Small weight modules of locally finite almost reductive Lie algebras

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Abstract

Let $\mathfrak{l}$ be a Lie algebra with a root decomposition and with semisimple commutator algebra. We assume that $\mathfrak{l}$ has a 3-grading compatible with the root decomposition. In this note we analyze the structure of the 2-graded weight modules of a 3-graded Lie algebra $\mathfrak{l}$. The classification results for such modules play a key role in the characterization of the locally finite split Lie algebras with faithful unitary highest weight modules because they arise in the description of such Lie algebras as semidirect sums of almost reductive Lie algebras with generalized Heisenberg algebras.

Introduction

The characterization of the locally finite split Lie algebras with faithful unitary highest weight modules in [Ne00b] shows that these Lie algebras are semidirect sums $\mathfrak{g} = \mathfrak{u} \rtimes \mathfrak{l}$, where $\mathfrak{l}$ is *almost reductive*, i.e., $[\mathfrak{l}, \mathfrak{l}]$ is semisimple (a direct sum of simple ideals), and $\mathfrak{u}$ is a *generalized Heisenberg algebra*, i.e., a two step nilpotent Lie algebra. Since the structure of both pieces $\mathfrak{u}$ and $\mathfrak{l}$ is quite well understood, the main point in understanding the structure of $\mathfrak{g}$ is to understand the action of $\mathfrak{l}$ on $\mathfrak{u}$.

The Lie algebra $\mathfrak{l}$ has a natural 3-grading $\mathfrak{l} = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1$, and we have $\mathfrak{u} = V \times \mathfrak{z}(\mathfrak{g})$, where $V = V^+ \oplus V^-$ is an $\mathfrak{l}$-module which is 2-graded in a way that is compatible with the 3-grading of $\mathfrak{l}$. The objective of this note is to describe the structure of 2-graded $\mathfrak{l}$-modules V.

Since the Lie algebra $\mathfrak{g}$ is assumed to have a root decomposition $\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha\in\Delta} \mathfrak{g}^\alpha$ with respect to a splitting Cartan subalgebra $\mathfrak{h}$, the $\mathfrak{l}$-module V is an *integrable weight module* of $\mathfrak{l}$ in the sense that it is the sum of the weight spaces for the Cartan subalgebra $\mathfrak{h}_\mathfrak{l} := \mathfrak{h} \cap \mathfrak{l}$ of $\mathfrak{l}$. Thus we have to consider 2-graded weight modules of 3-graded almost reductive Lie algebras. The key idea to analyze the structure of the $\mathfrak{l}$-module V is first to reduce matters to the case where $\mathfrak{l}$ is semisimple. Then $\mathfrak{l} = \mathfrak{l}_a \oplus \mathfrak{l}_b$, where $\mathfrak{l}_a$ is the ideal generated by $\mathfrak{l}_{\pm 1}$. Now V turns out to be a small weight module for the ideal $\mathfrak{l}_a$, which means that $\mu(\check\alpha) \in \{-1, 0, 1\}$ holds for all weights μ of V and roots α of $\mathfrak{l}_a$. Section I contains basic material on weight modules, and in Section II we completely describe the structure of small weight modules. In particular we show that small weight modules are semisimple and that the simple ones are highest weight modules. After reduction to the case of simple Lie algebras, we describe in Section III all those weights λ for which the corresponding integrable highest weight module $L(\lambda)$ is small (Theorem III.3). In Section IV we turn to the description of 2-graded modules. The possible 3-gradings of $\mathfrak{l}$ have been described in [NeSt99] (see also [Ne90]), and for simple 3-graded Lie algebras we classify the 2-graded simple modules in Theorem IV.11. The outcome of our analysis is that the 2-graded $\mathfrak{l}$-module V is a semisimple $\mathfrak{l}_a$-module, and each isotypic component $W \subseteq V$ is isomorphic to $L(\lambda) \otimes W_b$, where $L(\lambda)$ is a 2-graded simple highest weight module of a simple ideal of $\mathfrak{l}_a$, and W_b is an arbitrary weight module of $\mathfrak{l}_b$. Thus we have a complete description of the $\mathfrak{l}_a$-action on V, but there is essentially no information on the $\mathfrak{l}_b$-action. We conclude this paper with some remarks on infinite tensor products in Section V.

In this paper all Lie algebras are Lie algebras over a field $\mathbb{K}$ of characteristic 0.

I. Weight modules

In this section we discuss basic properties of weight modules of split Lie algebras which are almost reductive.

Definition I.1. (a) We call an abelian subalgebra $\mathfrak{h}$ of the Lie algebra $\mathfrak{g}$ a *splitting Cartan subalgebra* if $\mathfrak{h}$ is maximal abelian and the operators in $\operatorname{ad}\mathfrak{h}$ are simultaneously diagonalizable. If $\mathfrak{g}$ contains a splitting Cartan subalgebra, then it is called a *split Lie algebra.* This means that we have a *root decomposition*

$$\mathfrak{g} = \mathfrak{h} + \sum_{\alpha\in\Delta} \mathfrak{g}^\alpha,$$

where $\mathfrak{g}^\alpha = \{z \in \mathfrak{g} : (\forall x \in \mathfrak{h})[x,z] = \alpha(x)z\}$ and $\Delta := \{\alpha \in \mathfrak{h}^* \setminus \{0\} : \mathfrak{g}^\alpha \neq \{0\}\}$ is the corresponding *root system.*
(b) A subset $\Delta^+ \subseteq \Delta$ is called a *positive system* and its elements *positive roots* if $\Delta = \Delta^+ \dot\cup - \Delta^+$ and no non-trivial sum of positive roots is zero. This requirement implies in particular that $\Delta = -\Delta$ and that each positive system contains exactly one root of each set $\{\alpha, -\alpha\}$. We call a subset $\Sigma \subseteq \Delta$ *parabolic* if $\Sigma \cup -\Sigma = \Delta$ and $(\Sigma + \Sigma) \cap \Delta \subseteq \Sigma$ (cf. [Ne98, Def. I.6] for a discussion of this concept).
(c) We call a root $\alpha \in \Delta$ *integrable* if there exist $x_{\pm\alpha} \in \mathfrak{g}^{\pm\alpha}$ such that the subalgebra $\mathfrak{g}(x_\alpha, x_{-\alpha})$ generated by these two elements is three-dimensional simple and $\operatorname{ad} x_{\pm\alpha}$ are locally nilpotent operators on $\mathfrak{g}$. We write Δ_i for the set of integrable roots and observe that $\Delta_i = -\Delta_i$ follows from the symmetry in the definition of Δ_i. It can be shown that for all integrable roots α the root space $\mathfrak{g}^\alpha$ is one-dimensional and that the subalgebra $\mathfrak{g}(\alpha) := \mathfrak{g}^\alpha + \mathfrak{g}^{-\alpha} + [\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}]$ is isomorphic to $\mathfrak{sl}(2,\mathbb{K})$ (cf. [St99a, Prop. I.6]). The unique element $\check\alpha \in [\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}]$ with $\alpha(\check\alpha) = 2$ is called the *coroot* corresponding to α. We write $\check\Delta \subseteq \mathfrak{h}$ for the set of all coroots of integrable roots. The subgroup $\mathcal{W} \subseteq \mathrm{GL}(\mathfrak{h}^*)$ generated by the reflections r_α given by $r_\alpha.\beta = \beta - \beta(\check\alpha)\alpha$ is called the *Weyl group.*
(d) We call a Lie algebra $\mathfrak{g}$ *locally finite* if every finite subset of $\mathfrak{g}$ is contained in a finite-dimensional subalgebra. In [Ne00a, Th. VI.3], it was shown that if all roots are integrable, then $\mathfrak{g}$ is locally finite, so that [St99a, Th. IV.7, Lemma IV.8] show that the commutator algebra $[\mathfrak{g},\mathfrak{g}]$, which equals $\operatorname{span}\check\Delta + \sum_{\alpha\in\Delta}\mathfrak{g}^\alpha$ in this case, is a semisimple Lie algebra, i.e., a direct sum of simple ideals. If $\mathfrak{g}$ is finite-dimensional, then this is equivalent to $\mathfrak{g}$ being reductive. Therefore we call a Lie algebra $\mathfrak{g}$ for which the commutator algebra is semisimple *almost reductive.* ■

Throughout this paper $\mathfrak{g} = \mathfrak{h} + \sum_{\alpha\in\Delta}\mathfrak{g}^\alpha$ is a split $\mathbb{K}$-Lie algebra with $\Delta = \Delta_i$, i.e., $\mathfrak{g}$ is a locally finite almost reductive split Lie algebra (cf. [St99a, Th. III.19]).

Definition I.2. (a) For a $\mathfrak{g}$-module V and $\beta \in \mathfrak{h}^*$ we write $V^\beta := \{v \in V : (\forall X \in \mathfrak{h}) X.v = \beta(X)v\}$ for the *weight space of weight* β.
(b) Let V be a $\mathfrak{g}$-module and $0 \neq v \in V^\lambda$ an $\mathfrak{h}$-weight vector. We say that v is a *primitive element of* V (with respect to the positive system Δ^+) if $\mathfrak{g}^\alpha.v = \{0\}$ holds for all $\alpha \in \Delta^+$. A $\mathfrak{g}$-module V is called a *highest weight module* with highest weight λ (with respect to Δ^+) if it is generated by a primitive element of weight λ.
■

Proposition I.3. *Let $\mathfrak{g}$ be split Lie algebra and Δ^+ a positive system. Then, for each $\lambda \in \mathfrak{h}^*$ there exists a unique irreducible highest weight module $L(\lambda, \Delta^+)$, and each highest weight module V of highest weight λ with respect to Δ^+ has a unique maximal submodule M with $V/M \cong L(\lambda, \Delta^+)$.*

Proof. This is proved as Prop. IX.1.13 in [Ne99]. ■

If V is a $\mathfrak{g}$-module, then we write ρ_V for the corresponding representation of $\mathfrak{g}$ on V, and if, in particular, $V = L(\lambda, \Delta^+)$ is an irreducible highest weight module with respect to a positive system Δ^+, then we abbreviate $\rho_\lambda := \rho_{L(\lambda,\Delta^+)}$.

Definition I.4. (cf. [DiPe99]) (a) Let $\mathfrak{g}$ be an almost reductive split Lie algebra. A $\mathfrak{g}$-module V is called a *weight module* (with respect to $\mathfrak{h}$) if it is the sum of the $\mathfrak{h}$-weight spaces, where $\mathfrak{h} \subseteq \mathfrak{g}$ is a splitting Cartan subalgebra. We write $\mathcal{P}_V := \{\alpha \in \mathfrak{h}^* : V^\alpha \neq \{0\}\}$ for the set of $\mathfrak{h}$-weights of V.
(b) A weight module V is said to be
(1) *small* if for each $\mu \in \mathcal{P}_V$ and $\alpha \in \Delta$ we have $\mu(\check{\alpha}) \in \{-1, 0, 1\}$.
(2) *finite* if for each $\mu \in \mathcal{P}_V$ and each $\alpha \in \Delta$ the set $\{n \in \mathbb{Z} : \mu + n\alpha \in \mathcal{P}_V\}$ is finite.
(3) *integrable* if for each $\alpha \in \Delta$ and $x_\alpha \in \mathfrak{g}^\alpha$ the operator $\rho_V(x_\alpha)$ on V is locally nilpotent.
(c) If V is a weight module and $V^\alpha \subseteq V$ a weight space, then we identify its dual space $(V^\alpha)^*$ with the subspace of V^* consisting of all those linear functionals vanishing on $\sum_{\beta \in \mathcal{P}_V \setminus \{\alpha\}} V^\beta$. Now the subspace $V^\sharp := \bigoplus_{\alpha \in \mathcal{P}_V} (V^\alpha)^* \subseteq V^*$ is invariant under the natural action of $\mathfrak{g}$ on the algebraic dual space V^* given by $\rho_{V^*}(x).\alpha := -\alpha \circ \rho_V(x)$. It is called the *dual weight module* because it is a weight module and the largest with this property in V^*. ■

Lemma I.5. *Let V be a weight module.*
(i) *If V is small, then V is finite.*
(ii) *If V is finite, then it is integrable.*
(iii) *If V is integrable, then $\mathcal{P}_V$ is contained in the weight group $\mathcal{P} := \{\beta \in \mathfrak{h}^* : (\forall \alpha \in \Delta) \beta(\check{\alpha}) \in \mathbb{Z}\}$.*

Proof. (i) and (ii) are trivial consequences of the fact that $\alpha(\check{\alpha}) = 2$ and $\rho_V(x_\alpha).V^\mu \subseteq V^{\mu+\alpha}$, whereas (iii) follows from the representation theory of $\mathfrak{sl}(2, \mathbb{K})$. ■

Lemma I.6. *If V is an integrable weight module of $\mathfrak{g}$ and $\mathfrak{g}_0$ a finite-dimensional $\mathfrak{h}$-invariant subalgebra, then V is a locally finite $\mathfrak{g}_0$-module, i.e., every element generates a finite-dimensional submodule.*

Proof. Let $v_\mu \in V^\mu$ be a weight vector. For each root $\alpha \in \Delta_0 := \{\alpha \in \Delta : \mathfrak{g}^\alpha \subseteq \mathfrak{g}_0\}$ we choose a non-zero vector $x_\alpha \in \mathfrak{g}^\alpha$ and thus obtain a vector space basis of $[\mathfrak{h}, \mathfrak{g}_0]$. Let $\Delta_0 = \{\alpha_1, \ldots, \alpha_n\}$. Then the Poincaré–Birkhoff–Witt Theorem implies that

$$W := \sum_{\mathbf{m} \in \mathbb{N}_0^n} \mathbb{K} \rho_V(x_{\alpha_1})^{m_1} \cdots \rho_V(x_{\alpha_n})^{m_n}.v_\mu$$

is a $\mathfrak{g}_0$-invariant subspace. Moreover, we see by induction, using the local nilpotence of the operators $\rho_V(x_\alpha)$, that there exist $c_1, \ldots, c_n \in \mathbb{N}$ such that

$$W = \sum_{\mathbf{m} \leq \mathbf{c} \in \mathbb{N}_0^n} \mathbb{K}\, \rho_V(x_{\alpha_1})^{m_1} \cdots \rho_V(x_{\alpha_n})^{m_n}.v_\mu,$$

and hence that W is finite-dimensional (cf. [MoPi95, p.125] for a similar argument for $\mathfrak{sl}(2, \mathbb{K})$). ■

Remark I.7. Applied to the subalgebras $\mathfrak{g}(\alpha) := \mathfrak{g}^\alpha + \mathfrak{g}^{-\alpha} + [\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}]$, the preceding lemma implies that each integrable weight module of $\mathfrak{g}$ is a locally finite module of $\mathfrak{g}(\alpha) \cong \mathfrak{sl}(2, \mathbb{K})$. This implies that the set $\mathcal{P}_V$ is invariant under the corresponding reflection r_α. Thus, for each integrable weight module V the set $\mathcal{P}_V$ is invariant under the Weyl group $\mathcal{W}$ (cf. [Bou90, Ch. 8, no. 7.1, Cor. 2]). Moreover, the representation theory of $\mathfrak{sl}(2, \mathbb{K})$ shows that for each $\mu \in \mathcal{P}_V$ the set

$$\{n \in \mathbb{Z} \colon \mu + n\alpha \in \mathcal{P}_V\}$$

is an uninterrupted string of integers. ■

Lemma I.8. *A highest weight module V of highest weight λ is integrable if and only if λ is dominant integral, i.e.,*

$$\lambda(\check{\alpha}) \in \mathbb{N}_0 \quad \textit{for all} \quad \alpha \in \Delta^+.$$

Every integrable highest weight module V is simple, i.e., $V \cong L(\lambda, \Delta^+)$.

Proof. (cf. [DiPe99, Th. 5]) Let $v_\lambda \in V$ be a primitive element. To see that V is integrable if and only if $\lambda(\check{\alpha}) \in \mathbb{N}_0$ for all $\alpha \in \Delta^+$, we first note that if V is integrable and $\alpha \in \Delta^+$, then the $\mathfrak{g}(\alpha)$-module $U(\mathfrak{g}(\alpha)).v_\lambda$ is finite-dimensional, so that $\lambda(\check{\alpha}) \in \mathbb{N}_0$. If, conversely, this condition is satisfied, then v_λ is a $\mathfrak{g}(\alpha)$-finite element (it generates a finite-dimensional $\mathfrak{g}(\alpha)$-submodule), so that the local finiteness of the action of $\mathfrak{g}(\alpha)$ on $\mathfrak{g}$ implies that the set of $\mathfrak{g}(\alpha)$-finite elements in V is a $\mathfrak{g}$-submodule of V containing v_λ, so that V is a locally finite $\mathfrak{g}(\alpha)$-module. Therefore the corresponding operators $\rho_V(x_{\pm\alpha})$ are locally nilpotent for $x_{\pm\alpha} \in \mathfrak{g}^{\pm\alpha}$. This proves that V is integrable.

Suppose that V is integrable and let $M \subseteq V$ be the maximal proper submodule (Proposition I.3). If $M \neq \{0\}$, then there exists an $\mathfrak{h}$-invariant finite-dimensional semisimple subalgebra $\mathfrak{g}_0 \subseteq \mathfrak{g}$ such that $V_0 := U(\mathfrak{g}_0).v_\lambda$ intersects M non-trivially (cf. [St99a, Prop. V.5]). Now V_0 is an integrable highest weight module of $\mathfrak{g}_0$, hence finite-dimensional and simple (cf. [Bou90, Chap. 8, no. 7.2, Th. 1]). This contradicts $V_0 \neq V_0 \cap M \neq \{0\}$. ■

Theorem I.9. (Classification of integrable highest weight modules) *For each weight* $\lambda \in \mathcal{P} = \{\mu \in \mathfrak{h}^* \colon (\forall \alpha \in \Delta)\mu(\check{\alpha}) \in \mathbb{Z}\}$ *there exists a positive system* Δ^+ *such that* λ *is dominant integral. If* $\widetilde{\Delta}^+$ *is another positive system for which* λ *is dominant integral, then*

$$L(\lambda, \Delta^+) \cong L(\lambda, \widetilde{\Delta}^+), \tag{1.1}$$

so that we may write $L(\lambda) := L(\lambda, \Delta^+)$. *Furthermore*

$$L(\lambda) \cong L(\mu) \qquad \iff \qquad \mu \in \mathcal{W}.\lambda.$$

Proof. These facts follow essentially from the discussion in Section I of [Ne98], where the unitary highest weight modules of the corresponding complex Lie algebras have been classified in the same manner. For the sake of completeness, we include the key arguments.

For $\lambda \in \mathcal{P}$ we consider the subset $\Sigma_\lambda := \{\alpha \in \Delta \colon \lambda(\check{\alpha}) \geq 0\}$ and observe that this is a parabolic system of Δ (cf. [Ne98, Lemma I.18]) because parabolic systems of Δ and $\check{\Delta}$ are in one-to-one correspondence via the map $\alpha \mapsto \check{\alpha}$ (cf. [Bou90]). In view of [Ne98, Cor. I.10], the parabolic system Σ_λ contains a positive system Δ^+, and now λ is Δ^+-dominant integral.

The weight set $\mathcal{P}_{L(\lambda,\Delta^+)}$ is contained in $\lambda - \mathbb{N}_0[\Delta^+] \subseteq \lambda - \mathbb{Z}[\Delta]$, where $\mathbb{N}_0[\Delta^+]$ denotes the set of finite sums of elements of Δ^+ and $\mathbb{Z}[\Delta]$ denotes the additive subgroup of $\mathfrak{h}^*$ generated by Δ. Now the formula

$$\mathcal{P}_{L(\lambda,\Delta^+)} = \operatorname{conv}(\mathcal{W}.\lambda) \cap (\lambda + \mathbb{Z}[\Delta]), \tag{1.2}$$

where we view $\mathcal{P}$ as a subset of the real vector space $\mathbb{R} \otimes_{\mathbb{Z}} \mathcal{P}$, shows that the weight set $\mathcal{P}_{L(\lambda,\Delta^+)}$ does not depend on the positive system. If V is a simple $\mathfrak{g}$-module with

$$\lambda \in \mathcal{P}_V \subseteq \lambda - \mathbb{N}_0[\Delta^+],$$

then $V \cong L(\lambda, \Delta^+)$ follows from the fact that each non-zero element $v_\lambda \in V^\lambda$ is a primitive element with respect to Δ^+ because $\alpha \notin -\mathbb{N}_0[\Delta^+]$ for each $\alpha \in \Delta^+$ (this follows from the definition of a positive system). Now (1.1) follows from $\mathcal{P}_{L(\lambda,\Delta^+)} = \mathcal{P}_{L(\lambda,\widetilde{\Delta}^+)}$.

In the following we write

$$\operatorname{Ext}(C) = \{x \in C \colon (y, z \in C, \lambda \in]0,1[, x = \lambda y + (1-\lambda)z) \Rightarrow x = y = z\}$$

for the set of *extreme points* of a convex set C. If $L(\lambda) \cong L(\mu)$, then the equality of the weight sets $\mathcal{P}_{L(\lambda)} = \mathcal{P}_{L(\mu)}$ implies that

$$\begin{aligned} \mathcal{W}.\lambda &= \operatorname{Ext}(\operatorname{conv} \mathcal{W}.\lambda) = \operatorname{Ext}\left(\operatorname{conv} \mathcal{P}_{L(\lambda)}\right) \\ &= \operatorname{Ext}\left(\operatorname{conv} \mathcal{P}_{L(\mu)}\right) = \operatorname{Ext}(\operatorname{conv} \mathcal{W}.\mu) = \mathcal{W}.\mu \end{aligned}$$

(cf. [Ne98, Th. I.11]), so that $\mu \in \mathcal{W}.\lambda$. If, conversely, $\mu \in \mathcal{W}.\lambda$, then $\mu + \mathbb{Z}[\Delta] = \lambda + \mathbb{Z}[\Delta]$, and (1.2) lead to $\mathcal{P}_{L(\lambda)} = \mathcal{P}_{L(\mu)}$, so that the observation in the preceding paragraph implies that $L(\lambda) \cong L(\mu)$. ■

II. Small weight modules

In this section we discuss the special class of small weight modules.

Proposition II.1. *If V is a small weight module, then for each weight $\lambda \in \mathcal{P}_V$ and $0 \neq v_\lambda \in V^\lambda$ the submodule $U(\mathfrak{g}).v_\lambda$ is an integrable highest weight module isomorphic to $L(\lambda)$.*

Proof. Using Theorem I.9, we find a positive system Δ^+ such that λ is dominant integral with respect to Δ^+. Let $\alpha \in \Delta^+$. Then $(\lambda + \alpha)(\check{\alpha}) \geq 2$ implies that $\lambda + \alpha \notin \mathcal{P}_V$, and hence that each non-zero weight vector $v_\lambda \in V^\lambda$ is a primitive element for $\mathfrak{g}$ with respect to Δ^+. We conclude that $W := U(\mathfrak{g}).v_\lambda$ is an integrable highest weight module, hence simple by Lemma I.8, and therefore $W \cong L(\lambda)$. ■

Corollary II.2. (a) *Each small weight module V is a semisimple $\mathfrak{g}$-module.*
(b) *Every simple small weight module is an integrable highest weight module.*

Proof. (a) In view of Proposition II.1, the module V is a sum of simple submodules, hence a semisimple module.
(b) This follows directly from Proposition II.1. ■

Remark II.3. That simple small weight modules are highest weight modules relies heavily on the smallness requirement. The weaker condition $\mu(\check{\alpha}) \in \{\pm 2, \pm 1, 0\}$ for all $\mu \in \mathcal{P}_V$ and $\alpha \in \Delta$ is not sufficient to conclude that a simple weight module V is a highest weight module.

To see this, we consider the Lie algebra $\mathfrak{g} := \mathfrak{gl}(\mathbb{N}, \mathbb{K})$ as the union of the subalgebras $\mathfrak{g}_n := \mathfrak{gl}(2n, \mathbb{K})$, $n \in \mathbb{N}$, and fix the standard positive system $\Delta^+ := \{\varepsilon_j - \varepsilon_k : j < k\}$. For each $n \in \mathbb{N}$ we consider the dominant integral weight

$$\lambda_n := (\underbrace{1, 1, \ldots, 1}_{n \text{ times}}, \underbrace{-1, -1, \ldots, -1}_{n \text{ times}})$$

with respect to $\Delta_n^+ := \Delta_n \cap \Delta^+$ and $\Delta_n := \{\alpha \in \Delta : \mathfrak{g}^\alpha \subseteq \mathfrak{g}_n\}$. Then the set $\mathcal{P}_{L(\lambda_n)}$ of weights of the highest weight module $L(\lambda_n, \Delta_n^+)$ is given by

$$\mathcal{P}_{L(\lambda_n)} = \Big\{ \sum_{j=1}^{2n} a_j \varepsilon_j : a_j \in \{-1, 0, 1\}, \sum_{j=1}^{2n} a_j = 0 \Big\},$$

as follows easily from $\mathcal{P}_{L(\lambda_n)} = \mathrm{conv}(\mathcal{W}_n.\lambda_n) \cap (\lambda_n + \mathbb{Z}[\Delta_n])$, where $\Delta_n \subseteq \mathfrak{h}_n^*$ denotes the roots of $\mathfrak{g}_n$. In particular each weight $\alpha \in \mathcal{P}_{L(\lambda_n)}$ can be written as

$$\alpha = \sum_{j\in N_1} \varepsilon_j - \sum_{j\in N_2} \varepsilon_j, \quad \text{where} \quad |N_1| = |N_2| \leq n \quad \text{and} \quad N_1 \cap N_2 = \emptyset.$$

We see in particular that λ_{n-1} is a weight of $\mathcal{P}_{L(\lambda_n)}$, and that the corresponding weight space generates a $\mathfrak{g}_{n-1}$-submodule of highest weight λ_{n-1}. Using a fixed choice of embeddings

$$L(\lambda_n, \mathfrak{g}_n) \hookrightarrow L(\lambda_{n+1}, \mathfrak{g}_{n+1}), \qquad n \in \mathbb{N},$$

we obtain a simple weight module $V := \lim\limits_{\longrightarrow} L(\lambda_n, \mathfrak{g}_n)$ of $\mathfrak{g}$. The weight system of this module is given by

$$\mathcal{P}_V = \bigcup_{n\in\mathbb{N}} \mathcal{P}_{L(\lambda_n)} = \Big\{ \sum_{j=1}^m a_j\varepsilon_j : m \in \mathbb{N}, a_j \in \{-1,0,1\}, \sum_{j=1}^m a_j = 0 \Big\}.$$

If $\alpha \in \mathcal{P}_V$ is an extreme point of $\mathrm{conv}(\mathcal{P}_V)$, then there exists an $n \in \mathbb{N}$ with $\alpha = \sum_{j=1}^{2n} a_j\varepsilon_j \in \mathcal{P}_{L(\lambda_n)}$. Then $\alpha \in \mathrm{Ext}(\mathrm{conv}\,\mathcal{P}_{\lambda_n}) = \mathcal{W}_n.\lambda_n$. This means that $|\{j : a_j = 1\}| = n$. Then α is not extremal in $\mathrm{conv}(\mathcal{P}_{\lambda_{n+1}})$, hence not in $\mathrm{conv}(\mathcal{P}_V)$. This contradiction shows that $\mathrm{Ext}\big(\mathrm{conv}(\mathcal{P}_V)\big) = \emptyset$ holds in the real vector space $\mathbb{R} \otimes_{\mathbb{Z}} \mathcal{P}$, and hence that V is not a highest weight module (cf. [Ne98, Cor. I.14]).■

Proposition II.4. *For an integrable weight module V the following are equivalent:*

(1) *V is small.*

(2) *For each $x_\alpha \in \mathfrak{g}^\alpha$, $\alpha \in \Delta$, we have $\rho_V(x_\alpha)^2 = 0$.*

If, in addition, V is simple, then these conditions are equivalent to

(3) *$\mathcal{W}$ acts transitively on $\mathcal{P}_V$ and V is a highest weight module.*

Proof. (1) $\Rightarrow$ (2): Since $\langle \mathcal{P}_V, \check{\alpha}\rangle \subseteq \{0, 1, -1\}$ and $\alpha(\check{\alpha}) = 2$, we have $\rho_V(x_\alpha)^2 = 0$.
(2) $\Rightarrow$ (1): Let $\mu \in \mathcal{P}_V$. If $|\mu(\check{\alpha})| > 1$, then the subspace $\sum_{n\in\mathbb{Z}} V^{\mu+n\alpha}$ contains $\mathfrak{g}(\alpha)$-submodules of dimension > 2, so that $\rho_V(x_\alpha)^2 \neq 0$. This means that if V is not small, then (2) is not satisfied.
(2) $\Rightarrow$ (3): If V is simple, then $V \cong L(\lambda)$ for each $\lambda \in \mathcal{P}_V$ (Proposition II.1). Hence $L(\lambda) \cong L(\mu)$ for each $\mu \in \mathcal{P}_V$, so that Theorem I.9 implies that $\mu \in \mathcal{W}.\lambda$.
(3) $\Rightarrow$ (2): We view $\mathcal{P}_V$ as a subset of the real vector space $\mathbb{R} \otimes_{\mathbb{Z}} \mathcal{P}$. Then (3) implies that every weight $\mu \in \mathcal{P}_V$ is an extreme point of $\mathrm{conv}(\mathcal{P}_V)$. Hence for each $\mu \in \mathcal{P}_V$ the weight string $(\mu + \mathbb{Z}\alpha) \cap \mathcal{P}_V$ is of length ≤ 2, and this implies that $|\mu(\check{\alpha})| \leq 1$. ■

Corollary II.5. *If V is a small weight module, then the isotypic submodules of V are in one-to-one correspondence with the $\mathcal{W}$-orbits in $\mathcal{P}_V$. For each such orbit $\mathcal{W}.\lambda \subseteq \mathcal{P}_V$ the subspace $\sum_{w\in\mathcal{W}} V^{w.\lambda}$ is an isotypic submodule isomorphic to $L(\lambda)\otimes V^\lambda$, where V^λ is viewed as a trivial $\mathfrak{g}$-module.*

Proof. Let $\lambda \in \mathcal{P}_V$. Since each non-zero weight vector $v \in V^\lambda$ generates a simple integrable highest weight module $W_v \cong L(\lambda)$, and $W_v^\lambda = \mathbb{K}v$, we see that we have an inclusion

$$L(\lambda)\otimes V^\lambda \cong L(\lambda)\otimes \mathrm{Hom}_{\mathfrak{g}}(L(\lambda), V) \hookrightarrow V, \qquad v\otimes D \mapsto D(v).$$

Since $\mathcal{W}$ acts transitively on $\mathcal{P}_{L(\lambda)}$ (Proposition II.4(3)), we see that the image of the above inclusion map coincides with the subspace $\sum_{w\in\mathcal{W}} V^{w.\lambda}$. The remaining assertions are clear. ■

Lemma II.6. *Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ be a direct sum decomposition of $\mathfrak{g}$.*

(i) *If V is a simple $\mathfrak{g}$-module which contains a simple $\mathfrak{g}_1$-submodule V_1 with $\mathrm{End}_{\mathfrak{g}_1}(V_1) = \mathbb{K}\mathbf{1}$, then there exists a simple $\mathfrak{g}_2$-module V_2 such that $V \cong V_1 \otimes V_2$.*

(ii) *If V_1 is a simple $\mathfrak{g}_1$-module with $\mathrm{End}_{\mathfrak{g}_1}(V_1) = \mathbb{K}\mathbf{1}$ and V_2 a simple $\mathfrak{g}_2$-module, then $V := V_1 \otimes V_2$ is a simple $\mathfrak{g}$-module with $\mathrm{End}_{\mathfrak{g}}(V) \cong \mathbf{1}\otimes \mathrm{End}_{\mathfrak{g}_2}(V_2)$.*

Proof. (i) The subspace $V' := \sum_{D\in U(\mathfrak{g}_2)} \rho_V(D).V_1$ is a $\mathfrak{g}$-submodule of V, and each subspace $\rho_V(D).V_1$ either is zero or a simple $\mathfrak{g}_1$-submodule isomorphic to V_1. In view of the simplicity of V, we conclude that $V = V'$ and hence that V is a semisimple isotypic $\mathfrak{g}_1$-module. Therefore there exists a trivial $\mathfrak{g}_1$-module V_2 with $V \cong V_1 \otimes V_2$ as $\mathfrak{g}_1$-modules. In view of [Ne99, Lemma IX.4.7], the assumption $\mathrm{End}_{\mathfrak{g}_1}(V_1) = \mathbb{K}\mathbf{1}$ implies that

$$\mathrm{End}_{\mathfrak{g}_1}(V) \cong \mathrm{End}_{\mathfrak{g}_1}(V_1\otimes V_2) = \mathbf{1}\otimes \mathrm{End}(V_2).$$

Hence $\rho_V(\mathfrak{g}_2) \subseteq \mathrm{End}_{\mathfrak{g}_1}(V)$ implies that there exists a homomorphism $\rho_{V_2}\colon \mathfrak{g}_2 \to \mathrm{End}(V_2)$ with $\rho_V(X) = \mathbf{1}\otimes\rho_{V_2}(X)$ for $X \in \mathfrak{g}_2$. This proves that $V \cong V_1 \otimes V_2$ as $\mathfrak{g}$-modules, and the simplicity of V implies that V_2 is a simple $\mathfrak{g}_2$-module.
(ii) Jacobson's Density Theorem ([La74, Th. XVII.3.2]) implies that for each finite dimensional subspace $E \subseteq V_1$ we have

$$\mathrm{Hom}(E, V_1) = \rho_{V_1}(U(\mathfrak{g}_1))\,|_E.$$

For each linearly independent subset $\{e_1, \ldots, e_n\} \subseteq V_1$ we thus obtain elements $D_j \in U(\mathfrak{g}_1)$ with $D_j.e_k = \delta_{kj} e_j$.

Now let $0 \neq z := \sum_{j=1}^n x_j \otimes y_j \in V_1 \otimes V_2$ and assume w.l.o.g. that the x_j are linearly independent and $y_j \neq 0$ for each j. We have to show that z generates $V_1\otimes V_2$ as a $\mathfrak{g}$-module. With Jacobson's Density Theorem we obtain an element $D \in U(\mathfrak{g}_1)$ with $D.x_j = 0$ for $j = 2, \ldots, n$ and $D.x_1 = x_1$. Therefore the $\mathfrak{g}$-submodule of V generated by z contains the element $D.z = x_1 \otimes y_1$. Now $U(\mathfrak{g}_1)D.z = U(\mathfrak{g}_1).x_1 \otimes y_1 = V_1 \otimes y_1$ and further $U(\mathfrak{g}_2).(V_1 \otimes y_1) = V_1 \otimes U(\mathfrak{g}_2).y_1 = V_1 \otimes V_2$. The second assertion follows from the argument in the proof of (i). ■

The following theorem provides crucial information on the rough structure of weight modules of a direct sum Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ which are small for $\mathfrak{g}_1$. The essential information is that these modules are tensor products of modules of $\mathfrak{g}_1$ and $\mathfrak{g}_2$, where the $\mathfrak{g}_1$-module is small and therefore well behaved, whereas there is no further information available on the $\mathfrak{g}_2$-module.

Theorem II.7. (Factorization Theorem) *Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ and V a simple $\mathfrak{g}$-module which is a small weight module for $\mathfrak{g}_1$. Then there exists an integrable highest weight module $L(\lambda)$ of $\mathfrak{g}_1$ and a simple $\mathfrak{g}_2$-module V_2 with $V \cong L(\lambda) \otimes V_2$.*

Proof. According to Corollary II.2, V is a semisimple $\mathfrak{g}_1$-module, hence contains a simple submodule V_1. Now $V_1 \cong L(\lambda)$ for some $\lambda \in \mathfrak{h}_1^*$ (Proposition II.1), so that $\mathrm{End}_{\mathfrak{g}_1}(V_1) = \mathbb{K}\mathbf{1}$. Therefore Lemma II.6 applies. ■

Remark II.8. Let V be a weight module of the commutator algebra $\mathfrak{g}_0 := [\mathfrak{g}, \mathfrak{g}]$ with respect to the splitting Cartan subalgebra $\mathfrak{h}_0 := \mathfrak{h} \cap [\mathfrak{g}, \mathfrak{g}] = \mathrm{span}\, \check{\Delta}$ and recall that $\check{\Delta}$ separates the points of $\mathrm{span}\, \Delta$ (cf. [St99a, Prop. III.7]). Then we may identify the group $\mathbb{Z}[\Delta]$ with a subset of $\mathfrak{h}_0^*$. In this sense, for every weight $\mu \in \mathcal{P}_V \subseteq \mathfrak{h}_0^*$ the subspace $V(\mu) := \sum_{\alpha \in \mathbb{Z}[\Delta]} V^{\mu+\alpha}$ is a submodule, and V is the direct sum of such submodules. Therefore we may assume that $\mathcal{P}_V \subseteq \mu + \mathbb{Z}[\Delta]$. Now we extend μ to an element $\widetilde{\mu} \in \mathfrak{h}^*$ and define an action on the weight space $V^{\mu+\alpha}$, $\alpha \in \mathbb{Z}[\Delta]$, by $x.v := (\widetilde{\mu}(x) + \alpha(x))v$ for $x \in \mathfrak{h}$. One directly verifies that we thus obtain a representation of the whole Lie algebra $\mathfrak{g}$ on V which has the same submodules. In this sense all classification problems for weight modules of $\mathfrak{g}$ can directly be reduced to modules of the semisimple commutator algebra $\mathfrak{g}_0$.

We also observe that if V is a simple weight module for $\mathfrak{g}$ and $\lambda \in \mathcal{P}_V$, then $\mathcal{P}_V \subseteq \lambda + \mathbb{Z}[\Delta]$ implies that each $\mathfrak{g}_0$-submodule is adapted to the weight decomposition, showing that V is a simple $\mathfrak{g}_0$-module. ■

III. The classification of simple small modules

In this section we describe the classification of simple small weight modules. Since all small weight modules are semisimple (Corollary II.2), this yields a description of all small weight modules. First we reduce the situation to the case of simple Lie algebras.

Lemma III.1. *Let* $[\mathfrak{g},\mathfrak{g}] = \bigoplus_{j\in J}\mathfrak{g}_j$ *be the decomposition into simple ideals,* $\mathfrak{h}_j := \mathfrak{h}\cap\mathfrak{g}_j$, $\lambda\in\mathcal{P}$, *and* $\lambda_j := \lambda|_{\mathfrak{h}_j}$. *Then the integrable highest weight module* $L(\lambda)$ *is small if and only if all the integrable highest weight modules* $L(\lambda_j)$ *of the ideals* $\mathfrak{g}_j$ *are small.*

Proof. Let $v_\lambda \in L(\lambda)$ be a primitive element. Since the $\mathfrak{g}_j$-submodule $U(\mathfrak{g}_j).v_\lambda$ is isomorphic to $L(\lambda_j)$, all these modules are small if $L(\lambda)$ is small.

If, conversely, all the modules $L(\lambda_j)$ are small and $\alpha\in\Delta$, then there exists a $j\in J$ with $\mathfrak{g}^\alpha\subseteq\mathfrak{g}_j$. Now

$$L(\lambda) = U\big(\bigoplus_{i\neq j}\mathfrak{g}_i\big)U(\mathfrak{g}_j).v_\lambda$$

implies that $\mathcal{P}_{L(\lambda)}(\check\alpha) = \mathcal{P}_{L(\lambda_j)}(\check\alpha) \subseteq \{-1,0,1\}$ because $U\big(\bigoplus_{i\neq j}\mathfrak{g}_i\big)$ commutes with $\check\alpha$, hence preserves its eigenspaces. Therefore $L(\lambda)$ is small. ∎

In view of the preceding lemma, it suffices to classify the simple small modules for simple Lie algebras. We will see in Proposition V.2 below how these modules can be put together to a module of the big Lie algebra $\mathfrak{g}$ by an infinite tensor product construction.

Definition III.2. A root system of a locally finite split semisimple Lie algebra is called a *root system of semisimple type*. It is called *irreducible* if the corresponding Lie algebra is simple. ∎

According to [NeSt99] (see also [Ka73], [KaKi75]), for each infinite cardinal represented by a set J, there exist (up to linear equivalence) exactly four irreducible root systems of semisimple type A_J, B_J, C_J and D_J described below. These root systems still make sense for finite sets J, where we assume that $|J|\geq 2$ for A_J and B_J, $|J|\geq 3$ for C_J, and $|J|\geq 4$ for D_J. We also have the exceptional finite root systems E_6, E_7, E_8, F_4 and G_2 (cf. [Bou90, Ch. 8]). We call the root systems of types A–D root systems of classical type.

Here we realize the root systems of classical type in the subspace $\mathbb{Q}^J\subseteq\mathbb{K}^J$ which is the dual space of the $\mathbb{Q}$-vector space $\mathbb{Q}^{(J)}$ with the canonical basis $(e_j)_{j\in J}$. We write $\varepsilon_j\in\mathbb{Q}^J$ for the elements of the corresponding "dual basis" determined by $\varepsilon_j(e_k)=\delta_{jk}$. Then

$$\begin{aligned}
A_J &:= \{\varepsilon_j-\varepsilon_k : j,k\in J, j\neq k\},\\
B_J &:= \{\pm\varepsilon_j, \pm\varepsilon_j\pm\varepsilon_k : j,k\in J, j\neq k\},\\
C_J &:= \{\pm 2\varepsilon_j, \pm\varepsilon_j\pm\varepsilon_k : j,k\in J, j\neq k\}, \textit{and}\\
D_J &:= \{\pm\varepsilon_j\pm\varepsilon_k : j,k\in J, j\neq k\}.
\end{aligned}$$

Our notation for A_J is such that $A_{\{1,\ldots,n\}} \cong A_{n-1}$ in the finite-dimensional notation. For a more detailed discussion of the corresponding infinite-dimensional Lie algebras we refer to [NeSt99].

In the following we call a weight $\lambda \in \mathcal{P}$ *small* if $L(\lambda)$ is a small weight module. For a subset $M \subseteq J$ we put $\varepsilon_M := \sum_{j\in M} \varepsilon_j \in \mathbb{Q}^J \cong (\mathbb{Q}^{(J)})^*$ and consider it as the linear functional on $\mathbb{Q}^{(J)}$ given by $\varepsilon_M(x) = \sum_{j\in M} x_j$ (all these sums are finite). We write $M^c := J \setminus M$ for the complement of M in J.

Theorem III.3. (Classification of small weight modules) *In the following we represent a weight $\lambda \in \mathcal{P} \subseteq \mathfrak{h}^*$ as a function $J \to \mathbb{K}$, i.e., as an element of $\mathbb{K}^J$. We assume that $\lambda \neq 0$.*

(A_J) *A weight λ of A_J is small if and only if it can be represented as $\lambda = \varepsilon_M$ for a subset $M \subseteq J$. Its Weyl group orbit is given by*

$$\mathcal{W}.\lambda = \{\varepsilon_N \colon |M \setminus N| = |N \setminus M| < \infty\}.$$

(B_J) *A weight λ of B_J is small if and only if there exists a subset $M \subseteq J$ with $\lambda = \frac{1}{2}(\varepsilon_M - \varepsilon_{M^c})$. Its Weyl group orbit is given by*

$$\mathcal{W}.\lambda = \Big\{\frac{1}{2}(\varepsilon_N - \varepsilon_{N^c}) \colon |M \setminus N|, |N \setminus M| < \infty\Big\}.$$

(C_J) *The small weights of C_J are $\pm\varepsilon_j$, $j \in J$. They form a single $\mathcal{W}$-orbit.*

(D_J) *The small weights for D_J are the weights $\pm\varepsilon_j$, $j \in J$, which form a single $\mathcal{W}$-orbit, and the weights $\lambda = \frac{1}{2}(\varepsilon_M - \varepsilon_{M^c})$ whose $\mathcal{W}$-orbits are given by*

$$\mathcal{W}.\lambda = \Big\{\frac{1}{2}(\varepsilon_N - \varepsilon_{N^c}) \colon |M \setminus N|, |N \setminus M| < \infty, |M \setminus N| - |N \setminus M| \in 2\mathbb{Z}\Big\}.$$

(E_n) *For E_6 there exist two $\mathcal{W}$-orbits of small weights and for E_7 there is one. For the other exceptional root systems there is none.*

Proof. (A_J) Suppose that λ is small and observe that this implies in particular that there exists an $m \in \mathbb{K}$ such that $\lambda(J) \subseteq m + \mathbb{Z}$, and $\lambda(J) - m$ is a bounded subset of $\mathbb{Z}$. Therefore we may assume that $\lambda(J) \subseteq m - \mathbb{N}_0$. Replacing m by $m-n$ for a suitable $n \in \mathbb{N}_0$, we even may assume that $m \in \lambda(J)$. Now $\lambda(\check{\alpha}) \in \{0, 1, -1\}$ for $\alpha = \varepsilon_j - \varepsilon_k$, $j \neq k$, implies that $\lambda(J) \subseteq \{m, m-1\}$, i.e., there exists a subset $M \subseteq J$ with $\lambda = m\varepsilon_M + (m-1)\varepsilon_{M^c}$. Subtracting the constant function $m-1$ on J does not change the represented weight, so that we obtain $\lambda = \varepsilon_M$. Conversely, it is clear that ε_M is a small weight. The description of the $\mathcal{W}$-orbit follows directly from the fact that $\mathcal{W}$ acts as the group $S_{(J)}$ of finite permutations of the set J.

(B_J) From $\check{\varepsilon}_j = 2e_j$ we derive that $\lambda(J) \subseteq \{0, \pm\frac{1}{2}\}$, but since λ is assumed to be non-zero, the integrality implies that $\lambda(J) \subseteq \{\pm\frac{1}{2}\}$, i.e., $\lambda = \frac{1}{2}(\varepsilon_M - \varepsilon_{M^c})$ for $M := \lambda^{-1}(\frac{1}{2})$. That, conversely, all these weights are small is clear. The description of the $\mathcal{W}$-orbit follows from the fact that $\mathcal{W}$ acts as the group $\{\pm 1\}^{(J)} \rtimes S_{(J)}$ of finite signed permutations on $\mathbb{K}^J$.

(C_J) From $(2\varepsilon_j)\check{} = e_j$ we derive that $\lambda(J) \subseteq \{0, \pm 1\}$, and the smallness implies that $|\lambda^{-1}(\{\pm 1\})| \leq 1$ and $\{\pm 1\} \not\subseteq \lambda(J)$. Hence $\lambda = \pm\varepsilon_j$ for some $j \in J$. The description of the $\mathcal{W}$-orbit follows from the fact that $\mathcal{W}$ acts as the group $\{\pm 1\}^{(J)} \rtimes S_{(J)}$ of finite signed permutations on $\mathbb{K}^J$.

(D_J) It is clear that the weights $\pm\varepsilon_j$, $j \in J$, are small. These are the only small weights for which $\lambda_j \neq 0$ holds for only one $j \in J$. Next we assume that $\lambda_j, \lambda_k \neq 0$ holds for some $j \neq k \in J$. Then we get $\lambda(J) \subseteq \{\pm\frac{1}{2}\}$ and therefore $\lambda = \frac{1}{2}(\varepsilon_M - \varepsilon_{M^c})$ for some subset $M \subseteq J$. The description of the $\mathcal{W}$-orbit follows from the fact that $\mathcal{W}$ acts as the subgroup of those elements $(a, \sigma) \in \{\pm 1\}^{(J)} \rtimes S_{(J)}$ for which the set $\{j \in J \colon a_j = -1\}$ has even cardinality.

(E_n) See [Bou90, Ch. 8, no. 7.3]. ■

Remark III.4. For the applications that we have in mind, we will also need information on whether for a small highest weight module $L(\lambda)$ the corresponding operators $\rho_\lambda(x)$, $x \in \mathfrak{g}$, are of finite rank. Since the Lie algebra $\mathfrak{g}$ corresponding to an irreducible root systems is simple and the operators of finite rank in $\mathfrak{gl}(L(\lambda))$ form a Lie algebra ideal, the following are equivalent:

(1) All operators $\rho_\lambda(x)$, $x \in \mathfrak{g}$, are of finite rank.

(2) There exists a root $\alpha \in \Delta$ such that $\rho_\lambda(\check{\alpha})$ is of finite rank, which means that the set $\{\mu \in \mathcal{W}.\lambda \colon \mu(\check{\alpha}) \neq 0\}$ is finite.

Condition (2) can easily be checked for the weights λ showing up in Theorem III.3. Of course, we may assume that J is infinite, because otherwise all modules $L(\lambda)$ are finite-dimensional.

(A_J) If λ is constant, then $L(\lambda) \cong L(0) \cong \mathbb{K}$ is trivial. If λ is not constant and M and M^c contain at least two elements, then one easily finds an $\check{\alpha}$ such that $\rho_\lambda(\check{\alpha})$ has infinite rank. If $|M| = 1$, then we obtain $\lambda = \varepsilon_j$ for $M = \{j\}$. Therefore $L(\lambda) \cong \mathbb{K}^{(J)}$ is the identical representation for which all the operators are of finite rank. For $|M^c| = 1$ we obtain the dual weight module which also has this property (cf. Definition I.4(c)).

For general M the functional $\lambda = \sum_{j \in M} \varepsilon_j$ is the highest weight of the representation of $\mathfrak{g} = \mathfrak{sl}(J, \mathbb{K})$ on the space $\Lambda^{(M)}(\mathbb{K}^{(J)})$ which we describe in Section V below. If M is finite, then this space is the $|M|$-th exterior power $\Lambda^{|M|}(\mathbb{K}^{(J)})$ with the basis elements $e_{j_1} \wedge \ldots \wedge e_{j_k}$, where $k = |M|$.

(B_J) If λ is small, then $\lambda = \frac{1}{2}(\varepsilon_M - \varepsilon_{M^c})$ and each $\mathcal{W}$-conjugate of λ has this form. For $j \in J$ the relation $\mu(\check{\varepsilon}_j) \neq 0$ for each $\mu \in \mathcal{W}.\lambda$ therefore shows that $\rho_\lambda(\check{\varepsilon}_j)$ has infinite rank.

Up to automorphisms of the corresponding Lie algebra $\mathfrak{g}$, respectively the root system (cf. [St99b]), we may assume that $\lambda = \frac{1}{2}\sum_{j \in J} \varepsilon_j$. This is the highest weight of the spin representation of $\mathfrak{g}$ on the space $\Lambda(\mathbb{K}^{(J)}) := \bigoplus_{k=0}^{\infty} \Lambda^k(\mathbb{K}^{(J)})$ (see Sect. V in [Ne98]).

(C_J) Let $J^\pm := J \dot\cup -J$, where $-J$ is a copy of the set J whose elements we denote by $-j$, $j \in J$. On $\mathbb{K}^{(J^\pm)}$ we consider the skew-symmetric bilinear form given by

$$\Omega(v,w) = \sum_{j\in J}(v_j w_{-j} - v_{-j} w_j)$$

and put

$$\mathfrak{sp}(J,\mathbb{K}) := \{X \in \mathfrak{gl}(J^\pm,\mathbb{K})\colon (\forall v,w \in \mathbb{K}^{(J^\pm)})\, \Omega(X.v,w) + \Omega(v,X.w) = 0\}.$$

This Lie algebra is simple split with splitting Cartan subalgebra

$$\mathfrak{h}_1 = \mathrm{span}_{\mathbb{K}}\{E_{jj} - E_{-j,-j} : j \in J\},$$

where E_{jk}, $j,k \in J$, denote the canonical matrix units. The corresponding root system is of type C_J, where $\varepsilon_j(E_{kk} - E_{-k-k}) = \delta_{jk}$ for $j,k \in J$.

In this case the only small module is the identical representation of $\mathfrak{sp}(J,\mathbb{K})$ on $\mathbb{K}^{(J^\pm)}$ which is a representation by finite rank operators.

(D_J) On $\mathbb{K}^{(J^\pm)}$ we consider the symmetric bilinear form given by

$$\beta(v,w) = \sum_{j\in J}(v_j w_{-j} + v_{-j} w_j)$$

and put

$$\mathfrak{o}(J^\pm,\mathbb{K}) := \{X \in \mathfrak{gl}(J^\pm,\mathbb{C})\colon (\forall v,w \in \mathbb{K}^{(J^\pm)})\, \beta(X.v,w) + \beta(v,X.w) = 0\}.$$

This Lie algebra is simple split with splitting Cartan subalgebra $\mathfrak{h}_1 = \mathrm{span}_{\mathbb{K}}\{E_{jj} - E_{-j,-j} : j \in J\}$, where E_{jk}, $j,k \in J$, denote the canonical matrix units. The corresponding root system is of type D_J, where $\varepsilon_j(E_{kk} - E_{-k-k}) = \delta_{jk}$ for $j,k \in J$.

The weights $\lambda = \pm\varepsilon_j$ correspond to the identical representation of $\mathfrak{o}(J^\pm,\mathbb{K})$ on $\mathbb{K}^{(J^\pm)}$ which is a representation by finite rank operators. Next we assume that $\lambda = \frac{1}{2}(\varepsilon_M - \varepsilon_{M^c})$. Then either M or M^c is infinite, and we assume that M is. Pick $a \neq b \in M$. Then we can exchange every element in $M \setminus \{a,b\}$ with a fixed element $c \in M^c$ and thus obtain infinitely many weights in $\mathcal{W}.\lambda$ which are non-zero on the coroot of $\varepsilon_a + \varepsilon_b$. Therefore the operators $\rho_\lambda(x)$, $x \in \mathfrak{g}$, are of infinite rank.

The standard spin representation on $\Lambda(\mathbb{K}^{(J)})$ decomposes for each $m \in J$ as the direct sum of representations of highest weight

$$\frac{1}{2}\sum_{j\in J}\varepsilon_j \quad \text{and} \quad \Big(\frac{1}{2}\sum_{j\in J}\varepsilon_j\Big) - \varepsilon_m.$$

The preceding discussion shows that the only simple small modules V for which the operators $\rho_V(x)$, $x \in \mathfrak{g}$, are of finite rank are the identical representation for A_J, C_J and D_J and the dual of the identical representation for A_J. For C_J and D_J the identical representation is self-dual as a weight representation. ∎

Problems III. Classify the (simple) finite weight modules of semisimple locally finite split Lie algebras. For the case of countably dimensional Lie algebras $\mathfrak{g}$, this has been done in [DiPe99]. Important questions in this context are: Are finite weight modules always semisimple? Are simple weight modules restricted to semisimple subalgebras always semisimple modules? ∎

IV. 2-graded weight modules

For the description of those locally finite involutive Lie algebras which admit a faithful unitary highest weight representation, we have to study the following type of modules (cf. [Ne00b]). We consider a 3-graded locally finite split almost reductive Lie algebra

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1,$$

where $\mathfrak{g}_0$ contains a splitting Cartan subalgebra $\mathfrak{h}$, and we are interested in graded weight modules V of the graded Lie algebra $\mathfrak{g}$ which are 2-graded in the sense that $V = V^- \oplus V^+$, where

$$\mathfrak{g}_0.V^\pm \subseteq V^\pm, \quad \mathfrak{g}_{\pm 1}.V^\mp \subseteq V^\pm \quad \text{and} \quad \mathfrak{g}_{\pm 1}.V^\pm = \{0\}.$$

Since $\mathfrak{g}_0$ is assumed to contain a splitting Cartan subalgebra $\mathfrak{h}$, the root decomposition of $\mathfrak{g}$ leads to a disjoint decomposition $\Delta = \Delta_{-1} \dot\cup \Delta_0 \dot\cup \Delta_1$, where $\Delta_{-1} = -\Delta_1$.

Example IV.1. A typical example of such a gradation arises as follows. Let J be a set and $\mathfrak{g} := \mathfrak{gl}(J, \mathbb{K})$ be the Lie algebra of finite $J \times J$-matrices. We further fix a subset $M \subseteq J$. We thus obtain a direct sum decomposition of the space $V := \mathbb{K}^{(J)} = V^+ \oplus V^- = \mathbb{K}^{(M)} \oplus \mathbb{K}^{(J \setminus M)}$. Writing the elements of $\mathfrak{g}$ accordingly as 2×2-block matrices $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we get the gradation $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}$ with

$$\mathfrak{g}_1 = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \right\}, \quad \mathfrak{g}_0 = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \right\} \quad \text{and} \quad \mathfrak{g}_{-1} = \left\{ \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} \right\}.$$ ∎

Lemma IV.2. *If* $\mathfrak{g}$ *is simple, then* $\mathfrak{g}_1$ *is a simple* $\mathfrak{g}_0$*-module and* $\mathfrak{g}_0 = [\mathfrak{g}_1, \mathfrak{g}_{-1}]$*.*

Proof. Let $\{0\} \neq W \subseteq \mathfrak{g}_1$ be a $\mathfrak{g}_0$-invariant subspace. Then W is invariant under the subalgebra $\mathfrak{g}_0 + \mathfrak{g}_1$, so that the ideal generated by W is given by

$$W + [\mathfrak{g}_{-1}, W] + \big[\mathfrak{g}_{-1}, [\mathfrak{g}_{-1}, W]\big] \subseteq \mathfrak{g}_1 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}.$$

We conclude that $W = \mathfrak{g}_1$ and $\mathfrak{g}_0 = [\mathfrak{g}_{-1}, \mathfrak{g}_1]$. It follows that $\mathfrak{g}_1$ is a simple $\mathfrak{g}_0$-module. ■

Lemma IV.3. *There exists a unique linear functional* $f\colon \mathrm{span}_{\mathbb{K}}[\Delta] \to \mathbb{K}$ *with* $\Delta_{\pm 1} = f^{-1}(\pm 1)$ *and* $\Delta_0 = f^{-1}(0)$*.*

Proof. We may w.l.o.g. assume that $\mathfrak{g}$ is simple, because we can put the functions f corresponding to the simple ideals of $[\mathfrak{g}, \mathfrak{g}]$ together. Moreover, we assume that $\mathfrak{g}_1 \neq \{0\}$, otherwise we may take $f = 0$. Now $\Sigma := \Delta_0 \cup \Delta_1$ is a parabolic system. Let $\lambda \in \Delta_1$. We claim that $\lambda \not\in \mathrm{span}\, \Delta_0$. Suppose that this is false and that

$$\lambda = \sum_{j=1}^{k} m_j \alpha_j \tag{4.1}$$

with $\alpha_j \in \Delta_0$ and $m_1, \ldots, m_k \in \mathbb{K}$. Then there exists a finite-dimensional semisimple $\mathfrak{h}$-invariant subalgebra $\mathfrak{a} \subseteq \mathfrak{g}$ with $\mathfrak{a} \cap \mathfrak{g}_1 \neq \{0\}$ and $\lambda, \alpha_1, \ldots, \alpha_k \in \Delta_{\mathfrak{a}}$ (cf. [St99a, Prop. V.5]). Since $\Sigma_{\mathfrak{a}} := \Sigma \cap \Delta_{\mathfrak{a}}$ is a parabolic system in the finite root system $\Delta_{\mathfrak{a}}$, the fact that $\lambda \in \Sigma_{\mathfrak{a}} \setminus -\Sigma_{\mathfrak{a}}$ implies

$$\lambda \not\in \mathrm{span}(\Sigma_{\mathfrak{a}} \cap -\Sigma_{\mathfrak{a}}).$$

This contradicts (4.1), and we conclude that $\mathbb{K}\lambda \cap \mathrm{span}\, \Delta_0 = \{0\}$. Hence there exists a linear functional $f\colon \mathrm{span}_{\mathbb{K}} \Delta \to \mathbb{K}$ with $f(\lambda) = 1$ and $\Delta_0 \subseteq \ker f$.

Since $\mathfrak{g}_1$ is a simple $\mathfrak{g}_0$-module (Lemma IV.2), for each $\lambda \in \Delta_1$ we have $\Delta_1 \subseteq \lambda + \mathbb{Z}[\Delta_0]$, showing that $f(\Delta_1) = \{1\}$, and we likewise see that $f(\Delta_{-1}) = \{-1\}$. The uniqueness of f follows trivially from the requirements. ■

Definition IV.4. Let Δ be a root system of semisimple type. A partition $\Delta = \Delta_{-1} \dot\cup \Delta_0 \dot\cup \Delta_1$ is called a 3*-grading* if $\mathfrak{g}_{\pm 1} := \sum_{\alpha \in \Delta_{\pm 1}} \mathfrak{g}^\alpha$ and $\mathfrak{g}_0 := \mathfrak{h} + \sum_{\alpha \in \Delta_0} \mathfrak{g}^\alpha$ defines a 3-grading of $\mathfrak{g}$ (cf. [NeSt99] and [Ne90]). It is clear that each 3-grading is completely determined by the set Δ_1. ■

Proposition IV.5. *The sets Δ_1 corresponding to 3-gradings of the root systems $\Delta = A_J, B_J, C_J, D_J$ are given by*

(A_J) $A_J(M)_1 = \{\varepsilon_j - \varepsilon_k : j \in M, k \notin M\}$, *where $M \subseteq J$ is a subset.*

(B_J) $B_J(m)_1 = \{\varepsilon_m\} \cup \{\varepsilon_m \pm \varepsilon_j : j \neq m\}$, *where $m \in J$.*

(C_J) $C_J(M)_1 = \{\varepsilon_j - \varepsilon_k : j \in M, k \notin M\} \cup \{\varepsilon_j + \varepsilon_k : j, k \in M\} \cup \{-\varepsilon_j - \varepsilon_k : j, k \notin M\}$, *where $M \subseteq J$ is a subset.*

(D_J) $D_J(m)_1 = \{\varepsilon_m \pm \varepsilon_j : j \neq m\} = B_J(m)_1 \cap D_J$, *where $m \in J$, or by $D_J(M)_1 = C_J(M)_1 \cap D_J$.*

The corresponding functions f are given by $f(\alpha) = \alpha(e_m)$ for $B_J(m)_1$ and $D_J(m)_1$, and by

$$f(\alpha) = \frac{1}{2}\Big(\sum_{j \in M} \alpha(e_j) - \sum_{j \notin M} \alpha(e_j) \Big)$$

for $A_J(M)_1$, $C_J(M)_1$ and $D_J(M)_1$.

Proof. [NeSt99, Prop. VII.2]. ■

Having described the possible 3-gradings of irreducible root systems, we now turn to the description of the corresponding 2-graded modules.

Lemma IV.6. *The subspace $\mathfrak{g}_a := \mathfrak{g}_1 + \mathfrak{g}_{-1} + [\mathfrak{g}_1, \mathfrak{g}_{-1}] \subseteq [\mathfrak{g}, \mathfrak{g}]$ is an ideal of $\mathfrak{g}$.*

Proof. This is a trivial calculation. ■

If $\mathfrak{g}$ is semisimple, then we can write it as $\mathfrak{g} = \mathfrak{g}_a \oplus \mathfrak{g}_b$, where $\mathfrak{g}_b$ is an ideal of $\mathfrak{g}$ contained in $\mathfrak{g}_0$. In the following $\mathfrak{g}$ denotes an almost reductive locally finite split Lie algebra with 3-grading.

Proposition IV.7. *Let V be a 2-graded weight module of $\mathfrak{g}$. Then the following assertions hold:*

(i) *If V is simple and $V^\pm \neq \{0\}$, then $V^\pm$ are simple $\mathfrak{g}_0$-modules with $\mathfrak{g}_{\pm 1}.V^\mp = V^\pm$ and $V^\pm = V^{\mathfrak{g}_{\pm 1}} := \{v \in V : \rho_V(\mathfrak{g}_{\pm 1}).v = \{0\}\}$.*

(ii) *V is a small weight module for the ideal $\mathfrak{g}_a$.*

(iii) *For $\alpha, \beta \in \Delta_1$ and $\lambda \in \mathcal{P}_{V^+}$ with $\lambda(\check{\alpha}) = \lambda(\check{\beta}) = 1$, we have $\alpha(\check{\beta}) > 0$.*

(iv) *V is a semisimple $\mathfrak{g}_a$-module.*

Proof. (i) Let $W \subseteq V^+$ be a non-zero $\mathfrak{g}_0$-submodule. Then the Poincaré–Birkhoff–Witt Theorem implies that

$$U(\mathfrak{g}).W = W + \rho_V(\mathfrak{g}_{-1}).W \subseteq V^+ \oplus V^-,$$

so that the simplicity of V leads to $W = V^+$ and $V^- = \rho_V(\mathfrak{g}_{-1}).V^+$. This proves that V^+ is a simple $\mathfrak{g}_0$-module. Likewise we see that V^- is a simple module and that $V^+ = \mathfrak{g}_1.V^-$.

In view of the definition of a 2-graded module, we have $V^+ \subseteq V^{\mathfrak{g}_1}$, so that $V^{\mathfrak{g}_1} = V^+ + (V^- \cap V^{\mathfrak{g}_1})$. The subspace $V^- \cap (V^{\mathfrak{g}_1})$ is annihilated by $\mathfrak{g}_1$ and

$\mathfrak{g}_{-1}$ and invariant under $\mathfrak{g}_0$, hence a submodule. Now the simplicity of V yields $V^{\mathfrak{g}_1} \cap V^- = \{0\}$, so that $V^{\mathfrak{g}_1} = V^+$. Likewise we get $V^- = V^{\mathfrak{g}_{-1}}$.
(ii) If $\alpha \in \Delta_1$, then the $\mathfrak{g}(\alpha)$-module generated by a weight vector $v_\lambda \in V^\lambda \cap V^+$ is a simple module of $\mathfrak{g}(\alpha) \cong \mathfrak{sl}(2,\mathbb{K})$ of dimension $\lambda(\check{\alpha})+1$. Since V is 2-graded, we have $x_{-\alpha}^2.v_\lambda = 0$ for each $x_{-\alpha} \in \mathfrak{g}^{-\alpha}$. Thus $\lambda(\check{\alpha}) \in \{0,1\}$ holds for each $\alpha \in \Delta_1$.

If $\beta \in \Delta_{\mathfrak{g}_a,0}$, then $\mathfrak{g}_a \cap \mathfrak{g}_0 = [\mathfrak{g}_1, \mathfrak{g}_{-1}]$ (Lemma IV.6) implies that there exists an $\alpha \in \Delta_1$ with $\beta(\check{\alpha}) \neq 0$. We now have

$$\underbrace{(r_\beta.\lambda)(\check{\alpha})}_{\in\{0,1\}} = \underbrace{\lambda(\check{\alpha})}_{\in\{0,1\}} - \lambda(\check{\beta})\beta(\check{\alpha}).$$

Hence $\lambda(\check{\beta})\beta(\check{\alpha}) \in \{-1,0,1\}$ and since $\beta(\check{\alpha})$ is a non-zero integer, we see that $\lambda(\check{\beta}) \in \{-1,0,1\}$ holds for all $\beta \in \Delta_{\mathfrak{g}_a,0}$. We conclude that $\lambda(\check{\alpha}) \in \{0,\pm 1\}$ for all $\alpha \in \Delta_{\mathfrak{g}_a}$, i.e. that V is a small $\mathfrak{g}_a$-module.
(iii) Since $\lambda(\check{\alpha}) = 1$, the functional $r_\alpha.\lambda = \lambda - \alpha$ is a weight of V^-, and therefore $\mathfrak{g}^{-\beta}.V^- = \{0\}$ yields $0 \geq (r_\alpha.\lambda)(\check{\beta}) = \lambda(\check{\beta}) - \alpha(\check{\beta}) = 1 - \alpha(\check{\beta})$, so that $\alpha(\check{\beta}) > 0$.
(iv) follows directly from Corollary II.2. ■

Lemma IV.8. *If $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$ is a direct sum of graded Lie algebras and V is a simple 2-graded weight module with $\rho_V(\mathfrak{a}_1) \neq \{0\}$, then $\rho_V(\mathfrak{b}_1) = \{0\}$.*

Proof. The assumption $\rho_V(\mathfrak{a}_1) \neq \{0\}$ implies that $\mathfrak{a}_1.V^- \subseteq V^+$ is non-zero. Moreover, it is a $\mathfrak{g}_0$-submodule, and therefore coincides with V^+ (Proposition IV.7(i)). We conclude that $\mathfrak{b}_{-1}.V^+ = \mathfrak{a}_1\mathfrak{b}_{-1}.V^- = \{0\}$, so that $\mathfrak{b}_{-1}$ annihilates V, and likewise $\mathfrak{b}_1$ annihilates V. ■

Proposition IV.9. *Suppose that $\mathfrak{g}$ is almost reductive, V a simple 2-graded weight module, and $\mathfrak{a} \trianglelefteq \mathfrak{g}_a$ a simple ideal with $\rho_V(\mathfrak{a}_1) \neq \{0\}$. Then all other simple ideals $\mathfrak{b} \trianglelefteq \mathfrak{g}_a$ act trivially on V.*

Proof. This is an immediate consequence of Lemma IV.8 applied to the semisimple ideal $[\mathfrak{g},\mathfrak{g}]$ of $\mathfrak{g}$ for which V is a simple module (see Remark II.8). ■

In view of Proposition IV.7(iv), to describe the 2-graded weight modules of $\mathfrak{g}$, it suffices to describe those which are isotypic for $\mathfrak{g}_a$. These can be written as $V = V_1 \otimes V_2$, where V_2 is a simple weight module of $\mathfrak{g}_b$ and V_1 is a simple 2-graded weight module of $\mathfrak{g}_a$. Suppose that V_1 is non-trivial as a $\mathfrak{g}_1$-module. In view of Proposition IV.9, there exists a unique simple ideal $\mathfrak{g}_V \trianglelefteq \mathfrak{g}_a$ acting non-trivially on V_1, and all others act trivially. If, conversely, V_1 is a simple 2-graded weight module of a simple ideal $\mathfrak{g}_V \trianglelefteq \mathfrak{g}_a$, and V_2 a simple $\mathfrak{g}_b$-module, then $\mathrm{End}_{\mathfrak{g}_V}(V_1) = \mathbb{K}\mathbf{1}$ follows from the fact that V_1 is an integrable highest weight module, and Lemma II.6(ii) implies that $V := V_1 \otimes V_2$ is a simple 2-graded weight module of $\mathfrak{g}$, where the gradation of V is given by $V^\pm := V_1^\pm \otimes V_2$.

Therefore we are essentially left with the problem to determine those small weight modules of a simple 3-graded Lie algebra $\mathfrak{g}$ which are 2-graded. The following lemma provides a handy criterion.

Lemma IV.10. *Let $f\colon \operatorname{span}\Delta \to \mathbb{K}$ be a linear functional defining the 3-grading of Δ in the sense of* Lemma IV.3*, and $V = L(\lambda)$ an integrable highest weight module. Then the following are equivalent:*

(1) *V is 2-graded.*

(2) *$f(\mathcal{P}_{L(\lambda)} - \lambda)$ is a two-element set.*

Proof. (1) $\Rightarrow$ (2): Suppose that $V = V^+ \oplus V^-$ is a 2-graded $\mathfrak{g}$-module. Then $V^\pm$ are simple $\mathfrak{g}_0$-modules, so that the function f is constant on the subsets $\mathcal{P}_{V^\pm} \subseteq \mathcal{P}_V$.

(2) $\Rightarrow$ (1): Suppose that $f(\lambda - \mathcal{P}_\lambda) \subseteq \{m, M\}$, where $m < M$. Then

$$V^+ := \sum_{\mu \in \mathcal{P}_{L(\lambda)}, f(\lambda-\mu)=m} V^\mu \quad \text{and} \quad V^- := \sum_{\mu \in \mathcal{P}_{L(\lambda)}, f(\lambda-\mu)=M} V^\mu$$

yields a decomposition $V = V^+ \oplus V^-$ which is a 2-gradation of the $\mathfrak{g}$-module V. Note that $M = m + 1$ holds automatically. ■

Now it only remains to check the condition of Lemma IV.10 for the modules occurring in Theorem III.3, where the functional f is as in Proposition IV.5.

Theorem IV.11. (Classification of 2-graded simple modules) *For a simple 3-graded split Lie algebra $\mathfrak{g}$ the non-trivial simple 2-graded modules are the following:*

(A_J) *For $\Delta_1 = A_J(\{m\})_1$ all small modules $L(\varepsilon_N)$, $N \subseteq J$, are 2-graded.*

(A_J) *For $\Delta_1 = A_J(M)_1$ with $|M| > 1$ and $|M^c| > 1$ only the module $L(\varepsilon_j) \cong \mathbb{K}^{(J)}$ (not depending on $j \in J$), and the dual weight module $L(-\varepsilon_j)$ are 2-graded.*

(B_J) *For $\Delta_1 = B_J(m)_1$ only the spin representation on $L(\frac{1}{2}\varepsilon_J) \cong \Lambda(\mathbb{C}^{(J)})$ and the quasi-equivalent representations with highest weight $\lambda = \frac{1}{2}(\varepsilon_M - \varepsilon_{M^c})$, $M \subseteq J$, are 2-graded.*

(C_J) *For $\Delta_1 = C_J(M)_1$ only the identical representation on $L(\pm\varepsilon_j) \cong \mathbb{K}^{(J^\pm)}$ is 2-graded.*

(D_J) *For $\Delta_1 = D_J(m)_1$ only the two simple constituents of the spin representation on $\Lambda(\mathbb{K}^{(J)})$ and the corresponding quasi-equivalent representations are 2-graded.*

(D_J) *For $\Delta_1 = D_J(M)_1$ only the identical representation on $L(\pm\varepsilon_j) \cong \mathbb{K}^{(J^\pm)}$ and for $|J| = 4$ the module $L(\lambda) \cong \Lambda^{\text{odd}}(\mathbb{K}^4)$ with $\lambda = \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_4)$ is 2-graded.*

(E_n) *For the exceptional algebras there are no 2-graded modules.*

Proof. In view of [Ne99, Th. A.V.6], there exists no 2-graded simple modules for the exceptional algebras, so that we may assume that Δ is of classical type.

(A_J) For $\Delta_1 = A_J(\{m\})_1$ we have $f(\alpha) = \alpha(e_m)$ (Proposition IV.5). According to Theorem III.3, we may assume that $\lambda = \varepsilon_N$ for some subset $N \subseteq J$. Then the weight set of $L(\lambda)$ is given by

$$\mathcal{W}.\lambda = \{\varepsilon_K \colon |K \setminus N| = |N \setminus K| < \infty\}.$$

In view of $f(\mathcal{W}.\lambda) = \{0,1\}$, the module $L(\lambda)$ is 2-graded.

For $\Delta_1 = A_J(M)_1$ with $|M|, |M^c| > 1$ we have $f(\alpha) = \sum_{j\in M} \alpha(e_j)$ (Proposition IV.5). Again, we may assume that $\lambda = \varepsilon_N$ for some subset $N \subseteq J$. If $|N| = 1$, then

$$\mathcal{W}.\lambda = \{\varepsilon_j \colon j \in J\},$$

so that $f(\mathcal{W}.\lambda) = \{1,0\}$ implies that $L(\lambda)$ is 2-graded, and for $|N^c| = 1$ one argues similarly. Assume that $|N|, |N^c| > 1$. Then

$$\begin{aligned}\mathcal{W}.\lambda - \lambda &= \{\varepsilon_K - \varepsilon_N \colon |K \setminus N| = |N \setminus K| < \infty\}\\ &= \{\varepsilon_{K\setminus N} - \varepsilon_{N\setminus K} \colon |K \setminus N| = |N \setminus K| < \infty\}\\ &= \{\varepsilon_{M_1} - \varepsilon_{M_2} \colon |M_1| = |M_2| < \infty, M_1 \subseteq N^c, M_2 \subseteq N\}.\end{aligned}$$

Since we may assume that λ is Δ_1-dominant, we have $M \subseteq N$ or $N \subseteq M$ because otherwise there exists an $m \in M \setminus N$ and an $n \in N \setminus M$ which leads to $\lambda(e_m - e_n) = -1$.

Suppose first that $M \subseteq N$. Then $f(\varepsilon_{M_1} - \varepsilon_{M_2}) = -|M_2 \cap M|$, so that $-2 \in f(\mathcal{W}.\lambda - \lambda)$ implies that this set contains the three numbers $\{0,-1,-2\}$. If $N \subseteq M$, then

$$f(\varepsilon_{M_1} - \varepsilon_{M_2}) = |M_1| - |M_2 \cap M|,$$

so that $2 \in f(\mathcal{W}.\lambda - \lambda)$ implies that this set contains the three numbers $\{0,1,2\}$. In both cases we see that $L(\lambda)$ is not 2-graded.

(B_J) For $\Delta_1 = B_J(m)_1$ we have $f(\alpha) = \alpha(e_m)$ (Proposition IV.5). According to Theorem III.3, we may assume that $\lambda = \frac{1}{2}(\varepsilon_M - \varepsilon_{M^c})$ for some subset $M \subseteq J$. Then the weight set of $L(\lambda)$ is given by

$$\mathcal{W}.\lambda = \Big\{\frac{1}{2}(\varepsilon_N - \varepsilon_{N^c}) \colon |M \setminus N|, |N \setminus M| < \infty\Big\},$$

and we see that $f(\mathcal{W}.\lambda) = \{\pm\frac{1}{2}\}$, showing that $L(\lambda)$ is 2-graded. That all these modules are quasi-equivalent to the spin representation, which corresponds to $M = J$, follows from the fact that there exists an automorphism φ of $\mathfrak{g}$ with $\varphi(\mathfrak{h}) = \mathfrak{h}$ such that $(\varphi|_{\mathfrak{h}})^*\lambda = \frac{1}{2}\varepsilon_J$ (cf. [St99b]).

(C_J) For $\Delta_1 = C_J(M)_1$ we have $f(\alpha) = \frac{1}{2}\big(\sum_{j\in M} \alpha(e_j) - \sum_{j\in M^c} \alpha(e_j)\big)$. The only small module $L(\lambda)$ is the identical representation on $\mathbb{K}^{(J^\pm)}$ with the weight set $\mathcal{P}_{L(\lambda)} = \{\pm\varepsilon_j \colon j \in J\}$. Now $f(\mathcal{P}_{L(\lambda)}) = \{\pm\frac{1}{2}\}$ shows that $L(\lambda)$ is 2-graded.

(D_J) For $\Delta_1 = D_J(m)_1$ we have $f(\alpha) = \alpha(e_m)$ (Proposition IV.6). For $\lambda = \varepsilon_j$ we have $\mathcal{W}.\lambda = \{\pm\varepsilon_k : k \in J\}$, so that $f(\mathcal{W}.\lambda) = \{0, 1, -1\}$, and therefore $L(\lambda)$ is not 2-graded. Now we assume that $\lambda = \frac{1}{2}(\varepsilon_M - \varepsilon_{M^c})$ for some subset $M \subseteq J$. Then we weight set of $L(\lambda)$ is given by

$$\mathcal{W}.\lambda = \Big\{\frac{1}{2}(\varepsilon_N - \varepsilon_{N^c}) : |M \setminus N|, |N \setminus M| < \infty, |M \setminus N| - |N \setminus M| \in 2\mathbb{Z}\Big\},$$

and we see that $f(\mathcal{W}.\lambda) = \{\pm\frac{1}{2}\}$, showing that $L(\lambda)$ is 2-graded.

For $\Delta_1 = D_J(M)_1$ we have $f(\alpha) = \frac{1}{2}\big(\sum_{j \in M} \alpha(e_j) - \sum_{j \in M^c} \alpha(e_j)\big)$ which immediately shows that for $\lambda = \varepsilon_j$ the module $L(\lambda)$ is 2-graded. We note that, under the automorphism of the root system given by $\varepsilon_j \mapsto -\varepsilon_j$ for $j \in M$ and fixing the other ε_j's, this 3-grading is conjugate to the one defined by $M = J$, so that $\Delta_1 = D_J(J)_1$ and $f(\alpha) = \frac{1}{2}\sum_{j \in J} \alpha(e_j)$. Therefore we may assume that $M = J$.

We consider $\lambda = \frac{1}{2}(\varepsilon_N - \varepsilon_{N^c})$ for some subset $N \subseteq J$. Then we have

$$\begin{aligned} &\mathcal{W}.\lambda - \lambda \\ &= \Big\{\frac{1}{2}(\varepsilon_K - \varepsilon_{K^c} - \varepsilon_N + \varepsilon_{N^c}) : |K \setminus N|, |N \setminus K| < \infty, |K \setminus N| - |N \setminus K| \in 2\mathbb{Z}\Big\} \\ &= \Big\{\varepsilon_{K \setminus N} - \varepsilon_{N \setminus K} : |K \setminus N|, |N \setminus K| < \infty, |K \setminus N| - |N \setminus K| \in 2\mathbb{Z}\Big\} \end{aligned}$$

and

$$f(\varepsilon_{K \setminus N} - \varepsilon_{N \setminus K}) = \frac{1}{2}(|K \setminus N| - |N \setminus K|).$$

If $|J| \geq 5$, then we may w.l.o.g. assume that N contains at least four elements (otherwise we may add roots in Δ_1 to λ). Then there exists a $K \subseteq N$ with $N \setminus K = 4$, so that we obtain the value -2 for f. So let us assume that $|J| = 4$ and that $|N| < 4$. By the same argument as above, we may assume that $|N| = 3$. Then $|K \setminus N| \leq 1$ and $|N \setminus K| \leq 3$ yield $f(\varepsilon_{K \setminus N} - \varepsilon_{N \setminus K}) \in [-\frac{3}{2}, \frac{1}{2}]$ and therefore $f(\varepsilon_{K \setminus N} - \varepsilon_{N \setminus K}) \in \{-1, 0\}$. This proves that $L(\lambda)$ is 2-graded. It is the odd part $\Lambda^{\mathrm{odd}}(\mathbb{K}^4) = \Lambda^1(\mathbb{K}^4) \oplus \Lambda^3(\mathbb{K}^4)$ of the spin representation on $\Lambda(\mathbb{K}^4)$. ■

V. Infinite tensor products

Let $(V_i)_{i \in I}$ be a family of vector spaces and F denote the free vector space on the cartesian product $\prod_{i \in I} V_i$. A map $m: \prod_{i \in I} V_i \to W$ into a vector space W is called *multilinear* if it is linear in each argument provided that all other arguments are fixed. For $i \in I$ let $F_i \subseteq F$ be the subspace generated by elements of the type

$$(x', x_i + y_i) - (x', x_i) - (x', y_i), \quad \lambda(x', x_i) - (x', \lambda x_i), \quad x_i, y_i \in V_i, x' \in \prod_{j \neq i} V_j, \lambda \in \mathbb{K}.$$

We put $\bigotimes_{i \in I} V_i := F / \sum_i F_i$. Then we have a natural map $m: \prod_{i \in I} V_i \to \bigotimes_{i \in I} V_i$ which is multilinear, and one easily checks that each multilinear map $\prod_{i \in I} V_i \to W$ factors uniquely through m.

From this universal property, it follows immediately that for each collection of linear maps $A_i \in \mathrm{End}(V_i)$, we obtain a linear map

$$\otimes_{i \in I} A_i \in \mathrm{End}\big(\bigotimes_{i \in I} V_i\big) \quad \text{with} \quad \otimes_{i \in I} A_i.m((v_i)_{i \in I}) = m((A_i.v_i)_{i \in I})$$

because the right hand side defines a multilinear map $\prod_{i \in I} V_i \to \bigotimes_{i \in I} V_i$.

Let $\mathfrak{g} = \bigoplus_{i \in I} \mathfrak{g}_i$ be a direct sum of Lie algebras. If for each $i \in I$ the space V_i is a $\mathfrak{g}_i$-module, then $\bigotimes_{i \in I} V_i$ carries a natural $\mathfrak{g}$-module structure with the ideals $\mathfrak{g}_i$ acting by

$$\rho(x_i). \otimes_{j \in I} v_j = x_i.v_i \otimes (\otimes_{j \neq i} v_j).$$

Note that if $\mathfrak{g}_i = \mathfrak{d}$ for all $i \in I$, then this construction does *not* lead to a representation of $\mathfrak{d}$ on the tensor product space because, if I is infinite, then the diagonal algebra $\mathfrak{d}$ is not contained in the direct sum Lie algebra $\mathfrak{g}$.

Now suppose that I is a set and that $V_i = V$ for all $i \in I$. Then the restricted symmetric group $S_{(I)}$ acts on the space $T := \bigotimes_{i \in I} V_i$. We consider the subspace

$$U := \mathrm{span}\{\sigma.x - \varepsilon(\sigma)x : x \in T, \sigma \in S_{(I)}\} \subseteq T$$

and define

$$\Lambda^I(V) := T/U \quad \text{and} \quad \wedge_{i \in I} v_i := \otimes_{i \in I} v_i + U.$$

Then we have a natural map $\wedge: \prod_{i \in I} V_i \to \Lambda^I(V)$ which is multilinear and alternating, and it is easy to see that each alternating multilinear map $\prod_{i \in I} V_i \to W$ factors through $\wedge$.

The construction of a representation of a Lie algebra on subspaces of the space $\Lambda^I(V)$ is a bit subtle. To obtain this representation, we fix for each $i \in I$ an element $v_i \in V$ and consider the subspace $\Lambda^{(I)}(V) := \mathrm{span}\{\wedge_{i \in I} w_i : |\{i : w_i \neq v_i\}| < \infty\}$.

Now suppose that V is a module of the Lie algebra $\mathfrak{g}$ such that for each $x \in \mathfrak{g}$ the corresponding operator $\rho_V(x)$ annihilates all but finitely many of the v_i. Then for each $x \in \mathfrak{g}$ the operator

$$\rho(x). \wedge_{i \in I} w_i := \sum_{i \in I} x.w_i \wedge (\wedge_{j \neq i} w_j)$$

is defined because the sum on the right hand side is always finite. Note that the expression on the right hand side is not meant as a product in an algebra. It corresponds to writing the elements of a product set $\prod_{j\in I} X_j$ as $x = (x_j)_{j\in I} = (x_i, (x_j)_{j\neq i})$ for some $i \in I$. We thus obtain a representation ρ of $\mathfrak{g}$ on $\Lambda^{(I)}(V)$.

A typical example of such a situation is given by the canonical representation of $\mathfrak{gl}(J,\mathbb{K})$ on $V = \mathbb{K}^{(J)}$. Let $(e_j)_{j\in J}$ denote the canonical basis of V and $M \subseteq J$ be a subset. Then we obtain a representation of $\mathfrak{g} = \mathfrak{gl}(J,\mathbb{K})$ on the space $\Lambda^{(M)}(\mathbb{K}^{(J)})$ with $\lambda = \sum_{j\in M} \varepsilon_j$ as an extremal weight. The typical examples which are discussed in [KR87] are $J = \mathbb{Z}$ and $M = \{\ldots, m-2, m-1, m\}$ for $m \in \mathbb{Z}$.

Lemma V.1. *Let $\mathfrak{g} = \bigoplus_{j\in J} \mathfrak{g}_j$ be a direct sum of locally finite almost reductive split Lie algebras and V_j simple $\mathfrak{g}_j$-weight modules with $\mathrm{End}_{\mathfrak{g}_j}(V_j) = \mathbb{K}\mathbf{1}$. For each $j \in J$ we pick a non-zero weight vector $v_j \in V_j^{\alpha_j}$. Then the submodule $\widetilde{\bigotimes}_{j\in J} V_j \subseteq \bigotimes_{j\in J} V_j$ generated by the weight vector $v := \otimes_{j\in J} v_j$ of weight $\alpha = \sum_{j\in J} \alpha_j$ is a simple $\mathfrak{g}$-module.*

Proof. If $F \subseteq J$ is a finite subset and $\mathfrak{g}_F := \sum_{j\in F} \mathfrak{g}_j$, then the $\mathfrak{g}_F$-submodule V_F generated by v is isomorphic to $\bigotimes_{j\in F} V_j$ which is simple according to Lemma II.6(ii) applied inductively. We conclude that $\widetilde{\bigotimes}_{j\in J} V_j$ is an inductive limit of simple $\mathfrak{g}_F$-modules V_F and therefore a simple $\mathfrak{g}$-module. ∎

Proposition V.2. *For a simple highest weight module of the direct sum $\mathfrak{g} = \bigoplus_{j\in J} \mathfrak{g}_j$ of locally finite almost reductive split Lie algebras $\mathfrak{g}_j$ we have*

$$L(\lambda, \Delta^+, \mathfrak{g}) \cong \widetilde{\bigotimes}_{i\in I} L(\lambda_i, \Delta_i^+, \mathfrak{g}_i),$$

where $\lambda_i = \lambda|_{\mathfrak{h}_i}$ and $\mathfrak{h}_i := \mathfrak{h} \cap \mathfrak{g}_i$ is a splitting Cartan subalgebra of $\mathfrak{g}_i$. ∎

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References

[Bou90] Bourbaki, N., *Groupes et algèbres de Lie, Chapitres 7 et 8*, Masson, Paris, 1990.

[DiPe99] Dimitrov, I., and I. Penkov, *Weight modules of direct limit Lie algebras*, International Mathematics Research Notices **5** (1999), 223–249.

[KR87] Kac, V. G., and A. K. Raina, "Highest Weight Representations of Infinite Dimensional Lie Algebras," Advanced Series in Math. Physics, World Scientific, Singapore, 1987.

[Ka73] Kaplansky, I., *Infinite dimensional Lie algebras*, Scripta Math. **29** (1973), 237–241.

[KaKi75] Kaplansky, I., *Infinite dimensional Lie algebras II*, Linear and Multilinear Algebra **3** (1975/76), 61–65.

[La74] Lang, S., "Algebra," Addison Wesley Publ. Comp., London, 1974.

[MP95] Moody, R., and A. Pianzola, "Lie Algebras with Triangular Decompositions," Canad. Math. Soc. Series of Monographs and Advanced Texts, Wiley Interscience, 1995.

[Ne98] Neeb, K.–H., *Holomorphic highest weight representations of infinite dimensional complex classical groups*, J. reine ang. Math. **497** (1998), 171–222.

[Ne99] —, "Holomorphy and Convexity in Lie Theory," Expositions in Mathematics, de Gruyter Verlag, Berlin, 1999.

[Ne00a] —, *Integrable roots in split graded Lie algebras*, Journal of Algebra, to appear.

[Ne00b] —, *Locally finite Lie algebras with unitary highest weight representations*, in preparation.

[NeSt98] Neeb, K.–H., and N. Stumme, *On the classification of locally finite split simple Lie algebras*, submitted.

[Ne90] Neher, E., *Systemes de racines 3-gradués*, C. R. Acad. Sci. Paris, Ser. I, **310** (1990), 687–690.

[St99a] Stumme, N., *The structure of locally finite split Lie algebras*, Journal of Algebra **220** (1999), 664–693.

[St99b] —, "Locally Finite Split Lie Algebras," thesis, University of Technology Darmstadt, 1999.