A Comparison of the Plate Theories in the Sense of Kirchhoff–Love and Reissner–Mindlin

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Abstract

In this article we compare the two plate theories in the sense of Kirchhoff–Love and Reissner–Mindlin for several different settings of the physical system. We establish existence, uniqueness and regularity of solutions to the respective boundary and initial boundary value problems. Moreover, we give asymptotic expansions of the solutions in the limit of a vanishing plate thickness, $\varepsilon \to 0$, whenever this is possible. Finally, we compare the solutions in the sense of Kirchhoff–Love and Reissner–Mindlin in that very limit.

[1]The following results constitute a major part of a PhD thesis (see [9]).

Contents

1 Introduction

In this article we consider a thin elastic plate which is clamped at its lateral boundary and submitted to an exterior body force. Our starting point is the linear three dimensional elasticity theory. Here, the motion of the plate is described by the deformation mapping $\vec{\varphi} : \Omega_\varepsilon \times \mathbb{R} \longrightarrow \mathbb{R}^3$, where $\Omega_\varepsilon := \tilde{\Omega} \times (-\varepsilon, \varepsilon) \subset \mathbb{R}^3$ is the reference configuration. We assume that Hamilton's variational principle holds, i.e.

$$J[\vec{\varphi}] := \rho \int_0^T \int_{\Omega_\varepsilon} \left[\frac{1}{2} |\partial_t \vec{\varphi}|^2 - \Psi(\nabla \vec{\varphi}) + \vec{f}(\vec{x}) \cdot \vec{\varphi} \right] \mathrm{d}x\mathrm{d}t \quad \longrightarrow \quad \text{min.},$$

where the free energy Ψ is given by the linearized Lamé law (8). This yields the corresponding Euler–Lagrange equations (10) and natural boundary conditions (11).
Next, we consider plate theories. Here, we linearize the deformation $\vec{\varphi}$ with respect to the variable $x \in (-\varepsilon, \varepsilon)$, i.e. we make an ansatz of the form

$$\vec{\varphi}(\tilde{x}, x, t) := \vec{u}(\tilde{x}, t) + x\vec{v}(\tilde{x}, t),$$

see (12). Now, we insert this ansatz into the action functional $J[\vec{\varphi}]$ and apply Hamilton's variational principle, i.e. we vary $J[\vec{\varphi}]$ with respect to the independent components of $\vec{u}$ and $\vec{v}$. It turns out that this procedure is not unique. Actually, the various plate theories are distinguished by the different ways to consider the natural boundary conditions (11). In a *plate theory in the sense of Kirchhoff–Love* we assume that $\vec{u}$ and $\vec{v}$ are constrained by Kirchhoff's normal hypothesis (15) such that (14) holds. Then we obtain the Euler–Lagrange equations (21). On the other hand, in a *plate theory in the sense of Reissner–Mindlin* we assume that $\vec{u}$ and $\vec{v}$ are independent fields. Then we obtain the Euler–Lagrange equations (22).
The goal of this article is to compare these two plate theories for several different settings of the physical system, namely the static problem, the simplified dynamic problem and the full dynamic problem. We establish existence, uniqueness and regularity of solutions $(\vec{u}, \vec{v})$ to the respective problems. Moreover, for $\vec{w} = \vec{u}, \vec{v}$ we consider asymptotic expansions

with respect to ε, i.e.

$$\vec{w}(\tilde{x},t) = \sum_{k=0}^{\infty} \varepsilon^{2k} \vec{w}_k(\tilde{x},t).$$

Finally, we compare the respective solutions in the sense of Kirchhoff–Love and Reissner–Mindlin in the limit $\varepsilon \to 0$.
This article is purely analytic in character. A numerical comparison of the two plate theories can be found in [8]. Moreover, in this article we derive the basic equations of plate theory from three dimensional elasticity *postulating* a plate theoretical ansatz for the deformation. A rigorous study of the passage from three dimensional elasticity to plate theory can be found in [4] and [5], [6], [7].

2 Three–dimensional Elasticity

In this section we consider the three–dimensional elasticity theory for a thin plate. We postulate the existence of an action functional for the system such that we obtain the equations of motion together with natural boundary conditions from a variational principle. This description of the problem will underly the various plate theoretical approaches that we will introduce in the next section.

Throughout this article we will use the following notations for the elements of $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$:

$$a \in \mathbb{R}, \qquad \tilde{a} \in \mathbb{R}^2, \qquad \vec{a} \in \mathbb{R}^3 \qquad \text{and} \qquad \vec{a} = \begin{pmatrix} \tilde{a} \\ a \end{pmatrix}. \tag{1}$$

In particular the row vector of partial derivatives with respect to $\vec{x} \in \mathbb{R}^3$ is denoted by $\vec{\nabla}$ and the row vector of partial derivatives with respect to $\tilde{x} \in \mathbb{R}^2$ is denoted by $\tilde{\nabla}$. [2]
Furthermore, the unit vectors of $\mathbb{R}^3$ are denoted by $\hat{e}_i$ $(i = 1, 2, 3)$.

Now, let $\tilde{\Omega} \subset \mathbb{R}^2$ be a bounded domain with a smooth boundary $\partial\tilde{\Omega}$ and let $\varepsilon > 0$. We assume that the natural reference configuration of the plate is given by

$$\Omega_\varepsilon := \tilde{\Omega} \times (-\varepsilon, \varepsilon). \tag{2}$$

Then, the deformation of the plate is given by a function

$$\vec{\varphi} : \Omega_\varepsilon \times \mathbb{R} \longrightarrow \mathbb{R}^3 : (\vec{x}, t) \longmapsto \vec{\varphi}(\vec{x}, t). \tag{3}$$

We assume that the lateral boundary of the plate is clamped, i.e. we prescribe the following Dirichlet boundary conditions:

$$\vec{\varphi}(\vec{x}, t)\Big|_{\vec{x} \in \partial\tilde{\Omega} \times (-\varepsilon, \varepsilon)} = \vec{x}. \tag{4}$$

[2] As usual, for functions $g : \mathbb{R}^3 \times \mathbb{R} \longrightarrow \mathbb{R}^k$, $h : \mathbb{R}^2 \times \mathbb{R} \longrightarrow \mathbb{R}^k$ the Jacobian matrices with recpect to the space variables are denoted by $\nabla g(\vec{x}, t)$ and $\nabla h(\tilde{x}, t)$ respectively.

Furthermore, at time $t = 0$ we prescribe the following initial conditions:

$$\vec{\varphi}(\vec{x},0) = \vec{\varphi}_0(\vec{x}), \qquad \partial_t\vec{\varphi}(\vec{x},0) = \vec{\varphi}_1(\vec{x}) \tag{5}$$

where the functions $\vec{\varphi}_0, \vec{\varphi}_1 : \Omega_\varepsilon \longrightarrow \mathbb{R}^3$ will be chosen appropriately in the next section.

Next, we fix our constitutive Model. We assume that the mass density is given by a constant $\rho > 0$.
Furthermore, we assume that the specific body force is acting in 3–direction and does not depend on $x \in (-\varepsilon, \varepsilon)$ and $t \in \mathbb{R}$ explicitly, i.e. we assume that there is a function $f \in \mathcal{C}^\infty(\overline{\tilde{\Omega}}, \mathbb{R})$ such that the specific body force is given by

$$\vec{f} : \Omega_\varepsilon \longrightarrow \mathbb{R}^3 : \vec{x} \longmapsto \vec{f}(\vec{x}) := \begin{pmatrix} 0 \\ f(\tilde{x}) \end{pmatrix}. \tag{6}$$

Furthermore, we assume that the specific stress tensor is given by the linearized Lamé law

$$\mathcal{T} : \Omega_\varepsilon \times \mathbb{R} \longrightarrow \mathbb{R}^{3\times 3} : (\vec{x},t) \longmapsto \mathcal{T}(\vec{x},t) \tag{7a}$$

where

$$\mathcal{T}(\vec{x},t) := \lambda \operatorname{tr}\Big(\nabla\vec{\varphi}(\vec{x},t) - I\Big) I + \mu\Big[\Big(\nabla\vec{\varphi}(\vec{x},t) - I\Big) + \Big(\nabla\vec{\varphi}(\vec{x},t) - I\Big)^T\Big]. \tag{7b}$$

Then, the corresponding specific free energy reads

$$\Psi : \Omega_\varepsilon \times \mathbb{R} \longrightarrow \mathbb{R} : (\vec{x},t) \longmapsto \Psi(\vec{x},t) \tag{8a}$$

where

$$\Psi(\vec{x},t) := \frac{\lambda}{2} \operatorname{tr}\Big(\nabla\vec{\varphi}(\vec{x},t) - I\Big)^2 + \frac{\mu}{4}\Big|\Big(\nabla\vec{\varphi}(\vec{x},t) - I\Big) + \Big(\nabla\vec{\varphi}(\vec{x},t) - I\Big)^T\Big|^2. \tag{8b}$$

Here, $\lambda, \mu > 0$ denote the Lamé constants, $I \in \mathbb{R}^{3\times 3}$ denotes the unit matrix, and $|\cdot|$ denotes the Euklidian norm.

Next, we fix our dynamic principle. The action of the system for the time interval $[0,T] \subset \mathbb{R}$ is given by

$$J[\vec{\varphi}] := \rho \int_0^T \int_{\Omega_\varepsilon} \Big[\frac{1}{2}\Big|\partial_t\vec{\varphi}(\vec{x},t)\Big|^2 - \Psi(\vec{x},t) + \vec{f}(\vec{x}) \cdot \vec{\varphi}(\vec{x},t)\Big] \mathrm{d}x\mathrm{d}t. \tag{9}$$

An application of Hamilton's variational principle yields the corresponding Euler–Lagrange equations

$$\partial_t^2\vec{\varphi} - \lambda\vec{\nabla}\Big(\operatorname{div}\vec{\varphi}\Big) - \mu\Big[\vec{\nabla}\Big(\operatorname{div}\vec{\varphi}\Big) + \Delta\vec{\varphi}\Big] = \vec{f}(\vec{x}) \tag{10}$$

and the corresponding natural boundary conditions

$$\mathcal{T}(\vec{x},t)\hat{e}_3\Big|_{x=\pm\varepsilon} = 0. \tag{11}$$

Physically, equation (11) means that the normal stress vanishes on the upper and lower side of the plate.

3 Plate Theory

In this section we postulate a particular plate theoretical ansatz for the deformation $\vec{\varphi}$. Then, we derive the equations of motion for the plate from the variational principle of three–dimensional elasticity.

The basic assumption of plate theory is, that the deformation $\vec{\varphi}$ is linear with respect to $x \in (-\varepsilon, \varepsilon)$. Furthermore, we assume that during the process of deformation material points of the middle surface will be shifted only in 3–direction but not in 1,2–direction and that the plate thickness remains constant. This yields the following ansatz:

$$\vec{\varphi}(\vec{x},t) := \begin{pmatrix} \tilde{x} \\ u(\tilde{x},t) \end{pmatrix} + x \begin{pmatrix} \tilde{v}(\tilde{x},t) \\ 1 \end{pmatrix} \tag{12a}$$

where $u : \tilde{\Omega} \times \mathbb{R} \longrightarrow \mathbb{R}$ and $\tilde{v} : \tilde{\Omega} \times \mathbb{R} \longrightarrow \mathbb{R}^2$. $\tilde{v}$ is called the director of the plate.
In particular, we have

$$\partial_t \vec{\varphi}(\vec{x},t) = \begin{pmatrix} 0 \\ \partial_t u(\tilde{x},t) \end{pmatrix} + x \begin{pmatrix} \partial_t \tilde{v}(\tilde{x},t) \\ 0 \end{pmatrix}, \tag{12b}$$

$$\nabla \vec{\varphi}(\vec{x},t) - I = \begin{pmatrix} 0 & \tilde{v}(\tilde{x},t) \\ \nabla u(\tilde{x},t) & 0 \end{pmatrix} + x \begin{pmatrix} \nabla \tilde{v}(\tilde{x},t) & 0 \\ 0 & 0 \end{pmatrix}. \tag{12c}$$

Next, we fix our constitutive model. We insert (12) into (9) and carry out the integration with respect to $x \in (-\varepsilon, \varepsilon)$. Then, the action of the system reads

$$\begin{aligned} J_p[u,\tilde{v}] := 2\varepsilon\rho \int_0^T \int_{\tilde{\Omega}} \Big[\frac{1}{2}\Big(\partial_t u(\tilde{x},t)\Big)^2 - \Psi_0(\tilde{x},t) + f(\tilde{x},t)u(\tilde{x},t)\Big] \mathrm{d}x\mathrm{d}t \\ + \frac{2\varepsilon^3\rho}{3} \int_0^T \int_{\tilde{\Omega}} \Big[\frac{1}{2}\Big|\partial_t \tilde{v}(\tilde{x},t)\Big|^2 - \Psi_1(\tilde{x},t)\Big] \mathrm{d}x\mathrm{d}t \end{aligned} \tag{13a}$$

where

$$\Psi_0(\tilde{x},t) := \frac{\mu}{2}\Big|\tilde{\nabla} u(\tilde{x},t) + \tilde{v}(\tilde{x},t)\Big|^2, \tag{13b}$$

$$\Psi_1(\tilde{x},t) := \frac{\lambda}{2}\Big(\operatorname{div} \tilde{v}(\tilde{x},t)\Big)^2 + \frac{\mu}{4}\Big|\nabla \tilde{v}(\tilde{x},t) + \Big(\nabla \tilde{v}(\tilde{x},t)\Big)^T\Big|^2. \tag{13c}$$

From the variational principle of three dimensional elasticity we have obtained natural boundary conditions (11). Now, in a plate theory there are at least two different ways to deal with them and actually this distinguishes the various plate theories. One way to deal with the natural boundary conditions is to ignore them. This leads to a *plate theory in the sense of Reissner–Mindlin.* A second way to deal with the natural boundary conditions is to insert (12) into (7) and to replace (11) with

$$\frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} \mathcal{T}(\vec{x},t)\hat{e}_3 \mathrm{d}x = 0. \tag{14}$$

This yields Kirchhoff's normal hypothesis

$$\tilde{v}(\tilde{x},t)+\tilde{\nabla}u(\tilde{x},t)=0. \tag{15}$$

Physically, equation (14) means that the mean normal stress of the plate vanishes and this leads to a *plate theory in the sense of Kirchhoff–Love*.

Next, we fix the boundary and initial conditions. We insert (12) into (4). This yields the following boundary conditions:

$$u(\tilde{x},t)\Big|_{\tilde{x}\in\partial\tilde{\Omega}}=0, \qquad \tilde{v}(\tilde{x},t)\Big|_{\tilde{x}\in\partial\tilde{\Omega}}=0. \tag{16}$$

Now, we insert (12) into (5) and we assume that the initial data are compatible with both the ansatz (12) and Kirchhoff's normal hypothesis (15). Furthermore, we assume that at time $t=0$ an environment of the lateral boundary is in its natural state. This yields the following initial conditions:

$$u(\tilde{x},0)=u_0(\tilde{x}), \qquad \partial_t u(\tilde{x},0)=u_1(\tilde{x}), \tag{17a}$$

$$\tilde{v}(\tilde{x},0)=-\tilde{\nabla}u_0(\tilde{x}), \qquad \partial_t\tilde{v}(\tilde{x},0)=-\tilde{\nabla}u_1(\tilde{x}) \tag{17b}$$

where $u_1,u_2\in\mathcal{C}_0^\infty(\tilde{\Omega},\mathbb{R})$.

Next, we fix our dynamical principle. From three–dimensional elasticity we have Hamilton's variational principle postulating that the physical deformation $\vec{\varphi}$ is a critical point of the action functional J. Now, we apply this principle to the action functional (13).

Plate Theory in the Sense of Kirchhoff–Love

The basic assumption of a plate theory in the sense of Kirchhoff–Love is, that Kirchhoff's normal hypothesis (15) holds, i.e. that the director is constrained by

$$\tilde{v}^{\text{KL}}(\tilde{x},t):=-\tilde{\nabla}u^{\text{KL}}(\tilde{x},t). \tag{18}$$

We insert (18) into (16), (17). This yields the following boundary and initial conditions:

$$u^{\text{KL}}(\tilde{x},t)\Big|_{\tilde{x}\in\partial\tilde{\Omega}}=0, \qquad \tilde{\nabla}u^{\text{KL}}(\tilde{x},t)\Big|_{\tilde{x}\in\partial\tilde{\Omega}}=0, \tag{19a}$$

$$u^{\text{KL}}(\tilde{x},0)=u_0(\tilde{x}), \qquad \partial_t u^{\text{KL}}(\tilde{x},0)=u_1(\tilde{x}). \tag{19b}$$

Now, we insert (18) into (13). Then, the action of the system reads

$$\begin{aligned} J_p^{\text{KL}}[u^{\text{KL}}] := 2\varepsilon\rho\int_0^T\int_{\tilde{\Omega}}\Big[\frac{1}{2}\Big(\partial_t u^{\text{KL}}(\tilde{x},t)\Big)^2+f(\tilde{x},t)u^{\text{KL}}(\tilde{x},t)\Big]\mathrm{d}x\mathrm{d}t \\ +\frac{2\varepsilon^3\rho}{3}\int_0^T\int_{\tilde{\Omega}}\Big[\frac{1}{2}\Big|\partial_t\tilde{\nabla}u^{\text{KL}}(\tilde{x},t)\Big|^2-\Psi_1^{\text{KL}}(\tilde{x},t)\Big]\mathrm{d}x\mathrm{d}t \end{aligned} \tag{20a}$$

where

$$\Psi_1^{\text{KL}}(\tilde{x},t):=\frac{\lambda}{2}\Big(\Delta u^{\text{KL}}(\tilde{x},t)\Big)^2+\mu\Big|\nabla^2u^{\text{KL}}(\tilde{x},t)\Big|^2. \tag{20b}$$

An application of Hamilton's variational principle yields the corresponding Euler–Lagrange equation

$$\partial_t^2 u^{\mathrm{KL}} - \frac{\varepsilon^2}{3}\Big[\partial_t^2 \Delta u^{\mathrm{KL}} - (\lambda + 2\mu)\Delta\Delta u^{\mathrm{KL}}\Big] = f(\tilde{x}). \tag{21}$$

Plate Theory in the Sense of Reissner–Mindlin

The basic assumption of a plate theory in the sense of Reissner–Mindlin is, that the director is not constrained. In particular, Kirchhoff's normal hypothesis does not hold.

Then, the boundary and initial conditions for u^{RM}, $\tilde{v}^{\mathrm{RM}}$ are given by (16), (17). Furthermore, the action of the system is given by (13).
An application of Hamilton's variational principle yields the corresponding Euler–Lagrange equations

$$\partial_t^2 u^{\mathrm{RM}} - \mu\Big(\Delta u^{\mathrm{RM}} + \operatorname{div}\tilde{v}^{\mathrm{RM}}\Big) = f(\tilde{x}), \tag{22a}$$

$$\frac{\varepsilon^2}{3}\Big[\partial_t^2 \tilde{v}^{\mathrm{RM}} - (\lambda+\mu)\tilde{\nabla}\Big(\operatorname{div}\tilde{v}^{\mathrm{RM}}\Big) - \mu\Delta\tilde{v}^{\mathrm{RM}}\Big] + \mu\Big(\tilde{\nabla}u^{\mathrm{RM}} + \tilde{v}^{\mathrm{RM}}\Big) = 0. \tag{22b}$$

Comparison of the Initial Boundary Value Problems

Both, the plate theory in the sense or Kirchhoff–Love and the plate theory in the sense of Reissner–Mindlin, are assumed to be approximations to the same three–dimensional elasticity theory. [3] Therefore, the Kirchhoff–Love PDE system (18), (21) and the Reissner–Mindlin PDE system (22) should be in some sense approximately equivalent.

Now, we compare the PDE systems in the sense of Kirchhoff–Love and Reissner–Mindlin. For further use the next theorem is a little more general than it was necessary in this place.

Theorem 1 (Comparison Theorem)
Let $\alpha \in \mathcal{D}'(\tilde{\Omega}\times\mathbb{R},\mathbb{R})$ and $\tilde{\beta} \in \mathcal{D}'(\tilde{\Omega}\times\mathbb{R},\mathbb{R}^2)$ be distributions.
Then, in the distributional sense the following PDE systems are equivalent:

$$\partial_t^2 u - \mu\Big(\Delta u + \operatorname{div}\tilde{v}\Big) = f(\tilde{x}) + \alpha(\tilde{x},t), \tag{23a}$$

$$\frac{\varepsilon^2}{3}\Big[\partial_t^2 \tilde{v} - (\lambda+\mu)\tilde{\nabla}\Big(\operatorname{div}\tilde{v}\Big) - \mu\Delta\tilde{v}\Big] + \mu\Big(\tilde{\nabla}u + \tilde{v}\Big) = \tilde{\beta}(\tilde{x},t) \tag{23b}$$

[3] See [5], [6], [7].

and

$$
\begin{aligned}
&\partial_t^2 u - \frac{\varepsilon^2}{3}\Big[\partial_t^2 \Delta u - (\lambda+2\mu)\Delta\Delta u\Big] \\
&= f(\tilde{x}) + \alpha(\tilde{x},t) + \operatorname{div}\tilde{\beta}(\tilde{x},t) - \frac{\varepsilon^2}{3\mu}\Big[\partial_t^2\Big(\operatorname{div}\tilde{\beta}(\tilde{x},t)\Big) - (\lambda+2\mu)\Delta\Big(\operatorname{div}\tilde{\beta}(\tilde{x},t)\Big)\Big] \\
&\qquad + \frac{\varepsilon^4}{9\mu}\Big[\partial_t^4\Big(\operatorname{div}\tilde{v}\Big) - 2(\lambda+2\mu)\Delta\partial_t^2\Big(\operatorname{div}\tilde{v}\Big) + (\lambda+2\mu)^2\Delta\Delta\Big(\operatorname{div}\tilde{v}\Big)\Big], \qquad (24a)
\end{aligned}
$$

$$
\tilde{v} = -\tilde{\nabla} u + \frac{1}{\mu}\tilde{\beta}(\tilde{x},t) - \frac{\varepsilon^2}{3\mu}\Big[\partial_t^2 \tilde{v} - (\lambda+\mu)\tilde{\nabla}\Big(\operatorname{div}\tilde{v}\Big) - \mu\Delta\tilde{v}\Big]. \qquad (24b)
$$

In particular, the Kirchhoff–Love PDE system (18), (21) *and the Reissner–Mindlin PDE system* (22) *are equivalent up to higher order terms in* ε.

Proof.

Obviously, the equations (23b) and (24b) are equivalent. Furthermore, with the help of (23b), (24b) we find that both equations, (23a) and (24a), are equivalent to

$$\partial_t^2 u + \frac{\varepsilon^2}{3}\Big[\partial_t^2\Big(\operatorname{div}\tilde v\Big) - (\lambda+2\mu)\Delta\Big(\operatorname{div}\tilde v\Big)\Big] = f(\tilde x) + \alpha(\tilde x,t) + \operatorname{div}\tilde\beta(\tilde x,t). \tag{25}$$

□

The above theorem suggests that the Kirchhoff–Love solution $(u^{\mathrm{KL}}, \tilde v^{\mathrm{KL}})$ and the Reissner–Mindlin solution $(u^{\mathrm{RM}}, \tilde v^{\mathrm{RM}})$ coincide as $\varepsilon \to 0$. Nevertheless, since in the equations the terms of highest order in ε do also contain the highest order derivatives of u^{RM} and $\tilde v^{\mathrm{RM}}$ this result is not obvious.

In the remaining sections we will give a rigorous asymptotic analysis of the plate theories in the sense of Kirchhoff–Love and Reissner–Mindlin for various physical settings of the system.

4 The Static Problem

In this section we investigate the static problems in the sense of Kirchhoff–Love and Reissner–Mindlin, i.e. we consider time independent solutions $(u^{\mathrm{KL}}, \tilde v^{\mathrm{KL}})$, $(u^{\mathrm{RM}}, \tilde v^{\mathrm{RM}})$ to the respective partial differential equations and boundary conditions.
In particular, in order to obtain a suitable normalization we replace $f(\tilde x)$ by $\varepsilon^2 f(\tilde x)$.

The Static Problem in the Sense of Kirchhoff–Love

The static problem in the sense of Kirchhoff–Love reads

$$\frac{\lambda+2\mu}{3}\Delta\Delta u^{\mathrm{KL}} = f(\tilde x), \tag{26a}$$

$$u^{\mathrm{KL}}(\tilde x)\Big|_{\tilde x\in\partial\tilde\Omega} = 0, \qquad \tilde\nabla u^{\mathrm{KL}}(\tilde x)\Big|_{\tilde x\in\partial\tilde\Omega} = 0. \tag{26b}$$

Furthermore, Kirchhoff's nornal hypothesis reads

$$\tilde v^{\mathrm{KL}}(\tilde x) = -\tilde\nabla u^{\mathrm{KL}}(\tilde x). \tag{27}$$

Theorem 2 (Existence, Uniqueness, Regularity)
The static Kirchhoff–Love problem (26) *possesses a unique weak solution* $u^{KL} \in \mathcal{C}^\infty(\overline{\tilde\Omega}, \mathbb{R})$.
Proof.
This is a well known fact from elliptic theory. □

We remark that by our normalization the problem (26) is independent of ε and consequently the solution u^{KL} does not depend on ε either.

The Static Problem in the Sense of Reissner–Mindlin

The static problem in the sense of Reissner–Mindlin reads

$$-\mu\Big(\Delta u^{\mathrm{RM}} + \operatorname{div}\tilde{v}^{\mathrm{RM}}\Big) = \varepsilon^2 f(\tilde{x}), \tag{28a}$$

$$-\frac{\varepsilon^2}{3}\Big[(\lambda+\mu)\tilde{\nabla}\Big(\operatorname{div}\tilde{v}^{\mathrm{RM}}\Big) + \mu\Delta\tilde{v}^{\mathrm{RM}}\Big] + \mu\Big(\tilde{\nabla}u^{\mathrm{RM}} + \tilde{v}^{\mathrm{RM}}\Big) = 0, \tag{28b}$$

$$u^{\mathrm{RM}}(\tilde{x})\Big|_{\tilde{x}\in\partial\tilde{\Omega}} = 0, \qquad \tilde{v}^{\mathrm{RM}}(\tilde{x})\Big|_{\tilde{x}\in\partial\tilde{\Omega}} = 0. \tag{28c}$$

By theorem 1 the PDE system (28a), (28b) is equivalent to

$$\frac{\lambda+2\mu}{3}\Delta\Delta u^{\mathrm{RM}} = f(\tilde{x}) + \frac{\varepsilon^2(\lambda+2\mu)^2}{9\mu}\Delta\Delta\Big(\operatorname{div}\tilde{v}^{\mathrm{RM}}\Big), \tag{29a}$$

$$\tilde{v}^{\mathrm{RM}} = -\tilde{\nabla}u^{\mathrm{RM}} + \frac{\varepsilon^2}{3\mu}\Big[(\lambda+\mu)\tilde{\nabla}\Big(\operatorname{div}\tilde{v}^{\mathrm{RM}}\Big) + \mu\Delta\tilde{v}^{\mathrm{RM}}\Big]. \tag{29b}$$

Theorem 3 (Existence, Uniqueness, Regularity)
The static Reissner–Mindlin problem (28) *possesses a unique weak solution* $(u^{RM}, \tilde{v}^{RM}) \in \mathcal{C}^\infty(\overline{\tilde{\Omega}}, \mathbb{R}) \times \mathcal{C}^\infty(\overline{\tilde{\Omega}}, \mathbb{R}^2)$.
Proof.
This is a well known fact from elliptic theory. □

Next, we consider an asymptotic expansion of the solution $(u^{\mathrm{RM}}, \tilde{v}^{\mathrm{RM}})$ with respect to ε. We make the formal ansatz

$$u^{\mathrm{RM}}(\tilde{x}) := \sum_{k=0}^{\infty}\varepsilon^{2k}u_k^{\mathrm{RM}}(\tilde{x}), \qquad \tilde{v}^{\mathrm{RM}}(\tilde{x}) := \sum_{k=0}^{\infty}\varepsilon^{2k}\tilde{v}_k^{\mathrm{RM}}(\tilde{x}) \tag{30}$$

and insert it into the Reissner–Mindlin problem (28) where we use (29) instead of (28a), (28b). Then, we obtain

$$\frac{\lambda+2\mu}{3}\Delta\Delta u_0^{\mathrm{RM}} = f(\tilde{x}), \qquad \tilde{v}_0^{\mathrm{RM}} = -\tilde{\nabla}u_0^{\mathrm{RM}}, \tag{31a}$$

$$u_0^{\mathrm{RM}}(\tilde{x})\Big|_{\tilde{x}\in\partial\tilde{\Omega}} = 0, \qquad \tilde{v}_0^{\mathrm{RM}}(\tilde{x})\Big|_{\tilde{x}\in\partial\tilde{\Omega}} = 0 \tag{31b}$$

and

$$\Delta\Delta u_{k+1}^{\mathrm{RM}} = \frac{\lambda+2\mu}{3\mu}\Delta\Delta\Big(\operatorname{div}\tilde{v}_k^{\mathrm{RM}}\Big), \tag{32a}$$

$$\tilde{v}_{k+1}^{\mathrm{RM}} = -\tilde{\nabla}u_{k+1}^{\mathrm{RM}} + \frac{1}{3\mu}\Big[(\lambda+\mu)\tilde{\nabla}\Big(\operatorname{div}\tilde{v}_k^{\mathrm{RM}}\Big) + \mu\Delta\tilde{v}_k^{\mathrm{RM}}\Big], \tag{32b}$$

$$u^{\text{RM}}_{k+1}(\tilde{x})\Big|_{\tilde{x}\in\partial\tilde{\Omega}} = 0, \qquad \tilde{v}^{\text{RM}}_{k+1}(\tilde{x})\Big|_{\tilde{x}\in\partial\tilde{\Omega}} = 0. \tag{32c}$$

Now, let $\hat{m}(\tilde{x})$ be the positively oriented tangent unit vector on $\partial\tilde{\Omega}$ and let $\hat{n}(\tilde{x})$ be the outward normal unit vector on $\partial\tilde{\Omega}$. Then, with the help of (32b) and the first boundary condition in (32c) we can rewrite the second boundary condition in (32c) as

$$\frac{\partial u^{\text{RM}}_{k+1}}{\partial n}(\tilde{x})\Big|_{\tilde{x}\in\partial\tilde{\Omega}} = \frac{1}{3\mu}\hat{n}(\tilde{x})\cdot\Big[(\lambda+\mu)\tilde{\nabla}\Big(\operatorname{div}\tilde{v}^{\text{RM}}_k\Big) + \mu\Delta\tilde{v}^{\text{RM}}_k\Big]\Big|_{\tilde{x}\in\partial\tilde{\Omega}} \tag{32d}$$

and

$$\hat{m}(\tilde{x})\cdot\Big[(\lambda+\mu)\tilde{\nabla}\Big(\operatorname{div}\tilde{v}^{\text{RM}}_k\Big) + \mu\Delta\tilde{v}^{\text{RM}}_k\Big]\Big|_{\tilde{x}\in\partial\tilde{\Omega}} = 0. \tag{33}$$

Furthermore, let $n \in \mathbb{N}$ be fix. Then, we define a formal approximate solution to the static Reissner–Mindlin problem (28) by

$$u^{\text{RM}}_{app}(\tilde{x}) := \sum_{k=0}^{n} \varepsilon^{2k} u^{\text{RM}}_k(\tilde{x}), \qquad \tilde{v}^{\text{RM}}_{app}(\tilde{x}) := \sum_{k=0}^{n} \varepsilon^{2k} \tilde{v}^{\text{RM}}_k(\tilde{x}). \tag{34}$$

Theorem 4 (Asymptotic Expansion)

1. *The zero–order term in the formal expansion* (30) *of the Reissner–Mindlin solution is exactly the Kirchhoff–Love solution, i.e. we have*

$$(u^{RM}_0, \tilde{v}^{RM}_0) = (u^{KL}, \tilde{v}^{KL}). \tag{35}$$

2. *When we exclude the constraint* (33) *from* (32)*, then the remaining boundary value problem possesses a unique weak solution* $(u^{RM}_{k+1}, \tilde{v}^{RM}_{k+1}) \in \mathcal{C}^\infty(\overline{\tilde{\Omega}}, \mathbb{R}) \times \mathcal{C}^\infty(\overline{\tilde{\Omega}}, \mathbb{R}^2)$.

3. *For* $k = 0$ *the constraint* (33) *is satisfied if and only if*

$$\Delta u^{KL}(\tilde{x})\Big|_{\tilde{x}\in\partial\tilde{\Omega}} = C = \textit{constant}. \tag{36}$$

   Then, the solution to the static Reissner–Mindlin problem (28) *is given by*

$$u^{RM}(\tilde{x}) = u^{KL}(\tilde{x}) + \frac{\varepsilon^2(\lambda+2\mu)}{3\mu}\Big(C - \Delta u^{KL}(\tilde{x})\Big), \tag{37a}$$

$$\tilde{v}^{RM}(\tilde{x}) = -\tilde{\nabla} u^{KL}(\tilde{x}). \tag{37b}$$

   In particular, in case of radial symmetry the condition (36) *is satisfied.*

4. *The formal approximate solution* $(u^{RM}_{app}, \tilde{v}^{RM}_{app})$ *satisfies the following boundary value problem:*

$$-\mu\Big(\Delta u^{RM}_{app} + \operatorname{div}\tilde{v}^{RM}_{app}\Big) = \varepsilon^2 f(\tilde{x}) + \varepsilon^{2n+2}\alpha(\tilde{x}), \tag{38a}$$

$$-\frac{\varepsilon^2}{3}\Big[(\lambda+\mu)\tilde{\nabla}\Big(\operatorname{div}\tilde{v}^{RM}_{app}\Big) + \mu\Delta\tilde{v}^{RM}_{app}\Big] + \mu\Big(\tilde{\nabla}u^{RM}_{app} + \tilde{v}^{RM}_{app}\Big) = \varepsilon^{2n+2}\tilde{\beta}(\tilde{x}), \tag{38b}$$

$$u^{RM}(\tilde{x})\Big|_{\tilde{x}\in\partial\tilde{\Omega}} = 0, \qquad \hat{n}(\tilde{x})\cdot\tilde{v}^{RM}(\tilde{x})\Big|_{\tilde{x}\in\partial\tilde{\Omega}} = 0, \tag{38c}$$

$$\hat{m}(\tilde{x})\cdot\tilde{v}^{RM}(\tilde{x})\Big|_{\tilde{x}\in\partial\tilde{\Omega}} = \varepsilon^2\gamma(\tilde{x}) \tag{38d}$$

where

$$\alpha(\tilde{x}) := \frac{\lambda+2\mu}{3}\Delta\Big(\operatorname{div}\tilde{v}_n^{RM}\Big), \tag{39a}$$

$$\tilde{\beta}(\tilde{x}) := -\frac{1}{3}\Big[(\lambda+\mu)\tilde{\nabla}\Big(\operatorname{div}\tilde{v}_n^{RM}\Big) + \mu\Delta\tilde{v}_n^{RM}\Big]. \tag{39b}$$

$$\gamma(\tilde{x}) := \hat{m}(\tilde{x})\cdot\Big(\sum_{k=1}^{n}\varepsilon^{2k-2}\tilde{v}_k^{RM}(\tilde{x})\Big)\Big|_{\tilde{x}\in\partial\tilde{\Omega}} \tag{39c}$$

In particular, $(u_{app}^{RM}, \tilde{v}_{app}^{RM})$ *satisfies the Reissner–Mindlin PDE system* (28a), (28b) *up to terms of order* ε^{2n+2} *and the corresponding boundary conditions* (28c) *up to terms of order* ε^2*.*

Proof.

1. This follows from (26), (27) and (31).

2. For $\tilde{v}_k^{\mathrm{RM}} \in \mathcal{C}^\infty(\overline{\tilde{\Omega}}, \mathbb{R}^2)$ this is a well known fact from elliptic theory. Since by theorem 2 and point 1 we have $\tilde{v}_0^{\mathrm{RM}} = \tilde{v}^{\mathrm{KL}} \in \mathcal{C}^\infty(\overline{\tilde{\Omega}}, \mathbb{R}^2)$, the statement follows by induction.

3. For $k = 0$ the constraint (33) reads

$$\hat{m}(\tilde{x})\cdot\tilde{\nabla}\Big(\Delta u_0^{\mathrm{RM}}\Big)\Big|_{\tilde{x}\in\partial\tilde{\Omega}} = 0. \tag{40}$$

   By point 1 this is equivalent to (36). Then, the solution to the static Reissner–Mindlin problem (28) is obviously given by (37).

4. By construction the formal approximate solution $(u_{app}^{\mathrm{RM}}, \tilde{v}_{app}^{\mathrm{RM}})$ satisfies the boundary conditions (38c) and the following PDE system:

$$\frac{\lambda+2\mu}{3}\Delta\Delta u_{app}^{\mathrm{RM}} = f(\tilde{x}) + \frac{\varepsilon^2(\lambda+2\mu)^2}{9\mu}\Delta\Delta\Big(\operatorname{div}\tilde{v}_{app}^{\mathrm{RM}}\Big) - \frac{\varepsilon^{2n+2}(\lambda+2\mu)^2}{9\mu}\Delta\Delta\Big(\operatorname{div}\tilde{v}_n^{\mathrm{RM}}\Big), \tag{41a}$$

$$\tilde{v}_{app}^{\mathrm{RM}} = -\tilde{\nabla}u_{app}^{\mathrm{RM}} + \frac{\varepsilon^2}{3\mu}\Big[(\lambda+\mu)\tilde{\nabla}\Big(\operatorname{div}\tilde{v}_{app}^{\mathrm{RM}}\Big) + \mu\Delta\tilde{v}_{app}^{\mathrm{RM}}\Big] - \frac{\varepsilon^{2n+2}}{3\mu}\Big[(\lambda+\mu)\tilde{\nabla}\Big(\operatorname{div}\tilde{v}_n^{\mathrm{RM}}\Big) + \mu\Delta\tilde{v}_n^{\mathrm{RM}}\Big]. \tag{41b}$$

   With the help of theorem 1 we obtain (38a), (38b). Furthermore, (38d) follows from (31b).

□

Comparison of the Static Problems

In theorem 4 we have seen that the zero–order term in the formal expansion (30) of the Reissner–Mindlin solution coincides with the Kirchhoff–Love solution. On the other hand, in general condition (36) of the theorem does not hold and consequently the series does not converge. Furthermore, the theorem shows that in general the formal expansion (30) is not even an asymptotic expansion. It remains to show, that the Kirchhoff–Love solution $(u^{\mathrm{KL}}, \tilde{v}^{\mathrm{KL}})$ and the Reissner–Mindlin solution $(u^{\mathrm{RM}}, \tilde{v}^{\mathrm{RM}})$ coincide as $\varepsilon \to 0$.

Here and in the followig, $C, \hat{C}, \ldots > 0$ denote generic constants independent of the functions and parameters under consideration. Furthermore, as usual $H^s(\tilde{\Omega}, \mathbb{R}^k)$ $(s \in \mathbb{R})$ denotes the scale of L^2–Sobolev spaces.

Theorem 5 (Comparison Theorem)
Let $(u^{KL}, \tilde{v}^{KL})$ be the solution to the static Kirchhoff–Love problem (26), (27) and let $(u^{RM}, \tilde{v}^{RM})$ be the solution to the static Reissner–Mindlin problem (28).
Then, the following a–priori estimates hold for all $\varepsilon > 0$:

$$\left\|\tilde{\nabla} u^{RM} + \tilde{v}^{RM}\right\|_{L^2(\tilde{\Omega},\mathbb{R}^2)} \leq C\varepsilon^2 \|f\|_{H^{-1}(\tilde{\Omega},\mathbb{R})}, \tag{42a}$$

$$\|u^{RM} - u^{KL}\|_{H^2(\tilde{\Omega},\mathbb{R})} \leq C\Big(\varepsilon^2 \|f\|_{L^2(\tilde{\Omega},\mathbb{R})} + \varepsilon \|f\|_{H^{-1}(\tilde{\Omega},\mathbb{R})}\Big), \tag{42b}$$

$$\|u^{RM} - u^{KL}\|_{H^3(\tilde{\Omega},\mathbb{R})} \leq C\Big(\varepsilon^2 \|f\|_{H^1(\tilde{\Omega},\mathbb{R})} + \|f\|_{H^{-1}(\tilde{\Omega},\mathbb{R})}\Big), \tag{42c}$$

$$\|\tilde{v}^{RM} - \tilde{v}^{KL}\|_{H^1(\tilde{\Omega},\mathbb{R}^2)} \leq C\varepsilon \|f\|_{H^{-1}(\tilde{\Omega},\mathbb{R})}, \tag{42d}$$

$$\|\tilde{v}^{RM} - \tilde{v}^{KL}\|_{H^2(\tilde{\Omega},\mathbb{R}^2)} \leq C \|f\|_{H^{-1}(\tilde{\Omega},\mathbb{R})}. \tag{42e}$$

In particular, the Kirchhoff–Love solution $(u^{KL}, \tilde{v}^{KL})$ and the Reissner–Mindlin solution $(u^{RM}, \tilde{v}^{RM})$ coincide as $\varepsilon \to 0$.

Proof.
We define $r \in \mathcal{C}^\infty(\overline{\tilde{\Omega}}, \mathbb{R})$ and $\tilde{s} \in \mathcal{C}^\infty(\overline{\tilde{\Omega}}, \mathbb{R}^2)$ by

$$r(\tilde{x}) := u^{\mathrm{RM}}(\tilde{x}) - u^{\mathrm{KL}}(\tilde{x}), \qquad \tilde{s}(\tilde{x}) := \tilde{v}^{\mathrm{RM}}(\tilde{x}) - \tilde{v}^{\mathrm{KL}}(\tilde{x}). \tag{43}$$

Then, with the help of (26), (27) and (28) we find that $(r, \tilde{s})$ satisfies the following boundary value problem:

$$-\mu\Big(\Delta r + \operatorname{div}\tilde{s}\Big) = \frac{\varepsilon^2(\lambda + 2\mu)}{3}\Delta\Delta u^{\mathrm{KL}}, \tag{44a}$$

$$-\frac{\varepsilon^2}{3}\Big[(\lambda+\mu)\tilde{\nabla}\Big(\operatorname{div}\tilde{s}\Big) + \mu\Delta\tilde{s}\Big] + \mu\Big(\tilde{\nabla} r + \tilde{s}\Big) = -\frac{\varepsilon^2(\lambda+2\mu)}{3}\tilde{\nabla}\Delta u^{\mathrm{KL}}, \tag{44b}$$

$$r(\tilde{x})\Big|_{\tilde{x}\in\partial\tilde{\Omega}} = 0, \qquad \tilde{s}(\tilde{x})\Big|_{\tilde{x}\in\partial\tilde{\Omega}} = 0. \tag{44c}$$

We multiply (44a) by r in $L^2(\tilde{\Omega}, \mathbb{R})$ and (44b) by $\tilde{s}$ in $L^2(\tilde{\Omega}, \mathbb{R}^2)$ and then add the two equations. With the help of integration by parts we obtain

$$
\begin{aligned}
&\frac{\varepsilon^2}{3}\Big[(\lambda+\mu)\left\|\operatorname{div}\tilde{s}\right\|^2_{L^2(\tilde{\Omega},\mathbb{R})}+\mu\left\|\nabla\tilde{s}\right\|^2_{L^2(\tilde{\Omega},\mathbb{R}^{2\times 2})}\Big]+\mu\left\|\tilde{\nabla}r+\tilde{s}\right\|^2_{L^2(\tilde{\Omega},\mathbb{R}^2)}\\
&=-\frac{\varepsilon^2(\lambda+2\mu)}{3}\left\langle\tilde{\nabla}r+\tilde{s}\,\middle|\,\tilde{\nabla}\Delta u^{\mathrm{KL}}\right\rangle_{L^2(\tilde{\Omega},\mathbb{R}^2)}\\
&\leq\frac{\mu}{2}\left\|\tilde{\nabla}r+\tilde{s}\right\|^2_{L^2(\tilde{\Omega},\mathbb{R}^2)}+\frac{\varepsilon^4(\lambda+2\mu)^2}{18\mu}\left\|\tilde{\nabla}\Delta u^{\mathrm{KL}}\right\|^2_{L^2(\tilde{\Omega},\mathbb{R}^2)}.
\end{aligned}\tag{45}
$$

Furthermore, with the help of (26) and elliptic regularity theory we obtain the following a–priori estimates for u^{KL}:

$$
\left\|u^{\mathrm{KL}}\right\|_{H^{k+2}(\tilde{\Omega},\mathbb{R})}\leq C_k\left\|f\right\|_{H^{k-2}(\tilde{\Omega},\mathbb{R})}\qquad\forall k\in\mathbb{N}.\tag{46}
$$

Then, with the help of (45), (46) and Poincaré's inequality we obtain

$$
\left\|\tilde{s}\right\|_{H^1(\tilde{\Omega},\mathbb{R}^2)}\leq C\varepsilon\left\|f\right\|_{H^{-1}(\tilde{\Omega},\mathbb{R})},\qquad\left\|\tilde{\nabla}r+\tilde{s}\right\|_{L^2(\tilde{\Omega},\mathbb{R}^2)}\leq C\varepsilon^2\left\|f\right\|_{H^{-1}(\tilde{\Omega},\mathbb{R})}.\tag{47}
$$

This yields (42a) and (42d). Next, we rewrite (44b) as

$$
(\lambda+\mu)\tilde{\nabla}\Big(\operatorname{div}\tilde{s}\Big)+\mu\Delta\tilde{s}=\frac{3\mu}{\varepsilon^2}\Big(\tilde{\nabla}r+\tilde{s}\Big)+(\lambda+2\mu)\tilde{\nabla}\Delta u^{\mathrm{KL}}.\tag{48}
$$

Then, with the help of elliptic regularity theory and (46), (47) we obtain

$$
\left\|\tilde{s}\right\|_{H^2(\tilde{\Omega},\mathbb{R}^2)}\leq\hat{C}\Big(\frac{1}{\varepsilon^2}\left\|\tilde{\nabla}r+\tilde{s}\right\|_{L^2(\tilde{\Omega},\mathbb{R}^2)}+\left\|\tilde{\nabla}\Delta u^{\mathrm{KL}}\right\|_{L^2(\tilde{\Omega},\mathbb{R}^2)}\Big)\leq C\left\|f\right\|_{H^{-1}(\tilde{\Omega},\mathbb{R})}.\tag{49}
$$

This yields the second inequality in (42e). Next, we rewrite (44a) as

$$
-\mu\Delta r=\varepsilon^2 f(\tilde{x})+\mu\operatorname{div}\tilde{s}.\tag{50}
$$

Then, with the help of elliptic regularity theory and (47), (49) we obtain

$$
\begin{aligned}
\left\|r\right\|_{H^2(\tilde{\Omega},\mathbb{R})}&\leq\hat{C}\Big(\varepsilon^2\left\|f\right\|_{L^2(\tilde{\Omega},\mathbb{R})}+\left\|\tilde{s}\right\|_{H^1(\tilde{\Omega},\mathbb{R}^2)}\Big)\\
&\leq C\Big(\varepsilon^2\left\|f\right\|_{L^2(\tilde{\Omega},\mathbb{R})}+\varepsilon\left\|f\right\|_{H^{-1}(\tilde{\Omega},\mathbb{R})}\Big),
\end{aligned}\tag{51a}
$$

$$
\begin{aligned}
\left\|r\right\|_{H^3(\tilde{\Omega},\mathbb{R})}&\leq\hat{C}\Big(\varepsilon^2\left\|f\right\|_{H^1(\tilde{\Omega},\mathbb{R})}+\left\|\tilde{s}\right\|_{H^2(\tilde{\Omega},\mathbb{R}^2)}\Big)\\
&\leq C\Big(\varepsilon^2\left\|f\right\|_{H^1(\tilde{\Omega},\mathbb{R})}+\left\|f\right\|_{H^{-1}(\tilde{\Omega},\mathbb{R})}\Big).
\end{aligned}\tag{51b}
$$

This yields (42b) and (42c). □

5 The Simplified Dynamic Problem

In this section we consider the dynamic problems in the sense of Kirchhoff–Love and Reissner–Mindlin under the simplifying assumptions that the Lamé constants λ, μ are so

large and that ε is so small, that in the equations the terms of order ε^2 can be neglected in comparison with the terms of order $\varepsilon^2\lambda$, $\varepsilon^2\mu$, λ, μ and 1.
Furthermore, we assume that the external body force vanishes, i.e. $f(\tilde{x}) = 0$.

The Simpilfied Dynamic Problem in the Sense of Kirchhoff–Love

The simpilfied dynamic problem in the sense of Kirchhoff–Love reads

$$\partial_t^2 u^{\text{KL}} + \frac{\varepsilon^2(\lambda + 2\mu)}{3}\Delta\Delta u^{\text{KL}} = 0, \tag{52a}$$

$$u^{\text{KL}}(\tilde{x},t)\Big|_{\tilde{x}\in\partial\tilde{\Omega}} = 0, \qquad \tilde{\nabla} u^{\text{KL}}(\tilde{x},t)\Big|_{\tilde{x}\in\partial\tilde{\Omega}} = 0, \tag{52b}$$

$$u^{\text{KL}}(\tilde{x},0) = u_0(\tilde{x}), \qquad \partial_t u^{\text{KL}}(\tilde{x},0) = u_1(\tilde{x}). \tag{52c}$$

Furthermore, Kirchhoff's normal hypothesis reads

$$\tilde{v}^{\text{KL}}(\tilde{x},t) = -\tilde{\nabla} u^{\text{KL}}(\tilde{x},t). \tag{53}$$

Theorem 6 (Existence, Uniqueness, Regularity)
The simplified dynamic Kirchhoff–Love problem (52) *possesses a unique weak solution* $u^{KL} \in \mathcal{C}^\infty(\overline{\tilde{\Omega}} \times \mathbb{R}, \mathbb{R})$.
Proof.
This is a well known fact from the theory of evolution equations. □

Next, we consider an asymptotic expansion of the solution u^{KL} with respect to ε. We make the formal ansatz

$$u^{\text{KL}}(\tilde{x},t) := \sum_{k=0}^{\infty} \varepsilon^{2k} u_k^{\text{KL}}(\tilde{x},t) \tag{54}$$

and insert it into the Kirchhoff–Love problem (52). Then, we obtain

$$\partial_t^2 u_0^{\text{KL}} = 0, \tag{55a}$$

$$u_0^{\text{KL}}(\tilde{x},t)\Big|_{\tilde{x}\in\partial\tilde{\Omega}} = 0, \qquad \tilde{\nabla} u_0^{\text{KL}}(\tilde{x},t)\Big|_{\tilde{x}\in\partial\tilde{\Omega}} = 0, \tag{55b}$$

$$u_0^{\text{KL}}(\tilde{x},0) = u_0(\tilde{x}), \qquad \partial_t u_0^{\text{KL}}(\tilde{x},0) = u_1(\tilde{x}) \tag{55c}$$

and

$$\partial_t^2 u_{k+1}^{\text{KL}} + \frac{\lambda + 2\mu}{3}\Delta\Delta u_k^{\text{KL}} = 0, \tag{56a}$$

$$u_{k+1}^{\text{KL}}(\tilde{x},t)\Big|_{\tilde{x}\in\partial\tilde{\Omega}} = 0, \qquad \tilde{\nabla} u_{k+1}^{\text{KL}}(\tilde{x},t)\Big|_{\tilde{x}\in\partial\tilde{\Omega}} = 0, \tag{56b}$$

$$u_{k+1}^{\text{KL}}(\tilde{x},0) = 0, \qquad \partial_t u_{k+1}^{\text{KL}}(\tilde{x},0) = 0. \tag{56c}$$

Furthermore, let $n \in \mathbb{N}$ be fix. Then, we define a formal approximate solution to the simplified dynamic Kirchhoff–Love problem (52) by

$$u^{\mathrm{KL}}_{app}(\tilde{x},t) := \sum_{k=0}^{n} \varepsilon^{2k} u^{\mathrm{KL}}_k(\tilde{x},t). \tag{57}$$

Theorem 7 (Asymptotic Expansion)

1. *The recursion problem* (55), (56) *possesses a unique solution given by*

$$u^{KL}_k(\tilde{x},t) = \Big(-\frac{\lambda+2\mu}{3}\Big)^k \Big(\frac{t^{2k}}{(2k)!}\Delta^{2k}u_0(\tilde{x}) + \frac{t^{2k+1}}{(2k+1)!}\Delta^{2k}u_1(\tilde{x})\Big). \tag{58}$$

2. *Let $T>0$, $p,q \in \mathbb{N}$ and let $s \in \mathbb{N}$ be sufficiently large.*
   Then, the following a–priori estimate holds for all $\varepsilon > 0$:

$$\big\|u^{KL} - u^{KL}_{app}\big\|_{\mathcal{C}^p([-T,T],H^{2q}(\tilde{\Omega},\mathbb{R}))} \le C\varepsilon^{2n+2}\Big(\|u_0\|_{H^s(\tilde{\Omega},\mathbb{R})} + \|u_1\|_{H^s(\tilde{\Omega},\mathbb{R})}\Big). \tag{59}$$

   In particular, the formal expansion (54) *is actually an asymptotic expansion.*

Proof.

1. Obviously, the unique solution to the equations (55a), (55c) and (56a), (56c) is given by (58). Since u_0 and u_1 have compact support, the constraints (55b) and (56b) are also satisfied.

2. Let $m \in \mathbb{N}$. We define $r^{\mathrm{KL}}_m \in \mathcal{C}^\infty(\overline{\tilde{\Omega}} \times [-T,T],\mathbb{R})$ by

$$r^{\mathrm{KL}}_m(\tilde{x},t) := u^{\mathrm{KL}}(\tilde{x},t) - \sum_{k=0}^{m} \varepsilon^{2k} u^{\mathrm{KL}}_k(\tilde{x},t). \tag{60}$$

   Let $0 \le i \le 2m$. We show that $\partial_t^i r^{\mathrm{KL}}_m$ satisfies the following initial boundary value problem:

$$\partial_t^{i+2} r^{\mathrm{KL}}_m + \frac{\varepsilon^2(\lambda+2\mu)}{3}\Delta\Delta\partial_t^i r^{\mathrm{KL}}_m = -\frac{\varepsilon^{2m+2}(\lambda+2\mu)}{3}\Delta\Delta\partial_t^i u^{\mathrm{KL}}_m, \tag{61a}$$

$$\partial_t^i r^{\mathrm{KL}}_m(\tilde{x},t)\Big|_{\tilde{x}\in\partial\tilde{\Omega}} = 0, \qquad \tilde{\nabla}\partial_t^i r^{\mathrm{KL}}_m(\tilde{x},t)\Big|_{\tilde{x}\in\partial\tilde{\Omega}} = 0, \tag{61b}$$

$$\partial_t^i r^{\mathrm{KL}}_m(\tilde{x},0) = 0, \qquad \partial_t^{i+1} r^{\mathrm{KL}}_m(\tilde{x},0) = 0. \tag{61c}$$

   For $i=0$ the statement follows from (52) and the construction of u^{KL}_k.
   For $i = 1,\dots,2m$ the statement follows by induction, since by (58) we have

$$\Delta\Delta\partial_t^{i-1} u^{\mathrm{KL}}_m(\tilde{x},0) = 0. \tag{62}$$

Next, we multiply (61a) by $\partial_t^{i+1} r_m^{\text{KL}}$ in $L^2(\tilde{\Omega}, \mathbb{R})$ and integrate with respect to t. With the help of integration by parts we obtain for all $t \in [-T, T]$

$$
\begin{aligned}
&\frac{1}{2} \left\| \partial_t^{i+1} r_m^{\text{KL}}(\cdot, t) \right\|_{L^2(\tilde{\Omega},\mathbb{R})}^2 + \frac{\varepsilon^2(\lambda + 2\mu)}{6} \left\| \nabla^2 \partial_t^i r_m^{\text{KL}}(\cdot, t) \right\|_{L^2(\tilde{\Omega},\mathbb{R}^{2\times 2})}^2 \\
&= -\frac{\varepsilon^{2m+2}(\lambda + 2\mu)}{3} \int_0^t \left\langle \partial_t^{i+1} r_m^{\text{KL}}(\cdot, \tau) \middle| \Delta\Delta \partial_t^i u_m^{\text{KL}}(\cdot, \tau) \right\rangle_{L^2(\tilde{\Omega},\mathbb{R})} \mathrm{d}t \\
&\leq \frac{\varepsilon^{2m+2}(\lambda + 2\mu)}{3} \left\| \partial_t^{i+1} r_m^{\text{KL}} \right\|_{\mathcal{C}^0([-T,T],L^2(\tilde{\Omega},\mathbb{R}))} \left\| \Delta\Delta \partial_t^i u_m^{\text{KL}} \right\|_{L^1([-T,T],L^2(\tilde{\Omega},\mathbb{R}))} \\
&\leq \frac{1}{4} \left\| \partial_t^{i+1} r_m^{\text{KL}} \right\|_{\mathcal{C}^0([-T,T],L^2(\tilde{\Omega},\mathbb{R}))}^2 + C\varepsilon^{4m+4} \left\| \Delta\Delta \partial_t^i u_m^{\text{KL}} \right\|_{L^1([-T,T],L^2(\tilde{\Omega},\mathbb{R}))}^2 . \qquad (63)
\end{aligned}
$$

Now, let $s \in \mathbb{N}$ be sufficiently large. Then, with the help of (58), (63) and Poincaré's inequality we obtain

$$
\left\| \partial_t^{i+1} r_m^{\text{KL}} \right\|_{\mathcal{C}^0([-T,T],L^2(\tilde{\Omega},\mathbb{R}))} \leq C\varepsilon^{2m+2} \Big(\|u_0\|_{H^s(\tilde{\Omega})} + \|u_1\|_{H^s(\tilde{\Omega},\mathbb{R})} \Big), \qquad (64a)
$$

$$
\left\| \partial_t^{i} r_m^{\text{KL}} \right\|_{\mathcal{C}^0([-T,T],H^2(\tilde{\Omega},\mathbb{R}))} \leq C\varepsilon^{2m+1} \Big(\|u_0\|_{H^s(\tilde{\Omega})} + \|u_1\|_{H^s(\tilde{\Omega},\mathbb{R})} \Big). \qquad (64b)
$$

Next, we rewrite (61a) as

$$
\frac{\varepsilon^2(\lambda + 2\mu)}{3} \Delta\Delta \partial_t^i r_m^{\text{KL}} = -\partial_t^{i+2} r_m^{\text{KL}} - \frac{\varepsilon^{2m+2}(\lambda + 2\mu)}{3} \Delta\Delta \partial_t^i u_m^{\text{KL}}. \qquad (65)
$$

Let $j \in \mathbb{N}$ and $s \in \mathbb{N}$ be sufficiently large. Then, with the help of elliptic regularity theory and (58) we obtain

$$
\begin{aligned}
&\left\| \partial_t^i r_m^{\text{KL}} \right\|_{\mathcal{C}^0([-T,T],H^{2j}(\tilde{\Omega},\mathbb{R}))} \\
&\leq C\varepsilon^{-2} \left\| \partial_t^{i+2} r_m^{\text{KL}} \right\|_{\mathcal{C}^0([-T,T],H^{2j-4}(\tilde{\Omega},\mathbb{R}))} + C\varepsilon^{2m} \Big(\|u_0\|_{H^s(\tilde{\Omega})} + \|u_1\|_{H^s(\tilde{\Omega},\mathbb{R})} \Big). \quad (66)
\end{aligned}
$$

Furthermore, let $0 \leq l \leq 2m+1$. Then, with the help of (61c) we obtain

$$
\|r_m^{\text{KL}}\|_{\mathcal{C}^l([-T,T],H^{2j}(\tilde{\Omega},\mathbb{R}))} \leq C \left\| \partial_t^l r_m^{\text{KL}} \right\|_{\mathcal{C}^0([-T,T],H^{2j}(\tilde{\Omega},\mathbb{R}))} \qquad (67)
$$

Now, we choose $i = 2m$ in (64) and apply (66) inductively. Then, with the help of (67) we obtain

$$
\|r_m^{\text{KL}}\|_{\mathcal{C}^l([-T,T],H^{4m+2-2l}(\tilde{\Omega},\mathbb{R}))} \leq C\varepsilon^{l+1} \Big(\|u_0\|_{H^s(\tilde{\Omega})} + \|u_1\|_{H^s(\tilde{\Omega},\mathbb{R})} \Big). \qquad (68)
$$

Finally, we estimate $u^{\text{KL}} - u_{app}^{\text{KL}}$. By construction we have for $m \geq n$

$$
u^{\text{KL}} - u_{app}^{\text{KL}} = r_m^{\text{KL}} + \sum_{k=n+1}^{m} \varepsilon^{2k} u_k^{\text{KL}}. \qquad (69)
$$

Now, we choose m, l sufficiently large, $l \geq \max\{p, 2n+1\}$ and $4m + 2 - 2l \geq 2q$. Then, the desired statement (59) follows from (58), (68) and (69).

□

The Simplified Dynamic Problem in the Sense of Reissner–Mindlin

The simplified dynamic problem in the sense of Reissner–Mindlin reads

$$\partial_t^2 u^{\text{RM}} - \mu\Big(\Delta u^{\text{RM}} + \operatorname{div} \tilde{v}^{\text{RM}}\Big) = 0, \tag{70a}$$

$$-\frac{\varepsilon^2}{3}\Big[(\lambda+\mu)\tilde{\nabla}\Big(\operatorname{div} \tilde{v}^{\text{RM}}\Big) + \mu\Delta\tilde{v}^{\text{RM}}\Big] + \mu\Big(\tilde{\nabla} u^{\text{RM}} + \tilde{v}^{\text{RM}}\Big) = 0, \tag{70b}$$

$$u^{\text{RM}}(\tilde{x},t)\Big|_{\tilde{x}\in\partial\tilde{\Omega}} = 0, \qquad \tilde{v}^{\text{RM}}(\tilde{x},t)\Big|_{\tilde{x}\in\partial\tilde{\Omega}} = 0, \tag{70c}$$

$$u^{\text{RM}}(\tilde{x},0) = u_0(\tilde{x}), \qquad \partial_t u^{\text{RM}}(\tilde{x},0) = u_1(\tilde{x}). \tag{70d}$$

We remark that by our simplifying assumption the term $\partial_t^2 \tilde{v}^{\text{RM}}$ has vanished from equation (70b). Consequently, we can no longer impose the initial conditions (17b) on $\tilde{v}^{\text{RM}}$. Furthermore, from (70a) and $u_0 \in \mathcal{C}_0^\infty(\tilde{\Omega},\mathbb{R})$ we formally obtain

$$\partial_t^2 u^{\text{RM}}(\tilde{x},0)\Big|_{\tilde{x}\in\partial\tilde{\Omega}} = \mu\Big(\operatorname{div} \tilde{v}^{\text{RM}}(\tilde{x},0)\Big)\Big|_{\tilde{x}\in\partial\tilde{\Omega}}. \tag{71}$$

But, in general this contradicts the first boundary condition in (70c). Consequently, we can not expect that the regularity of the solution $(u^{\text{RM}}, \tilde{v}^{\text{RM}})$ will allow us to take the trace of $\partial_t^2 u^{\text{RM}}(\cdot,t)$ on $\partial\tilde{\Omega}$.

Theorem 8 (Existence, Uniqueness, Regularity)
The simplified dynamic Reissner–Mindlin problem (70) *possesses a unique weak solution* $(u^{RM}, \tilde{v}^{RM}) \in \bigcap_{k=0}^2 \mathcal{C}^k\big(\mathbb{R}, H^{2-k}(\tilde{\Omega},\mathbb{R})\big) \times \mathcal{C}^k\big(\mathbb{R}, H^{3-k}(\tilde{\Omega},\mathbb{R}^2)\big)$.
Proof.
We define a continuous linear operator $A : H^{k+1}(\tilde{\Omega},\mathbb{R}^2) \cap H_0^1(\tilde{\Omega},\mathbb{R}^2) \longrightarrow H^{k-1}(\tilde{\Omega},\mathbb{R}^2)$ by

$$A\tilde{w} := -\frac{\varepsilon^2}{3}\Big[(\lambda+\mu)\tilde{\nabla}\Big(\operatorname{div} \tilde{w}\Big) + \mu\Delta\tilde{w}\Big] + \mu\tilde{w}. \tag{72}$$

With the help of elliptic theory we find that A is bijective for all $k \in \mathbb{N}$. Now, we can rewrite the PDE system (70a), (70b) as

$$\tilde{v}^{\text{RM}} = -\mu A^{-1}(\tilde{\nabla} u^{\text{RM}}), \tag{73}$$

$$\partial_t^2 u^{\text{RM}} - \mu\Delta u^{\text{RM}} + \mu^2 \operatorname{div} A^{-1}\tilde{\nabla} u^{\text{RM}} = 0. \tag{74}$$

With the help of the theory of evolution equations we find that the initial boundary value problem (74), (70a), (70b) possesses a unique solution $u^{\text{RM}} \in \bigcap_{k=0}^2 \mathcal{C}^k\big(\mathbb{R}, H^{2-k}(\tilde{\Omega},\mathbb{R})\big)$. Furthermore, with the help of (73) we obtain $\tilde{v}^{\text{RM}} \in \bigcap_{k=0}^2 \mathcal{C}^k\big(\mathbb{R}, H^{3-k}(\tilde{\Omega},\mathbb{R})\big)$. □

Next, we consider an asymptotic expansion of the solution $(u^{\text{RM}}, \tilde{v}^{\text{RM}})$ with respect to ε. We make the formal ansatz

$$u^{\text{RM}}(\tilde{x},t) := \sum_{k=0}^{\infty} \varepsilon^{2k} u_k^{\text{RM}}(\tilde{x},t), \qquad \tilde{v}^{\text{RM}}(\tilde{x},t) := \sum_{k=0}^{\infty} \varepsilon^{2k} \tilde{v}_k^{\text{RM}}(\tilde{x},t) \tag{75}$$

and insert it into the Reissner–Mindlin problem (70). Then, we obtain

$$\partial_t^2 u_0^{\mathrm{RM}} = 0, \qquad \tilde{v}_0^{\mathrm{RM}} = -\tilde{\nabla} u_0^{\mathrm{RM}}, \tag{76a}$$

$$u_0^{\mathrm{RM}}(\tilde{x},t)\Big|_{\tilde{x}\in\partial\tilde{\Omega}} = 0, \qquad \tilde{v}_0^{\mathrm{RM}}(\tilde{x},t)\Big|_{\tilde{x}\in\partial\tilde{\Omega}} = 0, \tag{76b}$$

$$u_0^{\mathrm{RM}}(\tilde{x},0) = u_0(\tilde{x}), \qquad \partial_t u_0^{\mathrm{RM}}(\tilde{x},0) = u_1(\tilde{x}) \tag{76c}$$

and

$$\partial_t^2 u_{k+1}^{\mathrm{RM}} = \frac{\lambda+2\mu}{3}\Delta\Big(\operatorname{div}\tilde{v}_k^{\mathrm{RM}}\Big), \tag{77a}$$

$$\tilde{v}_{k+1}^{\mathrm{RM}} = -\tilde{\nabla} u_{k+1}^{\mathrm{RM}} + \frac{1}{3\mu}\Big[(\lambda+\mu)\tilde{\nabla}\Big(\operatorname{div}\tilde{v}_k^{\mathrm{RM}}\Big) + \mu\Delta\tilde{v}_k^{\mathrm{RM}}\Big], \tag{77b}$$

$$u_{k+1}^{\mathrm{RM}}(\tilde{x},t)\Big|_{\tilde{x}\in\partial\tilde{\Omega}} = 0, \qquad \tilde{v}_{k+1}^{\mathrm{RM}}(\tilde{x},t)\Big|_{\tilde{x}\in\partial\tilde{\Omega}} = 0, \tag{77c}$$

$$u_{k+1}^{\mathrm{RM}}(\tilde{x},0) = 0, \qquad \partial_t u_{k+1}^{\mathrm{RM}}(\tilde{x},0) = 0. \tag{77d}$$

Furthermore, let $n \in \mathbb{N}$ be fix. Then, we define a formal approximate solution to the simplified dynamic Reissner–Mindlin problem (70) by

$$u_{app}^{\mathrm{RM}}(\tilde{x},t) := \sum_{k=0}^{n} \varepsilon^{2k} u_k^{\mathrm{RM}}(\tilde{x},t), \qquad \tilde{v}_{app}^{\mathrm{RM}}(\tilde{x},t) := \sum_{k=0}^{n} \varepsilon^{2k} \tilde{v}_k^{\mathrm{RM}}(\tilde{x},t). \tag{78}$$

Theorem 9 (Asymptotic Expansion)

1. *The recursion problem* (76), (77) *possesses a unique solution given by*

$$u_0^{RM}(\tilde{x},t) = u_0(\tilde{x}) + t u_1(\tilde{x}), \tag{79a}$$

$$u_{k+1}^{RM}(\tilde{x},t) = \Big(\frac{\lambda+2\mu}{3}\Big)^{k+1} \sum_{l=0}^{k} \binom{k}{l} \frac{(-1)^{l+1}}{\mu^{k-l}} \times \Big(\frac{t^{2l+2}}{(2l+2)!}\Delta^{k+l+2}u_0(\tilde{x}) \frac{t^{2l+3}}{(2l+3)!}\Delta^{k+l+2}u_1(\tilde{x})\Big), \tag{79b}$$

$$\tilde{v}_k^{RM}(\tilde{x},t) = \Big(\frac{\lambda+2\mu}{3}\Big)^{k} \sum_{l=0}^{k} \binom{k}{l} \frac{(-1)^{l+1}}{\mu^{k-l}} \times \Big(\frac{t^{2l}}{(2l)!}\tilde{\nabla}\Delta^{k+l}u_0(\tilde{x}) \frac{t^{2l+1}}{(2l+1)!}\tilde{\nabla}\Delta^{k+l}u_1(\tilde{x})\Big). \tag{79c}$$

2. *Let $T > 0$ and let $s \in \mathbb{N}$ be sufficiently large. Then, the following a–priori estimate holds for all $\varepsilon > 0$:*

$$\big\|u^{RM} - u_{app}^{RM}\big\|_{\cap_{k=0}^{2}\mathcal{C}^k([-T,T],H^{2-k}(\tilde{\Omega},\mathbb{R}))} \le C\varepsilon^{2n+2}\Big(\|u_0\|_{H^s(\tilde{\Omega},\mathbb{R})} + \|u_1\|_{H^s(\tilde{\Omega},\mathbb{R})}\Big), \tag{80a}$$

$$\big\|\tilde{v}^{RM} - \tilde{v}_{app}^{RM}\big\|_{\cap_{k=0}^{2}\mathcal{C}^k([-T,T],H^{3-k}(\tilde{\Omega},\mathbb{R}^2))} \le C\varepsilon^{2n+2}\Big(\|u_0\|_{H^s(\tilde{\Omega},\mathbb{R})} + \|u_1\|_{H^s(\tilde{\Omega},\mathbb{R})}\Big). \tag{80b}$$

In particular, the formal expansion (75) *is actually an asymptotic expansion.*

Proof.

1. Obviously, the unique solution to the equations (76a), (76c) and (77a), (77b), (77d) is given by (79). Since u_0 and u_1 have compact support, the constraints (76b) and (77c) are also satisfied.

2. Let $m \in \mathbb{N}$. We define $(r_m^{\mathrm{RM}}, \tilde{s}_m^{\mathrm{RM}}) \in \bigcap_{k=0}^{2} \mathcal{C}^k\big(\mathbb{R}, H^{2-k}(\tilde{\Omega}, \mathbb{R})\big) \times \mathcal{C}^k\big(\mathbb{R}, H^{3-k}(\tilde{\Omega}, \mathbb{R}^2)\big)$ by

$$r_m^{\mathrm{RM}} := u^{\mathrm{RM}} - \sum_{k=0}^{m} \varepsilon^{2k} u_k^{\mathrm{RM}}, \qquad \tilde{s}_m^{\mathrm{RM}} := \tilde{v}^{\mathrm{RM}} - \sum_{k=0}^{m} \varepsilon^{2k} \tilde{v}_k^{\mathrm{RM}}. \tag{81}$$

By construction $(r_m^{\mathrm{RM}}, \tilde{s}_m^{\mathrm{RM}})$ satisfies the following initial boundary value problem:

$$\partial_t^2 r_m^{\mathrm{RM}} - \mu\Big(\Delta r_m^{\mathrm{RM}} + \operatorname{div} \tilde{s}_m^{\mathrm{RM}}\Big) = 0, \tag{82a}$$

$$- \frac{\varepsilon^2}{3}\Big[(\lambda+\mu)\tilde{\nabla}\Big(\operatorname{div} \tilde{s}_m^{\mathrm{RM}}\Big) + \mu\Delta\tilde{s}_m^{\mathrm{RM}}\Big] + \mu\Big(\tilde{\nabla} r_m^{\mathrm{RM}} + \tilde{s}_m^{\mathrm{RM}}\Big) = \varepsilon^{2m+2}\tilde{\beta}(\tilde{x}, t), \tag{82b}$$

$$r_m^{\mathrm{RM}}(\tilde{x}, t)\Big|_{\tilde{x}\in\partial\tilde{\Omega}} = 0, \qquad \tilde{s}_m^{\mathrm{RM}}(\tilde{x}, t)\Big|_{\tilde{x}\in\partial\tilde{\Omega}} = 0, \tag{82c}$$

$$r_m^{\mathrm{RM}}(\tilde{x}, 0) = 0, \qquad \partial_t r_m^{\mathrm{RM}}(\tilde{x}, 0) = 0 \tag{82d}$$

where

$$\tilde{\beta} := \frac{1}{3}\Big[(\lambda+\mu)\tilde{\nabla}\Big(\operatorname{div} \tilde{v}_m^{\mathrm{RM}}\Big) + \mu\Delta\tilde{v}_m^{\mathrm{RM}}\Big]. \tag{83}$$

We multiply (82a) by $\partial_t r_m^{\mathrm{RM}}(\cdot, t)$ in $L^2(\tilde{\Omega}, \mathbb{R})$ and (82b) by $\partial_t \tilde{s}_m^{\mathrm{RM}}(\cdot, t)$ in $L^2(\tilde{\Omega}, \mathbb{R}^2)$. Then, we add the two equations and integrate with respect to t. With the help of integration by parts we obtain for all $t \in [-T, T]$

$$\begin{aligned}
E(t) &:= \frac{1}{2}\left\|\partial_t r_m^{\mathrm{RM}}(\cdot, t)\right\|^2_{L^2(\tilde{\Omega},\mathbb{R})} + \frac{\mu}{2}\left\|\tilde{\nabla} r_m^{\mathrm{RM}}(\cdot, t) + \tilde{s}_m^{\mathrm{RM}}(\cdot, t)\right\|^2_{L^2(\tilde{\Omega},\mathbb{R}^2)} \\
&\quad + \frac{\varepsilon^2\lambda}{6}\left\|\operatorname{div} \tilde{s}_m^{\mathrm{RM}}(\cdot, t)\right\|^2_{L^2(\tilde{\Omega},\mathbb{R})} \\
&\quad + \frac{\varepsilon^2\mu}{12}\left\|\nabla\tilde{s}_m^{\mathrm{RM}}(\cdot, t) + \Big(\nabla\tilde{s}_m^{\mathrm{RM}}(\cdot, t)\Big)^T\right\|^2_{L^2(\tilde{\Omega},\mathbb{R}^{2\times 2})} \\
&= E(0) + \varepsilon^{2m+2}\int_0^t \Big\langle \partial_t\tilde{s}_m^{\mathrm{RM}}(\cdot, \tau)\Big|\tilde{\beta}(\cdot, \tau)\Big\rangle_{L^2(\tilde{\Omega},\mathbb{R}^2)} \,\mathrm{d}\tau \\
&= E(0) + \varepsilon^{2m+2}\Big\langle \tilde{\nabla} r_m^{\mathrm{RM}}(\cdot, t) + \tilde{s}_m^{\mathrm{RM}}(\cdot, t)\Big|\tilde{\beta}(\cdot, t)\Big\rangle_{L^2(\tilde{\Omega},\mathbb{R}^2)} \\
&\quad - \varepsilon^{2m+2}\Big\langle \tilde{s}_m^{\mathrm{RM}}(\cdot, 0)\Big|\tilde{\beta}(\cdot, 0)\Big\rangle_{L^2(\tilde{\Omega},\mathbb{R}^2)} \\
&\quad - \varepsilon^{2m+2}\int_0^t \Big\langle \tilde{\nabla} r_m^{\mathrm{RM}}(\cdot, \tau) + \tilde{s}_m^{\mathrm{RM}}(\cdot, \tau)\Big|\partial_t\tilde{\beta}(\cdot, \tau)\Big\rangle_{L^2(\tilde{\Omega},\mathbb{R}^2)} \,\mathrm{d}\tau \\
&\quad + \varepsilon^{2m+2}\int_0^t \Big\langle \partial_t r_m^{\mathrm{RM}}(\cdot, \tau)\Big|\operatorname{div}\tilde{\beta}(\cdot, \tau)\Big\rangle_{L^2(\tilde{\Omega},\mathbb{R})} \,\mathrm{d}\tau
\end{aligned}$$

$$\leq \frac{1}{2}E(t) + E(0) + \int_0^t E(\tau)\,\mathrm{d}\tau \\ + C\Big(\|\tilde{s}_m^{\mathrm{RM}}(\cdot,0)\|^2_{L^2(\tilde{\Omega},\mathbb{R}^2)} + \varepsilon^{4m+4} \left\|\tilde{\beta}\right\|^2_{\mathcal{C}^1([-T,T],H^1(\tilde{\Omega},\mathbb{R}^2))} \Big). \tag{84}$$

By construction we have

$$E(0) = \frac{\mu}{2}\|\tilde{s}_m^{\mathrm{RM}}(\cdot,0)\|^2_{L^2(\tilde{\Omega},\mathbb{R}^2)} + \frac{\varepsilon^2\lambda}{6}\|\mathrm{div}\,\tilde{s}_m^{\mathrm{RM}}(\cdot,0)\|^2_{L^2(\tilde{\Omega},\mathbb{R})} \\ + \frac{\varepsilon^2\mu}{12}\left\|\nabla\tilde{s}_m^{\mathrm{RM}}(\cdot,0) + \Big(\nabla\tilde{s}_m^{\mathrm{RM}}(\cdot,0)\Big)^T\right\|^2_{L^2(\tilde{\Omega},\mathbb{R}^{2\times 2})}. \tag{85}$$

Furthermore, $\tilde{s}_m^{\mathrm{RM}}(\cdot,0)$ satisfies the following boundary value problem:

$$-\frac{\varepsilon^2}{3}\Big[(\lambda+\mu)\tilde{\nabla}\Big(\mathrm{div}\,\tilde{s}_m^{\mathrm{RM}}(\cdot,0)\Big) + \mu\Delta\tilde{s}_m^{\mathrm{RM}}(\cdot,0)\Big] + \mu\tilde{s}_m^{\mathrm{RM}}(\cdot,0) \\ = \varepsilon^{2m+2}\tilde{\beta}(\tilde{x},0), \tag{86a}$$

$$\tilde{s}_m^{\mathrm{RM}}(\tilde{x},0)\Big|_{\tilde{x}\in\partial\tilde{\Omega}} = 0. \tag{86b}$$

With the help of elliptic regularity theory we obtain

$$\varepsilon^2\|\tilde{s}_m^{\mathrm{RM}}(\cdot,0)\|^2_{H^1(\tilde{\Omega},\mathbb{R}^2)} + \|\tilde{s}_m^{\mathrm{RM}}(\cdot,0)\|^2_{L^2(\tilde{\Omega},\mathbb{R}^2)} \leq C\varepsilon^{4m+4}\left\|\tilde{\beta}(\cdot,0)\right\|^2_{L^2(\tilde{\Omega},\mathbb{R}^2)}. \tag{87}$$

Now, we insert (85), (87) into (84). This yields

$$E(t) \leq 2\int_0^t E(\tau)\,\mathrm{d}\tau + C\varepsilon^{4m+4}\left\|\tilde{\beta}\right\|^2_{\mathcal{C}^1([-T,T],H^1(\tilde{\Omega},\mathbb{R}^2))}. \tag{88}$$

With the help of Gronwall's inequality we obtain

$$E(t) \leq C\varepsilon^{4m+4}\left\|\tilde{\beta}\right\|^2_{\mathcal{C}^1([-T,T],H^1(\tilde{\Omega},\mathbb{R}^2))} \qquad \forall t\in[-T,T]. \tag{89}$$

Furthermore, with the help of Poincaré's inequality and Korn's inequality we obtain

$$\left\|\tilde{\nabla}r_m^{\mathrm{RM}} + \tilde{s}_m^{\mathrm{RM}}\right\|_{\mathcal{C}^0([-T,T],L^2(\tilde{\Omega},\mathbb{R}))} \leq C\varepsilon^{2m+2}\left\|\tilde{\beta}\right\|_{\mathcal{C}^1([-T,T],H^1(\tilde{\Omega},\mathbb{R}^2))}, \tag{90a}$$

$$\|r_m^{\mathrm{RM}}\|_{\cap_{k=0}^1\mathcal{C}^k([-T,T],H^{1-k}(\tilde{\Omega},\mathbb{R}))} \leq C\varepsilon^{2m+1}\left\|\tilde{\beta}\right\|_{\mathcal{C}^1([-T,T],H^1(\tilde{\Omega},\mathbb{R}^2))}, \tag{90b}$$

$$\|\tilde{s}_m^{\mathrm{RM}}\|_{\mathcal{C}^0([-T,T],H^1(\tilde{\Omega},\mathbb{R}))} \leq C\varepsilon^{2m+1}\left\|\tilde{\beta}\right\|_{\mathcal{C}^1([-T,T],H^1(\tilde{\Omega},\mathbb{R}^2))}. \tag{90c}$$

Next, we formally differentiate the initial boundary value problem (82) with respect to t and note that

$$\partial_t^2 r_m^{\mathrm{RM}}(\tilde{x},0) = \mu\,\mathrm{div}\,\tilde{s}_m^{\mathrm{RM}}(\tilde{x},0). \tag{91}$$

Then, with the help of a Galerkin approximation procedure we can apply the same arguments as above. This yields

$$\left\| \tilde{\nabla} r_m^{\mathrm{RM}} + \tilde{s}_m^{\mathrm{RM}} \right\|_{\mathcal{C}^1([-T,T],L^2(\tilde{\Omega},\mathbb{R}))} \leq C\varepsilon^{2m+1} \left\| \tilde{\beta} \right\|_{\mathcal{C}^2([-T,T],H^1(\tilde{\Omega},\mathbb{R}^2))}, \tag{92a}$$

$$\| r_m^{\mathrm{RM}} \|_{\cap_{k=1}^2 \mathcal{C}^k([-T,T],H^{2-k}(\tilde{\Omega},\mathbb{R}))} \leq C\varepsilon^{2m} \left\| \tilde{\beta} \right\|_{\mathcal{C}^2([-T,T],H^1(\tilde{\Omega},\mathbb{R}^2))}, \tag{92b}$$

$$\| \tilde{s}_m^{\mathrm{RM}} \|_{\mathcal{C}^1([-T,T],H^1(\tilde{\Omega},\mathbb{R}))} \leq C\varepsilon^{2m} \left\| \tilde{\beta} \right\|_{\mathcal{C}^2([-T,T],H^1(\tilde{\Omega},\mathbb{R}^2))}. \tag{92c}$$

Next, we rewrite (82a) as

$$\Delta r_m^{\mathrm{RM}} = \frac{1}{\mu} \partial_t^2 r_m^{\mathrm{RM}} - \operatorname{div} \tilde{s}_m^{\mathrm{RM}}. \tag{93}$$

Then, with the help of elliptic regularity theory and (90), (92) we obtain

$$\begin{aligned} &\| r_m^{\mathrm{RM}} \|_{\mathcal{C}^0([-T,T],H^2(\tilde{\Omega},\mathbb{R}))} \\ &\leq \hat{C} \Big(\| \partial_t^2 r_m^{\mathrm{RM}} \|_{\mathcal{C}^0([-T,T],L^2(\tilde{\Omega},\mathbb{R}))} + \| \operatorname{div} \tilde{s}_m^{\mathrm{RM}} \|_{\mathcal{C}^0([-T,T],L^2(\tilde{\Omega},\mathbb{R}))} \Big) \\ &\leq C\varepsilon^{2m} \left\| \tilde{\beta} \right\|_{\mathcal{C}^2([-T,T],H^1(\tilde{\Omega},\mathbb{R}^2))}. \end{aligned} \tag{94}$$

Next, we rewrite (82b) as

$$-\frac{\varepsilon^2}{3} \Big[(\lambda+\mu) \tilde{\nabla} \Big(\operatorname{div} \tilde{s}_m^{\mathrm{RM}} \Big) + \mu \Delta \tilde{s}_m^{\mathrm{RM}} \Big] + \mu \tilde{s}_m^{\mathrm{RM}} = \varepsilon^{2m+2} \tilde{\beta}(\tilde{x},t) - \mu \tilde{\nabla} r_m^{\mathrm{RM}}. \tag{95}$$

Then, with the help of elliptic regularity theory and (92), (94) we obtain

$$\begin{aligned} &\varepsilon \| \tilde{s}_m^{\mathrm{RM}} \|_{\mathcal{C}^2([-T,T],H^1(\tilde{\Omega},\mathbb{R}^2))} + \| \tilde{s}_m^{\mathrm{RM}} \|_{\mathcal{C}^2([-T,T],L^2(\tilde{\Omega},\mathbb{R}^2))} \\ &\leq \hat{C} \Big(\varepsilon^{2m+2} \left\| \tilde{\beta} \right\|_{\mathcal{C}^2([-T,T],L^2(\tilde{\Omega},\mathbb{R}))} + \varepsilon^{-1} \| r_m^{\mathrm{RM}} \|_{\mathcal{C}^2([-T,T],L^2(\tilde{\Omega},\mathbb{R}^2))} \Big) \\ &\leq C\varepsilon^{2m-1} \left\| \tilde{\beta} \right\|_{\mathcal{C}^2([-T,T],H^1(\tilde{\Omega},\mathbb{R}))}, \end{aligned} \tag{96a}$$

$$\begin{aligned} &\varepsilon^2 \| \tilde{s}_m^{\mathrm{RM}} \|_{\cap_{k=0}^1 \mathcal{C}^k([-T,T],H^{3-k}(\tilde{\Omega},\mathbb{R}^2))} \\ &\leq \hat{C} \Big(\varepsilon^{2m+2} \left\| \tilde{\beta} \right\|_{\mathcal{C}^1([-T,T],H^1(\tilde{\Omega},\mathbb{R}))} + \left\| \tilde{\nabla} r_m^{\mathrm{RM}} + \tilde{s}_m^{\mathrm{RM}} \right\|_{\cap_{k=0}^1 \mathcal{C}^k([-T,T],H^{1-k}(\tilde{\Omega},\mathbb{R}))} \Big) \\ &\leq C\varepsilon^{2m} \left\| \tilde{\beta} \right\|_{\mathcal{C}^2([-T,T],H^1(\tilde{\Omega},\mathbb{R}))}. \end{aligned} \tag{96b}$$

Finally, we estimate $u^{\mathrm{RM}} - u_{app}^{\mathrm{RM}}$ and $\tilde{v}^{\mathrm{RM}} - \tilde{v}_{app}^{\mathrm{RM}}$. By construction we have

$$u^{\mathrm{RM}} - u_{app}^{\mathrm{RM}} = r_{n+2}^{\mathrm{RM}} + \sum_{k=n+1}^{n+2} \varepsilon^{2k} u_k^{\mathrm{RM}}, \qquad \tilde{v}^{\mathrm{RM}} - \tilde{v}_{app}^{\mathrm{RM}} = \tilde{s}_{n+2}^{\mathrm{RM}} + \sum_{k=n+1}^{n+2} \varepsilon^{2k} \tilde{v}_k^{\mathrm{RM}}. \tag{97}$$

Now, the desired statement (80) follows from (79), (83), (92), (94), (96) and (97).

□

Comparison of the Simplified Dynamic Problems

Theorem 10 (Comparison Theorem)
Let $(u^{KL}, \tilde{v}^{KL})$ be the solution to the simplified dynamic Kirchhoff–Love problem (52), (53), let $(u^{RM}, \tilde{v}^{RM})$ be the solution to the simplified dynamic Reissner–Mindlin problem (70) and let $s \in \mathbb{N}$ be sufficiently large.
Then, the following a–priori estimates hold for all $\varepsilon > 0$:

$$\left\| \tilde{\nabla} u^{RM} + \tilde{v}^{RM} \right\|_{\cap_{k=0}^{1} \mathcal{C}^k([-T,T],H^{1-k}(\tilde{\Omega},\mathbb{R}))} \leq C\varepsilon^2 \Big(\|u_0\|_{H^s(\tilde{\Omega},\mathbb{R})} + \|u_1\|_{H^s(\tilde{\Omega},\mathbb{R})} \Big), \tag{98a}$$

$$\|u^{RM} - u^{KL}\|_{\cap_{k=0}^{2} \mathcal{C}^k([-T,T],H^{2-k}(\tilde{\Omega},\mathbb{R}))} \leq C\varepsilon^4 \Big(\|u_0\|_{H^s(\tilde{\Omega},\mathbb{R})} + \|u_1\|_{H^s(\tilde{\Omega},\mathbb{R})} \Big), \tag{98b}$$

$$\|\tilde{v}^{RM} - \tilde{v}^{KL}\|_{\cap_{k=0}^{2} \mathcal{C}^k([-T,T],H^{3-k}(\tilde{\Omega},\mathbb{R}))} \leq C\varepsilon^2 \Big(\|u_0\|_{H^s(\tilde{\Omega},\mathbb{R})} + \|u_1\|_{H^s(\tilde{\Omega},\mathbb{R})} \Big). \tag{98c}$$

In particular, the Kirchhoff–Love solution $(u^{KL}, \tilde{v}^{KL})$ and the Reissner–Mindlin solution $(u^{RM}, \tilde{v}^{RM})$ coincide as $\varepsilon \to 0$.

Proof.
By (53), (58) and (79) we have

$$u_0^{\mathrm{KL}} = u_0^{\mathrm{RM}}, \qquad u_1^{\mathrm{KL}} = u_1^{\mathrm{RM}}, \qquad \tilde{v}_0^{\mathrm{KL}} = \tilde{v}_0^{\mathrm{RM}} = -\tilde{\nabla} u_0^{\mathrm{RM}}. \tag{99}$$

Now, the desired statement (98) follows from (59), (80). □

6 The Full Dynamic Problem

In this section we consider the full dynamic problems in the sense of Kirchhoff–Love and Reissner–Mindlin. As in the previous section we assume that the external body force vanishes, i.e. $f(\tilde{x}) = 0$.

The Full Dynamic Problem in the Sense of Kirchhoff–Love

The full dynamic problem in the sense of Kirchhoff–Love reads

$$\partial_t^2 u^{\mathrm{KL}} - \frac{\varepsilon^2}{3} \Big[\partial_t^2 \Delta u^{\mathrm{KL}} - (\lambda + 2\mu) \Delta\Delta u^{\mathrm{KL}} \Big] = 0, \tag{100a}$$

$$u^{\mathrm{KL}}(\tilde{x}, t)\Big|_{\tilde{x} \in \partial\tilde{\Omega}} = 0, \qquad \tilde{\nabla} u^{\mathrm{KL}}(\tilde{x}, t)\Big|_{\tilde{x} \in \partial\tilde{\Omega}} = 0, \tag{100b}$$

$$u^{\mathrm{KL}}(\tilde{x}, 0) = u_0(\tilde{x}), \qquad \partial_t u^{\mathrm{KL}}(\tilde{x}, 0) = u_1(\tilde{x}). \tag{100c}$$

Furthermore, Kirchhoff's normal hypothesis reads

$$\tilde{v}^{\mathrm{KL}}(\tilde{x}, t) := -\tilde{\nabla} u^{\mathrm{KL}}(\tilde{x}, t). \tag{101}$$

From (100) we formally obtain that $\partial_t^2 u^{\mathrm{KL}}(\cdot, 0)$ satisfies the following boundary value problem:

$$\partial_t^2 u^{\mathrm{KL}}(\cdot, 0) - \frac{\varepsilon^2}{3} \partial_t^2 \Delta u^{\mathrm{KL}}(\cdot, 0) = -\frac{\varepsilon^2 (\lambda + 2\mu)}{3} \Delta\Delta u_0^{\mathrm{KL}}(\tilde{x}), \tag{102a}$$

$$\partial_t^2 u^{\text{KL}}(\tilde{x},0)\Big|_{\tilde{x}\in\partial\tilde{\Omega}} = 0. \tag{102b}$$

But, in general this contradicts the second boundary condition in (100b). Consequently, we can not expect that the regularity of the solution u^{KL} will allow us to take the trace of $\partial_t^2 u^{\text{KL}}(\cdot,t)$ on $\partial\tilde{\Omega}$.

Theorem 11 (Existence, Uniqueness, Regularity)
The full dynamic Kirchhoff–Love problem (100) *possesses a unique weak solution* $u^{KL} \in \bigcap_{k=0}^{2} \mathcal{C}^k\big(\mathbb{R}, H^{3-k}(\tilde{\Omega},\mathbb{R})\big)$.
Proof.
Let $\mathfrak{H} := H_0^1(\tilde{\Omega},\mathbb{R})$, $\mathfrak{V} := H_0^2(\tilde{\Omega},\mathbb{R})$. With the help of Poincaré's inequality we can define equivalent scalar products on $\mathfrak{H},\mathfrak{V}$ by

$$\langle \mathfrak{u}|\mathfrak{v}\rangle_{\mathfrak{H}} := \langle \mathfrak{u}|\mathfrak{v}\rangle_{L^2(\tilde{\Omega},\mathbb{R})} + \frac{\varepsilon^2}{3}\left\langle \tilde{\nabla}\mathfrak{u}\middle|\tilde{\nabla}\mathfrak{v}\right\rangle_{L^2(\tilde{\Omega},\mathbb{R}^2)} \qquad \forall \mathfrak{u},\mathfrak{v}\in\mathfrak{H}, \tag{103a}$$

$$\langle \mathfrak{u}|\mathfrak{v}\rangle_{\mathfrak{V}} := \frac{\varepsilon^2(\lambda+2\mu)}{3}\left\langle \nabla^2\mathfrak{u}\middle|\nabla^2\mathfrak{v}\right\rangle_{L^2(\tilde{\Omega},\mathbb{R}^{2\times 2})} \qquad \forall \mathfrak{u},\mathfrak{v}\in\mathfrak{V}. \tag{103b}$$

Then, for test functions $y \in \mathcal{C}_0^1\big([0,\infty),\mathfrak{V}\big)$ the weak formulation of the Kirchhoff–Love problem (100) reads

$$-\int_0^\infty \langle \partial_t y(\cdot,t)|\partial_t u^{\text{KL}}(\cdot,t)\rangle_{\mathfrak{H}}\,\mathrm{d}t - \langle y(\cdot,0)|u_1\rangle_{\mathfrak{H}} + \int_0^\infty \langle y(\cdot,t)|u^{\text{KL}}(\cdot,t)\rangle_{\mathfrak{V}}\,\mathrm{d}t$$
$$= 0, \tag{104a}$$

$$u^{\text{KL}}(\tilde{x},0) = u_0(\tilde{x}). \tag{104b}$$

With the help of the theory of evolution equations we find that (104) possesses a unique solution $u^{\text{KL}} \in \mathcal{C}^0\big([0,\infty),\mathfrak{V}\big)\cap\mathcal{C}^1\big([0,\infty),\mathfrak{H}\big)$ and with the help of a time reversal argument we find that actually $u^{\text{KL}} \in \mathcal{C}^0\big(\mathbb{R},\mathfrak{V}\big)\cap\mathcal{C}^1\big(\mathbb{R},\mathfrak{H}\big)$.
Furthermore, the boundary value problem (102) possesses a unique weak solution $u_2 \equiv u^{\text{KL}}(\cdot,0) \in \mathfrak{H}$. But this is a compatibility condition to the Kirchhoff–Love problem (100). Consequently, with the help of the theory of evolution equations we find that actually $u^{\text{KL}} \in \mathcal{C}^1\big(\mathbb{R},\mathfrak{V}\big)\cap\mathcal{C}^2\big(\mathbb{R},\mathfrak{H}\big)$.
Next, we rewrite (100a) as

$$\frac{\varepsilon^2(\lambda+2\mu)}{3}\Delta\Delta u^{\text{KL}} = -\partial_t^2 u^{\text{KL}} + \frac{\varepsilon^2}{3}\partial_t^2\Delta u^{\text{KL}} =: \alpha(\tilde{x},t) \tag{105}$$

where $\alpha \in \mathcal{C}^0\big(\mathbb{R},H^{-1}(\tilde{\Omega},\mathbb{R})\big)$. With the help of elliptic regularity theory we find that $u^{\text{KL}} \in \mathcal{C}^0\big(\mathbb{R},H^3(\tilde{\Omega},\mathbb{R})\big)$. □

Next, we consider an asymptotic expansion of the solution u^{KL} with respect to ε. We make the formal ansatz

$$u^{\text{KL}}(\tilde{x},t) := \sum_{k=0}^{\infty} \varepsilon^{2k} u_k^{\text{KL}}(\tilde{x},t) \tag{106}$$

and insert it into the Kirchhoff–Love problem (100). Then, we obtain

$$\partial_t^2 u_0^{\mathrm{KL}} = 0, \tag{107a}$$

$$u_0^{\mathrm{KL}}(\tilde{x},t)\Big|_{\tilde{x}\in\partial\tilde{\Omega}} = 0, \qquad \tilde{\nabla} u_0^{\mathrm{KL}}(\tilde{x},t)\Big|_{\tilde{x}\in\partial\tilde{\Omega}} = 0, \tag{107b}$$

$$u_0^{\mathrm{KL}}(\tilde{x},0) = u_0(\tilde{x}), \qquad \partial_t u_0^{\mathrm{KL}}(\tilde{x},0) = u_1(\tilde{x}) \tag{107c}$$

and

$$\partial_t^2 u_{k+1}^{\mathrm{KL}} = \frac{1}{3}\Big[\partial_t^2 \Delta u_k^{\mathrm{KL}} - (\lambda+2\mu)\Delta\Delta u_k^{\mathrm{KL}}\Big], \tag{108a}$$

$$u_{k+1}^{\mathrm{KL}}(\tilde{x},t)\Big|_{\tilde{x}\in\partial\tilde{\Omega}} = 0, \qquad \tilde{\nabla} u_{k+1}^{\mathrm{KL}}(\tilde{x},t)\Big|_{\tilde{x}\in\partial\tilde{\Omega}} = 0, \tag{108b}$$

$$u_{k+1}^{\mathrm{KL}}(\tilde{x},0) = 0, \qquad \partial_t u_{k+1}^{\mathrm{KL}}(\tilde{x},0) = 0. \tag{108c}$$

Furthermore, let $n \in \mathbb{N}$ be fix. Then, we define a formal approximate solution to the full dynamic Kirchhoff–Love problem (100) by

$$u_{app}^{\mathrm{KL}}(\tilde{x},t) := \sum_{k=0}^{n} \varepsilon^{2k} u_k^{\mathrm{KL}}(\tilde{x},t). \tag{109}$$

Theorem 12 (Asymptotic Expansion)

1. *The recursion problem* (107), (108) *possesses a unique solution given by*

$$u_0^{KL}(\tilde{x},t) = u_0(\tilde{x}) + t u_1(\tilde{x}), \tag{110a}$$

$$\begin{aligned} u_{k+1}^{KL}(\tilde{x},t) = \frac{1}{3^{k+1}} \sum_{l=0}^{k} \binom{k}{l} (-1)^{l+1}(\lambda+2\mu)^{l+1} \\ \times \Big(\frac{t^{2l+2}}{(2l+2)!}\Delta^{l+k+2} u_0(\tilde{x}) + \frac{t^{2l+3}}{(2l+3)!}\Delta^{l+k+2} u_1(\tilde{x})\Big). \end{aligned} \tag{110b}$$

2. *Let $T>0$ and let $s \in \mathbb{N}$ be sufficiently large.*
   Then, the following a–priori estimate holds for all $\varepsilon > 0$:

$$\begin{aligned} &\big\|u^{KL} - u_{app}^{KL}\big\|_{\cap_{k=0}^{2} C^k([-T,T],H^{3-k}(\tilde{\Omega},\mathbb{R}))} \\ &\le C\varepsilon^{2n+2}\Big(\|u_0\|_{H^s(\tilde{\Omega},\mathbb{R})} + \|u_1\|_{H^s(\tilde{\Omega},\mathbb{R})}\Big). \end{aligned} \tag{111}$$

   In particular, the formal expansion (106) *is actually an asymptotic expansion.*

Proof.

1. Obviously, the unique solution to the equations (107a), (107c) and (108a), (108c) is given by (110). Since u_0 and u_1 have compact support, the constraints (107b) and (107b) are also satisfied.

2. Let $m \in \mathbb{N}$. We define $r_m^{\text{KL}} \in \bigcap_{k=0}^{2} \mathcal{C}^k\big(\mathbb{R}, H^{3-k}(\tilde{\Omega}, \mathbb{R})\big)$ by

$$r_m^{\text{KL}}(\tilde{x}, t) := u^{\text{KL}}(\tilde{x}, t) - \sum_{k=0}^{m} \varepsilon^{2k} u_k^{\text{KL}}(\tilde{x}, t). \tag{112}$$

By construction r_m^{KL} satisfies the following initial boundary value problem:

$$\partial_t^2 r_m^{\text{KL}} - \frac{\varepsilon^2}{3}\Big[\partial_t^2 \Delta r_m^{\text{KL}} - (\lambda + 2\mu)\Delta\Delta r_m^{\text{KL}}\Big] = \varepsilon^{2m+2}\alpha(\tilde{x}, t), \tag{113a}$$

$$r_m^{\text{KL}}(\tilde{x}, t)\Big|_{\tilde{x}\in\partial\tilde{\Omega}} = 0, \qquad \tilde{\nabla} r_m^{\text{KL}}(\tilde{x}, t)\Big|_{\tilde{x}\in\partial\tilde{\Omega}} = 0, \tag{113b}$$

$$r_m^{\text{KL}}(\tilde{x}, 0) = 0, \qquad \partial_t r_m^{\text{KL}}(\tilde{x}, 0) = 0 \tag{113c}$$

where

$$\alpha := \frac{1}{3}\Big[\partial_t^2 \Delta u_m^{\text{KL}} - (\lambda + 2\mu)\Delta\Delta u_m^{\text{KL}}\Big]. \tag{114}$$

We multiply (113) by $\partial_t r_m^{\text{KL}}(\cdot, t)$ in $L^2(\tilde{\Omega}, \mathbb{R})$ and integrate with respect to t. With the help of integration by parts we obtain for all $t \in [-T, T]$

$$\begin{aligned}
&\frac{1}{2}\left\|\partial_t r_m^{\text{KL}}(\cdot, t)\right\|_{L^2(\tilde{\Omega},\mathbb{R})}^2 + \frac{\varepsilon^2}{6}\left\|\partial_t \tilde{\nabla} r_m^{\text{KL}}(\cdot, t)\right\|_{L^2(\tilde{\Omega},\mathbb{R}^2)}^2 \\
&+ \frac{\varepsilon^2(\lambda + 2\mu)}{6}\left\|\nabla^2 r_m^{\text{KL}}(\cdot, t)\right\|_{L^2(\tilde{\Omega},\mathbb{R}^{2\times 2})}^2 \\
&= \varepsilon^{2m+2}\int_0^t \left\langle \partial_t r_m^{\text{KL}}(\cdot, t) \middle| \alpha(\cdot, t)\right\rangle_{L^2(\tilde{\Omega},\mathbb{R})} \,\mathrm{d}t \\
&\leq \varepsilon^{2m+2}\left\|\partial_t r_m^{\text{KL}}\right\|_{\mathcal{C}^0([-T,T],L^2(\tilde{\Omega},\mathbb{R}))}\left\|\alpha\right\|_{L^1([-T,T],L^2(\tilde{\Omega},\mathbb{R}))} \\
&\leq \frac{1}{4}\left\|\partial_t r_m^{\text{KL}}\right\|_{\mathcal{C}^0([-T,T],L^2(\tilde{\Omega},\mathbb{R}))}^2 + C\varepsilon^{4m+4}\left\|\alpha\right\|_{L^1([-T,T],L^2(\tilde{\Omega},\mathbb{R}))}^2.
\end{aligned} \tag{115}$$

With the help of (115) and Poincaré's inequality we obtain

$$\left\|\partial_t r_m^{\text{KL}}\right\|_{\mathcal{C}^0([-T,T],L^2(\tilde{\Omega},\mathbb{R}))} \leq C\varepsilon^{2m+2}\left\|\alpha\right\|_{L^1([-T,T],L^2(\tilde{\Omega},\mathbb{R}))}, \tag{116a}$$

$$\left\|\partial_t r_m^{\text{KL}}\right\|_{\mathcal{C}^0([-T,T],H^1(\tilde{\Omega},\mathbb{R}))} \leq C\varepsilon^{2m+1}\left\|\alpha\right\|_{L^1([-T,T],L^2(\tilde{\Omega},\mathbb{R}))}, \tag{116b}$$

$$\left\|r_m^{\text{KL}}\right\|_{\mathcal{C}^0([-T,T],H^2(\tilde{\Omega},\mathbb{R}))} \leq C\varepsilon^{2m+1}\left\|\alpha\right\|_{L^1([-T,T],L^2(\tilde{\Omega},\mathbb{R}))}. \tag{116c}$$

Next, we formally differentiate the initial boundary value problem (113) with respect to t and note that $\partial_t^2 r_m^{\text{KL}}(\cdot, 0)$ satisfies the following boundary value problem:

$$\partial_t^2 r_m^{\text{KL}}(\cdot, 0) - \frac{\varepsilon^2}{3}\partial_t^2 \Delta r_m^{\text{KL}}(\cdot, 0) = \varepsilon^{2m+2}\alpha(\tilde{x}, 0), \tag{117a}$$

$$\partial_t^2 r_m^{\mathrm{KL}}(\tilde{x},0)\Big|_{\tilde{x}\in\partial\tilde{\Omega}} = 0. \tag{117b}$$

With the help of elliptic regularity theory and (110) we obtain

$$\left\|\partial_t^2 r_m^{\mathrm{KL}}(\cdot,0)\right\|_{L^2(\tilde{\Omega},\mathbb{R})} + \varepsilon \left\|\partial_t^2 r_m^{\mathrm{KL}}(\cdot,0)\right\|_{H^1(\tilde{\Omega},\mathbb{R})} \le C\varepsilon^{2m+2} \left\|\alpha(\cdot,0)\right\|_{L^2(\tilde{\Omega},\mathbb{R}))}. \tag{118}$$

Now, with the help of a Galerkin approximation procedure we can apply the same arguments as above. This yields

$$\left\|\partial_t^2 r_m^{\mathrm{KL}}\right\|_{\mathcal{C}^0([-T,T],L^2(\tilde{\Omega},\mathbb{R}))} \le C\varepsilon^{2m+2} \left\|\alpha\right\|_{\mathcal{C}^0([-T,T],L^2(\tilde{\Omega},\mathbb{R}))}, \tag{119a}$$

$$\left\|\partial_t^2 r_m^{\mathrm{KL}}\right\|_{\mathcal{C}^0([-T,T],H^1(\tilde{\Omega},\mathbb{R}))} \le C\varepsilon^{2m+1} \left\|\alpha\right\|_{\mathcal{C}^0([-T,T],L^2(\tilde{\Omega},\mathbb{R}))}, \tag{119b}$$

$$\left\|\partial_t r_m^{\mathrm{KL}}\right\|_{\mathcal{C}^0([-T,T],H^2(\tilde{\Omega},\mathbb{R}))} \le C\varepsilon^{2m+1} \left\|\alpha\right\|_{\mathcal{C}^0([-T,T],L^2(\tilde{\Omega},\mathbb{R}))}. \tag{119c}$$

Next, we rewrite (113a) as

$$\frac{\varepsilon^2(\lambda+2\mu)}{3}\Delta\Delta r_m^{\mathrm{KL}} = -\partial_t^2 r_m^{\mathrm{KL}} + \frac{\varepsilon^2}{3}\partial_t^2\Delta r_m^{\mathrm{KL}} + \varepsilon^{2m+2}\alpha(\tilde{x},t). \tag{120}$$

Then, with the help of elliptic regularity theory and (119) we obtain

$$\begin{aligned}
&\left\|r_m^{\mathrm{KL}}\right\|_{\mathcal{C}^0([-T,T],H^3(\tilde{\Omega},\mathbb{R}))} \\
&\le \hat{C}\Big(\varepsilon^{-2}\left\|\partial_t r_m^{\mathrm{KL}}\right\|_{\mathcal{C}^0([-T,T],H^{-1}(\tilde{\Omega},\mathbb{R}))} + \left\|\partial_t r_m^{\mathrm{KL}}\right\|_{\mathcal{C}^0([-T,T],H^1(\tilde{\Omega},\mathbb{R}))} \\
&\qquad + \varepsilon^{2m}\left\|\alpha\right\|_{\mathcal{C}^0([-T,T],H^{-1}(\tilde{\Omega},\mathbb{R}))}\Big) \\
&\le C\varepsilon^{2m}\left\|\alpha\right\|_{\mathcal{C}^0([-T,T],L^2(\tilde{\Omega},\mathbb{R}))}.
\end{aligned} \tag{121}$$

Finally, we estimate $u^{\mathrm{KL}} - u_{app}^{\mathrm{KL}}$. By construction we have

$$u^{\mathrm{KL}} - u_{app}^{\mathrm{KL}} = r_{n+1}^{\mathrm{KL}} + \varepsilon^{2n+2}u_{n+1}^{\mathrm{KL}}. \tag{122}$$

Now, the desired statement (111) follows from (110), (114) (119), (121) and (122).

□

The Full Dynamic Problem in the Sense of Reissner–Mindlin

The full dynamic problem in the sense of Reissner–Mindlin reads

$$\partial_t^2 u^{\mathrm{RM}} - \mu\Big(\Delta u^{\mathrm{RM}} + \operatorname{div}\tilde{v}^{\mathrm{RM}}\Big) = 0, \tag{123a}$$

$$\frac{\varepsilon^2}{3}\Big[\partial_t^2\tilde{v}^{\mathrm{RM}} - (\lambda+\mu)\tilde{\nabla}\Big(\operatorname{div}\tilde{v}^{\mathrm{RM}}\Big) - \mu\Delta\tilde{v}^{\mathrm{RM}}\Big] + \mu\Big(\tilde{\nabla}u^{\mathrm{RM}} + \tilde{v}^{\mathrm{RM}}\Big) = 0, \tag{123b}$$

$$u^{\mathrm{RM}}(\tilde{x},t)\Big|_{\tilde{x}\in\partial\tilde{\Omega}} = 0, \qquad \tilde{v}^{\mathrm{RM}}(\tilde{x},t)\Big|_{\tilde{x}\in\partial\tilde{\Omega}} = 0, \tag{123c}$$

$$u^{\mathrm{RM}}(\tilde{x},0) = u_0(\tilde{x}), \qquad \partial_t u^{\mathrm{RM}}(\tilde{x},0) = u_1(\tilde{x}), \tag{123d}$$

$$\tilde{v}^{\mathrm{RM}}(\tilde{x},0) = -\tilde{\nabla}u_0(\tilde{x}), \qquad \partial_t\tilde{v}^{\mathrm{RM}}(\tilde{x},0) = -\tilde{\nabla}u_1(\tilde{x}). \tag{123e}$$

Theorem 13 (Existence, Uniqueness, Regularity)
The full dynamic problem in the sense of Reissner–Mindlin possesses a unique weak solution $(u^{RM}, \tilde{v}^{RM}) \in \mathcal{C}^\infty\big(\mathbb{R} \times \overline{\tilde{\Omega}}, \mathbb{R}\big) \times \mathcal{C}^\infty\big(\mathbb{R} \times \overline{\tilde{\Omega}}, \mathbb{R}^2\big)$.
Proof.
This is a well known fact from the theory of evolution equations. □

Next, we consider an asymptotic expansion of the solution $(u^{\mathrm{RM}}, \tilde{v}^{\mathrm{RM}})$ with respect to ε. We make the formal ansatz

$$u^{\mathrm{RM}}(\tilde{x}, t) := \sum_{k=0}^{\infty} \varepsilon^{2k} u_k^{\mathrm{RM}}(\tilde{x}, t), \qquad \tilde{v}^{\mathrm{RM}}(\tilde{x}, t) := \sum_{k=0}^{\infty} \varepsilon^{2k} \tilde{v}_k^{\mathrm{RM}}(\tilde{x}, t) \tag{124}$$

and insert it into the Reissner–Mindlin problem (123). Then, we obtain

$$\partial_t^2 u_0^{\mathrm{RM}} = 0, \qquad \tilde{v}_0^{\mathrm{RM}} = -\tilde{\nabla} u_0^{\mathrm{RM}}, \tag{125a}$$

$$u_0^{\mathrm{RM}}(\tilde{x}, t)\Big|_{\tilde{x} \in \partial\tilde{\Omega}} = 0, \qquad \tilde{v}_0^{\mathrm{RM}}(\tilde{x}, t)\Big|_{\tilde{x} \in \partial\tilde{\Omega}} = 0, \tag{125b}$$
$$u_0^{\mathrm{RM}}(\tilde{x}, 0) = u_0(\tilde{x}), \qquad \partial_t u_0^{\mathrm{RM}}(\tilde{x}, 0) = u_1(\tilde{x}), \tag{125c}$$
$$\tilde{v}_0^{\mathrm{RM}}(\tilde{x}, 0) = -\tilde{\nabla} u_0(\tilde{x}), \qquad \partial_t \tilde{v}_0^{\mathrm{RM}}(\tilde{x}, 0) = -\tilde{\nabla} u_1(\tilde{x}) \tag{125d}$$

and

$$\partial_t^2 u_{k+1}^{\mathrm{RM}} = \frac{\lambda + 2\mu}{3} \Delta\Big(\operatorname{div} \tilde{v}_k^{\mathrm{RM}}\Big) - \frac{1}{3}\partial_t^2\Big(\operatorname{div} \tilde{v}_k^{\mathrm{RM}}\Big), \tag{126a}$$
$$\tilde{v}_{k+1}^{\mathrm{RM}} = -\tilde{\nabla} u_{k+1}^{\mathrm{RM}} + \frac{1}{3\mu}\Big[(\lambda + \mu)\tilde{\nabla}\Big(\operatorname{div} \tilde{v}_k^{\mathrm{RM}}\Big) + \mu \Delta \tilde{v}_k^{\mathrm{RM}} - \partial_t^2 \tilde{v}_k^{\mathrm{RM}}\Big], \tag{126b}$$

$$u_{k+1}^{\mathrm{RM}}(\tilde{x}, t)\Big|_{\tilde{x} \in \partial\tilde{\Omega}} = 0, \qquad \tilde{v}_{k+1}^{\mathrm{RM}}(\tilde{x}, t)\Big|_{\tilde{x} \in \partial\tilde{\Omega}} = 0, \tag{126c}$$
$$u_{k+1}^{\mathrm{RM}}(\tilde{x}, 0) = 0, \qquad \partial_t u_{k+1}^{\mathrm{RM}}(\tilde{x}, 0) = 0, \tag{126d}$$
$$\tilde{v}_{k+1}^{\mathrm{RM}}(\tilde{x}, 0) = 0, \qquad \partial_t \tilde{v}_{k+1}^{\mathrm{RM}}(\tilde{x}, 0) = 0. \tag{126e}$$

Furthermore, let $n \in \mathbb{N}$ be fix. Then, we define a formal approximate solution to the full dynamic Reissner–Mindlin problem (123) by

$$u_{app}^{\mathrm{RM}}(\tilde{x}, t) := \sum_{k=0}^{n} \varepsilon^{2k} u_k^{\mathrm{RM}}(\tilde{x}, t), \qquad \tilde{v}_{app}^{\mathrm{RM}}(\tilde{x}, t) := \sum_{k=0}^{n} \varepsilon^{2k} \tilde{v}_k^{\mathrm{RM}}(\tilde{x}, t). \tag{127}$$

Theorem 14 (Asymptotic Expansion)

1. *The recursion problem* (125) *possesses a unique solution given by*

$$u_0^{RM}(\tilde{x}, t) = u_0(\tilde{x}) + t u_1(\tilde{x}), \tag{128a}$$
$$\tilde{v}_0^{RM}(\tilde{x}, t) = -\tilde{\nabla} u_0(\tilde{x}) - t\tilde{\nabla} u_1(\tilde{x}). \tag{128b}$$

In particular, the zero–order terms in the formal expansion (124) *of the Reissner–Mindlin solution are exactly the zero–order terms in the asymptotic expansion* (106) *of the Kirchhoff–Love solution, i.e. we have*

$$(u_0^{RM}, \tilde{v}_0^{RM}) = (u_0^{KL}, \tilde{v}_0^{KL}). \tag{129}$$

2. *When we exclude the constraint* (126e) *from* (126)*, then the remaining recursion problem possesses a unique solution given by*

$$\begin{aligned} &u_{k+1}^{RM}(\tilde{x}, t) \\ &= \int_0^t \int_0^{s_1} \Big[\frac{\lambda + 2\mu}{3} \Delta\Delta w_k(\tilde{x}, s_2) - \frac{1}{3}\partial_t^2\Big(\Delta w_k(\tilde{x}, s_2)\Big)\Big] \mathrm{d}s_2 \mathrm{d}s_1, && (130a) \\ &\tilde{v}_{k+1}^{RM}(\tilde{x}, t) = \tilde{\nabla} w_{k+1}(\tilde{x}, t) && (130b) \end{aligned}$$

where w_k *is given recursively by*

$$\begin{aligned} w_0(\tilde{x}, t) &= -u_0(\tilde{x}) - t u_1(\tilde{x}), && (131a) \\ w_{k+1}(\tilde{x}, t) &= \int_0^t \int_0^{s_1} \Big[\frac{1}{3}\partial_t^2\Big(\Delta w_k(\tilde{x}, s_2)\Big) - \frac{\lambda + 2\mu}{3} \Delta\Delta w_k(\tilde{x}, s_2)\Big] \mathrm{d}s_2 \mathrm{d}s_1 \\ &\quad + \frac{\lambda + 2\mu}{3\mu} \Delta w_k(\tilde{x}, t) - \frac{1}{3\mu}\partial_t^2 w_k(\tilde{x}, t). && (131b) \end{aligned}$$

3. *For* $k = 0$ *the constraint* (126e) *is satisfied if and only if*

$$\Delta u_0 = \textit{constant}, \qquad \Delta u_1 = \textit{constant}. \tag{132}$$

Then, the solution to the full dynamic Reissner–Mindlin problem (123) *is given by the zero–order terms in the formal expansion* (124)*, i.e. we have*

$$(u^{RM}, \tilde{v}^{RM}) = (u_0^{RM}, \tilde{v}_0^{RM}). \tag{133}$$

4. *The formal approximate solution* $(u_{app}^{RM}, \tilde{v}_{app}^{RM})$ *satisfies the following initial boundary value problem:*

$$\begin{aligned} &\partial_t^2 u_{app}^{RM} - \mu\Big(\Delta u_{app}^{RM} + \operatorname{div} \tilde{v}_{app}^{RM}\Big) = 0, && (134a) \\ &\frac{\varepsilon^2}{3}\Big[\partial_t^2 \tilde{v}_{app}^{RM} - (\lambda + \mu)\tilde{\nabla}\Big(\operatorname{div} \tilde{v}_{app}^{RM}\Big) - \mu\Delta \tilde{v}_{app}^{RM}\Big] + \mu\Big(\tilde{\nabla} u_{app}^{RM} + \tilde{v}_{app}^{RM}\Big) \\ &= \varepsilon^{2n+2}\tilde{\beta}(\tilde{x}, t), && (134b) \end{aligned}$$

$$\begin{aligned} &u_{app}^{RM}(\tilde{x}, t)\Big|_{\tilde{x}\in\partial\tilde{\Omega}} = 0, && \tilde{v}_{app}^{RM}(\tilde{x}, t)\Big|_{\tilde{x}\in\partial\tilde{\Omega}} = 0, && (134c) \\ &u_{app}^{RM}(\tilde{x}, 0) = u_0(\tilde{x}), && \partial_t u_{app}^{RM}(\tilde{x}, 0) = u_1(\tilde{x}), && (134d) \end{aligned}$$

$$\begin{aligned} &\tilde{v}_{app}^{RM}(\tilde{x}, 0) = -\tilde{\nabla} u_0(\tilde{x}) + \varepsilon^2 \tilde{\gamma}(\tilde{x}, 0), && (134e) \\ &\partial_t \tilde{v}_{app}^{RM}(\tilde{x}, 0) = -\tilde{\nabla} u_1(\tilde{x}) + \varepsilon^2 \partial_t \tilde{\gamma}(\tilde{x}, 0) && (134f) \end{aligned}$$

where

$$\tilde{\beta} := \frac{1}{3}\partial_t^2\left(\tilde{\nabla} w_n\right) - \frac{\lambda+2\mu}{3}\tilde{\nabla}\left(\Delta w_n\right), \tag{135a}$$

$$\tilde{\gamma} := \sum_{k=1}^{n} \varepsilon^{2k-2}\tilde{\nabla} w_k \tag{135b}$$

In particular, $(u_{app}^{RM}, \tilde{v}_{app}^{RM})$ *satisfies the Reissner–Mindlin PDE system* (123a), (123b) *up to terms of order* ε^{2n+2} *and the corresponding boundary and initial conditions* (123c), (123d) *exactly. Furthermore,* $(u_{app}^{RM}, \tilde{v}_{app}^{RM})$ *satisfies the corresponding initial conditions* (123e) *up to terms of order* ε^2.

Proof.

1. Obviously, the unique solution to the equations (125a), (125c) is given by (128). By construction and since u_0, u_1 have compact support, the constraints (125b) and (125d) are also satisfied. Furthermore, (129) follows from (101), (110a) and (128).

2. Obviously, the unique solution to the equations (126a), (126b), (126d) is given by (130), (131). Since u_0, u_1 have compact support, the constraint (126c) is also satisfied.

3. With the help of (130), (131) we obtain

$$\tilde{v}_1^{\mathrm{RM}}(\tilde{x},0) = -\frac{\lambda+2\mu}{3\mu}\tilde{\nabla}\Delta u_0(\tilde{x}), \qquad \partial_t \tilde{v}_1^{\mathrm{RM}}(\tilde{x},0) = -\frac{\lambda+2\mu}{3\mu}\tilde{\nabla}\Delta u_1(\tilde{x}). \tag{136}$$

   Consequently, (126e) is equivalent to (132). Furthermore, let (132) hold. Then, the unique solution to the full dynamic Reissner–Mindlin problem (123) is obviously given by (133).

4. The first statement (134) follows from the construction of $(u_{app}^{\mathrm{RM}}, \tilde{v}_{app}^{\mathrm{RM}})$ and (128), (130), (131). The second statement follows from a comparison of (123) and (134).

□

Comparison of the Full Dynamic Problems

In theorem 14 we have seen that the zero–order terms in the formal expansion (127) of the Reissner–Mindlin solution coincide with the zero–order terms in the asymptotic expansion (101), (106) of the Kirchhoff–Love solution.
On the other hand, in general condition (132) of the theorem does not hold and consequently the series in (127) do not converge. Furthermore, the theorem shows that in general (127) is not even an asymptotic expansion.
It remains to show, that the Kirchhoff–Love solution $(u^{\mathrm{KL}}, \tilde{v}^{\mathrm{KL}})$ and the Reissner–Mindlin solution $(u^{\mathrm{RM}}, \tilde{v}^{\mathrm{RM}})$ coincide as $\varepsilon \to 0$.

Theorem 15 (Comparison Theorem)
Let $(u^{KL}, \tilde{v}^{KL})$ be the solution to the full dynamic Kirchhoff–Love problem (100), (101), let $(u^{RM}, \tilde{v}^{RM})$ be the solution to the full dynamic Reissner–Mindlin problem (123) and let $s \in \mathbb{N}$ be sufficiently large.
Then, the following a–priori estimates hold for all $\varepsilon > 0$:

$$\left\| \tilde{\nabla} u^{RM} + \tilde{v}^{RM} \right\|_{\mathcal{C}^0([-T,T],L^2(\tilde{\Omega},\mathbb{R}))} \leq C\varepsilon^2 \Big(\|u_0\|_{H^s(\tilde{\Omega},\mathbb{R})} + \|u_1\|_{H^s(\tilde{\Omega},\mathbb{R})} \Big), \tag{137a}$$

$$\left\| \tilde{\nabla} u^{RM} + \tilde{v}^{RM} \right\|_{\mathcal{C}^1([-T,T],L^2(\tilde{\Omega},\mathbb{R}))} \leq C\varepsilon \Big(\|u_0\|_{H^s(\tilde{\Omega},\mathbb{R})} + \|u_1\|_{H^s(\tilde{\Omega},\mathbb{R})} \Big), \tag{137b}$$

$$\|u^{RM} - u^{KL}\|_{\mathcal{C}^1([-T,T],L^2(\tilde{\Omega},\mathbb{R}))} \leq C\varepsilon^2 \Big(\|u_0\|_{H^s(\tilde{\Omega},\mathbb{R})} + \|u_1\|_{H^s(\tilde{\Omega},\mathbb{R})} \Big), \tag{137c}$$

$$\|u^{RM} - u^{KL}\|_{\cap_{k=0}^1 \mathcal{C}^{2k}([-T,T],H^{2-2k}(\tilde{\Omega},\mathbb{R}))} \leq C\varepsilon \Big(\|u_0\|_{H^s(\tilde{\Omega},\mathbb{R})} + \|u_1\|_{H^s(\tilde{\Omega},\mathbb{R})} \Big), \tag{137d}$$

$$\|u^{RM} - u^{KL}\|_{\mathcal{C}^1([-T,T],H^1(\tilde{\Omega},\mathbb{R}))} \leq C \Big(\|u_0\|_{H^s(\tilde{\Omega},\mathbb{R})} + \|u_1\|_{H^s(\tilde{\Omega},\mathbb{R})} \Big), \tag{137e}$$

$$\|\tilde{v}^{RM} - \tilde{v}^{KL}\|_{\cap_{k=0}^1 \mathcal{C}^{k}([-T,T],H^{1-k}(\tilde{\Omega},\mathbb{R}))} \leq C\varepsilon \Big(\|u_0\|_{H^s(\tilde{\Omega},\mathbb{R})} + \|u_1\|_{H^s(\tilde{\Omega},\mathbb{R})} \Big), \tag{137f}$$

$$\|\tilde{v}^{RM} - \tilde{v}^{KL}\|_{\cap_{k=0}^2 \mathcal{C}^{k}([-T,T],H^{2-k}(\tilde{\Omega},\mathbb{R}))} \leq C \Big(\|u_0\|_{H^s(\tilde{\Omega},\mathbb{R})} + \|u_1\|_{H^s(\tilde{\Omega},\mathbb{R})} \Big). \tag{137g}$$

In particular, the Kirchhoff–Love solution $(u^{KL}, \tilde{v}^{KL})$ and the Reissner–Mindlin solution $(u^{RM}, \tilde{v}^{RM})$ coincide as $\varepsilon \to 0$.

Proof.
With the help of (101) and (110a), (111) we obtain

$$\|u^{\mathrm{KL}} - u_0 - tu_1\|_{\cap_{k=0}^2 \mathcal{C}^{k}([-T,T],H^{3-k}(\tilde{\Omega},\mathbb{R}))} \leq C\varepsilon^2 \Big(\|u_0\|_{H^s(\tilde{\Omega},\mathbb{R})} + \|u_1\|_{H^s(\tilde{\Omega},\mathbb{R})} \Big), \tag{138a}$$

$$\begin{aligned} &\left\| \tilde{v}^{\mathrm{KL}} + \tilde{\nabla} u_0 + t\tilde{\nabla} u_1 \right\|_{\cap_{k=0}^2 \mathcal{C}^{k}([-T,T],H^{2-k}(\tilde{\Omega},\mathbb{R}))} \\ &\leq C\varepsilon^2 \Big(\|u_0\|_{H^s(\tilde{\Omega},\mathbb{R})} + \|u_1\|_{H^s(\tilde{\Omega},\mathbb{R})} \Big). \end{aligned} \tag{138b}$$

Next, we define $(r^{\mathrm{RM}}, \tilde{s}^{\mathrm{RM}}) \in \mathcal{C}^\infty\big(\mathbb{R} \times \overline{\tilde{\Omega}}, \mathbb{R}\big) \times \mathcal{C}^\infty\big(\mathbb{R} \times \overline{\tilde{\Omega}}, \mathbb{R}^2\big)$ by

$$r^{\mathrm{RM}} := u^{\mathrm{RM}} - u_0 - tu_1, \qquad \tilde{s}^{\mathrm{RM}} := \tilde{v}^{\mathrm{RM}} + \tilde{\nabla} u_0 + t\tilde{\nabla} u_1. \tag{139}$$

By construction, $(r^{\mathrm{RM}}, \tilde{s}^{\mathrm{RM}})$ satisfies the following initial boundary value problem:

$$\partial_t^2 r^{\mathrm{RM}} - \mu\Big(\Delta r^{\mathrm{RM}} + \operatorname{div} \tilde{s}^{\mathrm{RM}}\Big) = 0, \tag{140a}$$

$$\begin{aligned} &\frac{\varepsilon^2}{3}\Big[\partial_t^2 \tilde{s}^{\mathrm{RM}} - (\lambda+\mu)\tilde{\nabla}\Big(\operatorname{div} \tilde{s}^{\mathrm{RM}}\Big) - \mu\Delta \tilde{s}^{\mathrm{RM}}\Big] + \mu\Big(\tilde{\nabla} r^{\mathrm{RM}} + \tilde{s}^{\mathrm{RM}}\Big) \\ &= -\frac{\varepsilon^2(\lambda+2\mu)}{3}\Big[\tilde{\nabla}\Big(\Delta u_0(\tilde{x})\Big) + t\tilde{\nabla}\Big(\Delta u_1(\tilde{x})\Big)\Big] =: \varepsilon^2 \tilde{\beta}(\tilde{x},t), \end{aligned} \tag{140b}$$

$$r^{\mathrm{RM}}(\tilde{x},t)\Big|_{\tilde{x}\in\partial\tilde{\Omega}} = 0, \qquad \tilde{s}^{\mathrm{RM}}(\tilde{x},t)\Big|_{\tilde{x}\in\partial\tilde{\Omega}} = 0, \tag{140c}$$

$$r^{\mathrm{RM}}(\tilde{x},0) = 0, \qquad \partial_t r^{\mathrm{RM}}(\tilde{x},0) = 0, \tag{140d}$$

$$\tilde{s}^{\mathrm{RM}}(\tilde{x},0) = 0, \qquad \partial_t \tilde{s}^{\mathrm{RM}}(\tilde{x},0) = 0. \tag{140e}$$

We multiply (140a) by $\partial_t r^{\mathrm{RM}}(\cdot,t)$ in $L^2(\tilde\Omega,\mathbb{R})$ and (140b) by $\partial_t \tilde s_m^{\mathrm{RM}}(\cdot,t)$ in $L^2(\tilde\Omega,\mathbb{R}^2)$. Then, we add the two equations and integrate with respect to t. With the help of integration by parts we obtain for all $t\in[-T,T]$

$$\begin{aligned}
E(t) :=& \frac{1}{2}\left\|\partial_t r^{\mathrm{RM}}(\cdot,t)\right\|^2_{L^2(\tilde\Omega,\mathbb{R})} + \frac{\mu}{2}\left\|\tilde\nabla r^{\mathrm{RM}}(\cdot,t)+\tilde s^{\mathrm{RM}}(\cdot,t)\right\|^2_{L^2(\tilde\Omega,\mathbb{R}^2)}\\
&+\frac{\varepsilon^2}{6}\left\|\partial_t \tilde s^{\mathrm{RM}}(\cdot,t)\right\|^2_{L^2(\tilde\Omega,\mathbb{R}^2)} + \frac{\varepsilon^2\lambda}{6}\left\|\operatorname{div}\tilde s^{\mathrm{RM}}(\cdot,t)\right\|^2_{L^2(\tilde\Omega,\mathbb{R})}\\
&+\frac{\varepsilon^2\mu}{12}\left\|\nabla \tilde s^{\mathrm{RM}}(\cdot,t)+\left(\nabla\tilde s^{\mathrm{RM}}(\cdot,t)\right)^T\right\|^2_{L^2(\tilde\Omega,\mathbb{R}^{2\times 2})}\\
=&\ \varepsilon^2\int_0^t \left\langle \partial_t\tilde s^{\mathrm{RM}}(\cdot,\tau)\middle|\tilde\beta(\cdot,\tau)\right\rangle_{L^2(\tilde\Omega,\mathbb{R}^2)}\,\mathrm{d}\tau\\
=&\ \varepsilon^2\left\langle \tilde\nabla r^{\mathrm{RM}}(\cdot,t)+\tilde s^{\mathrm{RM}}(\cdot,t)\middle|\tilde\beta(\cdot,t)\right\rangle_{L^2(\tilde\Omega,\mathbb{R}^2)}\\
&-\varepsilon^2\int_0^t \left\langle \tilde\nabla r^{\mathrm{RM}}(\cdot,\tau)+\tilde s^{\mathrm{RM}}(\cdot,\tau)\middle|\partial_t\tilde\beta(\cdot,\tau)\right\rangle_{L^2(\tilde\Omega,\mathbb{R}^2)}\,\mathrm{d}\tau\\
&+\varepsilon^2\int_0^t \left\langle \partial_t r^{\mathrm{RM}}(\cdot,\tau)\middle|\operatorname{div}\tilde\beta(\cdot,\tau)\right\rangle_{L^2(\tilde\Omega,\mathbb{R})}\,\mathrm{d}\tau\\
\le&\ \frac{1}{2}E(t)+\int_0^t E(\tau)\,\mathrm{d}\tau + C\varepsilon^4\left\|\tilde\beta\right\|^2_{\mathcal{C}^1([-T,T],H^1(\tilde\Omega,\mathbb{R}^2))}.
\end{aligned}\tag{141}$$

With the help of Gronwall's inequality we obtain

$$E(t)\le C\varepsilon^4\left\|\tilde\beta\right\|^2_{\mathcal{C}^1([-T,T],H^1(\tilde\Omega,\mathbb{R}^2))} \qquad \forall t\in[-T,T].\tag{142}$$

Furthermore, with the help of Poincaré's inequality and Korn's inequality we obtain

$$\left\|\tilde\nabla r^{\mathrm{RM}}+\tilde s^{\mathrm{RM}}\right\|_{\mathcal{C}^0([-T,T],L^2(\tilde\Omega,\mathbb{R}))}\le C\varepsilon^2\left\|\tilde\beta\right\|_{\mathcal{C}^1([-T,T],H^1(\tilde\Omega,\mathbb{R}^2))},\tag{143a}$$

$$\left\|r^{\mathrm{RM}}\right\|_{\mathcal{C}^1([-T,T],L^2(\tilde\Omega,\mathbb{R}))}\le C\varepsilon^2\left\|\tilde\beta\right\|_{\mathcal{C}^1([-T,T],H^1(\tilde\Omega,\mathbb{R}^2))},\tag{143b}$$

$$\left\|r^{\mathrm{RM}}\right\|_{\mathcal{C}^0([-T,T],H^1(\tilde\Omega,\mathbb{R}))}\le C\varepsilon\left\|\tilde\beta\right\|_{\mathcal{C}^1([-T,T],H^1(\tilde\Omega,\mathbb{R}^2))},\tag{143c}$$

$$\left\|\tilde s^{\mathrm{RM}}\right\|_{\cap_{k=0}^1\mathcal{C}^k([-T,T],H^{1-k}(\tilde\Omega,\mathbb{R}))}\le C\varepsilon\left\|\tilde\beta\right\|_{\mathcal{C}^1([-T,T],H^1(\tilde\Omega,\mathbb{R}^2))}.\tag{143d}$$

Next, we differentiate the initial boundary value problem (140) with respect to t and note that

$$\partial_t^2\tilde s^{\mathrm{RM}}(\tilde x,0)=3\tilde\beta(\tilde x,0).\tag{144}$$

Then, we can apply the same arguments as above. This yields

$$\left\|\tilde{\nabla} r^{\mathrm{RM}} + \tilde{s}^{\mathrm{RM}}\right\|_{\mathcal{C}^1([-T,T],L^2(\tilde{\Omega},\mathbb{R}))} \leq C\varepsilon \left\|\tilde{\beta}\right\|_{\mathcal{C}^2([-T,T],H^1(\tilde{\Omega},\mathbb{R}^2))}, \tag{145a}$$

$$\|r^{\mathrm{RM}}\|_{\mathcal{C}^2([-T,T],L^2(\tilde{\Omega},\mathbb{R}))} \leq C\varepsilon \left\|\tilde{\beta}\right\|_{\mathcal{C}^2([-T,T],H^1(\tilde{\Omega},\mathbb{R}^2))}, \tag{145b}$$

$$\|r^{\mathrm{RM}}\|_{\mathcal{C}^1([-T,T],H^1(\tilde{\Omega},\mathbb{R}))} \leq C \left\|\tilde{\beta}\right\|_{\mathcal{C}^2([-T,T],H^1(\tilde{\Omega},\mathbb{R}^2))}, \tag{145c}$$

$$\|\tilde{s}^{\mathrm{RM}}\|_{\cap_{k=1}^2 \mathcal{C}^k([-T,T],H^{2-k}(\tilde{\Omega},\mathbb{R}))} \leq C \left\|\tilde{\beta}\right\|_{\mathcal{C}^2([-T,T],H^1(\tilde{\Omega},\mathbb{R}^2))}. \tag{145d}$$

Next, we rewrite (140a) as

$$\Delta r^{\mathrm{RM}} = \frac{1}{\mu}\partial_t^2 r^{\mathrm{RM}} - \operatorname{div} \tilde{s}^{\mathrm{RM}}. \tag{146}$$

Then, with the help of elliptic regularity theory and (143), (145) we obtain

$$\begin{aligned} \|r^{\mathrm{RM}}\|_{\mathcal{C}^0([-T,T],H^2(\tilde{\Omega},\mathbb{R}))} &\leq \hat{C}\Big(\|\partial_t^2 r^{\mathrm{RM}}\|_{\mathcal{C}^0([-T,T],L^2(\tilde{\Omega},\mathbb{R}))} + \|\operatorname{div} \tilde{s}^{\mathrm{RM}}\|_{\mathcal{C}^0([-T,T],L^2(\tilde{\Omega},\mathbb{R}))} \Big) \\ &\leq C\varepsilon \left\|\tilde{\beta}\right\|_{\mathcal{C}^2([-T,T],H^1(\tilde{\Omega},\mathbb{R}^2))}. \end{aligned} \tag{147}$$

Next, we rewrite (140b) as

$$(\lambda+\mu)\tilde{\nabla}\Big(\operatorname{div} \tilde{s}^{\mathrm{RM}}\Big) + \mu\Delta\tilde{s}^{\mathrm{RM}} = \partial_t^2 \tilde{s}^{\mathrm{RM}} + \frac{3\mu}{\varepsilon^2}\Big(\tilde{\nabla} r^{\mathrm{RM}} + \tilde{s}^{\mathrm{RM}}\Big) - 3\tilde{\beta}(\tilde{x},t). \tag{148}$$

Then, with the help of elliptic regularity theory and (143), (145) we obtain

$$\begin{aligned} &\|\tilde{s}^{\mathrm{RM}}\|_{\mathcal{C}^0([-T,T],H^2(\tilde{\Omega},\mathbb{R}))} \\ &\leq \hat{C}\Big(\|\partial_t^2 \tilde{s}^{\mathrm{RM}}\|_{\mathcal{C}^0([-T,T],L^2(\tilde{\Omega},\mathbb{R}))} + \frac{1}{\varepsilon^2}\left\|\tilde{\nabla} r^{\mathrm{RM}} + \tilde{s}^{\mathrm{RM}}\right\|_{\mathcal{C}^0([-T,T],L^2(\tilde{\Omega},\mathbb{R}))} \\ &\qquad + \left\|\tilde{\beta}\right\|_{\mathcal{C}^0([-T,T],L^2(\tilde{\Omega},\mathbb{R}))}\Big) \\ &\leq C \left\|\tilde{\beta}\right\|_{\mathcal{C}^2([-T,T],H^1(\tilde{\Omega},\mathbb{R}^2))}. \end{aligned} \tag{149}$$

Now, the desired statement (137) follows from (143), (145) and (147), (149). □

References

[1] S. Antman, *Nonlinear Problems of Elasticity*, Springer, New York, 1995.

[2] E. Axelrad, *Schalentheorie*, Teubner, Stuttgart, 1983.

[3] E. Becker–W. Bürger, *Kontinuumsmechanik*, Teubner, Stuttgart, 1975.

[4] P. Ciarlet, *Mathmatical Elasticity II: Theory of Plates*, North–Holland, Amsterdam, 1997.

[5] P. Ciarlet–V. Lods, *Analyse Asymptotique des Coques Lineairement Elastiques 1*, C.R. Acad. Sci., Paris, I 318, 9, p. 863–868, 1994.

[6] P. Ciarlet–V. Lods, *Analyse Asymptotique des Coques Lineairement Elastiques 2*, C.R. Acad. Sci., Paris, I 319, 1, p. 95–100, 1994.

[7] P. Ciarlet–V. Lods, *Analyse Asymptotique des Coques Lineairement Elastiques 3*, C.R. Acad. Sci., Paris, I 319, 3, p. 299–304, 1994.

[8] P. Destuynder–M. Salaun, *Mathmatical Analysis of Thin Plate Models*, Springer, Paris, 1996.

[9] S. Ebenfeld, *Aspekte der Kontinua mit Mikrostruktur* (PhD thesis), Shaker, Aachen, 1998.

[10] J. Ericksen–C. Truesdell, *Exact Theory of Stress and Strain in Rods and Shells*, Arch. Ration. Mech. Analysis, vol. 1, p. 295–323, 1958.

[11] A. Green–P. Naghdi–W. Wainwright, *A General Theory of a Cosserat Surface*, Arch. Ration. Mech. Anal., vol. 20, p. 287–308, 1965.

[12] M. Gurtin, *An Introduction to Continuum Mechanics*, Academic Press, New York, 1981.

[13] J. Marsden–T. Hughes, *Topics in the Mathmatical Foundations of Elasticity*, Pitman, London, Heriot–Watt Symp., vol. 2, p. 30–285, 1978.