Infinite-dimensional groups and their representations

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In this paper we discuss some of the basic general notions and results which play a key role in the representation theory of infinite-dimensional Lie groups modeled over sequentially complete locally convex (s.c.l.c.) spaces. In the following each locally convex space will implicitly be assumed to be Hausdorff.

In the first section we review the basic facts on calculus in s.c.l.c. spaces. We choose the setup of s.c.l.c. spaces to ensure the existence of integrals of vector valued continuous functions on compact intervals which is the key to the Fundamental Theorem of Calculus. For the setting of Fréchet spaces these results can be found in [Ha82], but one readily notices that as soon as one has a Fundamental Theorem of Calculus the other results go through with the same proofs. The s.c.l.c. setting is also used in [Mi83]. Moreover, the setting of s.c.l.c. spaces is the natural general setting for holomorphic mappings between infinite-dimensional spaces (cf. [He89]). In particular we show that the usual notion of holomorphy is equivalent to being smooth with complex linear differential. In this section we also discuss Lie groups over s.c.l.c. spaces and how to define their Lie algebra. For the existence of an exponential function no general result is known, nevertheless in all known examples an exponential function seems to exist (cf. [Mi83, p. 1043]). Moreover the differential of the exponential function is given by the same formula as in the finite dimensional case ([Gr97]). A particularly interesting class of infinite-dimensional Lie groups are the direct limit Lie groups. For more details on such groups we refer to [NRW91], [NRW93], [NRW94] and [Gl99]. For more results on general s.c.l.c. Lie groups we refer to [Mi83] where one finds in particular a discussion of the class of "regular" Lie groups which is characterized by nice properties of the exponential function. A discussion of regular Lie groups in the "convenient setting" of [KM97a] can be found in [KM97b].

Section II consists of a collection of various results from functional analysis, in particular on dual spaces, which play a role in dealing with representations of infinite-dimensional groups. Since we are working with s.c.l.c. spaces, one has to make sure in many circumstances that the spaces obtained are in fact sequentially complete. This is where one needs some refined tools from functional analysis. In addition to completeness properties, we also discuss metrizability of dual spaces for certain natural topologies.

In Section III we show how the results from Section II can be used to define convenient spaces of smooth and holomorphic functions on infinite-dimensional manifolds in such a way that these spaces become s.c.l.c. spaces. We also analyze the natural actions of Fréchet Lie groups on these spaces which are naturally associated to smooth actions. In particular we show that a smooth action of a Fréchet semigroup S on a Fréchet manifold M induces a smooth action of S on $C^\infty(M, V)$ for every s.c.l.c. space V. We also derive a complex version of this result for holomorphic actions of complex semigroups on complex manifolds.

In Section IV these results are applied to define a derived representation of a representation (π, V) of an s.c.l.c. Lie group G on the subspace V^∞ of smooth vectors and to endow this space with a suitable complete locally convex topology inherited from $C^\infty(G, V)$ on which the action of G is smooth.

In the last Section V we then turn to a quite general setup for so called coherent state representations. Analytically these representations are characterized by the property that they can be realized in spaces of holomorphic sections of a homogeneous complex line bundle. On the geometric side this means that the action of G on the projective space of the dual space has a cyclic complex orbit. These concepts are well studied in the setting of Hilbert spaces and here we show that if one carefully distinguishes between the spaces and their duals, then one can generalize this correspondence to general s.c.l.c. spaces.

I. Calculus in locally convex spaces

In this section we explain briefly how calculus works in s.c.l.c. spaces. The main point is that one uses the appropriate notion of differentiability which for the special case of Banach spaces differs from Fréchet differentiability but which is more convenient in the setup of s.c.l.c. spaces. Our basic reference will be [Ha82], where one finds detailed proofs for the case of Fréchet spaces. One readily observes that once one has the Fundamental Theorem of Calculus, then the proofs of the Fréchet case carry over to a more general setup where one still requires smooth maps to be continuous (cf. also [Mi83]). A different approach to differentiability in infinite-dimensional spaces in the framework of the so called convenient setting can be found in [FK88] and [KM97a]. A central feature of this approach is that smooth maps are no longer required to be continuous, but for calculus over Fréchet spaces one finds the same class of smooth maps described by Hamilton and Milnor. Another approach which also gives up the continuity of smooth maps and requires only continuity on compact sets is discussed by E. G. F. Thomas in [Th96].

It is also interesting to note that since the Cauchy Integral Formula plays a similar role for holomorphic functions as the Fundamental Theorem of Calculus does for differentiable functions, the setting of s.c.l.c. spaces also seems to be the appropriate one for holomorphic mappings between infinite-dimensional spaces. We show in particular that these two concepts are related by the observation that the usual notion of holomorphy is equivalent to smoothness with complex linearity of the differential.

Then we turn to manifolds modeled over s.c.l.c. spaces. Due to the aforementioned relation between smooth and holomorphic functions, complex manifolds are special cases of real manifolds in any reasonable setting. One of our main objectives in this section is to discuss some of the most basic properties of Lie groups modeled over s.c.l.c. spaces. In particular we explain how to defined their Lie algebra and the adjoint representation. A major difficulty of the s.c.l.c. setup which does not arise for Banach Lie groups is that one cannot guarantee a priori that they have any exponential function. Thus one is forced in many places to argue without using an exponential functions.

Differentiable functions

Definition I.1. (a) Let X and Y be topological vector spaces, $U \subseteq X$ open and $f\colon U \to Y$ a continuous map. Then the *derivative of f at x in the direction of h* is defined as

$$df(x)(h) := \lim_{t \to 0} \frac{1}{t}\big(f(x+th) - f(x)\big)$$

whenever it exists. The function f is called *differentiable in x* if $df(x)(h)$ exists for all $h \in X$. It is called *continuously differentiable or C^1* if it is differentiable in all points of U and

$$df\colon U \times X \to Y, \quad (x,h) \mapsto df(x)(h)$$

is a continuous map.
(b) Higher derivatives are defined by

$$\begin{aligned} & d^n f(x)(h_1, \ldots, h_n) \\ := & \lim_{t \to 0} \frac{1}{t}\big(d^{n-1} f(x + th_n)(h_1, \ldots, h_{n-1}) - d^{n-1} f(x)(h_1, \ldots, h_{n-1})\big). \end{aligned}$$

The function f is called *n-times continuously differentiable or C^n* if

$$d^n f\colon U \times X^n \to Y, \quad (x, h_1, \ldots, h_n) \mapsto d^n f(x)(h_1, \ldots, h_n)$$

is a continuous map. We say that f is *smooth or C^∞* if it is C^n for all $n \in \mathbb{N}$.
(c) If X and Y are complex vector spaces, then the map f is called *holomorphic* if it is C^1 and for all $x \in U$ the map $df(x)\colon X \to Y$ is complex linear (cf. [Mi83, p. 1027]) ■

We note that if X and Y are Banach spaces, then the strong notion of continuous differentiability is weaker than the usual notion of continuous differentiability in Banach spaces which requires that the map $x \mapsto df(x)$ is continuous with respect to the operator norm. We will discuss this point below (Example I.6 and Theorem I.7). We also note that the existence of linear maps which are not continuous shows that the continuity of f does not follow from the differentiability of f because each linear map $f\colon X \to Y$ is differentiable in the sense of Definition I.1(a).

So far we did not use any special property of the topological vector spaces involved. To be able to develop a calculus on topological vector spaces which has at least the most basic properties of calculus in finite dimensions, we will have to make the assumption that the vector spaces under consideration are sequentially complete locally convex (s.c.l.c.) spaces.

The main point in making this assumption is to be able to integrate continuous curves $\gamma\colon [a,b] \to X$ in the sense that there exists a unique element $y := \int_a^b \gamma(t)dt \in X$ with

$$\omega(y) = \int_a^b \langle \omega, \gamma(t) \rangle \, dt$$

for all continuous linear functionals ω on X (cf. [He89, Prop. 1.2.3]).

We recall that a locally convex space X is called *quasicomplete* if each closed bounded subset of X is complete as a uniform space. Since Cauchy sequences form bounded sets, it is clear that completeness implies quasicompleteness and that quasicompleteness implies sequential completeness. For the existence of integrals of continuous functions $\gamma\colon C \to X$, where C is a compact space, the quasicompleteness of X is the appropriate assumption (cf. [Bou59, §1, no. 2, Cor. de Prop. 5; no. 6]).

Now we recall the precise statements of the most fundamental facts.

Lemma I.2. *The following assertions hold:*

(i) *If f is C^1 and $x \in U$, then $df(x)\colon X \to Y$ is a linear map, f is continuous, and if $x + th \in U$ holds for all $t \in [0,1]$, then*

$$f(x+h) = f(x) + \int_0^1 df(x+uh)(h) \, du.$$

(ii) *If f is C^n, then the functions $(h_1, \ldots, h_n) \mapsto d^n f(x)(h_1, \ldots, h_n)$, $x \in U$, are symmetric n-linear maps.*

Proof. (i) The first part is [Ha82, Th. 3.2.5] and the integral representation is [Ha82, Th. 3.2.2]. To see that f is continuous, let p be a continuous seminorm on Y and $\varepsilon > 0$. Then there exists a balanced 0-neighborhood $U_1 \subseteq X$ with $x + U_1 \subseteq U$ and $p\big(df(x+uh)(h)\big) < \varepsilon$ for $u \in [0,1]$ and $h \in U_1$. Hence

$$p\big(f(x+h) - f(x)\big) \leq \int_0^1 p\big(df(x+uh)(h)\big) \, du \leq \varepsilon,$$

and thus f is continuous.

(ii) [Ha82, Th. 3.6.2] ■

Proposition I.3. (The chain rule) *If X, Y and Z are s.c.l.c. spaces, $U \subseteq X$ and $V \subseteq Y$ are open, and $f_1 \colon U \to V$, $f_2 \colon V \to Z$ are C^1, then $f_2 \circ f_1 \colon U \to Z$ is C^1 with*

$$d(f_2 \circ f_1)(x) = df_2\big(f_1(x)\big) \circ df_1(x).$$

Proof. [Ha82, Th. 3.3.4] ∎

Proposition I.4. *If X_1, X_2 and Y are s.c.l.c. spaces, $X = X_1 \times X_2$, $U \subseteq X$ is open, and $f \colon U \to Y$ is continuous, then the partial derivatives*

$$d_1 f(x_1, x_2)(h) := \lim_{t \to 0} \frac{1}{t}\big(f(x_1 + th, x_2) - f(x_1, x_2)\big)$$

and

$$d_2 f(x_1, x_2)(h) := \lim_{t \to 0} \frac{1}{t}\big(f(x_1, x_2 + th) - f(x_1, x_2)\big)$$

exist and are continuous if and only if df exists and is continuous. In that case we have

$$df(x_1, x_2)(h_1, h_2) = d_1 f(x_1, x_2)(h_1) + d_2 f(x_1, x_2)(h_2).$$

Proof. [Ha82, Th. 3.4.3] ∎

Remark I.5. (a) If $f \colon X \to Y$ is a continuous linear map, then f is smooth with

$$df(x)(h) = f(h)$$

for all $x, h \in X$, and $d^n f = 0$ for $n \geq 2$.
(b) From (a) and Proposition I.4 it follows that a continuous k-linear map $m \colon X_1 \times \ldots \times X_k \to Y$ is continuously differentiable with

$$dm(x)(h_1, \ldots, h_k) = m(h_1, x_2, \ldots, x_k) + \cdots + m(x_1, \ldots, x_{k-1}, h_k).$$

Inductively one obtains that m is smooth with $d^{k+1} m = 0$.
(c) If $f \colon U \to Y$ is C^{n+1}, then Lemma I.2(ii) and Proposition I.4 imply that

$$\begin{aligned} d(d^n f)(x, h_1, \ldots, h_n)(y, k_1, \ldots, k_n) &= d^{n+1} f(x)(h_1, \ldots, h_n, y) \\ &+ d^n f(x)(k_1, h_2, \ldots, h_n) + \ldots + d^n f(x)(h_1, \ldots, h_{n-1}, k_n). \end{aligned}$$

It follows in particular that, whenever f is C^n, then f is C^{n+1} if and only if $d^n f$ is C^1.
(d) If $f \colon U \to Y$ is holomorphic, then the finite-dimensional theory shows that for each $h \in X$ the function $U \to Y, x \mapsto df(x)(h)$ is holomorphic. Hence $d^2 f(x)$ is complex bilinear and therefore $d(df)$ is complex linear. Thus $df \colon U \times X \to Y$ is also holomorphic. ∎

Differentiable functions on Banach spaces

In this subsection we discuss the relation between the notion of differentiability described in Definition I.1 and the notion of Fréchet differentiability in Banach spaces. In Example I.6 we will see that for maps between Banach spaces our C^1 concept differs from the concept of continuous Fréchet differentiability, and in Theorem I.7 we will show that smooth functions are also smooth in the Fréchet sense (the converse is obvious). For a more detailed discussion of several concepts of differentiability in Fréchet and Banach spaces we refer to [Ke74, p. 110].

Example I.6. Let $E := \{f \in C(\mathbb{R}) : (\forall x \in \mathbb{R}) f(x+1) = f(x)\}$ denote the Banach space of 1-periodic continuous functions on $\mathbb{R}$ endowed with the norm $\|f\|_E := \sup\{|f(x)| : x \in \mathbb{R}\}$. Further let $F := \{f \in E \cap C^1(\mathbb{R}) : f' \in E\}$ be endowed with the norm $\|f\|_F := \|f\|_E + \|f'\|_E$. We consider the map

$$m \colon X := \mathbb{R} \times F \to E, \quad (x, f) \mapsto f(x + \cdot).$$

We claim that in the sense of Definition I.1(a) this map is C^1, but that $\widetilde{dm} \colon X \to \mathcal{L}(X, E), x \mapsto \big(h \mapsto dm(x, h)\big)$, where $\mathcal{L}(X, E)$ denotes the Banach space of all continuous operators from X to E, is not continuous, i.e., m is not C^1 in the Fréchet sense.

We first show that the differential of m is given by

$$dm(x, f)(y, h) = f'(x + \cdot)y + h(x + \cdot).$$

In fact, for $s \in \mathbb{R}$ and $t \neq 0$ we have

$$\begin{aligned}
&\frac{1}{t}\big(m(x+ty, f+th)(s) - m(x,f)(s)\big) - f'(x+s)y - h(x+s) \\
=&\frac{1}{t}\big(f(x+ty+s) + th(x+ty+s) - f(x+s)\big) - f'(x+s)y - h(x+s) \\
=&\frac{1}{t}\big(f(x+ty+s) - f(x+s)\big) - f'(x+s)y + h(x+ty+s) - h(x+s) \\
=&\int_0^1 f'(x+s+uty)y \, du - f'(x+s)y + \int_0^1 h'(x+s+uty)ty \, du.
\end{aligned}$$

Now the facts that f' is uniformly continuous and that h' is bounded imply that this expression tends to 0 in E whenever $t \to 0$. This proves the formula for the differential of m.

Next we show that $dm \colon X \times X \to E$ is continuous. In fact, the continuity of $\mathbb{R} \times F \to E, (x, h) \mapsto h(x + \cdot)$ follows from

$$\begin{aligned}
\|h(x+\cdot) - h_1(x_1+\cdot)\|_E &\leq \|h(x+\cdot) - h(x_1+\cdot)\|_E + \|h(x_1+\cdot) - h_1(x_1+\cdot)\|_E \\
&\leq \|h'\|_E |x - x_1| + \|h - h_1\|_E.
\end{aligned}$$

So it remains to see that $(x, f) \mapsto f'(x + \cdot)$ is also continuous. We have

$$\|f'(x+\cdot) - f_1'(x_1+\cdot)\|_E \leq \|f'(x+\cdot) - f'(x_1+\cdot)\|_E + \|f' - f_1'\|_E,$$

so that the asserted continuity follows from the uniform continuity of f'.

To see that $\widetilde{d}m: X \to \mathcal{L}(X, E)$ is not continuous, we note that $d_2m(x,f)(h) = h(x+\cdot)$. If $\lambda_x.f = f(x+\cdot)$, then $x \neq x'$ implies that $\|\lambda_x - \lambda_{x'}\| = 2$. This shows that $(x, f) \mapsto d_2m(x, f) = \lambda_x$ is not continuous. ∎

Theorem I.7. *Let X and Y be Banach spaces, $U \subseteq X$ open, and $f: U \to Y$ a map. Then the following assertions hold:*

(i) *If f is C^2, then it is C^1 in the Fréchet sense.*

(ii) *f is C^∞ if and only if it is C^∞ in the Fréchet sense.*

Proof. (i) Let us fix $x \in U$ and suppose that the open δ-ball $U_\delta(x)$ about x is contained in U. We write $d^2f(x)(h)$ for the map $h_1 \mapsto d^2f(x)(h, h_1)$ in $\mathcal{L}(X, Y)$. We claim that there exists an $\varepsilon \in]0, \delta[$ such that the set

$$M_\varepsilon := \Big\{ \frac{1}{\sqrt{\|h\|}} d^2f(x+h)(h): 0 < \|h\| < \varepsilon \Big\}$$

is bounded. Suppose that this is not the case. Then there exists a sequence $h_n \to 0$ such that $\|d^2f(x+h_n)(h_n)\| \geq n\sqrt{\|h_n\|}$. For each $h_1 \in X$ we have

$$\frac{1}{\sqrt{\|h_n\|}} d^2f(x+h_n)(h_n)(h_1) = d^2f(x+h_n)\Big(\frac{h_n}{\sqrt{\|h_n\|}}, h_1\Big) \to 0$$

because $d^2f: U \times X^2 \to Y$ is continuous and $\frac{h_n}{\sqrt{\|h_n\|}} \to 0$. This contradicts the Banach-Steinhaus Theorem, and therefore one of the sets M_ε is bounded.

Now assume that $\|h\| < \varepsilon$ and that $\|d^2f(x+h)(h)\| \leq C\sqrt{\|h\|}$ for $\|h\| < \varepsilon$. Then

$$\begin{aligned}\|\widetilde{d}f(x+h) - \widetilde{d}f(x)\| &= \|\int_0^1 d^2f(x+uh)(h)\, du\| \leq \int_0^1 \|d^2f(x+uh)(uh)\| \frac{1}{u} du \\ &\leq \int_0^1 C\sqrt{\|uh\|}\frac{1}{u} du = C\sqrt{\|h\|}\int_0^1 u^{-\frac{1}{2}} du = 2C\sqrt{\|h\|}.\end{aligned}$$

We conclude that the map $\widetilde{d}f: U \to \mathcal{L}(X, Y)$ is continuous.

Furthermore we have

$$\begin{aligned}\|f(x+h) - f(x) - \widetilde{d}f(x)(h)\| &= \|\int_0^1 \widetilde{d}f(x+uh)(h) - \widetilde{d}f(x)(h)\, du\| \\ &\leq \sup\{\|\widetilde{d}f(x+h_1) - \widetilde{d}f(x)\|: \|h_1\| < \varepsilon\}\|h\|,\end{aligned}$$

and, in view of the continuity of $x \mapsto \widetilde{d}f(x)$, the expression on the right hand side is $o(\|h\|)$. This proves that f is C^1 in the Fréchet sense whenever it is C^2 in the sense of Definition I.1(a).
(ii) If f is C^∞ in the Fréchet sense, then it is trivially C^∞ in the sense of Definition I.1(a).

Suppose that f is C^∞. Then the map $df: U \times X \to Y$ is also C^∞, hence in particular C^2. Therefore (i) shows that the map

$$\widetilde{d}(df): U \times X \to \mathcal{L}(X^2, Y)$$

is continuous, hence in particular that $d^2 f: U \times X \to \mathcal{L}(X, Y)$ is continuous since $d^2 f(x)(h_1, h_2) = \widetilde{d}(df)(x, h_1)(h_2, 0)$. Now

$$\widetilde{d}f(x+h) - \widetilde{d}f(x) - d^2 f(x)(h) = \int_0^1 d^2 f(x+uh)(h) - d^2 f(x)(h) \, du$$

implies that $d^2 f$ can be viewed as $d(\widetilde{d}f)$. Iterating this argument, we conclude that the map $\widetilde{d}f: U \to \mathcal{L}(X, Y)$ is smooth in the sense of Definition I.1. Now we we can apply induction and obtain for all $n \in \mathbb{N}$ that the n^{th} Fréchet derivative of f is smooth, and therefore that f is smooth in the Fréchet sense. ■

Holomorphic functions

In this subsection we clarify the relation between several concepts of holomorphy for functions between s.c.l.c. spaces.

Definition I.8. Let X be a complex vector space.
(a) A subset $U \subseteq X$ is called *finitely open* if for all finite-dimensional affine subspaces $F \subseteq X$ the set $F \cap U$ is open in F.
(b) Let V be a sequentially complete locally convex space. A function f on a finitely open subset $U \subseteq X$ is called *Gateaux holomorphic* ((G)-*holomorphic*) if for each finite-dimensional affine subspace $F \subseteq X$ the function $f|_{F \cap U}$ is (weakly) holomorphic on $F \cap U$ (cf. [He89, Th. 2.1.3]). We write $\mathcal{G}(U, V)$ for the space of (G)-holomorphic V-valued functions on U. Note that, in view of Hartog's Theorem, a function is (G)-holomorphic if the above criterion is satisfied for all affine complex lines $F \subseteq X$.
(c) Suppose that X is a locally convex space. A (G)-holomorphic function $f: U \to V$ is called *Fréchet holomorphic* ((F)-*holomorphic*) if for each continuous seminorm p on V the function $p \circ f$ is locally bounded. We recall from [He89, Prop. 2.4.2(a)] that this property is equivalent to the continuity of the function f. ■

If X is of countable dimension and we write $X = \bigcup_{n\in\mathbb{N}} X_n$ with $X_n \subseteq X_{n+1}$ and $\dim X_n < \infty$, then X carries a natural LF space structure which is the finest locally convex topology on X (cf. [Tr67, Ex. 13.1]). The open sets in this topology are exactly the finitely open sets ([He89, Prop. 2.3.2]). If $\dim X > \aleph_0$, then the topology defined by the finitely open sets is no longer a vector spaces topology and therefore does not coincide with the finest locally convex topology (cf. [He89, Rem. 2.3.3]).

The notion of (G)-holomorphy is the weakest possible notion of holomorphy in infinite-dimensional spaces. Unfortunately it has the drawback that in general it even does not imply continuity. In this sense the "nice" holomorphic functions are the (F)-holomorphic functions. Note that (F)-holomorphy is preserved by passing to locally uniform limits. The relations between (F)-holomorphy and weak holomorphy are clarified for "nice" spaces in the following result.

Proposition I.9. *For a function $f: U \to V$ from an open subset U of a locally convex space X to the s.c.l.c. space V the following assertions hold:*

(i) *If X is metrizable, then f is* (F)*-holomorphic if and only if it is weakly* (F)*-holomorphic.*

(ii) *If X is the inductive limit of locally convex spaces $(X_n)_{n\in\mathbb{N}}$ such that the origin in X_n has a neighborhood which is relatively compact in X_{n+1}, then*

  (a) *f is* (F)*-holomorphic if and only if it is weakly* (F)*-holomorphic.*

  (b) *f is continuous if and only if all the functions $f|_{U\cap X_n}$ are continuous for all $n \in \mathbb{N}$.*

(iii) *If X is Baire, $f \in \mathcal{G}(U, V)$, and there exists a sequence of continuous functions $f_n: U \to V$ converging pointwise to f, then f is continuous, i.e.,* (F)*-holomorphic.*

Proof. (i), (ii)(a) [He89, Prop. 3.1.2]
(ii)(b) [He89, Prop. 1.5.1(b)]
(iii) [He89, Th. 2.4.4] ■

Proposition I.10. *For a function $f: U \to V$ the following are equivalent:*

(i) *f is holomorphic in the sense of* Definition I.1(c).

(ii) *f is* (F)*-holomorphic.*

(iii) *f is smooth with complex linear differentials $df(x)$, $x \in U$.*

Proof. (i) ⇒ (ii): If f is complex differentiable in the sense of Definition I.1(c), then f is (G)-holomorphic (differentiable functions on open domains in the complex plane are holomorphic), and continuous (Lemma I.2(i)), hence (F)-holomorphic.
(ii) ⇒ (iii): Suppose that f is (F)-holomorphic. We have to show that all its higher derivatives

$$d^n f: U \times E^n \to V, \quad (x, h_1, \ldots, h_n) \mapsto d^n f(x)(h_1, \ldots, h_n)$$

are continuous maps. It is clear that the (G)-holomorphy implies the (G)-holomorphy of $d^n f$ because a similar statement holds in finite dimensions. Moreover, the generalized Cauchy inequalities (cf. [He89, Th. 2.3.5]) imply that whenever f is locally bounded in the sense of Definition I.8(c), the same property is inherited by the functions

$$(x,h) \mapsto \widehat{d^n} f(x,h) := d^n f(x)(h,\ldots,h).$$

Next we use the formula

$$d^n f(x)(h_1,\ldots,h_n) = \frac{1}{2^n n!} \sum_{\varepsilon\in\{1,-1\}^n} (\varepsilon_1\cdots\varepsilon_n)\widehat{d^n} f(x)(\varepsilon_1 h_1 + \ldots + \varepsilon_n h_n)$$

(cf. [Na69, p.7]) to conclude that the function $d^n f$ is also locally bounded in the sense of Definition I.8(c), i.e., that $d^n f$ is (F)-holomorphic. It follows in particular that the functions $d^n f$ are continuous, hence that f is a smooth function.
(iii) $\Rightarrow$ (i): This is trivial since C^∞ implies C^1. ■

The following result clarifies the concept of (F)-holomorphy in the Banach setting.

Proposition I.11. *If X and V are complex Banach spaces, $U \subseteq X$ a domain, and $f\colon U \to V$ a function. Then the following assertions hold:*

(i) *If f is* (F)*-holomorphic, then f is complex Fréchet differentiable.*

(ii) *The function f is* (F)*-holomorphic if and only if it is Fréchet differentiable at each point $x \in U$.*

Proof. (i) ([HP57, Th. 3.17.1]) If f is (F)-holomorphic, then Proposition I.10 shows that f is smooth, hence f is Fréchet smooth (Theorem I.7).
(ii) [He89, Cor. 3.1.4] ■

Differentiable manifolds

Since we have a chain rule for differentiable maps between s.c.l.c. spaces, we can define smooth manifolds as one defines them in the finite-dimensional case (cf. [Ha82], [Mi83]). The underlying topological space is always required to be Hausdorff. Since locally convex spaces (which we always assume to be Hausdorff) are *regular* in the sense that each point has a neighborhood base consisting of closed sets, this property is inherited by manifolds modeled over these spaces (cf. [Mi83]). One also defines vector bundles and in particular the tangent bundle $TM \to M$ as usual.

Note that it is far more subtle to define a cotangent bundle because this requires an s.c.l.c. topology on the dual space of the underlying vector space and therefore depends on this topology. We will discuss topologies on the dual in Section II.

Let M and N be smooth manifolds modeled over s.c.l.c. spaces and $f\colon M \to N$ a smooth map. We write $Tf\colon TM \to TN$ for the corresponding map induced on the level of tangent vectors. Locally this map is given by

$$Tf(x,h) = \big(f(x), df(x)(h)\big),$$

where $df(p)\colon T_p(M) \to T_{f(p)}(N)$ denotes the differential of f in p. In view of Remark I.5(c), the tangent map Tf is also smooth if f is smooth. In the following we will always identify M with the zero section in TM. In this sense we have $Tf|_M = f$ with $Tf(M) \subseteq N \subseteq TN$.

A *vector field* on M is a smooth section of the tangent bundle $TM \to M$. We write $\mathcal{V}(M)$ for the space of all vector fields on M. If $f \in C^\infty(M)$ is a smooth function on M and $X \in \mathcal{V}(M)$, then we obtain a function on M via

$$(X.f)(p) := df(p)\big(X(p)\big).$$

Since locally $X(p) = \big(p, \widetilde{X}(p)\big)$, where $\widetilde{X}$ is a smooth function, we have $X.f = df \circ X$. Therefore the smoothness of $X.f$ follows from the smoothness of the maps $df\colon TM \to \mathbb{C}$ and $X\colon M \to TM$.

Lemma I.12. *If $X, Y \in \mathcal{V}(M)$, then there exists a vector field $[X,Y] \in \mathcal{V}(M)$ which is uniquely determined by the property that on each open subset $U \subseteq M$ we have*

$$[X,Y].f = X.(Y.f) - Y.(X.f) \tag{1.1}$$

for all $f \in C^\infty(U)$.

Proof. Locally the vector fields X and Y are given as $X(p) = \big(p, \widetilde{X}(p)\big)$ and $Y(p) = \big(p, \widetilde{Y}(p)\big)$. We define a vector field by

$$[X,Y]^\sim(p) := d\widetilde{Y}(p)\big(\widetilde{X}(p)\big) - d\widetilde{X}(p)\big(\widetilde{Y}(p)\big). \tag{1.2}$$

Then the smoothness of the right hand side follows from the chain rule. The requirement that (1.1) holds on continuous linear functionals determines $[X,Y]^\sim$ uniquely. Since an easy calculation shows that (1.2) defines in fact a smooth vector field on M (cf. Lemma I.14 below), the assertion follows because locally (1.1) is a consequence of the chain rule. ∎

Proposition I.13. $(\mathcal{V}(M), [\cdot,\cdot])$ *is a Lie algebra.*

Proof. The crucial part is to check the Jacobi identity. This follows from the observation that if $U \subseteq X$ is an open subset of an s.c.l.c. space, then the mapping

$$\Phi\colon \mathcal{V}(U) \to \mathrm{Der}\,\big(C^\infty(U)\big), \quad \Phi(X)(f) = X.f$$

is injective and satisfies $\Phi([X,Y]) = [\Phi(X), \Phi(Y)]$. Therefore the Jacobi identity in $\mathcal{V}(U)$ follows from the Jacobi identity in the associative algebra $\mathrm{End}\,\big(C^\infty(U)\big)$. ∎

For the applications to Lie groups we will need the following lemma.

Lemma I.14. *Let M and N be smooth manifolds and $\varphi\colon M \to N$ a smooth map. Suppose that $X_N, Y_N \in \mathcal{V}(N)$ and $X_M, Y_M \in \mathcal{V}(M)$ satisfy*

$$X_N\big(\varphi(p)\big) = d\varphi(p).X_M(p) \quad \text{ and } \quad Y_N\big(\varphi(p)\big) = d\varphi(p).Y_M(p)$$

for all $p \in M$, i.e., $X_N \circ \varphi = T\varphi \circ X_M$ and $Y_N \circ \varphi = T\varphi \circ Y_M$. Then $[X_N, Y_N] \circ \varphi = T\varphi \circ [X_M, Y_M]$.

Proof. It suffices to perform a local calculation. Therefore we may w.l.o.g. assume that $M \subseteq F$ is open, where F is a s.c.l.c. space and that N is an s.c.l.c. space. Then

$$[X_N, Y_N]\widetilde{\ }\big(\varphi(p)\big) = d\widetilde{Y}_N\big(\varphi(p)\big).\widetilde{X}_N\big(\varphi(p)\big) - d\widetilde{X}_N\big(\varphi(p)\big).\widetilde{Y}_N\big(\varphi(p)\big).$$

Next we note that our assumption implies that $\widetilde{Y}_N \circ \varphi = d\varphi \circ (\mathrm{id}_F \times \widetilde{Y}_M)$. Using the chain rule we obtain

$$d\widetilde{Y}_N\big(\varphi(p)\big)d\varphi(p) = d(d\varphi)\big(p, \widetilde{Y}_M(p)\big) \circ \big(\mathrm{id}_F, d\widetilde{Y}_M(p)\big)$$

which, in view of Remark I.5(c), leads to

$$\begin{aligned} d\widetilde{Y}_N\big(\varphi(p)\big).\widetilde{X}_N\big(\varphi(p)\big) &= d\widetilde{Y}_N\big(\varphi(p)\big)d\varphi(p).\widetilde{X}_M(p) \\ &= d(d\varphi)\big(p, \widetilde{Y}_M(p)\big) \circ \big(\mathrm{id}_F, d\widetilde{Y}_M(p)\big).\widetilde{X}_M(p) \\ &= d^2\varphi(p)\big(\widetilde{Y}_M(p), \widetilde{X}_M(p)\big) + d\varphi(p)\big(d\widetilde{Y}_M(p).\widetilde{X}_M(p)\big). \end{aligned}$$

Now the symmetry of the second derivative (Lemma I.2(ii)) implies that

$$[X_N, Y_N]\widetilde{\ }\big(\varphi(p)\big) = d\varphi(p)\big(d\widetilde{Y}_M(p).\widetilde{X}_M(p) - d\widetilde{X}_M(p).\widetilde{Y}_M(p)\big) = d\varphi(p)\big([X_M, Y_M]\widetilde{\ }(p)\big).$$

■

Infinite-dimensional Lie groups

In this subsection we consider *s.c.l.c. Lie groups*, i.e., Lie groups modeled over s.c.l.c. spaces. Basically we follow [Mi83]. Throughout this subsection G denotes such a Lie group, i.e., G is a smooth manifold which is a group such that multiplication and inversion are smooth maps. For $g \in G$ we write $\lambda_g\colon G \to G, x \mapsto gx$ for the left-multiplication with g and $\rho_g\colon G \to G, x \mapsto xg$ for the right-multiplication with g. Both are diffeomorphisms of G. Moreover, we write $m\colon G \times G \to G, (x, y) \mapsto xy$ for the multiplication map and $\eta\colon G \to G, x \mapsto x^{-1}$ for the Inversion.

Lemma I.15. *Let* $\mathfrak{g} := T_{\mathbf{1}}(G)$ *denote the tangent space in the identity. Then the mapping*

$$\Phi: G \times \mathfrak{g} \to TG, \quad (g, X) \mapsto d\lambda_g(\mathbf{1}).X$$

is a diffeomorphism.

Proof. First we note that for a product of two smooth manifolds M and N we have a canonical diffeomorphism $T(M \times N) \to TM \times TN$. Since the multiplication map $m: G \times G \to G$ is smooth, the same holds for its tangent map

$$Tm: T(G \times G) \cong TG \times TG \to TG.$$

In view of Proposition I.4, $dm(g, \mathbf{1})(0, X) = d\lambda_g(\mathbf{1}).X$. Therefore the smoothness of Φ follows from $\Phi(g, X) = Tm(g, X)$ for $(g, X) \in G \times T_{\mathbf{1}}(G) \subseteq T(G) \times T(G)$ and the fact that the restriction of Tm to $G \times T_{\mathbf{1}}(G) \subseteq TG \times TG$ is smooth.

To see that Φ^{-1} is also smooth, let $\pi: TG \to G$ denote the canonical projection. Then

$$\Phi^{-1}: TG \to G \times \mathfrak{g}, \quad v \mapsto \big(\pi(v), d\lambda_{\pi(v)^{-1}}\big(\pi(v)\big).v\big).$$

The maps

$$\alpha: TG \to TG \times TG, \quad v \mapsto \big(\pi(v), v\big) \in G \times TG$$

and $\widetilde{m}: G \times G \to G, (g_1, g_2) \mapsto g_1^{-1} g_2$ are smooth by the chain rule. Now

$$T(\widetilde{m}) \circ \alpha(v) = T(\widetilde{m})\big(\pi(v), v\big) = d_2\widetilde{m}\big(\pi(v), \pi(v)\big).v = d\lambda_{\pi(v)^{-1}}\big(\pi(v)\big).v$$

shows that Φ^{-1} is smooth. ■

The essential consequence of Lemma I.15 is that the tangent bundle of a Lie group is trivial, so that we can identify $\mathcal{V}(G)$ with $C^\infty(G, \mathfrak{g})$. We write $\mathcal{V}(G)^l \subseteq \mathcal{V}(G)$ for the subspace of *left invariant* vector fields, i.e., of those satisfying

$$X(g) = d\lambda_g(\mathbf{1}).X(\mathbf{1}) \tag{1.3}$$

for all $g \in G$. These are the vector fields that correspond to constant functions $G \to \mathfrak{g}$. We see in particular that each left invariant vector field is smooth, so that the mapping

$$\mathcal{V}(G)^l \to \mathfrak{g}, \quad X \mapsto X(\mathbf{1})$$

is a bijection. Moreover, Lemma I.14 implies that for $X, Y \in \mathcal{V}(G)^l$ we have

$$[X, Y](g) = d\lambda_g(\mathbf{1}).[X, Y](\mathbf{1}),$$

i.e., that $[X, Y] \in \mathcal{V}(G)^l$. Thus there exists a unique Lie bracket on $\mathfrak{g}$ satisfying

$$[X, Y](\mathbf{1}) = [X(\mathbf{1}), Y(\mathbf{1})]$$

for all left invariant vector fields on G.

Definition I.16. The Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$ is called *the Lie algebra of* G. ■

Definition I.17. Let G be a Lie group. Then for each $g \in G$ the map $I_g : G \to G, x \mapsto gxg^{-1}$, is a smooth automorphism, hence induces a continuous linear automorphism

$$\mathrm{Ad}(g) := dI_g(\mathbf{1}) : \mathfrak{g} \to \mathfrak{g}.$$

We thus obtain an action $G \times \mathfrak{g} \to \mathfrak{g}, (g, X) \mapsto \mathrm{Ad}(g).X$ called the *adjoint action* of G on $\mathfrak{g}$. ■

Proposition I.18. *For a Lie group* G *the following assertions hold:*

(i) $dm(g_1, g_2)(X_1, X_2) = d\rho_{g_2}(g_1).X_1 + d\lambda_{g_1}(g_2).X_2$ *and in particular we have* $dm(\mathbf{1}, \mathbf{1})(X_1, X_2) = X_1 + X_2$.

(ii) $d\eta(\mathbf{1}).X = -X$.

(iii) *The mapping* $Tm : TG \times TG \to TG$ *defines a Lie group structure on* TG *with identity element* $\Phi(\mathbf{1}, 0)$ *and inversion* $T\eta$. *More explicitly multiplication and inversion are given by*

$$\Phi(g_1, X_1) \cdot \Phi(g_2, X_2) = \Phi\big(g_1 g_2, \mathrm{Ad}(g_2)^{-1}.X_1 + X_2\big)$$

and $\Phi(g, X)^{-1} = \Phi\big(g^{-1}, -\mathrm{Ad}(g).X\big)$.

(iv) *If* $X_l : G \to TG$ *is a left invariant vector field with* $X_l(\mathbf{1}) = X$, *then* $X_r : g \mapsto -X_l(g)^{-1}$ *is a right-invariant vector field with* $X_r(\mathbf{1}) = X$. *The assignment* $\mathfrak{g} \to \mathcal{V}(G)^r, X \mapsto X_r$ *is an antiisomorphism of Lie algebras.*

(v) *If* $\sigma : G \times M \to M$ *is a smooth action of* G *on the smooth manifold* M, *then* $T\sigma : TG \times TM \to TM$ *is a smooth action of* TG *on* TM. *The assignment*

$$\dot\sigma : \mathfrak{g} \to \mathcal{V}(M), \qquad \text{with} \qquad \dot\sigma(X)(p) := -d\sigma(\mathbf{1}, p)(X, 0)$$

defines a homomorphism of Lie algebras.

Proof. (i) In view of Proposition I.4, we have

$$\begin{aligned} dm(g_1, g_2)(X_1, X_2) &= d_1 m(g_1, g_2)(X_1) + d_2 m(g_1, g_2)(X_2) \\ &= d\rho_{g_2}(g_1).X_1 + d\lambda_{g_1}(g_2).X_2. \end{aligned}$$

(ii) From $m \circ (\mathrm{id}_G \times \eta) = \mathbf{1}$, we derive $0 = dm(\mathbf{1}, \mathbf{1})\big(X, d\eta(\mathbf{1}).X\big) = X + d\eta(\mathbf{1}).X$ and hence the assertion.

(iii) Let $\varepsilon : G \to \{\mathbf{1}\}$ denote the constant map and $u : \{\mathbf{1}\} \to G$ the group morphism representing the identity element. Then the group axioms for G are encoded in the relations $m \circ (m \times \mathrm{id}) = m \circ (\mathrm{id} \times m)$ (associativity), $m \circ (\eta \times \mathrm{id}) = m \circ (\mathrm{id} \times \eta) = \varepsilon$ (inversion), and $m \circ (u \times \mathrm{id}) = m \circ (\mathrm{id} \times u) = \mathrm{id}$ (unit element). Using the functorial properties of T, we see that these properties carry over to the corresponding maps on TG and show that TG is a Lie group with multiplication Tm, inversion $T\eta$, and unit element $\Phi(\mathbf{1}, 0)$.

To derive an explicit formula for the multiplication in terms of the trivialization described in Lemma I.15, using (i), we calculate

$$\begin{aligned}\Phi(g_1,X_1)\cdot\Phi(g_2,X_2) &= dm(g_1,g_2)\big(d\lambda_{g_1}(\mathbf{1}).X_1, d\lambda_{g_2}(\mathbf{1}).X_2\big)\\ &= d\rho_{g_2}(g_1)d\lambda_{g_1}(\mathbf{1}).X_1 + d\lambda_{g_1}(g_2)d\lambda_{g_2}(\mathbf{1}).X_2\\ &= d\lambda_{g_1g_2}(\mathbf{1})\big(d\lambda_{g_2}^{-1}(g_2)d\rho_{g_2}(\mathbf{1}).X_1 + X_2\big)\\ &= \Phi\big(g_1g_2, \mathrm{Ad}(g_2)^{-1}.X_1 + X_2\big).\end{aligned}$$

The formula for the inversion follows directly from this formula.
(iv) In view of (ii) above, we have

$$X_r(g) = -d\eta(g^{-1}).X_l(g^{-1}) = -d\eta(g^{-1})d\lambda_{g^{-1}}(\mathbf{1}).X = -d\rho_g(\mathbf{1})d\eta(\mathbf{1}).X = d\rho_g(\mathbf{1}).X$$

and this proves the first part. The second part follows from Lemma I.14 which shows that

$$[X_r,Y_r](g) = d\eta(g^{-1}).[X_l,Y_l](g^{-1}) = d\eta(g^{-1}).[X,Y]_l(g^{-1}) = -[X,Y]_r(g).$$

(v) That $T\sigma$ defines an action of TG on TM follows in the same way as in (iii) above by applying T to the commutative diagrams defining a group action.

For the second part we pick $p \in M$ and write $\varphi_p: G \to M, g \mapsto g.p$ for the smooth orbit map of p. Then the equivariance of φ_p means that $\varphi_p \circ \rho_g = \varphi_{g.p}$. From that we derive

$$-d\varphi_p(g).X_r(g) = -d\varphi_p(g)d\rho_g(\mathbf{1}).X = -d\varphi_{g.p}(\mathbf{1}).X = \dot\sigma(X)(g.p).$$

Therefore Lemma I.14 and (iv) imply that

$$\dot\sigma([X,Y])(p) = -d\varphi_p(\mathbf{1})[X,Y]_r(\mathbf{1}) = d\varphi_p(\mathbf{1})[X_r,Y_r](\mathbf{1}) = [\dot\sigma(X),\dot\sigma(Y)](p). \quad\blacksquare$$

Remark I.19. If S is an s.c.l.c. semigroup, i.e., a manifold modeled over an s.c.l.c. space which is endowed with a smooth semigroup multiplication $m: S\times S \to S$, then Proposition I.18(iii) and (v) also hold in the following sense. The mapping $Tm: TS \times TS \to TS$ is an s.c.l.c. semigroup structure on the tangent bundle TS, and if $\sigma: M \times S \to M$ is a smooth right action of S on the manifold M, then $T\sigma: TM \times TS \to TM$ is a smooth right action of TS on the tangent bundle TM.■

II. Dual spaces of locally convex spaces

In the next section we will have to deal with topologies on function spaces which play a crucial role in representation theory. In this section we discuss the basic properties of the relevant topologies on the dual space of a locally convex space. In particular we discuss completeness of the dual space, metrizability, and the properties of the corresponding evaluation map $\eta: X \to X''$ given by $\eta(x)(\alpha) = \alpha(x)$.

Let X' denote the space of continuous linear functionals on the locally convex space X, *the topological dual.* If X^* denotes the set of all linear functionals $X \to \mathbb{C}$, then $X' \subseteq X^*$ is a subspace. There are several natural locally convex topologies on the space X'. We write X'_σ (X'_γ, X'_c, X'_b) for the space X' endowed with the *weak-$*$-topology*, i.e., the topology of pointwise convergence (the topology of uniform convergence on compact convex, compact, bounded subsets of X). The space X'_b is called the *strong dual.* Note that we have the following continuous bijections:

$$X'_b \to X'_c \to X'_\gamma \to X'_\sigma.$$

Before we turn to a closer investigation of the various dual spaces of locally convex spaces, we introduce an important class of locally convex spaces.

Definition II.1. Let X be a vector space which can be written as $X = \bigcup_{n=1}^\infty X_n$, where $X_n \subseteq X_{n+1}$ are subspaces of X which are endowed with the structures of locally convex spaces in such a way that the inclusion mappings $X_n \to X_{n+1}$ are topological embeddings. Then we obtain a locally convex vector topology on X by defining a seminorm p on X to be continuous if and only if its restriction to all the subspaces X_n is continuous. We call X the *strict inductive limit* of the spaces $(X_n)_{n\in\mathbb{N}}$. If, in addition, the spaces X_n are Fréchet spaces, then X is called an *LF space.* ■

A locally convex space X is called *barreled* if all lower semicontinuous seminorms on X are continuous. Geometrically this property can be interpreted as follows. A closed convex balanced subset of X is called a *barrel* if it is absorbing. Then X is barreled if and only if all barrels are 0-neighborhoods (cf. [He89, p.11]). Baire spaces are always barreled ([He89, Prop. 1.4.1]).

Proposition II.2. *If X is a strict inductive limit of the spaces $(X_n)_{n\in\mathbb{N}}$, then the following assertions hold:*

(i) *$X_n \hookrightarrow X$ is an embedding.*

(ii) *A linear map $f: X \to Y$, where Y is a locally convex space, is continuous if and only if its restriction to each X_n is continuous.*

If, in addition, all the spaces X_n are complete, then:

(iii) *Each* X_n *is closed in* X *and* X *is quasicomplete.*

(iv) *Any bounded subset of* X *is contained in some* X_n.

(v) *If the* X_n *are Baire spaces, then* X *is Baire if and only if* $X = X_n$ *holds for some* $n \in \mathbb{N}$.

(vi) *If* X *is an LF space, then* X *is complete and barreled.*

Proof. (i) [He89, Prop. 1.5.2]
(ii) This follows directly from the description of the topology by continuous seminorms.
(iii),(iv) [He89, Prop. 1.5.3]
(v) First we recall from (iii) that the subspaces X_n are closed. If $X \neq X_n$ holds for all $n \in \mathbb{N}$, then no X_n has an interior point. Therefore $X = \bigcup_{n=1}^{\infty} X_n$ shows that this cannot happen if X is a Baire space. If, conversely, $X = X_n$ for some $n \in \mathbb{N}$, then (i) implies that X is a Baire space.
(vi) For the completeness of X we refer to [Tr67, Th. 13.1]. Let p be a lower semicontinuous seminorm on X. Then the restrictions $p|_{X_n}$ are lower semicontinuous, hence continuous because Fréchet spaces are Baire spaces and therefore barreled. Thus p is continuous, and this shows that X is barreled. ∎

Metrizability

It is well known that for a normed space the strong dual space X'_b is a Banach space, hence that the category of Banach spaces is closed under taking dual spaces. This changes drastically for Fréchet spaces as we will see in Corollary II.7 below.

Definition II.3. Let X be a topological vector space. A subset $K \subseteq X$ is called *precompact* if for each 0-neighborhood $U \subseteq X$ there exists a finite subset $F \subseteq K$ with $K \subseteq F+U$. Note that if $\overline{X}$ denotes the completion of X ([Tr67, Th. 5.2]), then the precompactness of a subset $K \subseteq X$ is equivalent to the relative compactness of K as a subset of $\overline{X}$ (cf. [Tr67, Prop. 6.9]). ∎

Lemma II.4. *If* V *is a locally convex space and* $K \subseteq V$ *is a precompact set, then* $\overline{\mathrm{conv}(K)}$ *is precompact. If, in addition,* V *is quasicomplete, then* $\overline{\mathrm{conv}(K)}$ *is compact.*

Proof. First we use [Tr67, Prop. 7.11] to see that $\mathrm{conv}(K)$ and hence also $C := \overline{\mathrm{conv}(K)}$ is precompact (cf. [Tr67, Def. 6.3]). Further each precompact set is bounded. In fact, let U be a balanced convex 0-neighborhood in X. Then there exists a finite set $F \subseteq X$ with $C \subseteq F + U$ and $F \subseteq nU$ holds for some $n \in \mathbb{N}$, hence $C \subseteq nU + U \subseteq (n+1)U$. If V is quasicomplete, then the fact that C is closed and bounded implies that C is complete and therefore compact because it is precompact. ∎

For a subset B of a locally convex space we define its *polar*

$$\widehat{B} := \{\alpha \in X' : (\forall x \in B)|\alpha(x)| \leq 1\}$$

and for $C \subseteq X'$ we put

$$\widehat{C} := \{\alpha \in X : (\forall \alpha \in C)|\alpha(x)| \leq 1\}.$$

We recall the following basic properties of polar sets. They show in particular that the assignments $B \mapsto \widehat{B}$ and $C \mapsto \widehat{C}$ are mutually inverse bijections from the set of closed convex balanced subsets of X onto the set of weak-$*$-closed convex balanced subsets of X'.

Lemma II.5. (a) *$B \subseteq \widehat{C}$ if and only if $C \subseteq \widehat{B}$.*
(b) *$B \subseteq \widehat{\widehat{B}}$ and $\widehat{\widehat{B}}$ is the balanced convex closure of B.*
(c) *$C \subseteq \widehat{\widehat{C}}$ and $\widehat{\widehat{C}}$ is the balanced convex weak-$*$-closure of C.*
(d) *A closed convex balanced subset $B \subseteq X$ is a barrel if and only if $\widehat{B}$ is weak-$*$-bounded.*
(e) *A subset $B \subseteq X$ is bounded if and only if $\widehat{B}$ is absorbing.*
(f) *If $B \subseteq X$ is compact and convex, then $\widehat{\widehat{B}}$ is compact.*

Proof. (a) is trivial and (b), (c) are consequences of the Bipolar Theorem.
(d) B is a barrel if and only if it is absorbing. In view of $B = \widehat{\widehat{B}}$ this means that the function

$$\eta(x): \widehat{B} \to \mathbb{C}, \quad \alpha \mapsto \alpha(x)$$

is bounded for each $x \in X$. This in turn means that $\widehat{B}$ is weak-$*$-bounded.
(e) According to [He89, Prop. 1.4.2], a subset $B \subseteq X$ is bounded if and only if it is bounded for the weak topology on X which in turn is equivalent to the boundedness of all continuous linear functionals on B, i.e., that $\widehat{B}$ is absorbing.
(f) If $B \subseteq X$ is a compact convex set, then [Bou87, Ch. IV, §1, no. 1, Rem. 1] shows that $\widehat{\widehat{B}}$ is compact. In fact, it is closed and contained in the convex hull of the sets $\pm 2iB, \pm 2B$ which is compact. ∎

Proposition II.6. *Let X be a locally convex Baire space. Then the following assertions hold:*
 (i) *X_b' is metrizable if and only if X is normable.*
 (ii) *X_c' and X_σ' are metrizable if and only if $\dim X < \infty$.*

Proof. If X is finite-dimensional, then $X_\sigma' = X_c' = X_b'$ is metrizable, and if X is normable, then X_b' is a Banach space and in particular metrizable.
(a) Suppose that X_b' is metrizable. Then the there exists a countable basis $(U_n)_{n\in\mathbb{N}}$ of 0-neighborhoods in X_b'. The sets $\widehat{B} \subseteq X_b'$ for $B \subseteq X$ bounded form a neighborhood basis for 0. Hence there exist bounded sets $B_n \subseteq X$ with $\widehat{B_n} \subseteq U_n$.

Let $C_n := \widehat{\widehat{B_n}}$. Then $\widehat{C_n} = \widehat{B_n}$ shows that C_n is bounded because $\widehat{C_n}$ is absorbing (Lemma II.5(e)). Let $x \in X$. Then the evaluation functional

$$\eta(x): X'_b \to \mathbb{C}, f \mapsto f(x)$$

is continuous, i.e., $\widehat{\{x\}} = \{f \in X': |f(x)| \leq 1\}$ is a 0-neighborhood in X'. Thus we find $n \in \mathbb{N}$ with $\widehat{B_n} \subseteq \widehat{\{x\}}$. Now the Bipolar Theorem implies that $x \in \widehat{\widehat{\{x\}}} \subseteq \widehat{\widehat{B_n}} = C_n$ and therefore $X = \bigcup_{n \in \mathbb{N}} C_n$. Since the sets C_n are closed, the fact that X is a Baire space implies that one of the sets C_n has interior points. Hence $C_n - C_n$ is a bounded neighborhood of 0 in X, and therefore X is normable (cf. [He89, p.3]).
(b) Assume that X'_c is metrizable. Then the same argument as above shows that there exists a compact subset $K \subseteq X$ such that $C := \widehat{\widehat{K}}$ has interior points. Since C coincides with the closed balanced convex hull of K (Lemma II.5(b)), it is a precompact subset of X (Lemma II.4). Hence $C - C$ is a precompact 0-neighborhood. Therefore X is normable in such a way that the balls are precompact. Now the balls in the completion $\overline{X}$ of X are compact and therefore $\dim X \leq \dim \overline{X} < \infty$.
(c) If X'_σ is metrizable, then similar arguments as in (b) show that there exists a finite subset $F \subseteq X$ such that $\widehat{\widehat{F}}$ has interior points. But since $\operatorname{span} F$ is closed, it follows that $\widehat{\widehat{F}} \subseteq \operatorname{span} F$, whence $\dim X = \dim \operatorname{span} \widehat{\widehat{F}} < \infty$. ■

Corollary II.7. *If X is a Fréchet space, then X'_c is a Fréchet space if and only if $\dim X < \infty$.* ■

Semireflexivity

We recall that for a locally convex space X we have several natural topologies on the dual space leading to the following continuous bijections:

$$X'_b \xrightarrow{\alpha} X'_c \xrightarrow{\beta} X'_\gamma \xrightarrow{\gamma} X'_\sigma$$

which induce weak-$*$-continuous injective maps

$$(X'_\sigma)' \xrightarrow{\gamma'} (X'_\gamma)' \xrightarrow{\beta'} (X'_c)' \xrightarrow{\alpha'} (X'_b)'.$$

We write $\eta_\sigma: X \to (X'_\sigma)'$ for the evaluation map, and $\eta_\gamma := \gamma' \circ \eta_\sigma$, $\eta_c := \beta' \circ \eta_\gamma$, and $\eta_b := \alpha' \circ \eta_c$. The space X is called *semireflexive* if the map η_b is surjective, hence a bijection. Note that all these maps are injective with a weak-$*$-dense range.

Theorem II.8. *For a locally convex space the following assertions hold:*
(i) *The maps η_σ and η_γ are bijections.*
(ii) *If X is quasicomplete, then η_c is a bijection.*

(iii) *If X is semireflexive, then X is quasicomplete for the original topology and the weak topology.*

Proof. (i) We show that η_γ is surjective. Then η_σ is also surjective because γ' is injective.

If $C \subseteq X$ is a compact convex set, then $\hat{\hat{C}}$ is compact (Lemma II.5(f)). Hence the topology on X'_γ coincides with the topology of uniform convergence on balanced compact convex sets. If C is a balanced compact convex set, then C is also weakly compact and hence $\eta_\gamma(C) \subseteq (X'_\gamma)'$ is weak-$*$-compact. Each $\alpha \in (X'_\gamma)'$ is bounded on some set $\widehat{C} \subseteq X'$, hence contained in some set of the type $n\widehat{\widehat{\eta_\gamma(C)}} = n\eta_\gamma(\hat{\hat{C}}) \subseteq \eta_\gamma(X)$ (Bipolar Theorem). This proves that $\eta_\gamma(X) = (X'_\gamma)'$.
(ii) If X is quasicomplete and $C \subseteq X$ is compact, then $\overline{\mathrm{conv}(C)}$ is compact (Lemma II.4). Therefore the mapping $\beta\colon X'_c \to X'_\gamma$ is a homeomorphism, i.e., $X'_c = X'_\gamma$. Since η_γ is bijective according to (i), the surjectivity of $\eta_c = \beta' \circ \eta_\gamma$ follows.
(iii) (cf. [He89, Th. 1.1.2(e)]) Let $C \subseteq X$ be closed balanced convex and bounded. Then C is also weakly closed, and therefore $\eta_b(C) \subseteq \eta_b(X) = (X'_b)'$ is a weak-$*$-closed convex balanced subset. Since $\widehat{\eta_b(C)} = \widehat{C} \subseteq X'_b$ is a 0-neighborhood, the set $\eta_b(C)$ is weak-$*$-compact (Banach-Alaoglu Theorem). Hence C is weakly compact.

Now let $B \subseteq X$ be closed and bounded. Then its closed balanced convex hull C is also bounded, hence weakly compact and therefore in particular weakly complete. Further each Cauchy net in B for the original topology is a weak Cauchy net, hence converges weakly in B and therefore also in the strong topology because the closed convex neighborhoods of a point in X are also weakly closed. ∎

Proposition II.9. *Let X be a locally convex space.*

(i) *A subset $K \subseteq X'$ is equicontinuous if and only if its polar $\widehat{K} \subseteq X$ is a 0-neighborhood in X.*

(ii) *If K is equicontinuous, then*

(a) *K is weak-$*$-relatively compact.*

(b) *K is relatively compact in X'_c.*

(c) *K is strongly bounded.*

Furthermore (a), (b) *or* (c) *implies that K is weak-$*$-bounded, i.e., $\widehat{K} \subseteq X$ is a barrel. These properties are all equivalent if and only if X is barreled.*

(iii) *If X is barrelled, then the following properties are equivalent for $K \subseteq X'$:*

(a) *K is equicontinuous.*

(b) *K is bounded for one of the topologies X'_σ, X'_γ, X'_c or X'_b.*

(c) *K is relatively compact for one of the topologies X'_σ, X'_γ or X'_c.*

Proof. (i) This is more or less the definition of equicontinuity (cf. [Tr67, Prop. 32.7]).
(ii) ([He89, Th. 1.4.4]) If K is equicontinuous, then its balanced convex closure in the weak-$*$-topology of K has the same polar set $\widehat{K} \subseteq X$ (Lemma II.5(c)). So we

may w.l.o.g. assume that $K = \widehat{\widehat{K}}$. Since $\widehat{K}$ is a 0-neighborhood in X, the weak-$*$-compactness of $K = \widehat{\widehat{K}}$ follows from the Banach-Alaoglu Theorem. Now the topology of compact convergence and the weak-$*$-topology coincide on K ([Tr67, Prop. 32.5]), so that K is also compact in X'_c. If $B \subseteq X$ is bounded, then there exists $n \in \mathbb{N}$ with $B \subseteq n\widehat{K}$, i.e., $K \subseteq n\widehat{B}$. Hence K is strongly bounded. It is clear that (a), (b) or (c) implies that K is weak-$*$-bounded.

The equivalence of the stated properties is equivalent to the assertion that if K is weakly bounded then K is equicontinuous, i.e., that the barrel $\widehat{K}$ is a 0-neighborhood (Lemma II.5(d)). This is true if X is barreled, and if, conversely, X is not barreled and $B \subseteq X$ is a barrel which is not a 0-neighborhood, then its polar $\widehat{B} \subseteq X'$ is weakly bounded but not equicontinuous.
(iii)(a) $\Rightarrow$ (b): If K is equicontinuous, then (ii) implies that K is bounded in X'_b, hence also in the spaces X'_σ, X'_γ and X'_c.
(b) $\Rightarrow$ (c): If (b) holds, then K is in particular bounded in X'_σ, i.e., weak-$*$-bounded. Hence (ii) shows that it is also relatively compact in X'_c. Thus it is also compact as a subset of X'_γ and X'_σ.
(c) $\Rightarrow$ (a): If K is relatively compact for one of the topologies X'_σ, X'_γ or X'_c, then it is in particular weak-$*$-relatively compact, hence weak-$*$-bounded. As we have seen in the preceding argument, this implies that K is equicontinuous. ∎

Lemma II.10. *For a locally convex space X the following assertions hold:*

(i) *The mapping $\eta_c\colon X \to (X'_c)'_c$ is an open map onto $\eta_c(X)$.*

(ii) *The mapping $\eta_b\colon X \to (X'_b)'_b$ is an open map onto $\eta_b(X)$.*

(iii) *If X is barreled, then the maps $\eta_c\colon X \to (X'_c)'_c$ and $\eta_b\colon X \to (X'_b)'_b$ are embeddings.*

Proof. (i) If $U \subseteq X$ is a closed convex balanced 0-neighborhood, then $\widehat{U} \subseteq X'_c$ is closed and equicontinuous, hence compact in X'_c (Proposition II.9(ii)(b)). Therefore $\widehat{\widehat{U}} \subseteq (X'_c)'_c$ is a 0-neighborhood with $\widehat{\widehat{U}} \cap \eta_c(X) = \eta_c(U)$ (Bipolar Theorem). Thus η_c is open onto $\eta_c(X)$.
(ii) For a closed convex balanced 0-neighborhood $U \subseteq X$ the polar set $\widehat{U} \subseteq X'$ is equicontinuous and therefore strongly bounded (Proposition II.9(ii)(c)). Thus $\widehat{\widehat{U}} \subseteq (X'_b)'_b$ is a 0-neighborhood with $\widehat{\widehat{U}} \cap \eta_b(X) = \eta_b(U)$. Therefore η_b is open onto $\eta_b(X)$.
(iii) Suppose that X is barreled. If $K \subseteq X'_c$ is compact or $K \subseteq X'_b$, then it is equicontinuous (Proposition II.9(iii)), and therefore $\widehat{K} \subseteq X$ is a 0-neighborhood. Hence $\eta_c\colon X \to (X'_c)'_c$ and $\eta_b\colon X \to (X'_b)'_b$ are continuous maps. In view of (i) and (ii), this means that both are embeddings. ∎

Theorem II.11. (Reflexivity criterion for the c-topologies) *If X is a quasicomplete barreled space, then $\eta_c\colon X \to (X'_c)'_c$ is an isomorphism of topological vector spaces. This holds in particular if X is an LF space.*

Proof. Since X is quasicomplete, the surjectivity of η_c follows from Theorem II.8(ii). If, in addition, X is barreled, then Lemma II.10(iii) shows that η_c is an isomorphism of topological vector spaces.

To see that the assertion holds for LF spaces, we recall from Proposition II.2(vi) that they are complete and barreled. ■

Completeness properties of the dual space

Now we turn to the question whether a dual space X' is complete with respect to a given topology. The following lemma is the topological background for the completeness criteria.

Proposition II.12. (i) *Let X be a topological space satisfying the first axiom of countability and V be a (sequentially) complete locally convex space. Then the space $C(X,V)_c$ of continuous maps $X \to V$ is a (sequentially) complete locally convex space with respect to the topology of uniform convergence on compact subsets of X.*
(ii) *If X is an LF space and V is a (sequentially) complete locally convex space, then the space $\mathcal{L}(X,V)_c$ of continuous linear maps endowed with the topology of uniform convergence on compact subsets of X is a (sequentially) complete locally convex space.*
(iii) *If X is a Baire space and V is an s.c.l.c. space, then the space $\mathcal{L}(X,V)$ is sequentially complete with respect to any topology of uniform convergence on a system of subsets of X whose union is X.*

Proof. (i) That $C(X,V)_c$ is a locally convex space follows from the fact that its topology is defined by the seminorms

$$p_K(f) := \sup\{p\big(f(x)\big) : x \in K\},$$

where $K \subseteq X$ is a compact subset and $p: V \to \mathbb{R}^+$ is a continuous seminorm.

Let $\mathcal{F}$ be a Cauchy-Filter in $C(X,V)_c$. Since V is complete, $\mathcal{F}$ converges pointwise to a function $f: X \to V$. We claim that $\mathcal{F}$ converges uniformly on each compact subset K of X. In fact, let p be a continuous seminorm on V and $\varepsilon > 0$. Then there exists $F \in \mathcal{F}$ with $p_K(g-h) \leq \varepsilon$ for all $g, h \in F$. Since $f(x) \in \overline{\mathcal{F}(x)}$ holds for all $x \in K$, we conclude that $p_K(g-f) \leq \varepsilon$ for all $g \in F$. Hence $\mathcal{F} \to f$ holds uniformly on each compact subset $K \subseteq X$ and thus f is continuous on each compact subset of X.

If $(x_n)_{n\in\mathbb{N}}$ with $x_n \to x$ is a convergent sequence in X, then the set $\{x\} \cup \{x_n : n \in \mathbb{N}\}$ is compact. Since f is continuous on this set, it is continuous by our assumption on the space X. This proves that $C(X,V)_c$ is complete.

If V is sequentially complete, then similar arguments show that each Cauchy sequence in $C(X,V)_c$ converges, hence that $C(X,V)_c$ is sequentially complete.

(ii) ([Tr67, Cor. 32.2.4, p.345]) First we note that Fréchet spaces satisfy the assumption of (i). So let $(X_n)_{n\in\mathbb{N}}$ be a defining sequence for the topology on X. That $\mathcal{L}(X,V)_c$ is locally convex follows as in (i). If $\mathcal{F}$ is a Cauchy filter in $\mathcal{L}(X,V)_c$, then we see as in (i) that $\mathcal{F}$ converges pointwise to some function $f: X \to V$. Then f must be linear, and, in view of (i), f is continuous on each of the subspaces X_n, hence is continuous on X. This proves that $\mathcal{L}(X,V)_c$ is complete. If V is sequentially complete, then we see by a similar argument that $\mathcal{L}(X,V)_c$ is sequentially complete.
(iii) If $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{L}(X,V)$ for the topology of uniform convergence on a system $\mathcal{S}$ of subsets of X whose union is X, then the sequential completeness of V implies that f_n converges pointwise to a linear function $f: X \to V$. It follows in particular that f is (G)-holomorphic. Therefore the continuity of f follows from Proposition I.9(iii). Since (f_n) is a Cauchy sequence for the topology of uniform convergence on the sets in $\mathcal{S}$, we see that $f_n \to f$ holds uniformly on sets in $\mathcal{S}$. This proves that $\mathcal{L}(X,V)$ is sequentially complete with respect to the topology of uniform convergence on sets in $\mathcal{S}$. ■

Corollary II.13. (a) *If X is an LF space, then X'_c is a complete locally convex space.*
(b) *If X is a Baire space, then X'_σ, X'_γ, X'_c, and X'_b are sequentially complete.* ■

Note that in general one cannot expect that the dual X' is complete with respect to the topology of pointwise convergence. With respect to this topology the embedding $X'_\sigma \hookrightarrow X^*$ is a dense embedding if X^* carries the topology of pointwise convergence. Therefore X'_σ is not complete unless $X' = X^*$, i.e., each linear functional on X is continuous. This holds in particular for the finest locally convex topology on X, i.e., the topology for which all seminorms are continuous, and also for the weak topology defined by X^*.

Lemma II.14. *If X'_σ is quasicomplete, then the same holds for X'_γ, X'_c and X'_b.*
Proof. If $B \subseteq X'$ is closed and bounded for one of the topologies X'_γ, X'_c or X'_b, then B is also weak-$*$-bounded. Let $\mathcal{F}$ be a Cauchy filter in B. Then $\mathcal{F}$ converges to some element α in the weak-$*$-closure of B. Then $\mathcal{F}$ also converges to α in the original topology, and we see that $\alpha \in B$. This shows that B is complete, i.e., that X'_γ, X'_c and X'_b are quasicomplete. ■

Proposition II.15. *If X is barreled or semireflexive, then the spaces X'_σ, X'_γ, X'_c, and X'_b are quasicomplete.*
Proof. First we assume that X is barreled. In view of Lemma II.14, it suffices to show that X'_σ is quasicomplete. Let $B \subseteq X'_\sigma$ be closed and bounded. Then $\widehat{B}$ is a barrel (Lemma II.5(d)), hence a 0-neighborhood, and therefore Proposition II.9(ii) shows that B is weak-$*$-compact, hence in particular weak-$*$-complete.

If X is semireflexive, then X'_b is also semireflexive and therefore weakly quasicomplete ([He89, Th. 1.1.2(d)(e)] and Theorem II.8). Further $\eta_b(X) = (X'_b)'$, so that the weak topology on X'_b coincides with the weak-$*$-topology. Thus X'_σ is quasicomplete. ■

To clarify the relation between the assumptions in Proposition II.15, we note that a barreled space need not be semireflexive because there exist Banach spaces which are not reflexive. On the other hand one would not expect that the semireflexivity has strong implications for the topology on X because it only means that the map η_b is surjective. Nevertheless the following lemma shows that it has consequences for the strong dual.

Lemma II.16. *If X is semireflexive, then the strong dual X'_b is barreled. Furthermore the maps*

$$\widetilde{\eta}_b\colon X'_b \to ((X'_b)'_b)'_b \quad \textit{and} \quad \widetilde{\eta}_c\colon X'_b \to ((X'_b)'_c)'_c$$

are topological isomorphisms.

Proof. Let $C \subseteq X'_b$ be a barrel. Then C is convex and closed in X'_b, hence also weakly closed. Thus $\eta_b(X) = (X'_b)'$ shows that C is also weak-$*$-closed, and the Bipolar Theorem gives $\widehat{\widehat{C}} = C$. But $\widehat{C} \subseteq X$ is weakly bounded (Lemma II.5(e)), and so $\widehat{C}$ is bounded which in turn implies that $C = \widehat{\widehat{C}}$ is a 0-neighborhood in X'_b. This proves that X'_b is barreled.

Moreover X'_b is semireflexive and quasicomplete ([He89, Th. 1.1.2(d)(e)]), so Theorem II.11 implies that $\widetilde{\eta}_c$ is an isomorphism. Since X'_b is semireflexive and barreled, the assertion about $\widetilde{\eta}_b$ follows from Lemma II.10(iii). ■

III. Topologies on function spaces

To construct and analyze representations of infinite-dimensional Lie groups and semigroups one often has to consider representations in spaces of smooth functions on G. So one has to endow these function spaces with a suitable (sequentially) complete locally convex topology. The importance of these spaces comes from the fact that for smooth representations a dense subspace of the representation space V can be embedded in $C^\infty(G, V)$.

First we discuss the space $C^\infty(M, V)$ of smooth functions on M with values in an s.c.l.c. space V and show that this space carries a natural s.c.l.c. topology which is, roughly stated, the topology of uniform convergence of all derivatives on compact sets. The main point here is to use the appropriate interpretation of the higher derivatives that permits inductive arguments. We also show that smooth Lie group actions lead to smooth actions on the corresponding spaces of smooth functions.

Next we show that smooth mappings between open subsets of s.c.l.c. spaces induce smooth mappings on the level of function spaces. This result is crucial to show that groups of the type $C^\infty(M,G)$, M a compact manifold and G a finite dimensional Lie group are in fact Lie groups modeled over Fréchet spaces in the sense specified in Section I (cf. [Ne99]).

Finally we turn to the space of holomorphic functions on a complex manifold M over a Baire s.c.l.c. space with values in a s.c.l.c. space V and show that it is sequentially complete with respect to the topology of uniform convergence on compact subsets and that holomorphic semigroup actions lead to holomorphic actions on the corresponding spaces of holomorphic functions. Here the assumption that M is modeled on a Baire space, an assumption which is in particular satisfied for Fréchet spaces, is crucial for the sequential completeness of the space of holomorphic functions on M.

The space $C^\infty(M,V)$

Let V be a (sequentially) complete locally convex space. If M is a smooth Fréchet manifold, then we write $C^\infty(M,V)_c$ for the space $C^\infty(M,V)$ endowed with the topology of compact convergence. This topology on $C^\infty(M,V)$ need not be complete. Nevertheless, the space $C(M,V)_c$ is (sequentially) complete by Proposition II.12(i).

For $f \in C^\infty(M,V)$ we obtain a smooth function $df\colon T(M) \to V$, where we identify $T_v(V)$ with V in each point $v \in V$, and inductively we get smooth functions $d^{(n)}f\colon T^{(n)}(M) \to V$. Thus we obtain an embedding

$$C^\infty(M,V) \to \prod_{n=0}^\infty C^\infty\big(T^{(n)}(M),V\big)_c.$$

We endow $C^\infty(M,V)$ with the topology induced by the product topology via this embedding (cf. [Th95]). Note that if $M = X$ is a vector space, then $X'_c \to C^\infty(X,\mathbb{C})$ is a topological embedding.

Proposition III.1. *If M is a Fréchet manifold and V is a (sequentially) complete locally convex space, then the space $C^\infty(M,V)$ is a (sequentially) complete locally convex space.*

Proof. Let $(f_i)_{i\in I}$ be a Cauchy net in $C^\infty(M,V)$. Then Proposition II.12(i) implies the existence of continuous functions $F_n\colon T^{(n)}(M) \to V$ such that $d^{(n)}f_i \to F_n$ holds uniformly on each compact subset of $T^{(n)}(M)$.

Next we show that $f \in C^1(M,V)$. To do this, we may w.l.o.g. assume that M is an open subset of a Fréchet space X. Then the uniform convergence of $df_i \to F_1$ on compact sets implies for each sufficiently small $t \neq 0$ that

$$\begin{aligned}\frac{1}{t}\big(f(x+th)-f(x)\big) &= \lim_I \frac{1}{t}\big(f_i(x+th)-f_i(x)\big) = \lim_I \int_0^1 df_i(x+uth)(h)\,du \\ &= \int_0^1 F_1(x+uth)(h)\,du.\end{aligned}$$

Now the continuity of F_1 leads to

$$\lim_{t\to 0}\frac{1}{t}\big(f(x+th)-f(x)\big) = \lim_{t\to 0}\int_0^1 F_1(x+uth)(h)\,du = \int_0^1 F_1(x)(h)\,du = F_1(x)(h).$$

This proves that $f \in C^1(M,V)$ with $df = F_1$. By induction we now obtain $f \in C^n(M,V)$ and $d^{(n)}f = F_n$. Thus $f \in C^\infty(M,V)$ and $f_i \to f$ holds in $C^\infty(M,V)$.■

Before we proceed, we need a topological lemma.

Lemma III.2. *Let M and N be Hausdorff spaces and V a locally convex space. Then the following assertions hold:*

(i) *For $f \in C(M \times N, V)$ the map*

$$M \to C(N,V)_c, \quad x \mapsto \big(y \mapsto f(x,y)\big)$$

is continuous.

(ii) *If $\alpha\colon M \to N$ is continuous, then the map*

$$\alpha^*\colon C(N,V)_c \to C(M,V)_c, \quad f \mapsto f\circ\alpha$$

is continuous.

(iii) *Let S be a metrizable topological semigroup which acts continuously on M from the right. Then the action*

$$S \times C(M,V)_c \to C(M,V)_c, \quad (s,\varphi) \mapsto \big(x \mapsto \varphi(x.s)\big)$$

is continuous.

Proof. (i) First we recall that the topology on $C(N,V)$ coincides with the compact open topology (cf. [Bou71, §3, no. 4, Th. 10]). Let $K \subseteq N$ be compact and $U \subseteq V$ be open. We write $W(K,U) := \{h \in C(N,V)\colon h(K) \subseteq U\}$ for the corresponding fundamental open subset of $C(N,Y)_c$. Suppose that $f_x\colon y \mapsto f(x,y)$ is contained in $W(K,U)$, i.e., $\{x\}\times K \subseteq f^{-1}(U)$. Since $f^{-1}(U)$ is an open subset of $M\times N$ and $\{x\}\times K \subseteq M\times N$ is compact, there exists an open neighborhood $O \subseteq M$ of x such that $O\times K \subseteq f^{-1}(U)$. This means that $x \in O \subseteq \{p \in M\colon f_p \in W(K,U)\}$ which proves the assertion.

(ii) Let $K \subseteq M$ be compact, p a continuous seminorm on V, and $p_K(f) := \sup\{p\big(f(x)\big)\colon x \in K\}$ the corresponding seminorm on $C(M,V)_c$. These seminorms define the topology on this space. Now the set $\alpha(K)$ is compact and $p_K(\alpha^* f) \leq p_{\alpha(K)}(f)$ shows that the seminorms $p_K \circ \alpha^*$ are continuous for each choice of p and K, hence that α^* is continuous.

(iii) Let $s_n \to s$, $f_i \to f$ in $C(M,V)_c$, $K \subseteq M$ a compact subset, and p a continuous seminorm on V. Then the closure $\widetilde{K}$ of the set $\bigcup_{n=1}^\infty K.s_n$ is compact because it is the image of the compact set $K \times \{s, s_n : n \in \mathbb{N}\}$ under the action map. For $x \in K$ we have

$$\begin{aligned} p\big((s_n.f_i)(x) - (s.f)(x)\big) &= p\big(f_i(x.s_n) - f(x.s)\big) \\ &\leq p\big(f_i(x.s_n) - f(x.s_n)\big) + p\big(f(x.s_n) - f(x.s)\big) \\ &\leq p_{\widetilde{K}}(f_i - f) + p\big(f(x.s_n) - f(x.s)\big). \end{aligned}$$

Therefore the uniform continuity of f on $\widetilde{K}$ implies that $p_K(s_n.f_i - s.f) \to 0$. Hence $s_n.f_i \to s.f$ in $C(M, V)_c$. Thus the action of S on $C(M, V)_c$ is continuous.■

In the following lemma the assumption that M is Fréchet is made to insure that the space $C^\infty(M, V)$ is sequentially complete (Proposition III.1), a property needed to make calculus work (cf. Section I).

Lemma III.3. (i) *Let $\alpha: M \to N$ be a smooth map between Fréchet manifolds. Then the linear map*

$$\alpha^*: C^\infty(N, V) \to C^\infty(M, V), \quad f \mapsto f \circ \alpha$$

is continuous.
(ii) *Let M be a Fréchet manifold and $\pi_M: TM \to M$ the canonical projection. Then the assignment*

$$C^\infty(M, V) \to C^\infty(TM, TV) \cong C^\infty(TM, V)^2, \quad f \mapsto Tf = (f \circ \pi_M, df)$$

is an embedding of locally convex spaces.

Proof. (i) (cf. [Th95, Prop. 3]) For $f \in C^\infty(N, V)$ we have $d(f \circ \alpha) = df \circ T\alpha$ and inductively $d^{(n)}(f \circ \alpha) = d^{(n)} f \circ T^{(n)}\alpha$. Therefore the continuity of α^* follows from Lemma III.2(ii).
(ii) Since $d^{(n)}df = d^{(n+1)}f$ for $n \in \mathbb{N}$, it is clear that the map $C^\infty(M, V) \to C^\infty(TM, V), f \mapsto df$ is continuous. Since $C^\infty(M, V) \to C^\infty(TM, V), f \mapsto f \circ \pi_M$ is continuous according to (i), we see that $f \mapsto Tf$ is continuous.

If $\alpha: M \to TM$ is the natural embedding as the 0-section, then $(f \circ \pi_M) \circ \alpha = f$. Therefore (i) shows that the inverse $Tf \to f$ is also continuous. This proves that $f \mapsto Tf$ is an embedding. ■

In many applications the following theorem is a very efficient tool.

Theorem III.4. *Let M and N be Fréchet manifolds, $f \in C^\infty(M \times N, V)$, and $f_x(y) := f(x, y)$. Then the map*

$$\Phi: M \to C^\infty(N, V), \quad x \mapsto f_x$$

is smooth.

Proof. We prove the theorem in several steps. First we note that w.l.o.g. we may assume that M is an open subset of a Fréchet space X.
Claim 1: Φ is continuous. We have $(d^{(n)}f_x)(y) = d^{(n)}f(x,y)$. Therefore

$$T^{(n)}(M \times N) \cong T^{(n)}M \times T^{(n)}N$$

and Lemma III.2(i) show that

$$M \to C(T^{(n)}N, V)_c, \quad x \mapsto d^{(n)}f_x$$

is continuous. In view of the definition of the topology on $C^\infty(N,V)$, this proves that Φ is continuous.
Claim 2: The map

$$\Psi\colon M \times X \to C^\infty(N,V), \quad (x,h) \mapsto \big(y \mapsto d_1 f(x,y)(h)\big)$$

is continuous. This follows from Claim 1 and the fact that $d_1 f \in C^\infty(M \times X \times N, V)$ (cf. Lemma I.5(c)).
Claim 3: Φ is C^1 with $d\Phi(x)(h) = \Psi(x,h)$. First we note that for a sufficiently small $\varepsilon > 0$ the map

$$]-\varepsilon,\varepsilon[\times [0,1] \times M \times X \to C^\infty(N,V), \quad (t,u,x,h) \mapsto \Psi(x+uth,h)$$

is continuous by Claim 2. Therefore

$$]-\varepsilon,\varepsilon[\times M \times X \to C^\infty(N,V), \quad (t,x,h) \mapsto \int_0^1 \Psi(x+uth,h)\ du$$

is continuous and so

$$\lim_{t\to 0} \frac{1}{t}\big(\Phi(x+th) - \Phi(x)\big) = \lim_{t\to 0}\int_0^1 \Psi(x+uth,h)\ du = \int_0^1 \Psi(x,h)\ du = \Psi(x,h).$$

Thus $d\Phi(x)(h) = \Psi(x,h)$, and the continuity of Ψ implies that Φ is C^1.
Claim 4: Φ is smooth. Since $\Psi(x,h)(y) = d_1 f(x,y)(h)$ and

$$d_1 f \in C^\infty(M \ \times \ X \ \times \ N, V),$$

Claim 3 implies that $\Psi \in C^1$, hence that $\Phi \in C^2$. Proceeding inductively, we see that Φ is C^∞. ∎

In the following we call a Fréchet manifold S endowed with a smooth associative multiplication $S \times S \to S$ a *Fréchet semigroup.*

Theorem III.5. *If M is a Fréchet manifold and the Fréchet semigroup S acts smoothly on M via $\sigma: M \times S \to M$, then the action map $\widetilde{\sigma}: S \times C^\infty(M,V) \to C^\infty(M,V)$ given by $(s.f)(x) := f(x.s)$ is smooth.*

Proof. The partial derivative $d_2\widetilde{\sigma}$ with respect to the second argument is given by

$$d_2\widetilde{\sigma}(f,s)(h) = s.h = \widetilde{\sigma}(s,h)$$

because the linear mappings $f \mapsto s.f$ are continuous (Lemma III.3). To see that this maping is continuous means to show that the action of S on $C^\infty(M,V)$ is continuous. We recall that we have defined the topology on $C^\infty(M,V)$ via the embedding

$$C^\infty(M,V) \to \prod_{n=0}^{\infty} C^\infty\big(T^{(n)}(M),V\big)_c.$$

Therefore it suffices to prove the continuity of the action map for S on the spaces

$$C^\infty\big(T^{(n)}(M),V\big)_c.$$

This action comes from the action of S on the manifold $T^{(n)}(M)$. The natural map

$$T^{(n)}\sigma: T^{(n)}(M \times S) \to T^{(n)}(M)$$

is smooth. Comparing with the injection

$$T^{(n)}(M) \times S \hookrightarrow T^{(n)}(M) \times T^{(n)}(S) \cong T^{(n)}(M \times S),$$

we see that the action of S on $T^{(n)}(M)$ is smooth and in particular continuous. So the continuity of the action of S on $C^\infty\big(T^{(n)}(M),V\big)_c$ follows from Lemma III.2(iii).

Now we turn to the first partial derivative $d_1\widetilde{\sigma}$. We write $\pi_S: TS \to S$ and $\pi_M: TM \to M$ for the canonical projections, $\varphi_x: S \to M$, $s \mapsto x.s$ for the orbit map of $x \in M$, and $\rho_s: M \to M, x \mapsto x.s$ for the translation maps on M. For each f the smoothness of the map $s \mapsto s.f$ follows from the smoothness of the function $(s,x) \mapsto f(x.s) = (f \circ \sigma)(x,s)$ on $S \times M$ and Theorem III.4 which also implies that $d_1\widetilde{\sigma}(s,f).v = d_2(f \circ \sigma)(x,s).v$. To see that the partial derivative

$$d_1\widetilde{\sigma}: TS \times C^\infty(M,V) \to C^\infty(M,V)$$

is continuous, we will use the embedding $C^\infty(M,V) \to C^\infty(TM,TV), f \mapsto Tf$ from Lemma III.3(ii). According to Remark I.19, the smooth action $\sigma: M \times S \to M$ induces a smooth right action $T\sigma: TM \times TS \to TM$ so that the first part of the proof shows that the induced action map

$$TS \times C^\infty(TM,V) \to C^\infty(TM,V)$$

is continuous. If $\alpha: M \to TM$ is the 0-section, then we conclude with Lemma III.3(i) that the map

$$\begin{aligned}(v, f) &\mapsto (v, T.f) \mapsto v.Tf = Tf \circ T\sigma(\cdot, v) = T(f \circ \sigma)(\cdot, v) \\ &\mapsto T(f \circ \sigma)(\cdot, v) \circ \alpha \mapsto d(f \circ \sigma)(\cdot, v) \circ \alpha\end{aligned}$$

from $TS \times C^\infty(M, V) \to C^\infty(M, V)$ is continuous. Now

$$d(f \circ \sigma)(\cdot, v) \circ \alpha(x) = d(f \circ \sigma)(x, v) = d_2(f \circ \sigma)\big(x, \pi(v)\big).v = d_1\widetilde{\sigma}\big(\pi(v), f\big).v$$

shows that $d_1\widetilde{\sigma}$ is continuous.

We have shown that $d_1\widetilde{\sigma}$ and $d_2\widetilde{\sigma}$ are continuous, so that Proposition I.4 implies that $d\widetilde{\sigma}$ exists and is continuous, i.e., $\widetilde{\sigma} \in C^1$ with

$$d\widetilde{\sigma}(s, f)\big(v, h\big) = d_1\widetilde{\sigma}\big(\pi(v), f\big).v + s.h.$$

The fact that $\widetilde{\sigma}$ is C^1 implies in particular that $d_2\widetilde{\sigma}$ is C^1 and since $d_1\widetilde{\sigma}$ comes from the smooth action of TS on $C^\infty(TM, V)$, we conclude that this action is a C^1 map. But then $\widetilde{\sigma}$ is C^2. Proceeding inductively we see that $\widetilde{\sigma}$ is a smooth map.■

Smooth mappings between function spaces

In the preceding subsection we have seen how to topologise the space $C^\infty(M, V)$ of smooth functions on a Fréchet manifold M with values in an s.c.l.c. space. Let X and Y be s.c.l.c. spaces, $U \subseteq X$ an open subset, and $f: M \times U \to Y$ a smooth map. Then $C^\infty(M, U)$ is an open subset of the s.c.l.c. space $C^\infty(M, X)$, and

$$f_*: C^\infty(M, U) \to C^\infty(M, Y), \quad \gamma \mapsto f \circ (\mathrm{id}_M, \gamma)$$

is a well defined map. We will show that this map is smooth. First we consider a purely topological situation:

Lemma III.6. *If M is a topological space and $f: M \times U \to Y$ continuous, then the mapping*

$$f_*: C(M, U)_c \to C(M, Y)_c, \quad \gamma \mapsto f \circ (\mathrm{id}_M, \gamma)$$

is continuous.

Proof. First we recall that the topology of uniform convergence coincides with the compact open topology (cf. [Bou71, §3, no. 4, Th. 10]). Let $K \subseteq M$ be compact and $V \subseteq Y$ be open. We write $W(K, V) := \{h \in C(M, Y): h(K) \subseteq V\}$ for the corresponding fundamental open subset of $C(M, Y)_c$. Then

$$f_*^{-1}\big(W(K, V)\big) = \{\gamma \in C(M, U): (\mathrm{id}_M, \gamma)(K) \subseteq f^{-1}(V)\}.$$

To see that this set is open in the compact open topology, let γ_0 be contained in this set and choose for each $x \in K$ a compact neighborhood K_x of x in K and an open neighborhood $U_x \subseteq U$ of $\gamma_0(x)$ such that $\gamma_0(K_x) \subseteq U_x$ and $K_x \times U_x \subseteq f^{-1}(V)$. Then we find finitely many points $x_1, \ldots, x_n \in K$ such that the K_{x_j} cover K. Now each $\gamma \in C(M,U)$ with $\gamma(K_{x_j}) \subseteq U_{x_j}$ satisfies $(\mathrm{id}_M, \gamma)(K_{x_j}) \subseteq K_{x_j} \times U_{x_j} \subseteq f^{-1}(V)$. Hence

$$\bigcap_{j=1}^{n} W(K_{x_j}, U_{x_j}) \subseteq (f_*)^{-1}\big(W(K,V)\big)$$

proves the continuity of f_*. ■

Proposition III.7. *The map*

$$f_*: C^\infty(M,U) \to C^\infty(M,Y), \quad \gamma \mapsto f \circ (\mathrm{id}_M, \gamma)$$

is smooth.

Proof. First we show that f_* is continuous. For $\gamma \in C^\infty(M,X)$ the mapping $T\gamma: T(M) \to T(X) \cong X \times X$ can be split as $T\gamma(v_p) = \big(\gamma(p), d\gamma(p).v_p\big)$, where $d\gamma \in C^\infty(T(M),X)$. Inductively we obtain $d^{(n)}\gamma \in C^\infty(T^{(n)}M, X)$. In this sense $C^\infty(M,X)$ carries the topology induced by the embedding

$$C^\infty(M,X) \hookrightarrow \prod_{n=0}^{\infty} C^\infty\big(T^{(n)}(M), X\big)_c,$$

where the spaces on the right hand side carry the topology of uniform convergence on compact sets. We have

$$T(f_*\gamma) = T\big(f \circ (\mathrm{id}_M, \gamma)\big) = Tf \circ \big(\mathrm{id}_{TM}, T\gamma\big)$$

and thus $d(f_*\gamma) = df \circ \big(\mathrm{id}_{TM}, T\gamma\big)$. Inductively we obtain

$$d^{(n)}(f_*\gamma) = d^{(n)}f \circ \big(\mathrm{id}_{T^{(n)}M}, T^{(n)}\gamma\big). \tag{3.1}$$

In view of Lemma III.6, this shows that the maps $\gamma \to d^{(n)}(f_*\gamma)$ are continuous. We conclude that f_* is continuous.

Next we calculate the derivative of f_*. For each $x \in M$ we have

$$\begin{aligned} &\lim_{h\to 0} \frac{1}{h}\Big(f\big(x, (\gamma + h\eta)(x)\big) - f\big(x, \gamma(x)\big)\Big) \\ &= \lim_{h\to 0} \int_0^1 d_2 f\big(x, (\gamma + uh\eta)(x)\big)\big(\eta(x)\big)\ dx = df_2\big(x, \gamma(x)\big)\big(\eta(x)\big), \end{aligned}$$

where, in view of the continuity of the integrand, the limit on the left hand side exists uniformly on compact subsets of M. In view of (3.1), the same argument applies to the higher derivatives $d^{(n)}f_*$. So we see that $(df_*)(\gamma, \eta)$ exists and equals $d_2 f \circ (\mathrm{id}_M, \gamma, \eta) \in C^\infty(M,Y)$. This means that $d(f_*) = (d_2 f)_*: C^\infty(M, TU) \to C^\infty(M,Y)$. Using the first part of our proof, we now see that $d(f_*)$ is continuous, i.e., f_* is C^1. Since the map $d(f_*)$ can be written as $(d_2 f)_*$, it has the same structure as f_*, and iteration of the argument shows that f_* is smooth. ■

Corollary III.8. *If $f: U \to Y$ is a smooth map, then*

$$f_*: C^\infty(M, U) \to C^\infty(M, Y), \quad \gamma \mapsto f \circ \gamma$$

is smooth.

Proof. Put $\widetilde{f}(x, y) := f(y)$ and apply Proposition III.7. ∎

Applications to groups of continuous mappings

Remark III.9. (a) If F is an s.c.l.c. space and X a compact metric space, then $C(X, F)_c$ is an s.c.l.c. space with respect to the topology of uniform convergence (Propositition II.12(a)).
(b) If $U \subseteq F$ is an open subset, then $C(X, U)$ is an open subset of $C(X, F)_c$. Now let $U_j \subseteq F_j$, $j = 1, 2$, be open subsets of s.c.l.c. spaces and $\varphi: U_1 \to U_2$ a smooth map. We consider the map

$$\varphi_X: C(X, U_1) \to C(X, U_2), \quad \gamma \mapsto \varphi \circ \gamma.$$

Then φ_X is smooth. The continuity follows from Lemma III.6. For each $x \in X$ and $\gamma, \eta \in C(X, F_1)$ we have

$$\begin{aligned} \lim_{t \to 0} \frac{\varphi(\gamma(x) + t\eta(x)) - \varphi(\gamma(x))}{t} &= \lim_{t \to 0} \int_0^1 d\varphi(\gamma(x) + st\eta(x)).\eta(x)\, ds \\ &= d\varphi(\gamma(x)).\eta(x). \end{aligned}$$

Since the integrand is continuous in $[0, 1]^2 \times X$, the limit exists uniformly in X, hence in the space $C(X, F_2)$. Therefore $d\varphi_X(\gamma)(\eta)$ exists. Since $d\varphi: TU_1 \cong U_1 \times F_1 \to F_2$ is a continuous map, the first part of the proof shows that

$$d\varphi_X: C(X, TU_1) \cong C(X, U_1) \times C(X, F_1) \to C(X, F_2)$$

is continuous, so that φ_X is C^1. Iterating this argument shows that φ_X is C^∞. ∎

Proposition III.10. *If G is a Lie group and X is a compact metric space, then $C(X, G)_c$ is a Lie group with Lie algebra $C(X, \mathfrak{g})_c$.*

Proof. We use Remark III.9(b) to see that the inversion and multiplication in the canonical local charts are smooth. The remainder is a routine verification. ∎

Spaces of holomorphic functions

In this subsection we turn to spaces of holomorphic functions. In particular we show that holomorphic actions of complex Fréchet semigroups lead to holomorphic actions on the corresponding spaces of holomorphic functions, and that the inclusion $\mathrm{Hol}(M,V) \to C^\infty(M,V)$ is an embedding if $\mathrm{Hol}(M,V)$ carries the topology of uniform convergence on compact subsets. For refined investigations on topologies on spaces of holomorphic functions between Banach spaces we refer to [Na69].

In the following a *Baire manifold* is a manifold modeled over a s.c.l.c. Baire space.

Theorem III.11. *For a complex Baire manifold M the following assertions hold:*

(i) *If V is an s.c.l.c. space, then* $\mathrm{Hol}(M,V)$ *is s.c.l.c. with respect to the topology of uniform convergence on compact sets.*

(ii) *If, in addition, M is Fréchet and V is complete, then* $\mathrm{Hol}(M,V)$ *is complete.*

Proof. (i) Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathrm{Hol}(M,V)$. Since V is sequentially complete, this sequence converges uniformly on compact subsets of M to a function $f\colon M \to V$ (see the proof of Proposition II.12). It remains to show that f is holomorphic. For that we may w.l.o.g. assume that M is an open subset of a Baire space X. Since (G)-holomorphy is equivalent to weak (G)-holomorphy ([He89, Th. 2.1.3]), and for each $\alpha \in V'$ the function $\alpha \circ f\colon M \to \mathbb{C}$ is holomorphic on the intersection with each finite dimensional affine subspace, we see that $f \in \mathcal{G}(M,V)$. Now Proposition I.9(iii) implies that f is continuous, hence that f is (F)-holomorphic and therefore holomorphic (Proposition I.10).
(ii) (cf. [He71, p.79]) In view of Proposition II.12(i), it suffices to show that $\mathrm{Hol}(M,V)$ is closed in $C(M,V)_c$ because the latter space is complete. Suppose that $f_i \to f$, where f is continuous and the functions $f_i\colon M \to V$ are holomorphic. We have to show that f is holomorphic and, as in (i), we may w.l.o.g. assume that M is an open subset of a Fréchet space X. An argument similar to that in (i) implies that f is (G)-holomorphic, but then the continuity of f shows that $f \in \mathrm{Hol}(M,V)$. ■

Corollary III.12. *Let M and N be complex manifolds, where M is Fréchet. We write* $\mathrm{Hol}(M,N)_c$ *for the set of holomorphic maps $M \to N$ endowed with the compact open topology. Then the subspace* $\mathrm{Hol}(M,N)_c$ *is closed in* $C(M,N)_c$.

Proof. Since M is Fréchet, it is first countable, and therefore $C(M,N)_c$ is a complete uniform space. Now let $f \in C(M,N)$ and assume that $f_i \to f$ holds for $f_i \in \mathrm{Hol}(M,N)$ uniformly on compact subsets of M. We have to show that f is holomorphic. This is a local property, so that we may assume that M is an open subset of a Fréchet space F. In view of the continuity of f, it suffices to show that f is Gateaux-holomorphic, so that we may even assume that M is one-dimensional, hence locally compact (Proposition I.9). Let $x_0 \in M$ and fix a compact neighborhood K of x_0 and an open neighborhood $U \subseteq N$ of $f(x_0)$ which is diffeomorphic to an open subset of an s.c.l.c. space V. Then we may w.l.o.g. assume that $f_i(K) \subseteq U$ holds for all i, so that the same argument as in the proof of Theorem III.11(i) shows that f is holomorphic in a neighborhood of x_0. ∎

In the following the assumption that the manifolds under consideration are Baire is made to ensure that the spaces $\mathrm{Hol}(M,V)$ are sequentially complete (Theorem III.11(i)).

Proposition III.13. *Let M and N be complex Baire manifolds, $f\colon M \times N \to V$ holomorphic, and $f_x(y) := f(x,y)$. Then the map*

$$\Phi\colon M \to \mathrm{Hol}(N,V), \quad x \mapsto f_x$$

is holomorphic.

Proof. First the continuity of the map Φ follows from Lemma III.2(i). Next we note that we may w.l.o.g. assume that M is an open subset of a Baire space X.
Claim 1: The map

$$\Psi\colon M \times X \to \mathrm{Hol}(N,V), \quad (x,h) \mapsto \big(y \mapsto d_1 f(x,y)(h)\big)$$

is continuous. This follows from Lemma III.2(i) and the fact that

$$d_1 f \in \mathrm{Hol}(M \times \ X \times \ N, V)$$

(Remark I.5(d)).
Claim 2: Φ is C^1 with $d\Phi(x)(h) = \Psi(x,h)$. This is proved exactly as the corresponding assertion in the proof of Theorem III.4.

This shows that Φ is C^1 with complex linear differentials, i.e., that Φ is holomorphic. ∎

Theorem III.14. *Let M be a complex Baire manifold, S a complex Fréchet semigroup, and $M \times S \to M$ a holomorphic right action. Then the action*

$$S \times \mathrm{Hol}(M,V) \to \mathrm{Hol}(M,V)$$

with $\big(\pi(s).f\big)(x) = f(x.s)$ is holomorphic.

Proof. According to Lemma III.2(iii), the action of S on $\mathrm{Hol}(M,V) \subseteq C(M,V)_c$ is continuous.

For each $s \in S$ the map $\mathrm{Hol}(M,V) \to \mathrm{Hol}(M,V), f \mapsto s.f$ is continuous linear, hence holomorphic. Now let $f \in \mathrm{Hol}(M,V)$. Then the function defined by $\widetilde{f}(s,x) \mapsto f(x.s)$ is in $\mathrm{Hol}(S \times M, V)$. Hence the holomorphy of $S \to \mathrm{Hol}(M,V), s \mapsto s.f = \widetilde{f}_s$ follows from Proposition III.13. This proves that the action map is partially holomorphic in each argument. Now [He89, Prop. 2.3.8] implies that the action map is (G)-holomorphic, and finally the continuity implies that it is (F)-holomorphic, i.e., holomorphic (Proposition I.10). ∎

We have already seen in Proposition I.10 that holomorphic functions are in particular smooth, i.e., that $\mathrm{Hol}(M,V) \subseteq C^\infty(M,V)$ holds for each complex manifold M. We have endowed the space $\mathrm{Hol}(M,V)$ with the topology of compact convergence which could be coarser than the topology induced from $C^\infty(M,V)$ but it turns out that on $\mathrm{Hol}(M,V)$ the latter topology coincides with the original one.

Proposition III.15. *If M is manifold modeled over a s.c.l.c. space, then the inclusion $\mathrm{Hol}(M,V) \hookrightarrow C^\infty(M,V)$ is an embedding of locally convex spaces.*

Proof. It is clear that the topology $\mathrm{Hol}(M,V)$ inherits from $C^\infty(M,V)$ is finer than the original one. Therefore it suffices to show that the inclusion map is continuous. If f is holomorphic, then $df\colon TM \to V$ is also holomorphic. Therefore it suffices to show that $\mathrm{Hol}(M,V) \to \mathrm{Hol}(TM,V)$, $f \mapsto df$ is a continuous map. Then the assertion follows by induction.

Since each compact subset of TM is the union of finitely many pieces lying in coordinate neighborhoods, we may w.l.o.g. assume that M is an open subset of the s.c.l.c. space X. Let $x \in M$ and $h \in X$ with $x + zh \in M$ whenever $|z| \leq 1$. Then

$$df(x)(h) = \frac{1}{2\pi}\int_0^{2\pi} e^{-i\theta} f(x + e^{i\theta}h)\ d\theta.$$

For each continuous seminorm p on V we therefore have

$$p\big(df(x)(h)\big) \leq \sup_{|z|=1} p\big(f(x+zh)\big).$$

Let $K \subseteq TM \cong M \times X$ be a compact subset and w.l.o.g. $K = K_1 \times K_2$ with $K_1 \subseteq M$ and $K_2 \subseteq X$ compact and balanced. Then we find a balanced 0-neighborhood $V \subseteq X$ with $K_1 + V \subseteq M$ and $n \in \mathbb{N}$ with $K_2 \subseteq nV$. This means that for $(x,h) \in K$ we have $x + z\frac{h}{n} \in M$ whenever $|z| \leq 1$. Hence

$$p\big(df(x)(h)\big) = np\big(df(x)(\frac{h}{n})\big) \leq n \sup_{h \in \frac{1}{n}K_2} p\big(f(x+h)\big),$$

i.e., $p_K(df) \leq np_{K_1 + \frac{1}{n}K_2}(f)$. Since the set $K_1 + \frac{1}{n}K_2$ is compact, convergence in $\mathrm{Hol}(M,V)$ implies uniform convergence on this set, hence uniform convergence of df on K. This completes the proof. ∎

One of the main features of the representation theory of finite-dimensional Lie groups is that they have an exponential function which makes it possible to translate analytic problems on a Lie group G to algebraic problems on $\mathfrak{g}$ without loosing too much information. This works in particular quite well for representations with analytic or holomorphic orbit mappings. To obtain a suitable generalization to the infinite-dimensional setting, let us say that a smooth function $\exp\colon \mathfrak{g} \to G$ is an *exponential function for G* if for each $X \in \mathfrak{g}$ the curve $\gamma_X\colon t \mapsto \exp(tX)$ is an integral curve of the corresponding left invariant vector field $\widetilde{X} \in \mathcal{V}(G)$. Further we say that a Lie group G modeled over the s.c.l.c. space $\mathfrak{g}$ has a *good exponential function* if the closure $\overline{\exp(\mathfrak{g})}$ of the exponential image contains a neighborhood of the identity. If G is complex, we require, in addition, that the exponential function $\exp: \mathfrak{g} \to G$ is a holomorphic map. For a discussion of the exponential function for the class of regular Lie groups we refer to [KM97a]. We write $D_l(G) \subseteq \mathrm{End}\big(C^\infty(G)\big)$ for the unital algebra of all operators on $C^\infty(G)$ generated by the action of the left invariant vector fields. An element $D \in D_l(G)$ is called a *left invariant* differential operator on G.

Lemma III.16. (a) (Identity Theorem for Holomorphic Functions) *If M is connected and two functions $f, f' \in \mathrm{Hol}(M, V)$ coincide on a non-empty open subset of M, then $f = f'$.*
(b) *If G is a connected complex Lie group with a good exponential function and $f \in \mathrm{Hol}(G, V)$ with $(D.f)(\mathbf{1}) = 0$ for all $D \in D_l(G)$, then $f = 0$.*

Proof. (a) Since V is locally convex, the linear functionals on V separate the points, and so we may w.l.o.g. assume that $V = \mathbb{C}$. Let

$$D := \{x \in M: f(x) = f'(x)\}.$$

Then D is a closed subset of M which contains an open subset.

Since M is connected, it suffices to show that the interior D^0 of D is closed, i.e., that each point $x \in \overline{D^0}$ belongs to D^0. Choosing a local chart around x, we may w.l.o.g. assume that M is an open convex subset of the s.c.l.c. space X. Pick $y \in D^0$ and $x \in M$. Then we consider the holomorphic map $\varphi: \mathbb{C} \to X, z \mapsto x + z(y-x)$ and note that $f \circ \varphi$ and $f' \circ \varphi$ are holomorphic functions on $\varphi^{-1}(M)$ which coincide on an open neighborhood of y, hence also in 0 because $[0,1] \subseteq \varphi^{-1}(M)$. Thus $f(x) = f'(x)$, and therefore $D = M$ which completes the proof.
(b) For each $X \in \mathfrak{g}$ we obtain a holomorphic function $F: \mathbb{C} \to V, z \mapsto f(\exp zX)$. Inductively our assumption implies that

$$0 = (\widetilde{X}^n.f)(\mathbf{1}) = F^{(n)}(0).$$

Since F is holomorphic, we conclude that $F = 0$ and hence that $f|_{\exp \mathfrak{g}} = 0$. The assumption that G has a good exponential function now implies that f vanishes on a neighborhood of $\mathbf{1}$ and by (a) also on G. ■

IV. Representations of infinite-dimensional groups

Let V be an s.c.l.c. space and G a Lie group modeled over a s.c.l.c. space. In this section we will apply the results of Section III to define a derived representation of a representation (π, V) of G on the subspace V^∞ of smooth vectors and to endow this space with a suitable complete locally convex topology inherited from $C^\infty(G, V)$ on which the action of G is smooth. For many purposes it is irrelevant that G is a group and it will suffice to assume that it is an s.c.l.c. semigroup, i.e., a manifold modeled over an s.c.l.c. space with a smooth semigroup multiplication.

Definition IV.1. Let V be an s.c.l.c. space and S an s.c.l.c. semigroup.
(a) A *representation* (V, π) of S is a continuous action $S \times V \to V$ such that the mappings $\pi(s): v \mapsto s.v$ are linear and π denotes the corresponding homomorphism $\pi: S \to \mathcal{L}(V)$.
(b) If (V, π) is a representation of S, then a vector $v \in V$ is called *smooth* if the orbit map $S \to V, s \mapsto \pi(s).v$ is smooth. We write V^∞ for the subspace of smooth vectors. ■

The derived representation

Let (V, π) be a representation of the s.c.l.c. Lie group G, $v \in V^\infty$ and $\varphi_v: G \to V, g \mapsto \pi(g).v$, denote the corresponding orbit map. Then $d\varphi_v(\mathbf{1}): \mathfrak{g} \cong T_{\mathbf{1}}(G) \to V \cong T_v(V)$ is a continuous linear map. We define

$$d\pi(X).v := X.v := d\varphi_v(\mathbf{1}).X.$$

Lemma IV.2. *The prescription $\mathfrak{g} \times V^\infty \to V^\infty$ defines a representation of $\mathfrak{g}$ on V^∞.*

Proof. First we show that for $X \in \mathfrak{g}$ and $v \in V^\infty$ the element $X.v \in V$ is in fact contained in V^∞.

For $g \in G$ we have $\pi(g) \circ \varphi_v = \varphi_v \circ \lambda_g$ because the orbit map φ_v is equivariant with respect to left multiplications. Hence the chain rule implies

$$\pi(g) d\varphi_v(\mathbf{1}).X = d\varphi_v(g) d\lambda_g(\mathbf{1}).X.$$

Let $X_l \in \mathcal{V}(G)$ denote the left invariant vector field with $X_l(\mathbf{1}) = X$. Then the preceding calculation shows that

$$g \mapsto \pi(g)(X.v) = d\varphi_v(g).X_l(g) \tag{4.1}$$

is smooth since the map

$$T(\varphi_v) \circ X_l: G \to TV \cong V \times V, \quad g \mapsto \big(\pi(g).v, d\varphi_v(g).X_l(g)\big)$$

is smooth. This proves that $X.v \in V^\infty$.

It remains to show that $d\pi: \mathfrak{g} \to \mathrm{End}(V^\infty)$ is a homomorphism of Lie algebras. For $v \in V^\infty$ we obtain a map

$$\Phi_v: V' \to C^\infty(G), \quad \omega \mapsto \big(g \mapsto \langle \omega, \pi(g).v \rangle\big).$$

For $X \in \mathfrak{g}$, the corresponding left invariant vector field X_l, and $\omega \in V'$ the chain rule and (4.1) show that

$$\big(X_l.\Phi_v(\omega)\big)(g) = \langle \omega, d\varphi_v(g).X_l(g) \rangle = \langle \omega, \pi(g).(X.v) \rangle = \Phi_{X.v}(\omega)(g),$$

i.e., $X_l \circ \Phi_v = \Phi_{X.v}$. Therefore

$$\Phi_{[X,Y].v} = [X_l, Y_l] \circ \Phi_v = X_l \circ \Phi_{Y.v} - Y_l \circ \Phi_{X.v} = \Phi_{X.(Y.v)} - \Phi_{Y.(X.v)} = \Phi_{X.(Y.v) - Y.(X.v)}.$$

Evaluating this at $g = \mathbf{1}$ we obtain $\omega([X, Y].v) = \omega\big(X.(Y.v) - Y.(X.v)\big)$ for all $\omega \in V'$ and, since the continuous linear functionals on V separate the points, $[X, Y].v = X.(Y.v) - Y.(X.v)$. ■

Remark IV.3. If G is finite-dimensional, then Gårding's Theorem (cf. [Wa72, Prop. 4.4.1.1]) shows that V^∞ is a dense subspace of V. Another important fact on smooth vectors is Harish-Chandra's Theorem ([Wa72, Th. 4.4.2.1]) saying that if G is finite-dimensional and compact, $\widehat{G}$ is the set of equivalence classes of irreducible representations, and $P(\delta): V \to V$ the projection onto the isotypical component of type δ, then for each $v \in V^\infty$ the Fourier series

$$v = \sum_{\delta \in \widehat{G}} P(\delta).v$$

converges in V. ■

Lemma IV.4. *Let X be a topological space, S a metrizable topological semigroup acting continuously from the right on X, and V a (sequentially) complete locally convex space.*

(i) *If, in addition, X satisfies the first axiom of countability, then $C(X,V)_c$ is a (sequentially) complete locally convex space and we obtain a representation of S on this space by $(s.f)(x) := f(x.s)$.*

(ii) *If (π, V) is a representation of the s.c.l.c. group G, then the action $g.\alpha := \alpha \circ \pi(g^{-1})$ on the dual space V'_c is continuous. If, in addition, V is an LF-space, then we obtain a representation of G on V'_c.*

Proof. (i) The completeness follows from Proposition II.12(i), and the continuity of the action from Lemma III.2(iii).
(ii) Since V'_c is endowed with the topology of uniform convergence on compact subsets of V, Lemma III.2(iii) implies that the action of G on the space $V'_c \subseteq C(V, \mathbb{C})_c$ is continuous. If, in addition, V is an LF-space, then V'_c is complete by Corollary II.13, and we thus obtain a representation of G on this space. ■

Next we discuss an appropriate topology on the space V^∞ of smooth vectors. The key tool is Theorem III.5.

Proposition IV.5. *Let (π, V) be a continuous representation of the Fréchet semigroup S with identity element $\mathbf{1}$ on V and $V^\infty \subseteq V$ the space of smooth vectors. Via the map $v \mapsto \varphi_v: s \mapsto \pi(s).v$ we obtain a linear embedding $V^\infty \hookrightarrow C^\infty(S,V)$ which we use to define a locally convex topology on V^∞. Then the natural action of S on V^∞ defines a representation of S on V^∞ for which the action map $S \times V^\infty \to V^\infty$ is smooth.*

Proof. For $v \in V$ and $s, t \in S$ we have $\varphi_v(st) = \pi(st).v = \pi(s).\big(\pi(t).v\big) = \pi(s).\varphi_v(t)$, i.e., $\varphi_v: S \to V$ is equivariant. If, conversely, $\varphi: S \to V$ is a smooth equivariant map, then $\varphi(s) = s.\varphi(\mathbf{1})$ shows that $\varphi(\mathbf{1}) \in V^\infty$. Thus

$$V^\infty \cong C^\infty(S,V)^S = \{f \in C^\infty(S,V)) \colon (\forall s,t \in S) f(st) = \pi(s).f(t)\}$$

is a closed subspace of $C^\infty(S,V)$ because the representation of S on V is continuous, hence V^∞ is a complete locally convex space because S is Fréchet (Proposition III.1).

In view of Theorem III.5, the action map

$$S \times C^\infty(S, V) \to C^\infty(S, V), \quad (s, f) \mapsto s.f$$

with $(s.f)(x) = f(xs)$ is smooth. Since

$$(s.\varphi_v)(x) = \varphi_v(xs) = \pi(xs).v = \pi(x).\big(\pi(s).v\big) = \varphi_{\pi(s).v}(x),$$

this implies that the action of S on V^∞ is also smooth. ∎

Corollary IV.6. *If G is a Fréchet Lie group and (π, V) a continuous representation of G, then the action map*

$$\mathfrak{g} \times V^\infty \to V^\infty, \quad (X, v) \mapsto d\pi(X).v$$

is continuous.

Proof. If $\sigma: G \times V^\infty \to V^\infty$ denotes the action map, then $d\pi(X).v = d_1\sigma(\mathbf{1}, v)(X)$, so that the asserted continuity follows from $\sigma \in C^1$ (Proposition IV.5). ∎

Remark IV.7. (a) Note that Corollary IV.6 implies in particular that the operators

$$d\pi(X): V^\infty \to V^\infty$$

are continuous, hence that $\mathfrak{g}$ acts naturally on the dual space $V^{-\infty} := (V^\infty)'$ of continuous linear functionals on V^∞ by $(X.\alpha)(v) = -\alpha(X.v)$.
(b) With respect to the natural topology on V^∞ the inclusion map $V^\infty \to V$ is continuous because the evaluation map $C^\infty(G, V) \to V, f \mapsto f(\mathbf{1})$ is continuous. ∎

Example IV.8. Let G be a Lie group and $\mathrm{Ad}: G \to \mathrm{Aut}(\mathfrak{g})$ the adjoint representation. Then Ad is a representation of G on $\mathfrak{g}$ with a smooth action map.

In fact, since the action map can be written as $\mathrm{Ad}(g).X = dI_g(\mathbf{1}).X = d\Phi(g, \mathbf{1})(0, X)$, where $\Phi(g, x) = gxg^{-1}$, it is a restriction of the smooth map $T\Phi: T(G \times G) \to TG$, hence a smooth map. Thus the adjoint action of G is a representation in the sense of Definition IV.1 with $\mathfrak{g}^\infty = \mathfrak{g}$. Using Taylor expansions up to a certain order, one can show that the derived action $d\,\mathrm{Ad} = \mathrm{ad}$ is given by $\mathrm{ad}(X).Y = [X, Y]$. We refer to [Mi83, Sect. 5] for the details.

We give a direct proof for the case where G has enough smooth functions such that the representation $\mathfrak{g} \to \mathrm{Der}\,\big(C^\infty(G)\big)$ is injective. It follows in particular from the results in [Th95] that this is true if $\mathfrak{g}$ is a nuclear LF space.

Let $f \in C^\infty(G)$, $g \in G$, and $X \in \mathfrak{g}$. We write π for the natural representation of G on $C^\infty(G)$ given by $\big(\pi(g).f\big)(x) = f(g^{-1}.x)$. Passing to the derivative of the smooth map

$$\psi: G \to C^\infty(G), \quad h \mapsto \pi(g)\pi(h)\pi(g^{-1}).f = \pi(ghg^{-1}).f$$

yields

$$\pi(g)d\pi(Y)\pi(g^{-1}).f = d\pi\big(\operatorname{Ad}(g).Y\big).f.$$

In view of the smoothness of the map ψ, we see that we can take the derivative with respect to g in $\mathbf{1}$, and since f is arbitrary, we get

$$d\pi([X,Y]) = d\pi(X)d\pi(Y) - d\pi(Y)d\pi(X) = d\pi\big(d\operatorname{Ad}(X).Y\big).$$

If $d\pi$ is injective, then $d\operatorname{Ad} = \operatorname{ad}$ follows.

The above argument can be generalized to the setting where one only considers germs of smooth functions in $\mathbf{1}$. Then one does not have to worry about the existence of enough smooth function, and one can still show that the derivative of the map $G \to \mathfrak{g}, g \mapsto \operatorname{Ad}(g).X$ is $\operatorname{ad}(\cdot).X$ for every $X \in \mathfrak{g}$. ∎

In the next proposition we record an important application of the Identity Theorem for Holomorphic Functions (Lemma III.16(a)) to representation theory.

Proposition IV.9. *Let G be a connected complex Lie group with a good exponential function* $\exp\colon \mathfrak{g} \to G$ *and (π, V) a representation of G such that all orbit maps $G \to V, g \mapsto \pi(g).v$ are holomorphic. Then the following assertions hold:*

(i) *If $F \subseteq V$ is a subspace which is invariant under $\mathfrak{g}$, then its closure is invariant under G.*

(ii) *If $v \in V$ is annihilated by $\mathfrak{g}$, then v is fixed by G.*

Proof. (i) Let $\alpha \in F^\perp \subseteq V'$ be a continuous linear functional vanishing on F. For $v \in F$ we consider the function $f_v\colon G \to \mathbb{C}, g \mapsto \alpha(g.v)$, i.e., $f_v = \alpha \circ \varphi_v$, where φ_v is the orbit map. Then the calculation in the proof of Lemma IV.2 shows that for each $X \in \mathfrak{g}$ and the associated left invariant vector field X_l we have

$$(X_l.f_v)(g) = df_v(g).X_l(g) = \langle \alpha, d\varphi_v(g).X_l(g)\rangle = \langle \alpha, \pi(g)X.v\rangle = f_{X.v}(g),$$

i.e., $X_l.f_v = f_{X.v}$.

For $g = \mathbf{1}$ we now obtain $(X_l.f_v)(\mathbf{1}) = \alpha(X.v) = 0$. In view of $X.v \in F$, we can apply this argument inductively and thus obtain $(D.f_v)(\mathbf{1}) = 0$ for all $D \in D_l(G)$. Now Lemma III.16(b) implies that $f_v = 0$, hence that $\pi(G).v \subseteq \ker\alpha$ for all $\alpha \in F^\perp$. Next we use the Hahn-Banach Theorem to see that $\overline{F} = (F^\perp)^\perp$ from which we obtain $\pi(G).v \subseteq \overline{F}$. Since G acts by continuous operators on V, we conclude that $\overline{F}$ is invariant under G.

(ii) As above, we consider the function $f_v\colon g \mapsto \alpha(g.v) - \alpha(v)$ but now with an arbitrary element $\alpha \in V'$. Taking derivatives, we see that $X_l.f_v = f_{X.v} = 0$ for all $X \in \mathfrak{g}$ and therefore $\big(D_l(G).f_v\big)(\mathbf{1}) = 0$ because $f_v(\mathbf{1}) = 0$. So Lemma III.16(b) implies that $f_v = 0$, hence that $\alpha(g.v) = \alpha(v)$ for all $\alpha \in V'$ and $g \in G$. Since V' separates the points of V, the group G fixes v. ∎

V. Generalized coherent state representations

In this section we describe a general setup for so called coherent state representations. Analytically these representations are characterized by the property that they can be realized in spaces of holomorphic sections of a homogeneous complex line bundle. On the geometric side this means that the action of G on the projective space of the dual space has a cyclic orbit which is a complex manifold. These concepts are well studied in the setting of Hilbert spaces (cf. [Li95]) and here we show that if one carefully distinguishes between the spaces and their duals, then one can generalize this correspondence to general s.c.l.c. spaces.

In the first subsection we describe how to construct a natural complex line bundle on the projection space $\mathbb{P}(V)$ of an s.c.l.c. space. In the second subsection we then turn to group representations and show in particular that for finite-dimensional Lie groups the representations of G in an LF space which are generalized coherent state representations are precisely those on subspaces of the space of holomorphic sections of a homogeneous complex line bundle.

The line bundle over the projective space of a topological vector space

In this section V denotes an s.c.l.c. space and $\mathbb{P}(V)$ its projective space. We write $[v]$ for the element of $\mathbb{P}(V)$ which corresponds to the one-dimensional subspace generated by $v \in V \setminus \{0\}$. Furthermore we write $\mathrm{GL}(V)$ for the group of continuously invertible linear operators on V and V' for the topological dual of V.

Lemma V.1. *The group* $\mathrm{GL}(V)$ *acts transitively on*

(i) $V \setminus \{0\}$,

(ii) $\mathbb{P}(V)$,

(iii) $V' \setminus \{0\}$, *and*

(iv) $\mathbb{P}(V')$.

Proof. (i) Let $v, w \in V \setminus \{0\}$. If v and w are linearly dependent, then there exists $\lambda \in \mathbb{C}^\times \subseteq \mathrm{GL}(V)$ with $w = \lambda v$. We now assume that v and w are linearly independent. Since V is locally convex, there exists a continuous linear functional $\alpha \in V'$ with $\alpha(v + w) = 0$ and $\alpha(v - w) = 1$, i.e., $\alpha(v) = -\alpha(w) = \frac{1}{2}$. Then

$$\Phi(x) := x - 2\alpha(x)(v - w)$$

is a continuous reflection in the hyperplane $\ker \alpha$ satisfying $\Phi(v) = w$ and $\Phi^{-1} = \Phi$. It follows in particular that $\Phi \in \mathrm{GL}(V)$.
(ii) This is an immediate consequence of (i).

(iii) We endow V' with the weak-$*$-topology. If $\alpha, \beta \in V' \setminus \{0\}$ are linearly independent, then there exists $x \in V$ with $(\alpha + \beta)(x) = 0$ and $(\alpha - \beta)(x) = 1$. Therefore the same argument as in (i) works in this case.
(iv) This is a direct consequence of (iii). ∎

Proposition V.2. *The space $\mathbb{P}(V)$ carries the structure of a complex manifold modeled over closed hyperplanes of V. The charts are given by $(U_\alpha, \varphi_\alpha)_{\alpha \in V' \setminus \{0\}}$, where*

$$(5.1) \quad U_\alpha = \{[v] \in \mathbb{P}(V) : \alpha(v) \neq 0\} \quad \text{and} \quad \varphi_\alpha : U_\alpha \to \ker \alpha, \quad [v] \mapsto \frac{v}{\alpha(v)} - v_\alpha,$$

where $v_\alpha \in V$ is chosen with $\alpha(v_\alpha) = 1$.

Proof. First we note that the condition defining U_α makes sense because either α vanishes on the one-dimensional space $\mathbb{C}v$ or $\alpha(w) \neq 0$ holds for all $w \in \mathbb{C}v \setminus \{0\}$. According to Lemma V.1, for two different non-zero continuous functionals their kernels are isomorphic as topological vector spaces because they are conjugate under the group $\mathrm{GL}(V)$. Since these kernels are precisely the closed hyperplanes of V, we also see that two such hyperplanes are isomorphic.

Next we note that the inverse of φ_α is given by

$$\varphi_\alpha^{-1} : \ker \alpha \to U_\alpha, \quad v \mapsto [v + v_\alpha].$$

For $[v] \in U_\alpha \cap U_\beta$ and $w := \varphi_\beta([v])$ we have

$$\varphi_\alpha \circ \varphi_\beta^{-1}(w) = \frac{w + v_\beta}{\alpha(w + v_\beta)} - v_\alpha$$

which is a holomorphic map of an open subset of $\ker \beta$ to $\ker \alpha$. Hence the atlas given by the above charts defines on $\mathbb{P}(V)$ the structure of a complex manifold. ∎

We put $U_{\alpha\beta} := U_\alpha \cap U_\beta$ for $\alpha, \beta \in V' \setminus \{0\}$. We define functions

$$g_{\beta\alpha} : U_{\alpha\beta} \to \mathbb{C}^\times, \quad [v] \mapsto \frac{\alpha(v)}{\beta(v)}$$

and note that these functions satisfy $g_{\gamma\beta}([v]) \cdot g_{\beta\alpha}([v]) = g_{\gamma\alpha}([v])$ on $U_\alpha \cap U_\beta \cap U_\gamma$, i.e., the functions $g_{\alpha\beta}$ form a *system of transition functions* in the sense of [Hu94, Def. 5.2.4]. Next we construct a holomorphic line bundle $p : L_V \to \mathbb{P}(V)$ as follows. On the disjoint union

$$\widetilde{L}_V := \bigcup_{0 \neq \alpha \in V'} U_\alpha \times \mathbb{C} \times \{\alpha\}$$

we define an equivalence relation by

$$([v], z, \alpha) \sim ([v], g_{\beta\alpha}([v])z, \beta) = \Big([v], \tfrac{\alpha(v)}{\beta(v)} z, \beta\Big).$$

Proposition V.3. *The space* $L_V := \widetilde{L}_V/\sim$ *carries the structure of a complex line bundle over* $\mathbb{P}(V)$ *with projection*

$$q\colon L_V \to \mathbb{P}(V), \quad \big[[v], z, \alpha\big] \mapsto [v].$$

Proof. It is clear that L_V inherits the structure of a complex manifold because the transition functions are holomorphic and the sets $U_\alpha \times \mathbb{C} \times \{\alpha\}$ carry natural complex manifold structures.

The subset $q^{-1}(U_\alpha)$ is biholomorphically equivalent to $\ker\alpha \times \mathbb{C}$, where the charts are given by

$$\psi_\alpha\colon q^{-1}(U_\alpha) \to \ker\alpha \times \mathbb{C}, \quad \big[[v], z, \alpha\big] \mapsto \big(\varphi_\alpha([v]), z\big).$$

Note that for these coordinate charts we have

$$\begin{aligned}\psi_\beta \circ \psi_\alpha^{-1}(v, z) &= \psi_\beta\Big(\big[[v + v_\alpha], z, \alpha\big]\Big) = \psi_\beta\Big(\big[[v + v_\alpha], g_{\beta\alpha}([v + v_\alpha])z, \beta\big]\Big)\\ &= \psi_\beta\big(\big[[v + v_\alpha], \tfrac{z}{\beta(v+v_\alpha)}, \beta\big]\big) = \big(\varphi_\beta \circ \varphi_\alpha^{-1}(v), \tfrac{z}{\beta(v+v_\alpha)}\big).\end{aligned}$$

Since this map is holomorphic, we obtain another proof for the fact that L_V is a complex manifold. Moreover, the fact that this map is linear in the second argument shows that L_V is a holomorphic vector bundle with fiber $\mathbb{C}$, i.e., a holomorphic line bundle. ■

Theorem V.4. *The assignment*

$$s_\alpha([v]) := \big[[v], \tfrac{\alpha(v)}{\beta(v)}, \beta\big], \quad [v] \in U_\beta \tag{5.2}$$

yields a topological isomorphism $\eta\colon V'_c \to \Gamma(L_V)_c$, *where* $\Gamma(L_V)_c$ *denotes the space of holomorphic sections of* L_V *endowed with the topology of uniform convergence on compact subsets of* $\mathbb{P}(V)$.

Proof. First let $\alpha \in V'$. Then

$$\big[[v], \tfrac{\alpha(v)}{\beta(v)}, \beta\big] = \big[[v], g_{\gamma\beta}([v])\tfrac{\alpha(v)}{\beta(v)}, \gamma\big] = \big[[v], \tfrac{\alpha(v)}{\gamma(v)}, \gamma\big]$$

so that (5.2) defines in fact a section $\eta(\alpha)$ of L_V which is holomorphic. Now we show that the so obtained map $\eta\colon V' \to \Gamma(L_V)$ is a bijection. The subset

$$L_V^\times := \{\big[[v], z, \alpha\big]\colon z \neq 0, [v] \in \mathbb{P}(V), 0 \neq \alpha \in V'\},$$

of L_V is the complement of the zero section in L_V. We have a natural map

$$j\colon V \setminus \{0\} \to L_V^\times, \quad v \mapsto \big[[v], \tfrac{1}{\alpha(v)}, \alpha\big]$$

for $[v] \in U_\alpha$. For $[v] \in U_{\alpha\beta}$ we have

$$\big[[v], \tfrac{1}{\alpha(v)}, \alpha\big] = \big[[v], g_{\beta\alpha}([v])\tfrac{1}{\alpha(v)}, \beta\big] = \big[[v], \tfrac{1}{\beta(v)}, \beta\big].$$

The inverse of this map is given by

$$j^{-1}: L_V^\times \to V, \quad \big[[v], z, \alpha\big] \mapsto \tfrac{v}{z\alpha(v)},$$

for $[v] \in U_\alpha$, where we have to note that the expression on the right hand side is well defined because

$$\frac{v}{z\alpha(v)} = \frac{v}{g_{\beta\alpha}([v])z\beta(v)}.$$

Now let $s \in \Gamma(L_V)$ be a holomorphic section. Then we obtain a holomorphic function $\widetilde{s}: L_V^\times \to \mathbb{C}$ with $s\big(p(x)\big) = \widetilde{s}(x) \cdot x$. Note that $\widetilde{s}(\lambda x) = \frac{1}{\lambda}\widetilde{s}(x)$. Therefore the function $\widehat{s} := \widetilde{s} \circ j: V \setminus \{0\} \to \mathbb{C}$ is holomorphic and satisfies $\widehat{s}(\lambda x) = \lambda \widehat{s}(x)$ for all $\lambda \in \mathbb{C}^\times$. We claim that $\widehat{s}$ is the restriction of a continuous linear functional. If V is one-dimensional, then $\mathbb{P}(V)$ consists of one point and there is nothing to show. Let $W \subseteq V$ be a two-dimensional subspace. Then the restriction f of $\widehat{s}$ to $W \setminus \{0\}$ is a holomorphic function satisfying

$$f(\lambda v) = \lambda f(v), \quad 0 \neq v \in W, \lambda \in \mathbb{C}^\times. \tag{5.3}$$

Since $\{0\}$ is an isolated singularity of this function, Hartog's Theorem shows that f extends holomorphically to W. Now the Taylor expansion in the origin and (5.3) imply that f is linear. Thus the extension of $\widehat{s}$ by $\widehat{s}(0) := 0$ yields a linear functional $\widehat{s}$ on V. If $\widehat{s} \neq 0$, then $\ker \widehat{s}$ is a complex hyperplane with the property that $(V \setminus \{0\}) \cap \ker \widehat{s}$ is closed. Hence $\ker \widehat{s}$ is closed and therefore $\widehat{s}$ is continuous. Thus for each holomorphic section s there exists a continuous linear functional $\alpha \in V'$ such that

$$\begin{aligned} s([v]) &= \widetilde{s}\big(\big[[v], z, \beta\big]\big) \cdot \big[[v], z, \beta\big] = \alpha\big(j^{-1}\big(\big[[v], z, \beta\big]\big)\big) \cdot \big[[v], z, \beta\big] \\ &= \frac{\alpha(v)}{z\beta(v)} \cdot \big[[v], z, \beta\big] = \big[[v], \tfrac{\alpha(v)}{\beta(v)}, \beta\big], \end{aligned}$$

i.e., $s = s_\alpha$. This completes the proof of the bijectivity of η.

Now we show that η also is a topological isomorphism. We may w.l.o.g. assume that $V \neq \{0\}$. First we observe that the topology on V_c' coincides with the topology of uniform convergence on all compact subsets $C \subseteq V$ for which there exists a linear functional $\beta \in V'$ with $\inf \operatorname{Re} \beta(C) \geq 1$. In fact, if $C \subseteq V$ is a compact subset, then we pick $x \in V$ with $\operatorname{Re} \beta(x) > \max\big(1, 1 - \inf \operatorname{Re} \beta(C)\big)$. Then $\inf \operatorname{Re} \beta(C + x) = \inf \operatorname{Re} \beta(C) + \operatorname{Re} \beta(x) > 1$, and the uniform convergence on $C + x$ and x implies the uniform convergence on $C = (C + x) - x$. On the other hand, a covering argument using that the quotient map $p: V \setminus \{0\} \to \mathbb{P}(V), v \mapsto [v]$ is open and has local sections shows that every compact subset of $\mathbb{P}(V)$ is a finite union of compact subsets lying in some open subset U_β, $\beta \in V' \setminus \{0\}$.

Now let $C \subseteq V$ be a compact subset with $\inf \operatorname{Re} \beta(C) > 1$. Then $p(C) \subseteq \mathbb{P}(V)$ is a compact subset of $p(\{v \in V : \beta(v) \neq 0\}) = U_\beta$ and we have $\eta(\alpha)([v]) = \big[[v], \frac{\alpha(v)}{\beta(v)}, \beta\big]$ for $[v] \in U_\beta$. In view of $\inf |\beta(C)| > 1$, this formula implies that a net $(\alpha_j)_{j \in J}$ in V' converges uniformly on C if and only if the net $(\eta(\alpha_j))_{j \in J}$ of holomorphic sections of L_V converges uniformly on $p(C)$. Therefore η is a topological isomorphism $V'_c \to \Gamma(L_V)_c$. ■

Applications to representation theory

Definition V.5. A continuous representation (π, V) of G on an s.c.l.c. space V is called a *generalized coherent state representation* (GCS representation for short) if there exists $v \in V \setminus \{0\}$ such that

(1) v is cyclic,

(2) the homogeneous space $G/G_{[v]}$, where $G_{[v]} = \{g \in G : g.[v] = [v]\}$ carries the structure of a complex homogeneous space modeled over a Fréchet space such that the natural map $\eta: G/G_{[v]} \to \mathbb{P}(V), gG_{[v]} \mapsto g.[v]$ is holomorphic.

A vector $v \in V \setminus \{0\}$ satisfying (1) and (2) is called a GCS vector. ■

If $p: L \to M$ is a holomorphic line bundle over a Fréchet manifold M, then we endow the space $\Gamma(L)$ of holomorphic sections with the compact open topology which turns it into a complete locally convex space (cf. Theorem III.11). If V is a topological vector space, then we write V'_c for the topological dual of V endowed with the topology of uniform convergence on the compact subsets of V (cf. Section II).

Proposition V.6. *If (π, V) is a generalized coherent state representation, then the contragredient representation (π', V'_c) can be injected continuously into the natural representation of G on the space $\Gamma(L)$ of holomorphic sections of a holomorphic line bundle $p: L \to M$.*

Proof. Let $v \in V$ be a GCS vector and $M := G/G_{[v]}$. Then M carries the structure of a complex manifold such that the inclusion map

$$\eta: M \to \mathbb{P}(V), \quad gG_{[v]} \mapsto g.[v]$$

is holomorphic. Let $L_V \to \mathbb{P}(V)$ denote the line bundle from Proposition V.3. Then the pull back $L := \eta^* L_V$ is a holomorphic line bundle over M and thus we obtain a natural map

$$\psi: V' \cong \Gamma(L_V) \to \Gamma(L).$$

We claim that ψ is injective. So let $\alpha \in V'$ and suppose that $\psi(s_\alpha) = 0$. This means that the section s_α vanishes on $\eta(M) \subseteq \mathbb{P}(V)$. For $\beta \in V' \setminus \{0\}$ and $[w] \in U_\beta \subseteq \mathbb{P}(V)$ we have

$$s_\alpha([w]) := \big[[w], \tfrac{\alpha(w)}{\beta(w)}, \beta\big]. \qquad (5.4)$$

Hence s_α vanishes in $[w]$ if and only if $\alpha(w) = 0$. Therefore α vanishes on $G.v$, and the fact that v is cyclic implies that $\alpha = 0$, i.e., that ψ is injective.

To see that ψ is continuous, let $K \subseteq M$ be a compact subset. Then there exists a compact subset $C \subseteq V \setminus \{0\}$ with $\eta(K) = [C]$. Now convergence in V_c' implies uniform convergence on C, hence (5.4) shows that the corresponding sections converge uniformly on $K \subseteq M$. This proves that ψ is continuous. ■

Lemma V.7. *Let $p: L \to M$ be a holomorphic line bundle, M a complex Fréchet manifold, and $V \subseteq \Gamma(L)$ a closed subspace with the property that for each $x \in M$ the exists a holomorphic section $s \in V$ with $s(x) \neq 0$. Then the following assertions hold:*

(i) *The system $U_s := \{x \in M : s(x) \neq 0\}$, $s \in V \setminus \{0\}$, and the transition functions*

$$g_{ts}: U_s \cap U_t \to \mathbb{C}^\times, \quad x \mapsto \frac{s(x)}{t(x)}$$

define a line bundle over M which is isomorphic to L.

(ii) *Assume that V is a Fréchet space. For $x \in L^\times$ we define a holomorphic map $\gamma: L^\times \to V_c'$ by $s\big(p(x)\big) = \gamma(x)(s) \cdot x$. Then $\gamma(L^\times) \subseteq V_c' \setminus \{0\}$, and we obtain a holomorphic map*

$$\overline{\gamma}: M \to \mathbb{P}(V_c'), \quad p(x) \mapsto [\gamma(x)].$$

Furthermore the pull-back line bundle $\overline{\gamma}^* L_{V_c'}$ is isomorphic to L.

Proof. (i) We construct a holomorphic line bundle $q: E \to M$ as $\widetilde{E}/\sim$, where

$$\widetilde{E} := \bigcup_{0 \neq s \in V} U_s \times \mathbb{C} \times \{s\}$$

and

$$(x, z, s) \sim \big(x, g_{ts}(x) z, t\big) = \Big(x, \tfrac{s(x)}{t(x)} z, t\Big).$$

Then the projection $q: E \to M$ is given by $q([x, z, s]) = x$. To see that this bundle is isomorphic to L, we define a holomorphic mapping

$$\Phi: E \to L, \quad [x, z, s] \mapsto z \cdot s(x) \quad \text{for} \quad x \in U_s.$$

To see that Φ is well defined, we note that for $x \in U_s \cap U_t$ we have $[x, z, s] = \big[x, \frac{s(x)}{t(x)} z, t\big]$ and

$$z \cdot s(x) = \frac{s(x)}{t(x)} z \cdot t(x).$$

Hence Φ is a well defined holomorphic bundle map with $p \circ \Phi = q$.

Moreover, if $\Phi([x,z,s]) = \Phi([x',z',s'])$, then $x = p\big(\Phi(x)\big) = x' \in U_s \cap U_{s'}$, and $z \cdot s(x) = z' \cdot s'(x)$, i.e., $z' = \frac{s(x)}{s'(x)} z$. Hence Φ is bijective. Moreover, for $y \in p^{-1}(U_s)$ we have

$$\Phi^{-1}(y) = \Big[p(y), \frac{y}{s\big(p(y)\big)}, s\Big],$$

which shows that $\Phi^{-1}: L \to E$ is also holomorphic.

(ii) First we note that $V \to \mathbb{C}, s \mapsto \gamma(x)(s)$ is continuous, so that $\gamma(V) \subseteq V'$. We claim that γ is holomorphic. Since by assumption V is a Fréchet space, Corollary II.13 shows that V'_c is a complete locally convex space, and that the natural map $\eta_V: V \to (V'_c)'_c$ is surjective (Theorem II.8(ii)). Therefore each continuous linear functional on V'_c is given by evaluation in an element $s \in V$, and for each such s the mapping $x \mapsto \gamma(x)(s)$ is a holomorphic function on $L^\times$. This proves that γ is weakly holomorphic, hence that γ is holomorphic because V'_c is sequentially complete and M is Fréchet (Proposition I.9).

Since, by assumption, for each $x \in M$ there exists an $s \in V$ with $s(x) \neq 0$, we have $\gamma(L^\times) \subseteq V'_c \setminus \{0\}$. Moreover we have $\gamma(\lambda x) = \lambda^{-1}\gamma(x)$ for $\lambda \in \mathbb{C}^\times$, so that γ factors to a holomorphic map

$$\overline{\gamma}: M \to \mathbb{P}(V'_c), \quad p(x) \mapsto [\gamma(x)].$$

Let $E := \overline{\gamma}^* L_{V'_c}$ denote the pull-back line bundle with projection $q: E \to M$. Then $\overline{\gamma} \circ q = p_{V'_c} \circ \gamma$, and since the bundle $L_{V'_c}$ is defined by the transition functions

$$g_{\beta\alpha}([v]) = \frac{\alpha(v)}{\beta(v)} \quad \text{for} \quad \alpha(v), \beta(v) \neq 0, \alpha, \beta \in (V'_c)',$$

the bundle E is defined by the transition functions

$$g_{\beta\alpha}\big(p(x)\big) = \frac{\alpha\big(\gamma(x)\big)}{\beta\big(\gamma(x)\big)} \quad \text{for} \quad \alpha\big(\gamma(x)\big), \beta\big(\gamma(x)\big) \neq 0.$$

Using $\eta_c(V) = (V'_c)'_c$ (Theorem II.8(ii)), we write $\alpha = \eta_V(s)$ and $\beta = \eta_V(t)$ to obtain

$$g_{\beta\alpha}\big(p(x)\big) = \frac{\gamma(x)(s)}{\gamma(x)(t)} = \frac{s\big(p(x)\big)}{t\big(p(x)\big)} = g_{ts}\big(p(x)\big)$$

for $p(x) \in U_s \cap U_t$. Therefore (i) shows that the holomorphic line bundle E is isomorphic to L. ∎

For the remainder of this section we will restrict our attention to finite-dimensional Lie groups because we will need the differential geometric machinery describing complex structures and holomorphic sections in terms of the underlying real structure of the manifold.

Lemma V.8. *Let G be a finite-dimensional Lie group, H a closed subgroup, and suppose that the homogeneous space G/H is a complex manifold in such a way that G acts by holomorphic maps. Suppose further that M is a not necessarily finite-dimensional complex manifold on which G acts by holomorphic maps. If $\gamma: G/H \to M$ is a holomorphic equivariant map, $x_0 := \gamma(\mathbf{1}H)$, and G_{x_0} is the stabilizer of x_0, then $H \subseteq G_{x_0}$ and the homogeneous space G/G_{x_0} carries a unique complex structure such the quotient map $G/H \to G/G_{x_0}, gH \mapsto gG_{x_0}$ and the induced map $\overline{\gamma}: G/G_{x_0} \to M, gG_{x_0} \mapsto g.x_0$ are holomorphic.*

Proof. Let $\sigma: G \times M \to M$ denote the action of G on the complex manifold M and write $\mathcal{V}_{\rm hol}(M) \subseteq \mathcal{V}(M)$ for the Lie algebra of holomorphic vector fields on M. Then

$$\dot\sigma: \mathfrak{g} \to \mathcal{V}_{\rm hol}(M), \quad X \mapsto \big(p \mapsto -d\sigma(\mathbf{1}, p)(X, 0)\big)$$

is a homomorphism of Lie algebras. In fact, this follows easily from a local computation in coordinate charts.

We conclude that $\dot\sigma$ extends to a $\mathbb{C}$-linear homomorphism $\mathfrak{g}_{\mathbb{C}} \to \mathcal{V}_{\rm hol}(M)$ which we also denote by $\dot\sigma$. As the formula for the Lie bracket in local coordinates shows, the subspace

$$\mathfrak{a} := \{\mathcal{X} \in \mathcal{V}_{\rm hol}(M): \mathcal{X}(x_0) = 0\}$$

is a Lie subalgebra of $\mathcal{V}_{\rm hol}(M)$. Hence $\mathfrak{b} := \dot\sigma^{-1}(\mathfrak{a})$ is a complex subalgebra of $\mathfrak{g}_{\mathbb{C}}$. Moreover $\mathfrak{g}_{x_0} = \mathfrak{b} \cap \mathfrak{g}$ according to the fact that the G-orbit is an equivariant image of the finite-dimensional homogeneous manifold G/H. This can also be written as $\mathfrak{b} \cap \overline{\mathfrak{b}} = (\mathfrak{g}_{x_0})_{\mathbb{C}}$ for the complex conjugation $X \mapsto \overline{X}$ on $\mathfrak{g}_{\mathbb{C}}$. Further it is easy to see that $\mathrm{Ad}(G_{x_0}).\mathfrak{b} = \mathfrak{b}$.

The holomorphy of γ now implies that $d\gamma(\mathbf{1}H)T_{\mathbf{1}H}(G/H) = \dot\sigma(\mathfrak{g})(x_0)$ is a complex subspace of $T_{x_0}(M)$. This means that $\dot\sigma(\mathfrak{g})(x_0) = \dot\sigma(\mathfrak{g}_{\mathbb{C}})(x_0)$ which shows that

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} + \mathfrak{b}.$$

Thus we find for each $X \in \mathfrak{g}_{\mathbb{C}}$ an element $Y \in \mathfrak{g}$ and $Z \in \mathfrak{b}$ with $X = Y + Z$. Hence $X - \overline{X} = Z - \overline{Z} \in \mathfrak{b} + \overline{\mathfrak{b}}$, and therefore $i\mathfrak{g} \subseteq \mathfrak{b} + \overline{\mathfrak{b}}$ which in turns gives

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} + i\mathfrak{g} \subseteq i(\mathfrak{b} + \overline{\mathfrak{b}}) + \mathfrak{b} + \overline{\mathfrak{b}} = \mathfrak{b} + \overline{\mathfrak{b}}.$$

This completes the proof of

$$\mathrm{Ad}(G_{x_0}).\mathfrak{b} = \mathfrak{b}, \quad \mathfrak{b} \cap \overline{\mathfrak{b}} = (\mathfrak{g}_{x_0})_{\mathbb{C}}, \quad \text{and} \quad \mathfrak{b} + \overline{\mathfrak{b}} = \mathfrak{g}_{\mathbb{C}},$$

which, according to [Ki76, p. 203], is equivalent to the existence of a complex structure on G/G_{x_0} such that G acts by holomorphic mappings. More explicitly, this complex structure can be described by identifying the tangent space $T_{\mathbf{1}G_{x_0}}(G/G_{x_0}) \cong \mathfrak{g}/\mathfrak{g}_{x_0}$ with the complex vector space $\mathfrak{g}_{\mathbb{C}}/\mathfrak{b}$. From this description of the complex structure it follows that the canonical maps $G/H \to G/G_{x_0}$ and $G/G_{x_0} \to M$ are holomorphic because they are G-equivariant, smooth, and their differentials are complex linear in the base point. This completes the proof. ■

Proposition V.9. *Suppose that G is finite-dimensional and L is a holomorphic G-homogeneous line bundle. Then G acts on the Fréchet space $\Gamma(L)$ by $(g.s)(x) := g.s(g^{-1}.x)$. Let $\{0\} \neq V \subseteq \Gamma(L)$ be a closed invariant subspace. Then the representation of G on V'_c is a GCS representation.*

Proof. First we note that V inherits the structure of a Fréchet space. We claim that V satisfies the assumptions of Lemma V.7. Let $x \in M$. Since $V \neq \{0\}$, there exists $s \in V \setminus \{0\}$. Pick $y \in M$ with $s(y) \neq 0$. Then there exists $g \in G$ with $g.y = x$, and we see that $(g.s)(x) = g.s(y) \neq 0$. This means that V satisfies the assumptions of Lemma V.7, and thus $L \cong \overline{\gamma}^* L_{V'_c}$ holds for the natural holomorphic map $\overline{\gamma}: M \to \mathbb{P}(V'_c)$.

Moreover

$$\begin{aligned}\big(g.\gamma(x)\big)(s) \cdot x = \gamma(x)(g^{-1}.s) \cdot x = (g^{-1}.s)\big(p(x)\big) = g^{-1}.s\big(g.p(x)\big) \\ = g^{-1}.\gamma\big(g.p(x)\big)(s) \cdot (g.x) = \gamma\big(g.p(x)\big)(s) \cdot x,\end{aligned}$$

shows that $\gamma: L^\times \to V'_c \setminus \{0\}$ is G-equivariant and hence that $\overline{\gamma}$ is G-equivariant.

Pick $x_0 \in M$ and let $\overline{\gamma}(x_0) = [\alpha_0]$. Then the G-homogeneous space $G/G_{[\alpha_0]} \cong \overline{\gamma}(M)$ inherits the structure of a complex manifold because $\overline{\gamma}$ is holomorphic (Lemma V.8). Moreover, the natural map $G/G_{[\alpha_0]} \to \mathbb{P}(V'_c)$ is obtained by factorization of $\overline{\gamma}$ and therefore holomorphic. So, in view Definition V.5, it remains to prove that $\alpha_0 \in V'_c$ is a cyclic vector.

In fact, if α_0 is not cyclic, then $V \cong (V'_c)'$, and the Hahn-Banach Theorem imply the existence of $0 \neq s \in V$ vanishing on $G.\alpha_0$. This means that the section s of $\Gamma(L)$ vanishes on $G.x_0 = M$, contradicting $s \neq 0$. This completes the proof. ■

Theorem V.10. *If G is finite-dimensional, then a non-zero continuous representation (π, V) of G, where V is an LF space is a generalized coherent state representation if and only if the contragredient representation permits a continuous equivariant injection into $\Gamma(L)$ for a homogeneous line bundle $p: L \to M$.*

Proof. If (π, V) is a GCS representation, then Proposition V.6 shows that the contragredient representation permits a continuous equivariant injection into $\Gamma(L)$ for a homogeneous line bundle L.

Suppose, conversely, that $\psi: V'_c \to \Gamma(L)$ is a continuous equivariant injection. In view of Proposition V.9, the representation of G on $\Gamma(L)'_c$ is a GCS representation because this space contains $\psi(V'_c)$, hence is non-zero. The adjoint map $\psi': \Gamma(L)'_c \to (V'_c)'_c \cong V$ is continuous and G-equivariant. Let $\alpha_0 \in \Gamma(L)'_c$ be a GCS vector. We claim that $\psi'(\alpha_0)$ is a GCS vector in V.

First we show that it is cyclic. In fact, if it is not cyclic, then there exists a non-zero $\beta \in V'$ vanishing on $G.\psi'(\alpha_0) = \psi'(G.\alpha_0)$, i.e., $\psi(\beta)$ vanishes on $G.\alpha_0$, and thus $\psi(\beta) = \{0\}$ because α_0 is cyclic, contradicting the injectivity of ψ. Thus $\psi'(\alpha_0)$ is cyclic, and it follows in particular that $\psi'(\alpha_0) \neq 0$.

Now the fact that the natural map

$$\mathbb{P}\big(\Gamma(L)'_c\big) \setminus \psi(V'_c)^\perp \to \mathbb{P}(V), \quad [\alpha] \mapsto [\psi'(\alpha)]$$

is holomorphic and G-equivariant implies that $G/G_{[\psi'(\alpha_0)]}$ is a complex homogeneous G-space such that the natural map $G/G_{[\psi'(\alpha_0)]} \to \mathbb{P}(V)$ is holomorphic (Lemma V.8). This proves that (π, V) is a GCS representation. ■

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