SCATTERING OF ACOUSTICAL AND ELECTROMAGNETIC WAVES BY SOME CANONICAL OBSTACLES

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Abstract

A. Sommerfeld published in 1896 his habilitation thesis studying the diffraction of plane time–harmonic acoustical waves by a half–plane or a wedge. He used series expansions and reformulation by integrals. These *Sommerfeld integrals* where systematically used by Russian authors. Their approach is called the *Maliushinets method*. Western authors use preferably the *Wiener–Hopf method*, based on the Fourier transforms and *symbol factorization* of the related boundary integral equations.
Many new results, including electromagnetic and elastodynamic waves with different kind of boundary conditions, different media, plane surfaces, and aperiodic initial boundary conditions, using generalized eigenfunction expansions, have been achieved during the last decade. Still a lot of open problems exist. The paper reports on more recent results obtained together with coworkers.

MOS key words: Diffraction Theory, Canonical Domains, Wiener–Hopf method, Factorization.

1 Introduction

A. SOMMERFELD was one of the first who formulated and treated a *canonical boundary value problem* for the Helmholtz equation which governs time–harmonic scalar waves. He presented 1896 his famous Goettingen Habilitation thesis [21], dealing with half–planes and wedges. He used series expansions w.r.t. the polar angle and Riemann surface concepts to arrive at the solutions for the Dirichlet boundary value problem at the two faces $\Gamma_1 := \{(x,y) \in \mathbf{R^2} : x \geq 0, y = 0\}$ and $\Gamma_2 := \{(x,y) \in \mathbf{R^2} : x = r\cos\alpha, y = r\sin\alpha, r \geq 0\}$ with $\alpha = 0^+$ or $0 < \alpha \leq \pi/2$. The *Sommerfeld integrals* could be achieved also by the *Wiener–Hopf method* [16, 22]. Many papers have been written on more general situations, like semi–infinite wave–guides of different cross–sections or systems of a finite or infinite number of parallel screens even with very general linear boundary conditions or generalizing to elastodynamical wave scattering by screen–like semi–infinite cracks [1, 4, 13, 16]. Here we cannot evaluate all the enumerable papers on the subject but want to concentrate on some more recent results concerning electromagnetic boundary–transmissions problems.

2 The canonical boundary–transmissions problems

The simplest canonical boundary value problems apply to n–dimensional half–spaces $\mathbf{R}^{\mathbf{n}}_{\pm}$, $n \geq 2$, for linear, elliptic partial differential equations or systems of such with constant coefficients. The theory of these is now standard and can be found in many textbooks [6] applying a $(n-1)$–dimensional Fouriertransformation w.r.t. $\underline{x}' = (x_1, \ldots, x_{n-1})$ for $\varphi \in \mathcal{S}(\mathbf{R}^{\mathbf{n}})$ or $f \in \mathcal{S}'(\mathbf{R}^{\mathbf{n}})$, the Schwartz spaces:

$$\mathcal{F}_{\underline{x}' \to \underline{\xi}'} \varphi = \hat{\varphi}(\underline{\xi}', x_n) := \int_{\mathbf{R}^{\mathbf{n-1}}_{\underline{\mathbf{x}}'}} e^{i<\underline{x}', \underline{\xi}'>} \varphi(\underline{x}', x_n)\, d\underline{x}' \tag{1}$$

and $\hat{f} \in \mathcal{S}'(\mathbf{R}^{\mathbf{n-1}}_{\underline{\xi}'})$ defined by

$$(\hat{f}, \varphi) = (f, \hat{\varphi}), \quad \forall \varphi \in \mathcal{S} \tag{2}$$

acting as a continuous linear functional taking into account that $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ is a linear topological isomorphism. Then the $\mathcal{F}$–Transformation algebraizes the operations od differentiation and partial convolution. By means of the $(n-1)$–dimensional $\mathcal{F}$–transform the partial differential equations are reduced to ordinary differential equations containing the $\mathcal{F}$–variable $\underline{\xi}'$ in polynomial form.
Lets take as a simple example the Helmholtz equation in $\mathbf{R}^{\mathbf{n}}_{\pm}$ which leads to

Definition 1 (Boundary value problem (B.v.p.) for $\mathbf{R}^{\mathbf{n}}_{+}$:) *Find $u \in C^2(\mathbf{R}^{\mathbf{n}}_{+}) \cap C^1_0(\overline{\mathbf{R}^{\mathbf{n}}_{+}})$ (classically) or $\in H^1_{loc}(\mathbf{R}^{\mathbf{n}}_{+}, \Delta) \cap \mathcal{S}'(\mathbf{R}^{\mathbf{n}}_{\underline{\xi}'})$ (in the weak sense), s.th. for given $F \in C^1_0(\overline{\mathbf{R}^{\mathbf{n}}_{+}})$ or $\in L^2(\overline{\mathbf{R}^{\mathbf{n}}_{+}})$, $f \in C^1_0(\mathbf{R}^{\mathbf{n-1}}_{\underline{\mathbf{x}}'})$ or $\in H^{1/2-m}(\mathbf{R}^{\mathbf{n-1}}_{\underline{\mathbf{x}}'})$, $m = 0, 1$, and constant $k = k_1 + ik_2$ with $k_1 > 0$ and $k_2 \geq 0$*

$$(\Delta_n + k^2) u = F \qquad in \quad \mathbf{R}^{\mathbf{n}}_{+} \tag{3}$$

in the strong (pointwise) or weak (distributional) sense. Additionally u has to fulfil the Dirichlet boundary condition

$$B_0[u] := u|_{\mathbf{R}^{\mathbf{n-1}}_{\underline{\mathbf{x}}'}} = f_0 \in C^1_0(\mathbf{R}^{\mathbf{n-1}}_{\underline{\mathbf{x}}'}), \tag{4}$$

or the Neumann condition

$$B_1[u] := u|_{\mathbf{R}^{\mathbf{n-1}}_{\underline{\mathbf{x}}'}} = -\left.\frac{\partial}{\partial x_n} u\right|_{\mathbf{R}^{\mathbf{n-1}}_{\underline{\mathbf{x}}'}} = f_1 \in C^1_0(\mathbf{R}^{\mathbf{n-1}}_{\underline{\mathbf{x}}'}), \tag{5}$$

or the Impedance boundary condition

$$B_2[u] := \left(-\frac{\partial}{\partial x_n} + ip\right) u\Big|_{\mathbf{R}^{\mathbf{n-1}}_{\underline{\mathbf{x}}'}} = f_{imp} \in C_0(\mathbf{R}^{\mathbf{n-1}}_{\underline{\mathbf{x}}'}). \tag{6}$$

In the Sobolev space setting the boundary functions are assumed as traces from u s.th. $f_0 \in H^{1/2}_{loc}(\mathbf{R}^{\mathbf{n-1}}_{\underline{\mathbf{x}}'}) \cap \mathcal{S}'(\mathbf{R}^{\mathbf{n-1}}_{\underline{\mathbf{x}}'})$ and $f_1, f_{imp} \in H^{-1/2}_{loc}(\mathbf{R}^{\mathbf{n-1}}_{\underline{\mathbf{x}}'}) \cap \mathcal{S}'(\mathbf{R}^{\mathbf{n-1}}_{\underline{\mathbf{x}}'})$.

Remark 1 *It is well known, that one has to add further conditions on the behavior of u as $r := |\underline{x}| \to \infty$ in $x_n \geq 0$. If $k_2 > 0$ one can look e.g. for decaying functions or such from the smaller space $H^1(\mathbf{R}^{\mathbf{n}}_+; \Delta)$. For $k = k_1 > 0$ one puts the* Sommerfeld radiation condition

$$\frac{\partial}{\partial r} u - iku = o(r^{\frac{1-n}{2}}) \qquad as \quad r \to \infty \tag{7}$$

to guarantee uniqueness of these b.v.p.'s.

Now the second fundamental situation in diffraction theory is given by

Definition 2 (Transmission problem (T.p.) for $\mathbf{R}^{\mathbf{n}}_{\pm}$:) *For given $F = (F_+, F_-)^T \in C_0(\overline{\mathbf{R}^{\mathbf{n}}_+}) \times C_0(\overline{\mathbf{R}^{\mathbf{n}}_-})$ or $\in L^2(\mathbf{R}^{\mathbf{n}}_+) \times L^2(\mathbf{R}^{\mathbf{n}}_-)$, and $f \in C^1_0(\overline{\mathbf{R}^{\mathbf{n}}_{\underline{\mathbf{x}}}})$, $g \in C_0(\overline{\mathbf{R}^{\mathbf{n}}_{\underline{\mathbf{x}}}})$ (classically) or $f \in H^{1/2}_{loc}(\mathbf{R}^{\mathbf{n-1}}_{\underline{\mathbf{x}}'}) \cap \mathcal{S}'(\mathbf{R}^{\mathbf{n-1}}_{\underline{\mathbf{x}}'})$, $g \in H^{-1/2}_{loc}(\mathbf{R}^{\mathbf{n-1}}_{\underline{\mathbf{x}}'}) \cap \mathcal{S}'(\mathbf{R}^{\mathbf{n-1}}_{\underline{\mathbf{x}}'})$ (weakly) and constants $k^{\pm} = k^{\pm}_1 + ik^{\pm}_2$ with $k^{\pm}_1 > 0$ and $k^{\pm}_2 \geq 0$, $\varrho^{\pm}$ similar, find a function $u = (u_+, u_-)^T \in \left(C^2(\mathbf{R}^{\mathbf{n}}_+) \times C^2(\mathbf{R}^{\mathbf{n}}_-)\right) \cap \left(C^1_0(\overline{\mathbf{R}^{\mathbf{n}}_+}) \times C^1_0(\overline{\mathbf{R}^{\mathbf{n}}_-})\right)$ (classically) or $\in \left(H^1_{loc}(\mathbf{R}^{\mathbf{n}}_+; \Delta) \times H^1_{loc}(\mathbf{R}^{\mathbf{n}}_-; \Delta)\right) \cap \left(\mathcal{S}'(\overline{\mathbf{R}^{\mathbf{n}}_+}) \times \mathcal{S}'(\overline{\mathbf{R}^{\mathbf{n}}_-})\right)$ (weakly) s.th.*

$$(\Delta_n + (k^{\pm})^2) u^{\pm} = F^{\pm} \qquad in \quad \mathbf{R}^{\mathbf{n}}_{\pm} \tag{8}$$

in the strong or weak sense, respectively. Additionally the two transmission conditions *have to hold*

$$\mathrm{Tr}_0\, u := u^+|_{\mathbf{R}^{\mathbf{n-1}}_{\underline{\mathbf{x}}'}} - u^-|_{\mathbf{R}^{\mathbf{n-1}}_{\underline{\mathbf{x}}'}} = f, \tag{9}$$

$$\mathrm{Tr}_1\, u := \frac{1}{\varrho^+}\frac{\partial u^+}{\partial x_n}\Big|_{\mathbf{R}^{\mathbf{n-1}}_{\underline{\mathbf{x}}'}} - \frac{1}{\varrho^-}\frac{\partial u^-}{\partial x_n}\Big|_{\mathbf{R}^{\mathbf{n-1}}_{\underline{\mathbf{x}}'}} = g. \tag{10}$$

Remark 2 *In order to get a well–posed problem one has to add conditions concerning the behavior of the solution u for $r = |\underline{x}| \to \infty$ in $\mathbf{R}^{\mathbf{n}}_{\pm}$. They could be similar like in Remark 1: global L^2–integrability if $k^{\pm} \in \mathbf{C}^{++}$ or fulfilling the appropriate radiation conditions like (7) with different k's. For incident waves*

$$u^{\pm}_{inc}(\underline{x}) \sim e^{ik^{\pm}<\underline{e},\underline{x}>}, \tag{11}$$

with $|\underline{e}| = 1$ one arrives at the well–known Snellius law *at the interface $\partial\mathbf{R}^{\mathbf{n}}_+ = \partial\mathbf{R}^{\mathbf{n}}_- = \mathbf{R}^{\mathbf{n-1}}_{\underline{\mathbf{x}}'}$.*

Remark 3 *The above mentioned boundary and transmission conditions may be generalized to other differential equations and systems of them of elliptic type for smoothly bounded finite domains* $\Omega \subset \mathbf{R^n}$ *and* $\Omega' := \mathbf{R^n} \backslash \overline{\Omega}$. *There exists a fully developed theory making use of integral representations with fundamental solutions for the Lamé equations of linear elastodynamics, thermo–elastodynamics, and also for Maxwell's system — which for the stationary case is not even elliptic [2, 7, 9, 10, 14]*

There has been a growing interest by physicist and engineers in the behavior of structures having singularities in their geometry, like edges and vertices, under different constant or time–changing loads, like microwaves in radar technique or seismic waves in geophysics, but also the scattering and absorption of sound waves and pulses by traffic noise shielding walls. Nondestructive testing and tomography — as inverse scattering problems — rely very strongly on the well understood behavior of diffracted fields near the corners, at far distances, shortly after a wave hit the obstacle or a long time afterwards.
As the *main canonical obstacles* there are considered those with *semi–infinite* boundaries, like half- and quarter–planes, cones, octants, and pyramids. They model locally the most important geometrical singularities, not including cusps and needles. The actual physical behavior of the scattered fields depend also strongly on the types of boundary/transmission conditions like in Definitions (1) and (2), but there is now a great demand to improve the models of boundary material response to impinging waves. The combination of different materials to *composites* in elasticity or microwave electrical circuits lead to the mixed problems for thin layers of dielectrics and metal coatings on such substrates [20]. Of great interest are also the *chiral media*. One may find some informations about the state of the art in [14].
In the following two figures, you will see the basic canonical boundary –transmission problems for the scalar wave fields which arise in acoustics and for specially polarized electromagnetic waves having only one component of the electric or magnetic field vectors parallel to the infinite line–like edges. The goal in the scalar diffraction theory is to solve the initial boundary–transmission problems for the *d'Alembert wave equation*

$$\left(\frac{1}{c_j^2} \frac{\partial^2}{\partial t^2} - \Delta \right) U(\underline{x}, t) = F(\underline{x}, t) \tag{12}$$

in the *time half-cylinder* $\Omega_T := \Omega \times (0, T)$, $0 < T \leq \infty$ classically, i.e. $U \in C^2(\Omega_T) \cap C^1(\overline{\Omega_T})$ or in Sobolev–space valued functions with special weights w.r.t. time t to allow for a Laplace transformation

$$\tilde{U}(\underline{x}, s) := \int_0^\infty e^{-ts} U(\underline{x}, t)\, dt \tag{13}$$

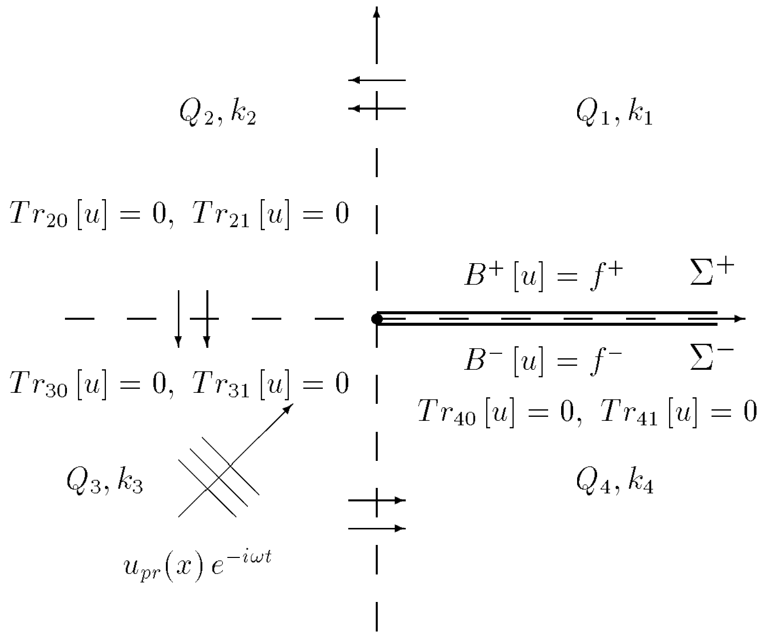


Figure 1: Sommerfeld problem with four media

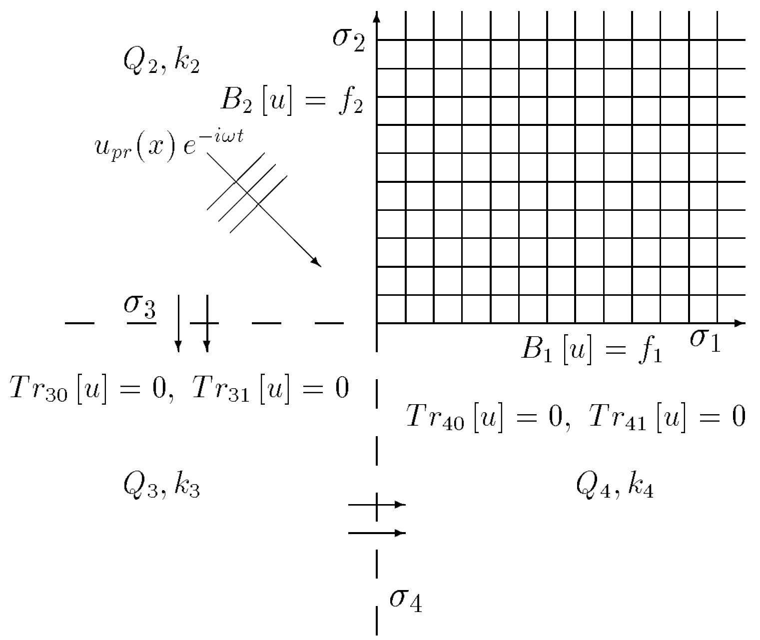


Figure 2: Wedge diffraction problem with three media

existing as parameter dependent Sobolev space functions from $H^1_{loc}(\Omega;\Delta)\cap\mathcal{S}'(\Omega;\Delta)$ when there are given (compatible) initial data

$$\begin{matrix} U(\underline{x},+0)=U_0(\underline{x}) \\ \frac{\partial}{\partial t}U(\underline{x},+0)=U_1(\underline{x}) \end{matrix} \quad \in \mathcal{S}'(\Omega) \tag{14}$$

The main question in this context are the short–time and the long–time behavior, particularly the validity of the *limiting amplitude principle* and the existence of the *general wave operators* of scattering theory.

Remark 4 *In the case of finite scatterers Ω such that $\Omega' = \mathbf{R^n}\backslash\overline{\Omega}$ is an* exterior domain *a lot is known about those questions [10], but in the case of semi–infinite obstacles, still a lot has to be done starting with the representation of the resolvents of the stationary boundary–transmission problems, particularly for the vectorial cases in electro- and elastodynamics.*

3 Electrodynamical Boundary–Transmission Problems

Let there be given a canonical domain $\Omega\subset\mathbf{R^3}$ with a piece–wise smooth boundary $\partial\Omega$ then we have

Definition 3 *If there are given $\underline{J}\in\mathbf{R^3}$, the electric current density, $\varrho\in\mathbf{R}$, the electrical charge density then we look for a quadruple of vectorfields $\in\mathbf{R^3}$, the electric, $\underline{E}$, the magnetic, $\underline{H}$, fieldstrength, and $\underline{D}$ the electric displacement and $\underline{B}$ the magnetic induction fulfilling the* Maxwell equations

$$\operatorname{curl}\underline{E}+\partial_t\underline{B}=\underline{0} \qquad \operatorname{div}\underline{B}=0 \tag{15}$$

$$\operatorname{curl}\underline{H}-\partial_t\underline{D}=\underline{J} \qquad \operatorname{div}\underline{D}=\varrho \tag{16}$$

Remark 5 *This system of eight partial differential equations has to be solved — in the classical or weak form — for $(\underline{x},t)\in\Omega_T:=\Omega\times(0,T)$ with additional boundary–transmission conditions on $\partial\Omega_t=\partial\Omega\times[0,T]$ where on Ω initial conditions are prescribed. Here the different types of boundary–transmission conditions will be formulated after specifying the* constitutive equations

Definition 4 *The material equations are*

$$\underline{D}(\underline{x},t) \;=\; \underline{\mathcal{D}}(\underline{E}(\underline{x},t),\underline{H}(\underline{x},t) \tag{17}$$

$$\underline{B}(\underline{x},t) \;=\; \underline{\mathcal{B}}(\underline{E}(\underline{x},t),\underline{H}(\underline{x},t) \tag{18}$$

which in general are nonlinear functions and even could depend on time history which could cause hysteresis effects. Neglecting these influences we arrive at the simplest situation of linear dependency

$$\begin{aligned} \underline{D}(\underline{x},t) &= \underline{\underline{\varepsilon}}\,\underline{E}(\underline{x},t) & (19)\\ \underline{B}(\underline{x},t) &= \underline{\underline{\mu}}\,\underline{H}(\underline{x},t) & (20)\\ \underline{J}(\underline{x},t) &= \underline{\underline{\sigma}}\,\underline{E}(\underline{x},t) \quad \textit{Ohm's law} & (21) \end{aligned}$$

$\underline{\underline{\varepsilon}}$, $\underline{\underline{\mu}}$ *and* $\underline{\underline{\sigma}}$ *are called respectively, the* permittivity, *the* permeability, *and the* conductivity *tensors. If they are multiples of the identity tensor* I_3 *the material is called* isotropic, *else* anisotropic.

By energy considerations with the *Poynting vector* it can be seen that for lossless media the relations

$$\underline{\underline{\varepsilon}} = \underline{\underline{\varepsilon}}^{\dagger} := (\underline{\underline{\varepsilon}}^{*})^{T} \qquad \text{and} \qquad \underline{\underline{\mu}} = \underline{\underline{\mu}}^{\dagger} = (\underline{\underline{\mu}}^{*})^{T} \tag{22}$$

hold, which specialize to $\Im\{\varepsilon\} = \Im\{\mu\}$ for $\underline{\underline{\varepsilon}} = \varepsilon I_3$ and $\underline{\underline{\mu}} = \mu I_3$ in the istropic case.

Definition 5 *If* $\underline{n}_s$ *denotes the unit normal vector to a surface* $\mathcal{S} \subset \mathbf{R^3}$ *being sufficiently smooth and* $\underline{U} \in \mathbf{R^3}$ *will be a vector field continuous on* $\mathcal{S}$ *— or having integrable traces* $\underline{U}|_{\mathcal{S}}$ *of fields* $\in H^m_{loc}(\Omega) \cap \mathcal{S}'(\Omega)$, $m \in \mathbf{N_0}$, $\mathcal{S} \subset \overline{\Omega}$, *then we have*

$$\underline{n}_s \wedge \underline{E}_{sc}|_{\mathcal{S}} = \underline{f}_t \qquad \textit{und} \qquad \underline{n}_s \wedge \underline{H}_{sc}|_{\mathcal{S}} = \underline{g}_t \tag{23}$$

as electric *and* magnetic boundary conditions *for the scattered fields given by*

$$\underline{E}_{sc} := \underline{E}_{tot} - \underline{E}_{inc}, \qquad \underline{H}_{sc} = \underline{H}_{tot} - \underline{H}_{inc} \tag{24}$$

which may be augmented by conditions for the normal components w.r.t. $\mathcal{S}$

$$< \underline{n}_s, \underline{\underline{\varepsilon}}\,\underline{E} > |_{\mathcal{S}} = f_n \qquad \textit{und} \qquad < \underline{n}_s, \underline{\underline{\mu}}\,\underline{H} > |_{\mathcal{S}} = g_n \tag{25}$$

These boundary conditions hold reasonably for electrically or magnetically ideal conducting surfaces at high–frequency, but have to be replaced by such who model better the absorption and reflection properties of the scatterer's surface $\mathcal{S}$. In the following sections we shall concentrate on the application of the so–called *Leontovich boundary conditions* of different orders, the simplest being given by

$$\underline{n}_s \wedge \underline{E}|_{\mathcal{S}} = \underline{\underline{Z}}\,\underline{n}_s \wedge (\underline{n}_s \wedge \underline{H})|_{\mathcal{S}} \tag{26}$$

with the *impedance matrix* $\underline{\underline{Z}} \in \mathbf{C^{3\times 3}}$. Many authors [8, 19, 20] studied this electromagnetic boundary value problem for smooth surfaces in the isotropic case. Like for acoustics or elastodynamics there are scattering problems for different, not ideally conducting media in $\mathbf{R^3}$–space leading to

Definition 6 *Let there be given at least two domains* Ω *and* $\Omega' \subset \mathbf{R^3}$ *such that* $\overline{\Omega} \cup \overline{\Omega}' = \mathbf{R^3}$ *and* $\Omega \cap \Omega' = \emptyset$ *but* $\mathcal{S} = \overline{\Omega} \cap \overline{\Omega}' \neq \emptyset$ *being an oriented piecewise smooth surface.* $\underline{\underline{\varepsilon}}$, $\underline{\underline{\varepsilon}}'$ *and* $\underline{\underline{\mu}}$, $\underline{\underline{\mu}}'$ *are permittivity and permeability tensors, respectively, continuously depending on* $\underline{x} \in \overline{\Omega}$ *and* $\underline{x} \in \overline{\Omega}'$, *and being strictly hermitean — here for* $\underline{\underline{\varepsilon}}(\underline{x})$ *—*

$$\underline{\xi}^T \underline{\underline{\varepsilon}}(\underline{x}) \underline{\xi} \geq c|\underline{\xi}|^2 \qquad \forall \underline{\xi} \in \mathbf{C^3}, \quad \underline{x} \in \Omega \tag{27}$$

then $\underline{E}$ *and* $\underline{H}$ *fulfil the* transmission conditions at the interface $\mathcal{S}$, *if*

$$\underline{n}_s \wedge \underline{E}_{tot} = \underline{n} \wedge \underline{E}'_{tot} \tag{28}$$

$$\underline{n}_s \wedge \underline{H}_{tot} = \underline{n} \wedge \underline{H}'_{tot} \tag{29}$$

$$< \underline{n}_s, \underline{\underline{\varepsilon}}\, \underline{E}_{tot} > = < \underline{n}_s, \underline{\underline{\varepsilon}}' \underline{E}' > \tag{30}$$

$$< \underline{n}_s, \underline{\underline{\mu}}\, \underline{H}_{tot} > = < \underline{n}_s, \underline{\underline{\mu}}' \underline{H}' > \tag{31}$$

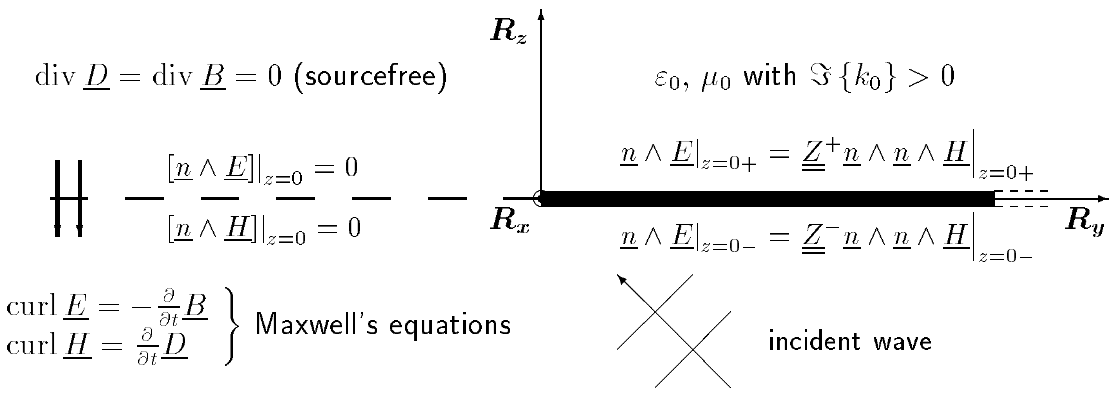
This problem has been discussed and solved by boundary integral equations [3] where a Silver–Müller radiation condition *has to be involved for* $\Omega' := \mathbf{R^3} \backslash \overline{\Omega}$ *an* exterior domain*:*

$$\begin{array}{l} \omega \underline{\underline{\mu}}' \left(\frac{x}{|\underline{x}|} \wedge \underline{H} \right) + k' \underline{E} = \mathcal{O}\left(\frac{1}{r^2}\right) \\ \omega \underline{\underline{\varepsilon}}' \left(\frac{x}{|\underline{x}|} \wedge \underline{E} \right) - k' \underline{E} = \mathcal{O}\left(\frac{1}{r^2}\right) \end{array} \qquad \text{for} \quad r = |\underline{x}| \to \infty \tag{32}$$

see also [7].

4 The Sommerfeld–Halfplane Problem for Anisotropic Dielectric Screens

Let's consider the following boundary–transmission problem (b.t.p.) represented by the figure 3



Figur 3: Sommerfeld Problem with anisotropic halfplane

If we split off the incident electromagnetic wave field assuming a harmonic time–dependance $\mathbf{E} = \underline{E}e^{-i\omega t}$ and $\mathbf{H} = \underline{H}e^{-i\omega t}$ with complex amplitude $\underline{E}$, $\underline{H} \in \mathbf{C^3}$ and frequency ω for the scattered field, we are lead to the following

Problem $\mathcal{S}$: Find a function $\underline{u}' := (\underline{E}, \underline{H})^T \in \mathbf{C^6}$ with $\underline{E}$, $\underline{H} \in [L^2(\mathbf{R^3})]^3$ and restrictions

$$\underline{u}'_\pm := \underline{u}'|_{\mathbf{R^3_\pm}} \in [H_{0,1}(\mathbf{R^3_\pm}) \times H_{1,0}(\mathbf{R^3_\pm}) \times L_2(\mathbf{R^3_\pm})]^2 \tag{33}$$

being weak solutions of Maxwell's equations in $\mathbf{R^3_\pm}$.

$$\operatorname{curl} \underline{E} - i\omega\mu_0 \underline{H} = \underline{0}, \qquad \operatorname{curl} \underline{H} + i\omega\varepsilon_0 \underline{E} = \underline{0} \tag{34}$$

and their traces on $z = 0$ fulfil the conditions

$$\underline{E},\, \underline{H} \in H_{-\frac{1}{2},\frac{1}{2}}(\mathbf{R^2}) \times H_{\frac{1}{2},-\frac{1}{2}}(\mathbf{R^2}) \times H_{-\frac{1}{2}}(\mathbf{R^2}) \tag{35}$$

and the *Leontovich–boundary conditions* on $\Sigma^\pm$

$$\pm \underline{e}_z \wedge \underline{E}^\pm = \underline{\underline{Z}}' \left(\underline{e}_z \wedge \left(\underline{e}_z \wedge \underline{H}^\pm \right) \right) + \underline{f}'_\pm \tag{36}$$

and the transmission conditions on $\Sigma^c_\pm$

$$\underline{e}_z \wedge \left(\underline{E}^+ - \underline{E}^- \right)\Big|_{z=0} = 0, \qquad \underline{e}_z \wedge \left(\underline{H}^+ - \underline{H}^- \right)\Big|_{z=0} = 0 \tag{37}$$

Here we have the Sommerfeld screen Σ and its complementary screen Σ^c given by

$$\Sigma,\, \Sigma^c := \left\{ (x, y, z) \in \mathbf{R^3};\, x \gtrless 0,\, y \in \mathbf{R},\, z = 0 \right\} \tag{38}$$

and the

Definition 7 *The* anisotropic Bessel potential spaces *are defined by*

$$H_{r,s}(\mathbf{R^3_\pm}) := \left\{ u \in L_2(\mathbf{R^3_\pm}) : \mathcal{F}^{-1}(1+\xi^2)^{r/2}(1+\eta^2)^{s/2}\mathcal{F}_{\substack{x \to \xi \\ y \to \eta}}\, u \in L_2(\mathbf{R^3_\pm}) \right\} \tag{39}$$

for all pairs $r, s \in \mathbf{R}$. For $r = s$ we write for short $H_s(\mathbf{R^3_\pm})$.

In the case of traces — which exist on $z = 0$ in the sense of distributions —we use the same notations with $\mathbf{R^2}$ replacing $\mathbf{R^3_\pm}$.
In order that boundary and transmission conditions fit together we have to claim

$$\underline{f}'_+ - \underline{f}'_- \in [\tilde{H}_{-1/2}(\Sigma)]^3 \tag{40}$$

having supports on $\Sigma \subset \mathbf{R^2}$.
Due to the interface $z = 0$, being $\mathbf{R^2}$, the third components $(\underline{f}'_\pm)_3$ vanish and we may put

$$\underline{f}'_\pm = (\underline{f}^\pm, 0)^T \qquad \text{and} \qquad \underline{\underline{Z}}'_\pm = \begin{pmatrix} \underline{\underline{Z}}^\pm & 0 \\ 0 & 0 \end{pmatrix} \tag{41}$$

with vectors $\underline{f}^{\pm} \in \mathbf{C^2}$ and tensors $\underline{\underline{Z}}' \in \mathbf{C^{2\times 2}}$.
Due to the special geometry we may reformulate the problem $\mathcal{S}$ by the following

Problem $\mathcal{P}$: Find a function $\underline{u} = (\underline{u}_1, \underline{u}_2)^T \in \mathbf{C^4}$ s.th. $\underline{u}_l \in [L_2(\mathbf{R^3})]^2$; $l = 1, 2$; with

$$\underline{u}^{\pm} = \underline{u}|_{\mathbf{R}^3_{\pm}} \in [H_{0,1}(\mathbf{R}^3_{\pm}) \times H_{1,0}(\mathbf{R}^3_{\pm})]^2 \tag{42}$$

which is a weak solution of

$$\frac{\partial}{\partial z}\underline{u} = \begin{pmatrix} 0 & -(i\omega\varepsilon_0)^{-1}(k_0^2 \underline{\underline{I}}_2 + \underline{\underline{D}})\underline{\underline{N}} \\ (i\omega\mu_0)^{-1}(k_0^2\underline{\underline{I}}_2 + \underline{\underline{D}})\underline{\underline{N}} & 0 \end{pmatrix} \underline{u} \tag{43}$$

in $\mathbf{R}^3_{\pm}$ with the differential matrix operator

$$\underline{\underline{D}} := \begin{pmatrix} \frac{\partial^2}{\partial x^2} & \frac{\partial^2}{\partial x \partial y} \\ \frac{\partial^2}{\partial x \partial y} & \frac{\partial^2}{\partial y^2} \end{pmatrix} \tag{44}$$

and the $\pi/2$–rotation operator

$$\underline{\underline{N}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{45}$$

and fulfils the *Leontovich–boundary conditions*

$$\pm \underline{\underline{N}}\, \underline{u}^{\pm}_{10} + \underline{\underline{Z}}^{\pm} \underline{u}^{\pm}_{20} = \underline{f}^{\pm} \qquad \text{on} \quad \Sigma^{\pm} \tag{46}$$

and the *transmission condition*

$$\underline{u}^{+}_0 - \underline{u}^{-}_0 = 0 \qquad \text{on} \quad \Sigma^c \tag{47}$$

for $\underline{u}^{\pm}_{l,0} \in [H_{-\frac{1}{2},\frac{1}{2}}(\mathbf{R^2}) \times H_{\frac{1}{2},-\frac{1}{2}}(\mathbf{R^2})]^2$; $l = 1, 2$; the traces of the electrical and magnetical tangential fields on $z = 0$. These data $\underline{f}^{\pm}$ have to fulfil the *compatibility condition*

$$\underline{f}^{+} - \underline{f}^{-} \in [\tilde{H}_{-\frac{1}{2}}(\Sigma)]^2. \tag{48}$$

It can be shown then [17] that the problems $\mathcal{S}$ and $\mathcal{P}$ are equivalent. After eliminating the normal components of the field vectors giving an equivalent 4×4–system for the four tangential components. Using then a 2D–Fouriertransformation w.r.t. (x, y) one gets

Theorem 1 (Representation of tangential components) *A vector function* $\underline{u} \in [L_2(\mathbf{R^3})]^4$ *with*

$$\underline{u}^{\pm} = \underline{u}|_{\mathbf{R}^3_{\pm}} \in [H_{0,1}(\mathbf{R}^3_{\pm}) \times H_{1,0}(\mathbf{R}^3_{\pm})]^2 \tag{49}$$

is a (weak) solution of problem $\mathcal{P}$ *iff it can be represented as*

$$\underline{u}(x, y, z) = \mathcal{F}^{-1}_{\substack{\xi \to x \\ \eta \to y}} \left\{ \underline{\hat{u}}^{+}_0(\xi, \eta) e^{-t(\xi,\eta)z} \chi_{+}(z) + \underline{\hat{u}}^{-}_0(\xi, \eta) e^{t(\xi,\eta)z} \chi_{-}(z) \right\} \tag{50}$$

where $\underline{\hat{u}}_0 = \mathcal{F}\underline{u}_0$, $\chi_\pm$ *the characteristic functions of* $\mathbf{R}_\pm$, *the square root* $t(\xi,\eta) := \sqrt{\xi^2+\eta^2-k_0^2}$ *for* $(\xi,\eta) \in \mathbf{R^2}$ *and* $\sqrt{\xi^2-k_0^2}$ *continuable into the cut* ζ*–plane with branch cuts* $\Gamma_\pm := \{\zeta \in \mathbf{C} : \zeta = \pm(k+i\tau), \tau \geq 0\}$, $k \in \mathbf{C}^{++}$. *The* $\mathcal{F}$*–transformed traces are given by*

$$\underline{\hat{u}}_0^\pm(\xi,\eta) = \hat{\alpha}_\pm(\xi,\eta)\begin{pmatrix} 1 \\ \frac{\xi\eta}{\xi^2-k_0^2} \\ 0 \\ \pm\frac{i\omega\varepsilon\, t(\xi,\eta)}{\xi^2-k_0^2}\end{pmatrix} + \hat{\beta}_\pm(\xi,\eta)\begin{pmatrix} 0 \\ \mp\frac{i\omega\mu\, t(\xi,\eta)}{\xi^2-k_0^2} \\ 1 \\ \frac{\xi\eta}{\xi^2-k_0^2}\end{pmatrix} \tag{51}$$

and the eigenvector factors $\hat{\alpha}_\pm$ *and* $\hat{\beta}_\pm$ *of the eigenvalue problem*

$$(\underline{\underline{M}} + it\underline{\underline{I}}_4)\underline{\hat{u}}_0 = 0 \tag{52}$$

with

$$\underline{\underline{M}} = \begin{pmatrix} 0 & 0 & \frac{\xi\eta}{\omega\varepsilon_0} & \frac{k_0^2-\xi^2}{\omega\varepsilon_0} \\ 0 & 0 & -\frac{k_0^2-\eta^2}{\omega\varepsilon_0} & -\frac{\xi\eta}{\omega\varepsilon_0} \\ -\frac{\xi\eta}{\omega\mu_0} & -\frac{k_0^2-\xi^2}{\omega\mu_0} & 0 & 0 \\ \frac{k_0^2-\eta^2}{\omega\mu_0} & \frac{\xi\eta}{\omega\mu_0} & 0 & 0 \end{pmatrix}. \tag{53}$$

If the tangential components on $z = 0$ are known, Maxwell's equations give

Theorem 2 (Representation formula for the normal components) *The two normal components of the electrical and magnetical field vectors from* $[L_2(\mathbf{R}^3_\pm)]^2$ *are given by*

$$\begin{aligned} v_1(x,y,z) &= \mathcal{F}^{-1}\left\{\frac{\xi t(\xi,\eta)}{\xi^2-k_0^2}\left(e^{-t(\xi,\eta)z}\mathcal{F}\alpha_+\chi_+(z) - e^{t(\xi,\eta)z}\mathcal{F}\alpha_-\chi_-(z)\right)\right\} \\ &- \mathcal{F}^{-1}\left\{\frac{i\omega\mu_0\eta}{\xi^2-k_0^2}\left(e^{-t(\xi,\eta)z}\mathcal{F}\beta_+\chi_+(z) - e^{t(\xi,\eta)z}\mathcal{F}\beta_-\chi_-(z)\right)\right\} \end{aligned} \tag{54}$$

and

$$\begin{aligned} v_2(x,y,z) &= \mathcal{F}^{-1}\left\{\frac{i\omega\varepsilon_0\eta}{\xi^2-k_0^2}\left(e^{-t(\xi,\eta)z}\mathcal{F}\alpha_+\chi_+(z) + e^{t(\xi,\eta)z}\mathcal{F}\alpha_-\chi_-(z)\right)\right\} \\ &+ \mathcal{F}^{-1}\left\{\frac{t(\xi,\eta)\xi}{\xi^2-k_0^2}\left(e^{-t(\xi,\eta)z}\mathcal{F}\beta_+\chi_+(z) - e^{t(\xi,\eta)z}\mathcal{F}\beta_-\chi_-(z)\right)\right\} \end{aligned} \tag{55}$$

Remark 6 *If* $\alpha_\pm, \beta_\pm \in H_{-\frac{1}{2},\frac{1}{2}}$ *then the traces* $\underline{u}_0^\pm \in H_{-\frac{1}{2},\frac{1}{2}}(\mathbf{R^2}) \times H_{\frac{1}{2},-\frac{1}{2}}(\mathbf{R^2})$ *which results from the* ξ,η*–orders of the anisotropic Fouriermultipliers in*

$$\underline{u}_0^\pm(x,y) = \mathcal{F}^{-1}\begin{pmatrix} 1 \\ \frac{\xi\eta}{\xi^2-k_0^2} \\ 0 \\ \pm\frac{i\omega\varepsilon t}{\xi^2-k_0^2}\end{pmatrix}\mathcal{F}\alpha_\pm + \mathcal{F}^{-1}\begin{pmatrix} 0 \\ \mp\frac{i\omega\mu_0 t}{\xi^2-k_0^2} \\ 1 \\ \frac{\xi\eta}{\xi^2-k_0^2}\end{pmatrix}\mathcal{F}\beta_\pm \tag{56}$$

Remark 7 *This Sommerfeld halfplane–problem may be generalized to higher order anisotropic Leontovich boundary conditions. On the other hand one gets the well–known scalar impedance boundary conditions of first and higher order as they were studied in [5, 11, 17].*

5 The Wiener–Hopf system for the anisotropic Leontovich–boundary value problem of a screen

Denoting by $H_{r,s}(\mathbf{R^2})$ for $r,s \in \mathbf{R}$ the closed subspace of $H_{r,s}(\mathbf{R^2})$ having support on

$$\overline{\mathbf{R^2_\pm}} := \left\{ (x,y) \in \mathbf{R^2} : y \gtrless 0 \right\} \tag{57}$$

and by

$$\underline{\phi}^+ \;:=\; \underline{u}_0^+ - \underline{u}_0^- \in [H_{-\frac{1}{2},\frac{1}{2}}(\mathbf{R^2}) \times [H_{\frac{1}{2},-\frac{1}{2}}(\mathbf{R^2})]^2 \tag{58}$$

$$= \mathcal{F}^{-1} \begin{pmatrix} 1 & 0 & -1 & 0 \\ \frac{\xi\eta}{\xi^2-k_0^2} & -\frac{i\omega\mu_0\, t(\xi,\eta)}{\xi^2-k_0^2} & -\frac{\xi\eta}{\xi^2-k_0^2} & -\frac{i\omega\mu_0\, t(\xi,\eta)}{\xi^2-k_0^2} \\ 0 & 1 & 0 & -1 \\ \frac{i\omega\varepsilon_0 t(\xi,\eta)}{\xi^2-k_0^2} & \frac{\xi\eta}{\xi^2-k_0^2} & \frac{i\omega\varepsilon_0 t(\xi,\eta)}{\xi^2-k_0^2} & -\frac{\xi\eta}{\xi^2-k_0^2} \end{pmatrix} \mathcal{F} \begin{pmatrix} \alpha_+ \\ \beta_+ \\ \alpha_- \\ \beta_- \end{pmatrix} \tag{59}$$

which is in $[H^+_{-\frac{1}{2},\frac{1}{2}}(\mathbf{R^2}) \times H^+_{\frac{1}{2},-\frac{1}{2}}(\mathbf{R^2})]^2$ if $\underline{\gamma} := (\alpha_+, \beta_+, \alpha_-, \beta_-)^T \in [H_{-\frac{1}{2},\frac{1}{2}}(\mathbf{R^2})]^4$. It's easy to see then that the four–vectors $\underline{\gamma}$ and $\underline{\phi}^+$ are connected via

$$\underline{\gamma} = B\underline{\phi}^+ \tag{60}$$

with the invertible matrix pseudo–differential operator

$$B : \mathcal{F}^{-1}\underline{\underline{\sigma}}_B\mathcal{F} : [H^+_{-\frac{1}{2},\frac{1}{2}}(\mathbf{R^2}) \times H^+_{\frac{1}{2},-\frac{1}{2}}(\mathbf{R^2})]^2 \to [H_{-\frac{1}{2},\frac{1}{2}}(\mathbf{R^2})]^4 \tag{61}$$

with the symbol matrix

$$\underline{\underline{\sigma}}_B = \frac{1}{2} \begin{pmatrix} 1 & 0 & -\frac{\xi\eta}{t_1} & \frac{\xi^2-k_0^2}{t_1} \\ \frac{\xi\eta}{t_2} & -\frac{\xi^2-k_0^2}{t_2} & 1 & 0 \\ -1 & 0 & -\frac{\xi\eta}{t_1} & \frac{\xi^2-k_0^2}{t_1} \\ \frac{\xi\eta}{t_2} & -\frac{\xi^2-k_0^2}{t_2} & -1 & 0 \end{pmatrix} \tag{62}$$

where $t_1 := i\omega\varepsilon_0\, t(\xi,\eta)$ and $t_2 := i\omega\mu_0\, t(\xi,\eta)$. If we write the impedance tensors in the form

$$\underline{\underline{Z}}^\pm := \begin{pmatrix} a^\pm & b^\pm \\ c^\pm & d^\pm \end{pmatrix} \tag{63}$$

we can transform the Leontovich conditions into the equation

$$r_+ C \underline{\gamma} = \underline{f} = (\underline{f}^+, \underline{f}^-)^T \tag{64}$$

with the pseudo–differential operator

$$C = \mathcal{F}^{-1} \underline{\underline{\sigma}}_C \mathcal{F} : [H_{-\frac{1}{2},\frac{1}{2}}(\mathbf{R^2})]^4 \to [H_{-\frac{1}{2}}(\mathbf{R^2})]^4 \tag{65}$$

having the symbol

$$\underline{\underline{\sigma}}_C := \begin{pmatrix} -\frac{\xi\eta}{\xi^2-k_0^2} + \frac{t_1 b^+}{\xi^2-k_0^2} & \frac{t_2}{\xi^2-k_0^2} + a^+ + \frac{b^+\xi\eta}{\xi^2-k_0^2} & 0 & 0 \\ 1 + \frac{t_1 d^+}{\xi^2-k_0^2} & c^+ + \frac{d^+\xi\eta}{\xi^2-k_0^2} & 0 & 0 \\ 0 & 0 & \frac{\xi\eta}{\xi^2-k_0^2} - \frac{t_1 b^-}{\xi^2-k_0^2} & \frac{t_2}{\xi^2-k_0^2} + a^- + \frac{b^-\xi\eta}{\xi^2-k_0^2} \\ 0 & 0 & -1 - \frac{t_1 d^-}{\xi^2-k_0^2} & c^- + \frac{d^-\xi\eta}{\xi^2-k_0^2} \end{pmatrix} \tag{66}$$

$\det \underline{\underline{\sigma}}_c \neq 0$ is the condition of invertibility which leads to the conditions for $\underline{\underline{Z}}(\xi, \eta)$:

$$\underline{\xi}' \underline{\underline{Z}}^{\pm} \underline{\xi}'^T + ik_0 Z_0 \left(\frac{\det \underline{\underline{Z}}^{\pm}}{Z_0^2} + 1 \right) t + (ik_0)^2 \operatorname{tr} \underline{\underline{Z}}^{\pm} \neq 0 \tag{67}$$

for $\underline{\xi}' \in \mathbf{R^2}$. In the 2D–case, $\xi = 0$, we have

$$t(\eta) \neq -\frac{ik_0}{2d^{\pm}} \left(Z_0 \det \underline{\underline{Z}}^{\pm} + \frac{1}{Z_0} \mp \sqrt{\left(Z_0 \det \underline{\underline{Z}}^{\pm} + \frac{1}{Z_0} \right)^2 - 4a^{\pm}d^{\pm}} \right) \tag{68}$$

In the classical isotropic case of impedance conditions $\underline{n} \wedge \underline{E} = \underline{\underline{Z}}\, \underline{n} \wedge (\underline{n} \wedge \underline{H})$ corresponding to

$$\frac{\partial E_n}{\partial n} - ik_0 \frac{Z}{Z_0} E_n = 0 \qquad \text{and} \qquad \frac{\partial H_n}{\partial n} - ik_0 \frac{Z_0}{Z} H_n = 0 \tag{69}$$

the necessary conditions read

$$t \neq \begin{cases} -i\omega\varepsilon_0 Z \\ -i\omega\frac{\mu_0}{Z} \end{cases} \tag{70}$$

Introducing the formula for $\underline{\gamma}$ (60) into (64) we arrive at a first relation of Wiener–Hopf type

$$r_+ C B \underline{\phi}^+ = \underline{f} \tag{71}$$

After adding and substracting pairwise the corresponding four equations one obtains the Wiener–Hopf equation

$$\tilde{W} \underline{\phi}^+ = \underline{\tilde{f}} \tag{72}$$

acting on the Dirichlet data jump across Σ with given data vector $\underline{\tilde{f}} := (\underline{f}^+ + \underline{f}^-, \underline{f}^+ - \underline{f}^-)^T$ and $\tilde{W} = r_+ W$ the restriction of the pseudo–differential operator

$$W := \mathcal{F}^{-1} \underline{\underline{\sigma}} \mathcal{F} : [H_{-\frac{1}{2},\frac{1}{2}}(\mathbf{R^2}) \times H_{\frac{1}{2},-\frac{1}{2}}(\mathbf{R^2})]^2 \to [H_{-\frac{1}{2}}(\mathbf{R^2})]^4 \tag{73}$$

with the Fouriersymbol

$$\underline{\underline{\sigma}} = \begin{pmatrix} \frac{a_+\xi\eta+b_+\left(\eta^2-k_0^2\right)}{ik_0Z_0t} & -1-\frac{a_+\left(\xi^2-k_0^2\right)+b_+\xi\eta}{ik_0Z_0t} & a_- & b_- \\ 1+\frac{c_+\xi\eta+d_+\left(\eta^2-k_0^2\right)}{ik_0Z_0t} & -\frac{c_+\left(\xi^2-k_0^2\right)+d_+\xi\eta}{ik_0Z_0t} & c_- & d_- \\ \frac{a_-\xi\eta+b_-\left(\eta^2-k_0^2\right)}{ik_0Z_0t} & -\frac{a_-\left(\xi^2-k_0^2\right)+b_-\xi\eta}{ik_0Z_0t} & Z_0\frac{\eta^2-k_0^2}{ik_0t}+a_+ & -Z_0\frac{\xi\eta}{ik_0t}+b_+ \\ \frac{c_-\xi\eta+d_-\left(\eta^2-k_0^2\right)}{ik_0Z_0t} & -\frac{c_-\left(\xi^2-k_0^2\right)+d_-\xi\eta}{ik_0Z_0t} & -Z_0\frac{\xi\eta}{ik_0t}+c_+ & Z_0\frac{\xi^2-k_0^2}{ik_0t}+d_+ \end{pmatrix} \tag{74}$$

$a_\pm$ etc. denote sum and difference of a etc., respectively. All of this gives

Theorem 3 (Equivalence) *Problem $\mathcal{P}$ is uniquely solvable iff the Wiener–Hopf operator $\tilde{W}$ is invertible. In that case it holds with the trace vector $\underline{\gamma}$ of $\underline{u}$*

$$\underline{\phi}^+ = B^{-1}\underline{\gamma} = \mathcal{F}^{-1}\begin{pmatrix} 1 & 0 & -1 & 0 \\ \frac{\xi\eta}{\xi^2-k^2} & -\frac{t_2}{\xi^2-k^2} & -\frac{\xi\eta}{\xi^2-k^2} & -\frac{t_2}{\xi^2-k^2} \\ 0 & 1 & 0 & -1 \\ \frac{t_1}{\xi^2-k^2} & \frac{\xi\eta}{\xi^2-k^2} & \frac{t_1}{\xi^2-k^2} & -\frac{\xi\eta}{\xi^2-k^2} \end{pmatrix}\mathcal{F}\underline{\gamma} \tag{75}$$

for the solution of the Wiener–Hopf equation. If, on the other hand , $\underline{\phi}^+$ is a solution of equation (74) the function $\underline{u}$ given by theorem 1 from which follows the data vector $\underline{\gamma}$ via $\underline{u}_0^\pm$ gives by

$$\underline{\gamma} = B\underline{\phi}^+ \tag{76}$$

a solution of problem $\mathcal{P}$.

Now it can be shown [17] that the system of Wiener–Hopf equations (74) is not uniquely solvable, since $\tilde{W}$ has a nontrivial kernel. (It is shown there for the special case of $\underline{\underline{Z}}^+ = \underline{\underline{Z}}^- = \underline{\underline{Z}}$ where the 4×4–symbol matrix $\underline{\underline{\sigma}}$ decouples into two 2×2–blocks.)
From the theory of impedance boundary value problems for the Sommerfeld half–plane in the case of the Helmholtz equation [5, 12] it is known that there exists an unique solution in smoother spaces $H_{1+\varepsilon}(\mathbf{R}^2_+)\times H_{1+\varepsilon}(\mathbf{R}^2_-)$. So we shall modify our Problem $\mathcal{P}$ to
Problem $\mathcal{P}_\varepsilon$: Find a function $\underline{u} := (\underline{u}_1, \underline{u}_2)^T \in [L_2(\mathbf{R}^3)]^4$ with

$$\underline{u}^\pm = \underline{u}|_{\mathbf{R}^3_\pm} \in \left[H_{\varepsilon,1+\varepsilon}\left(\mathbf{R}^3_\pm\right)\times H_{1+\varepsilon,\varepsilon}\left(\mathbf{R}^3_\pm\right)\right]^2, \tag{77}$$

being a weak solution to the modified Maxwell equations

$$\frac{\partial}{\partial z}\underline{u} = \begin{pmatrix} 0 & -(i\omega\varepsilon_0)^{-1}\left(k_0^2\underline{\underline{I}}_2+\underline{\underline{D}}\right)\underline{\underline{N}} \\ (i\omega\mu_0)^{-1}\left(k_0^2\underline{\underline{I}}_2+\underline{\underline{D}}\right)\underline{\underline{N}} & 0 \end{pmatrix}\underline{u} \tag{78}$$

with the matrix differential operator (44) and the rotational matrix $\underline{\underline{N}}$ in (45). The traces $\underline{u}^{\pm}_{l0} \in H_{-\frac{1}{2}+\varepsilon,\frac{1}{2}+\varepsilon}(\mathbf{R^2}) \times H_{\frac{1}{2}+\varepsilon,-\frac{1}{2}+\varepsilon}(\mathbf{R^2})$ of the electrical, $l = 1$, and the magnetic, $l = 2$, field tangential components on $z = 0$ fulfil the boundary (46) and transmission conditions (47) on Σ and Σ^c, respectively. The given screen data have to satify the compatibility condition

$$\underline{f}^+ - \underline{f}^- \in \left[\tilde{H}_{-\frac{1}{2}+\varepsilon}(\Sigma)\right]^2 \tag{79}$$

The representation theorems 1 and 2 remain valid. For $\varepsilon \neq n \in \mathbf{N_0}$ the compatibility condition holds trivially, since the spaces $[\tilde{H}_{-\frac{1}{2}+\varepsilon}(\Sigma)]^2$ coincide with $[H_{-\frac{1}{2}+\varepsilon}(\Sigma)]^2$ then.

The problem $\mathcal{P}_\varepsilon$ then may be equivalently reduced to a Wiener–Hopf system $\tilde{W}_\varepsilon \underline{\gamma} = \underline{f}$ with a pseudo–differential operator W_ε with the same symbol matrix $\underline{\underline{\sigma}}$ (74) such that

$$W_\varepsilon := \mathcal{F}^{-1}\underline{\underline{\sigma}}\mathcal{F} : [H^+_{-\frac{1}{2}+\varepsilon,\frac{1}{2}+\varepsilon}(\mathbf{R^2}) \times H^+_{\frac{1}{2}+\varepsilon,-\frac{1}{2}+\varepsilon}(\mathbf{R^2})]^2 \to [H_{-\frac{1}{2}+\varepsilon}(\mathbf{R^2})]^4 \tag{80}$$

is invertible iff the problem $\mathcal{P}_\varepsilon$ is uniquely solvable.

In the case of $\varepsilon \neq \mathbf{N_0}$ the system $\tilde{W}_\varepsilon \underline{\phi}^+ = \underline{\gamma}$ on Σ may be transformed into an equivalent 4×4–system of Wiener–Hopf type acting on $L_2(\cdot, \mathbf{R})^4$ with supports on $\overline{\mathbf{R}^+}$ and ξ acting as a parameter. Using Bessel potential operators

$$\Lambda_{+,\varepsilon} = \mathcal{F}^{-1}\underline{\underline{\sigma}}_{+,\varepsilon}\mathcal{F} : \left[H_{\frac{1}{2}+\varepsilon}(\cdot, \mathbf{R}) \times H_{-\frac{1}{2}+\varepsilon}(\cdot, \mathbf{R})\right]^2 \to [L_2(\cdot, \mathbf{R})]^4, \tag{81}$$

$$\Lambda_{-,\varepsilon} = \mathcal{F}^{-1}\underline{\underline{\sigma}}_{-,\varepsilon}\mathcal{F} : [L_2(\cdot, \mathbf{R})]^4 \to \left[H_{-\frac{1}{2}+\varepsilon}(\cdot, \mathbf{R})\right]^4, \tag{82}$$

with symbol matrices

$$\underline{\underline{\sigma}}_{+,\varepsilon} = \operatorname{diag}\left(t_+^{1+2\varepsilon}, t_+^{-1+2\varepsilon}, t_+^{1+2\varepsilon}, t_+^{-1+2\varepsilon}\right) \tag{83}$$

$$\underline{\underline{\sigma}}_{-,\varepsilon} = t_-^{1-2\varepsilon}\underline{\underline{I}}_4, \tag{84}$$

containing the factors from

$$t = \sqrt{\xi^2 + \eta^2 - k_0^2} = t_-(\cdot, \eta)t_+(\cdot, \eta) \qquad \text{w.r.t.} \quad \eta \tag{85}$$

After fixing $\xi = 0$ we can reduce to

$$\tilde{W}_0(\varepsilon) = \mathcal{P}^+ W_0(\varepsilon)\Big|_{[L_2^+(\cdot,\mathbf{R})]^4} : \left[L_2^+(\cdot, \mathbf{R})\right]^4 \to \left[L_2^+(\cdot, \mathbf{R})\right]^4, \tag{86}$$

with

$$W_0(\varepsilon) := \mathcal{F}^{-1}\underline{\underline{\sigma}}_\varepsilon\mathcal{F} : [L_2(\cdot, \mathbf{R})]^4 \to [L_2(\cdot, \mathbf{R})]^4 \tag{87}$$

and symbol matrix $\underline{\underline{\sigma}}_\varepsilon$ similar to (74) but now containing the factor $(t_+/t_-)^{1-2\varepsilon}$ in front and some places t_-/t_+. It follows

$$\underline{\underline{\sigma}}_\varepsilon = \left(\frac{t_+}{t_-}\right)^{1-2\varepsilon} \underline{\underline{\sigma}} \operatorname{diag}\left(t_+^{-2}, 1, t_+^{-2}, 1\right) \tag{88}$$

after all. The special case of 3D–Lifting onto L_2 when $\underline{\underline{Z}}_+ = \underline{\underline{Z}}_-$, which reduces to the decoupled 2×2–symbol matrices of the electrical, $\underline{\underline{\sigma}}_\varepsilon^e$, and the magnetic, $\underline{\underline{\sigma}}_\varepsilon^m$, part was treated in [17]. This simplifies further in the case of 2D–lifting onto $[L_2(\mathbf{R})]^2$. In the 3D–case we obtain

$$\tilde{W}_0(\varepsilon)\underline{\Phi}^+ = \tilde{\Lambda}_{-,\varepsilon}^{-1}\underline{f} \tag{89}$$

with

$$\underline{\Phi}^+ := \tilde{\Lambda}_{+,\varepsilon}\underline{\phi}^+, \tag{90}$$

where $\tilde{W}_0(\varepsilon)$ has a symbol with piecewise continous elements on $\dot{\mathbf{R}}$ having jumps at infinity due to $t_+/t_- \to \pm 1$ for $\eta \to \pm\infty$, ξ fixed.
So we may use the fact of the equivalence of Fredholmness of the operator $\tilde{W}_0(\varepsilon)$ and the general factorizability of its symbol [15] to get

Proposition 1 *The Wiener–Hopf operator $\tilde{W}_0(\varepsilon)$ is Fredholm iff*

$$\varepsilon \neq \frac{n-1}{2}, \qquad n \in \mathbf{N} \tag{91}$$

and has then

$$\operatorname{Ind} \tilde{W}_0(\varepsilon) = 0 \qquad \textit{for} \quad \frac{1}{4} \leq \varepsilon < \frac{1}{2} \tag{92}$$

Remark 8 *One has to show that the* augmented symbol matrix

$$\underline{\underline{\sigma}}'_\varepsilon = \begin{cases} \underline{\underline{\sigma}}_\varepsilon(\cdot,\eta+0)\mu + (1-\mu)\underline{\underline{\sigma}}_\varepsilon(\cdot,\eta) & : \quad \eta \in \mathbf{R},\ \mu \in [0,1] \\ \underline{\underline{\sigma}}_\varepsilon(\cdot,-\infty)\mu + (1-\mu)\underline{\underline{\sigma}}_\varepsilon(\cdot,\infty) & : \quad \mu \in [0,1] \end{cases} \tag{93}$$

has a nowhere vanishing determinant.

Remark 9 *In the twodimensional case of electrically or magnetically transverse polarized waves one ends up with the two scalar impedance boundary conditions. The corresponding 2×2–symbol matrices of the 2D–lifted Wiener–Hopf system runs as*

$$\left(\frac{t_+}{t_-}\right)^{1-2\varepsilon} \begin{pmatrix} -1 - \frac{\lambda_0^+ + \lambda_0^-}{2t} & \frac{\lambda_0^+ - \lambda_0^-}{2t}\frac{t_-}{t_+} \\ -\frac{\lambda_0^+ - \lambda_0^-}{2t} & \frac{t_-}{t_+} + \frac{\lambda_0^+ + \lambda_0^-}{2t}\frac{t_-}{t_+} \end{pmatrix} \tag{94}$$

which is well–known [5].

6 Factorization and explicit Representation of the scattered field

In this section we consider the 2D–problem with identical layers on both screens of the halfplane. The Impedanzmatrix now reads

$$\underline{\underline{Z}} = Z_0 \begin{pmatrix} \frac{a}{ik_0} & ik_0 b \\ \frac{c}{ik_0} & ik_0 d \end{pmatrix}. \tag{95}$$

Moreover we restrict ε to the intervall $\left[\frac{1}{4}, \frac{1}{2}\right)$. The Wiener–Hopf Operator $\tilde{W}_0(\varepsilon)$ is then a Fredholm operator with index zero. This yields our symbols for the electric Dirichlet–jumps

$$\tilde{W}_0^{\mathrm{e}}(\varepsilon) = \mathcal{P}^+\mathcal{F}^{-1}\underline{\underline{\sigma}}_\varepsilon^{\mathrm{e}}\mathcal{F} : \left[L_2^+(\mathbf{R})\right]^2 \to \left[L_2^+(\mathbf{R})\right]^2, \tag{96}$$

with

$$\underline{\underline{\sigma}}_\varepsilon^{\mathrm{e}} = \left(\frac{t_+}{t_-}\right)^{1-2\varepsilon} \begin{pmatrix} b\frac{t_-}{t_+} & -1-\frac{a}{t} \\ \left(\frac{1}{t}+d\right)\frac{t_-}{t_+} & -\frac{c}{t} \end{pmatrix} \tag{97}$$

and for the magnetic Dirichlet–jumps

$$\tilde{W}_0^{\mathrm{m}}(\varepsilon) = \mathcal{P}^+\mathcal{F}^{-1}\underline{\underline{\sigma}}_\varepsilon^{\mathrm{m}}\mathcal{F} : \left[L_2^+(\mathbf{R})\right]^2 \to \left[L_2^+(\mathbf{R})\right]^2 \tag{98}$$

$$\underline{\underline{\sigma}}_\varepsilon^{\mathrm{m}} = \left(\frac{t_+}{t_-}\right)^{1-2\varepsilon} \begin{pmatrix} \frac{Z_0}{ik_0}\left(1+\frac{a}{t}\right)\frac{t_-}{t_+} & ik_0Z_0b \\ \frac{Z_0}{ik_0}\frac{c}{t}\frac{t_-}{t_+} & ik_0Z_0\left(\frac{1}{t}+d\right) \end{pmatrix}. \tag{99}$$

Both symbols thus differ only by permutation of lines and columns and by a scalar factor:

$$\underline{\underline{\sigma}}_\varepsilon^{\mathrm{e}} = \underline{\underline{\sigma}}_\varepsilon^{\mathrm{m}} \begin{pmatrix} 0 & -\frac{ik_0}{Z_0} \\ \frac{1}{ik_0Z_0} & 0 \end{pmatrix} \qquad \text{und} \qquad \underline{\underline{\sigma}}_\varepsilon^{\mathrm{m}} = \underline{\underline{\sigma}}_\varepsilon^{\mathrm{e}} \begin{pmatrix} 0 & ik_0Z_0 \\ -\frac{Z_0}{ik_0} & 0 \end{pmatrix} \tag{100}$$

This yields

Theorem 4 (Factorization) *The Wiener–Hopf operator $W_0^{\mathrm{e/m}}(\varepsilon)$ is invertible for $\varepsilon \in \left[\frac{1}{4}, \frac{1}{2}\right)$. The inverses, under the assumption $\det \underline{\underline{Z}} = 0$, are given by*

$$\tilde{W}_{0,\mathrm{e}}^{-1}(\varepsilon) = \mathcal{F}^{-1}\left(\underline{\underline{\sigma}}_{\varepsilon,+}^{\mathrm{e/m}}\right)^{-1} P^+ \left(\underline{\underline{\sigma}}_{\varepsilon,-}^{\mathrm{e/m}}\right)^{-1} : \left[L_2^+(\mathbf{R})\right]^2 \to \left[L_2^+(\mathbf{R})\right]^2 \tag{101}$$

with the Cauchy–projector $P^+ = \mathcal{F}^{-1}\mathcal{P}\mathcal{F}$ and the inverted positive and negative factors of the symbol $\left(\underline{\underline{\sigma}}_{\varepsilon,\pm}^{\mathrm{e/m}}\right)^{-1}$, i.e.

$$\left(\underline{\underline{\sigma}}_{\varepsilon,+}^{\mathrm{e}}\right)^{-1} = \begin{pmatrix} t_+^{2\varepsilon}g_+^{-1} & t_+^{2\varepsilon}h_+g_+^{-1} \\ -\frac{a}{c}t_+^{2\varepsilon-2}g_+^{-1} & \left(1-\frac{a}{c}t_+^{-1}h_+g_+^{-1}\right)t_+^{2\varepsilon-1} \end{pmatrix} \tag{102}$$

and for the second factor

$$\left(\underline{\underline{\sigma}}_{\varepsilon,-}^{\mathrm{e}}\right)^{-1} = \begin{pmatrix} \left(t_-^{-1}g_-^{-1}+h_-\right)t_-^{1-2\varepsilon} & \frac{a}{c}h_-t_-^{1-2\varepsilon} \\ t_-^{1-2\varepsilon} & \frac{a}{c}t_-^{1-2\varepsilon} \end{pmatrix} \tag{103}$$

with the Wiener–factors $g_\pm \in \mathcal{W}^\pm$ of the function

$$g = b + \frac{a}{c}\left(\frac{1}{t} + \frac{a}{t^2}\right) = g_- g_+ . \tag{104}$$

The split $h = P^+ h + P^- h = h_+ + h_-$ for

$$h = \frac{1 + \frac{a}{t}}{g_- t_-} \tag{105}$$

is calculated via Cauchy–projectors. The operators for $\underline{\underline{\sigma}}_\varepsilon^{\mathrm{m}}$ we yield directly with the relation

$$\left(\underline{\underline{\sigma}}_\varepsilon^{\mathrm{m}}\right)^{-1} = \begin{pmatrix} 0 & -\frac{ik_0}{Z_0} \\ \frac{1}{ik_0 Z_0} & 0 \end{pmatrix} \left(\underline{\underline{\sigma}}_\varepsilon^{\mathrm{e}}\right)^{-1} . \tag{106}$$

We assume now, for the more general physical case $\det \underline{\underline{Z}}^\pm \neq 0$, that on both banks $\Sigma^\pm$ of the Sommerfeld screen Σ the impedance matrix is the same, i.e.

$$\underline{\underline{Z}}^+ = \underline{\underline{Z}}^- = \underline{\underline{Z}} = Z_0 \begin{pmatrix} \frac{1}{ik_0} a & ik_0 b \\ \frac{1}{ik_0} c & ik_0 d \end{pmatrix} \tag{107}$$

and assume additionally $\frac{\partial}{\partial x} = 0$ corresponding to $\xi = 0$ then we are lead to a block structure

$$\underline{\underline{\sigma}}_\varepsilon = \begin{pmatrix} \underline{\underline{\sigma}}_{\varepsilon,1} & 0 \\ 0 & \underline{\underline{\sigma}}_{\varepsilon,2} \end{pmatrix} \quad \in \mathbf{C}^{\mathbf{4}\times\mathbf{4}} \tag{108}$$

which decouples the 4×4–Wiener–Hopf system into one 2×2 for the electric and one for the magnetic tangential components. It can be proved that the corresponding Wiener–Hopf operator $\tilde{W}_0(\varepsilon)$ may be inverted by a Neumann series expansion for the inverse symbol

$$\underline{\underline{\sigma}}_\varepsilon^{-1} = \lambda^{-1} \sum_{n=0}^{\infty} \left(\underline{\underline{I}}_2 - \lambda^{-1} \underline{\underline{\sigma}}_\varepsilon\right)^n \tag{109}$$

if a certain inequality containing the data of $\underline{\underline{Z}}$ is fulfilled, which is too long to be written here [17].

7 Final remarks

In this paper a certain type of generalized Leontovich boundary conditions on the faces of a Sommerfeld half–plane type screen $\Sigma \subset \mathbf{R}^3$ was studied using the Fouriertransform technique w.r.t. the coordinates $(x, y) \in \mathbf{R}^2$. In general anisotropic situation on $\Sigma^\pm$ the electromagnetic field cannot be seperated into two fields, like for the simple boundary conditions $\underline{E}_{tan} = 0$ or $\underline{H}_{tan} = 0$ on $\Sigma^\pm$. The existing plane waves are calculated by solving an eigenvalue problem for a 4×4–matrix which reduces to a block matrix of 2×2–submatrices only in special cases.

Still open for the general anisotropic case is the explicit factorization of the lifted symbol matrix. The corresponding initial boundary–transmission problems are still challenging as well. One has to find the general resolvent operator with the representation of its kernel function by generalized eigenfunctions [18, 23] for the scalar case in electrodynamics or acoustics for the Dirichlet and Rawlin's mixed boundary conditions on $\Sigma^{\pm}$. The corresponding problems for arbitrary wedges K_α or octants $O_{++} \subset \mathbf{R^3}$ are unsolved, too.

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