New Results on Wave Diffraction by Canonical Obstacles

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Reaching back to A. Sommmerfeld's habilitation thesis [30] in 1896 mathematical diffraction theory started by formulating boundary and transmission problems for wave equations in canonical domains with semi-infinite boundaries, like planes, wedges, half-planes, cones, octants etc. During the last decade different boundary-transmission conditions were involved and explicit form solutions found using integral transforms and factorization techniques for Fourier symbols.

Recently some new results concerning aperiodic scalar initial boundary-transmission problems for half-planes and wedges are obtained using generalized plane wave representations and various electrodynamical time-harmonic boundary value problems are solved in explicit form now. They close in some sense a gap between the Wiener-Hopf and Maliuzhinets approach.

1. Introduction

The geometrically and analytically simplest diffraction problems arise in the following way:

Definition 1.1. Let there be given (at least) two piece-wise smoothly bounded domains $\Omega_1 = \Omega$ and $\Omega_2 = \Omega' := I\!R^n \setminus \bar{\Omega}$, $n = 2, 3$; constants k_1, $k_2 \in \mathbb{C}_{++}$, c_1, $c_2 \in I\!R_+$, and functions F_1 on Ω_1, F_2 on Ω_2 belonging to certain spaces $X(\Omega_1)$ and $X(\Omega_2)$,initial data $U_{01} \in Y_0(\Omega_1)$, $U_{02} \in Y_0(\Omega_2)$, $U_{11} \in Y_1(\Omega_1)$, $U_{12} \in Y_1(\Omega_2)$, boundary data $f_1 \in Z_1(\partial\Omega_1 \setminus \Gamma)$, $f_2 \in Z_2(\partial\Omega_2 \setminus \Gamma)$, and transmission data $g \in T_0(\Gamma)$, $h \in T_1(\Gamma)$, where $\Gamma := \partial\Omega_1 \cap \partial\Omega_2 \neq \emptyset$ is a piece-wise smooth (open) surface.

The initial boundary-transmission problem for the d'Alembert wave equations is the following:

Problem 1.2. Find $U_j(\underline{x}, t) \in C^2(\Omega_j \times (0,T)) \cap C^1(\overline{\Omega}_j \setminus K \times (0,T))$ or $\in H^1_{loc}(\Omega_j \times (0,T); \Delta)$, $j = 1, 2$; fulfilling additional conditions as $\underline{x} \to K$, the geometrical singular points of $\partial\Omega_1 \cup \partial\Omega_2$, and *radiation conditions* as $|\underline{x}| \to \infty$ if $T = \infty$. The $U_j(\underline{x}, t)$ are sought as solutions (strong or weak, respectively) of

$$(1.1) \qquad \left(\Delta_n - \frac{1}{c_j^2}\frac{\partial^2}{\partial t^2}\right) U_j(\underline{x}, t) = F_j(\underline{x}, t) \quad \text{in } \Omega_j \times (0,T),$$

with boundary conditions

(1.2) $$B_j[U_j](\underline{x},t) = f_j(\underline{x},t) \quad \text{on } \partial\Omega_j \times (0,T),$$

and transmission conditions on $\Gamma \times (0,T)$:

(1.3) $$T_{r_0}[U_1,U_2] := C_1[U_1] - C_1[U_2] = g(\underline{x},t)$$
(1.4) $$T_{r_1}[U_1,U_2] := C_2[U_1] - C_2[U_2] = h(\underline{x},t)$$

consisting of boundary (trace) values of U_j of the same type on $\Gamma \times (0,T)$.

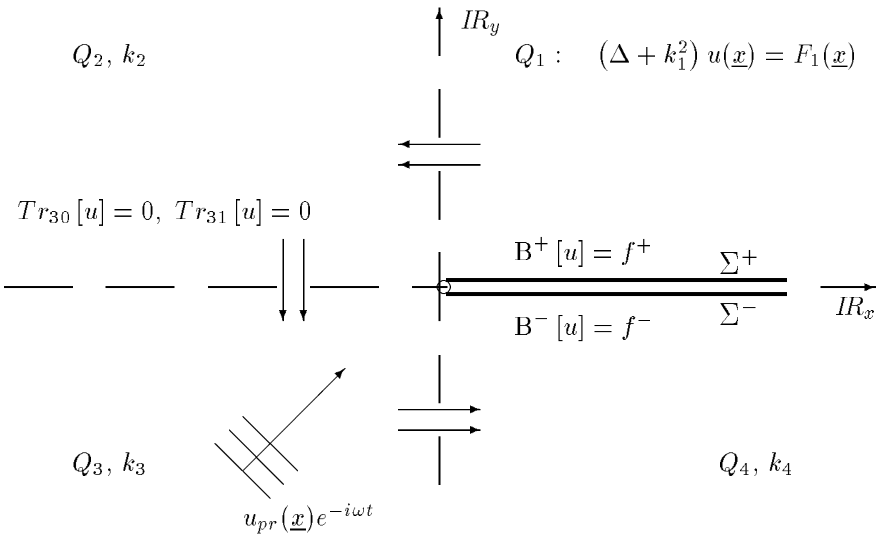

Fig. 1: 4-media Sommerfeld half-plane problem (2D-scalar)

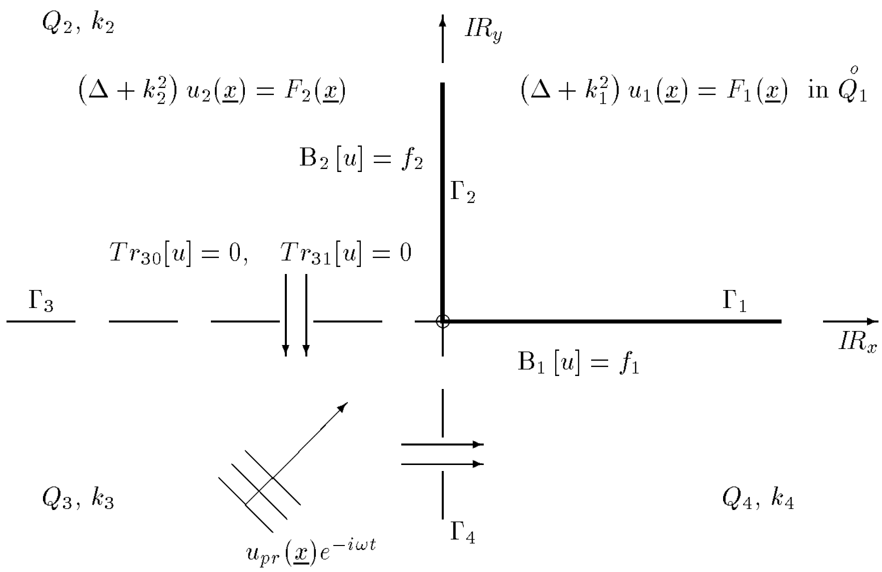

Fig. 2: Scattering of a primary wave by the wedge with cross-section Q_1, here coming from outside.

Remark 1.3. *Main canonical boundary transmission problems* are called those where $\Omega_1 = IR_-^n$, $\Omega_2 = IR_+^n$, $\partial\Omega_1 \cap \partial\Omega_2 = IR'^{\,n-1}$, $\Gamma = IR'^{\,n-1}_-$, $\Sigma = \partial\Omega_j \setminus \Gamma = IR'^{\,n-1}_+ := \{\underline{x}' = (x,y) \in IR^2 : \ x \geq 0,\ y \in IR\}$ leading to the *pure transmission problem* (see fig.1) or so-called *Sommerfeld half-plane problems* (see fig.1 even for four media) or the *wedge problem for two media* $\Omega_1 := W_\beta := \{(x,y,z) \in IR^3 : \ x = r\cos\varphi,\ z = r\sin\varphi, 0 < r < \infty,\ 0 < \varphi < \beta,\ y \in IR\}$, and $\Omega_2 := IR^3 \setminus \bar{\Omega}_1$. In the special case of $\beta = \frac{\pi}{2}$ one gets the *interior* and *exterior* wedge problem, if there are prescribed only boundary data on $\partial W_\beta = \partial\Omega_1 = \partial\Omega_2 = \{(x,y,z) \in IR^3 : \ x = r \geq 0,\ z = 0,\ y \in IR\} \cup \{(x,y,z) \in IR^3 : \ x = r\cos\beta,\ z = r\sin\beta,\ y \in IR,\ r \geq 0\}$. It is assumed $0 < \beta < \pi$ (see fig.3).

In the case of the general plane screen problem there is assumed $\Omega_1 = IR_-^3$, $\Omega_2 = IR_+^3$ with the common boundary $\partial\Omega_1 \cap \partial\Omega_2 = IR'^{\,2}$ divided into the screen Σ with Lipschitzian boundary curve γ and the IR^2- complement Σ', where transmission conditions of the type (1.3-1.4) have to hold. More general cases apply to obstacles

$\partial\Omega$ of domains $\Omega = \Omega_1$, like cones or pyramides, particularly octants $I\!R^3_{++}$.

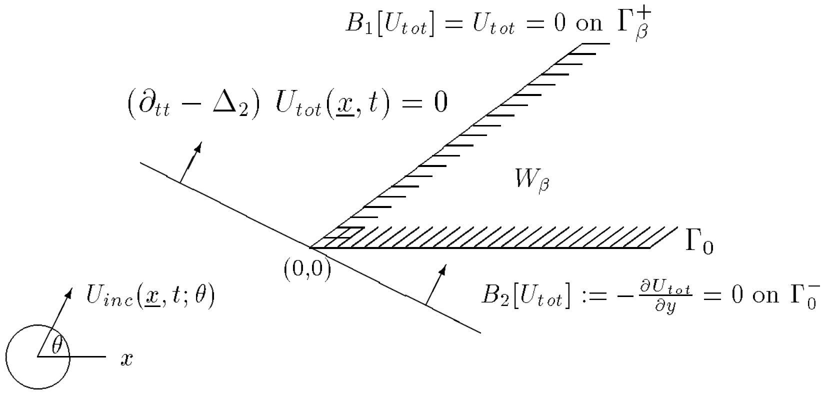

Fig.3: Plane wave falling upon wedge with opening angle β.

Remark 1.4. Instead of the scalar d'Alembert wave equations (1.1) being involved similar appropriate boundary-transmission problems arise for vectorial field equations like Maxwell's in electrodynamics, Lamé's in linear elastodynamics, and their generalizations to thermo- and viscoelastodynamics. Problems of this nature have been treated so far only for the time-harmonic case when no initial values are prescribed. The present authors in cooperation with C. Erbe, J. Mark, F.S. Teixeira and other members of the *Lisbon group* (see e.g. [4], [11], [12], [14]–[25], [32]–[34]) discussed many details during the last decade for half-planes and right-angled wedges. Here we shall concentrate only on two more recent results dealing with the *scalar aperiodic Sommerfeld half-plane* [24] and *wedge problem* [25] and the *anisotropic Leontovich boundary value problem for time-harmonic electromagnetic fields* [21].

2. The Aperiodic Diffraction of Plane Waves by Wedges or Sommerfeld Half-planes

We assume now (see fig.3) that a plane wave $U_{inc}(\underline{x}, t) = G(t - x\cos\theta - y\sin\theta)$ hits at $t = t_0 = 0$ the edge $(x, y) = (0, 0)$, $z \in I\!R$ of a wedge $W_\beta = \{(x, y, z) \in I\!R^3 : x = r\cos\alpha, y = r\sin\alpha,\ r \geq 0,\ 0 \leq \alpha \leq \beta,\ z \in I\!R\} \subset I\!R]^3$ inducing Dirichlet trace data on one face Γ^+_β, corresponding to $\alpha = \beta + 0$, and Neumann trace data on the other one Γ^-_0, corresponding to $\alpha = 2\pi - 0$. Let $\theta \in (0, \frac{\pi}{2})$ be the angle of incidence.

Remark 2.1. For simplicity we assume the velocity of propagation to be equal to one in the case of one medium. The corresponding boundary value problems for time-harmonic scalar wave fields have been treated before by many authors involving different methods. Here we mention only A. SOMMERFELD'S work from 1896 [30] and 1901 [31], and the Maliuzhinets method 1958 [10] which was used also for different boundary conditions on the wedge faces by T.B.A. SENIOR (e.g. in 1959 [26]), and in the book by him and J.L. VOLAKIS (1995) [27], and later by B. BUDAEV in his book [2] while the special case of a right-angle wedge was solved by the Wiener-Hopf factorization method for 2×2–Fourier symbols (see e.g. F.S. TEIXEIRA [34], and E. MEISTER, F.-O. SPECK and F.S. TEIXEIRA [19]). We would like to mention also the application of the Kantorovitch-Lebedev transform described in E. MEISTER'S review article [12].

Reducing the initial boundary value problem with homogeneous initial conditions for the total field by writing the incident wave field $U_{inc}(\underline{x},t) = G(t - r\cos(\theta-\alpha)$, where

$$G(t) := \chi_+(t)\cdot\int_0^t g(\tau)\,d\tau \tag{2.1}$$

with polar coordinates (r,α) in the xy–plane and Heaviside'step function $\chi_+(t)$, and a locally integrable time function g we obtain for the scattered field

$$U_{scat}(\underline{x},t) = V_{scat}(r,\alpha,t) := U_{tot}(\underline{x},t) - U_{inc}(\underline{x},t). \tag{2.2}$$

Problem 2.2. Find V_{scat} of the d'Alembert equation in polar coordinates

$$\left(\partial_{tt} - \partial_{rr} - \frac{1}{r}\partial_r - \frac{1}{r^2}\partial_{\alpha\alpha}\right)V_{scat}(r,\alpha,t) = 0 \tag{2.3}$$

for $0 < r < \infty$, $\beta < \alpha < 2\pi$, $t > 0$, with the initial conditions

$$V_{scat}(r,\alpha,t=0^+) = \partial_t V_{scat}(r,\alpha,t=0^+) = 0 \quad \text{in } \Omega \qquad \textbf{(I)} \tag{2.4}$$

and the (mixed) boundary conditions

$$V_{scat}(r,\alpha,t) = -G(t - r\cos(\theta-\alpha)) \quad \text{for } \alpha=\beta+0,\ t>0, \qquad \textbf{(D)} \tag{2.5}$$

$$\partial_y U(x,y=0^-,t) = \frac{1}{r}\partial_\alpha V_{scat}(r,\alpha,t) = -\frac{1}{r}\partial_\alpha G(t-r\cos(\theta-\alpha)) \qquad \textbf{(N)} \tag{2.6}$$

for $\alpha = 2\pi - 0$, $t > 0$, and bounded V_{scat} and square integrable ∇U_{scat} near the vertex (0,0) for $t > 0$.

We make use of the Laplace transformation w.r.t. $t \mapsto s$ and put a tilde on the top of the transformed functions. This gives

Problem 2.3. Find the transformed scattered field $\widetilde{V}_{scat}(r,\alpha;s)$ with the properties

$$\left(s^2 - \partial_{rr} - \frac{1}{r}\partial_r - \frac{1}{r^2}\partial_{\alpha\alpha}\right)\widetilde{V}_{scat}(r,\alpha;s) = 0, \quad \beta<\alpha<2\pi,\ \Re s > s_0, \tag{2.7}$$

(2.8) $\widetilde{V}_{scat}(r,\alpha;s) = -\mathrm{e}^{-sr\cos(\theta-\beta)}\dfrac{\widetilde{g}(s)}{s}$ for $\alpha = \beta + 0$, $\widetilde{\mathbf{(D)}}$

(2.9) $\dfrac{1}{r}\partial_\alpha \widetilde{V}_{scat}(r,\alpha;s) = s\sin(\theta-\beta)\,\mathrm{e}^{-sr\cos(\theta-\beta)}\dfrac{\widetilde{g}(s)}{s}$, $\alpha = 2\pi - 0$. $\widetilde{\mathbf{(N)}}$

Introducing

$$\widetilde{U}_{scat}(r,\alpha;s) = \frac{\widetilde{V}_{scat}(r,\alpha;s)}{\widetilde{G}(s)} = \frac{s\widetilde{V}_{scat}(r,\alpha;s)}{\widetilde{g}(s)} =: \widetilde{w}_{scat}(R,\alpha), \tag{2.10}$$

where $R = sr$ for $s > s_0 \geq 0$, we are led to the reduced problem

Problem 2.4. Find the reduced scattered field $\widetilde{w}_{scat}(R,\alpha)$ such that

(2.11) $\left(R^2 - \partial_{RR} + R\partial_R + \partial_{\alpha\alpha} - R^2\right)\widetilde{w}_{scat}(R,\alpha) = 0,\ R > 0,\ \beta < \alpha < 2\pi,$

(2.12) $\widetilde{w}_{scat}(R,\alpha) = -\mathrm{e}^{-R\cos(\theta-\beta)}$ for $\alpha = \beta + 0$, $\widetilde{\mathbf{(D)}}_*$

(2.13) $\partial_\alpha \widetilde{w}_{scat}(R,\alpha) = -\partial_\alpha\left(\mathrm{e}^{-R\cos(\theta-\alpha)}\right)$ for $\alpha = 2\pi - 0$. $\widetilde{\mathbf{(N)}}_*$

The reduction above leads to the following

Theorem 2.5. ***(Representation of the total field)****. The solution of the initial boundary value wedge problem 2.2 is given by the convolutional integral representation with Dirac's δ:*

$$\begin{aligned} V(r,\alpha,t;\theta,\beta) &= V_{inc}(r,\alpha,t;\theta) + \int_0^t G(t-\tau)\,U_{scat}(r,\alpha,\tau;\theta,\beta)\ d\tau \\ &= \int_0^t G(t-\tau)\ \left[\delta(\tau - r\cos(\theta-\alpha)) + U_{scat}(r,\alpha,\tau;\theta,\beta)\right]\ d\tau. \end{aligned} \tag{2.14}$$

The corresponding boundary conditions for $U = U_{scat}$ receive the generalized weak forms

$$U = -\frac{\delta\left(\frac{t}{r} - \cos(\theta-\alpha)\right)}{r}, \qquad \alpha = \beta + 0, \tag{2.15}$$

$$\partial_\alpha U = -\partial_\alpha\left(\frac{\delta\left(\frac{t}{r} - \cos(\theta-\alpha)\right)}{r}\right),\ \alpha = 2\pi - 0. \tag{2.16}$$

These formulae lead in a natural way to the following ansatz

$$U(r,\alpha,t) = \frac{u(\frac{t}{r},\alpha)}{r} = \frac{W(\varphi,\alpha)}{r}, \quad \frac{t}{r} = \cosh(\varphi), \quad \sinh\varphi = \sqrt{\frac{t^2}{r^2} - 1}, \quad \text{with } \varphi \geq 0. \tag{2.17}$$

In the special case of $\beta = 0$, the Sommerfeld half-plane case, the above formula (2.17) coincides with that one from the Fourier integral and Cagniard–de Hoop method (cf. e.g. J.D. Achenbach (1984)[1] and A.T. de Hoop (1958) [8] !). Inserting from (2.17) into the wave equation gives for $W(\varphi, \alpha)$ the equation

$$(2.18)\qquad -\frac{1}{r^3}\left(W_{\varphi\varphi} + 2\coth\varphi W_\varphi + W + W_{\alpha\alpha}\right) = 0,$$

from which follows the 2D–Laplace equation

$$(2.19)\qquad (W\sinh\varphi)_{\varphi\varphi} + (W\sinh\varphi)_{\alpha\alpha} = 0 \quad \text{in } \varphi > 0,\ \beta < \alpha < 2\pi.$$

Thus the wave potential of the diffracted field may be expressed by

$$(2.20)\qquad U_{diff}(r,\alpha,\varphi) = \frac{F_{diff}(\varphi,\alpha)}{r\,\sinh\varphi}, \quad r\,\sinh\varphi = \sqrt{t^2 - r^2},$$

with a real-valued solution to the 2D-Laplace equation

$$(2.21)\qquad F_{diff}(\varphi,\alpha) = \Re\left(F_1(\alpha + i\varphi) + F_2(-\alpha + i\varphi)\right).$$

This potential function has to fulfill the (homogeneous) boundary conditions **(D)** for $\alpha = \beta + 0$, and **(N)** for $\alpha = 2\pi - 0$. The field for $\varphi = 0$ corresponding to $r = t$, the cylindrical wave front, is unknown so far.

So we got the very strong tools to solve other time–dependent 2D diffraction problems making use of conformal mapping techniques in the (φ, α)–plane. Having here the upper half–strip $\mathcal{S}_\beta := \{z = \alpha + \mathrm{i}\varphi := (\alpha, \varphi) \in I\!R^2 :\ \beta < \alpha < 2\pi, \text{ and } \varphi > 0\}$ we may map this one by

$$(2.22)\qquad z' = \frac{2\pi}{2\pi - \beta}(z - \beta) = \alpha' + i\varphi'.$$

onto the strip $\mathcal{S}' = \mathcal{S}_0$, which corresponds to the above displayed reduction to the Rawlins DN–problem with a Sommerfeld half-plane Σ. K. Rottbrand solved this problem in a Sobolev space setting (1998) [25]. In his case it turned out that

$$(2.23)\qquad F_{diff}(\varphi',\alpha';\theta') = \Re\left[F_1(\varphi',\alpha';\theta') - F_1(\varphi',-\alpha';\theta')\right],$$

$$(2.24)\qquad F_1(\varphi',\alpha';\theta') = \frac{1}{4\pi}\left[\frac{1}{\cos\frac{\theta'+\alpha'+i\varphi'}{4}} - \frac{1}{\sin\frac{\theta'+\alpha'+i\varphi'}{4}}\right].$$

The factor $(4\pi)^{-1}$ in front has to be replaced by $(4\pi - 2\beta)^{-1}$ for the wedge which has the exterior region with angle $\gamma = 2\pi - \beta$. In his just mentioned paper Rottbrand also applied the conformal mapping

$$(2.25)\qquad z_0 = x + \mathrm{i}y = r\mathrm{e}^{\mathrm{i}\alpha} \mapsto z_3 = r^\mu \mathrm{e}^{\mathrm{i}\nu(\alpha-\beta)}$$

with $\mu > 0$ and $\nu = \frac{2\pi}{2\pi-\beta}$ from the complement W'_β of the wedge cross section onto the new complex plane cut along the positive real axis $x_3 \geq 0$. Putting $R = r^\mu$, $\alpha' = \nu(\alpha - \beta)$ and $U(r,\alpha,t) = W(R,\alpha',t)$ gives there for $t > 0$

$$\left(R^{\frac{2}{\mu}}\,\partial_{tt} - R^2\partial_{RR} - \mu R\partial_R - \nu^2\,\partial_{\alpha'\alpha'}\right) W(R,\alpha',t) = 0. \tag{2.26}$$

Choosing $\mu = 1$, $\varphi = \frac{\varphi'}{\nu}$, $t = r\cosh\varphi$ gives again the 2D–potential equation (cmp. (2.19)) $F_{\alpha'\alpha'} + F_{\varphi'\varphi'} = 0$.

The time-harmonic DD- and NN-Sommerfeld problems lead by the Wiener–Hopf scalar Fourier symbol factorization to the aperiodic solution involving the terms

$$F_1(\varphi',\alpha';\theta') = -\frac{1}{2\pi}\,\frac{1}{\sin\frac{\theta'+\alpha'+i\varphi'}{2}}, \tag{2.27}$$

generating the Green's functions through equation (2.23) , where in the Neumann case the difference there becomes a sum. For the corresponding wedge problems one has to replace $(2\pi)^{-1}$ by $(2\pi-\beta)^{-1}$. By some detailed investigations using the explicit factorization formulae for the half–plane problems with transmission conditions $aU_0^+ + bU_0^- = g_0^+$ and $cU_1^+ + dU_1^- = g_1^-$ on $\Sigma^\pm$ and 2×2–symbol matrices of the type

$$H_o(\xi,k;\lambda) = \begin{pmatrix} \frac{\sqrt{\xi-k}}{\sqrt{\xi+k}} & 1 \\ \lambda^{-1} & \frac{\sqrt{\xi+k}}{\sqrt{\xi-k}} \end{pmatrix}, \qquad \lambda \in \mathbb{C},\ \arg\sqrt{\lambda} \in (0,\pi), \tag{2.28}$$

where $\lambda := \frac{(a-b)(c-d)}{(a+b)(c+d)}$ (see E. Meister & F.-O. Speck (1987) [18], F.-O. Speck (1989) [32], or F.S. Teixeira (1990) [33] !) gives explicit factors from which the asymptotic behavior of the scattered field near $r := \sqrt{x^2+y^2} = 0$ may be calculated due to

$$\nabla u = \mathcal{O}(r^{\frac{\delta}{2}-1}) \quad \text{as } r \to 0, \tag{2.29}$$

$$\delta := \Re\left(\frac{i}{\pi}\log\frac{\sqrt{\lambda}+1}{\sqrt{\lambda}-1}\right), \quad 0 < \delta \leq 1. \tag{2.30}$$

Now it is possible to give formulae for wedge boundary problems too (see Rottbrand (1998) [25]!). There the author states also the the formulae for the representation of the scattered time–domain solution to the initial boundary wedge problem in Theorem 3.1 and 3.2, which are too lenghty to be repeated here. In the appendix he shows the use of the Cagniard–de Hoop method in the Laplace transform domain.

All details about the solution of Rawlins' mixed initial boundary value problem may be found in Rottbrand's papers (1998) [24, 25]. In a preceeding paper (1997) [23] this author derived the generalized eigenfunctions for this mixed problem in the case of real wave numbers.

3. Anisotropic Leontovich boundary conditions for electromagnetic wave–fields on a Sommerfeld half–plane

In microwave theory for printed electrical circuits or in antenna theory (see e.g. the book by V.P. & Yu.V. Shestopalov (1996) [29]) the study of the diffraction of electromagnetic waves at high frequencies ω by thin metallic strips on a dielectric substrate or, vice–versa, of such layers on an ideal conducting backing is of great interest. These physical and technical problems attracted also some time ago the interest of mathematicians (see e.g. the paper by C.H. Wilcox (1976) [35], or J.C. Guillot & Wilcox (1978) [7]!) and later also for anisotropic media in electro– and elastodynamics. It's impossible to mention all these papers, but let's point out that the Turkish scientists in Adana and Istanbul contributed a lot (cf. e.g. the article by A. Serbest (1996) [28]). Spectral theoretic investigations were done by D.S. Gilliam & J.R. Schulenberger in (1982) [5] and in their book (1986) [6].

Here we shall report on some new results concerning the scattering of electromagnetic waves by a thin layer of dielectric material backed by a perfectly conducting plane or an anisotropic half–space and the approximation by so–called *Leontovich boundary conditions* for the field near the boundary. We are following here the lines of A. Passow from his diploma thesis (1994) [20] and his recently submitted PhD–thesis (1998) [21] in Darmstadt. To get some idea about the boundary–transmission conditions look at fig.4.

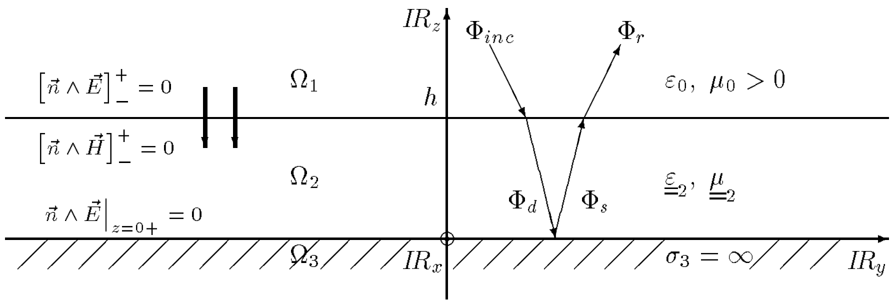

Fig. 4: Anisotropic dielectric layer on perfectly conducting half-space.

Problem 3.1. Let there be given the permittivity $\underline{\underline{\epsilon}}_1 = \epsilon_0 \underline{\underline{I}}_3$, $\underline{\underline{\epsilon}}_2$, and permeability tensors $\underline{\underline{\mu}}_1 = \mu_0 \underline{\underline{I}}_3$, $\underline{\underline{\mu}}_2$ in $\Omega_{1,2}$, respectively, an incident electromagnetic field $(\vec{E}_{inc}e^{-i\omega t}, \vec{H}_{inc}e^{-i\omega t})^T \in \mathbb{C}^6$ from Ω_1. Find the total field $(\vec{E}_{tot}e^{-i\omega t,}, \vec{H}_{tot}e^{-i\omega t,}) \in C^1(\Omega_T)$, where $\Omega_T = (\Omega_1 \cup \Omega_2) \times (0,T)$, $0 < T \leq \infty$, or in more general mathematical form $(\vec{E}_{scat}, \vec{H}_{scat})^T\Big|_{\Omega_1} \in \left(H^1_{loc}(\Omega_1)\right)^6$,

$(\vec{E}_{scat}, \vec{H}_{scat})^T\Big|_{\Omega_2} \in \left(H^1(\Omega_1)\right)^6$ with additional conditions (see e.g. R. LEIS' book (1986) [9]!) fulfilling Maxwell's equations in strong (or weak) form

$$\text{curl}\vec{E}_j - \text{i}\omega\underline{\underline{\mu}}_j\vec{H}_j = 0, \quad \text{curl}\vec{H}_j - \text{i}\omega\underline{\underline{\epsilon}}_j\vec{E}_j = 0, \tag{3.1}$$

$$\text{div}(\underline{\underline{\mu}}_j\vec{H}_j) = 0, \quad \text{div}(\underline{\underline{\epsilon}}_j\vec{E}_j) = 0, \tag{3.2}$$

in case of charge and current free spaces Ω_j. Additionally the fields have to fulfil the boundary condition

$$\vec{n}\wedge\vec{E}_{tot2}\Big|_{z=0^+} = 0 \quad \text{or} \quad \vec{n}\wedge\vec{H}_{tot2}\Big|_{z=0^+} = 0 \tag{3.3}$$

with $\vec{n} = -\vec{e}_2 = (0,0,-1)^T$, and the transmission conditions

$$[\vec{n}\wedge\vec{E}_{tot}]_{z=h} := \vec{n}\wedge\vec{E}_{tot1}\Big|_{z=h^+} - \vec{n}\wedge\vec{E}_{tot2}\Big|_{z=h^-} = 0, \tag{3.4}$$

$$[\vec{n}\wedge\vec{H}_{tot}]_{z=h} := \vec{n}\wedge\vec{H}_{tot1}\Big|_{z=h^+} - \vec{n}\wedge\vec{H}_{tot2}\Big|_{z=h^-} = 0. \tag{3.5}$$

From Maxwells' equations and the constitutive linear laws $\vec{D} = \underline{\underline{\epsilon}}\vec{E}$ and $\vec{B} = \underline{\underline{\mu}}\vec{H}$ with piece–wise continuous $\underline{\underline{\epsilon}}$ and $\underline{\underline{\mu}}$ tensors follow

$$[\vec{n}\cdot\vec{D}]_{z=h} = 0, \quad \text{and} \quad [\vec{n}\cdot\vec{B}]_{z=h} = 0, \tag{3.6}$$

which have to be augmented by the *Silver–Müller radiation conditions* in the free space part Ω_1:

$$\omega\epsilon_0\frac{\underline{x}}{|\underline{x}|}\wedge\vec{E}_{scat} - k_0\vec{H}_{scat} = \mathbf{o}(|\underline{x}|^{-1}), \quad \text{as } |\underline{x}|\to\infty, \tag{3.7}$$

$$\omega\mu_0\frac{\underline{x}}{|\underline{x}|}\wedge\vec{H}_{scat} + k_0\vec{E}_{scat} = \mathbf{o}(|\underline{x}|^{-1}), \quad \text{as } |\underline{x}|\to\infty, \tag{3.8}$$

where we put $k_0 := \frac{\omega}{c_0}$ with $c_0 := (\epsilon_0\mu_0)^{-1/2}$ for the free space wave number and propagation velocity, respectively.

To solve this special boundary–transmission problem it makes sense to apply the 2D–Fourier transformation $\mathcal{F}$ in the distributional sense for $\mathcal{S}' = \mathcal{S}'(I\!R^2_{\underline{x}'})$ defined as usual by

$$\widehat{\varphi}(\underline{\xi}') = \widehat{\varphi}(\xi,\eta) := \mathcal{F}_{\underline{\xi}'\mapsto\underline{x}'}\varphi = \int\limits_{I\!R^2_{\underline{x}'}} \text{e}^{i(\underline{x}',\underline{\xi}')}\varphi(x,y)\,\text{d}x\text{d}y \tag{3.9}$$

for all $\varphi(\underline{x}') \in \mathcal{S}(I\!R^2_{\underline{x}'})$, the Schwartz space of rapidly decaying C^∞– functions, and for $u \in \mathcal{S}'$ the Fourier transform is defined by $(\widehat{u},\varphi) := (u,\widehat{\varphi})$ for all $\varphi \in \mathcal{S}$.

Then the usual rules known for $L_1(I\!R^2)$–functions extend and we arrive at the $\mathcal{F}$–transformed equations (see e.g. A. PASSOW's PhD-thesis (1998)[21], Chap.5 !) if we write $\underline{u} := (\underline{u}_e, \underline{u}_m)^T \in [L^2_{loc}(I\!R^3_+)]^4$ with $\underline{u}^j := \underline{u}|_{\Omega_j} \in [H^1_{loc}(\Omega_j)]^4 \cap [\mathcal{S}'(I\!R^2_{\underline{x}'})]^4$, $j = 1, 2$, being a weak solution of

$$(3.10) \qquad \frac{\partial}{\partial z}\underline{u}^1 = \begin{pmatrix} 0 & -\omega \underline{\underline{N}}\,\mu_0 + \frac{1}{\omega\epsilon_0}\nabla_s\nabla_s^T\underline{\underline{N}} \\ \omega\underline{\underline{N}}\,\epsilon_0 - \frac{1}{\omega\mu_0}\nabla_s\nabla_s^T\underline{\underline{N}} & 0 \end{pmatrix} \underline{u}^1$$

for $z > h$ in Ω_1, where

$$(3.11) \qquad \underline{\underline{N}} := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \nabla_s := (\frac{\partial}{\partial x}, \frac{\partial}{\partial y})^T,$$

$$(3.12) \qquad \nabla_s\nabla_s^T =: \underline{\underline{D}} = \begin{pmatrix} \frac{\partial^2}{\partial x^2} & \frac{\partial^2}{\partial x\partial y} \\ \frac{\partial^2}{\partial y\partial x} & \frac{\partial^2}{\partial y^2} \end{pmatrix}$$

and a solution of

$$(3.13) \qquad \underline{u}^2 = \underline{\underline{M}}(\nabla_s)\,\underline{u}^2, \quad \underline{\underline{M}}(\nabla_s) :=$$

$$\begin{pmatrix} \frac{1}{\mu_z}\underline{\underline{N}}\mu_{sz}\nabla_s^T\underline{\underline{N}} - \frac{1}{\varepsilon_z}\nabla_s\varepsilon_{zs} & -\omega\underline{\underline{N}}\mu_s + \frac{\omega}{\mu_z}\underline{\underline{N}}\mu_{sz}\mu_{zs} + \frac{1}{\omega\varepsilon_z}\nabla_s\nabla_s^T\underline{\underline{N}} \\ \omega\underline{\underline{N}}\varepsilon_s - \frac{\omega}{\varepsilon_z}\underline{\underline{N}}\varepsilon_{sz}\varepsilon_{zs} - \frac{1}{\omega\mu_z}\nabla_s\nabla_s^T\underline{\underline{N}} & \frac{1}{\varepsilon_z}\underline{\underline{N}}\varepsilon_{sz}\nabla_s^T\underline{\underline{N}} - \frac{1}{\mu_z}\nabla_s\mu_{zs} \end{pmatrix}$$

for $0 < z < h$ in Ω_2, where for the anisotropic case in Ω_2 is written

$$(3.14) \qquad \underline{\underline{\epsilon}}_2 := \begin{pmatrix} \epsilon_s & \epsilon_{sz} \\ \epsilon_{zs} & \epsilon_z \end{pmatrix}, \quad \underline{\underline{\mu}}_2 := \begin{pmatrix} \mu_s & \mu_{sz} \\ \mu_{zs} & \mu_z \end{pmatrix},$$

with the additional boundary condition

$$(3.15) \qquad \underline{u}_e^1\big|_{z=0^+} = 0$$

and the transmission conditions on $\Gamma:\ z = h$:

$$(3.16) \qquad [\underline{u}]|_{z=h} := \underline{u}^1 - \underline{u}^2 = \underline{f} \in [H^{\frac{1}{2}}_{loc}(\Gamma)]^4 \cap [\mathcal{S}'(\Gamma)]^4.$$

We are able now to formulate the

Theorem 3.2. ***(Representation formula).*** *The function $\underline{u} \in [H_1(I\!R^3_+)]^6$ with $\underline{u} := (\underline{u}_e, \underline{u}_m)^T = \underline{u}^{1,2} := \underline{u}|_{\Omega_{1,2}} \in \mathbb{C}^6$ is a solution to problem 3.1 iff it has the representation*

$$\begin{aligned} \underline{u}(x,y,z) &= \chi_+(z)\chi_+(d-z)\sum_{\ell=1}^{4}\mathcal{F}^{-1}_{\underline{\xi}'\to\underline{x}'}\left\{\widehat{c}_{\ell,1}(\xi,\eta)e^{-\lambda_{\ell,1}(\xi,\eta)z}\mathcal{F}\vec{v}_{\ell,1}\right\} \\ (3.17) \qquad &+ \chi_+(z-d)\sum_{\ell=1}^{2}\mathcal{F}^{-1}_{\underline{\xi}'\to\underline{x}'}\left\{\widehat{c}_{\ell,2}(\xi,\eta)e^{-\lambda_{\ell,2}(\xi,\eta)z}\mathcal{F}\vec{v}_{l,2}\right\} \end{aligned}$$

for $(x,y,z) \in I\!R^3_+$ *with the characteristic function* χ_+ *on* $I\!R_+$. *The coefficients* $\widehat{c}_{\ell,1}$ *and* $\widehat{c}_{\ell,2}$ *are to be determined from the linear system*

$$\text{(3.18)}\quad \begin{pmatrix} \hat{f} \\ 0 \end{pmatrix} = \begin{pmatrix} e^{-\lambda_{1,1}d}\mathcal{F}\vec{v}_{1,1} & \cdots & e^{-\lambda_{4,1}d}\mathcal{F}\vec{v}_{4,1} & -e^{-\lambda_{1,2}d}\mathcal{F}\vec{v}_{1,2} & -e^{-\lambda_{2,2}d}\mathcal{F}\vec{v}_{2,2} \\ \mathcal{F}\vec{v}^e_{1,1} & \cdots & \mathcal{F}\vec{v}^e_{1,4} & 0 & 0 \end{pmatrix} \begin{pmatrix} c_{1,1} \\ \vdots \\ c_{4,1} \\ c_{1,2} \\ c_{2,2} \end{pmatrix}$$

The coefficients $\widehat{c}_{\ell,1}$, $\widehat{c}_{\ell,2}$, six in total, represent the waves going in z–direction, $\widehat{c}_{\ell,1}$ particularly are the reflexion coefficients in the upper domain Ω_1: $z > 0$. The eigenvalues $\lambda_{\ell,1} \in \mathbb{C}_{++}$ and the others $\lambda_{\ell,2} \in \mathbb{C}_{++}$, and eigenvectors $\vec{v}_{\ell,1}$ and $\vec{v}_{\ell,2}$ are due to the $\mathcal{F}$–transformed system of partial differential equations of order one

$$\text{(3.19)}\qquad \frac{\partial}{\partial z}\mathcal{F}\underline{u}^{1,2} = \mathcal{F}\{\underline{\underline{M}}^{1,2}(\nabla_s)\,\underline{u}^{1,2}\} = \underline{\underline{M}}^{1,2}\,(-i\xi')\,\mathcal{F}\underline{u}^{1,2}$$

corresponding for 1 to $z > h$ and for 2 to $0 < z < h$. It has been shown by A. Passow [20], [21], that this boundary transmission problem can be approximated step by step for $z > h$ by a set of so-called *anisotropic impedance* or *Leontovich boundary conditions* on $z = 0^+$ if h is small. Here we formulate only that one of order zero in the case of anisotropy for a Sommerfeld half-plane.

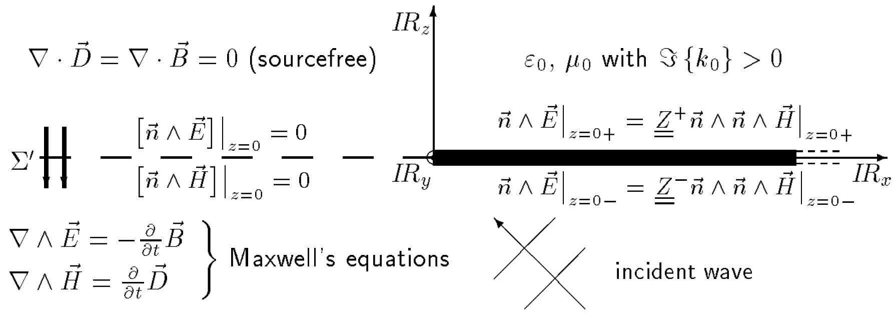

Fig.5: Anisotropic Sommerfeld problem with Leontovich cs. on screen Σ.

Due to the different behavior of the components of the electrical field $\vec{E}$ and the magnetic field $\vec{H}$ othogonal and parallel to the edge $x = z = 0$, $y \in I\!R$ of the screen $\Sigma \subset I\!R^3$ it is useful to introduce anisotropic Sobolev spaces according to

Definition 3.3. The anisotropic *Bessel potential* spaces $H_{r,s}(I\!R^3_\pm)$ are defined for all $r, s \in I\!R$ by

$$H_{r,s}(I\!R^3_\pm) := \left\{ u \in L_2(I\!R^3_\pm) :\ \mathcal{F}^{-1}_{\underline{\xi}'\mapsto\underline{x}'}(1+\xi^2)^{r/2}(1+\eta^2)^{s/2}\mathcal{F}_{\underline{x}'\mapsto\underline{\xi}'}u \in L_2(I\!R^3_\pm) \right\}. \tag{3.20}$$

In the case of isotropic spaces $r = s$ we write only $H_s(I\!R^3_\pm)$. A corresponding notation is used for the trace spaces on $I\!R'^2$.

We arrive at

Problem 3.4. Find a function $\underline{u}' := (\vec{E}, \vec{H})^T \in \mathbb{C}^6$ s.th. $\vec{E}, \vec{H} \in [L_2(I\!R^3)]^3$ with

$$\underline{u}'_\pm := \underline{u}'|_{I\!R^3_\pm} \in [H_{0,1}(I\!R^3_\pm) \times H_{1,0}(I\!R^3_\pm) \times L_2(I\!R^3_\pm)]^2 \tag{3.21}$$

which are weak solutions to Maxwell's equations

$$\operatorname{curl} \vec{E} - i\omega\mu_0 \vec{H} = \vec{0}, \qquad \operatorname{curl} \vec{H} + i\omega\epsilon_0 \vec{E} = \vec{0} \tag{3.22}$$

in $I\!R^3_+ \cup I\!R^3_-$ fulfilling the Leontovich boundary conditions on $\Sigma^\pm$

$$\pm\, \vec{e}_z \wedge \vec{E}^\pm = \underline{\underline{Z}}' \left(\vec{e}_z \wedge \left(\vec{e}_z \wedge \vec{H}^\pm\right)\right) + \underline{f}'_\pm \tag{3.23}$$

and the transmission conditions

$$\vec{e}_z \wedge \left(\vec{E}^+ - \vec{E}^-\right)\Big|_{z=0} = 0, \qquad \vec{e}_z \wedge \left(\vec{H}^+ - \vec{H}^-\right)\Big|_{z=0} = 0 \tag{3.24}$$

on the complementary screen $\Sigma'_\pm = I\!R'^2 \setminus \Sigma$ containing the trace values $\vec{E}^\pm_0$, $\vec{H}^\pm_0$ of the electrical and magnetical field vectors on the plane $I\!R'^2_{\underline{x}'}$ of the screen Σ. Due to trace theorems it holds

$$\vec{E}^\pm_0,\ \vec{H}^\pm_0 \in H_{-\frac{1}{2},\frac{1}{2}}(I\!R'^2) \times H_{-\frac{1}{2}}(I\!R'^2), \tag{3.25}$$

and the data on $\Sigma^\pm$ have to fulfil the compatibility condition of compact support

$$\underline{f}'_+ - \underline{f}'_- \in [\tilde{H}_{-\frac{1}{2}}(\Sigma)]^4 \tag{3.26}$$

The Leontovich boundary condition for the plane $\Sigma \subset I\!R'^2$ has only two components and the impedance matrices may be split into

$$\underline{\underline{Z}}'^\pm = \begin{pmatrix} \underline{\underline{Z}}^\pm & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.27}$$

We arrive then at

Problem 3.5. Find a solution $\underline{u} = (\underline{u}_1, \underline{u}_2)^T$ with $u_l \in [L_2(I\!R^3)]^2$; $l = 1, 2$; s.th.

$$\underline{u}^\pm = \underline{u}|_{I\!R^3_\pm} \in [H_{0,1}(I\!R^3_\pm) \times H_{1,0}(I\!R^3_\pm)]^2 \tag{3.28}$$

which are weak solution to

$$\frac{\partial}{\partial z}\underline{u} = \begin{pmatrix} 0 & -(i\omega\varepsilon_0)^{-1}(k_0^2 \underline{\underline{I}}_2 + \underline{\underline{D}})\underline{\underline{N}} \\ (i\omega\mu_0)^{-1}(k_0^2 \underline{\underline{I}}_2 + \underline{\underline{D}})\underline{\underline{N}} & 0 \end{pmatrix} \underline{u} \tag{3.29}$$

in $I\!R^3_\pm$ with $\underline{\underline{D}}$ and $\underline{\underline{N}}$ given through (3.11–3.12). The solution $\underline{u}$ has to fulfil the *Leontovich boundary conditions*

$$\pm\,\underline{\underline{N}}\,\underline{u}_{10}^{\pm} + \underline{\underline{Z}}^{\pm}\,\underline{u}_{20}^{\pm} = \underline{f}^{\pm} \qquad \text{on} \quad \Sigma^{\pm} \tag{3.30}$$

and the *transmission conditions*

$$\underline{u}_0^{+} - \underline{u}_0^{-} = 0 \qquad \text{on} \quad \Sigma' \tag{3.31}$$

with the traces $\underline{u}_{10}^{\pm}$, $\underline{u}_{20}^{\pm}$ of the electrical and magnetical tangential components on the plane $I\!R'^{\,2}_{\underline{x}'}$ involving the data

$$\underline{f}^{+} - \underline{f}^{-} \in [\tilde{H}_{-\frac{1}{2}}(\Sigma)]^2. \tag{3.32}$$

Now it can be shown that the problems 3.4 and 3.5 are equivalent and further it is true

Theorem 3.6. ***(Representation of tangential components).*** *A function $\underline{u} \in [L_2(I\!R^3)]^4$ with*

$$\underline{u}^{\pm} = \underline{u}|_{I\!R^3_\pm} \in [H_{0,1}(I\!R^3_\pm) \times H_{1,0}(I\!R^3_\pm)]^2 \tag{3.33}$$

is a solution of problem $\mathcal{P}$ iff it may be represented by

$$\underline{u}(x,y,z) = \mathcal{F}^{-1}_{\underline{\xi}'\mapsto\underline{x}'}\left\{\widehat{\underline{u}}_0^{+}(\xi,\eta)e^{-t(\xi,\eta)z}\chi_+(z) + \widehat{\underline{u}}_0^{-}(\xi,\eta)e^{t(\xi,\eta)z}\chi_-(z)\right\} \tag{3.34}$$

where the trace values transformed are given through

$$\widehat{\underline{u}}_0^{\pm}(\xi,\eta) = \widehat{\alpha}_\pm(\xi,\eta)\begin{pmatrix} 1 \\ \frac{\xi\eta}{\xi^2-k_0^2} \\ 0 \\ \pm\frac{i\omega\varepsilon\, t(\xi,\eta)}{\xi^2-k_0^2}\end{pmatrix} + \widehat{\beta}_\pm(\xi,\eta)\begin{pmatrix} 0 \\ \mp\frac{i\omega\mu\, t(\xi,\eta)}{\xi^2-k_0^2} \\ 1 \\ \frac{\xi\eta}{\xi^2-k_0^2}\end{pmatrix} \tag{3.35}$$

and $t(\xi,\eta) := \sqrt{\xi^2+\eta^2-k_0^2}$ for $(\xi,\eta) \in I\!R^2$ the characteristic square root with branch cuts from $\zeta = \pm k$ to $\pm i\infty$ in the complex ζ–plane for $\sqrt{\zeta^2-k^2}$.

From Maxwell's equations one can calculate the remaining normal components H_z, E_z from the electrical and magnetical tangential fields, respectively, leading to

Theorem 3.7. ***(Representation of normal components).*** *Denoting the pair of normal components by $\underline{v} \in [L_2(I\!R^3)]^2$ and their restrictions to $I\!R^3_\pm$ there hold the representations*

$$\begin{aligned} v_1(x,y,z) \;&=\; \mathcal{F}^{-1}\left\{\frac{\xi t(\xi,\eta)}{\xi^2-k_0^2}\left(e^{-t(\xi,\eta)z}\mathcal{F}\alpha_+\chi_+(z) - e^{t(\xi,\eta)z}\mathcal{F}\alpha_-\chi_-(z)\right)\right\} \\ &-\; \mathcal{F}^{-1}\left\{\frac{i\omega\mu_0\eta}{\xi^2-k_0^2}\left(e^{-t(\xi,\eta)z}\mathcal{F}\beta_+\chi_+(z) - e^{t(\xi,\eta)z}\mathcal{F}\beta_-\chi_-(z)\right)\right\} \end{aligned} \tag{3.36}$$

$$v_2(x,y,z) = \mathcal{F}^{-1}\left\{\frac{i\omega\varepsilon_0\eta}{\xi^2-k_0^2}\left(e^{-t(\xi,\eta)z}\mathcal{F}\alpha_+\chi_+(z)+e^{t(\xi,\eta)z}\mathcal{F}\alpha_-\chi_-(z)\right)\right\}$$

$$(3.37)\qquad + \mathcal{F}^{-1}\left\{\frac{t(\xi,\eta)\xi}{\xi^2-k_0^2}\left(e^{-t(\xi,\eta)z}\mathcal{F}\beta_+\chi_+(z)-e^{t(\xi,\eta)z}\mathcal{F}\beta_-\chi_-(z)\right)\right\}$$

for $z \in I\!R_+$ *with the characteristic functions* $\chi_\pm(z)$.

For details of the proofs see A. Passow's PhD-thesis at Darmstadt (1998) [21].

We shall transform the mixed boundary transmission problem into a system of Wiener–Hopf integral equations for the tangential components $\underline{\phi}_+ := \underline{u}_0^+ - \underline{u}_0^- \in [H_{-\frac{1}{2},\frac{1}{2}}(I\!R^2)\times H_{\frac{1}{2},-\frac{1}{2}}(I\!R^2)]^2$ having supports on Σ. The Dirichlet trace data (3.35) for problem 3.5 are connected to $\underline{\phi}^+$ by $\underline{\gamma} = B\underline{\phi}^+$, where $\underline{\gamma} := (\alpha_+,\beta_+,\alpha_-,\beta_-)^T$ with the invertible pseudodifferential operator

$$(3.38)\qquad B : \mathcal{F}^{-1}\underline{\underline{\sigma}}_B\mathcal{F} : [H^+_{-\frac{1}{2},\frac{1}{2}}(I\!R^2)\times H^+_{\frac{1}{2},-\frac{1}{2}}(I\!R^2)]^2 \mapsto [H_{-\frac{1}{2},\frac{1}{2}}(I\!R^2)]^4$$

with its Fouriersymbol matrix

$$(3.39)\qquad \underline{\underline{\sigma}}_B(\xi,\eta) = \frac{1}{2}\begin{pmatrix} 1 & 0 & -\frac{\xi\eta}{t_1} & \frac{\xi^2-k_0^2}{t_1} \\ \frac{\xi\eta}{t_2} & -\frac{\xi^2-k_0^2}{t_2} & 1 & 0 \\ -1 & 0 & -\frac{\xi\eta}{t_1} & \frac{\xi^2-k_0^2}{t_1} \\ \frac{\xi\eta}{t_2} & -\frac{\xi^2-k_0^2}{t_2} & -1 & 0 \end{pmatrix}$$

with the square roots $t_1 := i\omega\varepsilon_0\, t(\xi,\eta)$ and $t_2 := i\omega\mu_0\, t(\xi,\eta)$.

Now the Dirichlet data are uniquely connected with the ansatz vector $\underline{\gamma}$ and the jumps by $r_+C\underline{\underline{\gamma}} = \underline{f}$, r_+ the restriction to Σ. C again is a pseudodifferential operator

$$(3.40)\qquad C = \mathcal{F}^{-1}\underline{\underline{\sigma}}_c\mathcal{F} : [H_{-\frac{1}{2},\frac{1}{2}}(I\!R^2)]^4 \mapsto [H_{-\frac{1}{2}}(I\!R^2)]^4$$

with its symbol matrix $\underline{\underline{\sigma}}_C :=$

$$\begin{pmatrix} -\frac{\xi\eta}{\xi^2-k_0^2}+\frac{t_1b^+}{\xi^2-k_0^2} & \frac{t_2}{\xi^2-k_0^2}+a^++\frac{b^+\xi\eta}{\xi^2-k_0^2} & 0 & 0 \\ 1+\frac{t_1d^+}{\xi^2-k_0^2} & c^++\frac{d^+\xi\eta}{\xi^2-k_0^2} & 0 & 0 \\ 0 & 0 & \frac{\xi\eta}{\xi^2-k_0^2}-\frac{t_1b^-}{\xi^2-k_0^2} & \frac{t_2}{\xi^2-k_0^2}+a^-+\frac{b^-\xi\eta}{\xi^2-k_0^2} \\ 0 & 0 & -1-\frac{t_1d^-}{\xi^2-k_0^2} & c^-+\frac{d^-\xi\eta}{\xi^2-k_0^2} \end{pmatrix}$$

(3.41)

with coefficients from the impedance tensors

$$(3.42)\qquad \underline{\underline{Z}}^\pm = \begin{pmatrix} a^\pm & b^\pm \\ c^\pm & d^\pm \end{pmatrix}.$$

In order to get an invertible operator C the determinant of its symbol has to be different from zero for all $(\xi,\eta) \in I\!R^2$. One can show that a necessary condition for this is given by

$$(\xi,\eta)\underline{\underline{Z}}^{\pm}(\xi,\eta)^T + ik_0 Z_0\left(\frac{\det \underline{\underline{Z}}^{\pm}}{Z_0^2}+1\right)t + (ik_0)^2\,\mathrm{tr}\,\underline{\underline{Z}}^{\pm} \neq 0. \tag{3.43}$$

Putting $\xi = 0$ in the 2D–case this leads to

$$t(\eta) \neq -\frac{ik_0}{2d^{\pm}}\left(Z_0\det\underline{\underline{Z}}^{\pm} + \frac{1}{Z_0} \mp \sqrt{\left(Z_0\det\underline{\underline{Z}}^{\pm}+\frac{1}{Z_0}\right)^2 - 4a^{\pm}d^{\pm}}\right). \tag{3.44}$$

The isotropic case of $\vec{n}\wedge\vec{E} = \underline{\underline{Z}}\vec{n}\wedge\left(\vec{n}\wedge\vec{H}\right)$ drops down to classical impedance boundary conditions (with $\underline{\underline{Z}} = Z\underline{\underline{I}}_3$) for the (scalar) normal components

$$\frac{\partial E_n}{\partial n} - ik_0\frac{Z}{Z_0}E_n = 0 \qquad \text{and} \qquad \frac{\partial H_n}{\partial n} - ik_0\frac{Z_0}{Z}H_n = 0 \tag{3.45}$$

with the invertibility condition $t \neq -i\omega\varepsilon_0 Z$ and $t \neq -i\omega\frac{\mu_0}{Z}$ for all $(\xi,\eta) \in I\!R^2$. Combining the formulae above we get the pre–Wiener-Hopf operator for $r_+CB\underline{\Phi}^+ = \underline{f}$ whose pre-symbol $\underline{\underline{\sigma}}_{CB}$ will not be written down here. But that one which results after pair–wise addition and subtraction of the four equations involved in the last equation to give

$$\widetilde{W}\underline{\Phi}^+ := r_+W|_X = \underline{\widetilde{f}} := (\underline{f}^+ + \underline{f}^-, \underline{f}^+ - \underline{f}^-)^T \tag{3.46}$$

acting on $X := [H^+_{-\frac{1}{2},\frac{1}{2}}(I\!R^2) \times H^+_{\frac{1}{2},-\frac{1}{2}}(I\!R^2)]^2 \to [H_{-\frac{1}{2}}(\Sigma)]^2 \times [\widetilde{H}_{-\frac{1}{2}}(\Sigma)]^2$ with $\underline{\underline{\sigma}}_W :=$

$$\begin{pmatrix} \frac{a_+\xi\eta+b_+(\eta^2-k_0^2)}{ik_0Z_0t} & -1-\frac{a_+(\xi^2-k_0^2)+b_+\xi\eta}{ik_0Z_0t} & a_- & b_- \\ 1+\frac{c_+\xi\eta+d_+(\eta^2-k_0^2)}{ik_0Z_0t} & -\frac{c_+(\xi^2-k_0^2)+d_+\xi\eta}{ik_0Z_0t} & c_- & d_- \\ \frac{a_-\xi\eta+b_-(\eta^2-k_0^2)}{ik_0Z_0t} & -\frac{a_-(\xi^2-k_0^2)+b_-\xi\eta}{ik_0Z_0t} & Z_0\frac{\eta^2-k_0^2}{ik_0t}+a_+ & -Z_0\frac{\xi\eta}{ik_0t}+b_+ \\ \frac{c_-\xi\eta+d_-(\eta^2-k_0^2)}{ik_0Z_0t} & -\frac{c_-(\xi^2-k_0^2)+d_-\xi\eta}{ik_0Z_0t} & -Z_0\frac{\xi\eta}{ik_0t}+c_+ & Z_0\frac{\xi^2-k_0^2}{ik_0t}+d_+ \end{pmatrix} \tag{3.47}$$

where terms $x_\pm := \frac{1}{2}(x^+ \pm x^-)$ are of elements of $\underline{\underline{Z}}$. We arrive then at

Theorem 3.8. *Problem 3.5 is uniquely solvable iff the Wiener–Hopf operator* $\widetilde{W}$ *is invertible. Furthermore there holds*

(i) If $\underline{u}$ *is a solution to problem 3.5 with its traces* $\underline{u}^{\pm}$ *described by the ansatz vector* $\underline{\gamma}$ *then* $\underline{\Phi}^+ = B^{-1}\underline{\gamma}$ *is a solution to the Wiener–Hopf equation (3.46).*

(ii) If $\underline{\Phi}^+$ is a solution to (3.46) then the function $\underline{u}$ given by the representation formulae (3.34) with (3.35) substituted and $\underline{\gamma} = B\underline{\Phi}^+$ introduced is a solution of problem 3.5.

It can be shown, even in the case of equal impedance matrices $\underline{\underline{Z}}^+ = \underline{\underline{Z}}^- = \underline{\underline{Z}}$, that the Wiener–Hopf system is not invertible due to a non–trivial kernel of the operator on the prescribed space X (see [21]). In order to get rid of the situation we shall act as in the isotropic case of scalar impedance boundary conditions (cf. Dos Santos et al. (1989) [3]) when smoother spaces were introduced. So we transform into

Problem 3.9. For given $\underline{f}^\pm \in \left[H_{-\frac{1}{2}+\varepsilon}(\Sigma)\right]^2$ s.th. $\underline{f}^+ - \underline{f}^- \in \left[\tilde{H}_{-\frac{1}{2}+\varepsilon}(\Sigma)\right]^2$, $0 < \varepsilon < 1$, find a function $\underline{u} := (u_1, u_2)^T \in [L_2(I\!R^3)]^4$ with

$$\underline{u}^\pm = \underline{u}|_{I\!R^3_\pm} \in \left[H_{\varepsilon,1+\varepsilon}\left(I\!R^3_\pm\right) \times H_{1+\varepsilon,\varepsilon}\left(I\!R^3_\pm\right)\right]^2, \tag{3.48}$$

which is a weak solution to the modified Maxwell equations

$$\frac{\partial}{\partial z}\underline{u} = \begin{pmatrix} 0 & -(i\omega\varepsilon_0)^{-1}\left(k_0^2\underline{\underline{I}}_2 + \underline{\underline{D}}\right)\underline{\underline{N}} \\ (i\omega\mu_0)^{-1}\left(k_0^2\underline{\underline{I}}_2 + \underline{\underline{D}}\right)\underline{\underline{N}} & 0 \end{pmatrix}\underline{u} \tag{3.49}$$

with the 2×2 matrices $\underline{\underline{D}}$, $\underline{\underline{N}}$ defined in (3.11–3.12), and fulfils the boundary (3.30) and transmission conditions (3.31) of problem 3.5 now with the traces for the electrical, $\ell = 1, 2$, and the magnetical $\ell = 1, 2$, tangential components on $z = 0$:

$$\underline{u}^\pm_{\ell 0} \in \tilde{H}_{-\frac{1}{2}+\varepsilon,\frac{1}{2}+\varepsilon}(I\!R^2) \times \tilde{H}_{\frac{1}{2}+\varepsilon,-\frac{1}{2}+\varepsilon}(I\!R^2). \tag{3.50}$$

We are led by the same arguments as before to a theorem of equivalence which is not repeated here.

If we assume $0 < \varepsilon < 1$ we have $[\tilde{H}_{-\frac{1}{2}+\varepsilon}(\Sigma)]^2$ and the corresponding Wiener–Hopf operator $\widetilde{W}_\varepsilon$ acts on $[H^+_{-\frac{1}{2}+\varepsilon,\frac{1}{2}+\varepsilon}(I\!R^2) \times H^+_{\frac{1}{2}+\varepsilon,-\frac{1}{2}+\varepsilon}(I\!R^2)]^2 \to [H_{-\frac{1}{2}+\varepsilon}(\Sigma)]^4$ in which case the system of pseudodifferential equations may be reduced equivalently to a system of Wiener–Hopf equations on $[L_2^+(\cdot, I\!R)]^4$ with supports on $I\!R^+$ with η as a parameter.

Using Bessel potential operators $\Lambda_{\pm,\varepsilon}$ with symbols $\underline{\underline{\sigma}}_{\pm,\varepsilon}(\xi,\eta)$ given by

$$\underline{\underline{\sigma}}_{+,\varepsilon} = \operatorname{diag}\left(t_+^{1+2\varepsilon}, t_+^{-1+2\varepsilon}, t_+^{1+2\varepsilon}, t_+^{-1+2\varepsilon}\right) \tag{3.51}$$

$$\underline{\underline{\sigma}}_{-,\varepsilon} = t_-^{1-2\varepsilon}\underline{\underline{I}}_4, \tag{3.52}$$

where $t = \sqrt{\xi^2 + \eta^2 - k_0^2} = t_-(\xi,\eta)\, t_+(\xi,\eta)$ and

$$t_\pm(\xi,\eta) := \sqrt{\xi \pm i\sqrt{\eta^2 - k_0^2}}, \quad \Re\sqrt{\eta^2 - k_0^2} \geq 0, \tag{3.53}$$

having branch cuts in the complex ζ–plane from $\zeta = \pm i\sqrt{\eta^2 - k_0^2}$ to $\pm i\infty$, s.th. t_+ and t_- are holomorphically extendable into the complex halves $\mathbb{C}_{\zeta+}$, $\mathbb{C}_{\zeta-}$, respectively, the new Wiener–Hopf operator is given by

$$\widetilde{W}_0(\varepsilon) = \mathcal{P}^+ W_0(\varepsilon) = \mathcal{P}^+ \mathcal{F}^{-1} \underline{\underline{\sigma}}_\varepsilon \cdot \mathcal{F} \tag{3.54}$$

acting on $[L_2^+(\cdot, I\!R)]^4$ with a 4×4 Fouriersymbol matrix $\underline{\underline{\sigma}}_\varepsilon$ similar to (3.47), given now by

$$\underline{\underline{\sigma}}_\varepsilon = \underline{\underline{\sigma}}_{-,\varepsilon}^{-1} \, \underline{\underline{\sigma}}_W \, \underline{\underline{\sigma}}_{+,\varepsilon}^{-1} = \left(\frac{t_+}{t_-}\right)^{1-2\varepsilon} \underline{\underline{\sigma}}_W \; \mathrm{diag}\,\left(t_+^{-2}, 1, t_+^{-2}, 1\right). \tag{3.55}$$

In the special cases of equal impedance matrices $\underline{\underline{Z}}^+ = \underline{\underline{Z}}^-$, and for 2D dependence, the symbol matrices simplify appreciably and decouple into 2×2 Wiener–Hopf symbols, separately for the electrical and magnetical tangential components (see corollaries 3,4,5 in [21]!). The main result is now

Theorem 3.10. *The equivalent lifted Wiener–Hopf equation*

$$\widetilde{W}(\varepsilon)\, \underline{\Phi}^+ = \widetilde{\Lambda}_{-,\varepsilon}^{-1}\, \underline{f} \quad \textit{with} \quad \underline{\Phi}^+ := \widetilde{\Lambda}_{+,\varepsilon}\, \underline{\phi}^+ \tag{3.56}$$

has a piece-wise continuous symbol on $\dot{I\!R}_\xi$ for every fixed $\eta \in I\!R$ and is Fredholm iff $\varepsilon \neq (n-1)/2$, $n \in I\!N$, and has Ind $\widetilde{\underline{\underline{W}}}_0(\epsilon) = 0$ for $1/4 \leq \varepsilon < 1/2$.

Remark 3.11. The proof is omitted here but see [21]. It relies on the well-known fact (see e.g. Mikhlin & Prössdorf [13]) that $\widetilde{W}_0(\varepsilon)$ is a Fredholm operator iff $\det \underline{\underline{\sigma}}'_\varepsilon(\xi, \eta; \mu) \neq 0$ for $(\xi, \eta; \mu) \in \dot{I\!R} \times I\!R \times [0,1]$ with $\underline{\underline{\sigma}}'_\varepsilon$ given through

$$\underline{\underline{\sigma}}'_\varepsilon(\xi, \eta; \mu) = \begin{cases} \underline{\underline{\sigma}}_\varepsilon(\xi + 0, \eta)\mu + (1-\mu)\underline{\underline{\sigma}}_\varepsilon(\xi, \eta) & : \ \xi, \eta \in I\!R,\ \mu \in [0,1] \\ \underline{\underline{\sigma}}_\varepsilon(-\infty, \eta)\mu + (1-\mu)\underline{\underline{\sigma}}_\varepsilon(\infty, \eta) & : \ \eta \in I\!R,\ \mu \in [0,1]. \end{cases}$$

If we assume identical behavior of both sides of the Sommerfeld screen Σ and write for the impedance matrix

$$\underline{\underline{Z}} = Z_0 \begin{pmatrix} \frac{a}{ik_0} & ik_0 b \\ \frac{c}{ik_0} & ik_0 d \end{pmatrix}, \tag{3.57}$$

the 4×4 symbol matrix $\underline{\underline{\sigma}}_\varepsilon$ gets the diagonal 2×2 block structure $\underline{\underline{\sigma}}_\varepsilon = \mathrm{Diag}(\underline{\underline{\sigma}}_{\varepsilon,1}, \underline{\underline{\sigma}}_{\varepsilon,2})$ containing the 2×2 blocks

$$\underline{\underline{\sigma}}_{\varepsilon,1} = \left(\frac{t_+}{t_-}\right)^{1-2\varepsilon} \begin{pmatrix} b\frac{t_-}{t_+} & -1 - \frac{a}{t} \\ \left(\frac{1}{t} + d\right)\frac{t_-}{t_+} & -\frac{c}{t} \end{pmatrix}, \tag{3.58}$$

$$\underline{\underline{\sigma}}_{\varepsilon,2} = \left(\frac{t_+}{t_-}\right)^{1-2\varepsilon} \begin{pmatrix} \frac{Z_0}{ik_0}\left(1+\frac{a}{t}\right)\frac{t_-}{t_+} & ik_0 Z_0 b \\ \frac{Z_0}{ik_0}\frac{c}{t}\frac{t_-}{t_+} & ik_0 Z_0\left(\frac{1}{t}+d\right) \end{pmatrix}. \tag{3.59}$$

Since the measure of the set $\left\{\xi \in I\!R : \lambda_{max}[(\underline{\underline{I}} - \underline{\underline{\sigma}}_\epsilon(\xi))^*(\underline{\underline{I}} - \underline{\underline{\sigma}}_\epsilon(\xi))] > 1\right\} \neq \emptyset$ a theorem by E. Meister & F.-O. Speck (1979)[17] cannot be used to guarantee the invertibility but there are sufficient conditions on the coefficients a_+, b_+, c_+, d_+ given in theorem 7 of Passow's PhD–thesis [21] which allow to calculate the inverse operator $[\widetilde{W}_o(\epsilon)]^{-1}$ by a Neumann series of its symbol $\underline{\underline{\sigma}}_\epsilon^{-1} = \theta^{-1}\left(\underline{\underline{I}} - (\underline{\underline{I}} - \theta^{-1}\underline{\underline{\sigma}}_\epsilon)\right)^{-1}$ with a suitable θ s.th. $\left\|\underline{\underline{I}} - \theta^{-1}\underline{\underline{\sigma}}_\epsilon\right\| < 1$ in an appropriate matrix norm.

4. Concluding remarks

A general inversion procedure by matrix factorization of the 4×4 lifted Fourier symbol is not yet known up to know. There exist only results for above mentioned special cases of isotropic and 2D situations reducing to the scalar Leontovich boundary conditions for the normal components of the electrical and magnetic fields at the Sommerfeld half–plane. Higher order Leontovich conditions on Σ have still to be investigated when they lead to Fredholm mixed boundary value problems for smoother solutions in $I\!R^3_\pm$ and corresponding smoother data and boundary traces on Σ and its complement Σ' in $I\!R^2_{\underline{x}'}$. A further important step in the study of boundary–transmission problems for electromagnetic waves is to study the diffraction by plane polygonial screens like a quarter–plane $\subset I\!R^2_{\underline{x}'}$, or by an octant $I\!R^3_{++}$ with different generalized anisotropic Leontovich boundary conditions on the three faces.

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