) \cdot \nabla v \, dx \right| \leq C\Big(\|\nabla p\|_q \|\nabla \eta\|_s + \|\nabla p_1\|_r \Big) \|\nabla v\|_{L^{r'}(U)}$$

for $\frac{1}{s} = \frac{1}{r} - \frac{1}{q}$. Now $C_0^\infty(\mathbb{R}^n)$ is dense in $\widehat{W}_{r'}^1(U)$, hence $\eta p \in \widehat{W}_r^1(U)$. But $p = \eta_0 p + \eta_+ p + \eta_- p$ and therefore $p \in \widehat{W}_r^1(\Omega)$.

Since $\dot{W}_q^1(U) = \widehat{W}_q^1(U)$, $1 < q < \infty$ for $U = \Omega_0, \mathbb{R}^n_\pm$, the above proof applies also to the $\dot{W}$-spaces. □

Theorem 5 *Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$ be an aperture domain with $\partial\Omega \in C^1$ and $n' < q < n$.*

i) The following topological and algebraical decomposition holds true:

$$L^q(\Omega)^n = \mathcal{J}_q(\Omega) \oplus \mathcal{G}_q(\Omega). \tag{18}$$

ii) If $\mathcal{P}_q$ denotes the associated Helmholtz projection onto $\mathcal{J}_q(\Omega)$, then

$$(\mathcal{P}_q)' = \mathcal{P}_{q'}$$

with respect to the duality pairing $\langle\,\cdot\,,\,\cdot\,\rangle$. Moreover, the dual space $\mathcal{J}_q(\Omega)'$ can be identified with $\mathcal{J}_{q'}(\Omega)$ in this sense.

iii) The Helmholtz projection $\mathcal{P}_q$ is independent of q: If $u \in L^q(\Omega)^n \cap L^r(\Omega)^n$ for $n' < q,\ r < n$, then $\mathcal{P}_q u = \mathcal{P}_r u$.

iv) If $n \geq 2$ and $1 < q,\ r < \infty$, then i) – iii) apply to P_q, $J_q(\Omega)$, $G_q(\Omega)$ as well.

Proof. For $u \in L^q(\Omega)^n$ define $\mathcal{P}_q u = u - \nabla p$, where $p \in \widehat{W}_q^1(\Omega)$ is the unique solution of

$$\big\langle \nabla p, \nabla v \big\rangle = \big\langle u, \nabla v \big\rangle \tag{19}$$

for $v \in \widehat{W}_{q'}^1(\Omega)$, see Theorem 4. Then $\mathcal{P}_q$ is linear and continuous. Furthermore, $\mathcal{P}_q u \in \mathcal{G}_{q'}(\Omega)^\perp$ with respect to $\langle\,\cdot\,,\,\cdot\,\rangle$. It remains to show that $\mathcal{G}_{q'}(\Omega)^\perp = \mathcal{J}_q(\Omega)$. Because $\mathcal{J}_q(\Omega)$ is reflexive as a closed subspace of $L^q(\Omega)^n$, it is enough to prove $\mathcal{J}_q^1(\Omega)^\perp = \mathcal{G}_{q'}(\Omega)$ leading to

$$\mathcal{G}_{q'}(\Omega)^\perp = \mathcal{J}_q^1(\Omega)^{\perp\perp} = \overline{\mathcal{J}_q^1(\Omega)}^{\,\|\cdot\|_q} = \mathcal{J}_q(\Omega). \tag{20}$$

To show this, let $v \in \mathcal{J}_q^1(\Omega)^\perp \subset L^{q'}(\Omega)^n$, i.e. $\langle v, u\rangle = 0$ for $u \in \mathcal{J}_q^1(\Omega)$. Then $v = \nabla p \in G_{q'}(\Omega)$ by [8], Theorem III.1.1. Now Theorem 3 yields $[p] = 0$, hence $\nabla p \in \mathcal{G}_{q'}(\Omega)$.

If, on the other hand, $\nabla p \in \mathcal{G}_{q'}(\Omega)$, then $\langle \nabla p, u\rangle = 0$ for $u \in \mathcal{J}_q^1(\Omega)$ by the same Theorem, hence $\nabla p \in \mathcal{J}_q^1(\Omega)^\perp$. This proves *i)*.

To show *ii)*, let $u \in L^q(\Omega)^n$ and $v \in L^{q'}(\Omega)^n$. Then $\mathcal{P}_q u = u - \nabla p$ with $\nabla p \in \mathcal{G}_q(\Omega) = \mathcal{J}_{q'}(\Omega)^\perp$ and therefore

$$\big\langle u, \mathcal{P}_{q'} v \big\rangle = \big\langle \mathcal{P}_q u, \mathcal{P}_{q'} v \big\rangle.$$

Analogously $\langle \mathcal{P}_q u, v\rangle = \big\langle \mathcal{P}_q u, \mathcal{P}_{q'} v\big\rangle$. This yields $(\mathcal{P}_q)' = \mathcal{P}_{q'}$.

Define $\Upsilon : \mathcal{J}_{q'}(\Omega) \to \mathcal{J}_q(\Omega)'$ via the duality pairing by $\big(\Upsilon v\big)(u) = \langle v, u\rangle$ for $u \in \mathcal{J}_q(\Omega)$. If $\Upsilon v = 0$, then $\langle v, \mathcal{P}_q u\rangle = \langle v, u\rangle = 0$ for $u \in L^q(\Omega)^n$, showing that Υ is injective.

Let $F \in \mathcal{J}_q(\Omega)'$. Then by Hahn-Banach's theorem there exists $f \in \big(L^q(\Omega)^n\big)' = L^{q'}(\Omega)^n$ with $F(u) = \langle f, u\rangle = \big\langle \mathcal{P}_{q'} f, u\big\rangle$ for $u \in \mathcal{J}_q(\Omega)$. Now $\mathcal{P}_{q'} f \in \mathcal{J}_{q'}(\Omega)$, hence Υ is surjective. This proves $\mathcal{J}_q(\Omega)' = \mathcal{J}_{q'}(\Omega)$.

The property *iii)* follows from the definition of $\mathcal{P}_q$ and Theorem 4.

Let $n \geq 2$ and $1 < q < \infty$. For $u \in L^q(\Omega)^n$ define $P_q u = u - \nabla p$, where $\nabla p \in \dot{W}^1_q(\Omega)$ is the unique solution of (19) for $v \in \dot{W}^1_{q'}(\Omega)$. Then the proof of *iv)* is similar to the one above. □

Let $n' < q < n$ and $\nabla \pi \in G_q(\Omega)$ be the function defined by (17). Then obviously $\nabla \pi \in \mathcal{G}_{q'}(\Omega)^{\perp} = \mathcal{J}_q(\Omega)$. Hence,

$$\nabla \pi \in \mathcal{J}_q(\Omega) \cap G_q(\Omega).$$

Combining this with (18) yields the decomposition

$$L^q(\Omega)^n = \overbrace{J_q(\Omega) \oplus \underbrace{\operatorname{span}\{\nabla \pi\}}^{\mathcal{J}_q(\Omega)} \oplus \mathcal{G}_q(\Omega)}_{G_q(\Omega)}. \tag{21}$$

4 The Stokes Operator

The Helmholtz projections of the last section are used to define the Stokes operators for the aperture domain:

Definition 6 *Let $\Omega \subset \mathbb{R}^n$ be an aperture domain with $\partial\Omega \in C^{1,1}$.*

i) For $n \geq 3$ and $n' < q < n$ the Stokes operator associated with a prescribed pressure drop is defined by

$$\mathcal{A}_q = \mathcal{P}_q(-\Delta) : D(\mathcal{A}_q) \subset \mathcal{J}_q(\Omega) \to \mathcal{J}_q(\Omega),$$

where

$$D(\mathcal{A}_q) = \Big\{ u \in W^2_q(\Omega)^n : u|_{\partial\Omega} = 0,\ \operatorname{div} u = 0, \Big\}.$$

ii) For $n \geq 2$ and $1 < q < \infty$ the Stokes operator associated with a prescribed flux is defined by

$$A_q = P_q(-\Delta) : D(A_q) \subset J_q(\Omega) \to J_q(\Omega),$$

where

$$D(A_q) = \Big\{ u \in W^2_q(\Omega)^n : u|_{\partial\Omega} = 0,\ \operatorname{div} u = 0,\ \Phi(u) = 0 \Big\}.$$

In order to show that the Stokes operators generate bounded analytic semigroups, the following resolvent system has to be analysed:

$$\lambda u - \Delta u + \nabla p = f \quad \text{in } \Omega, \tag{22a}$$

$$\operatorname{div} u = 0 \quad \text{in } \Omega, \tag{22b}$$

$$u|_{\partial\Omega} = 0. \tag{22c}$$

This was done by Farwig and Sohr in [4], [5]. Their results are used to prove the next theorem.

Theorem 6 *Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$ be an aperture domain with $\partial\Omega \in C^{1,1}$, $0 < \varepsilon < \pi/2$ and $n' < q < n$.*

i) For every $f \in \mathcal{J}_q(\Omega)$ and

$$\lambda \in \sum\nolimits_{\varepsilon} = \Big\{ z \in \mathbb{C} : |\arg z| < \pi - \varepsilon \Big\}$$

there exists a unique solution $u \in D(\mathcal{A}_q)$ of the equation

$$\lambda u + \mathcal{A}_q u = f. \tag{23}$$

This solution satisfies the estimate

$$|\lambda| \|u\|_q + \|\mathcal{A}_q u\|_q \leq C_\varepsilon \|f\|_q. \tag{24}$$

Therefore, $\lambda + \mathcal{A}_q$ has a bounded inverse in $\mathcal{J}_q(\Omega)$ with

$$\|\lambda(\mathcal{A}_q + \lambda)^{-1}\| \leq C_\varepsilon. \tag{25}$$

ii) For $u \in D(\mathcal{A}_q)$

$$\|u\|_{W^2_q(\Omega)} \leq C \|(\mathcal{A}_q + I)u\|_q \tag{26}$$

and

$$\|\nabla^2 u\|_q \leq C \|\mathcal{A}_q u\|_q. \tag{27}$$

iii) $(\mathcal{A}_q)' = \mathcal{A}_{q'}$ with respect to the duality pairing $\langle \cdot , \cdot \rangle$.

iv) $\mathcal{A}_q$ is injective and $\mathcal{R}(\mathcal{A}_q)$ is dense in $\mathcal{J}_q(\Omega)$.

v) Let $n \geq 2$ and $1 < q < \infty$. Then i) – iv) applies to A_q, $J_q(\Omega)$ respectively, where (27) is valid only for $1 < q < n$.

Proof. By [4], Theorem 1.2 for every $\lambda \in \sum_\varepsilon$, $f \in \mathcal{J}_q(\Omega)$ there is a unique solution $(u, \nabla p) \in W^2_q(\Omega)^n \times L^q(\Omega)^n$ of the resolvent equations (22) with $[p] = 0$. This solution satisfies

$$|\lambda| \|u\|_q + \|\nabla^2 u\|_q \leq C_\varepsilon \|f\|_q.$$

Obviously $u \in D(\mathcal{A}_q)$ is the unique solution of (23) and (24), (25) follow from the preceding estimate.

Moreover, since $f = (\mathcal{A}_q + \lambda)u$, the estimate (26) follows by setting $\lambda = 1$. Letting $\lambda \to 0$ yields (27), since the estimate is uniform in $\lambda > 0$.

To show *iii)*, let $u \in D(\mathcal{A}_q)$ and $v \in D(\mathcal{A}'_q) \subset \mathcal{J}_{q'}(\Omega)$. Then by definition there exists $f \in \mathcal{J}_{q'}(\Omega)$ with $\langle(\mathcal{A}_q + I)u, v\rangle = \langle u, f\rangle$. Because $\mathcal{A}_{q'} + I$ is surjective, there is a $w \in D(\mathcal{A}_{q'})$ with $(\mathcal{A}_{q'} + I)w = f$. Hence,

$$\big\langle(\mathcal{A}_q + I)u, v\big\rangle = \big\langle u, (\mathcal{A}_{q'} + I)w\big\rangle.$$

By Theorem 5 there exist $\nabla p \in \mathcal{G}_q(\Omega)$, $\nabla q \in \mathcal{G}_{q'}(\Omega)$ such that $\mathcal{A}_q u = -\Delta u + \nabla p$ and $\mathcal{A}_{q'} w = -\Delta w + \nabla q$ respectively. Now (20) and integration by parts yield

$$\begin{aligned}\big\langle u, (\mathcal{A}_{q'} + I)w\big\rangle &= \big\langle u, w - \Delta w + \nabla q\big\rangle = \big\langle u - \Delta u + \nabla p, w\big\rangle \\ &= \big\langle(\mathcal{A}_q + I)u, w\big\rangle.\end{aligned}$$

Since $\mathcal{A}_q + I$ is surjective, $v = w \in D(\mathcal{A}_{q'})$ and

$$\big\langle\mathcal{A}_q u, v\big\rangle = \big\langle u, \mathcal{A}_{q'} v\big\rangle.$$

The injectivity of $\mathcal{A}_q$ follows from (27), whereas $\overline{\mathcal{R}(A_q)} = \mathcal{N}(A_{q'})^\perp = \mathcal{J}_q(\Omega)$ by *iv)* and Hahn-Banach's theorem.

The proof for A_q, $n \geq 2$, $1 < q < \infty$ is similar to the proof above. One considers the resolvent equations (22) with $\Phi(u) = 0$ instead of $[p] = 0$, for details see [5]. The only problem is the injectivity of A_q for $n \leq q < \infty$: Let $u \in D(A_q)$ with $A_q u = 0$. Then u is a weak solution of the homogeneous stationary Stokes equations with prescribed flux in the sense of [6]. Hence, $u = 0$ and A_q is injective. □

5 The Stokes Equations

Now the well-known theory of analytic semigroups is used to analyse the Stokes equations

$$\begin{aligned} u_t - \Delta u + \nabla p &= f && \text{in } \Omega \times (0,T), && \text{(28a)} \\ \operatorname{div} u &= 0 && \text{in } \Omega \times (0,T), && \text{(28b)} \\ u|_{\partial\Omega} &= 0 && \text{on } \partial\Omega \times (0,T), && \text{(28c)} \\ u(0) &= u_0 && \text{in } t = 0. && \text{(28d)} \end{aligned}$$

Therefore it is assumed that the right-hand side f is locally Hölder continuous in time with exponent $\kappa \in (0,1)$, i.e. $f \in C^{\kappa}\big(0,T,L^q(\Omega)^n\big)$.

Theorem 7 *Let $\Omega \subset \mathbb{R}^n$ be an aperture domain with $\partial\Omega \in C^{1,1}$ and $0 < \kappa < 1$. Moreover, let $u_0 \in \mathcal{J}_q(\Omega)$ and $f \in C^{\kappa}\big(0,T;L^q(\Omega)^n\big) \cap L^1\big(0,T;L^q(\Omega)^n\big)$ be given.*

i) For $n \geq 2$ and $1 < q \leq n'$ there is a unique solution

$$\begin{aligned}\big(u,\nabla p\big) \in \Big(C^{1,\kappa}\big(0,T;L^q(\Omega)^n\big) \cap C^{\kappa}\big(0,T;W^2_q(\Omega)^n\big)\Big)&\\ \times C^{\kappa}\big(0,T;L^q(\Omega)^n\big)&\end{aligned} \tag{29}$$

of the Stokes equations (28).

ii) For $n \geq 2$, $n' < q < \infty$ and $\alpha \in C^{1,\kappa}(0,T) \cap W^1_1(0,T)$ such that $\Phi(u_0) = \alpha(0)$, there is a unique solution of the Stokes equations (28) *with*

$$\Phi(u) = \alpha.$$

iii) For $n \geq 3$, $n' < q < n$ and $\beta \in C^{\kappa}(0,T) \cap L^1(0,T)$ there is a unique solution of the Stokes equations (28) *with*

$$[p] = \beta.$$

Proof. Let $(u,\nabla p)$ be a solution of the Stokes equations with $\Phi(u) = \alpha$. Applying P_q to (28a) shows that $v = u - \alpha\chi \in C^{1,\kappa}\big(0,T;J_q(\Omega)\big) \cap C^{\kappa}\big(0,T;D(A_q)\big)$ is a solution of

$$v_t + A_q v = P_q\Big(f - \alpha_t\chi + \alpha\Delta\chi\Big), \tag{30a}$$

$$v(0) = u_0 - \alpha(0)\chi. \tag{30b}$$

It is well known that this initial value problem has a unique solution $v \in C^{1,\kappa}\big(0,T;J_q(\Omega)\big) \cap C^{\kappa}\big(0,T;D(A_q)\big)$. Setting $u = v + \alpha\chi$ yields

$$u_t - \Delta u - f = -\nabla p \in G_q(\Omega),$$

by the properties of the Helmholtz decomposition. Hence, $(u,\nabla p)$ is the solution of the Stokes equations with $\Phi(u) = \alpha$. This proves *ii)*. Prescribing $\alpha = 0$ the proof of *i)* is similar.

To prove *iii)*, let $(u,\nabla p)$ be a solution of the Stokes equations with $[p] = \beta$. Applying $\mathcal{P}_q$ to (28a) shows that $u \in C^{1,\kappa}\big(0,T;\mathcal{J}_q(\Omega)\big) \cap C^{\kappa}\big(0,T;D(\mathcal{A}_q)\big)$ is a solution of

$$u_t + \mathcal{A}_q u = \mathcal{P}_q f - [p]\nabla\pi, \tag{31a}$$

$$u(0) = u_0. \tag{31b}$$

On the other hand, for the solution u of this initial value problem,

$$u_t - \varDelta u + [p]\nabla\pi - f = -\nabla p_0 \in \mathcal{G}_q(\varOmega),$$

by the properties of the Helmholtz decomposition.

Hence, $(u, \nabla p)$ with $\nabla p = [p]\nabla\pi + \nabla p_0$ is the unique solution of (28) with $[p] = \beta$. □

Following [2], it is possible to show estimates of maximal Hölder regularity as well. For maximal L^2 -regularity see [7].

6 The Navier-Stokes Equations

The construction of solutions of the Navier-Stokes equations (1) is based on their abstract integral formulation

$$v(t) = e^{-tA}v_0 + \int_0^t e^{-(t-s)A}P\Big(g - \big(u\cdot\nabla u\big)\Big)(s)\,ds\,. \tag{32}$$

Looking at the Stokes equations, it is clear how to deal with the additional boundary conditions: If a pressure drop $[p] = \beta$ is prescribed, then $A = \mathcal{A}_q$, $P = \mathcal{P}_q$, $g = f - [p]\nabla\pi$ and $u = v$. On the other hand, if a flux $\varPhi(u) = \alpha$ is given, then $A = A_q$, $P = P_q$, $g = f - \alpha_t\chi + \alpha\varDelta\chi$ and $u = v + \alpha\chi$.

By means of this identification the two different boundary conditions can be dealt with simultaneously, where in the latter case $n' < q < n$ is assumed.

Solutions of the integral equation (32) are constructed via a fixed point iteration, see [9]. In order to estimate the nonlinear term $P(u\cdot\nabla u)$, the boundedness of the imaginary powers of the Stokes operator is assumed in [9]. Up to now this has not been proved for aperture domains. Nevertheless, the nonlinear term can still be estimated, yielding a slightly weaker result.

For these reasons, fractional powers of positive operators and complex interpolation theory are needed. In order to fix notations a short summary is given, cf. [13]:

Interpolation theory: Let X_0, X_1 be complex Banach spaces, both continuously embedded in a complex linear Hausdorff space $\mathcal{X}$; then $\{X_0, X_1\}$ is said to be an interpolation couple. For such an interpolation couple the spaces $X_0 \cap X_1$ and $X_0 + X_1$, equipped with their natural norms, are also Banach spaces.

Let $S = \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$. Then $\mathcal{F}(X_0, X_1)$ denotes the space of analytic functions $f : S \to X_0 + X_1$ such that $f(j + it) : \mathbb{R} \to X_j$, $j = 0, 1$ is continuous and bounded. Provided with the norm

$$\|f\|_{\mathcal{F}(X_0,X_1)} = \max_{j=0,1}\left\{\sup_{t\in\mathbb{R}}\big\|f(j+it)\big\|_{X_j}\right\}$$

$\mathcal{F}(X_0, X_1)$ is a Banach space.

For $0 < \theta < 1$ the interpolation space $X_\theta = \big[X_0, X_1\big]_\theta$ is defined by

$$X_\theta = \Big\{x = f(\theta) : f \in \mathcal{F}(X_0, X_1)\Big\}.$$

Endowed with the norm

$$\|x\|_\theta = \inf\Big\{\|f\|_{\mathcal{F}(X_0,X_1)} : f(\theta) = x\Big\}$$

it becomes a Banach space.

Let $\{Y_0, Y_1\}$ be another interpolation couple and let $T : X_0 + X_1 \to Y_0 + Y_1$ be a linear operator such that $T : X_j \to Y_j$, $j = 0, 1$ is bounded. Then X_θ, Y_θ have the interpolation property, i.e. $T : X_\theta \to Y_\theta$ is bounded.

It is well known that the Bessel potential spaces $H^s_q(\mathbb{R}^n)$ for $s > 0$, $1 < q < \infty$ are interpolation spaces with

$$\big[H^{s_0}_{q_0}(\mathbb{R}^n), H^{s_1}_{q_1}(\mathbb{R}^n)\big]_\theta = H^s_q(\mathbb{R}^n),$$

where

$$s = (1-\theta)s_0 + \theta s_1, \quad \frac{1}{q} = (1-\theta)\frac{1}{q_0} + \theta\frac{1}{q_1}.$$

This applies to $H^s_q(\Omega)$ whenever $\Omega \subset \mathbb{R}^n$ has the extension property, i.e. whenever there is a linear bounded operator $\mathcal{E} : H^s_q(\Omega) \to H^s_q(\mathbb{R}^n)$, $s > 0$, $1 < q < \infty$ such that $\mathcal{E}u\big|_\Omega = u$. If Ω is an aperture domain with $d > 0$, the extension property holds, see [1].

Fractional powers: Let X be a Banach space and $A : D(A) \subset X \to X$ a densely defined positive operator, i.e.

$$\big\|(A+\lambda)^{-1}\big\| \le \frac{K}{1+\lambda}, \qquad \lambda \ge 0.$$

Then for $z \in \mathbb{C}$, $0 < \operatorname{Re} z < 1$ the fractional power A^{-z} is defined by

$$A^{-z} = \frac{\sin \pi z}{\pi} \int_0^\infty \lambda^{-z}(A+\lambda)^{-1}\, d\lambda\,.$$

This operator is continuous and its norm can be estimated by

$$\big\|A^{-z}\big\| \le \left|\frac{\sin \pi z}{\sin \pi \operatorname{Re} z}\right|. \tag{33}$$

Furthermore, it can be shown that A^{-z} extends to an analytic semigroup for $\operatorname{Re} z > 0$.

Since A^{-z} is injective, one can define the closed operators

$$A^z = (A^{-z})^{-1} : D(A^z) \subset X \to X,$$

with dense domain $D(A^z) = \{x = A^{-z}y : y \in X\}$.

Suppose A allows the definition of bounded imaginary powers A^{it}, $t \in \mathbb{R}$, where $\|A^{it}\| \leq Ce^{\gamma|t|}$ for some constants C, $\gamma > 0$. Then for $m \in \mathbb{N}$ and $0 < \theta < 1$

$$\big[X, D(A^m)\big]_\theta = D(A^{\theta m})$$

with respect to the graph norm. Without the boundedness of the imaginary powers, only the following can be shown:

Theorem 8 *Let A be a densely defined postive Operator on X, $m \in \mathbb{N}$ and $0 < \theta < 1$. Then for $\alpha_- < \theta m < \alpha_+$*

$$D(A^{\alpha_+}) \hookrightarrow \big[X, D(A^m)\big]_\theta \hookrightarrow D(A^{\alpha_-}). \tag{34}$$

Proof. For $x \in D(A^m)$ let

$$f(z) = e^{(z-\theta)^2} A^{-(z-\theta)m} x.$$

By (33) and the boundedness of A^{-1}

$$\|A^{-z}\| \leq C_\varepsilon e^{\pi|\operatorname{Im} z|},$$

for $0 < \varepsilon < \operatorname{Re} z < m$. Consequently,

$$\begin{aligned}
\big\|f(it)\big\|_X &= \big\|e^{(it-\theta)^2} A^{-(\varepsilon+itm)} A^{\theta m+\varepsilon} x\big\|_X \\
&\leq C_\varepsilon \big\|A^{\theta m+\varepsilon} x\big\|_X, \\
\big\|f(1+it)\big\|_{D(A^m)} &= \big\|e^{(1+it-\theta)^2} A^{-m} A^{-(\varepsilon+itm)} A^{\theta m+\varepsilon} x\big\|_{D(A^m)} \\
&\leq C_\varepsilon \big\|A^{\theta m+\varepsilon} x\big\|_X.
\end{aligned}$$

Choosing ε small enough yields $f \in F\big(X, D(A^m)\big)$. Because $f(\theta) = x$

$$\big\|x\big\|_{[X,D(A^m)]_\theta} \leq \big\|f\big\|_{F(X,D(A))} \leq C_\varepsilon \big\|A^{\theta m+\varepsilon} x\big\|_X.$$

This applies to $x \in D(A^{\theta m+\varepsilon})$ by density. Since ε can be chosen arbitrarily small,

$$D(A^{\alpha_+}) \hookrightarrow \big[X, D(A^m)\big]_\theta.$$

To show the reverse embedding let $x \in D(A^m)$ and $f \in F\big(X, D(A)\big)$ with $f(\theta) = x$. Moreover, let $J_k = k(A+k)^{-1}$, $k \in \mathbb{N}$ be the Yosida approximation and $\varepsilon > 0$. Then

$$g(z) = e^{(z-\theta)^2} A^{zm-\varepsilon} J_k^m f(z)$$

is analytic on S with $g(\theta) = A^{\theta m-\varepsilon} J_k^m x$. Hence, by the theorem of the three lines,

$$\big\| A^{\theta m-\varepsilon} J_k^m x \big\| \leq C_\varepsilon \big\| J_k^m f \big\|_{F(X,D(A^m))}.$$

Letting $k \to \infty$ and taking the infimum over all f with $f(\theta) = x$ yields

$$\big\| A^{\theta m-\varepsilon} x \big\| \leq C_\varepsilon \big\| f \big\|_{[X,D(A^m)]_\theta},$$

because $J_k^m x \to x$. This estimate applies to $x \in \big[X, D(A^m)\big]_\theta$ by density. As ε can be chosen arbitrarily small,

$$\big[X, D(A^m)\big]_\theta \hookrightarrow D(A^{\alpha-}).$$

□

The above results are used to prove the following Sobolev type embeddings: Since $(\mathcal{A}_q + I)$, $n' < q < n$ is positive, the fractional powers of the Stokes operator are well defined. By the previous theorem, (26) and complex interpolation

$$\|u\|_{H_q^{2s}(\Omega)} \leq C \|(\mathcal{A}_q + I)^\alpha u\|_q, \tag{35}$$

where $u \in D(\mathcal{A}_q^\alpha)$ and $0 < s < \alpha \leq 1$. Applying the Sobolev embedding theorem yields

$$\|u\|_r \leq C \|(\mathcal{A}_q + I)^\alpha u\|_q, \qquad \frac{1}{q} \geq \frac{1}{r} > \frac{1}{q} - \frac{2\alpha}{n}. \tag{36}$$

By iteration this estimate applies to all $\alpha > 0$. Similary follows

$$\|\nabla u\|_r \leq C \|(\mathcal{A}_q + I)^{\alpha+1/2} u\|_q, \qquad \frac{1}{q} \geq \frac{1}{r} > \frac{1}{q} - \frac{2\alpha}{n}, \tag{37}$$

for all $\alpha > 0$. The corresponding estimates for A_q, $1 < q < \infty$ are valid, too.

For $n' < q < n$ and $\alpha > 0$ one defines

$$D\big(\mathcal{A}_q^{-\alpha}\big) = D\big(\mathcal{A}_{q'}^\alpha\big)'.$$

Since $\big(\mathcal{A}_q^\alpha\big)' = \mathcal{A}_{q'}^\alpha$ it follows that $D(\mathcal{A}_q^{-\alpha})$ is the completion of $\mathcal{J}_q(\Omega)$ with respect to the norm $\big\| \big(\mathcal{A}_q + I\big)^{-\alpha} \cdot \big\|_q$. Hence, the operators $\mathcal{A}_q$ and $e^{-t\mathcal{A}_q}$ can be uniquely extended to $D\big(\mathcal{A}_q^{-\alpha}\big)$. For $1 < q < \infty$ the spaces $D\big(A_q^{-\alpha}\big)$ enjoy the same properties.

After these preparations the following estimate of the nonlinear term can be shown:

Theorem 9 *Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$ be an aperture domain with $\partial\Omega \in C^{1,1}$ and $d > 0$. Furthermore, let $n' < q < n$ and $0 \leq \delta < \frac{1}{2} + \frac{n}{2}\big(1 - \frac{1}{q}\big)$. Then*

$$\big\|(\mathcal{A}_q + I)^{-\delta}\mathcal{P}_q(u \cdot \nabla v)\big\|_q \leq C\big\|(\mathcal{A}_q + I)^{\theta}u\big\|_q\big\|(\mathcal{A}_q + I)^{\rho}\big\|_q \qquad (38)$$

with a constant $C = C(\delta, \theta, \rho, q)$, provided that ρ, $\theta \geq 0$, $\delta + \rho > \frac{1}{2}$ and $\delta + \theta + \rho > \frac{n}{2q} + \frac{1}{2}$. If $n \geq 2$ and $1 < q < \infty$, then $\mathcal{A}_q$ can be substituted by A_q.

Proof. For $\delta = 0$ the Hölder inequality yields

$$\big\|\mathcal{P}_q(u \cdot \nabla v)\big\|_q \leq C\big\|u\big\|_r\big\|\nabla v\big\|_s, \qquad \frac{1}{q} = \frac{1}{r} + \frac{1}{s},$$

where by assumption r, s can be chosen to satisfy

$$\frac{1}{r} > \frac{1}{q} - \frac{2\theta}{n}, \qquad \frac{1}{s} > \frac{1}{q} - \frac{2\rho - 1}{n}.$$

Hence, (36) and (37) show that

$$\big\|\mathcal{P}_q(u \cdot \nabla v)\big\|_q \leq C\big\|(\mathcal{A}_q + I)^{\theta}u\big\|_q\big\|(\mathcal{A}_q + I)^{\rho}v\big\|_q.$$

Now let $\frac{1}{2} < \delta < \frac{1}{2} + \frac{n}{2}\big(1 - \frac{1}{q}\big)$. Choose $m \in \mathbb{N}$ large enough such that $\nabla(vw) \in \mathcal{G}_{q'}(\Omega)$ for $v \in D(\mathcal{A}_q^m)$ and $w \in D(\mathcal{A}_{q'}^m)$. Hence, for $u \in D(\mathcal{A}_q^{\theta}) \cap \mathcal{J}_q^1(\Omega)$ Theorem 3 implies

$$0 = \int_{\Omega} u \cdot \nabla\big(vw\big)\,dx\; = \int_{\Omega}\big(u \cdot \nabla v\big)w\,dx\; + \int_{\Omega}\big(u \cdot \nabla w\big)v\,dx\,.$$

The Hölder inequality yields

$$\big|\big\langle\mathcal{P}_q(u \cdot \nabla v), w\big\rangle\big| \leq C\big\|u\big\|_r\big\|v\big\|_s\big\|\nabla w\big\|_{\sigma}, \qquad \frac{1}{r} + \frac{1}{s} + \frac{1}{\sigma} = 1,$$

where by assumption r, s, σ can be chosen to satisfy

$$\frac{1}{r} > \frac{1}{q} - \frac{2\theta}{n}, \qquad \frac{1}{s} > \frac{1}{q} - \frac{2\rho}{n}, \qquad \frac{1}{\sigma} > \frac{1}{q'} - \frac{2\delta - 1}{n} > 0.$$

By (36) and (37) follows that

$$\big|\big\langle\mathcal{P}_q(u \cdot \nabla v), w\big\rangle\big| \leq C\big\|(\mathcal{A}_q + I)^{\theta}u\big\|_q\big\|(\mathcal{A}_q + I)^{\rho}v\big\|_q\big\|(\mathcal{A}_{q'} + I)^{\delta}w\big\|_{q'}.$$

This estimate applies to $u \in D(\mathcal{A}_q^{\theta})$, $v \in D(\mathcal{A}_q^{\rho})$ and $w \in D(\mathcal{A}_{q'}^{\delta})$ by density, proving (38) for $\delta > \frac{1}{2}$.

For $0 < \delta \leq \frac{1}{2}$ the theorem follows by interpolation: Choose $0 < \tilde{\delta} < \delta$ and $0 < \tilde{\rho} < \rho$ such that $\tilde{\delta}$, $\tilde{\rho}$, θ fulfil the above assumptions. Then choose $\rho_0 = \tilde{\rho} + \tilde{\delta}$, $\delta_0 = 0$ and $\frac{1}{2} < \delta_1 < \frac{1}{2} + \frac{n}{2}\big(1 - \frac{1}{q}\big)$, $\rho_1 \geq 0$ with $\rho_1 + \delta_1 = \tilde{\rho} + \tilde{\delta}$.

Due to the previous results, $v \mapsto \mathcal{P}_q(u \cdot \nabla v)$ can be regarded as a linear bounded operator $T : D(\mathcal{A}_q^{\rho_0}) \to \mathcal{J}_q(\Omega)$ and $T : D(\mathcal{A}_q^{\rho_1}) \to D(\mathcal{A}_q^{-\delta_1})$. By complex interpolation

$$T : \big[D(\mathcal{A}_q^{\rho_0}), D(\mathcal{A}_q^{\rho_1})\big]_\mu \longrightarrow \big[J_q(\Omega), D(\mathcal{A}_q^{-\delta_1})\big]_\mu$$

is bounded.

Choose $\mu = \tilde{\delta}/\delta_1$. Since $\delta > \tilde{\delta} = \mu\delta_1$ and $\rho > \tilde{\rho} = (1-\mu)\rho_0 + \mu\rho_1$, Theorem 8 shows that

$$D(A_q^\rho) \hookrightarrow \big[D(A_q^{\rho_0}), D(A_q^{\rho_1})\big]_\mu, \qquad \big[J_q(\Omega), D(A_q^{-\delta_1})\big]_\mu \hookrightarrow D(A_q^{-\delta}).$$

Therefore

$$T : D(A_q^\rho) \longrightarrow D(A_q^{-\delta})$$

is bounded, proving the assertion for $0 < \delta \leq \frac{1}{2}$. The proof for A_q, where $n \geq 2$, $1 < q < \infty$ is analogous. □

Now similar to [9] the following existence and uniqueness theorem can be proven:

Theorem 10 *Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$ be an aperture domain with $\partial\Omega \in C^{1,1}$ and $d > 0$.*

i) (Existence) Fix γ and choose $n' < q < n$, $\delta \geq 0$ such that

$$\frac{n}{2q} - \frac{1}{2} < \gamma < 1, \qquad -\gamma < \delta < 1 - |\gamma|.$$

Assume that $u_0 \in D(\mathcal{A}_q^\gamma)$ and that $\|(\mathcal{A}+I)^{-\delta}\mathcal{P}_q g\|_q$ is continuous on $(0,T)$ and satisfies

$$\|(\mathcal{A}_q + I)^{-\delta}\mathcal{P}_q g(t)\|_q = o(t^{\gamma+\delta-1}) \text{ as } t \to 0.$$

Then there is a local solution $u(t)$ of (32) such that

(a) $u \in C\big([0,T_*]; D(\mathcal{A}_q^\gamma)\big)$, $u(0) = u_0$,

(b) $u \in C\big((0,T_*]; D(\mathcal{A}_q^\alpha)\big)$ for some $T_* > 0$,

(c) $\|\mathcal{A}^\alpha u(t)\|_q = o(t^{\gamma-\alpha})$ as $t \to 0$ for all $\gamma < \alpha < 1-\delta$,

ii) (Uniqueness) Any solution of (32) satisfying (a) and

(b') $u \in C\big((0,T_*]; D(\mathcal{A}_q^\beta)\big)$,

(c') $\|\mathcal{A}^\alpha u(t)\|_q = o(t^{\gamma-\beta})$ for some $|\gamma| < \beta$

is unique.

iii) If $n \geq 2$ and $1 < q < \infty$, then the Theorem applies to A_q as well.

If the given data are sufficiently smooth, the solutions of the preceding theorem are solutions of the Navier-Stokes equations:

Theorem 11 *Let $\Omega \subset \mathbb{R}^n$ be an aperture domain with $\partial\Omega \in C^{1,1}$ and $d > 0$. Moreover, let $u_0 \in \mathcal{J}_q(\Omega)$ and $f \in L^1\big(0,T;L^q(\Omega)^n\big)$ be locally Hölder continuous.*

i) For $n \geq 2$ and $1 < q \leq n'$ there is a solution of the Navier Stokes equations (1) *on* $(0,T^*)$.

ii) Let $n \geq 2$ and $n' < q < \infty$. Then for every $\alpha \in W^1_1(0,T)$ such that α_t is locally Hölder continuous and $\Phi(u_0) = \alpha$, there is a solution of the Navier Stokes equations (1) *on* $(0,T^*)$ *with* $\phi(u) = \alpha$.

iii) Let $n \geq 3$ and $n' < q < n$. Then for every $\beta \in L^1(0,T)$ being locally Hölder continuous, there is a solution of the Navier-Stokes equations (1) *on* $(0,T^*)$ *with* $[p] = \beta$.

Proof. To prove *i)* let u be the solution of the Navier-Stokes equations constructed in Theorem 10.*i)*. Then $u \in C^\kappa\big(0,T^*,D(A_q^\alpha)\big)$ for $\alpha + \kappa < 1$ by [9], Proposition 2.4. Hence, $P_q(u\cdot\nabla u)$ is locally Hölder continuous and by the well-known theory of analytic semigroups, u is a solution of (1).

The proof of *ii)* and *iii)* is analogous.

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