# Borel–Weil theory for loop groups

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# Introduction

Let $K$ be a compact Lie group and $LK := C^\infty(\mathbb{S}^1, K)$ the group of smooth loops with values in $K$. This is a group under pointwise multiplication and it carries the structure of a Lie group modeled over the Fréchet space $L\mathfrak{k} := C^\infty(\mathbb{S}^1, \mathfrak{k})$ of smooth loops with values in the Lie algebra of $\mathfrak{k}$. These notes grew out of a reworking of the proof of the Borel–Weil theory for loop groups as it is presented in the book of Pressley and Segal ([PS86]). Our main objective is to develop the techniques which are relevant for this theory in the setting of the Fréchet groups of smooth loops from an analytic point of view. We will describe an intrinsic construction of the irreducible positive energy representations which does not refer to embeddings of loop groups into infinite-dimensional classical Banach Lie groups as in [GW84] and [Ner83]. For an algebraic version of a Borel–Weil Theorem for general Kac–Moody groups, considered as algebraic groups of infinite type, we refer to [Ka85b, p. 192]. Generalizations of Bott–Borel–Weil theory to direct limits of Lie groups are discussed in [NRW99]. A realization of the spin representation of the group

$O(\infty, \mathbb{C})$ in a Fréchet space of holomorphic sections is constructed by Neretin in [Ner87].

In Section I we recall some basic notation from the theory of compact groups and their Lie algebras. In Section II we briefly discuss the loop group $LK$, where $K$ is a compact group, its complexification and certain central extensions. We then turn to the root decomposition of the loop algebra $L\mathfrak{k}$ in Section III, and in Section IV we discuss some general aspects of the representation theory of loop groups. Section V is dedicated to some rather general constructions of representations of involutive semigroups on pre-Hilbert spaces. We include this material to make the construction of a certain (pre-)Hilbert space of holomorphic functions in Section VI more transparent. The heart of this article is Section VI, where we explain how the finite-dimensional Borel–Weil theory can be extended to loop groups. The basic idea is that the irreducible representations of positive energy (maybe after passing to a dense subspace) can be realized as the holomorphic sections of a certain line bundle over a complex homogeneous space of $LK$. Then the parametrization of these representations is derived from a characterization of those line bundles which have non-zero holomorphic sections. Finally we discuss in Section VII how these constructions can be used to analyze more general positive energy representations.

## I. Compact groups

In this section we collect the basic material concerning compact groups that we will need in the following. In particular we introduce the notation that will be used later on.

In the following we will write $\mathbb{S}^1 := \mathbb{R}/2\pi\mathbb{Z}$ for the circle as a real smooth manifold. If the circle is considered as a group, we will write $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$.

Let $\mathfrak{k}$ be a compact Lie algebra and $\mathfrak{t} \subseteq \mathfrak{k}$ be a Cartan subalgebra. Then the set $\operatorname{ad}\mathfrak{t}$ is commutative and diagonalizable on the complexification $\mathfrak{k}_{\mathbb{C}}$ of $\mathfrak{k}$. Thus we obtain the *root decomposition*

$$\mathfrak{k}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Delta_{\mathfrak{k}}} \mathfrak{k}_{\mathbb{C}}^{\alpha},$$

where

$$\mathfrak{k}_{\mathbb{C}}^{\alpha} = \{X \in \mathfrak{k}_{\mathbb{C}} \colon (\forall Y \in \mathfrak{t})[Y, X] = i\alpha(Y)X\} \quad \text{and} \quad \Delta_{\mathfrak{k}} = \{\alpha \in \mathfrak{t}^* \setminus \{0\} \colon \mathfrak{k}_{\mathbb{C}}^{\alpha} \neq \{0\}\}.$$

Let $K$ be a connected Lie group with Lie algebra $\mathfrak{k}$ and $T := \exp \mathfrak{t}$ the subgroup corresponding to $\mathfrak{t}$. If $K$ is compact, then $T$ is a maximal torus in $K$. We identify the *character group* $\widehat{T} := \operatorname{Hom}(T, \mathbb{T})$ with a subset of $\mathfrak{t}^*$ by associating to each continuous character $\chi \colon T \to \mathbb{T}$ the linear functional $\lambda = -id\chi(\mathbf{1}) \in \mathfrak{t}^*$ satisfying

$$\chi(\exp X) = e^{i\lambda(X)} \tag{1.1}$$

for all $X \in \mathfrak{t}$. Since the Lie algebra $\mathfrak{t}$ of $T$ can be identified with the set of one-parameter subgroups $\mathbb{R} \to T$ we can in particular identify the group $\check{T} := \mathrm{Hom}(\mathbb{T}, T)$ with a subset of $\mathfrak{t}$ by associating to each $\gamma \in \check{T}$ the uniquely determined element $X_\gamma \in \mathfrak{t}$ with

$$\gamma(\theta) = \exp(\theta X_\gamma)$$

for all $\theta \in \mathbb{T} \cong \mathbb{R}/(2\pi\mathbb{Z})$. In this sense we obtain the identification

$$\check{T} \cong \{X \in \mathfrak{t} \colon \exp(2\pi X) = \mathbf{1}\}.$$

An important tool which will also be crucial in the infinite-dimensional setting are certain subalgebras of $\mathfrak{k}$ which are isomorphic to the three-dimensional compact Lie algebra $\mathfrak{su}(2) \cong \mathfrak{so}(3, \mathbb{R})$. Let $\alpha \in \Delta_\mathfrak{k}$ and $X \mapsto \overline{X}$ denote complex conjugation on $\mathfrak{k}_\mathbb{C}$. Then we choose for each $\alpha \in \Delta_\mathfrak{k}$ an element $e_\alpha \in \mathfrak{k}_\mathbb{C}^\alpha$ in such a way that $e_{-\alpha} = \overline{e_\alpha}$ and put $ih_\alpha := [e_\alpha, e_{-\alpha}]$. Here $h_\alpha \in \mathfrak{t} \cap [\mathfrak{k}_\mathbb{C}^\alpha, \mathfrak{k}_\mathbb{C}^{-\alpha}]$ is uniquely determined by the requirement that $\alpha(h_\alpha) = 2$ and called the *coroot* associated to $\alpha$. We then have

$$[h_\alpha, e_{\pm\alpha}] = \pm 2i e_{\pm\alpha}.$$

We call $(e_\alpha, e_{-\alpha}, h_\alpha)$ a *basic* $\mathfrak{su}_2$*-triple*. This is justified by the fact that

$$\mathfrak{k}(\alpha) := \mathbb{R}h_\alpha + \mathbb{R}(e_\alpha + e_{-\alpha}) + \mathbb{R}i(e_\alpha - e_{-\alpha})$$

is a subalgebra of $\mathfrak{k}$ isomorphic to $\mathfrak{su}(2)$. For $\mathfrak{k} = \mathfrak{su}(2)$ the corresponding basis elements are given by

$$e_\alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_{-\alpha} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad \text{and} \quad h_\alpha = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

**Lemma I.1.** *If $\kappa$ is an invariant $\mathbb{C}$-bilinear form on $\mathfrak{k}_\mathbb{C}$, then $2\kappa(e_\alpha, e_{-\alpha}) = \kappa(h_\alpha, h_\alpha)$.*

**Proof.** The invariance of $\kappa$ implies that

$$2\kappa(e_\alpha, e_{-\alpha}) = \kappa(-i[h_\alpha, e_\alpha], e_{-\alpha}) = -i\kappa(h_\alpha, [e_\alpha, e_{-\alpha}]) = \kappa(h_\alpha, h_\alpha). \qquad \blacksquare$$

## II. Loop groups and their central extensions

Before we turn to the specific case of the loop group of a compact group, we first study the Fréchet–Lie group structure on the group $C^\infty(M, G)$, where $G$ is a finite-dimensional Lie group and $M$ is a compact manifold.

## Groups of smooth maps

**Theorem II.1.** *If $M$ is a compact manifold, then the group*

$$\mathcal{D}(M,G) := C^\infty(M,G)$$

*is a Fréchet–Lie group and its Lie algebra is $\mathcal{D}(M,\mathfrak{g}) := C^\infty(M,\mathfrak{g})$ endowed with its natural Fréchet structure. The exponential function*

$$\exp\colon \mathcal{D}(M,\mathfrak{g}) \to \mathcal{D}(M,G), \quad \gamma \mapsto \exp\circ\gamma$$

*is a smooth map which maps a 0-neighborhood in $\mathcal{D}(M,\mathfrak{g})$ diffeomorphically onto a 0-neighborhood in the group $\mathcal{D}(M,G)$, and we have the Trotter-Product-Formula:*

$$\exp(\gamma+\eta) = \lim_{n\to\infty} \Big(\exp\big(\tfrac{1}{n}\gamma\big)\exp\big(\tfrac{1}{n}\eta\big)\Big)^n.$$

*If, in addition, $G$ is a complex Lie group, then $\mathcal{D}(M,G)$ is a complex Lie group and the exponential function is holomorphic.*

**Proof.** First we note that the spaces $\mathcal{D}(M,\mathfrak{g})$ are nuclear Fréchet spaces (cf. [Alb93, Sect. 1.2]). The topology defined on these spaces is the same as the topology described in [Ne00a, Sect. II].

Next we have to describe the manifold structure on $\mathcal{D}(M,G)$. In [Alb93] it is shown how $\mathcal{D}(M,G)$ can be given the structure of a topological group such that the exponential function $\exp\colon \mathcal{D}(M,\mathfrak{g}) \to \mathcal{D}(M,G)$ is a local homeomorphism. Hence $\mathcal{D}(M,G)$ carries the structure of a topological manifold modeled over the complete locally convex space $\mathcal{D}(M,\mathfrak{g})$.

Let $U \subseteq \mathfrak{g}$ be an open convex 0-neighborhood such that the exponential function $\exp\colon U \to \exp(U) \subseteq G$ is a diffeomorphism and the Campbell–Hausdorff series

$$X * Y = X + Y + \tfrac{1}{2}[X,Y] + \ldots$$

converges for $X, Y \in U$. Let further $V \subseteq U$ be a symmetric 0-neighborhood with $V * V \subseteq U$. Then we obtain charts

$$\varphi_g\colon \mathcal{D}(M,V) \to \mathcal{D}(M,G), \quad \gamma \mapsto g\cdot(\exp\circ\gamma).$$

Suppose that two chart neighborhoods $g\exp\big(\mathcal{D}(M,V)\big)$ and $h\exp\big(\mathcal{D}(M,V)\big)$ intersect. Then

$$\begin{aligned} g^{-1}h &\in \exp\big(\mathcal{D}(M,V)\big)\exp\big(\mathcal{D}(M,V)\big)^{-1} \\ &= \exp\big(\mathcal{D}(M,V) * \mathcal{D}(M,V)\big) \subseteq \exp\big(\mathcal{D}(M,V*V)\big) \\ &\subseteq \exp\big(\mathcal{D}(M,U)\big). \end{aligned}$$

Let $g^{-1}h = \exp \circ \alpha$, $\alpha \in \mathcal{D}(M, U)$. Then

$$\varphi_g^{-1}\varphi_h(\eta) = (\exp|_U)^{-1}\big(g^{-1}h \exp\circ\eta\big) = (\exp|_U)^{-1}\big(\exp\circ\alpha \cdot \exp\circ\eta\big)$$
$$= (\exp|_U)^{-1}\big(\exp\circ(\alpha * \eta)\big) = \alpha * \eta.$$

The mapping

$$\lambda_\alpha^*: M \times V \to U, \quad (p, X) \mapsto \alpha(p) * X$$

is smooth. Therefore Corollary III.8 in [Ne00a] shows that $\eta \mapsto \alpha * \eta, \mathcal{D}(M, V) \to \mathcal{D}(M, U)$ is smooth. This proves that the charts $\varphi_g$, $g \in \mathcal{D}(M, G)$, have smooth transition functions, hence define the structure of a smooth manifold on $\mathcal{D}(M, G)$.

Applying [Ne00a, Cor. III.8] to the Campbell–Hausdorff multiplication

$$V \times V \to U, \quad (X, Y) \mapsto X * Y = X + Y + \tfrac{1}{2}[X, Y] + \ldots,$$

we see that the map

$$\mathcal{D}(M, V) \times \mathcal{D}(M, V) \to \mathcal{D}(M, U), \quad (\alpha, \beta) \mapsto \alpha * \beta$$

is smooth. Moreover, $\alpha \mapsto -\alpha$ is a smooth involution of $\mathcal{D}(M, U)$. From that it easily follows that multiplication and inversion in $\mathcal{D}(M, G)$ are smooth mappings, i.e., that $\mathcal{D}(M, G)$ is a Lie group.

To see that the exponential function is smooth, let $\gamma \in \mathcal{D}(M, \mathfrak{g})$. Then there exists an open 0-neighborhood $C \subseteq \mathfrak{g}$ such that $\exp\big(-\gamma(p)\big)\exp\big(\gamma(p) + X\big) \in \exp(U)$ for all $X \in C$. We put

$$f: M \times C \to \mathfrak{g}, \quad (p, X) \mapsto (\exp|_U)^{-1}\Big(\exp\big(-\gamma(p)\big)\exp\big(\gamma(p) + X\big)\Big)$$

For $\eta \in \mathcal{D}(M, C)$ we now have

$$\varphi_{\exp\circ\gamma}^{-1}\big(\exp\circ(\gamma + \eta)\big) = f \circ (\mathrm{id}_M, \eta).$$

Hence the smoothness of this map follows from [Ne00a, Prop. III.7], and we conclude that the exponential function is smooth.

To prove the Trotter-Product-Formula, we consider the analytic map

$$f: U \times U \times ]-1, 1[ \to \mathfrak{g}, \quad (X, Y, t) \mapsto \begin{cases} \frac{1}{t}(tX * tY) & \text{for } t \neq 0 \\ X + Y & \text{for } t = 0. \end{cases}$$

Note that the analyticity follows from the fact that if $X * Y = \sum_{n=1}^\infty h_n(X, Y)$ denotes the expansion of the Campbell–Hausdorff product into homogeneous terms, then

$$f(X, Y, t) = \sum_{n=1}^\infty t^{2n-1} h_n(X, Y).$$

Using [Ne00a, Cor. III.8], we see that the map

$$C^\infty(M,U)\times C^\infty(M,U)\times C^\infty(M,]-1,1[)\to C^\infty(M,\mathfrak{g}),\quad (\alpha,\beta,\gamma)\mapsto f\circ(\alpha,\beta,\gamma)$$

is smooth and therefore in particular continuous. Fixing $\alpha$ and $\beta$, we conclude that

$$\alpha+\beta=\lim_{n\to\infty} f\circ(\alpha,\beta,\tfrac{1}{n}) \tag{2.1}$$

holds in $C^\infty(M,\mathfrak{g})$. Now the Trotter-Product-Formula is proved as follows. For $\gamma,\eta\in C^\infty(M,\mathfrak{g})$ we use the compactness of $M$ to find $m\in\mathbb{N}$ with $\frac{1}{m}\gamma,\frac{1}{m}\eta\in C^\infty(M,U)$. Then

$$\exp(\gamma+\eta)=\exp\left(\tfrac{1}{m}\gamma+\tfrac{1}{m}\eta\right)^m,$$

and we can apply (2.1) to the right hand side.

If, in addition, $G$ is a complex Lie group, then transition functions, multiplication and inversion are smooth maps with complex linear differentials, i.e., holomorphic (cf. [Ne00a, Sect. I]). Thus $\mathcal{D}(M,G)$ is a complex Lie group with holomorphic exponential function. ■

**Remark II.2.** The curves $\mathbb{R}\to\mathcal{D}(M,G), t\mapsto\exp(t\gamma)$ are one-parameter groups and the curves $t\mapsto g\exp(t\gamma)$ are integral curves of the left invariant vector field corresponding to the element $\gamma\in\mathcal{D}(M,\mathfrak{g})$. ■

**Corollary II.3.** *If $G$ is a connected finite-dimensional Lie group, then the loop group $LG:=C^\infty(\mathbb{S}^1,G)$ is a Fréchet–Lie group and $L\mathfrak{g}:=C^\infty(\mathbb{S}^1,\mathfrak{g})$ its Lie algebra. If, in addition, $G$ is a complex Lie group, then the same holds for $LG$.* ■

**Remark II.4.** The groups described in Theorem II.1 are special cases of the groups $\mathcal{D}(M,G)$ discussed in [Alb93], where $G$ is a connected Lie group and $M$ a not necessarily compact manifold. Here $\mathcal{D}(M,G)\subseteq C^\infty(M,G)$ denotes the subgroup of those smooth functions having compact support in the sense that the set $\{m\in M\colon f(m)\neq\mathbf{1}\}$ has compact closure in $M$. This group is topologized as a topological group locally homeomorphic to the space $\mathcal{D}(M,\mathfrak{g})$ of $\mathfrak{g}$-valued test functions which is a nuclear LF space.

The representation theory of these groups seems to be much more involved if $M$ is non-compact because in this case it is not clear whether $\mathcal{D}(M,G)$ is a $K$-space, i.e., functions on $\mathcal{D}(M,G)$ are continuous if they are continuous on all compact subsets. This makes it harder to put complete topologies on function spaces on these groups (cf. [Ne00a, Sect. III]). ■

**Central extensions of loop groups**

In this section $K$ denotes a compact connected Lie group. We write $LK = C^\infty(\mathbb{S}^1, K)$ for the associated *loop group*, i.e., the group of all smooth maps with values in $K$, where the group structure is given by pointwise multiplication. First we discuss this group as an infinite-dimensional Lie group modeled on a Fréchet space in the sense of Section I in [Ne00a].

As before, we identify $\mathbb{S}^1$ with $\mathbb{R}/2\pi\mathbb{Z}$, hence identify functions on $\mathbb{S}^1$ with $2\pi$-periodic functions on $\mathbb{R}$.

To each $K$-invariant symmetric bilinear form $\kappa$ on $\mathfrak{k}$ we associate the skew-symmetric bilinear form on $L\mathfrak{k}$ given by

$$\omega(\xi,\eta) := \frac{1}{2\pi}\int_0^{2\pi} \kappa\big(\xi(\theta), \eta'(\theta)\big)\, d\theta.$$

If $\kappa$ is $\mathfrak{k}$-invariant, then $\omega$ is a cocycle, hence defines a central extension

$$\{0\} \to \mathbb{R} \to \widetilde{L}\mathfrak{k} \to L\mathfrak{k} \to \{0\}, \tag{2.2}$$

where $\widetilde{L}\mathfrak{k} = L\mathfrak{k} \oplus \mathbb{R}$ is endowed with the bracket

$$[(\xi,s),(\eta,t)] = \big([\xi,\eta], \omega(\xi,\eta)\big)$$

(cf. [PS86, p. 39]). If $K$ is simply connected, then a corresponding central extension

$$\{\mathbf{1}\} \to \mathbb{T}_c \cong \mathbb{T} \longrightarrow \widetilde{L}K \overset{q}{\longrightarrow} LK \to \{\mathbf{1}\} \tag{2.3}$$

exists if and only if

$$\kappa(h_\alpha, h_\alpha) \in 2\mathbb{Z} \tag{2.4}$$

(cf. Section I for the notation) is satisfied for all roots $\alpha \in \Delta_{\mathfrak{k}}$ (see Theorem II.5 below). For a discussion of the case where $K$ is not simply connected we refer to [PS86, p.55].

The group $\mathbb{T}_r := \mathbb{T}$ acts by rotating loops $\gamma \in LK$ via

$$(R_\theta.\gamma)(t) = \gamma(t-\theta).$$

Since the cocycle $\omega$ on $L\mathfrak{k}$ is invariant under this action, this action lifts to an action by automorphisms on $\widetilde{L}K$ and we obtain a semidirect product group $\widehat{L}K := \mathbb{T}_r \ltimes \widetilde{L}K$ which is a central extension of the semidirect product $L_rK := \mathbb{T}_r \ltimes LK$. (Corollary II.20 below). Its Lie algebra $\widehat{L}\mathfrak{k}$ is a semidirect sum $\mathfrak{t}_r \ltimes \widetilde{L}\mathfrak{k}$, where $\mathfrak{t}_r$ denotes the Lie algebra of the group $\mathbb{T}_r$.

We identify the group $K$ with the subgroup of constant loops in $LK$ and accordingly $\mathfrak{k}$ with a Lie subalgebra of $L\mathfrak{k}$. The triviality of the cocycle $\omega$ on $\mathfrak{k}$ implies that the Lie algebra $\mathfrak{k}$ lifts to a subalgebra (also denoted $\mathfrak{k}$) of $\widetilde{L}\mathfrak{k}$. If $K$ is simply connected, then we conclude that there exists a continuous homomorphism $K \to \widetilde{L}K$ which is injective because $q$ maps the image isomorphically to the group $K$ of constant loops. Thus we can identify $K$ with a subgroup of the central extension $\widetilde{L}K$.

If $\mathfrak{k}_{\mathbb{C}}$ is the complexification of $\mathfrak{k}$ and $\kappa$ denotes the complex bilinear extension of the bilinear form on $\mathfrak{k}$, then the complex bilinear extension of $\omega$ to $L\mathfrak{k}_{\mathbb{C}}$ is given by the same formula and defines a central extension

$$\{0\} \to (\mathfrak{t}_c)_{\mathbb{C}} \cong \mathbb{C} \to \widetilde{L}\mathfrak{k}_{\mathbb{C}} \to L\mathfrak{k}_{\mathbb{C}} \to \{0\}. \tag{2.5}$$

If $\kappa$ satisfies condition (2.4) and $K$ is simply connected, then we use Theorem II.5 to see that we also obtain a central extension on the level of the complex groups, where the homomorphisms are holomorphic:

$$\{\mathbf{1}\} \to (\mathbb{T}_c)_{\mathbb{C}} \cong \mathbb{C}^\times \longrightarrow \widetilde{L}K_{\mathbb{C}} \xrightarrow{\ q\ } LK_{\mathbb{C}} \to \{\mathbf{1}\} \tag{2.6}$$

(cf. also [PS86, p.90] and the remarks at the end of Section II.3 in [Wu00]). By the same argument as above, we see that $K_{\mathbb{C}}$ can be realized as a subgroup of $\widetilde{L}K_{\mathbb{C}}$.

### Appendix IIa: Central extensions and semidirect products

In this appendix we use the results on central extensions of general infinite-dimensional Lie groups in [Ne00b] to derive the existence of the central extensions $\widetilde{L}K$ and $\widetilde{L}K_{\mathbb{C}}$, and also the existence of the semidirect product Lie group $\widehat{L}K \cong \mathbb{T}_r \ltimes \widetilde{L}K$ discussed above.

**Theorem II.5.** *If $K$ is a compact simple simply connected Lie group and $\kappa\colon \mathfrak{k} \times \mathfrak{k} \to \mathbb{R}$ an invariant symmetric bilinear form with $\kappa(h_\alpha, h_\alpha) \in 2\mathbb{Z}$ for all $\alpha \in \Delta_{\mathfrak{k}}$, then the 2-cocycle*

$$\omega(\xi, \eta) := \frac{1}{2\pi}\int_0^{2\pi} \kappa(\xi(t), \eta'(t))\, dt$$

*of the Lie algebra $L\mathfrak{k}$ corresponds to a central group extension*

$$\mathbb{T} \cong \mathbb{R}/2\pi\mathbb{Z} \hookrightarrow \widetilde{L}K \twoheadrightarrow LK.$$

*In this case we also obtain a central extension of complex Lie groups*

$$\mathbb{T}_{\mathbb{C}} \cong \mathbb{C}^\times \hookrightarrow \widetilde{L}K_{\mathbb{C}} \twoheadrightarrow LK_{\mathbb{C}}.$$

**Proof.** Since the second homotopy group of a finite-dimensional Lie group $K$ vanishes, we have

$$\pi_0(K) = \pi_1(K) = \pi_2(K) = \{\mathbf{1}\} \quad \text{and} \quad \pi_3(K) \cong \mathbb{Z}$$

([Mi95, Ths. 3.7 and 3.9]). With $\Omega K := \{\gamma \in LK : \gamma(0) = \mathbf{1}\}$ we have a semidirect decomposition $LK \cong \Omega K \rtimes K$, so that

$$\pi_k(\Omega K) \cong \pi_{k+1}(K), \quad k \in \mathbb{N}_0$$

(cf. [Br93, Cor. VII.4.4]) leads to

$$\pi_k(LK) = \pi_k(\Omega K \rtimes K) \cong \pi_k(\Omega K) \times \pi_k(K) \cong \pi_{k+1}(K) \times \pi_k(K),$$

so that $LK$ is connected and simply connected with $\pi_2(LK) \cong \mathbb{Z}$. Therefore we have to show that the left invariant 2-form $\Omega$ on $LK$ with $\Omega_{\mathbf{1}} = \omega$ satisfies

$$\int_{\mathbb{S}^2} \gamma^*\Omega \in 2\pi\mathbb{Z},$$

where $\gamma: \mathbb{S}^2 \to LK$ is a smooth representative of a generator of $\pi_2(LK)$ (cf. [Ne00b, Ths. IV.12(b) and V.7]). Since the inclusion $K \hookrightarrow K_{\mathbb{C}}$ is a homotopy equivalence, the same holds for the inclusion $LK \hookrightarrow LK_{\mathbb{C}}$, so that the condition for the existence of a central extension of the complex group $LK_{\mathbb{C}}$ is the same.

Next we prove the proposition for the special case $K = \mathrm{SU}(2) \cong \mathbb{S}^3$. On the Lie algebra $\mathfrak{k} = \mathfrak{su}(2)$ we consider the operator norm corresponding to the complex euclidean norm on $\mathbb{C}^2$. On $\mathfrak{su}(2)$ this norm is given by the scalar product $\frac{1}{2}\kappa(x,y) := -\frac{1}{2}\operatorname{tr}(xy)$ (note that the unit balls in $\mathfrak{su}(2)$ are invariant under the adjoint action, hence euclidean balls). Let $B_3 \subseteq \mathfrak{su}(2)$ denote the closed unit ball and idenitfy $\mathbb{S}^2$ with $\partial B_3$. We consider the smooth map

$$\gamma: \mathbb{S}^2 \to L\,\mathrm{SU}(2), \quad \gamma(x)(t) := e^{tx} = \exp(tx).$$

It is easy to see that $\|x\| = 1$ for $x \in \mathfrak{su}(2)$ leads to $e^{2\pi x} = \mathbf{1}$. The map $\mathbb{S}^3 \to \mathrm{SU}(2)$ corresponding to $\gamma$ is induced by $B_3 \to \mathrm{SU}(2), x \mapsto e^{2\pi x}$, which maps $\partial B_3$ to $\mathbf{1}$, hence leads to a map $\mathbb{S}^3 \to \mathrm{SU}(2)$. It is not hard to verify (counting inverse images of points) that this map is of degree 2, hence its homotopy class corresponds to the element $2 \in \mathbb{Z} \cong \pi_3(\mathrm{SU}(2))$. Now the isomorphism $\pi_3(\mathrm{SU}(2)) \cong \pi_2(L\,\mathrm{SU}(2))$ from above implies that $[\gamma] = 2 \in \mathbb{Z} \cong \pi_2(L\,\mathrm{SU}(2))$. We claim that the left invariant 2-form $\Omega$ on $L\,\mathrm{SU}(2)$ satisfies

$$\int_{\mathbb{S}^2} \gamma^*\Omega = 4\pi.$$

To facilitate this computation, we first observe that the Lie algebra cocycle $\omega$ on $L\mathfrak{su}(2)$ is $\mathrm{Ad}(\mathrm{SU}(2))$-invariant, so that $\Omega$ on $L\,\mathrm{SU}(2)$ is invariant under conjugation by $\mathrm{SU}(2)$. For $g \in \mathrm{SU}(2)$ we further have $\gamma(\mathrm{Ad}(g).x) = g\gamma(x)g^{-1}$, showing that

$\gamma^*\Omega$ is an invariant 2-form on $\mathbb{S}^2$, hence determined by the value in a fixed element $x_0$.

In $\mathfrak{su}(2)$ there exist elements $y_0, z_0$ with the brackets

$$[x_0, y_0] = 2z_0, \quad [x_0, z_0] = -2y_0 \quad \text{and} \quad [y_0, z_0] = 2x_0$$

and such that $(x_0, y_0, z_0)$ is an orthonormal basis with respect to $\kappa$. We have to evaluate

$$(\gamma^*\Omega)(x_0)(y_0, z_0) = \Omega(\gamma(x_0))(d\gamma(x_0).y_0, d\gamma(x_0).z_0).$$

Writing $\lambda_g(x) := gx$ for group elements, the element

$$d\gamma(x_0).y_0 \in T_{\gamma(x_0)}(L\,\mathrm{SU}(2))$$

is given by

$$(d\gamma(x_0).y_0)(t) = d\exp(tx_0)(ty_0) = d\lambda_{\exp(tx_0)}(\mathbf{1})f(\mathrm{ad}\,tx_0).ty_0,$$

where $f$ is the holomorphic function given by $f(z) = \frac{1-e^{-z}}{z}$. With

$$\xi(t) := f(\mathrm{ad}\,tx_0).ty_0 \quad \text{and} \quad \eta(t) := f(\mathrm{ad}\,tx_0).tz_0$$

we therefore obtain

$$(\gamma^*\Omega)(x_0)(y_0, z_0) = \omega(\xi, \eta).$$

We compute

$$\begin{aligned}
\xi(t) = f(\mathrm{ad}\,tx_0).ty_0 &= \frac{\mathbf{1} - e^{-\mathrm{ad}\,tx_0}}{\mathrm{ad}\,tx_0}.ty_0 \\
&= \frac{\mathbf{1} - \cosh(\mathrm{ad}\,tx_0)}{\mathrm{ad}\,tx_0}.ty_0 + \frac{\sinh(\mathrm{ad}\,tx_0)}{\mathrm{ad}\,tx_0}.ty_0 \\
&= \frac{\cos(2t) - 1}{2}z_0 + \frac{\sin(2t)}{2}y_0.
\end{aligned}$$

Applying $\frac{1}{2}\,\mathrm{ad}\,x_0$, we also get

$$\eta(t) = f(\mathrm{ad}\,tx_0).tz_0 = \frac{1 - \cos(2t)}{2}y_0 + \frac{\sin(2t)}{2}z_0,$$

and hence

$$\eta'(t) = \sin(2t)y_0 + \cos(2t)z_0.$$

This leads to

$$\kappa(\xi(t), \eta'(t)) = \sin^2(2t) + \big(\cos^2(2t) - \cos(2t)\big) = 1 - \cos(2t).$$

Integration of this expression leads to

$$\omega(\xi,\eta) = \frac{1}{2\pi}\int_0^{2\pi} \kappa(\xi(t),\eta'(t))\,dt = 1.$$

This proves that $\gamma^*\Omega$ is the volume form on $\mathbb{S}^2$, and hence that

$$\int_{\mathbb{S}^2} \gamma^*\Omega = \mathrm{vol}(\mathbb{S}^2) = 4\pi,$$

which, in view of $[\gamma] = 2 \in \mathbb{Z} \cong \pi_2(LK)$, we had to show.

Let $\alpha \in \Delta_{\mathfrak{k}}$ be a long root and $\mathfrak{k}(\alpha) \subseteq \mathfrak{k}$ the corresponding $\mathfrak{su}(2)$-subalgebra (cf. Section I). Then the corresponding homomorphism inclusion $\mathrm{SU}(2) \cong \mathbb{S}^3 \to K$ represents a generator of $\pi_3(K)$ ([Bo58]), so that the corresponding map

$$\gamma_\alpha \colon \mathbb{S}^2 \to LK$$

represents twice a generator of $\pi_2(LK)$. Therefore the preceding calculation shows that for $\kappa$ on $\mathfrak{k}$ as above, we get

$$\int_{\mathbb{S}^2} \gamma_\alpha^*\Omega = 2\pi \cdot \kappa(h_\alpha, h_\alpha) \in 4\pi\mathbb{Z}$$

if and only if $\kappa(h_\alpha, h_\alpha) \in 2\mathbb{Z}$ holds for one and hence for all long roots $\alpha$.

If $\beta \in \Delta_k$ is a short root, then $h_\beta$ is a long coroot, so that $\kappa(h_\beta, h_\beta) \in \mathbb{N}\kappa(h_\alpha, h_\alpha) \subseteq 2\mathbb{Z}$. Therefore the requirement that $\kappa(h_\alpha, h_\alpha) \in 2\mathbb{Z}$ holds for all long roots is equivalent to the same requirement for all roots. ∎

## Appendix IIb: Smoothness of group actions

We consider the rotation action $\mathbb{T}_r \times LK \to LK$. In view of Theorem III.5 in [Ne00a], the action of $\mathbb{T}_r$ on $L\mathfrak{k}$ is smooth. Now we also show that the action on $LK$ is smooth.

**Lemma II.6.** *Let $M$ be a compact manifold, $K$ a finite-dimensional Lie group and $G \times M \to M$ a smooth Lie group action on $M$. Then the action of $G$ on $C^\infty(M,K)$ given by $(g.f)(x) := f(g^{-1}.x)$ is smooth.*

**Proof.** Step 1: In view of Theorem III.5 in [Ne00a], the action of $G$ on the Fréchet space $C^\infty(M,\mathfrak{k})$ is smooth. Since the exponential function of $C^\infty(M,K)$ is a local diffeomorphism around 0, we obtain an open $\mathbf{1}$-neighborhood $U \subseteq K$ such that the action of $G$ on $C^\infty(M,U)$ is smooth.

Step 2: Fix $f \in C^\infty(M,K)$ and write $\sigma \colon G \times M \to M$ for the smooth action map. We claim that $G \to C^\infty(M,K), g \mapsto g.f$ is a smooth map. Since each

$g \in G$ acts by a diffeomorphism on $C^\infty(M,K)$, it suffices to verify smoothness in a neighborhood of $\mathbf{1}$. The smoothness of the left multiplications on $C^\infty(M,K)$ shows that it even suffices to show that the map $g \mapsto f^{-1} \cdot (g.f)$ is smooth in a neighborhood of $\mathbf{1}$. Since $M$ is compact, $f$ is uniformly continuous, and there exists an open neighborhood $U$ of $\mathbf{1}$ in $G$ and an open neighborhood $V$ of 0 in $\mathfrak{k}$ such that $\exp|_{V_\mathfrak{k}}: V_\mathfrak{k} \to \exp(V_\mathfrak{k})$ is a diffeomorphism and $f(x)^{-1} f(g^{-1}.x) \in \exp(V_\mathfrak{k})$ holds for all $g \in U$ and $x \in M$. Now it suffices to see that the map

$$U \to C^\infty(\mathfrak{k}), \quad g \mapsto (\exp|_{V_\mathfrak{k}})^{-1}(f(x)^{-1} f(g^{-1}.x))$$

is smooth, which follows from Theorem III.4 in [Ne00a]. This completes the proof of Step 2.

Step 3: For $f \in C^\infty(M,K)$ the set $f \cdot C^\infty(M,U)$ is an open neighborhood of $f$, and the arguments above show that the map

$$G \times \big(f \cdot C^\infty(M,U)\big) \to C^\infty(M,K), \quad (g.f, x) \mapsto (g.f) \cdot (g.x)$$

is smooth. This eventually proves that the action of $G$ on $C^\infty(M,K)$ is smooth.■

**Corollary II.7.** *Let $G$ act on $C^\infty(M,K)$ by a smooth action of $M$ as above. Then the semidirect product group $C^\infty(M,K) \rtimes G$ is a Lie group with Lie algebra $C^\infty(M,\mathfrak{k}) \rtimes \mathfrak{g}$.*

**Proof.** The preceding lemma implies that the inversion and the multiplication of this group are smooth maps, showing that the product manifold $C^\infty(M,K) \times G$ is a Lie group with the natural semidirect product structure. ■

**Corollary II.8.** *The natural rotation action of $\mathbb{T}_r$ on $LK$ is smooth, so that we obtain a semidirect product Lie group $L_r K := \mathbb{T}_r \ltimes LK$.* ■

### Appendix IIc: Lifting automorphisms to central extensions

In the following we will use the concept of an infinite-dimensional Lie group described in detail in [Ne00a]. In this context central extensions of Lie groups are always assumed to have a smooth local section. Let $Z \hookrightarrow \widehat{G} \twoheadrightarrow G$ be a central extension of the connected Lie group by the connected abelian group $Z$ which can be written as $\mathfrak{z}/\pi_1(Z)$. This means that $Z$ is a quotient of a sequentially complete locally convex space $\mathfrak{z}$ modulo a discrete subgroup which can then be identified with $\pi_1(Z)$. Since the quotient map $q: \widehat{G} \to G$ has a smooth local section, the corresponding Lie algebra homomorphism $\widehat{\mathfrak{g}} \to \mathfrak{g}$ has a continuous linear section, hence is defined by a continuous cocycle $\omega \in Z^2_c(\mathfrak{g},\mathfrak{z})$ in the sense that

$$\widehat{\mathfrak{g}} \cong \mathfrak{g} \oplus_\omega \mathfrak{z}$$

with the bracket $[(x,z),(x',z')] = ([x,x'],\omega(x,x'))$. From [Ne00b] we recall the period homomorphism $\mathrm{per}_\omega\colon \pi_2(G) \to \mathfrak{z}$ of $\omega$ which on smooth representatives $\gamma\colon \mathbb{S}^2 \to G$ of elements of $\pi_2(G)$ is given by $\mathrm{per}_\omega([\gamma]) = \int_{\mathbb{S}^2} \gamma^*\Omega$, where $\Omega$ is the $\mathfrak{z}$-valued left invariant 2-form on $G$ with $\Omega_{\mathbf{1}} = \omega$ ([Ne00b, Th. IV.12]).

We recall from [Ne00b, Prop. IV.2] that central Lie group extensions as above can always be written as

$$\widehat{G} \cong G \times_f Z,$$

where $f \in Z^2_s(G,Z)$, the group cocycles $f\colon G \times G \to Z$ which are smooth in a neighborhood of $(\mathbf{1},\mathbf{1})$. Two such cocycles $f_1$, $f_2$ define equivalent extensions if and only if their difference is of the form $h(gg')h(g)^{-1}h(g')^{-1}$, where $h\colon G \to Z$ is smooth in an identity neighborhood. The abelian group all these functions is called $B^2_s(G,Z)$, and the quotient group $H^2_s(G,Z) := Z^2_s(G,Z)/B^2_s(G,Z)$ now parametrizes the equivalence classes of central $Z$-extensions of $G$ with smooth local sections ([Ne00b, Remark IV.4]). On the Lie algebra level we likewise have the space $Z^2_c(\mathfrak{g},\mathfrak{z})$ of continuous linear 2-cocycles $\omega\colon \mathfrak{g}\times\mathfrak{g} \to \mathfrak{z}$, the subspace $B^2_c(\mathfrak{g},\mathfrak{z})$ of coboundaries, i.e., the cocycles of the form $(x,x') \mapsto \alpha([x,x'])$, where $\alpha\colon \mathfrak{g} \to \mathfrak{z}$ is a continuous linear map. The quotient space $H^2_c(\mathfrak{g},\mathfrak{z}) := Z^2_c(\mathfrak{g},\mathfrak{z})/B^2_c(\mathfrak{g},\mathfrak{z})$ classifies the central $\mathfrak{z}$-extensions of $\mathfrak{g}$ with continuous linear sections. For more details on all that we refer to [Ne00b].

For a Lie group $G$ we write $\mathrm{Aut}(G)$ for the group of Lie group automorphisms of $G$. We also define

$$\mathrm{Aut}_Z(\widehat{G}) := \{\gamma \in \mathrm{Aut}(\widehat{G})\colon f\,|_Z = \mathrm{id}_Z\}.$$

Then we have a natural homomorphism

$$\eta\colon \mathrm{Aut}_Z(\widehat{G}) \to \mathrm{Aut}(G), \quad \eta(\gamma)(q(g)) = q(\gamma(g)),$$

where $q\colon \widehat{G} \to G$ is the quotient map of the central extension.

**Lemma II.9.** *To each $f \in \mathrm{Hom}(G,Z)$ we assign the element of $\mathrm{Aut}_Z(\widehat{G})$ given by $\widehat{f}(g) := gf(q(g))$. Then*

$$\ker\eta = \{\widehat{f}\colon f \in \mathrm{Hom}(G,Z)\} \cong \mathrm{Hom}(G,Z).$$

**Proof.** For each $f \in \mathrm{Hom}(G,Z)$ we have

$$\widehat{f}(g_1g_2) = g_1g_2f(q(g_1)q(g_2)) = g_1f(q(g_1))g_2f(q(g_2)) = \widehat{f}(g_1)\widehat{f}(g_2),$$

showing that $\widehat{f} \in \mathrm{Aut}_Z(G)$. It is clear that $\widehat{f} \in \ker\eta$.

If, conversely, $\gamma \in \ker\eta$, then

$$f\colon G \to Z, \quad q(g) \mapsto \gamma(g)g^{-1}$$

is well defined, and it is easy to verify that $f$ is a group homomorphism with $\gamma = \widehat{f}$. ∎

**Corollary II.10.** *If $G$ has no nontrivial homomorphisms into abelian groups, then $\eta$ is injective.* ■

In view of [Ne00b, Cor. III.20], the preceding condition is equivalent to the density of the commutator algebra $[\mathfrak{g}, \mathfrak{g}]$ in $\mathfrak{g}$.

**Lemma II.11.** *Let $f \in Z^2_s(G, Z)$ be a group cocycle which is smooth in a neighborhood of $(\mathbf{1}, \mathbf{1})$ and for which $\widehat{G}$, as an abstract group, not as a manifold, is isomorphic to the group $G \times_f Z$ with the multiplication*

$$(g, z)(g', z') = (gg', zz'f(g, g'))$$

*and the map $G \to \widehat{G}, g \mapsto (g, \mathbf{1})$ is smooth in an identity neighborhood. Then $\gamma \in \mathrm{Aut}(G)$ is contained in the image of $\eta$ if and only if the cocycle $\gamma^* f := f \circ (\gamma, \gamma)$ is equivalent to $f$, i.e., there exists a function $h: G \to Z$, smooth on an identity neighborhood, such that*

$$(\gamma^* f)(g, g') f(g, g')^{-1} = h(gg')h(g)^{-1}h(g')^{-1}, \quad g, g' \in G. \tag{2.7}$$

**Proof.** Let us first assume that $\gamma = \eta(\widehat{\gamma})$. Writing $\widehat{G}$ as $G \times_f Z$, the automorphism $\widehat{\gamma}$ has the form

$$\widehat{\gamma}(g, z) = (\gamma(g), zh(g)),$$

where $h: G \to Z$ is a function which is smooth in an identity neighborhood. The condition that $\widehat{\gamma}$ is an automorphism of $\widehat{G}$ is equivalent to the relation (2.7).

If, conversely, (2.7) is satisfied, then the formula above defines an element $\widehat{\gamma} \in \mathrm{Aut}_Z(\widehat{G})$ which is smooth in an identity neighborhood, and since $\widehat{G}$ is connected, it is smooth on $\widehat{G}$, hence an isomorphism of Lie groups. ■

**Lemma II.12.** *If $\gamma \in \mathrm{Aut}(G)$ is contained in the range of $\eta$, then there exists a continuous linear map $\alpha: \mathfrak{g} \to \mathfrak{z}$ such that $(\gamma^*\omega)(x, y) := \omega(\gamma.x, \gamma.y)$ satisfies*

$$(\gamma^*\omega - \omega)(x, y) = \alpha([x, y]), \tag{2.8}$$

*i.e., $[\gamma^*\omega] = [\omega]$ in $H^2_c(\mathfrak{g}, \mathfrak{z})$. If $G$ is simply connected, then the preceding condition is also sufficient for $\gamma$ to be in the range of $\eta$.*

**Proof.** If $\gamma = \eta(\widehat{\gamma})$, then $\widehat{\gamma}$ induces an automorphism of the Lie algebra $\widehat{\mathfrak{g}} \cong \mathfrak{g} \oplus_\omega \mathfrak{z}$ with the bracket

$$[(x, z), (x', z')] = ([x, x'], \omega(x, x')).$$

Hence

$$\widehat{\gamma}.(x, z) = (\gamma.x, z + \alpha(x)),$$

where $\alpha: \mathfrak{g} \to \mathfrak{z}$ is a continuous linear map. The relation (2.8) means that such a map is a Lie algebra automorphism.

If, conversely, (2.8) is satisfied by $\gamma$, then we use the exact sequence for Lie group extensions to see that the natural map

$$\mathrm{Ext}(G,Z) \cong H^2_s(G,Z) \to \mathrm{Ext}(\mathfrak{g},\mathfrak{z}) \cong H^2_c(\mathfrak{g},\mathfrak{z})$$

is injective. Moreover, it is equivariant with respect to the action of $\mathrm{Aut}(G)$ on $H^2_s(G,Z)$, resp., $H^2_c(\mathfrak{g},\mathfrak{z})$. Therefore $[\gamma^*\omega] = [\omega]$ in $H^2_c(\mathfrak{g},\mathfrak{z})$ implies that $[\gamma^* f] = [f]$ in $H^2_s(G,Z)$ if $f \in Z^2_s(G,Z)$ represents $\widehat{G}$ as in Lemma II.11. ■

**Corollary II.13.** *If $G$ is simply connected and $\omega \in Z^2_c(\mathfrak{g},\mathfrak{z})$ is a Lie algebra cocycle corresponding to the Lie algebra extension $\mathfrak{z} \hookrightarrow \widehat{\mathfrak{g}} \twoheadrightarrow \mathfrak{g}$, then an automorphism $\gamma \in \mathrm{Aut}(G)$ lifts to an automorphism $\widehat{\gamma} \in \mathrm{Aut}_Z(\widehat{G})$ if and only if $[\gamma^*\omega] = [\omega]$, i.e., if the corresponding automorphism of $\mathfrak{g}$ lifts to an automorphism of $\widehat{\mathfrak{g}}$ fixing $\mathfrak{z}$ pointwise.*

**Proof.** This is a direct consequence of Lemma II.12. ■

If $G$ is not simply connected, then it has non-trivial central $Z$-extensions with trivial Lie algebra extension. These are discussed in the following lemma.

**Lemma II.14.** *If $\widehat{G}$ is of the form*

$$\widehat{G} = (\widetilde{G} \times Z)/\Gamma(\varphi^{-1}),$$

*where $q_G\colon \widetilde{G} \to G$ is the universal covering group of $G$, $\pi_1(G) \cong \ker q_G$ is identified with a subgroup of $\widetilde{G}$, $\varphi\colon \pi_1(G) \to Z$ is a homomorphism, and*

$$\Gamma(\varphi^{-1}) := \{(d, \varphi(d)^{-1})\colon d \in \pi_1(G)\}$$

*the graph of $\varphi^{-1}$ (pointwise inverse), then an automorphism $\gamma \in \mathrm{Aut}(G)$ is in the range of $\eta$ if and only if $(\varphi \circ \pi_1(\gamma)) \cdot \varphi^{-1}$ extends to a smooth homomorphism $\widetilde{G} \to Z$.*

**Proof.** Let $\widetilde{\gamma}$ be the canonical lift of $\gamma$ to $\widetilde{G}$ (cf. [Ne00b, Lemma II.3]). The canonical map $\widetilde{G} \times Z \to \widehat{G}$ is a covering, and $\widetilde{G} \times \mathfrak{z}$ is the universal covering group of $\widehat{G}$. Therefore, if $\gamma = \eta(\widehat{\gamma})$, the automorphism $\widehat{\gamma}$ also lifts to some automorphism $\widetilde{\gamma}$ of $\widetilde{G} \times Z$ preserving the subgroup $\Gamma(\varphi^{-1})$. Then $\widetilde{\gamma}$ is of the form

$$\widetilde{\gamma}(g,z) = (\widetilde{\gamma}_0(g), zf(g)),$$

with $f \in \mathrm{Hom}(\widetilde{G}, Z)$. The condition that $\widetilde{\gamma}$ preserves $\Gamma(\varphi^{-1})$ means that $f$ extends $\varphi \cdot (\varphi \circ \pi_1(\gamma))^{-1}$. If, conversely, this condition is satisfied, then the above formula yields an automorpism $\widetilde{\gamma}$ on $\widetilde{G} \times Z$ preserving $\Gamma(\varphi^{-1})$ and hence factoring to the quotient group $\widehat{G}$. ■

With Lemma II.14 one can easily construct examples showing that in Corollary II.13 the assumption that $G$ is simply connected is crucial.

**Example II.15.** (a) We consider $G = \mathrm{SL}(2,\mathbb{R})$ with the automorphism $\gamma(g) = JgJ$ for $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then $\pi_1(G) \cong \pi_1(\mathrm{SO}(2,\mathbb{R})) \cong \mathbb{Z}$ and $\pi_1(\gamma) = -\mathrm{id}_{\mathbb{Z}}$. On the other hand $\widetilde{G}$ is perfect, so that $\mathrm{Hom}(\widetilde{G}, Z)$ is trivial for every abelian group. Therefore $\varphi \in \mathrm{Hom}(\mathbb{Z}, Z)$ satisfies the condition from Lemma II.14 if and only if $\varphi(d)^2 = \mathbf{1}$ for all $d \in \mathbb{Z}$, i.e., $2\mathbb{Z} \subseteq \ker \varphi$.

If $Z = \mathbb{T}$ and $\varphi\colon \mathbb{Z} \to \mathbb{T}$ is injective, then we obtain a central $\mathbb{T}$-extension $\widehat{G}$ of $G$ whose corresponding Lie algebra extension is trivial, but $\gamma$ does not lift to an element of $\mathrm{Aut}_Z(\widehat{G})$.

(b) Let

$$\widehat{\mathfrak{g}} := \left\{ \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ 0 & 0 & a_{23} & a_{24} \\ 0 & 0 & 0 & a_{34} \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathfrak{gl}(4,\mathbb{R})\colon a_{jk} \in \mathbb{R} \right\}.$$

Then $\widehat{\mathfrak{g}}$ contains the ideals

$$\widehat{\mathfrak{g}}_1 := \left\{ \begin{pmatrix} 0 & 0 & a_{13} & a_{14} \\ 0 & 0 & 0 & a_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\} = [\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}] \quad \text{and} \quad \widehat{\mathfrak{g}}_2 := \left\{ \begin{pmatrix} 0 & 0 & 0 & a_{14} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\} = \mathfrak{z}(\widehat{\mathfrak{g}})$$

satisfying $[\widehat{\mathfrak{g}}_1, \widehat{\mathfrak{g}}] \subseteq \mathfrak{z}(\widehat{\mathfrak{g}})$. We define $\mathfrak{g} := \widehat{\mathfrak{g}}/\mathfrak{z}(\widehat{\mathfrak{g}})$ and consider the central extension $\mathfrak{z}(\mathfrak{g}) \hookrightarrow \widehat{\mathfrak{g}} \twoheadrightarrow \mathfrak{g}$. The Lie algebra $\mathfrak{g}$ is a 2-step nilpotent Lie algebra, and $\mathrm{ad}\,\mathfrak{g} \cong \mathbb{R}^3$ is abelian. On the other hand, the adjoint action of $\widehat{\mathfrak{g}}$ on $\widehat{\mathfrak{g}}$ factors through an action of $\mathfrak{g}$ on $\widehat{\mathfrak{g}}$, where the image $\mathrm{ad}_{\widehat{\mathfrak{g}}}\,\mathfrak{g}$ of this Lie algebra is isomorphic to $\mathfrak{g}$. This means that the action of $\mathrm{ad}\,\mathfrak{g} \cong \mathbb{R}^3$ on $\mathfrak{g}$ does not lift to an action of the same Lie algebra on $\widehat{\mathfrak{g}}$ because the central extension

$$\mathrm{Hom}(\mathfrak{g}, \mathfrak{z}(\mathfrak{g})) \hookrightarrow \mathrm{Hom}(\mathfrak{g}, \mathfrak{z}(\mathfrak{g})) + \mathrm{ad}_{\widehat{\mathfrak{g}}}(\mathfrak{g}) \twoheadrightarrow \mathrm{ad}\,\mathfrak{g}$$

is non-trivial. From this one obtains an example of an $\mathbb{R}^3$-action on a simply connected group $G$ which does not lift to an action on a central extension $\widehat{G}$, even though the action of every element can be lifted. ■

## Appendix IId: Lifting automorphic group actions to central extensions

In the preceding subsection we have lifted automorphisms of $G$ to automorphisms of $\widehat{G}$. Now we consider automorphic actions of groups $R$ on $G$ and want to lift those to actions on $\widehat{G}$.

**Lemma II.16.** *Let $Z^\sharp := \mathfrak{z}/\operatorname{im}(\operatorname{per}_\omega)$. Then there exists a central Lie group extension*

$$Z^\sharp \hookrightarrow G^\sharp \xrightarrow{q^\sharp} \widetilde{G}$$

*corresponding to the cocycle $\omega$, and $G^\sharp$ is a universal covering group of $\widehat{G}$.*

**Proof.** In view of [Ne00b, Th. V.7], $\operatorname{im}(\operatorname{per}_\omega)$ is a subgroup of the discrete group $\pi_1(Z) \subseteq \mathfrak{z}$, so that $Z^\sharp := \mathfrak{z}/\operatorname{im}(\operatorname{per}_\omega)$ is a covering group of $Z \cong \mathfrak{z}/\pi_1(Z)$. The relation $\pi_2(G) \cong \pi_2(\widetilde{G})$ and the criterion Theorem V.7 in [Ne00b] imply the existence of a central extension

$$Z^\sharp \hookrightarrow G^\sharp \xrightarrow{q^\sharp} \widetilde{G}$$

corresponding to the cocycle $\omega$. Now $\pi_2(Z) = \{\mathbf{1}\}$ and the exact homotopy sequence of the bundle $G^\sharp \to \widetilde{G}$ lead to an exact sequence

$$\pi_2(G^\sharp) \to \pi_2(\widetilde{G}) \xrightarrow{\ \delta\ } \pi_1(Z^\sharp) \twoheadrightarrow \pi_1(G^\sharp).$$

Since $\delta = -\operatorname{per}_\omega$ ([Ne00b, Prop. VII.7]), it is surjective, which implies that $G^\sharp$ is simply connected. On the other hand, the construction of $G^\sharp$ implies that $G^\sharp$ and $\widehat{G}$ are locally isomorphic ([Ne00b, Lemmas IV.8, V.8]), so that $G^\sharp$ is the universal covering group of $\widehat{G}$. ∎

We assume that we have a smooth automorphic action of the Lie group $R$ on $G$, which leads to a semidirect product Lie group $G \rtimes R$. We are looking for sufficient conditions to lift the smooth action of $R$ on $G$ to a smooth action on $\widehat{G}$ which apply in particular to the rotation action of $\mathbb{T}_r$ on $LK$, where $K$ is a compact simple simply connected Lie group.

**Lemma II.17.** *The action of $R$ on $G$ lifts to a smooth action of $R$ on the simply connected covering group $\widetilde{G}$ of $G$.*

**Proof.** Since each automorphism of $G$ lift in a unique fashion to an automorphism of $\widetilde{G}$, the action of $R$ on $G$ directly leads to an action of $R$ on $\widetilde{G}$. That the action map is smooth follows easily by using local sections of the universal covering map $q_G \colon \widetilde{G} \to G$. ∎

**Theorem II.18.** (Lifting Theorem) *Let $\sigma_G \colon R \times G \to G$ be a smooth automorphic action of the Lie group $R$ on the connected Lie group $G$. Assume that $G$ is simply connected and that there exists a smooth function $\alpha \colon R \times \mathfrak{g} \to \mathfrak{z}$ with*

$$r^*\omega - \omega = \alpha(r, [\cdot,\cdot]), \quad r \in R \tag{2.9}$$

*and the cocycle condition*

$$\alpha(r_1 r_2, x) = \alpha(r_2, x) + \alpha(r_1, r_2.x), \quad r_1, r_2 \in R, x \in \mathfrak{g}. \tag{2.10}$$

*Then the action of $R$ on $G$ lifts uniquely to a smooth automorphic action of $R$ on $\widehat{G}$ such that the corresponding action of $R$ on $\widehat{\mathfrak{g}} \cong \mathfrak{g} \oplus_\omega \mathfrak{z}$ is given by*

$$r.(x,z) = (r.x, z + \alpha(r,x)), \quad r \in R, x \in \mathfrak{g}, z \in \mathfrak{z}.$$

*This action fixes the subgroup $Z$ of $\widehat{G}$ pointwise.*

**Proof.** First we turn to the action on the Lie algebra $\widehat{\mathfrak{g}}$. We define the a smooth map

$$\sigma_{\widehat{\mathfrak{g}}}\colon R \times \widehat{\mathfrak{g}} \to \widehat{\mathfrak{g}}, \quad (r,(x,z)) \mapsto (r.x, z + \alpha(r,x)).$$

Then (2.9) implies that each map $\sigma_{\widehat{\mathfrak{g}}}(r,\cdot)$ is an automorphism of $\widehat{\mathfrak{g}}$:

$$\begin{aligned} r.[(x,z),(x',z')] &= r.\big([x,x'],\omega(x,x')\big) = \big([r.x,r.x'],\omega(x,x') + \alpha(r,[x,x'])\big) \\ &= ([r.x,r.x'],\omega(r.x,r.x') = [r.(x,z), r.(x',z')], \end{aligned}$$

and (2.10) implies that $\sigma_{\widehat{\mathfrak{g}}}$ is an action of $R$ on $\widehat{\mathfrak{g}}$:

$$\begin{aligned} r_1.(r_2.(x,z)) &= ((r_1r_2).x, z + \alpha(r_2,x) + \alpha(r_1, r_2.x)) = ((r_1r_2).x, z + \alpha(r_1r_2,x)) \\ &= (r_1r_2).(x,z). \end{aligned}$$

Since every automorphism of $\widehat{G}$ is uniquely determined by the corresponding automorphism of $\widehat{\mathfrak{g}}$ (cf. [Mi83, Lemma 7.1]), there exists at most one automorphic action $\sigma_{\widehat{G}}$ of $R$ on $\widehat{G}$ corresponding to $\sigma_{\widehat{\mathfrak{g}}}$.

Now we reduce the problem to the case where $\widehat{G}$ is simply connected. Suppose that the theorem holds for the special situation where $\widehat{G}$ is simply connected. Then we consider the simply connected central extension $G^\sharp$ of $G$ discussed in Lemma II.16. The assertion of the theorem shows that the action of $R$ on $G$ lifts to a smooth action of $R$ on $G^\sharp$ fixing the elements of $Z^\sharp$ pointwise. Since $G$ is simply connected, the exact homotopy sequence of the bundle $q\colon \widehat{G} \to G$ implies that the natural map $\zeta\colon \pi_1(Z) \to \pi_1(\widehat{G})$ is surjective. Identifying $\pi_1(\widehat{G})$ with the kernel of the universal covering map $G^\sharp \to \widehat{G}$, we see that $\zeta$, viewed as a homomorphism $\pi_1(Z) \to G^\sharp$, is a composition of the maps

$$\pi_1(Z) \to \pi_1(Z)/\pi_1(Z^\sharp) \subseteq Z^\sharp \to G^\sharp.$$

Therefore the subgroup $\pi_1(\widehat{G})$ of $G^\sharp$ is contained in $Z^\sharp$ and therefore fixed pointwise by $R$. Hence the action of $R$ on $G^\sharp$ factors to a smooth action of $R$ on $\widehat{G} \cong G^\sharp/\pi_1(\widehat{G})$. In view of the preceding argument, we may from now on assume that $\widehat{G}$ is simply connected.

Next we consider the local situation in a suitable small neighborhood of the identity in $\widehat{G}$. In $\widehat{G}$ we have an open $\mathbf{1}$-neighborhood of the form $U \times Z \subseteq \widehat{G}$, where the multiplication is given for $x, x', xx' \in U$ by

$$(x,z)(x',z') = (xx', zz'f^Z(x,x'))$$

for a local smooth cocycle $f^Z: U \times U \to Z$ ([Ne00b, Lemma IV.8]). To see how this description of the multiplication can be used to obtain an action of $R$, we have to recall the construction of the local cocycle $f^Z$ from the Lie algebra cocycle $\omega$ (cf. [Ne00b, Lemma IV.8]). Let us assume that, in addition, $U \subseteq G$ is diffeomorphic to a convex subset in $\mathfrak{g}$. Using the Poincaré Lemma ([Ne00b, Lemma III.3]), we write $\Omega|_U = d\theta$ for a $\mathfrak{z}$-valued 1-form $\theta$ on $U$ with $\theta_{\mathbf{1}} = 0$. Then, on an open symmetric $\mathbf{1}$-neighborhood $W \subseteq U$ with $W^2 \subseteq U$ and also diffeomorphic to a convex set, we determine the function

$$f: W \times W \to \mathfrak{z}$$

by

$$df(x, \cdot) = \lambda_x^* \theta|_W - \theta|_W.$$

Now let $r \in R$ and $W_1 \subseteq W$ be open and diffeomorphic to a convex set with $r.W_1 \subseteq W$. Let $\alpha_r$ be the left invariant $\mathfrak{z}$-valued 1-form on $G$ with $\alpha_r(\mathbf{1}) = \alpha(r, \cdot)$. Then (2.9) implies that

$$r^*\Omega - \Omega = -d\alpha_r.$$

On $W_1$ we therefore have $d(r^*\theta - \theta + \alpha_r) = 0$, so that there exists a unique function $h_r: W_1 \to \mathfrak{z}$ with $h_r(\mathbf{1}) = 0$ and $dh_r = r^*\theta - \theta + \alpha_r$.

On $W_1 \times W_1$ we consider the function $(r^*f)(x, y) := f(r.x, r.y)$ and put $f_x := f(x, \cdot)$. Then $(r^*f)_x = r^*(f_{r.x})$, so that on $W_1$ we have

$$d\big((r^*f)_x\big) = r^*df_{r.x} = r^*(\lambda_{r.x}^*\theta - \theta) = \lambda_x^* r^*\theta - r^*\theta.$$

Now the left invariance of $\alpha_r$ leads to

$$\begin{aligned} d((r^*f)_x - f_x) &= \lambda_x^*(r^*\theta - \theta) - (r^*\theta - \theta) \\ &= \lambda_x^*(r^*\theta - \theta + \alpha_r) - (r^*\theta - \theta + \alpha_r) \\ &= \lambda_x^* dh_r - dh_r = d(\lambda_x^* h_r - h_r). \end{aligned}$$

In view of the normalizations $f_x(\mathbf{1}) = 0 = h_r(\mathbf{1})$, we therefore obtain

$$(r^*f)(x, y) - f(x, y) = h_r(xy) - h_r(y) - h_r(x) \tag{2.11}$$

for $x, y$ near to $\mathbf{1}$.

Let $q_Z: \mathfrak{z} \to Z$ be the quotient map, $f^Z := q_Z \circ f$ and $h_r^Z := q_Z \circ h_r$. Then we have a $\mathbf{1}$-neighborhood of the form $W_2 \times Z$ in $\widehat{G}$, where $W_2 \subseteq W_1$, and the multiplication is given by

$$(g, z)(g', z') = (gg', zz' f^Z(g, g')).$$

Pick an open symmetric connected $\mathbf{1}$-neighborhood $W_3 \subseteq W_2$ with $r.W_3 \subseteq W_2$ and (2.11) holding on $W_3$. Then the map

$$\sigma_{\widehat{G}}(r): W_3 \times Z \to W_2 \times Z \subseteq \widehat{G}, \quad (g, z) \mapsto (r.g, z h_r^Z(g)) \tag{2.12}$$

is a smooth homomorphism of local groups. Using Lemma II.3 in [Ne00b] and the simple connectedness of $\widehat{G}$, we see that $\sigma_{\widehat{G}}(r)$ extends to a smooth homomorphism $\widehat{G} \to \widehat{G}$. The derivative of this automorphism in $\mathbf{1} \in \widehat{G}$ is given by

$$\begin{aligned} d\sigma_{\widehat{G}}(r)(\mathbf{1})(x,z) &= (r.x, z + dh_r^Z(\mathbf{1})(x)) = (r.x, z + dh_r(\mathbf{1})(x)) \\ &= (r.x, z + \alpha(r,x) + \theta(\mathbf{1})(r.x) - \theta(\mathbf{1})(x)) \\ &= (r.x, z + \alpha(r,x)) = \sigma_{\widehat{\mathfrak{g}}}(r,x). \end{aligned}$$

This proves that every automorphism $\sigma_{\widehat{\mathfrak{g}}}(r)$, $r \in R$, integrates to a smooth endomorphism $\sigma_{\widehat{G}}(r)$ of $\widehat{G}$. The uniqueness of this extension implies that $\sigma_{\widehat{G}}(r_1 r_2) = \sigma_{\widehat{G}}(r_1)\sigma_{\widehat{G}}(r_2)$ for $r_1, r_2 \in R$, hence in particular that each $\widehat{\sigma}_{\widehat{G}}(r)$ is an automorphism of $\widehat{G}$. Let $\sigma_{\widehat{G}}\colon R \times \widehat{G} \to \widehat{G}$ denote the corresponding action of $R$ on $\widehat{G}$.

It remains to show that this action is smooth. Since $R$ acts by smooth automorphism on $\widehat{G}$, it suffices to show that the action is smooth in a neighborhood of $(\mathbf{1},\mathbf{1})$ and that all orbits maps $R \to \widehat{G}$ are smooth in a neighborhood of $\mathbf{1}$. Since the latter property can be derived from the first one, it remains to see that the action is smooth in a neighborhood of $(\mathbf{1},\mathbf{1})$. To this end, we slightly adjust the choices of $W_1$ and $W_3$ above. First we choose an open $\mathbf{1}$-neighborhood $V$ in $R$ and $W_1$ such that, in addition, $V.W_1 \subseteq W$. Likewise we choose $V_1 \subseteq V$ and $W_3 \subseteq W_2$ with $V_1.W_3 \subseteq W_2$. Then the function $(r,x) \mapsto h_r(x)$ is defined on $V \times W_1$, and the construction of $h_r$ with the Poincaré Lemma implies that this function is smooth in a neighborhood of $(\mathbf{1},\mathbf{1})$ (cf. [Ne00b, Lemma III.3]). This implies that the action map $\sigma_{\widehat{G}}(r,x)$ is smooth on a neighborhood of $(\mathbf{1},\mathbf{1})$ contained in $V_1 \times W_3$, and this completes the proof. ■

**Corollary II.19.** *Let $\sigma_G\colon R \times G \to G$ be a smooth automorphic action of the Lie group $R$ on the connected Lie group $G$. Assume that $G$ is simply connected and that $r^*\omega = \omega$ for all $r \in R$. Then the action of $R$ on $G$ lifts uniquely to a smooth automorphic action of $R$ on $\widehat{G}$ such that the corresponding action of $R$ on $\widehat{\mathfrak{g}} \cong \mathfrak{g} \oplus_\omega \mathfrak{z}$ is given by*

$$r.(x,z) = (r.x, z), \quad r \in R, x \in \mathfrak{g}, z \in \mathfrak{z}.$$

*This action fixes the subgroup $Z$ of $\widehat{G}$ pointwise.*

**Proof.** We apply Theorem II.18 with $\alpha = 0$. ■

**Corollary II.20.** *If $K$ is a compact simple simply connected Lie group and $\widetilde{L}K$ is the central extension from* Theorem II.5*, then the rotation action of $\mathbb{T}_r$ on $LK$ lifts to a smooth action of $\mathbb{T}_r$ on $\widetilde{L}K$. The same holds for the complex groups $LK_{\mathbb{C}}$, resp., $\widetilde{L}K_{\mathbb{C}}$.*

**Proof.** Since $\pi_2(K) = \pi_1(K) = \{\mathbf{1}\}$, the group $LK$ is simply connected. Further the $\mathbb{T}_r$-action on $L\mathfrak{k}$ fixes the cocycle $\omega$, so that Corollary II.19 applies. ■

**Remark II.21.** (a) If the commutator algebra $[\mathfrak{g}, \mathfrak{g}]$ is dense in $\mathfrak{g}$ ($\mathfrak{g}$ is topologically perfect), then in (2.9) the continuous linear map $\alpha_r := \alpha(r, \cdot): \mathfrak{g} \to \mathfrak{z}$ is uniquely determined by $r^*\omega - \omega = -d\alpha_r$. Therefore

$$-d\alpha_{r_1 r_2} = (r_1 r_2)^*\omega - \omega = r_2^*(r_1^*\omega - \omega) + r_2^*\omega - \omega = -r_2^* d\alpha_{r_1} - d\alpha_{r_2}$$

implies the relation (2.10). In this sense (2.10) is only needed if $\mathfrak{g}$ is not topologically perfect.
(b) If $\widehat{G}$ is a regular Lie group in the sense of [Mi83], then every automorphism of $\widehat{\mathfrak{g}}$ integrates to a unique automorphism of $\widehat{G}$ ([Mi83, Th. 8.1]). In our context it does not make sense to work with this additional assumption because we anyway need the more explicit information obtained in the proof of Theorem II.18 to show that the action is smooth.
(c) We consider the situation of Corollary II.19, where $\omega$ is fixed by $R$. Then we have a Lie group $G_R := G \rtimes R$ defined by the smooth action of $R$ on $G$, and Corollary II.19 provides a central extension $\widehat{G}_R \cong \widehat{G} \rtimes R$. It is interesting to try to construct this group and therefore the smooth action of $R$ on $\widehat{G}$ more directly as a central Lie group extension.

We write $\widehat{\mathfrak{g}} = \mathfrak{g} \oplus_\omega \mathfrak{z}$, and since $\omega$ is $R$-invariant, we obtain a natural action of $R$ on $\widehat{\mathfrak{g}}$ by $r.(x, z) := (r.x, z)$. The derived representation of this smooth action leads to a continuous action $\mathfrak{r} \times \widehat{\mathfrak{g}} \to \widehat{\mathfrak{g}}$ by derivations (cf. [Ne00a]). Let $\widehat{\mathfrak{g}}_R := \widehat{\mathfrak{g}} \rtimes \mathfrak{r}$ denote the corresponding Lie algebra. Then $\mathfrak{z} \subseteq \widehat{\mathfrak{g}}_R$ is central with the topological complement $\mathfrak{g} \times \mathfrak{r}$, and the corresponding cocycle of the central extension $\widehat{\mathfrak{g}}_R \to \mathfrak{g}_R$ is given by

$$\omega_R(x + y, x' + y') := \omega(x, x'), \quad x, x' \in \mathfrak{g}, y, y' \in \mathfrak{r}.$$

That this formula defines a cocycle can also verified more directly. It is clear that $\omega_R$ is continuous and skew symmetric. To see that it is a cocycle, we have to verify that for $a, b, c \in \mathfrak{g}_R$ the alternating expression

$$\omega_R(a, [b, c]) + \omega_R(b, [c, a]) + \omega_R(c, [a, b])$$

vanishes. For $a, b, c \in \mathfrak{g}$ this follows from the cocycle property of $\omega$. If $a \in \mathfrak{r}$ and $b, c \in \mathfrak{g}$, then

$$\omega_R(a, [b, c]) + \omega_R(b, [c, a]) + \omega_R(c, [a, b]) = \omega(b, [c, a]) + \omega([b, a], c) = -(a.\omega)(b, c) = 0$$

because $\omega$ is $R$-invariant, which implies that $\mathfrak{r}.\omega = 0$ in $Z_c^2(\mathfrak{g}, \mathfrak{z})$. If two or three among $a, b, c$ are in $\mathfrak{r}$, then each summand vanishes. Hence $\omega_R \in Z_c^2(\mathfrak{g}_R, \mathfrak{z})$.

One now might try to construct a group $\widehat{G} \rtimes R$ as a central extension of $G_R$ corresponding to $\omega_R$. Since $\pi_2(G_R) \cong \pi_2(G) \times \pi_2(R)$ and the restriction of the left invariant 2-form $\Omega_R$ to the subgroup $R$ of $G_R$ vanishes, the image of the period map $\mathrm{per}_{\omega_R}: \pi_2(G_R) \to \mathfrak{z}$ coincides with the image of $\mathrm{per}_\omega$, hence is contained in $\pi_1(Z)$. What is not clear in this situation is how to lift the $G$-action on $\mathfrak{g}$ to an action on $\widehat{\mathfrak{g}}_R$. If this could be done, then the criteria in [Ne00b] would imply the existence of a central Lie group extension of $G_R$ corresponding to the Lie algebra $\widehat{\mathfrak{g}}_R$. ■

## III. Root decompositions

In this section we will consider the root decomposition of the Lie algebra $\widehat{L}\mathfrak{k}_{\mathbb{C}}$. We will use the same notation as in Section I for the finite-dimensional Lie algebra $\mathfrak{k}$, we write $\mathfrak{t}_{\mathfrak{k}}$ for a Cartan subalgebra, and $T_K = \exp \mathfrak{t}_{\mathfrak{k}}$ for the corresponding maximal torus.

Let $z^k\colon \mathbb{S}^1 \to \mathbb{C}$ denote the function given by $z^k(\theta) := e^{ik\theta}$ and note that for $X \in \mathfrak{k}_{\mathbb{C}}$ the function $z^k X\colon \mathbb{S}^1 \to \mathfrak{k}_{\mathbb{C}}$ defines an element of $L\mathfrak{k}_{\mathbb{C}}$. The elements of the subalgebra

$$L_{\mathrm{pol}}\mathfrak{k}_{\mathbb{C}} := \mathbb{C}[z, \tfrac{1}{z}] \otimes \mathfrak{k}_{\mathbb{C}} = \sum_{k\in\mathbb{Z}} z^k \mathfrak{k}_{\mathbb{C}}$$

are called *polynomial loops* (with values in $\mathfrak{k}_{\mathbb{C}}$).

**Lemma III.1.** *$L_{\mathrm{pol}}\mathfrak{k}_{\mathbb{C}}$ is a dense subalgebra of $L\mathfrak{k}_{\mathbb{C}}$. Moreover, for $\gamma \in L\mathfrak{k}_{\mathbb{C}}$ its Fourier series*

$$\gamma = \sum_{n\in\mathbb{Z}} z^n \widehat{\gamma}(n), \qquad \textit{where} \qquad \widehat{\gamma}(n) = \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta}\gamma(\theta)\, d\theta$$

*converges in the Fréchet topology of $L\mathfrak{k}_{\mathbb{C}}$.*

**Proof.** It is clear that the second assertion is much sharper than the first one, so it suffices to prove the second one.

It is easily seen that the rotation action of $\mathbb{T}_r$ on $L\mathfrak{k}_{\mathbb{C}}$ defines a smooth action, i.e., for each element $\gamma \in L\mathfrak{k}_{\mathbb{C}}$ the orbit map $\mathbb{T}_r \to L\mathfrak{k}_{\mathbb{C}}$ is smooth (cf. [Ne00a, Th. III.5]). Therefore the convergence of the Fourier series follows from Harish-Chandra's Theorem ([Wa72, Th. 4.4.2.1]) because $\gamma \mapsto z^n\widehat{\gamma}(n)$ is the projection onto an isotypical $\mathbb{T}_r$-submodule. The latter fact follows directly from the observation that for $\gamma = z^m X$ we have

$$\widehat{\gamma}(n) = \frac{1}{2\pi}\int_0^{2\pi} e^{im\theta}e^{-in\theta}\, d\theta X = \delta_{n,m}X. \qquad \blacksquare$$

**Remark III.2.** A similar assertion as in Lemma III.1 applies to the $\mathbb{T}_r$-action on the extended Lie algebra $\widetilde{L}\mathfrak{k}_{\mathbb{C}}$. ■

The subgroup $T_r := \mathbb{T}_r \times T_K \subseteq L_r K$ acts on $L\mathfrak{k}_{\mathbb{C}}$ via the adjoint action. To identify the corresponding weights, we identify the character group $\widehat{T_r}$ with

$$\widehat{\mathbb{T}_r} \times \widehat{T_K} \cong \mathbb{Z} \times \widehat{T_K} \subseteq \mathbb{Z} \times \mathfrak{t}^*$$

as in Section I. Then the set of weights occuring in the adjoint representation is given by

$$\{(k,\alpha)\colon k \in \mathbb{Z}, \alpha \in \Delta_{\mathfrak{k}} \cup \{0\}\},$$

where

$$(L\mathfrak{k}_{\mathbb{C}})^{(k,\alpha)} = z^k \mathfrak{k}_{\mathbb{C}}^{\alpha}, \quad \alpha \in \Delta_{\mathfrak{k}}, \quad \text{and} \quad (L\mathfrak{k}_{\mathbb{C}})^{(k,0)} = z^k \mathfrak{t}_{\mathbb{C}}.$$

We write $\Delta_{L\mathfrak{k}} := \{(k,\alpha) \neq (0,0) \colon k \in \mathbb{Z}, \alpha \in \Delta_{\mathfrak{k}} \cup \{0\}\}$ for the set of *roots of* $L\mathfrak{k}_{\mathbb{C}}$ with respect to $\mathfrak{t}_r \oplus \mathfrak{t}$. The Lie algebra $\mathfrak{t}_{L\mathfrak{k}} := \mathfrak{t}_r \oplus \mathfrak{t}$ can be identified with the space $\mathbb{R}\frac{d}{dt} \oplus \mathfrak{t} \cong \mathbb{R} \oplus \mathfrak{t}$, where $\frac{d}{dt} = (1,0)$, and accordingly we have $R_\theta = \exp(-\theta\frac{d}{dt})$, so that

$$R_\theta.z^k = e^{-ik\theta}.z^k \quad \text{and} \quad \frac{d}{dt}z^k = ikz^k.$$

### The Weyl group

Let $\mathcal{W} := N_K(T)/T$ denote the *Weyl group* of $K$. Similarly we define the *affine Weyl group*

$$\mathcal{W}_{\text{aff}} := N_{L_rK}(T_r)/T_r,$$

and note that $T_r = Z_{L_rK}(T_r)$, i.e., $T_r$ is maximal abelian in $L_rK$. In fact, a loop commuting with $\mathbb{T}_r$ must be constant, and if, in addition, it commutes with $T_K$, the fact that $T = Z_K(T)$ implies that its value is in $T$.

Since the group $\check{T}_K := \text{Hom}(\mathbb{T}, T_K)$ consists in particular of smooth loops, it can be identified with a subgroup of $LT_K \subseteq L_rK$.

**Proposition III.3.** *The group* $\mathcal{W}_{\text{aff}}$ *is isomorphic to the semidirect product* $\check{T}_K \rtimes \mathcal{W}$*, and it acts on* $\mathfrak{t}_{L\mathfrak{k}}$ *via*

$$(Z,w).(t,X) = (t, w.X - tZ)$$

*for* $Z \in \check{T}_K$ *and* $w \in \mathcal{W}$*.*

**Proof.** The first part follows from [PS86, p.71]. To verify the formula for the action, it clearly suffices to compute the action of elements of $\check{T}_K$. So let $\gamma \in \check{T}_K$ be given by $\gamma(\theta) = \exp(\theta Z)$, $Z \in \mathfrak{t}_{\mathfrak{k}}$.

We have

$$\gamma \exp(t,0)\gamma^{-1} = \gamma R_{-t}\gamma^{-1} = R_{-t}(R_t\gamma R_{-t}\gamma^{-1}) = \exp(t,0)((R_t.\gamma)\gamma^{-1})$$

with

$$((R_t.\gamma)\gamma^{-1})(\theta) = \gamma(\theta - t)\gamma^{-1}(\theta) = \gamma(\theta - t)\gamma(-\theta) = \gamma(-t) = \exp(-tZ).$$

Thus $\text{Ad}(\gamma).(t,0) = (t, -tZ)$, and the assertion follows. ∎

It follows in particular from Proposition III.3 that the affine subspace $\{1\} \times \mathfrak{t}_{\mathfrak{k}}$ is invariant under the action of the affine Weyl group, and that the action on this hyperplane induces an affine action on $\mathfrak{t}_{\mathfrak{k}}$ given by

$$(Z, w).X = w.X - Z.$$

Note that $Z \in \check{T}_K \subseteq \mathfrak{t}_{\mathfrak{k}}$ acts by translation in the opposite direction. Then the kernel of a root $(k, \alpha) \in \Delta_{LK}$ corresponds to the affine hyperplane

$$H_{k,\alpha} := \{X \in \mathfrak{t}_{\mathfrak{k}} \colon \alpha(X) = -k\}.$$

The set $\bigcup_{\alpha\in\Delta_{\mathfrak{k}},k\in\mathbb{Z}} H_{k,\alpha}$ is called the *diagram of* $LK$. If $\Delta_{\mathfrak{k}}^+ \subseteq \Delta_{\mathfrak{k}}$ is a positive system, then the set

$$C_0 := \{X \in \mathfrak{t}_{\mathfrak{k}} : (\forall \alpha \in \Delta_{\mathfrak{k}}^+)\, 0 < \alpha(X) < 1\}$$

is called a *fundamental alcove*. It is a fundamental domain for the affine action of $\mathcal{W}_{\text{aff}}$ on $\mathfrak{t}_{\mathfrak{k}}$ ([PS86, Prop. 5.1.4]). A root $\underline{\alpha} = (\alpha, k)$ is said to be *positive* if $\underline{\alpha}(\{1\} \times C_0) \subseteq \mathbb{R}^+$ which is equivalent to $\alpha(C_0) \subseteq [-k, \infty[$. Therefore the set $\Delta_{L\mathfrak{k}}^+$ of positive roots is given by

$$\Delta_{L\mathfrak{k}}^+ = \{(k, \alpha) \colon (k > 0) \text{ or } (k = 0, \alpha \in \Delta_{\mathfrak{k}}^+)\}.$$

A root $\underline{\alpha} = (k, \alpha)$ is called *simple* if $H_{k,\alpha}$ contains a *wall* of the fundamental alcove $C_0$. To each root $\underline{\alpha} = (k, \alpha)$ corresponds an affine reflection on $\mathfrak{t}_{\mathfrak{k}}$ in the hyperplane $H_{k,\alpha}$ which is given by the formula

$$s_{\underline{\alpha}}(X) = s_\alpha(X) - kh_\alpha, \quad \text{i.e.,} \quad s_{\underline{\alpha}} = (kh_\alpha, s_\alpha) \in \widehat{T}_K \rtimes \mathcal{W}$$

is the decomposition according to Proposition III.3.

**Remark III.4.** If $\mathfrak{k}$ is simple, $\alpha_1, \ldots, \alpha_r$ are the fundamental roots of $\mathfrak{k}$ with respect to $\Delta_{\mathfrak{k}}^+$, and $\alpha_0 \in \Delta_{\mathfrak{k}}^+$ is the highest root, then

$$(0, \alpha_1), \ldots, (0, \alpha_r), (1, -\alpha_0)$$

is a system of simple roots in $\Delta_{L\mathfrak{k}}^+$ (cf. [PS86, p. 73]). ■

### Root decomposition of the central extension

We write

$$T_{\widehat{LK}} := \mathbb{T}_r \times T_K \times \mathbb{T}_c \subseteq \widehat{L}K,$$

for the maximal torus in $\widehat{L}K$, where $\mathbb{T}_c$ stands for the central torus defining the central extension from Theorem II.5. We identify the character group $\widehat{T}_{\widehat{L}K}$ of $T_{\widehat{L}K}$ with $\widehat{\mathbb{T}}_r \times \widehat{T}_K \times \widehat{\mathbb{T}}_c \cong \mathbb{Z} \times \widehat{T}_K \times \mathbb{Z} \subseteq \mathbb{Z} \times \mathfrak{t}_{\mathfrak{k}}^* \times \mathbb{Z}$. Since $\mathbb{T}_c$ acts trivially via the adjoint action, we can identify the root system $\Delta_{L\mathfrak{k}} \cong \Delta_{L\mathfrak{k}} \times \{0\}$ with a subset of $\mathfrak{t}_{L_r\mathfrak{k}}^*$.

Identifying $L\mathfrak{k}_{\mathbb{C}}$ with a vector subspace of $\widetilde{L}\mathfrak{k}_{\mathbb{C}}$, we have

$$(\widetilde{L}\mathfrak{k}_{\mathbb{C}})^{(k,\alpha)} = z^k \mathfrak{k}_{\mathbb{C}}^{\alpha}, \quad \alpha \in \Delta_{\mathfrak{k}}, \quad \text{and} \quad (\widetilde{L}\mathfrak{k}_{\mathbb{C}})^{(k,0)} = z^k \mathfrak{t}_{\mathbb{C}}.$$

Via the adjoint representation the affine Weyl group $\mathcal{W}_{\mathrm{aff}}$ can also be identified with $N(T_{\widehat{L}K})/T_{\widehat{L}K}$. It acts on $\widehat{T}_{\widehat{L}K}$ by

$$Z.(k, \lambda, h) = \Big(k + \lambda(Z) + \tfrac{1}{2} h\kappa(Z, Z), \lambda + hZ^*, h\Big)$$

for $Z \in \check{T}_K$, where $Z^* \in \mathfrak{t}_{\mathfrak{k}}$ is defined by the embedding $\check{T}_K \to \widehat{T}_K$, $Z \mapsto Z^*$ which is well defined because $\widehat{T}_K \cong \mathrm{Hom}(\check{T}_K, \mathbb{Z})$ and $\kappa(\check{T}_K, \check{T}_K) \subseteq \mathbb{Z}$ ([PS86, Prop. 4.9.5]).

Let $\underline{\alpha} = (k, \alpha)$ denote a root of $\widetilde{L}\mathfrak{k}_{\mathbb{C}}$. We put

$$e_{\underline{\alpha}} := z^k e_\alpha \quad \text{and} \quad e_{-\underline{\alpha}} := \overline{e_{\underline{\alpha}}} = z^{-k} e_{-\alpha}.$$

Then

$$[e_{\underline{\alpha}}, e_{-\underline{\alpha}}] = \big([e_\alpha, e_{-\alpha}], \omega(e_{\underline{\alpha}}, e_{-\underline{\alpha}})\big),$$

where

$$\begin{aligned} \omega(e_{\underline{\alpha}}, e_{-\underline{\alpha}}) &= \frac{1}{2\pi} \int_0^{2\pi} \kappa(z^k e_\alpha, (-ik) z^{-k} e_{-\alpha}) \, d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} (-ik) \kappa(e_\alpha, e_{-\alpha}) \, d\theta = -i \frac{k}{2} \kappa(h_\alpha, h_\alpha) \end{aligned}$$

(Lemma I.1). Therefore

$$[e_{\underline{\alpha}}, e_{-\underline{\alpha}}] = i\Big(h_\alpha, -\tfrac{k}{2}\kappa(h_\alpha, h_\alpha)\Big).$$

Putting

$$h_{\underline{\alpha}} := \Big(h_\alpha, -\tfrac{k}{2}\kappa(h_\alpha, h_\alpha)\Big),$$

we see that $(e_{\underline{\alpha}}, e_{-\underline{\alpha}}, h_{\underline{\alpha}})$ is a basic $\mathfrak{su}_2$-triple (cf. Section I). The corresponding reflection $s_{\underline{\alpha}}$ acts on $(\mathfrak{t}_{\widehat{L}K})^*$ via

$$s_{\underline{\alpha}}(\underline{\lambda}) = \underline{\lambda} - \langle \underline{\lambda}, h_{\underline{\alpha}} \rangle \underline{\alpha}.$$

We write $\widetilde{L}K(\underline{\alpha})$ for the corresponding three-dimensional subgroup of $\widetilde{L}K$.

Having introduced the relevant notation, we can give a simple proof for the necessity of condition (2.4).

**Lemma III.5.** *The condition $\kappa(h_\alpha, h_\alpha) \in 2\mathbb{Z}$ for all $\alpha \in \Delta_{\mathfrak{k}}$ is necessary for the existence of the central extension $\widetilde{L}K$.*

**Proof.** Suppose that we have a central extension

$$\{\mathbf{1}\} \to \mathbb{R} \longrightarrow \widetilde{L}K \xrightarrow{\ q\ } LK \to \{\mathbf{1}\}$$

corresponding to the given central extension on the level of Lie algebras. For each root $\underline{\alpha} \in \Delta_{L\mathfrak{k}}$ the corresponding three-dimensional subgroup $\widetilde{L}K(\underline{\alpha})$ is compact and isomorphic to $\mathrm{SU}(2)$ or $\mathrm{SO}(3,\mathbb{R})$. Therefore

$$\mathbf{1} = \exp 2\pi h_{\underline{\alpha}} = \Big(\exp 2\pi h_\alpha, \exp(-k\pi\kappa(h_\alpha, h_\alpha))\Big).$$

If we apply this to a root $\underline{\alpha} = (0, \alpha)$, we see that $\exp(2\pi h_{\underline{\alpha}}) = \mathbf{1}$, so that $\exp(-k\pi\kappa(h_\alpha, h_\alpha)) = 1$ holds for all $k \in \mathbb{N}$. This implies that $\kappa(h_\alpha, h_\alpha) \in 2\mathbb{Z}$. ∎

## IV. Representations of loop groups

**Definition IV.1.** Let $V$ be a complete complex locally convex space. Then a *representation* $(\pi, V)$ of a topological group $G$ on $V$ is a group homomorphism $\pi: G \to \mathrm{GL}(V)$ for which the mapping

$$G \times V \to V, \quad (g, v) \mapsto \pi(g).v$$

is continuous, i.e., the action of $G$ on $V$ is continuous.

If $G$ is a Lie group (modeled on a sequentially complete locally convex space, cf. [Ne00a, Sect. I]), then a vector $v \in V$ is called *smooth* if the orbit mapping $G \to V, g \mapsto \pi(g).v$ is smooth. We write $V^\infty$ for the space of smooth vectors in $V$. A representation $(\pi, V)$ is said to be *smooth* if $V^\infty$ is dense in $V$.

As we have seen in [Ne00a, Lemma IV.2], we obtain a representation of the Lie algebra $\mathfrak{g}$ of $G$ on $V^\infty$ by putting

$$d\pi(X).v := d\varphi_v(\mathbf{1}).X,$$

where $\varphi_v(g) := \pi(g).v$ denotes the orbit map of $v$. If $G$ is real, then the representation $d\pi$ of $\mathfrak{g}$ on $V^\infty$ naturally extends to a complex linear representation of the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}}$ on $V^\infty$. ∎

**Definition IV.2.** (a) Let $(\pi, V)$ be a continuous representation of the group $\widehat{L}K$. The subspace

$$V(k) := \{v \in V : (\forall \theta \in \mathbb{R})\ R_\theta.v = e^{-ik\theta}.v\}$$

is called the *energy subspace of degree* $k$. We note that the Peter–Weyl Theorem applied to the representation $\pi|_{\mathbb{T}_r}$ on $V$ implies that the *finite energy subspace*

$$\check{V} := \sum_{k \in \mathbb{Z}} V(k)$$

is dense in $V$ and that there exist continuous projections

$$p_k : V \to V(k), \quad v \mapsto \frac{1}{2\pi} \int_0^{2\pi} e^{ik\theta} R_\theta.v \, d\theta.$$

Note that the integrals make sense because $V$ is a complete locally convex space. Moreover, if $v \in V^\infty$ is a smooth vector, then the Fourier series $v = \sum_{k \in \mathbb{Z}} p_k(v)$ converges in $V$ according to Harish-Chandra's Theorem (cf. [Wa72, Th. 4.4.2.1]).
(b) If $V$ is a module of the Lie algebra $\widehat{L}\mathfrak{k}_{\mathbb{C}}$, we call $V(k) := \{v \in V : X_r.v = ikv\}$, where $X_r := \frac{d}{dt}$ is the canonical basis element of $\mathfrak{t}_r$, the *energy subspace of degree* $k$, and $\check{V} := \sum_{k \in \mathbb{Z}} V(k)$ the *finite energy subspace.*
(c) We say that $(\pi, V)$ is a *representation of positive energy* if $V(k) = 0$ for $k < 0$.■

**Example IV.3.** If $V = \widetilde{L}\mathfrak{k}_{\mathbb{C}}$, where the representation of $\widetilde{L}K$ is the adjoint representation, then $V(k) = z^k \mathfrak{k}_{\mathbb{C}}$ for $k \neq 0$, $V(0) = (\mathfrak{t}_r)_{\mathbb{C}} \oplus (\mathfrak{t}_c)_{\mathbb{C}} \oplus \mathfrak{k}_{\mathbb{C}}$, and therefore $\check{V} = \widetilde{L}_{\mathrm{pol}}\mathfrak{k}_{\mathbb{C}}$. ■

**Lemma IV.4.** *For every representation $(\pi, V)$ of $\widetilde{L}K$ the space $\check{V}^\infty$ of smooth vectors of finite energy is invariant under $\widetilde{L}_{\mathrm{pol}}\mathfrak{k}_{\mathbb{C}}$.*

**Proof.** First we show that the action map $\widetilde{L}\mathfrak{k}_{\mathbb{C}} \otimes V^\infty \to V^\infty$ is equivariant with respect to the action of the torus $\mathbb{T}_r$ on both sides. In fact, since $\mathbb{T}_r$ acts on $\widetilde{L}K$ by conjugation, it is clear that the action map $\widetilde{L}K \times V \to V$ is $\mathbb{T}_r$-equivariant. This means that for each $v \in V$ and $\theta \in \mathbb{R}$ we have

$$\varphi_v \circ R_\theta = \pi(R_\theta) \circ \varphi_{R_{-\theta}.v},$$

where $\varphi_w(g) := \pi(g).w$ denotes the orbit map of $w$. Taking derivatives in $\mathbf{1}$, this leads to

$$(R_\theta.X).v = \pi(R_\theta)\big(X.(R_{-\theta}.v)\big).$$

This proves our assertion because of the complex linearity of the action map. We conclude that

$$\widetilde{L}\mathfrak{k}_{\mathbb{C}}(n).V^\infty(m) \subseteq V^\infty(n+m)$$

holds for all $n, m \in \mathbb{Z}$. Since $\widetilde{L}_{\mathrm{pol}}\mathfrak{k}_{\mathbb{C}} = \sum_{k \in \mathbb{Z}} \widetilde{L}\mathfrak{k}_{\mathbb{C}}(k)$, this proves the assertion. ■

We obtain a more refined picture by looking at the representation of the larger torus subgroup $T_{\widehat{L}K}$. We have the weight spaces $V_{(k,\lambda,h)} \subseteq V(k)$ corresponding to the weight $\underline{\lambda} = (k, \lambda, h)$. We write

$$\mathcal{P}_V := \{\underline{\lambda} \in \widehat{T}_{\widehat{L}K} : V_{\underline{\lambda}} \neq \{0\}\}$$

for the *set of all weights* of $V$.

**Definition IV.5.** If the representation $(\pi, V)$ is irreducible or, more generally, generated by a $\mathbb{T}_c$-eigenvector, then the invariance of the larger $\mathbb{T}_c$-eigenspaces under $\widehat{L}K$ shows that there exists $h \in \mathbb{Z}$ with

$$\mathcal{P}_V \subseteq \mathbb{Z} \times \widehat{T}_K \times \{h\}.$$

In this case $h$ is called the *level of the representation* $(\pi, V)$. ■

### Lowest weight vectors and antidominant weights

**Definition IV.6.** (a) Let $V$ be a module of the Lie algebra $\widehat{L}\mathfrak{k}_{\mathbb{C}}$. A non-zero weight vector $v_{\underline{\lambda}} \in V_{\underline{\lambda}}$ is called a *lowest weight vector* if

$$(L\mathfrak{k}_{\mathbb{C}})^{\underline{\alpha}}.v_{\underline{\lambda}} = \{0\}$$

for all $\underline{\alpha} \in \Delta^-_{L\mathfrak{k}}$. In this case $\underline{\lambda}$ is called the corresponding *lowest weight.* If $(\pi, V)$ is a representation of $\widehat{L}K$, then a lowest weight vector means a lowest weight vector for the derived representation of the Lie algebra $\widehat{L}\mathfrak{k}_{\mathbb{C}}$ on $V^\infty$.
(b) A module $V$ of $\widehat{L}\mathfrak{k}_{\mathbb{C}}$ is said to be a *lowest weight module* if it is generated by a lowest weight vector. A representation $(V, \pi)$ of $\widehat{L}K$ is called a *lowest weight representation* if it contains a lowest weight vector generating $V$.
(c) A weight $\underline{\lambda}$ satisfying $\langle \underline{\lambda}, h_{\underline{\alpha}} \rangle \in -\mathbb{N}_0$ for all $\underline{\alpha} \in \Delta^+_{L\mathfrak{k}}$ is called *antidominant* (with respect to the positive system $\Delta^+_{L\mathfrak{k}}$). ■

The following observation is crucial for the whole theory.

**Proposition IV.7.** (i) *If $(\pi, V)$ is a smooth representation of $\widehat{L}K$ with positive energy and $V(0) \neq \{0\}$, then $V(0)$ contains a lowest weight vector.*
(ii) *Each lowest weight $\underline{\lambda}$ is antidominant.*

**Proof.** (i) According to [Ne00a, Prop. IV.5], the representation of $G$ on the complete locally convex space $V^\infty$ is continuous. Since this applies in particular to the subgroup $\mathbb{T}_r$, it follows that the projection

$$p_0 \colon V \to V(0), \quad v \mapsto \frac{1}{2\pi} \int_0^{2\pi} R_\theta.v \, d\theta$$

satisfies

$$p_0(V^\infty) \subseteq V^\infty(0) \subseteq V^\infty.$$

In view of the smoothness assumption on $(\pi, V)$, the subspace $V^\infty(0)$ which is invariant under the compact group $K$ (the constant loops) is dense in $V(0)$.

The Peter–Weyl Theorem implies the existence of a finite-dimensional irreducible $K$-subspace $F \subseteq V^\infty(0)$. Let $v_{\underline{\lambda}} \in F$ be a lowest weight vector for the representation of $\mathfrak{k}_{\mathbb{C}}$ on $F$. Since $\mathbb{T}_c$ commutes with $K$, we may w.l.o.g. assume that $v_{\underline{\lambda}} \in V_{(0,\lambda,h)}$. Then the fact that

$$(\widetilde{L}\mathfrak{k}_{\mathbb{C}})^{(k,\alpha)}.v_{\underline{\lambda}} \in V_{(k,\lambda+\alpha,h)} \subseteq V(k) = \{0\}$$

for $k < 0$ implies that $v_{\underline{\lambda}}$ is a lowest weight vector for $\widetilde{L}\mathfrak{k}_{\mathbb{C}}$.
(ii) If $\underline{\lambda}$ is a lowest weight of a representation $(\pi, V)$, then $\mathfrak{sl}_2$-theory applied to the group $\widetilde{L}K(\underline{\alpha})$ shows that $\langle \underline{\lambda}, h_{\underline{\alpha}} \rangle \in -\mathbb{N}_0$ because otherwise $e_{-\underline{\alpha}}.v_\lambda \neq \{0\}$. ∎

**Remark IV.8.** (a) Another possibility to prove Proposition IV.7(ii) is to note that the fact that the representation of the Lie algebra $\widetilde{L}\mathfrak{k}(\underline{\alpha})$ integrates to a representation of the corresponding group implies that

$$s_{\underline{\alpha}}(\underline{\lambda}) = \underline{\lambda} - \langle \underline{\lambda}, h_{\underline{\alpha}} \rangle \underline{\alpha} \in \mathcal{P}_V$$

is also a weight in the smallest $\widetilde{L}\mathfrak{k}(\underline{\alpha})$-invariant subspace containing $v_{\underline{\lambda}}$.

If $\underline{\lambda} = (n_0, \lambda, h)$ is a weight with minimal $n_0$, then we see that for each root $\underline{\alpha} = (k, \alpha, 0)$ with $k > 0$ we have $\langle \underline{\lambda}, h_{\underline{\alpha}} \rangle \leq 0$. Further the $\mathcal{W}$-orbit of $\underline{\lambda}$ contains an element which is antidominant for all positive roots of the type $\underline{\alpha} = (0, \alpha, 0)$ which then yields the existence of an antidominant weight.
(b) We want to make the antidominance condition more explicit. So let $\underline{\lambda} = (n, \lambda, h)$. Then the antidominance means that

$$\langle \underline{\lambda}, h_{\underline{\alpha}} \rangle = \langle \lambda, h_\alpha \rangle - \frac{hk}{2}\kappa(h_\alpha, h_\alpha) \leq 0$$

for all $\underline{\alpha} = (k, \alpha, 0) \in \Delta^+_{L\mathfrak{k}}$. This means that $\lambda(h_\alpha) \leq 0$ for all $\alpha \in \Delta_{\mathfrak{k}}^+$ and that

$$\lambda(h_\alpha) \leq \frac{hk}{2}\kappa(h_\alpha, h_\alpha) \tag{4.1}$$

for all $k \in \mathbb{N}$ and $\alpha \in \Delta_{\mathfrak{k}}$. Using $\Delta_{\mathfrak{k}} = -\Delta_{\mathfrak{k}}$, we see that (4.1) leads to

$$h\kappa(h_\alpha, h_\alpha) \geq 2\max\{\lambda(h_\alpha), \lambda(h_{-\alpha})\} = 2\max\{\lambda(h_\alpha), -\lambda(h_\alpha)\} = 2|\lambda(h_\alpha)| \geq 0,$$

and hence to

$$h\kappa(h_\alpha, h_\alpha) \geq 0. \tag{4.2}$$

For $\underline{\alpha} = (1, -\alpha)$, $\alpha \in \Delta_{\mathfrak{k}}^+$, we see that whenever (4.2) is satisfied, then $\underline{\lambda}$ is antidominant if and only if

$$-\frac{h}{2}\kappa(h_\alpha, h_\alpha) \leq \lambda(h_\alpha) \leq 0 \tag{4.3}$$

holds for all $\alpha \in \Delta_{\mathfrak{k}}^+$. ∎

For each fixed $h$ condition (4.3) specifies a finite set of integral linear functionals on $\mathfrak{t}_{\mathfrak{k}}$ and for $h = 0$ the only functional satisfying this condition is $\lambda = 0$. As the following proposition shows, this has serious consequences for the representation theory of the group $\widehat{L}K$.

**Proposition IV.9.** *For a lowest weight representation $(\pi, V)$ of $\widehat{L}K$ of lowest weight $\underline{\lambda}$ the following assertions hold:*

(i) *If $v_{\underline{\lambda}}$ is a lowest weight vector and $\underline{\alpha}$ is a root, then $v_{\underline{\lambda}}$ generates a finite-dimensional irreducible $\widetilde{L}K(\underline{\alpha})$-representation of lowest weight $\underline{\lambda}(h_{\underline{\alpha}})$.*

(ii) *If $\underline{\lambda} = (n, \lambda, 0)$, i.e., if $(\pi, V)$ is a representation of level 0, then $(\pi, V)$ is the trivial representation on the identity component of $\widetilde{L}K$. If, in addition, $K$ is simply connected, then it is given by a character of $\mathbb{T}_r$.*

**Proof.** (i) Since $v_{\underline{\lambda}}$ is assumed to be a smooth vector, we can write out its Fourier series with respect to the compact group $\widetilde{L}K(\underline{\alpha})$:

$$v_{\underline{\lambda}} = \sum_{m \in \mathbb{N}_0} v_{\underline{\lambda}}(m),$$

where we have identified the set of equivalence classes of irreducible representations of $\widetilde{L}K(\underline{\alpha})$ with a subset of $\mathbb{N}_0$ in such a way that $m \in \mathbb{N}_0$ corresponds to the representation with lowest weight $\mu_m$ satisfying $\mu_m(h_{\underline{\alpha}}) = -m$.

The facts that $v_{\underline{\lambda}}$ is a smooth vector for $\widetilde{L}K(\underline{\alpha})$ and that the projections onto the isotypical components are continuous entail that the vectors $v_{\underline{\lambda}}(m)$ are also smooth with respect to $\widetilde{L}K(\underline{\alpha})$. Then these vectors have to be lowest weight vectors for $\widetilde{L}K(\underline{\alpha})$ and therefore $h_{\underline{\alpha}}.v_{\underline{\lambda}}(m) = -imv_{\underline{\lambda}}(m)$. In view of the fact that $h_{\underline{\alpha}}.v_{\underline{\lambda}} = i\underline{\lambda}(h_{\underline{\alpha}})v_{\underline{\lambda}}$, the uniqueness of the Fourier expansion shows that

$$v_{\underline{\lambda}} = v_{\underline{\lambda}}(m) \quad \text{for} \quad m = -\underline{\lambda}(h_{\underline{\alpha}}),$$

and the assertion follows.

(ii) We have seen in Remark IV.8 that the antidominance of $\underline{\lambda}$ and $h = 0$ imply that $\lambda = 0$. Therefore (i) implies that for each root $\underline{\alpha} \in \Delta_{L\mathfrak{k}}$ the $\widetilde{L}K(\underline{\alpha})$-submodule generated by $v_{\underline{\lambda}}$ is a lowest weight module with trivial lowest weight, hence a trivial module. We conclude that the three-dimensional groups $\widetilde{L}K(\underline{\alpha})$ fix the lowest weight vector $v_{\underline{\lambda}}$.

On the other hand $v_{\underline{\lambda}}$ is fixed by the torus $T_{\widetilde{L}K}$. Since each element $X \in \widetilde{L}_{\mathrm{pol}}\mathfrak{k}$ is a finite sum of elements contained in a three-dimensional subalgebra $\widetilde{L}\mathfrak{k}(\underline{\alpha})$, an application of the Trotter-Product-Formula (Theorem II.1) shows that

$$\exp X.v_{\underline{\lambda}} = v_{\underline{\lambda}}$$

holds for all $X \in \widetilde{L}_{\mathrm{pol}}\mathfrak{k}$, hence for all $X \in \widetilde{L}\mathfrak{k}$ because $\widetilde{L}_{\mathrm{pol}}\mathfrak{k}$ is dense (cf. Lemma III.1).

We further know that the group $\widehat{L}K$ is connected whenever $K$ is simply connected (cf. [PS86, p.48]). Since the central circle $\mathbb{T}_c$ acts trivially, we may assume that we have a representation of the group $L_rK \cong \widehat{L}K/\mathbb{T}_c$. Now the fact that the exponential function of $LK$ is a local diffeomorphism (Theorem II.1) implies that the identity component of $L_r K = \mathbb{T}_r \ltimes LK$ is generated by the image of the exponential function. From that it follows that $v_{\underline{\lambda}}$ is fixed by the whole group $(\widetilde{L}K)_0$, i.e., $(\pi, V)$ is a one-dimensional representation whenever $K$ is simply connected. ■

The following result shows that the non-trivial lowest weight representation of $\widehat{L}K$ do not factor to the quotient $L_r K$, hence that the central extension of this group is necessary to obtain non-trivial lowest weight representations.

**Corollary IV.10.** *Each lowest weight representation of $L_r K$ is one-dimensional.*

**Proof.** Let $(\pi, V)$ be a lowest weight representation of $L_r K$ which can also be considered as a representation of the central extension $\widehat{L}K$ which is trivial on the center. Then Proposition IV.9 shows that $(\pi, V)$ is a one-dimensional representation. ■

**Remark IV.11.** In the special case where $\mathfrak{k}$ is simple and $\kappa$ is *the fundamental invariant form* which is normalized by $\kappa(h_{\alpha_0}, h_{\alpha_0}) = 2$ for the highest root $\alpha_0$ (cf. [PS86, p.49]), then $\underline{\lambda} = (0, \lambda, h)$ is antidominant if and only if $\lambda$ is antidominant and $\lambda(h_{\alpha_0}) \geq -h$. The set of all integral functionals satisfying this condition can be represented as $\sum_{k=0}^{l} n_k \varpi_k$, $n_k \in \mathbb{N}_0$, where

$$\underline{\varpi_0} = (0, 0, 1) \quad \text{and} \quad \underline{\varpi_k} = \big(0, -\varpi_k, \varpi_k(h_{\alpha_0})\big), \quad k = 1, \ldots, l,$$

where $\varpi_1, \ldots, \varpi_l$ denote the fundamental weights of $\mathfrak{k}$. The above weights are the dual basis to

$$\underline{h_0} = (0, h_{\alpha_0}, -1) \quad \text{and} \quad \underline{h_k} = (0, h_{\alpha_k}, 0), \quad k = 1, \ldots, l. \qquad ■$$

### The Casimir operator

In this subsection we will describe how to define a Casimir operator for the Lie algebras $\widetilde{L}_{\rm pol}\mathfrak{k}_{\mathbb{C}}$ which will be used later on to show that certain contravariant hermitian forms on lowest weight modules of this Lie algebra are positive definite.

We recall that the *Casimir operator* of $\mathfrak{k}$ associated to an invariant non-degenerate positive definite bilinear form $\kappa$ on $\mathfrak{k}$ is given by

$$\Omega_{\mathfrak{k}} = -\tfrac{1}{2} \sum_{j=1}^{N} e_j^* e_j,$$

where $e_1, \ldots, e_N$ denotes a basis of $\mathfrak{k}$ and $e_1^*, \ldots, e_N^*$ denotes the dual basis of $\mathfrak{k}$ with respect to $\kappa$. Then $\Omega_{\mathfrak{k}}$ does not depend on the chosen basis and is a central element in the enveloping algebra $\mathcal{U}(\mathfrak{k})$. This implies in particular that $\Omega_{\mathfrak{k}}$ acts as a scalar multiple of the identity in every finite-dimensional irreducible representation $(\pi_\lambda, V_\lambda)$ of highest weight $\lambda$. According to [PS86, Prop. 9.4.2], the corresponding scalar is given by

$$c_\lambda = \tfrac{1}{2}\big(\|\lambda - \rho\|^2 - \|\rho\|^2\big) = \tfrac{1}{2}\|\lambda\|^2 - \kappa(\lambda, \rho),$$

where $\rho = \frac{1}{2}\sum_{\alpha \in \Delta_{\mathfrak{k}}^+} \alpha$.

To find an operator which has similar properties for the Lie algebra $\widetilde{L}_{\rm pol}\mathfrak{k}$, let $e_1, \ldots, e_N$ form an orthonormal basis in $\mathfrak{k}$, write $e_j^n := z^n e_j$, and let $c \in \mathbb{R}$ denote the eigenvalue of the Casimir operator $\Omega_{\mathfrak{k}}$ in the adjoint representation acting on $\mathfrak{k}_{\mathbb{C}}$. If $\mathfrak{k}$ is simple, then we define the *Casimir operator* of $\widetilde{L}_{\rm pol}\mathfrak{k}_{\mathbb{C}}$ by

$$\Omega := \Omega_0 + (I + ic)\frac{d}{dt},$$

where $I = (0,0,1) \in \mathfrak{t}_c$ is the generator of the center and

$$\Omega_0 = -\sum_{j=1}^N \sum_{n>0} e_j^n e_j^{-n} - \frac{1}{2}\sum_{j=1}^N e_j^2 = -\sum_{j=1}^N \sum_{n>0} e_j^n e_j^{-n} + \Omega_{\mathfrak{k}}.$$

If $\mathfrak{k} = \mathfrak{k}_1 \oplus \ldots \oplus \mathfrak{k}_m$ is the decomposition into the center $\mathfrak{k}_1$ and the simple ideals $\mathfrak{k}_j$, $j \geq 2$, then we obtain an action of the $m$-dimensional torus $\mathbb{T}_r := \mathbb{T}^m$ on $L\mathfrak{k} \cong \bigoplus_{j=1}^m L\mathfrak{k}_j$ by rotations in each summand. Lifting this action to the central extension $\widetilde{L}\mathfrak{k}$ we obtain a larger Lie algebra

$$\widetilde{\mathfrak{t}}_r \ltimes \widetilde{L}\mathfrak{k} \cong (\mathbb{R}^m) \ltimes \widetilde{L}\mathfrak{k}.$$

In this sense we define the Casimir operator of $\widetilde{L}\mathfrak{k}$ as

$$\Omega := \Omega_0 + \sum_{j=1}^m (I + ic_j)\frac{d}{dt_j}, \qquad \text{where} \qquad \frac{d}{dt} := \sum_{j=1}^m \frac{d}{dt_j},$$

where $c_j$ are the eigenvalues of the Casimir operators on the ideals $\mathfrak{k}_j$.

**Lemma IV.12.** *If $(\pi_{\underline{\lambda}}, V_{\underline{\lambda}})$ is a lowest weight module of the Lie algebra $\widetilde{\mathfrak{t}}_r \ltimes \widetilde{L}\mathfrak{k}_{\mathbb{C}}$ with lowest weight $\underline{\lambda} = (\underline{n}, \lambda, \underline{h})$, $\underline{h} = (h, \ldots, h)$, then the Casimir operator $\Omega$ acts on $V$ by $c_{\underline{\lambda}}\mathbf{1}$, where*

$$c_{\underline{\lambda}} = \frac{1}{2}\|\lambda\|^2 - \kappa(\lambda, \rho) - \sum_{j=1}^m (h + c_j) n_j = c_\lambda - \sum_{j=1}^m (h + c_j) n_j.$$

*If we put $\underline{\rho} = (0, \rho, -\underline{c})$, where $\underline{c} = (c_1, \ldots, c_m)$, then this can also be written as*

$$c_{\underline{\lambda}} = \frac{1}{2}\big(\|\underline{\lambda} - \underline{\rho}\|^2 - \|\underline{\rho}\|^2\big),$$

*where the scalar product on $\widetilde{\mathfrak{t}}_r \oplus \mathfrak{t}_{\mathfrak{k}} \oplus \mathbb{R}^k$ is given by*

$$\kappa_c\big((\underline{n}, \lambda, \underline{h}), (\underline{n}', \lambda', \underline{h}')\big) = \kappa(\lambda, \lambda') - \langle \underline{n}, \underline{h}' \rangle - \langle \underline{n}', \underline{h} \rangle.$$

**Proof.** [PS86, Prop. 9.4.9] ∎

In the following we call a hermitian form $\langle \cdot, \cdot \rangle$ on a module of a real Lie algebra $\mathfrak{g}$ *invariant* if $\langle X.v, w\rangle = -\langle v, X.w\rangle$ for all $v, w \in V$, $X \in \mathfrak{g}$. Note that this means that it is *contravariant* for $\mathfrak{g}_{\mathbb{C}}$ in the sense that

$$\langle X.v, w\rangle = -\langle v, \overline{X}.w\rangle$$

holds for all $v, w \in V$, $X \in \mathfrak{g}_{\mathbb{C}}$, with respect to the natural extension of the representation of $\mathfrak{g}$ to $\mathfrak{g}_{\mathbb{C}}$.

**Theorem IV.13.** (Garland's Theorem) *If $(\pi_{\underline{\lambda}}, V_{\underline{\lambda}})$ is a lowest weight module of $\widetilde{\mathfrak{t}}_r \ltimes \widetilde{L}_{\rm pol}\mathfrak{k}_{\mathbb{C}}$ such that*

(i) *$\underline{\lambda}$ is antidominant, and*

(ii) *for each $\underline{\alpha} \in \Delta_{\mathfrak{k}}$ the representation of $\widetilde{L}\mathfrak{k}(\underline{\alpha})$ integrates to a representation of the associated simply connected group,*

*then each contravariant hermitian form on $V_{\underline{\lambda}}$ which is positive definite on a lowest weight vector $v_{\underline{\lambda}}$ is positive definite.*

**Proof.** By tensoring $\pi$ with an appropriate one-dimensional representation of $\widetilde{\mathfrak{t}}_r$, we may w.l.o.g. assume that $\underline{\lambda} = (0, \lambda, h)$.

Since $V := V_{\underline{\lambda}}$ is a lowest weight module, we conclude that $V = \sum_{m\in\mathbb{N}_0} V(m)$ and that $V(0)$ is a lowest weight module of $\mathfrak{k}_{\mathbb{C}}$ with lowest weight $\lambda$. Since the set of weights of $V(0)$ in $\mathfrak{t}$ is invariant under the Weyl group $\mathcal{W}$, the module $V(0)$ is finite-dimensional, hence an irreducible module for $\mathfrak{k}_{\mathbb{C}}$. Moreover, the Poincaré-Birkhoff-Witt Theorem shows that the $\mathfrak{k}_{\mathbb{C}}$-submodules $V(m)$, $m \in \mathbb{N}$, are finite-dimensional.

We show by induction over $k$ that the hermitian form on $V$ is positive definite on the submodule $V(k)$. Since the form is contravariant for $\widetilde{\mathfrak{t}}_r \oplus \mathfrak{t}_{\mathfrak{k}}$, it suffices to check positivity on the weight spaces $V_{\underline{\mu}}$ for $\widetilde{\mathfrak{t}}_r \oplus \mathfrak{t}$, where $\underline{\mu} = (\underline{m}, \mu, h)$ and $\sum_{j=1}^m m_j = k$. Since these weight spaces can be decomposed as orthogonal sums, where each piece is contained in an irreducible $\mathfrak{k}_{\mathbb{C}}$-submodule on which, according to the uniqueness of the form on irreducible submodules, the form is either positive or negative, it suffices to assume that $v_{\underline{\mu}} \in V^{\underline{\mu}}$ is a non-zero weight vector which is contained in an irreducible $\mathfrak{k}_{\mathbb{C}}$-submodule. We may even assume that $v_{\underline{\mu}}$ is a lowest weight vector for the representation of $\mathfrak{k}_{\mathbb{C}}$ on this irreducible submodule.

If $k = 0$, then the positivity of the form on $V(0)$ follows from the positivity on $v_{\underline{\lambda}}$ and the irreducibility of $V(0)$ as a $\mathfrak{k}_{\mathbb{C}}$-module which implies the uniqueness of the form up to a real scalar multiple. Thus we may assume that $k > 0$. We recall that for each root $\underline{\alpha} \in \Delta_{\mathfrak{k}}$ the representation of $\widetilde{L}\mathfrak{k}(\underline{\alpha})$ integrates to a representation of the associated simply connected group, so that for each reflection $s_{\underline{\alpha}} \in \mathcal{W}_{\rm aff}$ there exists an operator $\widetilde{s}_{\underline{\alpha}}$ on $V$ leaving the form invariant and which has the property that

$$\widetilde{s}_{\underline{\alpha}}.V^{\underline{\mu}} = V^{s_{\underline{\alpha}}.\underline{\mu}}.$$

If $\underline{\alpha} \in \Delta^+_{L\mathfrak{k}} \setminus \Delta_{\mathfrak{k}}$, then $\underline{\alpha}$ has positive energy. If $\underline{\mu}(h_{\underline{\alpha}}) > 0$, then

$$s_{\underline{\alpha}}.\underline{\mu} = \underline{\mu} - \langle \underline{\mu}, h_{\underline{\alpha}}\rangle\underline{\alpha}$$

has lower energy. Hence our induction proves that

$$\langle v_{\underline{\mu}}, v_{\underline{\mu}}\rangle = \langle \widetilde{s}_{\underline{\alpha}}.v_{\underline{\mu}}, \widetilde{s}_{\underline{\alpha}}.v_{\underline{\mu}}\rangle > 0.$$

Thus we may from now on assume in addition that $\underline{\mu}(h_{\underline{\alpha}}) \leq 0$ for all $\underline{\alpha} \in \Delta^+_{L\mathfrak{k}} \setminus \Delta_{\mathfrak{k}}$. Since our assumption that $v_{\underline{\mu}}$ is a $\mathfrak{k}_{\mathbb{C}}$-lowest weight vector implies that $\underline{\mu}(h_\alpha) \leq 0$ holds for all $\alpha \in \Delta^+_{\mathfrak{k}}$, we see that $\underline{\mu}$ is an antidominant weight.

Now we have

$$\Omega_{\mathfrak{k}}.v_{\underline{\mu}} = \tfrac{1}{2}\big(\|\mu - \rho\|^2 - \|\rho\|^2\big) v_{\underline{\mu}}$$

and so we obtain with

$$\Omega = -\sum_{j=1}^N \sum_{n>0} e_j^n e_j^{-n} + \Omega_{\mathfrak{k}} + \sum_{j=1}^m (I + ic_j)\frac{d}{dt_j},$$

and Lemma IV.12 that

$$\begin{aligned} c_{\underline{\lambda}}\langle v_{\underline{\mu}}, v_{\underline{\mu}}\rangle =& \langle \Omega.v_{\underline{\mu}}, v_{\underline{\mu}}\rangle = \Big(c_\mu - \sum_{j=1}^m (h + c_j)m_j\Big)\langle v_{\underline{\mu}}, v_{\underline{\mu}}\rangle - \sum_{j=1}^N \sum_{n>0} \langle e_j^n e_j^{-n}.v_{\underline{\mu}}, v_{\underline{\mu}}\rangle \\ =& c_{\underline{\mu}}\langle v_{\underline{\mu}}, v_{\underline{\mu}}\rangle + \sum_{j=1}^N \sum_{n>0} \langle e_j^{-n}.v_{\underline{\mu}}, e_j^{-n}.v_{\underline{\mu}}\rangle \geq c_{\underline{\mu}}\langle v_{\underline{\mu}}, v_{\underline{\mu}}\rangle \end{aligned}$$

because $\langle e_j^{-n}.v_{\underline{\mu}}, e_j^{-n}.v_{\underline{\mu}}\rangle \geq 0$ by the induction hypothesis. To show that $\langle v_{\underline{\mu}}, v_{\underline{\mu}}\rangle$ is non-negative, it now suffices to prove that $c_{\underline{\lambda}} > c_{\underline{\mu}}$. In the notation from above we have

$$2(c_{\underline{\lambda}} - c_{\underline{\mu}}) = \|\underline{\lambda} - \underline{\rho}\|^2 - \|\underline{\mu} - \underline{\rho}\|^2 = -\langle \underline{\lambda} + \underline{\mu} - 2\underline{\rho}, \underline{\mu} - \underline{\lambda}\rangle.$$

This expression is positive because $\underline{\mu} - \underline{\lambda}$ is a sum of positive roots, and

$$\langle \underline{\lambda} + \underline{\mu} - 2\underline{\rho}, \underline{\alpha}\rangle < 0$$

follows for each simple root $\alpha$ from the antidominance of $\underline{\lambda}$ and $\underline{\mu}$ together with the fact that $\langle \underline{\rho}, \underline{\alpha}\rangle = 1$.

This proves that $\langle v_{\underline{\mu}}, v_{\underline{\mu}}\rangle \geq 0$. If $\langle v_{\underline{\mu}}, v_{\underline{\mu}}\rangle = 0$, then we have equality in the above chain of inequalities and hence

$$\langle e_j^{-n}.v_{\underline{\mu}}, e_j^{-n}.v_{\underline{\mu}}\rangle = 0$$

for all $j = 1, \ldots, N$ and $n > 0$. Thus $v_{\underline{\mu}}$ is a lowest weight vector for the whole Lie algebra $\widetilde{L}_{\rm pol}\mathfrak{k}_{\mathbb{C}}$. Since $\Omega$ acts by the scalar $c_{\underline{\lambda}}$ on $V_{\underline{\lambda}}$ and by the scalar $c_{\underline{\mu}}$ on the lowest weight module generated by $v_{\underline{\mu}}$ (Lemma IV.12), we conclude that $c_{\underline{\lambda}} = c_{\underline{\mu}}$, contradicting the observation made above. This proves that $\langle\cdot,\cdot\rangle$ is positive definite on $V_{\underline{\lambda}}$. ∎

We will see in Theorem V.6 that the antidominance of $\underline{\lambda} \in \widehat{T}_{\widehat{L}K}$ implies the existence of a lowest weight module satisfying the assumptions of Theorem IV.13. In general not every lowest weight module with lowest weight $\underline{\lambda}$ has this property. For instance Verma modules do not (cf. [PS86]).

It is an interesting consequence of the assumptions (i) and (ii) in Theorem IV.13 that the lowest weight module $V_{\underline{\lambda}}$ is irreducible. Otherwise there would be a lowest weight module $V_{\underline{\mu}}$ properly contained in $V_{\underline{\lambda}}$, and this is exactly what we have shown to be impossible in the last part of the proof.

## V. Representations of involutive semigroups

Before we start with the detailed analysis of the positive energy representations of the group $\widehat{L}K$, we need some background from the abstract theory of representations of involutive semigroups. This background will make the constructions and results in Section VI more transparent.

**Definition V.1.** An *involutive semigroup* is a semigroup $S$ endowed with an involutive antiautomorphism $s \mapsto s^*$, which means that $(s^*)^* = s$ and $(st)^* = t^* s^*$ holds for $s, t \in S$. ■

**Example V.2.** (a) If $G$ is a group and $\tau$ is an involutive automorphism of $G$, then $g^* := \tau(g)^{-1}$ defines the structure of an involutive group on $G$. A particularly important case is $\tau = \mathrm{id}_G$.

The examples that will play a central role in Section VI are the groups $\widetilde{L}K_{\mathbb{C}}$ with $\gamma^* = \overline{\gamma}^{-1}$, where $g \mapsto \overline{g}$ denotes complex conjugation with respect to the real form $\widetilde{L}K$. Sometimes we will also consider the extended group $\mathbb{T}_r \ltimes \widetilde{L}K_{\mathbb{C}}$, with

$$(R_\theta, \gamma)^* = (R_{-\theta}, R_\theta.\gamma^*).$$

Note that on the subgroup $\widehat{L}K = \mathbb{T}_r \ltimes \widetilde{L}K$ this involution is the inversion.
(b) Let $\mathfrak{g}$ be a complex Lie algebra endowed with an involutive antilinear antiisomorphism $\omega: \mathfrak{g} \to \mathfrak{g}$. Then $\omega$ induces an involutive antilinear antiisomorphism $D \mapsto D^*$ on the enveloping algebra $\mathcal{U}(\mathfrak{g})$ satisfying $X^* = \omega(X)$ for all $X \in \mathfrak{g}$.

If $\mathfrak{g} = \mathfrak{h}_{\mathbb{C}}$ is the complexification of a real Lie algebra, then $\omega(X) = -\overline{X}$, where $\overline{X}$ denotes complex conjugation, defines an involutive antilinear antiisomorphism of $\mathfrak{g}$. Conversely, each involutive antilinear antiisomorphism $\omega$ defines the real form $\mathfrak{h} := \{X \in \mathfrak{g}: \omega(X) = -X\}$.

The example that will arise in Section VI is the Lie algebra $\mathfrak{g} = \widehat{L}\mathfrak{k}_{\mathbb{C}}$ with the real form $\widehat{L}\mathfrak{k}$.
(c) Let $V$ be a pre-Hilbert space. We write $B_0(V) \subseteq \mathrm{End}_{\mathbb{C}}(V)$ for the set of all linear operators $A$ on $V$ for which there exists an operator $A^\sharp \in \mathrm{End}_{\mathbb{C}}(V)$ with $\langle A.v, w\rangle = \langle v, A^\sharp.w\rangle$ for all $v, w \in V$. Note that such an operator $A^\sharp$ is uniquely

determined by this property whenever it exists. It is easy to see that $B_0(V)$ is an involutive semigroup with respect to the involution $A \mapsto A^\sharp$ and composition of operators (cf. [Ne99, Lemma II.3.2]). ∎

**Definition V.3.** (a) Let $M$ be a set. A function $Q: M \times M \to \mathbb{C}$ is called a *positive definite kernel* if for each finite subset $\{x_1, \ldots, x_n\} \subseteq M$ the matrix $\big(Q(x_i, x_j)\big)_{i,j=1,\ldots,n}$ is positive semidefinite. This condition is equivalent to the following one (cf. [Ne99, Th. I.1.6]): There exists a Hilbert space $\mathcal{H} \subseteq \mathbb{C}^M$ with continuous point evaluations represented by the functions $Q_x: y \mapsto Q(y,x)$, i.e., $f(x) = \langle f, Q_x\rangle$ for all $f \in \mathcal{H}$. Then $Q$ is called the *reproducing kernel* of $\mathcal{H}$ and since $\mathcal{H}$ is, as a subspace of $\mathbb{C}^M$, uniquely determined by $Q$, we put $\mathcal{H}_Q := \mathcal{H}$ and call it the *reproducing kernel space* associated to $Q$. The dense subspace of $\mathcal{H}_Q$ spanned by the functions $Q_x$, $x \in M$, is denoted $\mathcal{H}_Q^0$.
(b) A function $\varphi: S \to \mathbb{C}$ on an involutive semigroup $S$ is called *positive definite* if the kernel $Q: S \times S \to \mathbb{C}$ defined by $Q(s,t) := \varphi(st^*)$ is positive definite.
(c) If we have a left action $S \times M \to M$ of an involutive semigroup $S$ on the set $M$, then a function $Q: M \times M \to \mathbb{C}$ is called an *invariant kernel* if

$$Q(s.x, y) = Q(x, s^*.y)$$

for $x, y \in M$ and $s \in S$. This terminology is inspired by the group case where $g^* = g^{-1}$, so that invariance means that $Q(g.x, g.y) = Q(x,y)$ for all $g \in G$.
(d) If $(S, *)$ is an involutive semigroup, then a morphism of involutive semigroups $\pi: S \to B_0(V)$ is called *a hermitian representation* of $S$ on the pre-Hilbert space $V$. This means that $\pi$ is a homomorphism of semigroups and that $\pi(s)^\sharp = \pi(s^*)$ for all $s \in S$. ∎

The following lemma relates the invariance of a positive definite kernel to the existence of certain hermitian representations ([Ne99, Prop. II.4.3]).

**Proposition V.4.** *Let $Q$ be a positive definite kernel on the set $M$ and $S \times M \to M$ a left action of the involutive semigroup $S$. Then $Q$ is invariant if and only if the action of $S$ on $\mathbb{C}^M$ given by $s.f(x) := f(s^*.x)$ leaves the space*

$$\mathcal{H}_Q^0 = \operatorname{span}\{Q_x : x \in M\}$$

*invariant and defines on this spaces a hermitian representation $(\pi, \mathcal{H}_Q^0)$. In this case we have $\pi(s).Q_x = Q_{s.x}$ for $x \in X$ and $s \in S$.* ∎

**Remark V.5.** We note that for each hermitian representation $(\pi, V)$ of the involutive semigroup $S$ and $v \in V$ the function defined by $\varphi_v(s) := \langle \pi(s).v, v\rangle$ is positive definite.

If, conversely, $S$ has an identity element $\mathbf{1}$ and $\varphi$ is a positive definite function on $S$, then the kernel defined by $Q(s,t) := \varphi(st^*)$ is positive definite and invariant under the left action of $S$ on $S$ given by $s.x := xs^*$. Since $\varphi = Q_{\mathbf{1}} \in \mathcal{H}_Q^0$ is contained in the corresponding pre-Hilbert space on which the action of $S$ is given by $(s.f)(x) := f(xs)$, we obtain

$$\langle s.\varphi, \varphi\rangle = \langle s.\varphi, Q_{\mathbf{1}}\rangle = (s.\varphi)(\mathbf{1}) = \varphi(s). \qquad \blacksquare$$

So far these concepts do not refer to any topology or differentiable structure on the semigroups or the spaces involved. Now we turn to the additional properties of the representation that will be available if the kernels of the actions have additional regularity properties.

**Definition V.6.** Let $M$ be a complex manifold (modeled over a sequentially complete locally convex space). We write $\overline{M}$ for the same manifold endowed with the opposite complex structure, i.e., the identity $\mathrm{id}_M: M \to \overline{M}$ is an antiholomorphic map.

A kernel $Q: M \times M \to \mathbb{C}$ is called *holomorphic* if it is holomorphic as a function $M \times \overline{M} \to \mathbb{C}$. This means that it is holomorphic in the first and antiholomorphic in the second argument. $\blacksquare$

**Proposition V.7.** *Let $Q$ be a continuous positive definite kernel on the topological space $M$ satisfying the first countability axiom. Then the following assertions hold:*

(i) *The Hilbert space $\mathcal{H}_Q$ consists of continuous functions on $M$ and the inclusion $\mathcal{H}_Q \to C(M)$ is continuous if $C(M)$ is endowed with the topology of uniform convergence on compact subsets of $M$.*

(ii) *If $G \times M \to M$ is an action of the topological group $G$ on $M$ leaving $Q$ invariant, then $(g.f)(x) := f(g^{-1}.x)$ defines a unitary representation of $G$ on $\mathcal{H}_Q$ which is continuous in the sense that the map $G \times \mathcal{H}_Q \to \mathcal{H}_Q$ is continuous.*

**Proof.** (i) Since $Q$ is continuous, we find for each compact subset $C \subseteq M$ a constant $c > 0$ with $Q(x,x) \leq c$ for all $x \in C$. For $f \in \mathcal{H}_Q$ we then have

$$|f(x)| = |\langle f, Q_x\rangle| \leq \|f\| \cdot \|Q_x\| = \|f\|\sqrt{\langle Q_x, Q_x\rangle} = \|f\|\sqrt{Q(x,x)} \leq \sqrt{c}\|f\|.$$

This proves that the mapping $\mathcal{H}_Q \to \mathbb{C}^M$ is continuous with respect to the topology of uniform convergence on compact subsets of $M$ on the space $\mathbb{C}^M$. For each $x \in M$ the function $Q_x: y \mapsto Q(y,x)$ is continuous. Therefore the statement follows from the closedness of $C(M)$ in $\mathbb{C}^M$ which is the same as the completeness of $C(M)$ (cf. [Ne00a, Prop. II.12(i)]).
(ii) For each pair $x, y \in M$ the function

$$G \to \mathbb{C}, \quad g \mapsto \langle g.Q_x, Q_y\rangle = \langle Q_{g.x}, Q_y\rangle = Q(y, g.x)$$

is continuous and since $G$ acts by isometries on $\mathcal{H}_Q$, it follows that the representation $G \to U(\mathcal{H}_Q)$ is continuous if $U(\mathcal{H}_Q)$ is endowed with the weak operator topology which on $U(\mathcal{H}_Q)$ coincides with the strong operator topology. Hence it suffices to show that with respect to this topology the action map

$$U(\mathcal{H}_Q) \times \mathcal{H}_Q \to \mathcal{H}_Q, \quad (g, v) \mapsto g.v$$

is continuous. In fact, suppose that $v_n \to v$ and $g_i \to g$. Then

$$\|g_i.v_n - g.v\| \leq \|g_i.(v_n - v)\| + \|(g_i - g).v\| = \|v_n - v\| + \|(g_i. - g).v\| \to 0.$$

This completes the proof. ∎

**Proposition V.8.** *If $Q$ is a holomorphic positive definite kernel on the complex Fréchet manifold $M$, then $\mathcal{H}_Q$ consists of holomorphic functions on $M$ and the inclusion $\mathcal{H}_Q \to \mathrm{Hol}(M)$ is continuous if $\mathrm{Hol}(M)$ is endowed with the topology of uniform convergence on compact subsets of $M$.*

**Proof.** Since $Q$ is holomorphic, it is in particular continuous, and Proposition V.7 applies and shows that $\mathcal{H}_Q \subseteq C(M)$ and that $\mathcal{H}_Q \to C(M)$ is continuous with respect to the topology of uniform convergence on compact sets. Now the assertion follows from the observation that the dense subspace $\mathcal{H}_Q^0$ consists of holomorphic functions and the closedness of $\mathrm{Hol}(M)$ in $\mathbb{C}^M$, which is the same as the completeness of $\mathrm{Hol}(M)$ (cf. [Ne00a, Th. III.9]). ∎

## VI. Borel–Weil theory

In this section we turn to the Borel–Weil theory for loop groups. This means that we study representations that can be realized in certain homogeneous complex line bundles for loop groups. One of the main points of these constructions is that once the appropriate geometric information on this homogeneous space is available, then everything works quite analogous to the finite-dimensional case.

First we explain how to construct certain complex line bundles parametrized by the characters $\underline{\lambda}$ of the torus $T_{\widehat{LK}}$. We then study the corresponding representation of $\widehat{LK}$ in the space $\Gamma_{\underline{\lambda}}$ of holomorphic sections. Finally we derive a criterion for the corresponding space to be non-zero (Theorem VI.8). In Section VII we will see that in some sense the representations obtained by this construction exhaust all irreducible representations with positive energy.

We recall how the fundamental homogeneous space $LK/T_K$ of the loop group $LK$ can be realized as a homogeneous space of the complexified loop group $LK_{\mathbb{C}}$.

**Definition VI.1.** Let $B_0^+ \subseteq K_{\mathbb{C}}$ be the Borel subgroup with the Lie algebra $\mathfrak{b}_0^+ := \mathfrak{t}_{\mathbb{C}} \oplus \sum_{\alpha \in \Delta_{\mathfrak{k}}^+} \mathfrak{k}_{\mathbb{C}}^{\alpha}$, i.e., the Borel subalgebra of $\mathfrak{k}_{\mathbb{C}}$ corresponding to the positive system $\Delta_{\mathfrak{k}}^+$. We write $N_0^\pm \subseteq K_{\mathbb{C}}$ for the nilpotent subgroup corresponding to the nilpotent Lie algebra

$$\mathfrak{n}_0^\pm = \sum_{\alpha \in \Delta_{\mathfrak{k}}^\pm} \mathfrak{k}_{\mathbb{C}}^\alpha.$$

In $LK_{\mathbb{C}}$ we consider the subgroup $B^+$ consisting of all smooth boundary values of holomorphic maps $\gamma: \{z \in \mathbb{C}: |z| < 1\} \to K_{\mathbb{C}}$ with $\gamma(0) \in B_0^+$. Its Lie algebra is given by

$$\mathfrak{b}^+ := \Big\{ \sum_{k=0}^{\infty} z^k a_k \in L\mathfrak{k}_{\mathbb{C}}: a_k \in \mathfrak{k}_{\mathbb{C}}, a_0 \in \mathfrak{b}_0^+ \Big\}.$$

Likewise we consider the subgroup $N^- \subseteq LK_{\mathbb{C}}$ consisting of all smooth boundary values of holomorphic maps $\gamma: \{z \in \mathbb{C}: |z| > 1\} \cup \{\infty\} \to K_{\mathbb{C}}$ with $\gamma(\infty) \in N_0^-$. Its Lie algebra is given by

$$\mathfrak{n}^- := \Big\{ \sum_{k \leq 0} z^k a_k \in L\mathfrak{k}_{\mathbb{C}}: a_k \in \mathfrak{k}_{\mathbb{C}}, a_0 \in \mathfrak{n}_0^- \Big\}.$$

Similarly one defines the subgroup $N^+ := \{\gamma \in B^+ : \gamma(0) \in N_0^+\}$ with the Lie algebra

$$\mathfrak{n}^+ = \Big\{ \sum_{k \geq 0} z^k a_k \in L\mathfrak{k}_{\mathbb{C}}: a_k \in \mathfrak{k}_{\mathbb{C}}, a_0 \in \mathfrak{n}_0^+ \Big\}.$$

The appendix of [GW84] contains a detailed discussion of this group.

The inclusions $N_0^\pm \hookrightarrow N^\pm$ (as constant maps) are homotopy equivalences via the map

$$H: N^\pm \times [0,1] \to N_0^\pm, \quad (\gamma, t) \mapsto (z \mapsto \gamma(tz)), \quad |z| \leq 1.$$

Since $N_0^\pm$ are unipotent, hence contractible, it follows that the groups $N^\pm$ are contractible and in particular simply connected. ∎

**Proposition VI.2.** *The following assertions hold:*

(i) *The exponential functions of the subgroups $B^+$ and $N^-$ are local diffeomorphisms in 0.*

(ii) *The multiplication map $N^- \times B^+ \to LK_{\mathbb{C}}$ is a diffeomorphism onto an open subset of $LK_{\mathbb{C}}$.*

(iii) $B^+ \cap LK = T_K$.

(iv) *The group $LK_{\mathbb{C}}$ acts transitively on $LK/T_K$, which leads to a diffeomorphism $Y := LK_{\mathbb{C}}/B^+ \cong LK/T_K$.*

**Proof.** [PS86, Th. 8.7.2]. ∎

We write $q\colon \widetilde{L}K_{\mathbb{C}} \to LK_{\mathbb{C}}$ for the quotient mapping defining the central extension (2.6). Let $\widetilde{K}_{\mathbb{C}} := q^{-1}(K_{\mathbb{C}})$ and $\widetilde{T}_{\mathbb{C}} := q^{-1}(T_{\mathbb{C}})$. Then

$$\{\mathbf{1}\} \to (\mathbb{T}_c)_{\mathbb{C}} \cong \mathbb{C}^{\times} \longrightarrow \widetilde{K}_{\mathbb{C}} \xrightarrow{\ q\ } K_{\mathbb{C}} \to \{\mathbf{1}\}$$

is a central extension of the connected complex group $K_{\mathbb{C}}$ whose Lie algebra cocycle is trivial. If $K$ and hence $K_{\mathbb{C}}$ is simply connected, then this extension splits, and $\widetilde{K}_{\mathbb{C}} \cong K_{\mathbb{C}} \times (\mathbb{T}_c)_{\mathbb{C}}$, where $K_{\mathbb{C}} \cong (\widetilde{K}_{\mathbb{C}}, \widetilde{K}_{\mathbb{C}})$ is the commutator subgroup. In this case we also have

$$\widetilde{T}_{\mathbb{C}} \cong T_{\mathbb{C}} \times (\mathbb{T}_c)_{\mathbb{C}} \cong (T \times \mathbb{T}_c)_{\mathbb{C}}.$$

Now let $\underline{\lambda} \in \widehat{T}_{\widehat{L}K}$ be a character which is trivial on the rotation group $\mathbb{T}_r$. Then $\underline{\lambda}$ extends to a holomorphic character of the complexification $\widetilde{T}_{\mathbb{C}} = \widetilde{T}\exp(i\widetilde{\mathfrak{t}})$. Since the following diagram which is defined by the evaluation and inclusion morphisms in the bottom row and the corresponding pullbacks in the top row is commutative, we have a holomorphic homomorphism $\widetilde{B}^+ \to \widetilde{T}_{\mathbb{C}}$ which permits us to extend holomorphic characters from $\widetilde{T}_{\mathbb{C}}$ to $\widetilde{B}^+$ by pulling them back.

$$\begin{array}{ccccccc}
\widetilde{B}^+ & \longrightarrow & \widetilde{B}_0^+ & \longrightarrow & \widetilde{T}_{\mathbb{C}} & \hookrightarrow & \widetilde{L}K_{\mathbb{C}} \\
\big\downarrow q & & \big\downarrow q & & \big\downarrow q & & \big\downarrow q \\
B^+ & \xrightarrow{\ eval\ } & B_0^+ = N_0^+ \rtimes T_{\mathbb{C}} & \longrightarrow & T_{\mathbb{C}} & \hookrightarrow & LK_{\mathbb{C}}.
\end{array}$$

**Definition VI.3.** In view of the preceding discussion, we may consider $\underline{\lambda}$ as a holomorphic character of the group $\widetilde{B}^+$. Let

$$L_{\underline{\lambda}} := \widetilde{L}K_{\mathbb{C}} \times_{\widetilde{B}^+} \mathbb{C} \to \widetilde{L}K_{\mathbb{C}}/\widetilde{B}^+ \cong LK_{\mathbb{C}}/B^+ = Y.$$

denote the associated holomorphic line bundle, i.e., the quotient of $\widetilde{L}K_{\mathbb{C}} \times \mathbb{C}$ modulo the action of $\widetilde{B}^+$ which is given by

$$b.(g,z) = \big(gb^{-1}, \underline{\lambda}(b).z\big).$$

The coordinates on this bundle can be obtained by using Proposition VI.2(ii). Since the central subgroup $\mathbb{C}^{\times} = (\mathbb{T}_c)_{\mathbb{C}}$ of $\widetilde{L}K_{\mathbb{C}}$ is contained in $\widetilde{B}^+$, we obtain a natural isomorphism

$$\widetilde{L}K_{\mathbb{C}}/\widetilde{B}^+ \to LK_{\mathbb{C}}/B^+.$$

■

**Lemma VI.4.** *The natural holomorphic action of the complex group $\widetilde{L}K_{\mathbb{C}}$ on the bundle $L_{\underline{\lambda}}$ can be extended to the group $\mathbb{T}_r \ltimes \widetilde{L}K_{\mathbb{C}}$ by holomorphic automorphisms via*

$$R_\theta.[g,z] := [R_\theta.g, z].$$

**Proof.** First we observe that $\underline{\lambda}(R_\theta b R_{-\theta}) = \underline{\lambda}(b)$ for all $b \in \widetilde{B}^+$ follows from the fact that $\mathbb{T}_r$ acts trivially on $T \times \mathbb{T}_c$. We let $\mathbb{T}_r$ act on $\widetilde{L}K_{\mathbb{C}} \times \mathbb{C}$ by $R_\theta.(g,z) :=$

$(R_\theta.g, z)$, where the action of $\mathbb{T}_r$ on $\widetilde{L}K_{\mathbb{C}}$ is the canonical lift of the rotation action on $LK_{\mathbb{C}}$ (cf. Section II). Then

$$\begin{aligned} R_\theta.\big(b.(g,z)\big) &= R_\theta.\big(gb^{-1}, \underline{\lambda}(b).z\big) = \big(R_\theta g b^{-1} R_{-\theta}, \underline{\lambda}(b).z\big) \\ &= \big(R_\theta g R_{-\theta} R_\theta b^{-1} R_{-\theta}, \underline{\lambda}(R_\theta.b).z)\big) = (R_\theta.b).\big(R_\theta.(g,z)\big) \end{aligned}$$

implies that the action of $\mathbb{T}_r$ on $\widetilde{L}K_{\mathbb{C}} \times \mathbb{C}$ factors to an action on the bundle $L_{\underline{\lambda}}$ given by $R_\theta.[g,z] := [R_\theta.g, z]$. This is an action by holomorphic automorphisms. We thus obtain an action of the group $\mathbb{T}_r \ltimes \widetilde{L}K_{\mathbb{C}}$ by holomorphic automorphisms of the bundle $L_{\underline{\lambda}}$ over $LK_{\mathbb{C}}/B^+$. ∎

**Lemma VI.5.** *Let $\Gamma_{\underline{\lambda}}$ denote the space of all holomorphic sections of $L_{\underline{\lambda}}$. We identify this space with a space of holomorphic functions on $LK_{\mathbb{C}}$ by assigning to a section $s\colon \widetilde{L}K_{\mathbb{C}} \to L_{\underline{\lambda}}$ the function $f_s \in \mathrm{Hol}(\widetilde{L}K_{\mathbb{C}})$ defined by*

$$s(g\widetilde{B}^+) = [g, f_s(g)].$$

*Then a holomorphic function $f$ on $\widetilde{L}K_{\mathbb{C}}$ defines a section of $L_{\underline{\lambda}}$ if and only if*

$$[gb^{-1}, f(gb^{-1})] = [g, f(g)]$$

*for all $b \in \widetilde{B}^+$ and $g \in \widetilde{L}K_{\mathbb{C}}$, and this is equivalent to*

$$f(gb^{-1}) = \underline{\lambda}(b)f(g)$$

*for $g \in \widetilde{L}K_{\mathbb{C}}$, $b \in \widetilde{B}^+$.* ∎

**Proposition VI.6.** *We endow the space $\mathrm{Hol}(\widetilde{L}K_{\mathbb{C}})$ with the topology of uniform convergence on compact sets. Then the following assertions hold:*

(i) *The space $\mathrm{Hol}(\widetilde{L}K_{\mathbb{C}})$ is a complete locally convex space, the natural action of $\mathbb{T}_r \ltimes \widetilde{L}K_{\mathbb{C}}$ induced by the action on $\widetilde{L}K_{\mathbb{C}}$ is continuous, and the action map*

$$\widetilde{L}K_{\mathbb{C}} \times \mathrm{Hol}(\widetilde{L}K_{\mathbb{C}}) \to \mathrm{Hol}(\widetilde{L}K_{\mathbb{C}})$$

*is holomorphic.*

(ii) *The subspace $\Gamma_{\underline{\lambda}} \subseteq \mathrm{Hol}(\widetilde{L}K_{\mathbb{C}})$ is a closed left invariant subspace, hence a complete locally convex space.*

**Proof.** (i) The completeness of $\mathrm{Hol}(\widetilde{L}K_{\mathbb{C}})$ follows from [Ne00a, Th. III.11] because $\widetilde{L}K_{\mathbb{C}}$ is modeled over the Fréchet space $\widetilde{L}\mathfrak{k}_{\mathbb{C}}$. The remaining assertions follow from [Ne00a, Th. III.14].

(ii) This follows from the fact that the functions $f \in \Gamma_{\underline{\lambda}}$ are characterized by the condition that for all $b \in \widetilde{B}^+$ and $g \in \widetilde{L}K_{\mathbb{C}}$ we have $f(gb^{-1}) = \underline{\lambda}(b)f(g)$. ∎

**Remark VI.7.** We note that multiplying the character $\underline{\lambda}$ with a character $\chi \in \widehat{\mathbb{T}}_r$ of the group $\mathbb{T}_r$ of rotations corresponds to tensoring the corresponding representation of $\widehat{L}K$ with the one-dimensional representation defined by the characer of $\chi$ of $\mathbb{T}_r \cong \widehat{L}K/\widetilde{L}K$. Therefore we may w.l.o.g. assume in the following that $\underline{\lambda}$ is a character of the form $(0, \lambda, h)$, i.e., trivial on $\mathbb{T}_r$. ∎

**Theorem VI.8.** *If $\underline{\lambda} = (0, \lambda, h)$ and $\Gamma_{\underline{\lambda}} \neq \{0\}$, then the following assertions hold:*

(i) *$\underline{\lambda}$ is antidominant.*

(ii) *The representation of $\widehat{L}K$ on $\Gamma_{\underline{\lambda}}$ is a positive energy representation.*

(iii) *$\Gamma_{\underline{\lambda}}(0)$ is an irreducible representation of $K \times \mathbb{T}_c$ of lowest weight $(\lambda, h)$.*

(iv) *If $\underline{\mu}$ is a weight of $\Gamma_{\underline{\lambda}}$, then $\underline{\mu} - \underline{\lambda}$ is a sum of positive roots.*

(v) *The module $\Gamma_{\underline{\lambda}}$ is of finite type in the sense that* $\dim \Gamma_{\underline{\lambda}}(n) < \infty$ *holds for all $n \in \mathbb{N}$.*

(vi) *$\Gamma_{\underline{\lambda}}$ contains up to scalar multiple exactly one lowest weight vector $f$. As a function on $\widetilde{L}K_{\mathbb{C}}$, this function is characterized by*

$$f(n^- t n^+) = \underline{\lambda}(t) f(\mathbf{1}) \tag{6.1}$$

*for $n^- \in N^-$, $t \in \widetilde{T}_{\mathbb{C}}$ and $n^+ \in N^+$.*

**Proof.** (i) Let $\underline{\alpha} = (k, \alpha) \in \Delta^+_{L\mathfrak{k}}$ and $K_{\mathbb{C}}(\underline{\alpha}) \subseteq \widetilde{L}K_{\mathbb{C}}$ the corresponding 3-dimensional complex subgroup. Since $\Gamma_{\underline{\lambda}}$ is a left invariant subspace of $\mathrm{Hol}(\widetilde{L}K_{\mathbb{C}})$, our assumption implies that it contains a function $f$ with $f(\mathbf{1}) \neq 0$.

The subalgebra $\mathfrak{b}(\underline{\alpha}) := \mathbb{C}h_{\underline{\alpha}} + \mathfrak{k}^{\alpha}_{\mathbb{C}} z^k$ is a Borel subalgebra of the three-dimensional simple complex Lie algebra $\mathfrak{k}_{\mathbb{C}}(\underline{\alpha})$. Let $B(\underline{\alpha})$ denote the corresponding Borel subgroup of $K_{\mathbb{C}}(\underline{\alpha})$. Then the restriction of $\Gamma_{\underline{\lambda}}$ to $K_{\mathbb{C}}(\underline{\alpha})$ is non-zero and contained in the space

$$\Gamma(\underline{\alpha}) := \{f \in \mathrm{Hol}\left(K_{\mathbb{C}}(\underline{\alpha})\right) : (\forall b \in B(\underline{\alpha})) f(gb^{-1}) = \underline{\lambda}(b) f(g)\}.$$

Now the finite-dimensional Borel–Weil Theory shows that the fact that this space is non-zero implies that $\underline{\lambda}(h_{\underline{\alpha}}) \in -\mathbb{N}_0$. We conclude that $\underline{\lambda}$ is antidominant.

(ii) If $N^-$ is the subgroup defined above, then we will need the fact that $U := N^- B^+/B^+ \subseteq Y := LK_{\mathbb{C}}/B^+$ is an open dense subset and that $N^-$ can also be identified with a subgroup of the central extension $\widetilde{L}K_{\mathbb{C}}$ because the cocycle defining the central extension is trivial on its Lie algebra $\mathfrak{n}^-$, and the group $N^-$ is simply connected, so that $\widetilde{N}^- \cong \mathbb{C}^\times \times N^-$. Here we refer to [Ne00b, Th. V.4] for the fact that central extensions of simply connected Lie groups with trivial Lie algebra cocycles are trivial.

Now we consider the following sequence of maps:

$$\Gamma_{\underline{\lambda}} \xrightarrow{\ \alpha\ } \mathrm{Hol}(N^-) \xrightarrow{\ \beta\ } \mathrm{Hol}(\mathfrak{n}^-) \xrightarrow{\ \gamma\ } \prod_{p \geq 0} S^p(\mathfrak{n}^-)', \tag{6.2}$$

where $\alpha(f) = f|_{N^-}$ is the restriction map, $\beta(f) = f \circ \exp_{N^-}$, where $\exp\colon \mathfrak{n}^- \to N^-$ is the holomorphic exponential function of the group $N^-$ which is a local diffeomorphism (Proposition VI.2) and $\gamma$ is defined by the Taylor expansion of a holomorphic function on the complex Fréchet space $\mathfrak{n}^-$, where $S^p(\mathfrak{n}^-)'$ denotes the vector space of symmetric $p$-linear continuous maps $(\mathfrak{n}^-)^p \to \mathbb{C}$. Let $\Phi := \gamma \circ \beta \circ \alpha$. Since $U = N^-B^+/B^+$ is open in $Y = LK_{\mathbb{C}}/B^+$, the set $N^-\widetilde{B}^+ \subseteq \widetilde{L}K_{\mathbb{C}}$ is open. Hence the maps $\alpha$ and $\beta$ are injective. Moreover, if $f \in \mathrm{Hol}(\mathfrak{n}^-)$, then $f$ is uniquely determined by its Taylor expansion because this holds for the restriction of $f$ to each one-dimensional subspace. Therefore $\Phi$ is a $\mathbb{T}_r$-equivariant injection

$$\Gamma_{\underline{\lambda}} \to \prod_{p \geq 0} S^p(\mathfrak{n}^-)'.$$

Next we explain how the action of the torus $T_{\widehat{LK}}$ on $\Gamma_{\underline{\lambda}}$ can also be seen on the spaces on the right hand side of (6.2). If $s$ is a section of $L_{\underline{\lambda}}$ and $f$ the corresponding holomorphic function on $\widetilde{L}K_{\mathbb{C}}$, then $t \in T_{\widehat{LK}}$ acts on $s$ via

$$\begin{aligned}(t.s)(gB^+) &:= t.s(t^{-1}gB^+) = t.\big(s(t^{-1}gtB^+)\big)\\ &= t.[t^{-1}gt, f(t^{-1}gt)] = [gt, f(t^{-1}gt)] = [g, \underline{\lambda}(t)f(t^{-1}gt)].\end{aligned}$$

Thus $\alpha$ is equivariant with respect to the action of $T_{\widehat{LK}}$ on $\mathrm{Hol}(N^-)$ given by

$$(t.f)(n) := \underline{\lambda}(t)f(t^{-1}nt).$$

Similarly $\beta$ is equivariant with respect to the action of $T_{\widehat{LK}}$ on $\mathrm{Hol}(\mathfrak{n}^-)$ given by

$$(t.f)(X) := \underline{\lambda}(t)f\big(\mathrm{Ad}(t^{-1}).X\big).$$

Since the spaces $S^p(\mathfrak{n}^-)'$ are subspaces of $\mathrm{Hol}(\mathfrak{n}^-)$, this formula also defines an action of $T_{\widehat{LK}}$ on these spaces, hence on their cartesian products, and we see that $\beta$ and $\gamma$ are equivariant with respect to these actions.

To see that the module $\Gamma_{\underline{\lambda}}$ is of positive energy, it now suffices to see that the module on the right hand side has this property because $\Phi$ is injective and $\mathbb{T}_r$-equivariant. In view of $\underline{\lambda} = (0, \lambda, h)$, for $t \in \mathbb{T}_r$ we have

$$(t.f)(X) := f\big(\mathrm{Ad}(t^{-1}).X\big).$$

The $\mathfrak{t}_r$-weights for the action on the space

$$\mathfrak{n}^- = \mathfrak{n}_0^- + \sum_{k<0} \mathfrak{k}_{\mathbb{C}} z^k$$

are contained in $-\mathbb{N}_0$, so that the corresponding action on the dual has weights in $\mathbb{N}_0$. So the weights on the spaces $S^p(\mathfrak{n}^-)'$ are sums of non-negative integers and

therefore non-negative integers. This shows that the representation of $\mathbb{T}_r$ on the product of the space $S^p(\mathfrak{n}^-)'$ has only non-negative weights. This proves that $\Gamma_{\underline{\lambda}}$ is a positive energy module.
(iii) The $\mathbb{T}_r$-equivariance of $\Phi$ implies that it maps $\Gamma_{\underline{\lambda}}(0)$ into

$$\Big(\prod_{p\in\mathbb{N}_0} S^p(\mathfrak{n}^-)'\Big)(0) = \prod_{p\in\mathbb{N}_0} S^p(\mathfrak{n}^-)'(0) \cong \prod_{p\in\mathbb{N}_0} S^p(\mathfrak{n}_0^-)'(0).$$

We conclude that each function $f \in \Gamma_{\underline{\lambda}}(0)$ restricts to a function on $N^-$ which factors over the evaluation morphism $N^- \to N_0^-, \gamma \mapsto \gamma(\infty)$ (cf. [Ne00a, Prop. IV.9(ii)]). Hence the $K \times \mathbb{T}_c$-representation on the space $\Gamma_{\underline{\lambda}}(0)$ can be realized on the space $\mathrm{Hol}(N_0^-)$, where it corresponds to the action of $K \times \mathbb{T}_c$ on the space $\Gamma^0_{(\lambda,h)}$ of holomorphic sections of the bundle

$$L_{(\lambda,h)} = K_{\mathbb{C}} \times_{B_0^+} \mathbb{C} \to K_{\mathbb{C}}/B_0^+.$$

The Borel–Weil Theorem for finite-dimensional groups now implies that the representation of $K \times \mathbb{T}_c$ on $\Gamma^0_{(\lambda,h)}$ is irreducible with lowest weight $(\lambda, h)$. Since $\Gamma_{\underline{\lambda}}(0)$ embeds into this space in a $(K \times \mathbb{T}_c)$-equivariant way, we conclude that the embedding is surjective, which proves (iii).
(iv) In view of the $T_{\widehat{LK}}$-equivariance of $\Phi$, it suffices to prove the corresponding statement for the representation of $T_{\widehat{LK}}$ on $\prod_{p\in\mathbb{N}} S^p(\mathfrak{n}^-)'$.

The weights for the adjoint action of $T_{\widehat{LK}}$ on the symmetric algebra $S(\mathfrak{n}^-)$ are given as sums of negative roots in $\Delta^-_{L\mathfrak{k}}$. Hence the weights on the dual space are sums of positive roots and so the same holds for the weights on the product $\prod_{p\in\mathbb{N}} S^p(\mathfrak{n}^-)'$. This proves (iv).
(v) First we note that the subspace $\Gamma_{\underline{\lambda}}(n)$ is $K_{\mathbb{C}}$-invariant for all $n \in \mathbb{N}$. Since the group $K$ is compact, the Big Peter–Weyl Theorem ([HoMo98, Th. 3.51]) applies to the representation of $K$ on this space and shows that the sum of all irreducible finite-dimensional subspaces is dense. The representations of $K_{\mathbb{C}}$ on these spaces are lowest weight representations. Thus each one contains a one-dimensional subspace of $N_0^-$-fixed points. Therefore to show that $\Gamma_{\underline{\lambda}}(n)$ is finite-dimensional, it suffices to show that the subspace $\Gamma_{\underline{\lambda}}(n)^{N_0^-}$ of $N_0^-$-fixed vectors in this space is finite-dimensional.

To see how this space looks like, we follow the action of $N_0^-$ through the mappings $\alpha$, $\beta$ and $\gamma$. The action of $N^-$ on $\mathrm{Hol}(N^-)$ which makes $\alpha$ equivariant is simply the action by left translations $(n.f)(x) := f(n^{-1}x)$. Hence the $N_0^-$-fixed points are the functions which are invariant on the $N_0^-$-right cosets $N_0^- n$, $n \in N^-$. Since $N^- \cong N_1^- \rtimes N_0^-$ is a semidirect product, where $N_1^- \subseteq N^-$ is the kernel of the natural map to $N_0^-$, the $N_0^-$-fixed points correspond to functions on the group $N_1^-$ with Lie algebra

$$\mathfrak{n}_1^- = \sum_{k<0} \mathfrak{k}_{\mathbb{C}} z^k \cong \mathfrak{k}_{\mathbb{C}} \otimes z^{-1}\mathbb{C}[z^{-1}].$$

Form that we conclude that the weights of $\mathbb{T}_r$ on this space lie in $-\mathbb{N}$, hence that the weights of $\mathbb{T}_r$ on $S^p(\mathfrak{n}_1^-)'$ are contained in $p + \mathbb{N}$. Therefore

$$\Big(\prod_{p \in \mathbb{N}_0} S^p(\mathfrak{n}_1^-)'\Big)(n) = \prod_{p \in \mathbb{N}_0} S^p(\mathfrak{n}_1^-)'(n) \cong \sum_{p=0}^{n} S^p(\mathfrak{n}_1^-)'(n).$$

Further

$$S^p(\mathfrak{n}_1^-) \subseteq (\mathfrak{n}_1^-)^{\otimes p} = (\mathfrak{k}_{\mathbb{C}})^{\otimes p} \otimes \big(z^{-1}\mathbb{C}[z^{-1}]\big)^{\otimes p},$$

and

$$\big((\mathfrak{k}_{\mathbb{C}})^{\otimes p} \otimes \big(z^{-1}\mathbb{C}[z^{-1}]\big)^{\otimes p}\big)(-n) \cong (\mathfrak{k}_{\mathbb{C}})^{\otimes p} \otimes P_n,$$

where $P_n$ is linearly isomorphic to the space of all polynomials in $z_1^{-1}, \ldots, z_p^{-1}$ of degree $n$. Since this space is finite-dimensional, the spaces $S^p(\mathfrak{n}_1^-)'(n)$, $p \leq n$, are also finite-dimensional, and thus $\Gamma_{\underline{\lambda}}(n)^{N_0^-}$ is finite-dimensional because it embeds in a finite sum of these spaces.

(vi) If $f \in \Gamma_{\underline{\lambda}}$ is a lowest weight vector, then on the open subset $N^-\widetilde{B}^+ = N^-\widetilde{T}_{\mathbb{C}}N^+ \subseteq \widetilde{L}K_{\mathbb{C}}$ this function satisfies (6.1) because $\mathfrak{n}^-.f = \{0\}$ implies that $f$ is constant on the cosets $N^-g$ of $N^-$ (cf. [Ne00a, Prop. IV.9(ii)]). We conclude in particular that $f$ is uniquely determined by $f(\mathbf{1})$.

To prove the existence of $f$, we recall from (iii) that the representation of $K \times \mathbb{T}_c$ on $\Gamma_{\underline{\lambda}}(0)$ is an irreducible representation of lowest weight $(\lambda, h)$. Using Proposition IV.6, we find a lowest weight vector $f \in \Gamma_{\underline{\lambda}}(0)$ for $\widehat{L}K$. This proves (vi). ■

**Theorem VI.9.** *If $\Gamma_{\underline{\lambda}} \neq \{0\}$, then the following assertions hold:*

(i) *The module $\Gamma_{\underline{\lambda}}$ is essentially unitary in the sense that there exists a dense $\widehat{L}K$-invariant reproducing kernel Hilbert space $\mathcal{H}_{\underline{\lambda}} \subseteq \Gamma_{\underline{\lambda}}$ on which $\widehat{L}K$ acts continuously by an irreducible unitary representation.*

(ii) *The representation of $\widehat{L}K$ on $\Gamma_{\underline{\lambda}}$ is irreducible.*

(iii) *The antidual $\Gamma_{\underline{\lambda}}^\sharp$, i.e., the space of all continuous antilinear functionals on $\Gamma_{\underline{\lambda}}$, can be identified with a subspace of $\mathcal{H}_{\underline{\lambda}}$ in such a way that it contains the dense subspace $\mathcal{H}_{\underline{\lambda}}^0$ spanned by the point evaluations.*

**Proof.** In view of Remark VI.7, we may w.l.o.g. assume that $\underline{\lambda} = (0, \lambda, h)$. Let $\Gamma(0) := \Gamma_{\underline{\lambda}}(0) \subseteq \Gamma := \Gamma_{\underline{\lambda}}$ denote the energy subspace of degree 0. Using Theorem VI.8(vi), we choose a lowest weight function $f \in \Gamma(0)$ which is normalized by $f(\mathbf{1}) = 1$.

As a crucial piece of the proof, we will show below that the function $f$ is a holomorphic positive definite function on the complex group $\widetilde{L}K_{\mathbb{C}}$ endowed with the antiholomorphic involution given by $g^* := \overline{g}^{-1}$, where $\overline{g}$ denotes complex conjugation with respect to the real subgroup $\widetilde{L}K$. The differential of this involution is an antilinear involution on the Lie algebra level which is denoted $X \mapsto X^*$. For

$X \in \widetilde{L}\mathfrak{k}$ we have $X^* = -X$. Since $\underline{\lambda}(t^*) = \overline{\underline{\lambda}(t)}$ for all $t \in \widetilde{T}_{\mathbb{C}}$, and $(N^+)^* = N^-$, formula (6.1) implies that

$$f(g^*) = \overline{f(g)} \tag{6.3}$$

for all $g \in \widetilde{N}^- \widetilde{B}^+$ and hence for all $g \in \widetilde{L}K_{\mathbb{C}}$ because both sides are antiholomorphic function which agree on an open subset of $\widetilde{L}K_{\mathbb{C}}$ (cf. [Ne00a, Lemma III.13(ii)]).

For $b \in \widetilde{B}^+$ this implies that

$$(\overline{b}.f)(g) = f(b^*g) = \overline{f(g^*b)} = \overline{\underline{\lambda}(b^{-1})f(g^*)} = \overline{\underline{\lambda}(b^{-1})}f(g),$$

i.e.,

$$\overline{b}.f = \overline{\underline{\lambda}(b^{-1})}f. \tag{6.4}$$

Now let $\Gamma^\sharp$ denote the antidual of the locally convex space $\Gamma$, i.e., the space of all continuous antilinear functionals $\Gamma \to \mathbb{C}$. We define a map

$$\beta \colon \Gamma^\sharp \to \mathrm{Hol}(\widetilde{L}K_{\mathbb{C}}), \quad \beta(\alpha)(g) := \alpha(\overline{g}.f).$$

Since the orbit map $\widetilde{L}K_{\mathbb{C}} \to \Gamma, g \mapsto g.f$ is holomorphic (Proposition VI.6) and $g \mapsto \overline{g}$ is antiholomorphic, the functions $\beta(\alpha)$ are indeed holomorphic functions on $\widetilde{L}K_{\mathbb{C}}$. We claim that $\beta \colon \Gamma^\sharp \to \Gamma$ is an $\widetilde{L}K_{\mathbb{C}}$-equivariant map, where $\widetilde{L}K_{\mathbb{C}}$ acts on $\Gamma^\sharp$ via $(g.\alpha)(v) := \alpha(g^*.v)$. For the continuity of this action we refer to [Ne00a, Lemma IV.4(ii)].

The inclusion $\beta(\alpha) \in \Gamma$ follows from the observation that for $b \in \widetilde{B}^+$ and $g \in \widetilde{L}K_{\mathbb{C}}$ we have in view of (6.4):

$$\beta(\alpha)(gb^{-1}) = \alpha(\overline{g}\overline{b}^{-1}.f) = \alpha\big(\overline{g}\overline{\underline{\lambda}(b)}.f\big) = \underline{\lambda}(b)\alpha\big(\overline{g}.f\big) = \underline{\lambda}(b)\beta(\alpha)(g).$$

The equivariance of $\beta$ follows from

$$\big(g.\beta(\alpha)\big)(x) = \beta(\alpha)(g^{-1}x) = \alpha(\overline{g}^{-1}\overline{x}.f) = (g.\alpha)(\overline{x}.f) = \beta(g.\alpha)(x)$$

for $g \in \widetilde{L}K_{\mathbb{C}}$. The continuity of the representation on $\Gamma$ implies that for each compact subset $C \subseteq \widetilde{L}K_{\mathbb{C}}$ the subset $C.f \subseteq \Gamma$ is compact (cf. Proposition IV.1), hence that $\beta$ is continuous.

Now we can define a sesquilinear form on $\Gamma^\sharp$ by

$$\langle \alpha, \alpha' \rangle := \alpha\big(\beta(\alpha')\big).$$

Note that this form is sesquilinear because $\beta$ is linear and $\alpha$ is antilinear. Our major goal is to show that this form turns $\Gamma^\sharp$ into a pre-Hilbert space such that the representation of $\widetilde{L}K_{\mathbb{C}}$ is hermitian (cf. Definition IV.3(d)).

First we show that this form is hermitian. For $g \in \widetilde{L}K_{\mathbb{C}}$ let $\delta_g \in \Gamma^\sharp$ be defined by $\delta(g)(f) = \overline{f(g)}$. Then

$$\begin{aligned}\langle \delta_g, \delta_{g'} \rangle &= \delta_g\big(\beta(\delta_{g'})\big) = \overline{\beta(\delta_{g'})(g)} = \overline{\delta_{g'}(\overline{g}.f)} \\ &= (\overline{g}.f)(g') = f(g^* g') = \overline{f(g'^* g)} = \overline{\langle \delta_{g'}, \delta_g \rangle}.\end{aligned}$$

We endow the space $\Gamma^\sharp$ with the topology of compact convergence. Then the form $\langle \cdot, \cdot \rangle$ is separately continuous in each argument. In fact, the continuity in the first argument is trivial, whereas the continuity in the second argument follows from the continuity of $\beta$.

The subset $\{\delta_g : g \in \widetilde{L}K_{\mathbb{C}}\} \subseteq \Gamma^\sharp$ spans a dense subspace because its annihilator in $\Gamma$ is trivial and $\Gamma$ can be identified with the antidual of $\Gamma^\sharp$ ([Ne00a, Th. II.8(ii)]). This proves that

$$\overline{\langle \delta_g, \alpha' \rangle} = \langle \alpha', \delta_g \rangle$$

holds for all $g \in \widetilde{L}K_{\mathbb{C}}$, $\alpha' \in \Gamma^\sharp$, and applying the same argument a second time, that

$$\overline{\langle \alpha, \alpha' \rangle} = \langle \alpha', \alpha \rangle$$

for $\alpha, \alpha' \in \Gamma^\sharp$, i.e., that $\langle \cdot, \cdot \rangle$ is a hermitian form on $\Gamma^\sharp$.

That the representation of $\widetilde{L}K_{\mathbb{C}}$ on $\Gamma^\sharp$ is hermitian with respect to this form follows from

$$\langle g.\alpha, \alpha' \rangle = (g.\alpha)\big(\beta(\alpha')\big) = \alpha\big(g^*.\beta(\alpha')\big) = \alpha\big(\beta(g^*.\alpha')\big) = \langle \alpha, g^*.\alpha' \rangle$$

for $\alpha, \alpha' \in \Gamma^\sharp$ and $g \in \widetilde{L}K_{\mathbb{C}}$, where we have used that $\beta$ is equivariant.

The hardest part is to show that the hermitian form that we have constructed on $\Gamma^\sharp$ is positive definite. This is the point where we have to use the essential pieces of information from Lie algebra representation theory and in particular Garland's Theorem. We write $\Gamma^\sharp(n) \subseteq \Gamma^\sharp$ for the energy subspace of degree $n$ with respect to the action of $\mathbb{T}_r$ defined by $(t.\alpha)(v) := \alpha(t^{-1}.v)$. Then the sesquilinear pairing $\Gamma^\sharp \times \Gamma \to \mathbb{C}$ is invariant under $\mathbb{T}_r$ and so we see that $\langle \Gamma^\sharp(n), \Gamma(m) \rangle \neq \{0\}$ implies $n = m$. Since all the spaces $\Gamma(n)$ are finite-dimensional (Theorem VI.8(v)) and their sum is a dense subspace of $\Gamma$, we conclude that $\Gamma^\sharp(n) \cong \Gamma(n)^\sharp$, hence that $\Gamma^\sharp(n)$ is finite-dimensional.

The Lie algebra $\widetilde{L}\mathfrak{k}_{\mathbb{C}}$ acts naturally on $\mathrm{Hol}(\widetilde{L}K_{\mathbb{C}})$ by right-invariant vector fields which is compatible with the derived action of the Lie algebra on $\Gamma$ (cf. Proposition VI.6). Since the elements of the Lie algebra act by continuous endomorphisms of $\Gamma = \Gamma^\infty$, where the latter space carries the subspace topology of $C^\infty(\widetilde{L}K_{\mathbb{C}}, \Gamma)$ (cf. [Ne00a, Rem. IV.7]), they also act on $\Gamma^\sharp \subseteq \Gamma^{-\infty}$ by $(X.\alpha)(v) := \alpha(X^*.v)$, where $X^* = -\overline{X}$ and $\Gamma^{-\infty} := (\Gamma^\infty)^\sharp$ denotes the space of all continuous antilinear functionals on $\Gamma^\infty$. Since for this action the map

$$\widetilde{L}\mathfrak{k}_{\mathbb{C}} \otimes \Gamma^{-\infty} \to \Gamma^{-\infty}$$

is also $\mathbb{T}_r$-equivariant, it follows as in Lemma IV.4 that the finite energy subspace $\check{\Gamma}^{-\infty}$ is invariant under the subalgebra $\widetilde{L}_{\rm pol}\mathfrak{k}_{\mathbb{C}}$. Note that the finite dimensionality of the subspaces $\Gamma^{-\infty}(n) \cong \Gamma(n)^\sharp$ implies that $\Gamma^{-\infty}(n) = \Gamma^\sharp(n)$, hence that $\check{\Gamma}^{-\infty} = \check{\Gamma}^\sharp$.

We write $F \subseteq \check{\Gamma}^\sharp$ for the $\widetilde{L}_{\rm pol}\mathfrak{k}_{\mathbb{C}}$-submodule generated by $\delta_{\mathbf{1}}$. From

$$(t.\delta_{\mathbf{1}})(h) = \delta_{\mathbf{1}}(t^{-1}.h) = \overline{(t^{-1}.h)(\mathbf{1})} = \overline{h(\mathbf{1})} = \delta_{\mathbf{1}}(h)$$

for $t \in \mathbb{T}_r$ and $h \in \Gamma$ we conclude that $\delta_{\mathbf{1}} \in \Gamma^\sharp(0)$, hence that $F$ is a $\mathbb{T}_r$-submodule of $\check{\Gamma}^\sharp$.

We claim that $F$ is dense in $\Gamma^\sharp$. First we show that $\overline{F}$ is $\widetilde{L}K_{\mathbb{C}}$-invariant. The annihilator $F^\perp$ of $F$ in $\Gamma$ is clearly invariant under the Lie algebra $\widetilde{L}_{\rm pol}\mathfrak{k}_{\mathbb{C}}$. Since for each $v \in \Gamma$ the orbit map $\widetilde{L}\mathfrak{k}_{\mathbb{C}} \to \Gamma, X \mapsto X.v = d\varphi_v(\mathbf{1})(X)$, where $\varphi_v\colon g \mapsto \pi(g).v$ is the orbit map, is continuous ([Ne00a, Cor. IV.6]), the density of $\widetilde{L}_{\rm pol}\mathfrak{k}_{\mathbb{C}}$ in $\widetilde{L}\mathfrak{k}_{\mathbb{C}}$ (Lemma III.1) implies that for each $v \in F^\perp$ we also have $\widetilde{L}\mathfrak{k}_{\mathbb{C}}.v \subseteq F^\perp$, i.e., that $F^\perp$ is invariant under $\widetilde{L}\mathfrak{k}_{\mathbb{C}}$. In view of the fact that $\widetilde{L}K_{\mathbb{C}}$ has a good exponential function (Theorem II.1), and the orbit maps of elements in $\Gamma \subseteq \mathrm{Hol}(\widetilde{L}K_{\mathbb{C}})$ are holomorphic, [Ne00a, Th. III.11] implies that $F^\perp$ is also invariant under the group $\widetilde{L}K_{\mathbb{C}}$. Further duality implies that $\overline{F} = (F^\perp)^\perp$ is invariant under $\widetilde{L}K_{\mathbb{C}}$ (cf. [Ne00a, Th. II.8(ii)]). Since $F \subseteq \check{\Gamma}$ contains the functional $\delta_{\mathbf{1}}\colon f \mapsto \overline{f(\mathbf{1})}$ which clearly is cyclic for $\widetilde{L}K_{\mathbb{C}}$ because the annihilator of its orbit is trivial, we see that $F^\perp = \{0\}$ and thus $F$ is dense in $\Gamma^\sharp$.

If there exists an $n \in \mathbb{N}_0$ with $\Gamma^\sharp(n) \not\subseteq F$, then we find a non-zero element $v \in \Gamma(n) \cap F^\perp = \{0\}$. So we even see that $F = \check{\Gamma}^\sharp$, i.e., the finite energy subspace of $\Gamma^\sharp$ is a cyclic $\widetilde{L}\mathfrak{k}_{\mathbb{C}}$-module.

We want to apply Garland's Theorem (Theorem IV.13) to see that the hermitian form on $F = \check{\Gamma}^\sharp$ is positive definite. For that we have to note that by a similar argument as for the rotation group $\mathbb{T}_r$ the action of the $m$-dimensional torus $\widetilde{\mathbb{T}}_r$ on $LK$ lifts to an action on $\widetilde{L}K_{\mathbb{C}}$ leaving $\widetilde{B}^+$ and the character $\underline{\lambda}$ invariant. Hence it induces an action on the bundle $L_{\underline{\lambda}}$ and also on the space $\Gamma_{\underline{\lambda}}$ of holomorphic sections. In this sense we can think of $\check{\Gamma}^\sharp$ as a lowest weight module of the extended Lie algebra $\widetilde{\mathfrak{t}}_r \ltimes \widetilde{L}_{\rm pol}\mathfrak{k}_{\mathbb{C}}$. Now Garland's Theorem applies and shows that the form on $\check{\Gamma}^\sharp$ is positive definite.

This has several consequences. The kernel of $\beta$ is a closed $\mathbb{T}_r$-invariant subspace of $\Gamma^\sharp$, so that

$$\ker\beta \cap \check{\Gamma}^\sharp = \sum_{n\in\mathbb{N}} (\ker\beta)(n)$$

is dense in $\ker\beta$. On the other hand $\beta(\alpha) = 0$ means that $\langle \alpha, \Gamma^\sharp\rangle = 0$. Therefore the positive definiteness of the hermitian form on $\check{\Gamma}^\sharp$ entails that $\ker\beta$ does not intersect $\check{\Gamma}^\sharp$, hence that $\beta$ is injective. The fact that $\beta$ is injective on each finite energy subspace $\Gamma^\sharp(n)$ entails in particular that $\beta\big(\Gamma^\sharp(n)\big) = \Gamma(n)$ for all $n \in \mathbb{N}_0$, and thus $\beta(\check{\Gamma}^\sharp) = \check{\Gamma}$. Further $\beta(\delta_{\mathbf{1}})(g) = \delta_{\mathbf{1}}(\overline{g}.f) = \overline{f(g^*)} = f(g)$, i.e., $\beta(\delta_{\mathbf{1}}) = f$ which shows that $\check{\Gamma}$ is a cyclic $\widetilde{L}_{\rm pol}\mathfrak{k}_{\mathbb{C}}$-module generated by the function $f$.

To see that $\langle \cdot, \cdot \rangle$ is positive definite on $\Gamma^\sharp$, let $\alpha \in \Gamma^\sharp$ and $\alpha_n \in \Gamma^\sharp(n)$ its restriction to the subspace $\Gamma(n)$. Then $\beta(\alpha) = \sum_{n \in \mathbb{N}_0} \beta(\alpha)_n$ converges in $\Gamma$ because it is a smooth vector for the representation of $\mathbb{T}_r$ (cf. [Wa72, Th. 4.4.2.1]), and so

$$\langle \alpha, \alpha \rangle = \alpha\big(\beta(\alpha)\big) = \sum_{n \in \mathbb{N}_0} \alpha(\beta(\alpha_n)) = \sum_{n \in \mathbb{N}_0} \langle \alpha_n, \alpha_n \rangle \geq 0$$

because the restriction of $\langle \cdot, \cdot \rangle$ to $\check{\Gamma}^\sharp$ is positive definite. The preceding formula also shows that $\check{\Gamma}^\sharp$ is dense in the Hilbert space completion of $\Gamma^\sharp$ with respect to $\langle \cdot, \cdot \rangle$, and therefore that $\check{\Gamma}^\sharp$ is also dense in $\Gamma^\sharp$ with respect to the corresponding norm.

For $g \in \widetilde{L}K_{\mathbb{C}}$ we have $(g.\delta_{\mathbf{1}})(h) = \delta_{\mathbf{1}}(g^*.h) = \overline{h(\overline{g})}$, i.e., $g.\delta_{\mathbf{1}} = \delta_{\overline{g}}$. Therefore

$$\langle g.\delta_{\mathbf{1}}, \delta_{\mathbf{1}} \rangle = \langle \delta_{\overline{g}}, \delta_{\mathbf{1}} \rangle = \delta_{\overline{g}}\big(\beta(\delta_{\mathbf{1}})\big) = \delta_{\overline{g}}(f) = \overline{f(\overline{g})} = f(g^{-1}).$$

Hence $f(s^*t) = \langle t^{-1}\overline{s}.\delta_{\mathbf{1}}, \delta_{\mathbf{1}} \rangle = \langle \overline{s}.\delta_{\mathbf{1}}, \overline{t}.\delta_{\mathbf{1}} \rangle$, and we see that $f$ is a positive definite function on the involutive group $\widetilde{L}K_{\mathbb{C}}$. Let $\mathcal{H}_{\underline{\lambda}} \subseteq \mathrm{Hol}(\widetilde{L}K_{\mathbb{C}})$ denote the corresponding reproducing kernel Hilbert space with kernel $Q(s,t) := f(t^*s)$ (cf. Proposition IV.8). This kernel $Q$ on $\widetilde{L}K_{\mathbb{C}}$ is left invariant, i.e.,

$$Q(sx, y) = Q(x, s^*y)$$

for $s, x, y \in \widetilde{L}K_{\mathbb{C}}$. Therefore the left translations $(g.f)(x) := f(g^{-1}x)$ define a natural hermitian representation of $\widetilde{L}K_{\mathbb{C}}$ on $\mathcal{H}^0_{\underline{\lambda}}$ which, according to Proposition IV.7, yields a continuous unitary representation of $\widehat{L}K$ on the Hilbert space $\mathcal{H}_{\underline{\lambda}}$. The functions $Q_s$, $s \in \widetilde{L}K_{\mathbb{C}}$, are given by $t \mapsto Q(t,s) = f(s^*t) = (\overline{s}.f)(t)$, i.e., $Q_s = \overline{s}.f$. From the left invariance of $\Gamma$ and $f \in \Gamma$ we now obtain $\mathcal{H}^0_{\underline{\lambda}} \subseteq \Gamma$, and since $\Gamma$ is closed (Proposition VI.6(ii)) and $\mathcal{H}_{\underline{\lambda}} \to \mathrm{Hol}(\widetilde{L}K_{\mathbb{C}})$ is continuous (Proposition IV.8), we see that $\mathcal{H}_{\underline{\lambda}} \subseteq \Gamma$ with a continuous inclusion map.

Next we claim that $\beta(\Gamma^\sharp) \subseteq \mathcal{H}_{\underline{\lambda}}$. Let $\gamma \colon \Gamma^\sharp \to \mathcal{H}^\sharp_{\underline{\lambda}} \cong \mathcal{H}_{\underline{\lambda}}$ denote the antiadjoint of the continuous inclusion map $\mathcal{H}_{\underline{\lambda}} \hookrightarrow \Gamma$. For each $s \in \widetilde{L}K_{\mathbb{C}}$ we then have

$$\gamma(\alpha)(s) = \langle \gamma(\alpha), \overline{s}.f \rangle = \alpha(\overline{s}.f) = \beta(\alpha)(s).$$

This shows that

$$\beta(\Gamma^\sharp) = \gamma(\Gamma^\sharp) \subseteq \mathcal{H}_{\underline{\lambda}}.$$

Now we can show that the representation on $\Gamma$ is irreducible. Let $W \subseteq \Gamma$ be a closed $\widetilde{L}K_{\mathbb{C}}$-invariant subspace. If $\delta_{\mathbf{1}}(W) = \{0\}$, then $W \subseteq \delta_{\mathbf{1}}{}^{\perp}$ and the fact that $\delta_{\mathbf{1}} \in \Gamma^\sharp$ is cyclic shows that $W = \{0\}$. If $\delta_{\mathbf{1}}(W) \neq \{0\}$, then $W(0) \neq \{0\}$, and since $\Gamma(0)$ is an irreducible $K$-module (Theorem VI.8(iii)), we see that $f \in V(0) = W(0)$. Then $\check{\Gamma} = \mathcal{U}(\widetilde{L}_{\mathrm{pol}}\mathfrak{k}_{\mathbb{C}}).f \subseteq W$, and therefore $\check{\Gamma} \subseteq W$ and the closedness of $W$ finally show that $W = \Gamma$. The same proof shows that the representation of $\widetilde{L}K$ on the Hilbert space $\mathcal{H}_{\underline{\lambda}}$ is irreducible. ∎

**Corollary VI.10.** *Every holomorphic function on* $\widetilde{L}K_{\mathbb{C}}/\widetilde{B}^+ \cong LK_{\mathbb{C}}/B^+$ *is constant.*

**Proof.** For $\underline{\lambda} = 0$ we have

$$\Gamma_{\underline{\lambda}} \cong \mathrm{Hol}(\widetilde{L}K_{\mathbb{C}}/\widetilde{B}^+).$$

Now the subspace of constant functions is a $\widehat{L}K$-invariant subspace, and, on the other hand, Theorem VI.9(ii) asserts that $\widehat{L}K$ acts irreducibly on $\Gamma_{\underline{\lambda}}$. This implies the assertion. ∎

**Remark VI.11.** In the proof of Theorem VI.9 we have seen that the complex group acts on the spaces $\Gamma_{\underline{\lambda}}$ and therefore on the antidual spaces $\Gamma_{\underline{\lambda}}^\sharp$. Moreover, the construction of the Hilbert space $\mathcal{H}_{\underline{\lambda}}$ also gave us a hermitian representation of $\widetilde{L}K_{\mathbb{C}}$ on the dense subspace of $\mathcal{H}_{\underline{\lambda}}$ generated by the point evaluations. The whole framework is described by the triple

$$\Gamma_{\underline{\lambda}}^\sharp \hookrightarrow \mathcal{H}_{\underline{\lambda}} \hookrightarrow \Gamma_{\underline{\lambda}}.$$

As the general theory of hermitian representations shows, an element $g \in \widetilde{L}K_{\mathbb{C}}$ acts continuously on this pre-Hilbert space if and only if it leaves the Hilbert space $\mathcal{H}_{\underline{\lambda}}$ invariant (cf. [Ne99, Prop. II.4.9]). For an element $\exp iX$, $X \in \mathfrak{t}_{\mathfrak{k}}$, this happens if and only if the set of weights is bounded from below on $X$. ∎

Up to this point we have always studied the spaces $\Gamma_{\underline{\lambda}}$ under the assumptions that they are non-zero. Now we will show that the necessary condition of the antidominance of $\underline{\lambda}$ that we have encountered in Theorem VI.8 is sufficient for the non-triviality of the spaces $\Gamma_{\underline{\lambda}}$.

**Theorem VI.12.** *For* $\underline{\lambda} \in \widehat{T}_{\widehat{L}K}$ *the space* $\Gamma_{\underline{\lambda}}$ *is non-zero if and only if the weight* $\underline{\lambda}$ *is antidominant.*

**Proof.** If $\Gamma_{\underline{\lambda}}$ is non-zero, then Theorem VI.8(i) states that $\underline{\lambda}$ is antidominant.

Suppose that $\underline{\lambda}$ is antidominant. We have to show that there exists a non-zero function in $\Gamma_{\underline{\lambda}}$. We will construct such a function $f$ with the additional property that $f$ is $N^-$-invariant.

Let $w \in N_{LK}(T)$ be an element representing the corresponding element $[w]$ of the Weyl group $\mathcal{W}_{\mathrm{aff}} \cong \check{T} \rtimes \mathcal{W}_{\mathfrak{k}}$. Then the Bruhat decomposition of $LK_{\mathbb{C}}$ yields a decomposition

$$Y = LK_{\mathbb{C}}/B^+ = \bigcup_{[w]\in\mathcal{W}_{\mathrm{aff}}} w.U,$$

where $U = N^-B^+/B^+ \subseteq Y$ is the open subset, and $w.U = (wN^-w^{-1}).y_w$ with $y_w = wB^+ \in Y$ are other open domains in $Y$ over which the bundle is trivial ([PS86, Th. 8.7.2]). We write

$$wN^-w^{-1} = N_w^- A_w$$

with

$$N_w^- = N^- \cap wN^-w^{-1} \quad \text{and} \quad A_w = N^+ \cap wN^-w^{-1},$$

where $A_w$ is a finite-dimensional nilpotent group whose dimension is the length $l([w])$ of the Weyl group element $[w]$ ([PS86, Th. 8.7.2]). We are looking for a family of holomorphic functions $f_w\colon wN^-w^{-1} \to \mathbb{C}$ which define a holomorphic section of the bundle $L_{\underline{\lambda}} \to Y$ in the sense that for each $w$ the function $f_w$ is obtained from trivializing the bundle $L_{\underline{\lambda}}$ over the open set $wN^-w^{-1}.y_w = wN^-.y_{\mathbf{1}} = w.U$. Since, in addition, we want the functions $f_w$ to be $N_w^-$-invariant, these functions will be determined by their values on the finite-dimensional group $A_w$.

From the Bruhat decomposition we know that

$$\dim A_w = \operatorname{codim}_{N^-} N_w^- = \operatorname{codim}_Y(N_w^-.y_w) = \operatorname{codim}_Y(N^-.y_w),$$

where $N_w^-.y_w$ is a stratum of the Bruhat decomposition ([PS86, Th. 8.7.2]). We also use this reference to see that

$$w.U \cap \Big( \bigcup_{l([w'])<l([w])} w'.U \Big) = w.U \setminus N^-.y_w = (wNw^{-1}).y_w \setminus N^-.y_w$$

and that the action of $A_w$ on the domain $w.U$ yields a diffeomorphism with $A_w \times N^-.y_w \cong A_w \times N_w^-$. By induction we thus have a function on the domain

$$w.U \setminus N^-.y_w \cong (A_w \setminus \{\mathbf{1}\}) \times N_w^-.$$

Now we construct the function $f_w$ on the group $A_w$ by induction over the length $l([w])$ of the Weyl group element $[w]$. We put $f_{\mathbf{1}} = 1$ on $N^- \cong U = N^-.y_{\mathbf{1}}$, where $A_{\mathbf{1}} = \{\mathbf{1}\}$.

If $l([w]) = 1$, i.e., if $w = s_{\underline{\alpha}}$ is a reflection corresponding to a simple root $\underline{\alpha}$, then we can calculate explicitly in $\mathrm{SL}(2,\mathbb{C})$. Here $e_{\underline{\alpha}}$ corresponds to the matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, so that

$$A_w = \exp(\mathbb{C}e_{\underline{\alpha}}) \cong \Big\{ \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} : z \in \mathbb{C} \Big\}$$

and $\exp(ze_{\underline{\alpha}}).y_w \notin U$ if and only if $z = 0$. From the relation

$$\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ z^{-1} & 1 \end{pmatrix} \begin{pmatrix} -z & 0 \\ 0 & -z^{-1} \end{pmatrix} \begin{pmatrix} 1 & -z^{-1} \\ 0 & 1 \end{pmatrix}$$

in $\mathrm{SL}(2,\mathbb{C})$ we obtain

$$\exp(ze_{\underline{\alpha}})w = \exp(-z^{-1}e_{-\underline{\alpha}})h_{\underline{\alpha}}(-z)\exp(-z^{-1}e_{\underline{\alpha}}),$$

where $h_{\underline{\alpha}}(z)$ denotes the image of the matrix $\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix}$ under the natural morphism $\mathrm{SL}(2,\mathbb{C}) \to \widetilde{L}K_{\mathbb{C}}(\underline{\alpha})$. We recall from (1.1) in Section I that

$$\underline{\lambda}(h_{\underline{\alpha}}(z)) = z^{\underline{\lambda}(h_{\underline{\alpha}})}.$$

Now we observe that the point

$$[\exp(ze_{\underline{\alpha}})w, x] \in \widetilde{L}K_{\mathbb{C}} \times_{\widetilde{B}^+} \mathbb{C} = L_{\underline{\lambda}}$$

is the same as

$$[\exp(ze_{\underline{\alpha}})w, x] = [\exp(-z^{-1}e_{-\underline{\alpha}}), \underline{\lambda}\big(h_{\underline{\alpha}}(-z)\big)x] = [\exp(-z^{-1}e_{-\underline{\alpha}}), (-z)^{\underline{\lambda}(h_{\underline{\alpha}})}x].$$

In particular we obtain for $z \neq 0$:

$$\begin{aligned} &[\exp(-z^{-1}e_{-\underline{\alpha}}), 1] = [\exp(-z^{-1}e_{-\underline{\alpha}}), f_{\mathbf{1}}\big(\exp(-z^{-1}e_{-\underline{\alpha}})\big)] \\ &= [\exp(ze_{\underline{\alpha}})w, f_w\big(\exp(ze_{\underline{\alpha}})\big)] = [\exp(-z^{-1}e_{-\underline{\alpha}}), (-z)^{\underline{\lambda}(h_{\underline{\alpha}})} f_w\big(\exp(ze_{\underline{\alpha}})\big)], \end{aligned}$$

i.e.,

$$f_w\big(\exp(ze_{\underline{\alpha}})\big) = (-z)^{-\langle \underline{\lambda}, h_{\underline{\alpha}}\rangle}.$$

In view of $\underline{\lambda}(h_{\underline{\alpha}}) \in -\mathbb{N}_0$, the function on the right hand side is a polynomial, hence extends from $\mathbb{C}^\times$ to the whole complex plane.

If $l(w) > 1$, then $\dim A_w \geq 2$ and we can use Hartog's Theorem to extend the holomorphic function given on the finite-dimensional complex manifold $A_w \setminus \{\mathbf{1}\}$ to a holomorphic function on $A_w$ which we then use to obtain a left $N_w^-$-invariant function $f_w$ on $wNw^{-1}$ which is consistent with all the functions $f_{w'}$ for $l([w']) < l([w])$. By the uniqueness of analytic extension, this also implies that this function is consistent with other functions $f_{w'}$ with $l([w']) = l([w])$ which have already been constructed. This completes the proof. ■

Theorem VI.12 rounds off the picture presented in this section in the sense that it is a characterization of those cases where the space $\Gamma_{\underline{\lambda}}$ is non-trivial. Together with the structural information on these spaces obtained in Theorem VI.8 and VI.9, we have quite a good picture of the representations in the spaces $\Gamma_{\underline{\lambda}}$.

## VII. Consequences for general representations

In this section we will see how Theorems VI.8, VI.9, and VI.12 from Section VI can be used to obtain information on the general positive energy representations of the groups $\widetilde{L}K$ and $\widehat{L}K$. More precisely, we will see that for any smooth irreducible positive energy representation a dense subspace can be embedded into some $\Gamma_{\underline{\lambda}}$.

**Proposition VII.1.** *Let $(\pi, V)$ be a smooth positive energy representation of $\widehat{L}K$, $V^{-\infty}$ the antidual of $V^\infty$, and $\varepsilon \in V^{-\infty}$ a lowest weight vector of weight $\underline{\lambda}$ for the representation of $\widehat{L}\mathfrak{k}_{\mathbb{C}}$. Then the prescription $\Phi(v)(g) := \overline{\varepsilon(g^{-1}.v)}$ defines a continuous equivariant map*

$$\Phi: V^\infty \to \Gamma_{\underline{\lambda}} \subseteq C^\infty(\widehat{L}K).$$

**Proof.** We recall that $V^{-\infty}$ denotes the space of all continuous antilinear functionals on the locally convex space $V^\infty$. The continuity of the inclusion map $V^\infty \to V$ ([Ne00a, Rem. IV.7(b)]) then yields a natural map $V^\sharp \to V^{-\infty}$. Since the representation $(\pi, V)$ was assumed to be smooth, i.e., $V^\infty$ is dense in $V$, this map is injective, and so we can think of $V^\sharp$ as a subspace of $V^{-\infty}$. Then we have a natural action of the Lie algebra $\widehat{L}\mathfrak{k}_{\mathbb{C}}$ on $V^{-\infty}$ by $(X.\alpha)(v) := \alpha(X^*.v)$, where $X^* = -\overline{X}$ (cf. [Ne00a, Cor. IV.6]).

Next we note that $\widehat{L}K$ acts continuously on $V^{-\infty}$ by $(g.\alpha)(v) := \alpha(g^{-1}.v)$ (cf. [Ne00a, Lemma IV.4(ii), Prop. IV.5]). We may w.l.o.g. assume that 0 is the minimal non-zero energy degree (Remark VI.7). Then the subspace $V^{-\infty}(0) \cong V^\infty(0)^\sharp$ is invariant under the action of the compact subgroup $K$. Let $M \subseteq V(0)^{-\infty}$ be an irreducible subspace for $K$ and $\varepsilon \in M$ be a lowest weight vector of lowest weight $\underline{\lambda} = (0, \lambda, h)$ for the corresponding representation of $\mathfrak{k}_{\mathbb{C}}$ with respect to $\Delta_{\mathfrak{k}}^+$. Then we define a linear map

$$\Phi: V^\infty \to C^\infty(\widehat{L}K) \quad \text{ by } \quad \Phi(v)(g) := \overline{\varepsilon(g^{-1}.v)}$$

which is continuous because inversion induces a continuous endomorphism of $C^\infty(\widehat{L}K, V)$, the topology on $V^\infty$ is defined by the embedding into $C^\infty(\widehat{L}K, V)$, and for each continuous linear functional $\alpha$ on $V$ the corresponding natural map $C^\infty(\widehat{L}K, V) \to C^\infty(\widehat{L}K), f \mapsto \alpha \circ f$ is continuous.

For $t \in T_{\widehat{L}K}$ we further have

$$\Phi(v)(gt^{-1}) = \overline{\varepsilon(tg^{-1}.v)} = \overline{(t^{-1}.\varepsilon)(g^{-1}.v)} = \overline{\underline{\lambda}(t^{-1})\varepsilon(g^{-1}.v)} = \underline{\lambda}(t)\Phi(v)(g).$$

This shows that $\Phi(v) \in C^\infty(\widehat{L}K)$ represents a smooth section of the bundle $\Gamma_{\underline{\lambda}} \to LK/T_K \cong Y$.

Now the construction of $\varepsilon$ and the fact that $(\pi, V)$ has positive energy implies that $\varepsilon \in V^{-\infty}$ is a lowest weight vector for the representation of the Lie algebra $\widehat{L}\mathfrak{k}_{\mathbb{C}}$. If $\widetilde{X}$ is the left invariant vector field with $\widetilde{X}(\mathbf{1}) = X$, then

$$\begin{aligned}\widetilde{X}.\Phi(v)(g) &= \frac{d}{dt}\Big|_{t=0} \Phi(v)(g \exp tX) = \frac{d}{dt}\Big|_{t=0} \overline{\varepsilon\big(\exp(-tX)g^{-1}.v\big)} \\ &= -\overline{\varepsilon\big(X.(g^{-1}.v)\big)} = \overline{(X.\varepsilon)(g^{-1}.v)}\end{aligned}$$

and by complex linear extension, it follows that

$$\widetilde{X}.\Phi(v)(g) = \overline{(X^*.\varepsilon)(g^{-1}.v)}$$

for all $X \in \widehat{L\mathfrak{k}}_{\mathbb{C}}$. For $X \in \widetilde{\mathfrak{b}}^+$ this leads to

$$\widetilde{X}.\Phi(v) = \overline{\underline{\lambda}(X^*)}\Phi(v).$$

These differential equations characterize the holomorphic section of the bundle $\Gamma_{\underline{\lambda}} \to L\widetilde{K}_{\mathbb{C}}/\widetilde{B}^+$ among the smooth sections (cf. [PS86, p.222]), whence $\Phi(V^\infty) \subseteq \Gamma_{\underline{\lambda}}$. ∎

Let us assume that $V$ is an LF space ([Ne00a, Def. II.1]) and that $\varepsilon$ is contained in the subspace $V^\sharp \subseteq V^{-\infty}$ of those functionals which extend continuously to $V$. If we endow $\Gamma_{\underline{\lambda}} \subseteq \mathrm{Hol}(\widetilde{L}K_{\mathbb{C}})$ with the subspace topology, i.e., the topology of uniform convergence of compact subsets of $\widetilde{L}K_{\mathbb{C}}$ resp. $\widetilde{L}K$, then $\Phi$ is continuous. In fact, for the element $\varepsilon \in V^\sharp$ the orbit map $\widetilde{L}K \to V^\sharp$ is continuous with respect to the topology of compact convergence ([Ne00a, Lemma IV.4(ii)]). Hence the image of a compact subset of $\widetilde{L}K$ is a compact subset of $V^\sharp$, therefore equicontinuous because $V$ is an LF space and hence barreled ([Ne00a, Prop. II.2(vi), II.11(iii)]), so that convergence in $V$ implies uniform convergence on this compact set.

**Corollary VII.2.** *For each smooth irreducible positive energy representation there exists a dense invariant subspace $V_0 \subseteq V$ which injects in an equivariant way in the irreducible unitary representation $(\pi_{\underline{\lambda}}, \mathcal{H}_{\underline{\lambda}})$ on the Hilbert subspace $\mathcal{H}_{\underline{\lambda}} \subseteq \Gamma_{\underline{\lambda}}$.*

**Proof.** According to Proposition VII.1, we have a continuous injection $\Phi: V^\infty \to \Gamma_{\underline{\lambda}}$. Then $\Phi$ maps the dense finite energy space $\check{V}^\infty$ into $\check{\Gamma}_{\underline{\lambda}} = \check{\mathcal{H}}_{\underline{\lambda}}$. This shows that $V_0 := \Phi^{-1}(\mathcal{H}_{\underline{\lambda}}) \subseteq V^\infty$ is a dense invariant subspace. This proves the assertion. ∎

If $(\pi, V)$ is a smooth positive energy representation and $\varepsilon_{\underline{\lambda}} \in V^\sharp$ is a lowest weight vector with respect to the action of the Lie algebra on $V^\sharp$, then the ray $[\varepsilon_{\underline{\lambda}}] \in \mathbb{P}(V^\sharp)$ yields an equivariant map

$$\widetilde{L}K/\widetilde{T} \cong LK/T \to \mathbb{P}(V^\sharp), \quad gT \mapsto g.[\varepsilon_{\underline{\lambda}}].$$

Thus one obtains a realization of the representation in the space $\Gamma_{\underline{\lambda}}$. This is the geometric picture corresponding to the construction of Proposition VII.1. For a more detailed discussion of this construction we refer to Section V in [Ne00a].

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