Existence of solutions to a class of quasi-static problems in viscoplasticity theory

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Abstract

We prove existence of solutions for quasi-static initial-boundary value problems to a class of constitutive equations with internal variables. This class consists of constitutive equations of monotone type with positive definite free energy. They model the deformation behavior of metallic bodies. The existence theorem is proved by reduction of the initial-boundary value problem to an abstract evolution equation with a time dependent maximal monotone evolution operator, and by application of known existence results for such evolution equations. The proofs are sketched. At the end an example is given for a constitutive equation satisfying the hypotheses of the existence theorem.

1 Introduction

Let $\Omega \subseteq \mathbb{R}^3$ be the set of material points of a solid body. History dependent deformation behavior of this body at small strains can be modeled by the equations

$$-\operatorname{div}_x T(x,t) = b(x,t), \tag{1.1}$$

$$T(x,t) = D(\varepsilon(\nabla_x u(x,t)) - Bz(x,t)), \tag{1.2}$$

$$\frac{\partial}{\partial t} z(x,t) \in f(\varepsilon(\nabla_x u(x,t)), z(x,t)), \tag{1.3}$$

which must hold for $(x,t) \in \Omega \times [0,\infty)$. The solution must satisfy the initial condition

$$z(x,0) = z^{(0)}(x), \quad x \in \Omega, \tag{1.4}$$

and either the Dirichlet boundary condition

$$u(x,t) = \gamma_D(x,t), \quad (x,t) \in \partial\Omega \times (0,\infty), \tag{1.5}$$

or the Neumann boundary condition

$$T(x,t)n(x) = \gamma_N(x,t), \quad (x,t) \in \partial\Omega \times (0,\infty). \tag{1.6}$$

Here $u(x,t) : \Omega \times [0,\infty) \to \mathbb{R}^3$ denotes the displacement of the material point labeled x at time t. With the 3×3–matrix $\nabla_x u(x,t)$ of first order derivatives of u with respect to the components x_1, x_2, x_3 of x and with the transposed matrix $(\nabla_x u(x,t))^T$ the strain tensor is defined by

$$\varepsilon(\nabla_x u(x,t)) = \frac{1}{2}\Big(\nabla_x u(x,t) + (\nabla_x u(x,t))^T\Big).$$

It belongs to $\mathcal{S}^3$, the set of symmetric 3×3–matrices. $T : \Omega \times [0,\infty) \to \mathcal{S}^3$ is the Cauchy stress tensor, and $z \in \Omega \times [0,\infty) \to \mathbb{R}^N$ is the vector of internal variables. With $z \mapsto Bz = \varepsilon_p : \mathbb{R}^N \to \mathcal{S}^3$ we denote a linear mapping, which yields the plastic strain tensor $\varepsilon_p(x,t) \in \mathcal{S}^3$ as a function of the vector $z(x,t)$. If we identify $\mathcal{S}^3$ with $\mathbb{R}^6$, we can consider the six components of ε_p to be internal variables. Then ε_p is a part of z, and B is the projection to those components of z which form ε_p.

$D : \mathcal{S}^3 \to \mathcal{S}^3$ is a linear, symmetric, positive definite mapping, the elasticity tensor, $b : \Omega \times [0,\infty) \to \mathbb{R}^3$ is a given volume force, $\gamma_D : \partial\Omega \times [0,\infty) \to \mathbb{R}^3$ is a given boundary displacement, $\gamma_N : \partial\Omega \times [0,\infty) \to \mathbb{R}^3$ is a given traction at the boundary, and $f : \mathcal{S}^3 \times \mathbb{R}^N \to 2^{\mathbb{R}^N}$ is a given nonlinear function. Finally, $n(x)$ in the Neumann boundary condition denotes the exterior unit normal to $\partial\Omega$ at x.

The inelastic behavior modeled by the constitutive equations (1.2) and (1.3) is determined by the function f. These constitutive equations must be taken from a class, which is restrictive enough to preserve all the characteristic properties of the inelastic behavior of metalls, but is large enough to include the variants of this behavior shown by different metals and alloys. A class with interesting mathematical and thermodynamical properties is formed by the constitutive equations of generalized standard materials defined by B. Halphen and Nguyen Quoc Son in [6]. This class includes important constitutive equations like the Prandtl-Reuss law, but it is too small to allow the modelling of the inelastic behavior of most metals. This is shown in [1] by studying a number of constitutive equations used in engineering. Therefore in [1] the larger class of constitutive equations of monotone type is introduced. From it, a still larger class is constructed using the method of transformation of interior variables.

The constitutive equation (1.2) and (1.3) are of monotone type if (1.3) is of the form

$$z_t(x,t) \in g\Big(-\rho\nabla_z\, \psi(\varepsilon(\nabla_x u(x,t)), z(x,t))\Big) \tag{1.7}$$

with a monotone vector field $g : \mathbb{R}^N \to 2^{\mathbb{R}^N}$ satisfying $0 \in g(0)$, with the constant mass density $\rho > 0$, and with the free energy ψ being a positive definite or positive semi-definite quadratic form

$$\rho\psi(\varepsilon, z) = \frac{1}{2}[D(\varepsilon - Bz)] \cdot (\varepsilon - Bz) + \frac{1}{2}(Lz) \cdot z\,. \tag{1.8}$$

L is a symmetric, positive definite or positive semi-definite $N \times N$–matrix. We remark that for the majority of constitutive equations developed in engineering, including the Prandtl-Reuss law, ψ is only positive semi-definite, cf. [1].

For the existence theory of initial-boundary value problems to constitutive equations of monotone type it is a fundamental difference, whether ψ is positive definite or only positive semi-definite. In [1] it is proved that if ψ is positive definite and the vector field g in (1.7) is maximal monotone, then the dynamic problem, where (1.1) is replaced by

$$\rho u_{tt}(x,t) - \operatorname{div}_x T(x,t) = b(x,t),$$

has to every $b \in \bigcap_{T_e>0} L^2(\Omega \times (0,T_e))$ a unique solution existing for all time. To prove this, it is shown that the initial-boundary value problem can be written as an evolution equation $w_t + Aw = F$ to a maximal monotone operator A in the Hilbert space $L^2(\Omega)$. The existence result is immediately obtained from the classical theory of these evolution equations.

For the dynamic initial-boundary value problem to positive semi-definite ψ such a complete theory does not exist. The solution must be sought in other Banach spaces and has less regularity. For recent results we refer to [3]. Since the Prandtl-Reuss model and also the Norton-Hoff model are constitutive models with positive semi-definite free energy, the results proved in [2, 8] yield examples of existence theorems for quasi-static initial-boundary value problems to positive semi-definite ψ.

The aim of this article is to prove existence of solutions for the quasi-static problem in the case of positive definite free energy ψ. To this end it is shown that the initial-boundary value problem can be written as an evolution equation

$$z_t + A(t)z = 0, \quad z(0) = z^{(0)} \tag{1.9}$$

in the Hilbert space $L^2(\Omega, \mathbb{R}^N)$. Different from the dynamic problem, the evolution equation is homogeneous even if the right hand side b in (1.1) differs from zero, but this right hand side introduces a time dependence of the evolution operator $A(t)$. The theory for such non-autonomous evolution equations developed in [7, 4, 5], for example, yields existence of a solution if $A(t)$ is maximal monotone for every t and if the time dependence of $A(t)$ is restricted by a certain condition. Several such conditions have been found. One of these is the condition C stated below, which is well adapted to our situation and can easily be verified. In [5] it is proved that if the operators $A(t)$ satisfy this condition, then there is a solution of the evolution equation (1.9) on a time interval $[0, T_e)$. It turns out that for this condition to be satisfied restrictions must be imposed on the right hand side b of (1.1), on the boundary data γ_D or γ_N (save load conditions), and on the monotone vector field g:

Condition C: Let $T_e > 0$, let X be a real Banach space with norm $\|\cdot\|$, and let $A(t) : X \to X$ be an m-accretive operator with domain $\Delta(A(t)) \equiv \Delta$, independently of t. For $\lambda > 0$ let

$$J_\lambda(t) = (I + \lambda A(t))^{-1},$$

and assume that there are $y_0 \in X$ and $\lambda_0 > 0$ such that

$$\sup_{\substack{0<\lambda<\lambda_0 \\ 0\le t<T_e}} \|J_\lambda(t)y_0\| < \infty. \tag{1.10}$$

Moreover, assume that there exist a measurable function $h : [0, T_e) \to X$, of bounded variation, and a nondecreasing continuous function $\Theta : [0, \infty) \to [0, \infty)$ such that

$$\|A(t)y - A(s)y\| \le \|h(t) - h(s)\|\, \Theta(\|y\|)\, (1 + \|A(t)y\|) \tag{1.11}$$

for all $y \in \Delta$ and all $0 \le s, t < T_e$.

Statement of the main result. To state the existence theorem obtained in this way we need two definitions, which we give first. We assume in the following that $\Omega \subseteq \mathbb{R}^3$ is a bounded open set with Lipschitz boundary. T_e is a positive constant, the time of existence. $H_1(\Omega, \mathbb{R}^3)$ denotes the Hilbert space of functions in $L^2(\Omega, \mathbb{R}^3)$ with quadratically integrable first derivatives.

Definition 1.1 *Let $\mathcal{C}_D$ or $\mathcal{C}_N$, respectively, be the class of all functions $(b, \gamma) : [0, T_e) \to L^2(\Omega, \mathbb{R}^3) \times L^2(\partial\Omega, \mathbb{R}^3)$ such that for all $t \in [0, T_e)$ there is a weak solution $(\hat{u}(t), \hat{T}(t)) \in H_1(\Omega, \mathbb{R}^3) \times L^2(\Omega, \mathcal{S}^3)$ of the boundary value problem*

$$-\operatorname{div}_x \hat{T}(x,t) = b(x,t), \qquad x \in \Omega, \tag{1.12}$$

$$\hat{T}(x,t) = D\,\varepsilon(\nabla_x \hat{u}(x,t)), \quad x \in \Omega, \tag{1.13}$$

with the Dirichlet or Neumann boundary condition, respectively,

$$\hat{u}(x,t) = \gamma(x,t), \quad \textit{or} \quad \hat{T}(x,t)n(x) = \gamma(x,t), \quad x \in \partial\Omega, \tag{1.14}$$

for which $\varepsilon(\nabla_x \hat{u}(t))$ *belongs to* $L^\infty(\Omega, \mathcal{S}^3)$ *and for which the function*

$$t \mapsto \varepsilon(\nabla_x \hat{u}(t)) : [0, T_e) \to L^\infty(\Omega, \mathcal{S}^3)$$

is of bounded variation.

We note that the Neumann problem is solvable only if the functions b and γ satisfy the identity

$$\int_\Omega b(x,t) \cdot (a + \omega \times x) dx + \int_{\partial\Omega} \gamma(x,t) \cdot (a + \omega \times x) dS_x = 0$$

for all $a, \omega \in \mathbb{R}^3$ and all $t \in [0, T_e)$. The function $a + \omega \times x$ is an infinitesimal rigid motion. The solution of the Dirichlet problem is unique. The solution of the Neumann problem is unique only up to infinitesimal rigid motions, but since $\varepsilon(\nabla_x(a+\omega\times x)) = 0$, to given b and γ the function $t \mapsto \varepsilon(\nabla_x \hat{u}(t))$ is unique also for the Neumann problem.

From the regularity theory for elliptic systems it follows that if $\partial\Omega$ is sufficiently smooth, then $\mathcal{C}_D$ contains the class of all functions $(b, \gamma) : [0, T_e) \to H_1(\Omega) \times H_{5/2}(\partial\Omega)$ of bounded variation, and $\mathcal{C}_N$ contains the class of all functions $(b, \gamma) : [0, T_e) \to H_1(\Omega) \times H_{3/2}(\partial\Omega)$ of bounded variation, for example.

Definition 1.2 *A function*

$$(u, T, z) \in C([0, T_e), H_1(\Omega, \mathbb{R}^3) \times L^2(\Omega, \mathcal{S}^3) \times L^2(\Omega, \mathbb{R}^N))$$

is called mild solution of the initial-boundary value problem consisting of the equations

$$-\mathrm{div}_x\, T = b \tag{1.15}$$

$$T = D(\varepsilon(\nabla_x u) - Bz), \tag{1.16}$$

$$z_t = g(-\rho \nabla_z \psi(\varepsilon(\nabla_x u), z)) \tag{1.17}$$

on $\Omega \times [0, T_e)$*, of the initial condition*

$$z(x, 0) = z^{(0)}(x), \quad x \in \Omega, \tag{1.18}$$

and either of the Dirichlet boundary condition

$$u(x,t) = \gamma(x,t), \quad (x,t) \in \partial\Omega \times (0, T_e), \tag{1.19}$$

or of the Neumann boundary condition

$$T(x,t)n(x) = \gamma(x,t), \quad (x,t) \in \partial\Omega \times (0, T_e), \tag{1.20}$$

if (u, T, z) *can be approximated in the following sense:*

To every $\overline{T} < T_e$ *there is a sequence of partitions* $P^n = \{0 = t_0^n < \ldots < t_{k(n)}^n\}$ *and a sequence of functions* $(u^n, T^n, z^n) : [0, t_{k(n)}^n] \to H_1(\Omega, \mathbb{R}^3) \times L^2(\Omega, \mathcal{S}^3) \times L^2(\Omega, \mathbb{R}^3)$ *such that*

(i) $\overline{T} \le t_{k(n)}^n < T_e$,

(ii) $\lim_{n\to\infty} \max_{1\le k\le k(n)} (t_k^n - t_{k-1}^n) = 0$,

(iii) z^n *is constant on* $(t_{k-1}^n, t_k^n]$,

(iv) $(u^n, T^n, z^n)(t)$ *is a weak solution of the boundary value problems* (1.15), (1.16), (1.19) *or* (1.15),(1.16), (1.20) *for every* t *and satisfies*

$$\frac{z^n(t_k^n) - z^n(t_{k-1}^n)}{t_k^n - t_{k-1}^n} = g\Big(-\rho\nabla_z \psi(\varepsilon(\nabla_x u^n(t_k^n)), z^n(t_k^n))\Big),$$
$$k = 1,\ldots,k(n), \qquad n = 1,2,3\ldots,$$

(v) $z^n(0) = z^{(0)}$,

(vi) *if* $\|\cdot\|$ *denotes the norm of* $H_1(\Omega,\mathbb{R}^3) \times L^2(\Omega,\mathcal{S}^3) \times L^2(\Omega,\mathbb{R}^N)$, *then*

$$\lim_{n\to\infty} \sup_{0\le t\le T} \|(u^n(t),T^n(t),z^n(t)) - (u(t),T(t),z(t))\| = 0.$$

Let Δ_D and Δ_N, respectively, denote the sets of all $z \in L^2(\Omega,\mathbb{R}^N)$ with

$$g(-\rho\nabla_z \psi(\varepsilon(\nabla u), z)) \in L^2(\Omega,\mathbb{R}^N)\,.$$

Here $u \in H_1(\Omega,\mathbb{R}^3)$ is determined as weak solution of the Dirichlet boundary value problem (1.15), (1.16), (1.19), or Neumann boundary value problem (1.15), (1.16), (1.20), respectively, with this z and with $b \equiv \gamma \equiv 0$ inserted.

$\Delta = \Delta_D$ or $\Delta = \Delta_N$ is the t–independent domain of the operator $A(t)$ from (1.9). In the following we write $\mathcal{C}$ and Δ if a statement holds for the Dirichlet and Neumann boundary condition. $\mathcal{C},\ \Delta$ can be replaced by $\mathcal{C}_D,\ \Delta_D$ or $\mathcal{C}_N,\ \Delta_N$.

Now we formulate the main

Theorem 1.3 *Let* $g : \mathbb{R}^N \to \mathbb{R}^N$ *be a maximal monotone vector field with* $g(0) = 0$. *Assume that to every* $C_1 > 0$ *there is* $C_2 > 0$ *such that*

$$|g(z + B^T\tau) - g(z)| \le C_2(|g(z)| + 1)|\tau| \tag{1.21}$$

for all $z \in \mathbb{R}^N$ *and all* $\tau \in \mathcal{S}^3$ *with* $|\tau| \le C_1$. *Assume moreover, that the symmetric matrix* L *in* (1.8) *is positive definite.*

Then to all $(b,\gamma) \in \mathcal{C}$ *and* $z^{(0)} \in \Delta$ *there is a mild solution* (u,T,z) *of the initial-boundary value problem* (1.15)–(1.19) *with Dirichlet boundary condition or* (1.15)–(1.18), (1.20) *with Neumann boundary condition. The component* z *of this solution satisfies* $z(t) \in \Delta$ *for* t *a.e.*

The solution of the Dirichlet problem is unique. If (u_0,T,z) *is a mild solution of the Neumann problem, then all mild solutions are obtained in the form* $(u,T,z) = (u_0,T,z) + (w,0,0)$, *where* $w(x,t) = a(t) + \omega(t) \times x$ *with* $a,\omega \in C([0,T_e),\mathbb{R}^3)$. *For every* t *the function* $x \mapsto w(x,t)$ *is an infinitesimal rigid motion.*

We remark that L is positive definite if and only if the free energy ψ is positive definite, cf. [1, p. 48]. A class of functions satisfying the condition (1.21) is given in the following

Lemma 1.4 *Assume that the function* $\tau \mapsto g(z + B^T\tau) : \mathcal{S}^3 \to \mathbb{R}^N$ *is differentiable and that there is a constant* C *with*

$$|\nabla_\tau g(z + B^T\tau)| \le C(|g(z + B^T\tau)| + 1) \tag{1.22}$$

for all $z \in \mathbb{R}^N$ *and* $\tau \in \mathcal{S}^3$. *Then* (1.21) *is satisfied.*

In the remainder we proceed as follows: The proof of Theorem 1.3 is sketched in Sections 2 and 3. For simplicity, we only consider the case of Dirichlet boundary conditions. In Section 2 we discuss the reduction of the initial-boundary value problem to an evolution equation, and in Section 3 we sketch the verification of condition C. In particular, we give the detailed proof of the inequality (1.11). In Section 4 we present an example for constitutive laws satisfying the assumptions of Theorem 1.3. The bibliography contains only a small number of articles and books. For other references we must refer to the literature cited in these articles and books.

2 Reduction to an evolution equation

We denote the scalar product of two matrices $\sigma, \tau \in \mathcal{S}^3$ by

$$\sigma \cdot \tau = \sum_{i,j=1}^{3} \sigma_{ij} \tau_{ij} \, .$$

With this notation, the scalar products and norms on $L^2(\Omega, \mathbb{R}^n)$ and on $L^2(\Omega, \mathcal{S}^3)$ are given by

$$(\sigma, \tau)_\Omega = \int_\Omega \sigma(x) \cdot \tau(x) \, dx, \quad \|\sigma\|_\Omega = \|\sigma\|_{0,\Omega} = (\sigma, \sigma)_\Omega^{1/2}.$$

Since $D : \mathcal{S}^3 \to \mathcal{S}^3$ is symmetric and positive definite, a second scalar product on $L^2(\Omega, \mathcal{S}^3)$ is defined by

$$[\sigma, \tau]_\Omega = (D\sigma, \tau)_\Omega \, .$$

The associated norm $[\sigma, \sigma]_\Omega^{1/2} = (\int_\Omega (D\sigma(x)) \cdot \sigma(x) dx)^{1/2}$ is equivalent to the norm $\|\sigma\|_\Omega$. By $\overset{\circ}{H}_1(\Omega, \mathbb{R}^3)$ we denote the closure in $H_1(\Omega, \mathbb{R}^3)$ of the set of infinitely differentiable functions with compact support contained in Ω.

As preparation for the reduction of the initial-boundary value problem to an evolution equation we must study the Dirichlet boundary value problem

$$-\operatorname{div} T(x) = 0, \qquad x \in \Omega, \tag{2.1}$$

$$T(x) = D(\varepsilon(\nabla u(x)) - Bz(x)), \quad x \in \Omega, \tag{2.2}$$

$$u(x) = 0, \qquad x \in \partial\Omega. \tag{2.3}$$

Let the linear subspace $\mathcal{D}$ of $L^2(\Omega, \mathcal{S}^3)$ be defined by

$$\mathcal{D} = \{\varepsilon(\nabla u) \mid u \in \overset{\circ}{H}_1(\Omega, \mathbb{R}^3)\} \, . \tag{2.4}$$

It follows from Korn's inequality (cf. [9, pp. 278]) that $\mathcal{D}$ is a closed subspace of $L^2(\Omega, \mathcal{S}^3)$. Therefore there is a projection operator $P : L^2(\Omega, \mathcal{S}^3) \to L^2(\Omega, \mathcal{S}^3)$ onto $\mathcal{D}$, which is orthogonal with respect to the scalar product $[\sigma, \tau]_\Omega$.

Lemma 2.1 (i) *Let Ω be open and bounded and let $z \in L^2(\Omega, \mathbb{R}^N)$. Let $u \in \overset{\circ}{H}_1(\Omega, \mathbb{R}^3)$ be the unique weak solution of the boundary value problem* (2.1)–(2.3)*. Then $\varepsilon = \varepsilon(\nabla u)$ satisfies*

$$\varepsilon = PBz \, .$$

(ii) *The mapping $B^T DPB : L^2(\Omega, \mathbb{R}^N) \to L^2(\Omega, \mathbb{R}^N)$ is symmetric with respect to the scalar product $(z, \hat{z})_\Omega$.*

With this lemma we can reduce the initial-boundary value problem (1.15)–(1.19) to an evolution equation. Note first that (1.8) yields

$$-\rho\nabla_z\psi(\varepsilon,z) = B^T D(\varepsilon - Bz) - Lz = B^T D\varepsilon - Mz,$$

with the symmetric $N \times N$–matrix $M = B^T DB + L$. Therefore (1.17) can be written as

$$z_t = g(B^T D\varepsilon(\nabla_x u) - Mz)\,. \tag{2.5}$$

Now assume that the pair of functions (b,γ) with b from (1.15) and γ from (1.19) belongs to $\mathcal{C}$. Assume moreover that (u,T,z) is a mild solution of the initial-boundary value problem. As a consequence of conditions (iv) and (vi) of Definition 1.2, this implies that $(u,T,z)(t) \in \overset{\circ}{H}_1(\Omega,\mathbb{R}^3) \times L^2(\Omega,\mathcal{S}^3) \times L^2(\Omega,\mathbb{R}^N)$ for all $t \in [0,T_e)$ and that $u(t)$ is a weak solution of the Dirichlet boundary value problem formed by the equations (1.15), (1.16) and (1.19), where $z(t)$ from the mild solution is considered to be given.

Let $\hat u(t) \in \overset{\circ}{H}_1(\Omega,\mathbb{R}^3)$ be the unique weak solution of the Dirichlet boundary value problem (1.12)–(1.14) with $b(t)$ from (1.15) and $\gamma(t)$ from (1.19) inserted, and let $\tilde u(t)$ be the weak solution of the Dirichlet boundary value problem (2.1)–(2.3) with $z(t)$ from the given solution inserted. Then $\tilde u(t) + \hat u(t)$ is also a solution of the boundary value problem (1.15), (1.16), (1.19). Whence, $u(t) = \tilde u(t) + \hat u(t)$, since the solution is unique. Lemma 2.1 thus implies that the function $\varepsilon(\nabla_x u)$ in (2.5) satisfies

$$\varepsilon(\nabla_x u(t)) = PBz(t) + \varepsilon(\nabla_x \hat u(t))\,.$$

Insertion of this equality into (2.5) yields the evolution equation

$$z_t(t) = g\Big((B^T DPB - M)z(t) + \hat b(t)\Big) = -A(t)z(t)\,, \tag{2.6}$$

where

$$\Big(x \mapsto [\hat b(t)](x) = B^T D\varepsilon(\nabla_x \hat u(x,t))\Big) \in L^2(\Omega,\mathbb{R}^N)\,. \tag{2.7}$$

(2.6) is an evolution equation for z on the Hilbert space $L^2(\Omega,\mathbb{R}^N)$. The evolution operator $A(t) : \Delta(A(t)) \subseteq L^2(\Omega,\mathbb{R}^N) \to L^2(\Omega,\mathbb{R}^N)$ is defined by

$$A(t)z = -g\Big((B^T DPB - M)z + \hat b(t)\Big)\,, \tag{2.8}$$

with the domain

$$\Delta(A(t)) = \Big\{ z \in L^2(\Omega,\mathbb{R}^N) \,\Big|\, g((B^T DPB - M)z + \hat b(t)) \in L^2(\Omega,\mathbb{R}^N) \Big\}\,. \tag{2.9}$$

3 Proof of the existence theorem

In this section we sketch the verification of condition C. Note first that by Lemma 2.1 (ii) the linear mapping

$$M - B^T DPB = L + B^T DB - B^T DPB : L^2(\Omega,\mathbb{R}^N) \to L^2(\Omega,\mathbb{R}^N)$$

is symmetric with respect to the scalar product $(z,\hat z)_\Omega$. It is also positive definite, since

$$((M - B^T DPB)z, z)_\Omega = (Lz,z)_\Omega + (D(I-P)Bz, Bz)_\Omega$$
$$= (Lz,z)_\Omega + [(I-P)Bz, Bz]_\Omega = (Lz,z)_\Omega + [(I-P)Bz,(I-P)Bz]_\Omega$$
$$\geq (Lz,z)_\Omega \geq \mu\|z\|_\Omega^2$$

with $\mu > 0$. Here we used that the projector $I - P$ is orthogonal with respect to the scalar product $[\sigma, \tau]_\Omega$, and we used the assumption that L is positive definite. Hence,

$$\langle z, \hat{z} \rangle_\Omega = ((M - B^T DPB)z, \hat{z})_\Omega$$

is a scalar product on $L^2(\Omega, \mathbb{R}^N)$. Some well known considerations show that

$$\|M - B^T DPB\|^{-1} \langle z, z \rangle_\Omega \leq \|z\|_\Omega^2 \leq \|(M - B^T DPB)^{-1}\| \langle z, z \rangle_\Omega .$$

Whence, the associated norm

$$|\!|\!| z |\!|\!|_\Omega = \langle z, z \rangle_\Omega^{1/2}$$

is equivalent to $\|z\|_\Omega$.

Theorem 3.1 *Let $(b, \gamma) \in \mathcal{C}$ be a given function and let $g : \mathbb{R}^N \to \mathbb{R}^N$ be a maximal monotone vector field with $g(0) = 0$. Then for all $t \in [0, T_e)$ the following assertions hold:*

(i) *The operator $A(t)$ defined in (2.8) is monotone with respect to the scalar product $\langle z, \hat{z} \rangle_\Omega$.*

(ii) *For all $\lambda > 0$ the operator $I + \lambda A(t) : \Delta(A(t)) \to L^2(\Omega, \mathbb{R}^N)$ is surjective. Hence, $A(t)$ is maximal monotone.*

(iii) *For all $\lambda > 0$*

$$|\!|\!| (I + \lambda A(t))^{-1} 0 |\!|\!|_\Omega \leq \|(M - B^T DPB)^{-1}\|^{1/2} \, \|\hat{b}(t)\|_\Omega ,$$

with $\hat{b}$ defined in (2.7). Here $0 \in L^2(\Omega, \mathbb{R}^N)$ is the null function.

Proof of Theorem 1.3: To prove this theorem it must be shown that for $(b, \gamma) \in \mathcal{C}$ the operator family $A(t)$ satisfies condition C. In this condition we take for X the Hilbert space $L^2(\Omega, \mathbb{R}^N)$. From Theorem 3.1(ii) it follows that $A(t)$ is maximal monotone, hence m-accretive.

Next, the assertion (iii) of Theorem 3.1 immediately shows that for $(b, \gamma) \in \mathcal{C}$ the inequality (1.10) from condition C can be satisfied with the choice $y_0 = 0$.

Further, some computations show that as a consequence of (1.21), for functions $(b, \gamma) \in \mathcal{C}$ the domain of $A(t)$ is independent of t and satisfies $\Delta(A(t)) = \Delta$ for all $t \in [0, T_e)$, with the set Δ from Theorem 1.3.

Finally, to verify (1.11) for the operator family $A(t)$, note that for $(b, \gamma) \in \mathcal{C}$ the function $t \mapsto D\varepsilon(\nabla_x \hat{u}(t)) : [0, T_e) \to L^\infty(\Omega, \mathcal{S}^3)$ is of bounded variation. Thus, there exists C_1 with

$$\|D\varepsilon(\nabla_x \hat{u}(t))\|_{\infty,\Omega} \leq C_1 \tag{3.1}$$

for all t. Since $\hat{b}(s) - \hat{b}(t) = B^T(D\varepsilon(\nabla_x \hat{u}(s)) - D\varepsilon(\nabla_x \hat{u}(t)))$, and since (3.1) implies $\|D\varepsilon(\nabla_x \hat{u}(s)) - D\varepsilon(\nabla_x \hat{u}(t))\|_{\infty,\Omega} \leq 2C_1$, we infer from (1.21) that for a suitable constant C_3, for all $0 \leq s, t < T_e$ and all $z \in \Delta$

$$\begin{aligned}
&\|A(t)z - A(s)z\|_\Omega \\
&= \|g((B^T DPB - M)z + \hat{b}(t)) - g((B^T DPB - M)z + \hat{b}(s))\|_\Omega \qquad (3.2) \\
&\leq \|C_3(|g((B^T DPB - M)z + \hat{b}(t))| + 1)\, |D\varepsilon(\nabla_x \hat{u}(s)) - D\varepsilon(\nabla_x \hat{u}(t))| \,\|_\Omega \\
&\leq C_3(\|A(t)z\|_\Omega + \|1\|_\Omega)\, \|D\| \, \|\varepsilon(\nabla_x \hat{u}(s)) - \varepsilon(\nabla_x \hat{u}(t))\|_{\infty,\Omega} .
\end{aligned}$$

Setting $\rho(r) = \text{var}_{0\le t\le r}(t \mapsto \varepsilon(\nabla_x \hat{u}(t)))$ and $h(x,r) = \rho(r)w(x)$ with a function $w \in L^2(\Omega, \mathbb{R}^N)$ satisfying $\|w\|_\Omega = 1$, we obtain

$$\|\varepsilon(\nabla_x \hat{u}(s)) - \varepsilon(\nabla_x \hat{u}(t))\|_{\infty,\Omega} \le |\rho(s) - \rho(t)| = \|h(t) - h(s)\|_\Omega \,.$$

This inequality and (3.2) together yield (1.11) with $\Theta \equiv C_3\|D\| \max\{1, \|1\|_\Omega\}$. Therefore $A(t)$ satisfies condition C on $[0, T_e)$.

Now we can use the result proved in [5]: If condition C is satisfied, then to every $z^{(0)} \in \Delta$ there exists a unique mild solution $z \in C([0,T_e), L^2(\Omega, \mathbb{R}^N))$ of the initial value problem

$$z_t(t) + A(t)z(t) = 0, \quad z(0) = z^{(0)},$$

where this mild solution is characterized as follows:

To every $\overline{T} < T_e$ there is a sequence of partitions $P^n = \{0 = t_0^n < \ldots < t_{k(n)}^n\}$ and a sequence of step functions $z^n[0, t_{k(n)}^n] \to L^2(\Omega, \mathbb{R}^N)$ such that

$$\overline{T} \le t_{k(n)}^n < T_e, \tag{3.3}$$

$$\lim_{n\to\infty} \max_{1\le k\le k(n)} (t_k^n - t_{k-1}^n) = 0, \tag{3.4}$$

$$z^n \text{ is constant on } (t_{k-1}^n, t_k^n], \tag{3.5}$$

$$\frac{z^n(t_k^n) - z^n(t_{k-1}^n)}{t_k^n - t_{k-1}^n} + A(t_k^n)z^n(t_k^n) = 0\,, \tag{3.6}$$

$$z^n(0) = z^{(0)}. \tag{3.7}$$

$$\lim_{n\to\infty} \sup_{0\le t\le \overline{T}} \|z^n(t) - z(t)\|_\Omega = 0. \tag{3.8}$$

In [5] it is also shown that this mild solution satisfies $z(t) \in \Delta$ for t a.e.

We insert this mild solution $z(t)$ into (1.16). Then for every $t \in [0, T_e)$ the equations (1.15), (1.16), (1.19) define a Dirichlet boundary value problem on Ω. Let $(u(t), T(t))$ be the weak solution of this boundary value problem. The function (u, T, z) thus defined on $[0, T_e)$ is a mild solution of the initial-boundary value problem (1.15)–(1.19) in the sense of Definition 1.2. Sequences of partitions P^n and of functions (u^n, T^n, z^n) satisfying conditions (i)–(vi) of this definition are obtained by choosing sequences $\{P^n\}_{n=1}^\infty$ and $\{z^n\}_{n=1}^\infty$ satisfying (3.3)–(3.8), and by choosing for $(u^n(t), T^n(t))$ a weak solution of the boundary value problem (1.15), (1.16), (1.19) obtained after insertion of $z^n(t)$ into (1.16). The proof of Theorem 1.3 is complete.

4 Example

We present a simple example for a constitutive law satisfying the conditions of Theorem 1.3. We cannot give all the computations necessary to verify that these conditions are fulfilled, but must refer to [1] for detailed considerations and other, more complicated examples.

Let $\Gamma : [0,\infty) \to [0,\infty)$ be a continuously differentiable function with $\Gamma(0) = 0$ and with $\Gamma'(r) > 0$ for all $r > 0$. We consider the initial-boundary value problem to the equations

$$-\text{div}_x T \;=\; b, \tag{4.1}$$

$$T = D(\varepsilon(\nabla_x u) - \varepsilon_p), \tag{4.2}$$

$$\frac{\partial}{\partial t}\varepsilon_p = \Gamma(|P_0(T - k\varepsilon_p)|)\frac{P_0(T - k\varepsilon_p)}{|P_0(T - k\varepsilon_p)|}, \tag{4.3}$$

where $\varepsilon_p : \Omega \times [0, \infty) \to \mathcal{S}^3$ is the plastic strain tensor, where k is a positive constant, and where $P_0 : \mathcal{S}^3 \to \mathcal{S}^3$ is the orthogonal projector onto the subspace $\{\sigma \in \mathcal{S}^3 \mid \operatorname{trace}(\sigma) = 0\}$. If T is the stress, then $P_0 T$ is the stress deviator. For $\tau \in \mathcal{S}^3$ we set $|\tau|^2 = \sum_{i,j=1}^3 \tau_{ij}^2$.

By some computations the following assertions can be proved: (4.1)–(4.3) can be written in the form (1.15)–(1.17) if we identify $\mathcal{S}^3$ with $\mathbb{R}^6$ and set $z = \varepsilon_p$, hence $B = I$, and if we define g and the positive definite free energy ψ by

$$g(\tau) = \Gamma(|P_0\tau|)\frac{P_0\tau}{|P_0\tau|},$$

$$\rho\psi(\varepsilon, \varepsilon_p) = \frac{1}{2}[D(\varepsilon - \varepsilon_p)] \cdot (\varepsilon - \varepsilon_p) + \frac{1}{2}k|z|^2.$$

The continuous function $g : \mathbb{R}^6 \to \mathbb{R}^6$ is the gradient of a convex function. Hence, g is a maximal monotone vector field. Moreover, g satisfies the inequality (1.22) if

$$\Gamma'(r) \le c(\Gamma(r) + 1)$$

holds with a constant c. In this case all conditions of Theorem 1.3 are satisfied. One can for example choose $\Gamma(r) = C(\exp(\kappa r) - 1)$, or $\Gamma(r) = Cr^n$, where $C, \kappa > 0,\ n > 1$ are positive constants. With the last choice (4.3) becomes the Melan-Prager model, a modification of the Norton-Hoff law.

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