On Algebras in which every Vector Subspace is a Subalgebra

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Abstract

In this note we classify all those algebras over fields of characteristic different from 2 in which every vector subspace is a subalgebra.

Introduction

The author's investigations of algebras in which every subspace is a subalgebra have originated in topological incidence geometry. The study of topological affine translation planes motivates to consider topological vector spaces admitting a so-called stable spread, that is, a partition of the vector space into pairwise complementary subspaces which is compact in the respective Grassmann topology (cf. [2]). A slight generalisation leads to so-called stable translation structures which are constructed from Lie groups admitting a partition into closed subgroups of half dimension (cf. [3]). So the problem arises to find such Lie groups.

A rough classification of such groups is contained in [5]. It turns out that in almost all of the known examples the groups are almost abelian in the sense that the corresponding Lie algebra carries a vector space structure over some skewfield such that every vector subspace is a subalgebra (cf. [4]). So one is lead to classify those algebras. For real finite-dimensional Lie algebras a classification is contained in [1], see Theorem II.2.30. In the present note the respective classification problem is solved for arbitrary algebras over fields of characteristic $\neq 2$ without any restriction on the dimension and without assuming any additional structure except a bilinear multiplication on the underlying vector space.

Throughout, the ground field of the algebras under consideration is denoted by K and supposed to have characteristic $\neq 2$.

Definitions and Examples

1 Definition. We call an algebra A **almost abelian** if every vector subspace is a subalgebra, and we call it **abelian** if $xy = 0$ holds for all $x, y \in A$. A vector $x \in A$ is called **isotropic** if $x^2 = 0$; otherwise it is called **anisotropic**. An algebra is called **symplectic** if it consists only of isotropic vectors.

2 Examples. (a) Obviously, every abelian algebra is an almost abelian algebra.

(b) Let V be a vector space over K. On the vector space $K \times V$ we define a multiplication by

$$(a, v)(b, w) := (0, aw - bv).$$

Endowed with this multiplication $K \times V$ becomes an almost abelian algebra which we denote by $\mathrm{dil}(V)$. Almost abelian algebras of this type are called **dilatation algebras**.

(c) Again let V be a vector space over K and $k \in K$. On the vector spaces $K \times V$ and $K \times K \times V$ we define multiplications by

$$(a, v)(b, w) := b(1 - k)(a, v) + ak(b, w)$$

and

$$(a, b, v)(c, d, w) := (c(1 - k) - d)(a, b, v) + (ak + b)(c, d, w),$$

respectively. Endowed with these multiplications the considered vector spaces become almost abelian algebras which we denote by $\mathrm{fab}_1(V, k)$ and $\mathrm{fab}_2(V, k)$, respectively.

Obviously, an algebra is almost abelian if every 2-dimensional vector subspace is a subalgebra. Having this in mind it is easy to see that the aforementioned algebras are almost abelian. In our examples we have considered four types of algebras, namely abelian ones, dilatation algebras and algebras of type $\mathrm{fab}_1(V,k)$ and $\mathrm{fab}_2(V,k)$. Now we are going to investigate in which cases two of these algebras of different type can be isomorphic.

Abelian as well as dilatation algebras are symplectic algebras whereas algebras of type $\mathrm{fab}_1(V,k)$ and $\mathrm{fab}_2(V,k)$ are not. Moreover, one can say that a dilatation algebra is abelian if and only if it is 1-dimensional. Now we claim that two algebras $\mathrm{fab}_1(V,k)$ and $\mathrm{fab}_2(W,l)$ are never isomorphic. This follows from the fact that an algebra of type $\mathrm{fab}_1(V,k)$ contains an ideal of codimension 1 (namely $\{0\}\times V$) whereas an algebra of type $\mathrm{fab}_2(V,k)$ does not. (In our context an ideal of an algebra A is ment to be a vector subspace I that satisfies $AI\cup IA\subseteq I$.) The latter statement is obvious if $V=\{0\}$. In order to see it in general, we note that the existence of a 1-codimensional ideal in some algebra $\mathrm{fab}_2(V,k)$ would imply the existence of a 1-dimensional ideal in any 2-dimensional subalgebra of $\mathrm{fab}_2(V,k)$, contradicting the fact that $\mathrm{fab}_2(0,k)$ embeds into $\mathrm{fab}_2(V,k)$.

The Symplectic Case

First, we treat low-dimensional cases. Clearly, we are only interested in non-abelian algebras. So let A be a 2-dimensional non-abelian symplectic almost abelian algebra and let $x,y\in A$ be a basis for the vector space A. Then there exist $a,b\in K$ such that $z:=xy=ax+by$. By symplecticity we have $xy=-yx$ and therefore we can assume that $b\neq 0$ holds. Setting $u:=b^{-1}x$ and $v:=z$ we obtain $uv=v$ and the assignment $u\mapsto(1,0)$ and $v\mapsto(0,1)$ induces an isomorphism of A onto $\mathrm{dil}(K)$. Hence, any at most 2-dimensional symplectic almost abelian algebra is either abelian or isomorphic to $\mathrm{dil}(K)$.

3 Lemma. *Let A be a symplectic almost abelian algebra. If we have $\dim A>2$ then A contains a 2-dimensional abelian subalgebra.*

Proof. Let $x,y,z\in A$ be linearly independent and suppose that no two of these elements generate an abelian subalgebra. According to the previous considerations we can assume that $xy=x$. Since A is almost abelian, there exist $a,b\in K$ such that $xz=ax+bz$ and we obtain

$$Kx\oplus K(y+z)\ni x(y+z)=(a+1)x+bz.$$

This implies $b=0$ and thus $Kx\oplus K(ay-z)$ is a 2-dimensional abelian subalgebra. □

4 Proposition. *Let A be an almost abelian algebra, $B\subseteq A$ an abelian subalgebra, and suppose further that $\dim B>1$. Then for any $x\in A$ we have $xB\subseteq B$ and multiplication with x from the left is a homothety on B.*

Proof. The assertion is obvious if $A=B$ or if $x\in B$. So assume $B\neq A$ and pick $x\in A\setminus B$. Then for any $y\in B\setminus\{0\}$ and any $z\in B\setminus Ky$ we obtain

$$K(x+z)\oplus Ky\ni(x+z)y=xy\in Kx\oplus Ky,$$

and since x, y and z are linearly independent this implies $xy\in Ky$. Now the assertion follows by simple linear algebra. □

5 Theorem. *If A is a symplectic almost abelian algebra, then exactly one of the following holds:*

(i) *A is abelian.*

(ii) *A is isomorphic to some dilatation algebra.*

PROOF. According to our previous considerations, the assertion is true if $\dim A < 3$. In the sequel A is supposed to be non-abelian and of dimension at least 3. We choose an abelian subalgebra B of maximal dimension of A and consider elements $x, y \in A \setminus B$. By Lemma 3 we know that the dimension of B is at least 2. Therefore, Proposition 4 applies and we obtain the existence of $a, b \in K$ so that $xz = az$ and $yz = bz$ hold for any $z \in B$. This yields $(ay - bx)B = \{0\}$ and by maximality of B we get $ay - bx \in B$. Maximality of B also implies $a \neq 0$ and thus we have $y \in Kx + B$. So we see that B has codimension 1 and by rescaling we can achieve $a = 1$. Now it is obvious that A is isomorphic to $\mathrm{dil}(B)$. □

The General Case

As in the symplectic case, we first consider low-dimensional cases. If A is a non-symplectic almost abelian algebra we find an element $x \in A$ satisfying $x^2 = x$. Thus, if A is 1-dimensional it is isomorphic to $\mathrm{fab}_1(0, k)$ for any $k \in K$. In order to treat the 2-dimensional case we need a first result.

6 Lemma. *Let A be an almost abelian algebra, $x, y \in A$ linearly independent, and let $a, b \in K$ be such that $x^2 = ax$, and $y^2 = by$. Then we have $(x + y)^2 = (a + b)(x + y)$. In particular, if $B \subseteq A$ is a symplectic subalgebra, and if x is isotropic, then $B + Kx$ is a symplectic subalgebra of A.*

PROOF. As A is almost abelian we find $c, d \in K$ so that

$$(x + y)^2 = c(x + y) \qquad \text{and} \qquad (x - y)^2 = d(x - y).$$

Adding these two equations yields

$$2ax + 2by = (c + d)x + (c - d)y$$

and we obtain $c = a + b$, because of the linear independence of x and y. □

7 Lemma. *Let A be an almost abelian algebra and suppose that $\dim A > 1$. Then A contains at least one non-zero isotropic vector.*

PROOF. Choose $x, y \in A$ linearly independent and suppose that both x and y are non-isotropic vectors. By rescaling we can achieve $x^2 = x$ and $y^2 = -y$. Application of Lemma 6 then yields $(x + y)^2 = 0$. □

8 Lemma. *Any at most 2-dimensional non-symplectic almost abelian algebra is isomorphic to either one of the algebras $\mathrm{fab}_1(0, k)$, $\mathrm{fab}_2(0, k)$, or $\mathrm{fab}_1(K, k)$ for some $k \in K$.*

PROOF. We already know that any 1-dimensional non-symplectic almost abelian algebra is isomorphic to $\mathrm{fab}_1(0, k)$ for any $k \in K$, so let A be a 2-dimensional non-symplectic almost abelian algebra. Then, in view of Lemma 7, we find a basis $x, y \in A$ of the vector space A which satisfies $x^2 = x$ and $y^2 = 0$. Applying Lemma 6, we obtain $(x + y)^2 = x + y$, and thus

$yx = y - xy$. Since x and y form a basis for A there exist $a, b \in K$ be such that $xy = ax + by$, and by rescaling we can achieve $a \in \{-1, 0\}$. Now we consider the assignment $x \mapsto (1, 0)$ and $y \mapsto (0, 1)$. For $a = 0$ this assignment induces an isomorphism of A onto $\mathrm{fab}_1(K, b)$ and for $a = -1$ it induces an isomorphism onto $\mathrm{fab}_2(0, b)$. □

9 Proposition. *Every almost abelian algebra of positive dimension contains a symplectic subalgebra of codimension* 1.

PROOF. Let A be an almost abelian algebra. The assertion is trivial if A is symplectic, so we assume A to be non-symplectic. Let B be a maximal symplectic subalgebra of A. Suppose that $\dim A/B > 1$. By choosing a vector space complement of B in A, Lemma 7 yields the existence of some isotropic vector $x \in A \setminus B$. According to Lemma 6 this implies that $B + Kx$ is a symplectic subalgebra of A which properly contains B. Since this contradicts the maximality of B, we obtain $\dim A/B = 1$, and the assertion is proved. □

10 Theorem. *Let A be an almost abelian algebra. Then one of the following holds:*

(i) *A is abelian.*

(ii) *A is isomorphic to some dilatation algebra.*

(iii) *A is isomorphic to an algebra of type* $\mathrm{fab}_1(V, k)$ *or* $\mathrm{fab}_2(V, k)$.

PROOF. The symplectic case and the non-symplectic case where $\dim A < 3$ are covered by Theorem 5 and Lemma 8, respectively. So in the sequel we assume A to be non-symplectic and of dimension at least 3. According to Proposition 9 there exists a 1-codimensional symplectic subalgebra B of A, and since A is non-symplectic we find some $x \in A \setminus B$ satisfying $x^2 = x$. As in the proof of Lemma 8, we obtain $yx = y - xy$ for each $y \in B$. If B is abelian then we know by Proposition 4 that there exists $k \in K$ such that $xy = ky$ holds for each $y \in B$. In this case the assignment $x \mapsto (1, 0)$ and $y \mapsto (0, y)$ for each $y \in B$ induces an isomorphism of A onto $\mathrm{fab}_1(B, k)$.

If B is non-abelian, then, by Theorem 5, it is a dilatation algebra and we obtain that $C := \mathrm{span}_K\{yz \mid y, z \in B\}$ is an abelian subalgebra of A of codimension 2. Further, we find some $y \in B \setminus C$ such that $yw = w$ holds for each $w \in C$. Now for any $w \in C$ we have

$$K(x + y) \oplus Kw \ni (x + y)w = xw + w \in Kx \oplus Kw,$$

and thus $xw \in Kw$. The latter condition implies the existence of some $k \in K$ satisfying $xw = kw$ for each $w \in C$. Next we show that $xy = -x + ky$. In order to see this we note that

$$K(x + w) \oplus Ky \ni (x + w)y = xy - w \in -w + Kx \oplus Ky$$

implies $xy \in -x + Ky$; so there exists $l \in K$ such that $xy = -x + ly$. Since we have

$$Kx \oplus K(y + w) \ni x(y + w) = -x + ly + kw \in -x + Ky \oplus Kw$$

we obtain $k = l$, and now it is easy to check that the assignment $x \mapsto (1, 0, 0)$, $y \mapsto (0, 1, 0)$ and $w \mapsto (0, 0, w)$ for each $w \in C$ induces an isomorphism of A onto $\mathrm{fab}_2(C, k)$. □

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