Connections on fiber bundles and canonical extensions of differential forms

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Abstract

This article treats connections on fiber bundles $B = P \times_G F$ that are induced by a connection 1-form on the associated principal bundle P. Using horizontal lifts of vector fields it is shown which combinations of differential forms on the fiber F and on P canonically define differential forms on B. Local representations for these forms involving the gauge potentials and fields of the connection are given and lead to formulas for the exterior derivative. Finally the case of an abelian structure group, especially $G \cong \mathbb{S}^1$, is examined.

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1 Introduction

Let us start with principal bundles first. A connection Γ on a principal bundle $P(M, G)$ is a homogeneous vector subbundle $H(P)$ of the tangent bundle $T(P)$ that is complementary to the naturally given vertical bundle $V(P)$, i. e., $T(P) = H(P) \oplus V(P)$ and $(R_g)_\star H_p(P) = H_{R(p,g)}(P)$ with the free right action $R: P \times G \to P$ of the finite dimensional structure group G on P and the induced maps $R_g: P \to P$ and $R^p: G \to P$ given by $R_g(p) = R^p(g) := R(p, g)$, cf. [5, p. 276], [4, p. 63]. The connection is uniquely defined by its connection 1-form $\omega^\Gamma \in \mathcal{A}_1(P, \mathfrak{g})$, where $\mathfrak{g} = T_e(G)$ denotes the Lie algebra of G. Connection 1-forms are pseudotensorial, resp., equivariant, i. e., $R_g^\star \omega^\Gamma = \mathrm{Ad}(g^{-1})_\star \omega^\Gamma$ for all $g \in G$, and obey $\omega^\Gamma(\mathcal{R}_X) = X$ for all fundamental vector fields $\mathcal{R}_X$ with $X \in \mathfrak{g}$ and $(\mathcal{R}_X)_p := (dR^p)(X)$. Thus if Θ^L means the left canonical 1-form on G then $(R^p)^\star \omega^\Gamma = \Theta^L$ for all $p \in P$.

Let $h, v: \mathcal{D}^1(P) \to \mathcal{D}^1(P)$ denote the induced projections of vector fields onto the $C^\infty(P)$-modules $h\mathcal{D}^1(P)$, resp., $v\mathcal{D}^1(P)$ of sections of the bundles $H(P)$, resp., $V(P)$. Then ω^Γ is given by $\omega^\Gamma(\mathcal{X})(p) = \omega^\Gamma(v\mathcal{X})(p) := (dR^p)^{-1}(v_p\mathcal{X}_p)$. Reversely, given ω^Γ one recovers Γ by $v_p = dR^p \circ \omega^\Gamma_p: T_p(P) \to V_p(P)$ and $h_p = \mathrm{id} - v_p$. The projections of vector fields canonically define projections of forms $h, v: \mathcal{A}(P, V) \to \mathcal{A}(P, V)$ for every vector space V. Then the exterior covariant differentiation on $\mathcal{A}(P) \otimes V$ is given by $d^\Gamma \phi = (d\phi)h$ and $\Omega^\Gamma := d^\Gamma \omega^\Gamma$ is the curvature 2-form for Γ.

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Let $\pi: P \to M$ denote the global projection onto the base manifold and let $\mathcal{D}^\Gamma(P) := h\mathcal{D}^1(P)_{\mathrm{inv}}$ denote the $C^\infty(M)$-module of horizontal invariant vector fields, i. e, those vector fields $\mathcal{Y}$ with $\mathcal{Y} = h\mathcal{Y}$ and $(R_g)_\star\mathcal{Y} = \mathcal{Y}$ for all $g \in G$. The horizontal lift $\mathbb{L}: \mathcal{D}^1(M) \to h\mathcal{D}^1(P)_{\mathrm{inv}}$, which is uniquely defined by $d\pi_p(\mathbb{L}\mathcal{X})_p = \mathcal{X}_p$, is an isomorphism of $C^\infty(M)$-modules with inverse morphism $\pi_\star$. Just as every connection defines canonical lifts of vector fields on M, the reverse statement is also true. Every $\mathbb{L} \in \mathrm{Hom}_{C^\infty(M)}(\mathcal{D}^1(M), \mathcal{D}^1(P)_{\mathrm{inv}})$ with $\pi_\star \circ \mathbb{L} = \mathrm{id}_{\mathcal{D}^1(M)}$ uniquely defines a connection: if $\mathbb{L}_p$ denotes the local inverse of the differential $d\pi_p$, then the horizontal projection is given by $h_p := \mathbb{L}_p \circ d\pi_p$.

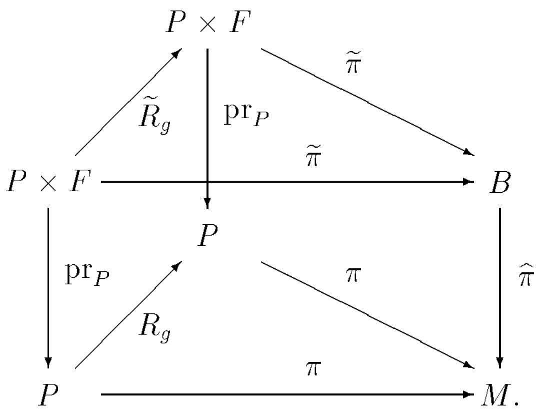
The following lemma on induced connections on principal bundles is well known:

Lemma 1.1 *Let $f: P'(M', G) \to P(M, G)$ be a G-equivariant mapping of principal bundles, i. e., $f \circ R'_g = R_g \circ f$ for all $g \in G$, then every connection Γ on P induces a unique connection Γ' on P', such that $f_\star$ maps horizontal subspaces of Γ' into horizontal subspaces of Γ.*

Instead of connections on principal bundles where we have the connection 1-form at hand, we are interested in connections on fiber bundles in general. So let $B(M, F, G) = P \times_G F$ denote any fiber bundle with fiber F associated with the principal bundle P. Recall its definition: if $L: G \times F \to F$ is a left effective Lie group action of the structure group on a manifold F we define a free right Lie group action $\widetilde{R}$ of G on the product manifold $P \times F$ as follows:

$$\widetilde{R}_g(p, f) := (R_g(p), L_{g^{-1}}(f)) \qquad \text{for all} \quad p \in P, f \in F, g \in G.$$

Now $B = P \times_G F$ denotes the quotient manifold by this action $\widetilde{R}$. Recall that for every fiber bundle $B(M, F, G)$, one can construct an associated principal bundle $P(M, G)$ by taking G as fiber, and then B can be obtained from P in the above way (up to equivalences). In the sequel, $\widetilde{\pi}: P \times F \to B$ will denote the canonical projection and $\widehat{\pi}: B \to M$ will denote the projection of the bundle B such that the following diagram commutes for every $g \in G$:



Suppose $\mathfrak{U} = \{U_\alpha\}_{\alpha \in A}$ is an open cover of M for which a bundle atlas for P exists. The bundle atlas consists of bundle charts (U_α, ψ_α) with local trivializations $\psi_\alpha = (\pi, \pi_\alpha): \pi^{-1}(U_\alpha) \to U_\alpha \times G$ and local projections $\pi_\alpha: \pi^{-1}(U_\alpha) \to G$. Then we also have local projections $\widehat{\pi}_\alpha: \widehat{\pi}^{-1}(U_\alpha) \to F$ and

$$\widehat{\pi} \circ \widetilde{\pi} = \pi \circ \mathrm{pr}_P, \qquad \widehat{\pi}_\alpha \circ \widetilde{\pi} = L \circ (\pi_\alpha \circ \mathrm{pr}_P, \mathrm{pr}_F). \tag{1}$$

In addition, $(P \times F)(B, G)$ is a principal bundle over B equivalent to the pullback bundle $\widehat{\pi}^\star P$ with right action $\widetilde{R}$ and cover $\widehat{\pi}^{-1}\mathfrak{U}$ of B.

Every connection on a principal bundle induces connections on all associated bundles. In the literature ([4, pp. 87 – 88], [5, p. 290]) we find the following definition:

Definition 1.2 *Every connection Γ on a principal bundle $P(M, G)$ induces splittings $T(B) = H(B) \oplus V(B)$ on any associated bundle $B(M, F, G) = P \times_G F$ with $H(B) := \widetilde{\pi}_\star(H(P) \times \{0\})$.*

The following lemma is also standard:

Lemma 1.3 *Let $\widehat{\Gamma}$ be a connection on $B(M, F, G)$ induced by Γ on the associated principal bundle $P(M, G)$. Every embedding $i: U \to M$ and every fiber preserving diffeomorphism of bundles $f: B(M, F, G) \to B'(M', F', G)$ induce connections $\widehat{\Gamma}|_U$ on $\widehat{\pi}^{-1}(U)$, resp., $\widehat{\Gamma}^f$ on B'. For every bundle chart $(U_\alpha, \widehat{\psi}_\alpha)$ the induced connection $(\widehat{\Gamma}|_{U_\alpha})^{\widehat{\psi}_\alpha}$ on $U_\alpha \times F$ coincides with the connection induced by $(\Gamma|_{U_\alpha})^{\psi_\alpha}$ on $U_\alpha \times G$.*

Yet from this approach the projections of the vector fields and of the forms cannot easily be read off. Thus the first task of this article is a slightly different approach to these induced connections in order to get formulas for the projections $\widehat{h}$ and $\widehat{v}$ on the bundle B. Using these formulas we will then be able to prove certain globality theorems for differential forms: Every form on P or F defines a form on $P \times F$ by use of the pullbacks $\mathrm{pr}_P^\star$, resp., $\mathrm{pr}_F^\star$. In which cases do these forms induce forms on B? For example, in this way every invariant form ϕ on F canonically defines a global vertical form "ϕv" on B. Locally this form is given by vertical projections of the pullbacks $\widehat{\pi}_\alpha^\star \phi$.

Of course every form on M defines a horizontal form on B via the pullback $\widehat{\pi}^\star$. We shall prove that these forms are the only horizontal forms one can obtain from forms on P. Also we will prove theorems for combinations of forms on P and on F.

In presenting local representatives for these differential forms (Section 4), we will then be able to give formulas for their exterior derivative. This is quite important since, e. g., the following diagram for an invariant form $\phi \in \mathcal{A}(F) \otimes V$ does *not* commute in general:

$$\begin{array}{ccc} \phi & \longmapsto & \phi v \\ \big\downarrow & & \big\downarrow \\ d\phi & \longmapsto & (d\phi)v \neq d(\phi v). \end{array}$$

Finally we will apply our results to bundles with abelian G, especially if $\mathfrak{g} \cong \mathbb{R}$.

2 Projections on $P \times F$ and on $P \times_G F$

For any left or right action $S = L, R$ of a Lie group G on a manifold P, let $\mathcal{S}: \mathfrak{g} \to \mathcal{D}^1(P)$ denote the Lie algebra (anti) homomorphism defined by $(S_X)_p := dS^p(X)$ (cf. the notation in the previous section), and let $\mathcal{S}': C^\infty(P, \mathfrak{g}) \to C^\infty(P)\mathcal{S}(\mathfrak{g}) \subseteq$

$\mathcal{D}^1(P)$ denote the induced $C^\infty(P)$-module homomorphism. (If G acts effectively on P, then $\mathcal{S}$ is injective and if G acts freely on P, then $\mathcal{S}'$ is an isomorphism of free $C^\infty(P)$-modules, cf. [2, Lemma 2.3].)

The following observation on the natural connection Γ^{nat} on trivial bundles is quite trivial:

Lemma 2.1 *We have natural lifts* $\mathbb{L}_h^{\mathrm{nat}}, \mathbb{L}_v^{\mathrm{nat}}: \mathcal{D}^1(P) \to \mathcal{D}^1(P \times F)$ *on the product manifold* $P \times F$ *with* $(\mathrm{pr}_P)_\star \circ \mathbb{L}_h^{\mathrm{nat}} = \mathrm{id}_{\mathcal{D}^1(P)}$ *and* $(\mathrm{pr}_F)_\star \circ \mathbb{L}_v^{\mathrm{nat}} = \mathrm{id}_{\mathcal{D}^1(F)}$, *which are injective homomorphisms of* $C^\infty(P)$*-modules, resp.,* $C^\infty(F)$*-modules and* LIE *algebras and obey* $(\widetilde{R}_g)_\star \circ \mathbb{L}_h^{\mathrm{nat}} = \mathbb{L}_h^{\mathrm{nat}} \circ (R_g)_\star$ *and* $(\widetilde{R}_g)_\star \circ \mathbb{L}_v^{\mathrm{nat}} = \mathbb{L}_v^{\mathrm{nat}} \circ (L_{g^{-1}})_\star$. *If* $i_f: P \to P \times F$ *and* $i_p: F \to P \times F$ *defined by* $i_f(p) = i_p(f) = (p, f)$, *denote the natural injections then* $(\mathbb{L}_h^{\mathrm{nat}}\mathcal{X})_{(p,f)} = (di_f)_p\mathcal{X}_p$ *and* $(\mathbb{L}_v^{\mathrm{nat}}\mathcal{Y})_{(p,f)} = (di_p)_f\mathcal{Y}_f$ *for all* $p \in P$, $f \in F$, $\mathcal{X} \in \mathcal{D}^1(P)$ *and* $\mathcal{D}^1(F)$.

We also have natural projections of vector fields $h^{\mathrm{nat}}, v^{\mathrm{nat}}: \mathcal{D}^1(P\times F) \to \mathcal{D}^1(P\times F)$ *with* $\mathcal{D}^1(P \times F) = h^{\mathrm{nat}}\mathcal{D}^1(P \times F) \oplus v^{\mathrm{nat}}\mathcal{D}^1(P \times F)$ *as a* $C^\infty(P \times F)$*-module and* $h^{\mathrm{nat}} \circ \mathbb{L}_h^{\mathrm{nat}} = \mathbb{L}_h^{\mathrm{nat}}$, $v^{\mathrm{nat}} \circ \mathbb{L}_h^{\mathrm{nat}} = 0$, *resp.,* $h^{\mathrm{nat}} \circ \mathbb{L}_v^{\mathrm{nat}} = 0$, $v^{\mathrm{nat}} \circ \mathbb{L}_v^{\mathrm{nat}} = \mathbb{L}_v^{\mathrm{nat}}$.

Since $\mathrm{pr}_P \circ \widetilde{R}_g = R_g \circ \mathrm{pr}_P$ *and* $\mathrm{pr}_F \circ \widetilde{R}_g = L_{g^{-1}} \circ \mathrm{pr}_F$ *for all* $g \in G$, *we have*

$$
\begin{aligned}
h^{\mathrm{nat}} \circ (\widetilde{R}_g)_\star &= (\bar{R}_g)_\star \circ h^{\mathrm{nat}} = (\widetilde{R}_g)_\star \circ h^{\mathrm{nat}}, \quad & h^{\mathrm{nat}} \circ \widetilde{\mathcal{R}}' \circ (\mathrm{pr}_P)^\star = \mathbb{L}_h^{\mathrm{nat}} \circ \mathcal{R}',\\
v^{\mathrm{nat}} \circ (\widetilde{R}_g)_\star &= (\bar{L}_{g^{-1}})_\star \circ v^{\mathrm{nat}} = (\widetilde{R}_g)_\star \circ v^{\mathrm{nat}}, \quad & v^{\mathrm{nat}} \circ \widetilde{\mathcal{R}}' \circ (\mathrm{pr}_F)^\star = -\mathbb{L}_v^{\mathrm{nat}} \circ \mathcal{L}',
\end{aligned}
$$

where $\bar{R}$ *and* $\bar{L}$ *denote the actions on* $P \times F$ *naturally induced by* R *and* L:

$$
\begin{aligned}
\bar{R}&: G \times P \times F \to G \times F, \quad & \bar{R}(g, p, f) = (R(g, p), f),\\
\bar{L}&: G \times P \times F \to G \times F, \quad & \bar{L}(g, p, f) = (p, L(g, f)).
\end{aligned}
$$

h and v induce projections h' and v' on $h^{\mathrm{nat}}\mathcal{D}^1(P\times F)$ such that $h' \circ \mathbb{L}_h^{\mathrm{nat}} = \mathbb{L}_h^{\mathrm{nat}} \circ h$, $v' \circ \mathbb{L}_h^{\mathrm{nat}} = \mathbb{L}_h^{\mathrm{nat}} \circ v$. Also a $C^\infty(P \times F)$-linear extension of ω^Γ on $h^{\mathrm{nat}}\mathcal{D}^1(P \times F)$ exists, which we denote by $\bar{\omega}^\Gamma$. Then $v' = \bar{\mathcal{R}}' \circ \bar{\omega}^\Gamma$ and $\bar{\mathcal{R}}' = h^{\mathrm{nat}}\widetilde{\mathcal{R}}'$. Note that the splitting of $T(P \times F)$ into $H(P \times F) = H(P) \times \{0\}$ and $V(P \times F) = V(p) \times \{0\} \oplus \{0\} \times T(F)$ corresponds to projections $h_{P\times F} := h' \circ h^{\mathrm{nat}}$ and $v_{P\times F} = \mathrm{id}_{\mathcal{D}^1(P\times F)} - h' \circ h^{\mathrm{nat}}$ with

$$
h_{P\times F} \circ (\widetilde{R}_g)_\star = (\widetilde{R}_g)_\star \circ h_{P\times F}, \qquad v_{P\times F} \circ (\widetilde{R}_g)_\star = (\widetilde{R}_g)_\star \circ v_{P\times F}.
$$

Yet these are not the only projections given on $P \times F$. Recall that $P \times F$ is a principal bundle over B equivalent to $\widehat{\pi}^\star P$. Now every connection Γ on P induces a connection $\widetilde{\Gamma} = \mathrm{pr}_P^\star\, \Gamma$ on $P \times F$ since pr_P is a G-equivariant mapping of principal bundles according to Lemma 2.1. We have $\widetilde{\omega}^{\widetilde{\Gamma}} = \mathrm{pr}_P^\star\, \omega^\Gamma = \bar{\omega}^\Gamma \circ h^{\mathrm{nat}}$ with

$$
\widetilde{\mathcal{R}}_g^\star \widetilde{\omega}^{\widetilde{\Gamma}} = \mathrm{Ad}(g^{-1})_\star \widetilde{\omega}^{\widetilde{\Gamma}}, \qquad \widetilde{\omega}^{\widetilde{\Gamma}} \circ \widetilde{\mathcal{R}}' = \mathrm{id}_{C^\infty(P\times F, \mathfrak{g})}\,.
$$

$\widetilde{\Gamma}$ defines projections and lifts on $(P\times F)(B, G)$, let us denote them by $\widetilde{h}$, $\widetilde{v}$, $\widetilde{\mathbb{L}}$. Then $\widetilde{v} := \widetilde{\mathcal{R}}' \circ \widetilde{\omega}^{\widetilde{\Gamma}} = \widetilde{\mathcal{R}}' \circ \bar{\omega}^\Gamma \circ h^{\mathrm{nat}} = \widetilde{\mathcal{R}}'\bar{\mathcal{R}}'^{-1} \circ v' \circ h^{\mathrm{nat}}$ and $\widetilde{h} = \mathrm{id}_{\mathcal{D}^1(P\times F)} - \widetilde{\mathcal{R}}'\bar{\mathcal{R}}'^{-1} \circ v' \circ h^{\mathrm{nat}}$. Thus $\widetilde{h} \circ \mathbb{L}_v^{\mathrm{nat}} = \mathbb{L}_v^{\mathrm{nat}}$ and $\widetilde{v} \circ \mathbb{L}_v^{\mathrm{nat}} = 0$. As for any connection on a principal bundle, we have

$$
\widetilde{h} \circ (\widetilde{R}_g)_\star = (\widetilde{R}_g)_\star \circ \widetilde{h}, \qquad \widetilde{v} \circ (\widetilde{R}_g)_\star = (\widetilde{R}_g)_\star \circ \widetilde{v}.
$$

Lemma 2.2 *Let $\Gamma \in \gamma(P(M,G))$, then the various projections on $\mathcal{D}^1(P \times F)$ obey*

$$\begin{array}{rclrclrcl}
h_{P\times F} \circ \tilde{v} &=& \tilde{v} \circ h_{P\times F} \;=\; 0, & h_{P\times F} \circ \tilde{h} &=& \tilde{h} \circ h_{P\times F} \;=\; h_{P\times F}, && \\
v_{P\times F} \circ \tilde{v} &=& \tilde{v} \circ v_{P\times F} \;=\; \tilde{v}, & v_{P\times F} \circ \tilde{h} &=& \tilde{h} \circ v_{P\times F} \;=\; \tilde{h} - h_{P\times F} &=& v^{\mathrm{nat}} \circ \tilde{h}, \\
h^{\mathrm{nat}} \circ \tilde{v} &=& v' \circ h^{\mathrm{nat}}, & \tilde{v} \circ h^{\mathrm{nat}} &=& \tilde{v}, & \tilde{v} \circ h' \circ h^{\mathrm{nat}} &=& 0, \\
h^{\mathrm{nat}} \circ \tilde{h} &=& h' \circ h^{\mathrm{nat}}, & \tilde{h} \circ h^{\mathrm{nat}} &=& h^{\mathrm{nat}} - \tilde{v}, & \tilde{h} \circ h^{\mathrm{nat}} \circ \tilde{h} &=& h^{\mathrm{nat}} \circ \tilde{h}, \\
v^{\mathrm{nat}} \circ \tilde{v} &=& \tilde{v} - v' \circ h^{\mathrm{nat}}, & \tilde{h} \circ v^{\mathrm{nat}} &=& v^{\mathrm{nat}}, & \tilde{v} \circ v^{\mathrm{nat}} &=& 0.
\end{array}$$

By Lemma 2.2, h^{nat}, $h_{P\times F}$ and $v_{P\times F}$ also act on $\mathcal{D}^{\tilde{\Gamma}}(P \times F)$ and

$$h^{\mathrm{nat}}|_{\mathcal{D}^{\tilde{\Gamma}}(P\times F)} = h_{P\times F}|_{\mathcal{D}^{\tilde{\Gamma}}(P\times F)} = \mathrm{id}_{\mathcal{D}^{\tilde{\Gamma}}(P\times F)} - v_{P\times F}|_{\mathcal{D}^{\tilde{\Gamma}}(P\times F)}.$$

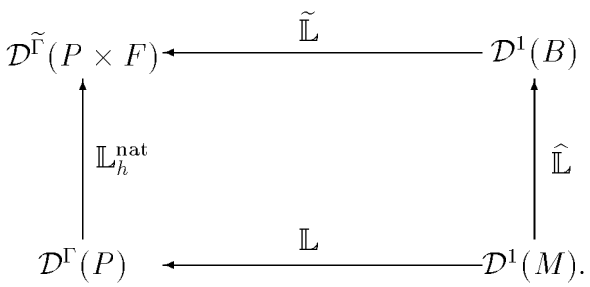
But $\tilde{\mathbb{L}}: \mathcal{D}^1(B) \to \mathcal{D}^{\tilde{\Gamma}}(P \times F)$ is a $C^\infty(B)$-module isomorphism with inverse morphism $\tilde{\pi}_\star$. This defines the desired projections $\hat{h}$, $\hat{v}$ on $\mathcal{D}^1(B)$

$$\hat{h} = \tilde{\pi}_\star h_{P\times F}\tilde{\mathbb{L}} = \tilde{\pi}_\star h^{\mathrm{nat}}\tilde{\mathbb{L}}, \;\; \hat{v} = \tilde{\pi}_\star v_{P\times F}\tilde{\mathbb{L}} = \tilde{\pi}_\star v^{\mathrm{nat}}\tilde{\mathbb{L}}, \quad \text{so } \mathcal{D}^1(B) = \hat{h}\mathcal{D}^1(B) \oplus \hat{v}\mathcal{D}^1(B).$$

Finally note that $\tilde{h}\mathbb{L}_h^{\mathrm{nat}}\mathbb{L} = \tilde{h}\mathbb{L}_h^{\mathrm{nat}}h\mathbb{L} = \tilde{h}h'h^{\mathrm{nat}}\mathbb{L}_h^{\mathrm{nat}}\mathbb{L} = h'h^{\mathrm{nat}}\mathbb{L}_h^{\mathrm{nat}}\mathbb{L} = \mathbb{L}_h^{\mathrm{nat}}\mathbb{L}$ by Lemma 2.2 and $(\tilde{R}_g)_\star\mathbb{L}_h^{\mathrm{nat}}\mathbb{L} = \mathbb{L}_h^{\mathrm{nat}}(R_g)_\star\mathbb{L} = \mathbb{L}_h^{\mathrm{nat}}\mathbb{L}$, so $\mathbb{L}_h^{\mathrm{nat}}\mathbb{L}: \mathcal{D}^1(M) \to \mathcal{D}^{\tilde{\Gamma}}(P \times F)$ and the horizontal lift $\hat{\mathbb{L}}: \mathcal{D}^1(M) \to \mathcal{D}^1(B)$ is well-defined by

$$\hat{\mathbb{L}} := \tilde{\pi}_\star \circ \mathbb{L}_h^{\mathrm{nat}} \circ \mathbb{L}, \qquad \text{i. e.} \quad \tilde{\mathbb{L}} \circ \hat{\mathbb{L}} = \mathbb{L}_h^{\mathrm{nat}} \circ \mathbb{L}.$$

This is illustrated by the following commutative diagram:



$\hat{h}\hat{\mathbb{L}} = \tilde{\pi}_\star h^{\mathrm{nat}}\mathbb{L}_h^{\mathrm{nat}}\mathbb{L} = \hat{\mathbb{L}}$ proves that $\hat{\mathbb{L}}$ maps into $\hat{h}\mathcal{D}^1(B)$, so $\hat{h}_b = \hat{\mathbb{L}}_b \circ d\tilde{\pi}_b$. Also

$$\begin{aligned}
\hat{h}[\hat{\mathbb{L}}\mathcal{X}, \hat{\mathbb{L}}\mathcal{Y}] &= \tilde{\pi}_\star h^{\mathrm{nat}}\tilde{h}[\tilde{\mathbb{L}}\hat{\mathbb{L}}\mathcal{X}, \tilde{\mathbb{L}}\hat{\mathbb{L}}\mathcal{Y}] = \tilde{\pi}_\star h'h^{\mathrm{nat}}[\mathbb{L}_h^{\mathrm{nat}}\mathbb{L}\mathcal{X}, \mathbb{L}_h^{\mathrm{nat}}\mathbb{L}\mathcal{Y}] = \tilde{\pi}_\star h'\mathbb{L}_h^{\mathrm{nat}}[\mathbb{L}\mathcal{X}, \mathbb{L}\mathcal{Y}] \\
&= \tilde{\pi}_\star\mathbb{L}_h^{\mathrm{nat}}h[\mathbb{L}\mathcal{X}, \mathbb{L}\mathcal{Y}] = \tilde{\pi}_\star\mathbb{L}_h^{\mathrm{nat}}\mathbb{L}[\mathcal{X}, \mathcal{Y}] = \hat{\mathbb{L}}[\mathcal{X}, \mathcal{Y}].
\end{aligned}$$

We have thus proved the following proposition:

Proposition 2.3 *The horizontal lift $\hat{\mathbb{L}}: \mathcal{D}^1(M) \to \hat{h}\mathcal{D}^1(B)$ is an injective homomorphism of $C^\infty(M)$-modules with $\tilde{\pi}_\star \circ \hat{\mathbb{L}} = \mathrm{id}_{\mathcal{D}^1(M)}$ and $\hat{h}[\hat{\mathbb{L}}\mathcal{X}, \hat{\mathbb{L}}\mathcal{Y}] = \hat{\mathbb{L}}[\mathcal{X}, \mathcal{Y}]$ for all $\mathcal{X}$, $\mathcal{Y} \in \mathcal{D}^1(M)$. $\hat{\mathbb{L}}$ is uniquely defined by $\tilde{\mathbb{L}}\hat{\mathbb{L}} = \mathbb{L}_h^{\mathrm{nat}}\mathbb{L}: \mathcal{D}^1(M) \to h^{\mathrm{nat}}\mathcal{D}^{\tilde{\Gamma}}(P \times F)$.*

Now what happens if $B = P$? One would expect that $\hat{h} = h$ and $\hat{\mathbb{L}} = \mathbb{L}$, and this is indeed true. We have the following commutative diagram:

$$\begin{array}{ccc} P\times G & \xrightarrow{\widetilde{\pi} = R\circ\tau_{PG}} & P \\ \Big\downarrow \mathrm{pr}_P & & \Big\downarrow \pi \\ P & \xrightarrow{\pi} & M \end{array}$$

On the left, $(P\times G)(P,G)$ is a trivial principal bundle with projection pr_P and right action $\widetilde{\rho} = \mathrm{id}\times\rho$. (Here and in the sequel, ρ_g and λ_g denote right and left multiplication with $g\in G$.) This bundle is the trivialization of the square of P, which is the bundle on the top of the diagram. We can identify $\widetilde{\pi}$ and $R\circ\tau_{PG}$, where $\tau_{PG}: P\times G\to G\times P$ is the natural morphism exchanging P and G. Thus $d\widetilde{\pi}_{(p,g)}(\mathcal{P}_g,\mathcal{X}_g) = dR_g\mathcal{P}_p + dR^p\mathcal{X}_g$. We will prove $\widehat{\mathbb{L}} = \mathbb{L}$, then both connections Γ and $\widehat{\Gamma}$ on P must coincide according to our statements in Section 1. For every $\mathcal{X}\in\mathcal{D}^1(M)$ and all $p\in P$ we have

$$(\widetilde{\pi}_\star\mathbb{L}_h^{\mathrm{nat}}\mathbb{L}\mathcal{X})_p = d\widetilde{\pi}_{(R(g,p),g^{-1})}((\mathbb{L}\mathcal{X})_{R(g,p)}, 0_{g^{-1}}) = dR_{g^{-1}}(\mathbb{L}\mathcal{X})_{R(g,p)} = (\mathbb{L}\mathcal{X})_p,$$

since $(R_{g^{-1}})_\star\mathbb{L}\mathcal{X} = \mathbb{L}\mathcal{X}$ for all $g\in G$. Thus $\widehat{\mathbb{L}} = \mathbb{L}$.

Projections of forms are defined as in the case of principal bundles:

Definition 2.4 *For any connection $\Gamma\in\gamma(P(M,G))$ and any $\omega_s\in\mathcal{A}_s(B,V)$, $s>0$, where B is an associated bundle $B(M,F,G) = P\times_G F$ and V is a vector space, we define horizontal and vertical projections $\omega_s\widehat{h}$, resp., $\omega_s\widehat{v}\in\mathcal{A}_s(B,V)$ by*

$$\begin{array}{rcll} \omega_s\widehat{h}(\mathcal{X}^1,\dots,\mathcal{X}^s) & := & \omega_s(\widehat{h}\mathcal{X}^1,\dots,\widehat{h}\mathcal{X}^s), & \textit{for all}\quad \mathcal{X}^i\in\mathcal{D}^1(B), \\ \omega_s\widehat{v}(\mathcal{X}^1,\dots,\mathcal{X}^s) & := & \omega_s(\widehat{v}\mathcal{X}^1,\dots,\widehat{v}\mathcal{X}^s), & \textit{for all}\quad \mathcal{X}^i\in\mathcal{D}^1(B). \end{array}$$

$\mathcal{A}(B,V)\widehat{h}\subseteq\mathcal{A}(B,V)$ and $\mathcal{A}(B,V)\widehat{v}\subseteq\mathcal{A}(B,V)$ (with $\mathcal{A}_0(B,V)\widehat{h} := \mathcal{A}_0(B,V)\widehat{v} := \mathcal{A}_0(B,V) = C^\infty(B,V)$) denote the $C^\infty(B)$-submodules of $\mathcal{A}(B,V)$ that contain these horizontal, resp., vertical V-valued forms.

Obviously $\mathcal{A}_1(B,V) = \mathcal{A}_1(B,V)\widehat{h}\oplus\mathcal{A}_1(B,V)\widehat{v}$ and $\widehat{h}$ and $\widehat{v}$ commute with exterior products: if $\varphi: Z\times W\to V$ is a bilinear mapping and $\wedge_\varphi$ denotes the induced exterior product of Z- and W-valued forms, then with $\alpha\in\mathcal{A}(B)\otimes Z$ and $\beta\in\mathcal{A}(B)\otimes W$,

$$(\alpha\wedge_\varphi\beta)\widehat{h} = \alpha\widehat{h}\wedge_\varphi\beta\widehat{h},\qquad (\alpha\wedge_\varphi\beta)\widehat{v} = \alpha\widehat{v}\wedge_\varphi\beta\widehat{v}.$$

In the sequel we will write $\wedge_{\mathfrak{g}}$ for the exterior product induced by $[,]:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$.

In the next section we will need a generalization of the exterior product of two differential forms: the operator $\bullet$ which is linear in its first argument and multilinear in its second argument and produces V-valued forms from $\mathrm{Hom}(\mathcal{T}(W),V)$-valued and W-valued forms ($\mathcal{T}(W)$ means the tensor algebra of W), cf. [1]. For $s>0$ let $E_j\in W$, $j=1,\dots,s$ and let $E_1\otimes\cdots\otimes E_s: \mathrm{Hom}(\bigotimes^s W,V)\to V$ denote the canonical evaluation morphism. For any differential form $\chi_r^s\in\mathcal{A}_r(M,\mathrm{Hom}(\bigotimes^s W,V))$ on a manifold M define $\chi_r^{E_1,\dots,E_s}\in\mathcal{A}_r(M,V)$ to be the push-out of χ_r^s under this morphism: $\chi_r^{E_1,\dots,E_s} := (E_1\otimes\cdots\otimes E_s)_\star\chi_r^s$, i. e., for all $x\in M$ and $\mathcal{X}^i\in\mathcal{D}^1(M)$, $i=1,\dots,r$,

$$(\chi_r^{E_1,\dots,E_s})_x(\mathcal{X}_x^1,\dots,\mathcal{X}_x^r) := (E_1\otimes\cdots\otimes E_s)\circ(\chi_r^s)_x(\mathcal{X}_x^1,\dots,\mathcal{X}_x^r).$$

Now let $\phi_p = \sum_{i=1}^m \phi^i \otimes E_i \in \mathcal{A}_p(M) \otimes W$ be a W-valued form, then we define a V-valued form $\chi_r^s \bullet \phi_p$ in the following way:

$$\chi_r^s \bullet \phi_p = \sum_{i_1,\dots,i_s=1}^{m} \chi_r^{E_{i_1},\dots,E_{i_s}} \wedge \phi^{i_1} \wedge \dots \wedge \phi^{i_s} \in \mathcal{A}(M,V).$$

Thus if $\chi_r^s \in \mathcal{A}_r(M) \otimes \mathrm{Hom}(\bigotimes^s W, V)$ then also $\chi_r^s \bullet \phi_p \in \mathcal{A}_{r+sp}(M) \otimes V$. Linear extension defines the operator $\bullet$ for $\chi \in \mathcal{A}(M, \mathrm{Hom}(\mathcal{T}(W), V))$.

Note that $\bullet$ is a generalization of $\wedge_\varphi$ for bilinear $\varphi: Z \times W \to V$ in the following sense: φ canonically defines $\varphi': Z \to \mathrm{Hom}(W,V)$. For any $\alpha_r \in \mathcal{A}_r(M) \otimes Z$ we thus have a push-out $\varphi'_\star \alpha_r \in \mathcal{A}_r(M) \otimes \mathrm{Hom}(W,V)$. Now if $\beta_p \in \mathcal{A}_p(M) \otimes W$ then $\alpha_r \wedge_\varphi \beta_p = (\varphi'_\star \alpha_r) \bullet \beta_p$.

For a bundle B, one easily checks that the projections $\widehat{h}$ and $\widehat{v}$ commute with $\bullet$: for $\chi \in \mathcal{A}(B, \mathrm{Hom}(\mathcal{T}(W), V))$ and $\phi_p \in \mathcal{A}_p(B) \otimes W$,

$$(\chi \bullet \phi_p)\widehat{h} = \chi\widehat{h} \bullet \phi_p\widehat{h}, \qquad (\chi \bullet \phi_p)\widehat{v} = \chi\widehat{v} \bullet \phi_p\widehat{v}.$$

Also since $\bullet$ behaves well under pullbacks and push-outs, one easily proves that $\bullet$ maps equivariant forms onto invariant forms (cf. [1, Lemma 7.1]):

Lemma 2.5 *Let $S: G \times P \to P$ be a* Lie *group action and $L: G \to \mathrm{Gl}(W)$ be a left representation. If $\phi_p \in \mathcal{A}_p(P) \otimes W$ and $\chi \in \mathcal{A}(P, \mathrm{Hom}(\mathcal{T}(W), V))$ are equivariant (i. e., $S_g^\star \phi_p = L(g^{-\,\mathrm{sgn}(S)})_\star \phi_p$ and $S_g^\star \chi = (L(g^{\mathrm{sgn}(S)})^\star)_\star \chi$ for all $g \in G$, where* $\mathrm{sgn}(S) := +1$ *for a right action and* $\mathrm{sgn}(S) := -1$ *for a left action), then $\chi \bullet \phi_p$ is invariant.*

E. g., if $\chi \in \mathcal{A}(P, \mathrm{Hom}(\mathcal{T}(\mathfrak{g}), V))_{\mathrm{equiv}}$ and $\phi_p \in \mathcal{A}_p(P, \mathfrak{g})_{\mathrm{equiv}}$, where equivariance is meant with respect to the adjoint action Ad on $\mathfrak{g}$, then $\chi \bullet \phi_p$ is invariant. (We only consider finite dimensional Lie groups, so $\mathcal{A}(P, \mathfrak{g}) = \mathcal{A}(P) \otimes \mathfrak{g}$.) Recall that the equivariant forms $\phi \in \mathcal{A}(P) \otimes W$ on a principal bundle P are also called pseudotensorial forms of type (L, W), while *horizontal* equivariant forms (like the curvature 2-form) are called tensorial forms of type (L, W). We will denote their modules by $\mathcal{A}^P(P, L, V)$ and $\mathcal{A}^T(P, L, V)$, resp., $\mathcal{A}^P(P, \mathfrak{g}) := \mathcal{A}^P(P, \mathrm{Ad}, \mathfrak{g})$ and $\mathcal{A}^T(P, \mathfrak{g}) := \mathcal{A}^T(P, \mathrm{Ad}, \mathfrak{g})$.

3 Generating forms on B from forms on F and P

For the trivial bundle $M \times F$ with $G = \{e\}$ we can extend vector fields and forms on M and F to the bundle using the natural projections pr_M and pr_F, resp., the natural injections i_f and i_x for $f \in F$ and $x \in M$. For arbitrary bundles only one global projection $\widehat{\pi}$ is given naturally and we only have "global" (with regard to F) injections $i_{\alpha,x}$ on every bundle chart. These enable us to define a vertical bundle $V(B)$ and a global horizontal lift of differential forms $\widehat{\pi}^\star: \mathcal{A}(M, V) \to \mathcal{A}(B, V)$. We have seen that it requires a connection as an additional structure to define $H(B)$ and horizontal lifts of vector fields on M onto the bundle.

Now we will be concerned with the "dual problem" to extend forms on the fiber to the bundle. Locally we can achieve this using the pullbacks $\widehat{\pi}_\alpha^\star$ of the local projections onto the fiber, but normally for $\phi \in \mathcal{A}(F, V)$, $\{\widehat{\pi}_\alpha^\star \phi \in \mathcal{A}(\pi^{-1}(U_\alpha), V)\}_{\alpha \in A}$

will not define a global form since in general on the overlaps $U_{\alpha\beta} := U_\alpha \cap U_\beta$ we will find $(\widehat{\pi}_\beta^\star \phi|_{\widehat{\pi}^{-1}(U_{\alpha\beta})}) \neq (\widehat{\pi}_\alpha^\star \phi|_{\widehat{\pi}^{-1}(U_{\alpha\beta})})$. In order to investigate how a given connection will define global forms, we can compute the transition functions and evaluate the projections of fields and forms locally. Let us postpone this access to the problem to Section 4. For now, we will again take the detour over $P \times F$ in order to derive global expressions for the extended forms.

Lemma 3.1 *$\mathcal{Y} \in \mathcal{D}^1(F)$ defines a vertical vector field $i_\star \mathcal{Y} = \widetilde{\pi}_\star \mathbb{L}_v^{\mathrm{nat}} \mathcal{Y} \in \mathcal{D}^1(B)$, such that locally $(i_\star \mathcal{Y})_{\psi_\alpha^{-1}(x,f)} = (d\psi_\alpha^{-1})_{(x,f)}(0_x, \mathcal{Y}_f)$ on $\pi^{-1}(U_\alpha)$, iff $\mathcal{Y}$ is invariant.*

Proof. We already saw that $\widetilde{\pi}_\star \mathbb{L}_v^{\mathrm{nat}} \mathcal{Y}$ defines a section of $\widetilde{\pi}^\star T(B)$. A section of $\widetilde{\pi}^\star T(B)$ is a section of $T(B)$ iff it is invariant under all $\widetilde{R}_g^\star$. But this is the case iff $\mathbb{L}_v^{\mathrm{nat}} \mathcal{Y} = (\widetilde{R}_{g^{-1}})_\star \mathbb{L}_v^{\mathrm{nat}} \mathcal{Y} = \mathbb{L}_v^{\mathrm{nat}} (L_g)_\star \mathcal{Y}$ for all $g \in G$. Since $\mathbb{L}_v^{\mathrm{nat}}$ is injective and $\widehat{h} \widetilde{\pi}_\star \mathbb{L}_v^{\mathrm{nat}} \mathcal{Y} = \widetilde{\pi}_\star h^{\mathrm{nat}} \mathbb{L}_v^{\mathrm{nat}} \mathcal{Y} = 0$, this yields our assertion. That $(i_\star \mathcal{Y})_{\psi_\alpha^{-1}(x,f)} = (d\psi_\alpha^{-1})_{(x,f)}(0_x, \mathcal{Y}_f)$ holds for all $x \in U_\alpha$ and $f \in F$, now follows from verticality and (1): $d\widehat{\pi}_\alpha \, d\widetilde{\pi}(\mathbb{L}_v^{\mathrm{nat}})_{(p,f)} \mathcal{Y}_f = dL^f \, d\pi_\alpha \, d\,\mathrm{pr}_P(\mathbb{L}_v^{\mathrm{nat}})_{(p,f)} \mathcal{Y}_f + dL_{\pi_\alpha(p)} \mathcal{Y}_f = \mathcal{Y}_{L(\pi_\alpha(p),f)}$. □

So the situation for M and F is not totally dual but involves L, and it is no surprise that, given a connection, we can only extend *invariant* forms $\phi \in \mathcal{A}(F,V)$ naturally onto the bundle. To see this, we observe that the only canonical way, how a differential form $\phi \in \mathcal{A}(F,V)$ acts on vector fields $\mathcal{Y}^i \in \mathcal{D}^1(B)$ is via

$$(\mathrm{pr}_F^\star \phi)(\ldots, \widetilde{\mathbb{L}} \mathcal{Y}^i, \ldots) = \widetilde{f} \in C^\infty(P \times F, V).$$

This defines a form on B if and only if we find $f \in C^\infty(B,V)$ for any $\mathcal{Y}^i \in \mathcal{D}^1(B)$, such that $\widetilde{f} = f \circ \widetilde{\pi}$. We note that the resulting form will be vertical since

$$(\mathrm{pr}_F)_\star \widetilde{\mathbb{L}} \widehat{v} \mathcal{Y}^i = (\mathrm{pr}_F)_\star \widetilde{\mathbb{L}} \widetilde{\pi}_\star v^{\mathrm{nat}} \widetilde{\mathbb{L}} \mathcal{Y}^i = (\mathrm{pr}_F)_\star v^{\mathrm{nat}} \widetilde{\mathbb{L}} \mathcal{Y}^i = (\mathrm{pr}_F)_\star \widetilde{\mathbb{L}} \mathcal{Y}^i.$$

Proposition 3.2 *$\phi \in \mathcal{A}(F,V)$ defines a vertical V-valued form on $B(M,F,G)$ iff ϕ is invariant under all $L_g^\star$. For such a ϕ and all $\mathcal{Y}^i \in \mathcal{D}^1(B)$ then there exists $f \in C^\infty(B,V)$ with*

$$(\mathrm{pr}_F^\star \phi)(\ldots, \widetilde{\mathbb{L}} \mathcal{Y}^i, \ldots) = f \circ \widetilde{\pi}.$$

Proof. According to the previous discussion, ϕ defines a form on B if and only if $(\mathrm{pr}_F^\star \phi)(\ldots, \widetilde{\mathbb{L}} \mathcal{Y}^i, \ldots) \in C^\infty(P \times F, V)$ is invariant under all $\widetilde{R}_g^\star$, i. e., if and only if $\widetilde{R}_g^\star[(\mathrm{pr}_F^\star \phi)(\ldots, \widetilde{\mathbb{L}} \mathcal{Y}^i, \ldots)] = (\widetilde{R}_g^\star \,\mathrm{pr}_F^\star \phi)(\ldots, (\widetilde{R}_{g^{-1}})_\star \widetilde{\mathbb{L}} \mathcal{Y}^i, \ldots) = (\mathrm{pr}_F^\star \, L_{g^{-1}}^\star \phi)(\ldots, \widetilde{\mathbb{L}} \mathcal{Y}^i, \ldots)$ for all $g \in G$ and $\mathcal{Y}^i \in \mathcal{D}^1(B)$. Obviously this relation holds if $\phi \in \mathcal{A}(F,V)$ is invariant. So let us assume, that ϕ is not invariant. Then we find $g \in G$, $f \in F$ and $\mathcal{X}^i \in \mathcal{D}^1(F)$ such that $(L_g^\star \phi)_f(\ldots, \mathcal{X}_f^i, \ldots) = \phi_{L(g,f)}(\ldots, dL_g \mathcal{X}_f^i, \ldots) \neq \phi_f(\ldots, \mathcal{X}_f^i, \ldots)$. Since only $\mathcal{X}_f^i$ are involved, we may assume that all $\mathcal{X}^i$ are invariant and thus define $\widetilde{\pi}_\star \mathbb{L}_v^{\mathrm{nat}} \mathcal{X}^i \in \mathcal{D}^1(B)$ by Lemma 3.1. For these vector fields on B we compute $\widetilde{\mathbb{L}} \widetilde{\pi}_\star \mathbb{L}_v^{\mathrm{nat}} \mathcal{X}^i = \widetilde{h} \mathbb{L}_v^{\mathrm{nat}} \mathcal{X}^i = \widetilde{h} v^{\mathrm{nat}} \mathbb{L}_v^{\mathrm{nat}} \mathcal{X}^i = v^{\mathrm{nat}} \mathbb{L}_v^{\mathrm{nat}} \mathcal{X}^i = \mathbb{L}_v^{\mathrm{nat}} \mathcal{X}^i$ and thus $(R_{g^{-1}}^\star \,\mathrm{pr}_F^\star \phi)(\ldots, \widetilde{\mathbb{L}} \widetilde{\pi}_\star \mathbb{L}_v^{\mathrm{nat}} \mathcal{X}^i, \ldots)(p,f) = (L_g^\star \phi)_f(\ldots, \mathcal{X}_f^i, \ldots) \neq \phi_f(\ldots, \mathcal{X}_f^i, \ldots) = (\mathrm{pr}_F^\star \phi)(\ldots, \widetilde{\mathbb{L}} \widetilde{\pi}_\star \mathbb{L}_v^{\mathrm{nat}} \mathcal{X}^i, \ldots)(p,f)$. So $(\mathrm{pr}_F^\star \phi)(\ldots, \widetilde{\mathbb{L}} \widetilde{\pi}_\star \mathbb{L}_v^{\mathrm{nat}} \mathcal{X}^i, \ldots)$ is not invariant under all $\widetilde{R}_g^\star$. Verticality was already proved above. □

Similar arguments hold for $\phi \in \mathcal{A}(P,V)$ acting on $\mathcal{Y}^i \in \mathcal{D}^1(B)$ via

$$(\mathrm{pr}_P^\star \phi)(\ldots, \widetilde{\mathbb{L}} \mathcal{Y}^i, \ldots) \in C^\infty(P \times F, V).$$

The resulting form will be horizontal because $(\mathrm{pr}_P)_\star \widetilde{\mathbb{L}} \widehat{h} = (\mathrm{pr}_P)_\star \widetilde{\mathbb{L}}$. Moreover, only ϕh is of interest: $(\mathrm{pr}_P)_\star \widetilde{\mathbb{L}} = (\mathrm{pr}_P)_\star h^{\mathrm{nat}} \widetilde{h} \widetilde{\mathbb{L}} = (\mathrm{pr}_P)_\star h' h^{\mathrm{nat}} \widetilde{\mathbb{L}} = h'(\mathrm{pr}_P)_\star \widetilde{\mathbb{L}}$, thus

$$(\mathrm{pr}_P^\star \phi)(\ldots, \widetilde{\mathbb{L}}\mathcal{Y}^i, \ldots) = (\mathrm{pr}_P^\star \phi h)(\ldots, \widetilde{\mathbb{L}}\mathcal{Y}^i, \ldots).$$

Proposition 3.3 *$\phi \in \mathcal{A}(P,V)$ defines a horizontal V-valued form on $B(M,F,G)$ iff $\phi h = \pi^\star \varphi$, $\varphi \in \mathcal{A}(M,V)$. For such a ϕ and all $\mathcal{Y}^i \in \mathcal{D}^1(B)$ we then have*

$$(\mathrm{pr}_P^\star \phi)(\ldots, \widetilde{\mathbb{L}}\mathcal{Y}^i, \ldots) = (\widehat{\pi}^\star \varphi)(\ldots, \mathcal{Y}^i, \ldots) \circ \widetilde{\pi}.$$

Proof. We already saw that only ϕh matters and that the resulting form is horizontal. Now $\phi h = \pi^\star \varphi$ iff $R_g^\star(\phi h) = \phi h$ for all $g \in G$, and analogously to the previous proof we can show that this suffices to define a form on B. But then

$$(\mathrm{pr}_P^\star \phi h)(\ldots, \widetilde{\mathbb{L}}\mathcal{Y}^i, \ldots) = (\widetilde{\pi}^\star \widehat{\pi}^\star \varphi)(\ldots, \widetilde{\mathbb{L}}\mathcal{Y}^i, \ldots) = (\widehat{\pi}^\star \varphi)(\ldots, \widetilde{\mathbb{L}}\mathcal{Y}^i, \ldots) \circ \widetilde{\pi}.$$

On the other hand, if there exists $g \in G$ with $R_g^\star \phi h \neq \phi h$, we can find invariant vector fields in $\mathcal{D}^\Gamma(P)$, i. e. $\mathcal{X}^i \in \mathcal{D}^1(M)$, such that $\phi h(\ldots, \mathbb{L}\mathcal{X}^i, \ldots) \circ R_g \neq \phi h(\ldots, \mathbb{L}\mathcal{X}^i, \ldots)$. So $(\mathrm{pr}_P^\star \phi)(\ldots, \widetilde{\mathbb{L}}\widehat{\mathbb{L}}\mathcal{X}^i, \ldots) \circ \widetilde{R}_g = \phi h(\ldots, \mathbb{L}\mathcal{X}^i, \ldots) \circ \mathrm{pr}_P \circ \widetilde{R}_g = \phi h(\ldots, \mathbb{L}\mathcal{X}^i, \ldots) \circ R_g \circ \mathrm{pr}_P \neq \phi h(\ldots, \mathbb{L}\mathcal{X}^i, \ldots) \circ \mathrm{pr}_P = (\mathrm{pr}_P^\star \phi)(\ldots, \widetilde{\mathbb{L}}\widehat{\mathbb{L}}\mathcal{X}^i, \ldots)$. Thus $(\mathrm{pr}_P^\star \phi)(\ldots, \widetilde{\mathbb{L}}\widehat{\mathbb{L}}\mathcal{X}^i, \ldots)$ does not define $f \in C^\infty(B,V)$. □

As a simple example that only the horizontal part of $\phi \in \mathcal{A}(P,V)$ counts and needs to be invariant, we compute

$$(\mathrm{pr}_P^\star \omega^\Gamma)(\widetilde{\mathbb{L}}\mathcal{Y}) = \widetilde{\omega}^{\widetilde{\Gamma}}(\widetilde{\mathbb{L}}\mathcal{Y}) = \widetilde{\mathcal{R}}'^{-1} \circ \widetilde{v}\widetilde{\mathbb{L}}\mathcal{Y} = 0. \tag{2}$$

Now we want to combine forms on F with forms on P. This could be done by an exterior product of the generated forms on $P \times F$. More generally, we will use the operator $\bullet$ instead.

Theorem 3.4 *If $\chi \in \mathcal{A}(F, \mathrm{Hom}(\mathcal{T}(\mathfrak{g}), V))_{\mathrm{equiv}}$ and $\phi \in \mathcal{A}_r^P(P, \mathfrak{g}) = \mathcal{A}_r(P, \mathfrak{g})_{\mathrm{equiv}}$, $r \in \mathbb{N}_0$, then $(\mathrm{pr}_F^\star \chi) \bullet (\mathrm{pr}_P^\star \phi) \in \mathcal{A}(P \times F, V)$ defines a V-valued form on B: for all vector fields $\mathcal{Y}^i \in \mathcal{D}^1(B)$ then there exists $f \in C^\infty(B,V)$ such that*

$$[(\mathrm{pr}_F^\star \chi) \bullet (\mathrm{pr}_P^\star \phi)](\ldots, \widetilde{\mathbb{L}}\mathcal{Y}^i, \ldots) = [(\mathrm{pr}_F^\star \chi) \bullet (\mathrm{pr}_P^\star \phi h)](\ldots, \widetilde{\mathbb{L}}\mathcal{Y}^i, \ldots) = f \circ \widetilde{\pi}.$$

$(\mathrm{pr}_F^\star \chi)$ defines the vertical and $(\mathrm{pr}_P^\star \phi)$ defines the horizontal part of the form.

Proof. Analogously to the previous proofs, we must show that for any $\mathcal{Y} \in \mathcal{D}^1(B)$, $[(\mathrm{pr}_F^\star \chi) \bullet (\mathrm{pr}_P^\star \phi)](\ldots, \widetilde{\mathbb{L}}\mathcal{Y}^i, \ldots) \in C^\infty(P \times F, V)$ is invariant. Again this means that $(\mathrm{pr}_F^\star \chi) \bullet (\mathrm{pr}_P^\star \phi) \in \mathcal{A}(P \times F, V)$ is invariant. $\mathrm{pr}_F \circ L_{g^{-1}} = \widetilde{R}_g \circ \mathrm{pr}_F$ and $\mathrm{pr}_P \circ R_g = \widetilde{R}_g \circ \mathrm{pr}_P$ yield that $\mathrm{pr}_F^\star \chi$ and $\mathrm{pr}_P^\star \phi$ are G-equivariant. Now Lemma 2.5 applies. □

All of these results are just special cases of the following theorem. If we replace $\mathfrak{g}$ by any vector space W with a left representation L', we may prove in total analogy for pseudotensorial forms of type (L', W) on P:

Theorem 3.5 *Let V, W be vector spaces, $L' \colon G \times W \to W$ a left representation and $\phi \in \mathcal{A}_r^P(P, L', W)$, $r \in \mathbb{N}_0$. If $\chi \in \mathcal{A}(F, \mathrm{Hom}(\mathcal{T}(W), V))_{\mathrm{equiv}}$ then $(\mathrm{pr}_F^\star \chi) \bullet (\mathrm{pr}_P^\star \phi) \in \mathcal{A}(P \times F, V)$ defines a V-valued form on B: for all vector fields $\mathcal{Y}^i \in \mathcal{D}^1(B)$ then there exists $f \in C^\infty(B,V)$ such that*

$$[(\mathrm{pr}_F^\star \chi) \bullet (\mathrm{pr}_P^\star \phi)](\ldots, \widetilde{\mathbb{L}}\mathcal{Y}^i, \ldots) = [(\mathrm{pr}_F^\star \chi) \bullet (\mathrm{pr}_P^\star \phi h)](\ldots, \widetilde{\mathbb{L}}\mathcal{Y}^i, \ldots) = f \circ \widetilde{\pi}.$$

$(\mathrm{pr}_F^\star \chi)$ defines the vertical and $(\mathrm{pr}_P^\star \phi)$ defines the horizontal part of the form.

Let us again consider the case $B = P$. Now $\mathcal{Y} \in \mathcal{D}^1(G)$ in Lemma 3.1 is invariant iff $\mathcal{Y}_g = d\lambda_g(X)$ for all $g \in G$ and $X \in \mathfrak{g}$. But then $(i_\star \mathcal{Y})_{\psi_\alpha^{-1}(x,g)} = (d\psi_\alpha^{-1})_{(x,g)}(0_x, d\lambda_g(X)) = (\mathcal{R}_X)_{\psi_\alpha^{-1}(x,g)}$, so the vector field generated by $\mathcal{Y} = \mathcal{L}_X \in \mathcal{D}^1_L(G)$ is the fundamental vector field $\mathcal{R}_X$. Recall that the connection 1-form ω^Γ and the left canonical 1-form $\Theta^L \in \mathcal{A}_1^L(G)$ are connected via $(R^p)^\star\omega^\Gamma = \Theta^L$ for all $p \in P$. According to Proposition 3.2, Θ^L defines a vertical $\mathfrak{g}$-valued 1-form "$\Theta^L v$" on P. Since $\Theta^L v$ is vertical, we may compute it by evaluating $(\Theta^L v)(\mathcal{R}_X)$. Now $(\mathrm{pr}_G^\star \Theta^L)(\widetilde{\mathbb{L}}\mathcal{R}_X) = (\mathrm{pr}_G^\star \Theta^L)(\widetilde{\mathbb{L}} i_\star \mathcal{L}_X) = (\mathrm{pr}_G^\star \Theta^L)(\mathbb{L}_v^{\mathrm{nat}}\mathcal{L}_X) = \Theta^L(\mathcal{L}_X) = X$. Thus $\Theta^L v = \omega^\Gamma$. Finally we can recover $\Omega^\Gamma \in \mathcal{A}_2^P(P, \mathfrak{g})$ using Theorem 3.4 with $\chi := \mathrm{Ad} \circ \eta \in C^\infty(G, \mathrm{Hom}(\mathfrak{g}, \mathfrak{g}))_{\mathrm{equiv}}$, where $\eta\colon G \to G$ means the inversion on G, since $\mathrm{pr}_G^\star(\mathrm{Ad} \circ \eta) \bullet (\mathrm{pr}_P^\star \Omega^\Gamma) = \widetilde{\pi}^\star\Omega^\Gamma$, cf. (7) and Corollary 4.6 below.

4 Local Evaluation of Connections

In order to compute the exterior derivatives of the generated V-valued forms on B in Proposition 3.2 and Theorem 3.4 we give local representations for these forms in this section. For this purpose we need to evaluate the local connections on $U_\alpha \times F$ that are induced by Γ due to Lemma 1.3 and thus to compute the local projections of fields and forms. Since we will be concerned with fiber bundles in general from now on, we will distinguish between π and $\widehat{\pi}$, h and $\widehat{h}$, $\mathbb{L}$ and $\widehat{\mathbb{L}}$, etc., only where necessary, but use $\pi\colon M \to B$, etc., for convenience.

We start our local evaluations by computing the change of bundle charts. For $U_{\alpha\beta} = U_\alpha \cap U_\beta \neq \emptyset$ let $T_{\beta\alpha} := (\psi_\beta|_{\pi^{-1}(U_{\alpha\beta})}) \circ (\psi_\alpha|_{\pi^{-1}(U_{\alpha\beta})})^{-1}\colon U_{\alpha\beta} \times F \to U_{\alpha\beta} \times F$ denote the maps for the change of bundle charts. If $g_{\beta\alpha}\colon U_{\alpha\beta} \to G$ are the transition functions then the maps $T_{\beta\alpha}$ are given by

$$T_{\beta\alpha} = (\mathrm{pr}_{U_{\alpha\beta}}, L \circ (g_{\beta\alpha} \circ \mathrm{pr}_{U_{\alpha\beta}}, \mathrm{pr}_F)) = L \circ (g_{\beta\alpha} \circ \mathrm{pr}_{U_{\alpha\beta}}, \mathrm{id}_{U_{\alpha\beta}\times F}), \tag{3}$$

where we have identified L with the induced action $\bar{L}$ on $U_{\alpha\beta} \times F$ from Lemma 2.1 ($P := U_{\alpha\beta}$). For $x \in U_{\alpha\beta} \neq \emptyset$ and $f \in F$ let $(X, F) \in T_x(M) \oplus T_f(F)$. Then (3) yields $(dT_{\beta\alpha})_{(x,f)}(X, F) = (X, dL_{g_{\beta\alpha}(x)}(F) + dL^f\, dg_{\beta\alpha}(X))$ and if f^α and $f^\beta \in F$ are related by $f^\beta = L(g_{\beta\alpha}(x), f^\alpha)$, then $(T_{\beta\alpha}^\star \omega^\beta)_{(x,f^\alpha)}(\ldots, (X^\alpha, F^\alpha)^i_{(x,f^\alpha)}, \ldots) =$

$$\omega^\beta_{(x,f^\beta)}(\ldots, (X^\alpha, dL_{g_{\beta\alpha}(x)}(F^\alpha) + dL^{f^\alpha} dg_{\beta\alpha}(X^\alpha)^i_{(x,f^\beta)}, \ldots) \text{ for all } \omega^\beta \in \mathcal{A}(U_{\alpha\beta} \times F, V).$$

In order to treat such expressions in terms of forms we already introduced the opeator $\circledcirc$ in [2] in the following way: for any $\omega_n \in \mathcal{A}_n(F, V)$ (resp., $U_\alpha \times F$, etc., instead of F), we define $L^i_\bullet \omega_n \in \mathcal{A}_{n-i}(F, \mathrm{Alt}_i(\mathfrak{g}, V))$, $i \leq n$, for all $\mathcal{X}^j \in \mathcal{D}^1(F)$, $E_k \in \mathfrak{g}$ and $f \in F$ by

$$[(L^i_\bullet \omega_n)(\mathcal{X}^1, \ldots, \mathcal{X}^{n-i})(f)](E_1, \ldots, E_i) := \tfrac{n!}{(n-i)!}\omega_n(\mathcal{L}^1, \ldots, \mathcal{L}^i, \mathcal{X}^1, \ldots, \mathcal{X}^{n-i})(f) \in V,$$

where $\mathcal{L}^i := \mathcal{L}_{E_i}$. Thus $L^i_\bullet \omega_n \in \mathcal{A}_{n-i}(F) \otimes \mathrm{Alt}_i(\mathfrak{g}, V)$ if $\omega_n \in \mathcal{A}_n(F) \otimes V$. For $i > n$ we put $L^i_\bullet \omega_n = 0$. In the case $i = 1$ we also define for $\chi^s_n \in \mathcal{A}_n(F, \mathrm{Hom}(\bigotimes^s \mathfrak{g}, V))$ using the symmetrization map $\mathrm{Sym}\colon \mathrm{Hom}(\bigotimes^s \mathfrak{g}, V) \to \mathrm{Sym}_s(\mathfrak{g}, V)$:

$$L^\vee_\bullet \chi^s_n := \mathrm{Sym}_\star(L_\bullet \chi^s_n) \in \mathcal{A}_{n-1}(F, \mathrm{Sym}_{s+1}(\mathfrak{g}, V)).$$

(Obviously $\mathrm{Sym}_\star(L^i_\bullet \chi^s_n) = 0$ for $i > 1$.) For any $\omega_n \in \mathcal{A}_n(F,V)$ and $\theta \in \mathcal{A}_1(F,\mathfrak{g})$, we define

$$\omega_n \circledcirc \theta := \sum_{i=0}^{n} \frac{(-1)^{i(n-i)}}{i!} (L^i_\bullet \omega_n) \bullet \theta \in \mathcal{A}_n(F,V).$$

Then a straightforward calculation yields (cf. [2, Lemma 9.2]) that

$$(\omega \circledcirc \theta)_f(\ldots,\mathcal{X}^i_f,\ldots) = \omega_f(\ldots,\mathcal{X}^i_f + (dL^f)_e\theta_f(\mathcal{X}^i_f),\ldots). \tag{4}$$

Thus we obtain from (3):

Proposition 4.1 *If L' is a representation of G on V and $\omega^\beta_n \in \mathcal{A}_n(U_{\alpha\beta}\times F,V)_{\mathrm{equiv}}$,*

$$T^\star_{\beta\alpha}\omega^\beta_n = [(L'\circ g_{\beta\alpha}\circ \mathrm{pr}_{U_{\alpha\beta}})\bullet \omega^\beta_n] \circledcirc (g_{\beta\alpha}\circ \mathrm{pr}_{U_{\alpha\beta}})^\star \Theta^L_G.$$

Corollary 4.2 *If $\chi \in \mathcal{A}_n(F,\mathrm{Hom}(\mathcal{T}(\mathfrak{g}),V))_{\mathrm{equiv}}$ then*

$$T^\star_{\beta\alpha}(\mathrm{pr}^\star_F\,\chi) = [(\mathrm{Ad}\circ g_{\beta\alpha}\circ \mathrm{pr}_{U_{\alpha\beta}})\bullet(\mathrm{pr}^\star_F\,\chi)] \circledcirc (g_{\beta\alpha}\circ \mathrm{pr}_{U_{\alpha\beta}})^\star\Theta^L_G.$$

If $\phi \in \mathcal{A}_n(F,V)_{\mathrm{inv}}$ then $T^\star_{\beta\alpha}(\mathrm{pr}^\star_F\,\phi) = (\mathrm{pr}^\star_F\,\phi) \circledcirc (g_{\beta\alpha}\circ \mathrm{pr}_{U_{\alpha\beta}})^\star\Theta^L_G$.

For $\mu \in \mathcal{A}(M,V)$ we obviously have $T^\star_{\beta\alpha}((\mathrm{pr}_{U_{\alpha\beta}})^\star\mu) = (\mathrm{pr}_{U_{\alpha\beta}})^\star\mu$.

In order to treat local projections, recall that the gauge potentials A^α and the gauge fields F^α of a connection Γ are given by

$$\mathrm{A}^\alpha := \sigma^\star_{\alpha,e}(\omega^\Gamma|_{\pi^{-1}(U_\alpha)}) \in \mathcal{A}_1(U_\alpha,\mathfrak{g}),\quad \mathrm{F}^\alpha := \sigma^\star_{\alpha,e}(\Omega^\Gamma|_{\pi^{-1}(U_\alpha)}) \in \mathcal{A}_2(U_\alpha,\mathfrak{g}), \tag{5}$$

where $\sigma_{\alpha,e}: U_\alpha \to \pi^{-1}(U_\alpha)$ are the local sections of the principal bundle onto the neutral element $e \in G$, i. e., $\sigma_{\alpha,e}(x) = \psi_\alpha(x,e)$. The collection of A^α and F^α determines ω^Γ and Ω^Γ completely (recall that η means the inversion on G):

$$\omega^\Gamma|_{\pi^{-1}(U_\alpha)} = (\mathrm{Ad}\circ\eta\circ\pi_\alpha)\bullet(\pi^\star \mathrm{A}^\alpha) + \pi^\star_\alpha\Theta^L, \tag{6}$$
$$\Omega^\Gamma|_{\pi^{-1}(U_\alpha)} = (\mathrm{Ad}\circ\eta\circ\pi_\alpha)\bullet(\pi^\star \mathrm{F}^\alpha), \tag{7}$$

and from $\eta\circ\pi_\beta\circ\sigma_{\alpha,e} = g_{\alpha\beta}$ one derives on $U_{\alpha\beta}\neq\emptyset$:

$$\mathrm{A}^\alpha|_{U_{\alpha\beta}} = (\mathrm{Ad}\circ g_{\alpha\beta})\bullet \mathrm{A}^\beta|_{U_{\alpha\beta}} + g^\star_{\beta\alpha}\Theta^L = (\mathrm{Ad}\circ g_{\alpha\beta})\bullet(\mathrm{A}^\beta|_{U_{\alpha\beta}} - g^\star_{\alpha\beta}\Theta^L), \tag{8}$$
$$\mathrm{F}^\alpha|_{U_{\alpha\beta}} = (\mathrm{Ad}\circ g_{\alpha\beta})\bullet \mathrm{F}^\beta|_{U_{\alpha\beta}}. \tag{9}$$

In general, for a tensorial form $\varphi \in \mathcal{A}^T(P,L,V)$ on a principal bundle $P(M,G)$, we define analogously to (5) for every bundle chart

$$\mathrm{P}^\alpha := \sigma^\star_{\alpha,e}(\varphi|_{\pi^{-1}(U_\alpha)}) \in \mathcal{A}(U_\alpha,V). \tag{10}$$

Then again the collection of P^α determines φ completely:

$$\varphi|_{\pi^{-1}(U_\alpha)} = (L\circ\eta\circ\pi_\alpha)\bullet(\pi^\star \mathrm{P}^\alpha), \tag{11}$$

and on $U_{\alpha\beta}\neq\emptyset$ the P^α transform according to

$$\mathrm{P}^\alpha|_{U_{\alpha\beta}} = (L\circ g_{\alpha\beta})\bullet \mathrm{P}^\beta|_{U_{\alpha\beta}}. \tag{12}$$

According to (6), ω^Γ is locally given by $\omega^\alpha_{(x,g)}(X,Y) = \mathrm{Ad}(g^{-1})\mathrm{A}^\alpha_x(X) + d\lambda_{g^{-1}}(Y)$ for all $x \in U_\alpha$, $g \in G$ and $(X,Y) \in T_x(U_\alpha) \oplus T_g(G)$. From $v = \mathcal{R}' \circ \omega^\Gamma$ we thus conclude that the induced local projections of vector fields on $U_\alpha \times G$ are:

$$v^\alpha_{(x,g)}(X,Y) = (0, (d\rho_g)_e \mathrm{A}^\alpha_x(X) + Y), \qquad h^\alpha_{(x,g)}(X,Y) = (X, -(d\rho_g)_e \mathrm{A}^\alpha_x(X)).$$

Finally the horizontal lifts $\mathbb{L}^\alpha \colon \mathcal{D}^1(U_\alpha) \to \mathcal{D}^1(U_\alpha \times G)$ are given by

$$\mathbb{L}^\alpha_{(x,g)}(X) = (X, -(d\rho_g)_e \mathrm{A}^\alpha_x(X)). \tag{13}$$

In order to compute v^α for associated bundles, we first need the connection on $P \times F$ for our construction in Section 2. By definition,

$$(d\widetilde{R}^{(p,f)})_e(Y) = ((dR^p)_e(Y), -(dL^f)_e(Y)) \qquad \text{for all} \quad p \in P,\ f \in F \quad \text{and} \quad Y \in \mathfrak{g},$$

$$\text{thus} \qquad (d\widetilde{R}^{(x,g,f)})^\alpha_e(Y) = (0, (d\lambda_g)_e(Y), -(dL^f)_e(Y)) \in T_x(U_\alpha) \oplus T_g(G) \oplus T_f(F).$$

With $\omega^\alpha_{(x,g)}$ from above, $\widetilde{v}^\alpha_{(x,g,f)}(X,Y,Z) = (d\widetilde{R}^{(x,g,f)})^\alpha_e \omega^\alpha_{(x,g)}(X,Y)$ yields

$$\begin{aligned}
\widetilde{v}^\alpha_{(x,g,f)}(X,Y,Z) &= (0, (d\rho_g)_e \mathrm{A}^\alpha_x(X) + Y, -(dL^f)_e[\mathrm{Ad}(g^{-1})\mathrm{A}^\alpha_x(X) + d\lambda_{g^{-1}}(Y)]),\\
\widetilde{h}^\alpha_{(x,g,f)}(X,Y,Z) &= (X, -(d\rho_g)_e \mathrm{A}^\alpha_x(X), +(dL^f)_e[\mathrm{Ad}(g^{-1})\mathrm{A}^\alpha_x(X) + d\lambda_{g^{-1}}(Y)] + Z).
\end{aligned}$$

A little computation then shows using $d\widehat{\pi}(X,Y,Z) = (X, (dL^f)_g Y + (dL_g)_f Z)$

$$\widetilde{\mathbb{L}}^\alpha_{(x,g,L(g^{-1},f))}(X,Z) = (X, -(d\rho_g)_e \mathrm{A}^\alpha_x(X), +(dL_{g^{-1}})_f[(dL^f)_e \mathrm{A}^\alpha_x(X) + Z]). \tag{14}$$

Thus we obtain from $\widehat{v} = \widetilde{\pi} v^{\mathrm{nat}} \widetilde{\mathbb{L}}$ the following lemma (now omitting "$\widehat{\ }$"):

Lemma 4.3 *Every connection Γ on an associated bundle $B = P(M,G) \times_G F$, that is defined by a collection of gauge potentials $\mathrm{A}^\alpha \in \mathcal{A}_1(U_\alpha, \mathfrak{g})$, induces the following projections for all $x \in U_\alpha$, $f \in F$ and $(X,Z) \in T_x(U_\alpha) \oplus T_f(F)$:*

$$v^\alpha_{(x,f)}(X,Z) = (0, (dL^f)_e \mathrm{A}^\alpha_x(X) + Z), \quad h^\alpha_{(x,f)}(X,Z) = (X, -(dL^f)_e \mathrm{A}^\alpha_x(X)). \tag{15}$$

The horizontal lifts $\mathbb{L}^\alpha \colon \mathcal{D}^1(U_\alpha) \to \mathcal{D}^1(U_\alpha \times F)$ are thus given by

$$\mathbb{L}^\alpha_{(x,f)}(X) = (X, -(dL^f)_e \mathrm{A}^\alpha_x(X)).$$

Observe that for $B = P$, we indeed recover the original connection. Our result is no less than surprising since replacing $d\rho_g$ by dL^f is the only canonical way to generalize a connection on $U_\alpha \times G$ to associated connections on $U_\alpha \times F$.

Finally we compute the local projections of forms. Lemma 4.3 yields

$$(\omega^\alpha v^\alpha)_{(x,f)}(\ldots, (X^i, Z^i), \ldots) = \omega^\alpha_{(x,f)}(\ldots, (0, (dL^f)_e \mathrm{A}^\alpha_x(X^i) + Z^i), \ldots)$$

for all $\omega^\alpha \in \mathcal{A}(U_\alpha \times F, V)$ and $(X^i, Z^i) \in T_x(U_\alpha) \oplus T_f(F)$ and we obtain from (4):

Lemma 4.4 *If $\phi \in \mathcal{A}_n(F,V)$ then on every local trivialization $U_\alpha \times F$:*

$$(\mathrm{pr}_F^\star \phi) v^\alpha = (\mathrm{pr}_F^\star \phi) \circledcirc (\mathrm{pr}_{U_\alpha}^\star \mathrm{A}^\alpha).$$

Thus for all $x \in U_\alpha$, $i^\star_{\alpha,x}[(\mathrm{pr}_F^\star \phi) v^\alpha] = \phi$: restriction to the fibers reproduces ϕ.

Now we can evaluate Propositions 3.2, 3.3 and Theorems 3.4 and 3.5 on the bundle charts. For $\phi \in \mathcal{A}^P(U_\alpha \times G, L', W)$ one derives using (13) and (14) that

$$\begin{aligned} ((\mathrm{pr}_{U_\alpha\times G})^\star\phi)(\ldots,\widetilde{\mathbb{L}}^\alpha_{(x,g,L(g^{-1},f))}(X^i,F^i),\ldots) &= \phi_{(x,g)}(\ldots,\mathbb{L}^\alpha_{(x,g)}(X^i),\ldots) \\ &= (\phi h)_{(x,g)}(\ldots,\mathbb{L}^\alpha_{(x,g)}(X^i),\ldots). \end{aligned}$$

Since we already proved invariance under $\widetilde{R}^\star_g$, we may restrict ourselves to $g = e$. If we define $\mathrm{P}^\alpha = \sigma^\star_{\alpha,e}\phi h \in \mathcal{A}(U_\alpha, W)$ as in (10), then (11) yields

$$(\phi h)_{(x,g)}(\ldots,\mathbb{L}^\alpha_{(x,g)}(X^i),\ldots) = \mathrm{P}^\alpha(\ldots,X^i,\ldots).$$

So the horizontal part $(\mathrm{pr}^\star_P\,\phi)$ of the form in Theorem 3.5 is locally just $(\widehat{\mathrm{pr}}^\star_{U_\alpha}\mathrm{P}^\alpha)$, resp., $(\widehat{\pi}^\star\mathrm{P}^\alpha)$.

Analogously for the vertical part $(\mathrm{pr}^\star_F\,\chi)$, again (14) and (15) yield that it is locally given by $(\widehat{\mathrm{pr}}^\star_F\chi)v^\alpha$, resp., $(\widehat{\pi}_\alpha\chi)v^\alpha$. So our results take the following form (again omitting "$\widehat{\ }$" for convenience):

Theorem 4.5 *Let Γ be a connection on a principal fiber bundle $P(M,G)$ with associated bundle $B(M,F,G)$, V,W any vector spaces and $L'\colon G\times W\to W$ a left representation. Let v^α denote the local vertical projections of V-valued forms induced by Γ on $U_\alpha\times F$, resp., $\pi^{-1}(U_\alpha)$ for all $\alpha\in A$. Then for any $\chi\in\mathcal{A}(F,\mathrm{Hom}(\mathcal{T}(W),V))_{\mathrm{equiv}}$ and any family $\{\mathrm{P}^\alpha\in\mathcal{A}(U_\alpha,W)\}_{\alpha\in A}$ with $\mathrm{P}^\alpha|_{U_{\alpha\beta}} = (L'\circ g_{\alpha\beta})\bullet\mathrm{P}^\beta|_{U_{\alpha\beta}}$ for all $U_{\alpha\beta}\neq\emptyset$,*

$$\begin{aligned} T^\star_{\beta\alpha}\{[(\mathrm{pr}^\star_F\,\chi)v^\beta]\bullet[(\mathrm{pr}_{U_\beta})^\star\mathrm{P}^\beta]\}|_{U_{\alpha\beta}\times F} &= \{[(\mathrm{pr}^\star_F\,\chi)v^\alpha]\bullet[(\mathrm{pr}_{U_\alpha})^\star\mathrm{P}^\alpha]\}|_{U_{\alpha\beta}\times F}, \quad \textit{resp.,} \\ \{[(\pi^\star_\beta\chi)v^\beta]\bullet(\pi^\star\mathrm{P}^\beta)\}|_{\pi^{-1}(U_{\alpha\beta})} &= \{[(\pi^\star_\alpha\chi)v^\alpha]\bullet(\pi^\star\mathrm{P}^\alpha)\}|_{\pi^{-1}(U_{\alpha\beta})}. \end{aligned}$$

Thus $\{[(\pi^\star_\alpha\chi)v^\alpha]\bullet(\pi^\star\mathrm{P}^\alpha)\in\mathcal{A}(\pi^{-1}(U_\alpha),V)\}_{\alpha\in A}$ defines a global form "$\chi v\bullet\mathrm{P}$" on B.

Corollary 4.6 *For any G-equivariant $\chi\in\mathcal{A}(F,\mathrm{Hom}(\mathcal{T}(\mathfrak{g}),V))$ and $\alpha,\beta\in A$*

$$\{[(\pi^\star_\beta\chi)v^\beta]\bullet(\pi^\star\mathrm{F}^\beta)\}|_{\pi^{-1}(U_{\alpha\beta})} = \{[(\pi^\star_\alpha\chi)v^\alpha]\bullet(\pi^\star\mathrm{F}^\alpha)\}|_{\pi^{-1}(U_{\alpha\beta})}.$$

Thus $\{[(\pi^\star_\alpha\chi)v^\alpha]\bullet(\pi^\star\mathrm{F}^\alpha)\}_{\alpha\in A}$ defines a global form "$\chi v\bullet\mathrm{F}$" on B.

Corollary 4.7 *If $\phi\in\mathcal{A}(F,V)$ is invariant then $\{(\mathrm{pr}^\star_F\,\phi)v^\alpha\in\mathcal{A}(U_\alpha\times F,V)\}_{\alpha\in A}$, resp., $\{(\pi^\star_\alpha\phi)v^\alpha\in\mathcal{A}(\pi^{-1}(U_\alpha),V)\}_{\alpha\in A}$ defines a global form $\phi v\in\mathcal{A}(B,V)$. If ϕ is invariant and locally vertical, then $\{\pi^\star_\alpha\phi\}_{\alpha\in A}$ is global.*

The opposite is not true in general, as the case of a trivial bundle with Lie group $G\neq\{e\}$ shows, where every invariant $\phi\in\mathcal{A}(F,V)$ defines a global but not necessarily vertical form $\pi^\star_\alpha\phi$ on the bundle (all $g^\star_{\beta\alpha}\Theta^L_G$ in Corollary 4.2 vanish). Nevertheless, the canonically generated form due to Proposition 3.2 is always vertical.

These local representations for the generated forms on B can be used to determine the exterior derivative d of the forms. Yet the evaluation of $d(\chi v)$ by Lemma 4.4 is quite annoying. Since this has already been worked out in detail for the general case $d[(\chi \odot \theta)\bullet\phi]$ in [2], we simply quote the result:

Theorem 4.8 *Let Γ be a connection on a principal fiber bundle $P(M,G)$ and let $B(M,F,G)$ be an associated bundle, V any vector space, $\chi^s_n\in\mathcal{A}_n(F)\otimes\mathrm{Hom}(\bigotimes^s\mathfrak{g},V)$ be G-equivariant and $\phi_n\in\mathcal{A}_n(F)\otimes V$ be invariant under G. Then*

$$\begin{aligned} d(\chi^s_n v\bullet\mathrm{F}) &= [(d\chi^s_n)v]^s_{n+1}\bullet\mathrm{F} \;+\; [(L_\bullet\chi^s_n)v]^{s+1}_{n-1}\bullet\mathrm{F}, \\ &= [(d\chi^s_n)v]^s_{n+1}\bullet\mathrm{F} \;+\; [(L^\vee_\bullet\chi^s_n)v]^{s+1}_{n-1}\bullet\mathrm{F}, \\ d(\phi_n v) &= (d\phi_n)v \;+\; [(L_\bullet\phi_n)v]^1_{n-1}\bullet\mathrm{F}. \end{aligned}$$

5 Bundles with Abelian Structure Group

As already stated in Lemma 2.1, the left action on the fiber $L: G \times F \to F$ naturally induces a left action on the product manifold $\bar{L}: G \times P \times F \to P \times F$, that is trivial in the factor P. Thus, besides $\widetilde{\mathcal{R}}'$, we also have a G-equivariant (with respect to $\bar{L}''$ and $\bar{L}_\star$) $C^\infty(P \times F)$-module homomorphism $\bar{\mathcal{L}}': C^\infty(P \times F, \mathfrak{g}) \to \mathcal{D}^1(P \times F)$ with $(\bar{L}_g)_\star \widetilde{\mathcal{R}}' = \widetilde{\mathcal{R}}' \bar{L}^\star_{g^{-1}}$ and $(\widetilde{R}_g)_\star \widetilde{\mathcal{L}}' = \widetilde{\mathcal{L}}' \widetilde{R}^\star_{g^{-1}}$. In addition, $\mathrm{pr}_P \circ \bar{L}_g = \mathrm{pr}_P$ yields

$$\begin{aligned} (\bar{L}_g)_\star \circ \tilde{v} &= \tilde{v} \circ (\bar{L}_g)_\star = \tilde{v} & (\bar{L}_g)_\star \circ \tilde{h} &= \tilde{h} \circ (\bar{L}_g)_\star = (\bar{L}_g)_\star - \tilde{v}, \\ (\bar{L}_g)_\star \circ v^{\mathrm{nat}} &= v^{\mathrm{nat}} \circ (\bar{L}_g)_\star, & (\bar{L}_g)_\star \circ h^{\mathrm{nat}} &= h^{\mathrm{nat}} \circ (\bar{L}_g)_\star. \end{aligned}$$

$\mathrm{pr}_P \circ \bar{L}^{(p,f)} = p$ yields $h^{\mathrm{nat}} \bar{\mathcal{L}}' = 0$, thus $\bar{\mathcal{L}}': C^\infty(P \times F, \mathfrak{g}) \to v^{\mathrm{nat}} \mathcal{D}^1(P \times F)$.

Now $\bar{L}$ defines an action on the quotient manifold $P \times_G F$ iff $\bar{L}_{h^{-1}} \circ \widetilde{R}_g \circ \bar{L}_h \in \widetilde{R}_G$ for all $g, h \in G$, where $\widetilde{R}_G := \{\widetilde{R}_g \in \mathrm{Diff}(P \times F)\}_{g \in G}$. Thus $\bar{L}_G < N_{\mathrm{Diff}(P \times F)}(\widetilde{R}_G)$: $\bar{L}_G$ needs to be a subgroup of the normalizer of $\widetilde{R}_G$ in $\mathrm{Diff}(P \times F)$. Even if G is abelian and $\widetilde{R}$ acts freely, this does not hold automatically, as the example of the action of $\mathbb{Z}_4$ on $\mathbb{R}^3 \backslash \{$"axes"$\}$ by $\frac{\pi}{2}$-rotations around different axes shows.

In our case $(\bar{L}_{h^{-1}} \circ \widetilde{R}_g \circ \bar{L}_h)(p, f) = \widetilde{R}_g(p, L_{gh^{-1}g^{-1}h}(f))$, thus

$$\bar{L} \text{ defines an action } \widehat{L}: G \times B \to B \qquad \Longleftrightarrow \qquad L_{G'} = \{\mathrm{id}_F\},$$

where G' means the commutator subgroup in G. This is equivalent to the requirement that G acts effectively only through its largest abelian factor group G/G'. Since we require G to act effectively itself, this means G is abelian.

According to the structure theorem for abelian Lie groups [3, p. 228], a connected Lie group G is abelian iff it is isomorphic to $\mathfrak{g}/\ker \exp$, thus iff G is isomorphic to $\mathbb{R}^m \times (\mathbb{S}^1)^n = \mathbb{R}^m \times (\mathbb{R}^n/\mathbb{N}^n)$ where $m, n \in \mathbb{N}_0$. Thus for any abelian Lie group we will write the group operation additively, with neutral element 0, and we will identify all tangent spaces $T_g(G)$ with $T_0(G)$ in a natural way, such that $d\lambda_g = d\rho_g: T_h(G) \to T_{h+g}(G)$ becomes the identity morphism for all $g, h \in G$.

In that case, $\widehat{L}_g \circ \tilde{\pi} = \tilde{\pi} \circ \bar{L}_g$ and $\hat{\pi} \circ \widehat{L}_g = \hat{\pi}$ (and thus $\hat{\pi} \circ \widehat{L}^b = \hat{\pi}(b)$), because

$$\hat{\pi} \circ \widehat{L}_g \circ \tilde{\pi} = \hat{\pi} \circ \tilde{\pi} \circ \bar{L}_g = \pi \circ \mathrm{pr}_P \circ \bar{L}_g = \pi \circ \mathrm{pr}_P = \hat{\pi} \circ \tilde{\pi}$$

and $\tilde{\pi}$ is surjective. Since $(\bar{L}_g)_\star$ commutes with $\tilde{h}$ and (for abelian G) commutes with $(\widetilde{R}_g)_\star$, it defines an action on $\mathcal{D}^{\widetilde{\Gamma}}(P \times F)$, i. e., $(\bar{L}_g)_\star \widetilde{\mathbb{L}} = \widetilde{\mathbb{L}}(\bar{L}_g)_\star$. This proves

$$(\widehat{L}_g)_\star \circ \hat{v} = \hat{v} \circ (\widehat{L}_g)_\star, \qquad (\widehat{L}_g)_\star \circ \hat{h} = \hat{h} \circ (\widehat{L}_g)_\star,$$

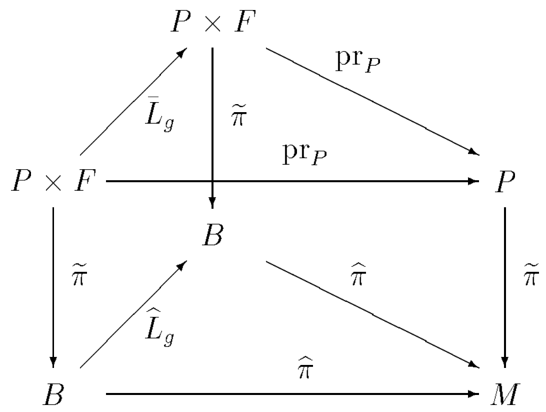
because $(\widehat{L}_g)_\star \hat{h} = (\widehat{L}_g)_\star \tilde{\pi}_\star h^{\mathrm{nat}} \widetilde{\mathbb{L}} = \tilde{\pi}_\star (\bar{L}_g)_\star h^{\mathrm{nat}} \widetilde{\mathbb{L}} = \tilde{\pi}_\star h^{\mathrm{nat}} (\bar{L}_g)_\star \widetilde{\mathbb{L}} = \hat{h}(\widehat{L}_g)_\star$. Finally $(\widehat{L}_g)_\star \widehat{\mathbb{L}} = \tilde{\pi}_\star (\bar{L}_g)_\star \mathbb{L}^{\mathrm{nat}}_h \mathbb{L} = \widehat{\mathbb{L}}$ and the horizontal lifts $\widehat{\mathbb{L}}$ are $\widehat{L}$-invariant. $\hat{h} \widehat{\mathcal{L}}' = 0$, because $\widehat{\mathcal{L}}': C^\infty(B, \mathfrak{g}) \to \hat{v} \mathcal{D}^1(B)$, since $\hat{\pi}_\star \circ d\widehat{L}^b = 0$ and $V_b(B)$ is the kernel of $d\hat{\pi}_b$. It is quite obvious that $\widehat{L}$ coincides with the following locally defined action:

Lemma 5.1 *For abelian G, we have a left action $\widehat{L}$ of G on the whole bundle:*

$$\widehat{L}(g, b) := \psi_\alpha^{-1}(\hat{\pi}(b), L(g, \hat{\pi}_\alpha(b))) \quad \textit{for all } b \in B, g \in G, \quad \textit{where} \quad \hat{\pi}(b) \in U_\alpha,$$

is then well-defined and fiber preserving: $\hat{\pi}(\widehat{L}(g, b)) = \hat{\pi}(b)$.

We thus get another diagram that commutes for every $g \in G$:



Suppose $f\colon B(M,F,G) \to B'(M',F',G)$ is a fiber preserving bundle diffeomorphism between two bundles with left actions L, resp., L' of the abelian LIE group G and $\widehat\Gamma$ is a connection on B induced by Γ on $P(M,G)$, such that $(\widehat L_g)_\star \circ \widehat h = \widehat h \circ (\widehat L_g)_\star$ for all $g \in G$. By Lemma 1.3, $\widehat\Gamma$ induces a connection $\Gamma' = \widehat\Gamma^f$ on B'. For this new connection, h', v' and $(\widehat L'_g)_\star$ need not commute on $\mathcal{D}^1(B')$. As an example, take $f = \mathrm{id}\colon M \times \mathbb{R} \to M \times \mathbb{R}$ and actions $L, L'\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ with $L(r,s) = e^r s$ and $L'(r,s) = r + s$. Then $h'(\mathcal{X}_x, \mathcal{Y}_s) = (\mathcal{X}_x, -sA_x\mathcal{X}_x)$ with $A \in \mathcal{A}_1(M)$ and $(\widehat L'_r)_\star h'(\mathcal{X}_x, \mathcal{Y}_s) = (\mathcal{X}_x, -sA_x\mathcal{X}_x)$, while $h'(\widehat L'_r)_\star(\mathcal{X}_x, \mathcal{Y}_s) = (\mathcal{X}_x, -rsA_x\mathcal{X}_x)$.

Analogously to Lemma 1.1, it is sufficient for commutativity of h', v' and $\widehat L'_g$ that f is G-equivariant. In fact, if B and B' are associated bundles over M and f is G-equivariant and induces the identity on M, then $\widehat\Gamma$ and Γ induce the same connection $\widehat\Gamma'$ on $B'(M,F',G)$.

For abelian G, the adjoint action on $\mathfrak{g}$ is trivial, which makes life easier in most cases. E. g., (8) and (9) read (we have $g^\star_{\beta\alpha}\Theta^L = dg_{\beta\alpha}$):

$$\mathrm{A}^\alpha|_{U_{\alpha\beta}} = \mathrm{A}^\beta|_{U_{\alpha\beta}} + dg_{\beta\alpha} = \mathrm{A}^\beta|_{U_{\alpha\beta}} - dg_{\alpha\beta}, \qquad \mathrm{F}^\alpha|_{U_{\alpha\beta}} = \mathrm{F}^\beta|_{U_{\alpha\beta}}, \tag{16}$$

and then ω^Γ and Ω^Γ are locally given by — cf. (6) and (7) —

$$\omega^\Gamma|_{\pi^{-1}(U_\alpha)} = \pi^\star\mathrm{A}^\alpha + d\pi_\alpha, \qquad \Omega^\Gamma|_{\pi^{-1}(U_\alpha)} = \pi^\star\mathrm{F}^\alpha. \tag{17}$$

Thus for abelian G, the collection of F^α defines a *global* 2-form $\mathrm{F} \in \mathcal{A}_2(M,\mathfrak{g})$, whose pullback is the curvature 2-form $\Omega^\Gamma = \pi^\star\mathrm{F}$.

Finally let us treat the one-dimensional case, $\mathfrak{g} \cong \mathbb{R}$. So $G = D \times G_1$ with a discrete abelian subgroup D and $G_1 \cong \mathbb{S}^1$ or $G_1 \cong \mathbb{R}$. Recall that if G is connected, nontrivial bundles only exist for $G \cong \mathbb{S}^1$, e. g. for the electromagnetic gauge group $G_{\mathrm{em}} \cong \mathrm{U}_1 \cong \mathbb{S}^1$.

So suppose $\mathfrak{g} = E\mathbb{R}$ with a basis vector $E \in \mathfrak{g}$, then the antisymmetry of differential forms yields that $L_\bullet^2\phi = 0$ for all $\phi \in \mathcal{A}_n(F,V)$. Thus Lemma 4.4 reads $(\mathrm{pr}_F^\star\,\phi)v^\alpha = (\mathrm{pr}_F^\star\,\phi) - (-1)^n[\mathrm{pr}_F^\star(L_\bullet\phi)] \bullet (\mathrm{pr}_{U_{\alpha\beta}}^\star\,\mathrm{A}^\alpha) = (\mathrm{pr}_F^\star\,\phi) + \frac{1}{E}(\mathrm{pr}_{U_{\alpha\beta}}^\star\,\mathrm{A}^\alpha) \wedge (\mathrm{pr}_F^\star\,\imath_{\mathcal{L}_E}\phi)$, where $\imath_{\mathcal{L}_E}$ is the inner product with $\mathcal{L}_E \in \mathcal{D}^1(F)$. Analogously, Corollary 4.2 takes the form $T^\star_{\beta\alpha}(\mathrm{pr}_F^\star\,\phi) = (\mathrm{pr}_F^\star\,\phi) + \frac{1}{E}(\mathrm{pr}_{U_{\alpha\beta}}^\star\,dg_{\beta\alpha}) \wedge (\mathrm{pr}_F^\star\,\imath_{\mathcal{L}_E}\phi)$ if $\phi \in \mathcal{A}(F,V)_{\mathrm{inv}}$. In that case, since $L_\bullet(L_\bullet\phi) = 0$, $\imath_{\mathcal{L}_E}\phi$ is vertical and global (it is invariant because Ad is trivial). Also recall from the homotopy identity for the LIE derivative, $L_{\mathcal{X}} = d\,\imath_{\mathcal{X}} + \imath_{\mathcal{X}}\,d$, that $d\imath_{\mathcal{L}_E}\phi + \imath_{\mathcal{L}_E}d\phi = L_{\mathcal{L}_E}\phi = 0$ if ϕ is invariant. Thus Corollary 4.7 and Theorem 4.8 prove:

Theorem 5.2 *Let Γ be a connection on $P(M,G)$ with abelian G, $\mathfrak{g} = E\mathbb{R} \cong \mathbb{R}$, $B(M,F,G)$ an associated bundle and V any vector space. For any $\phi \in \mathcal{A}_n(F,V)$ with $L_g^\star \phi = \phi$ for all $g \in G$ define $\nu \in \mathcal{A}_{n-1}(F,V)$ by $\nu = \imath_{\mathcal{L}_E}\phi$, i. e.*

$$\nu_f(\mathcal{Y}_f^1, \ldots, \mathcal{Y}_f^{n-1}) := n \cdot \phi_f(dL^f(E), \mathcal{Y}_f^1, \ldots, \mathcal{Y}_f^{n-1}) \quad \textit{for all } f \in F,\ \mathcal{Y}^i \in \mathcal{D}^1(F).$$

For any $U_\alpha \in \mathfrak{U}$ denote $\phi^\alpha := \pi_\alpha^\star \phi$, $\nu^\alpha := \pi_\alpha^\star \nu$. Then on all $U_{\alpha\beta} \neq \emptyset$

$$\begin{aligned} \phi^\alpha &= \phi^\beta + \frac{1}{E}\pi^\star dg_{\alpha\beta} \wedge \nu^\beta, \quad \phi^\alpha v = \phi^\alpha + \frac{1}{E}\pi^\star \mathrm{A}^\alpha \wedge \nu^\alpha = \phi^\beta + \frac{1}{E}\pi^\star \mathrm{A}^\beta \wedge \nu^\beta = \phi^\beta v, \\ \nu^\alpha &= \nu^\alpha v = \nu^\beta = \nu^\beta v. \end{aligned}$$

Thus ϕv and ν define global vertical invariant V-valued forms on B. The same holds for $(d\phi)v$ since $d\phi$ is also invariant, and we have

$$d(\phi v) = (d\phi)v + \frac{1}{E}\pi^\star \mathrm{F} \wedge \nu, \quad \textit{where} \quad (d\phi^\alpha)v = d\phi^\alpha - \frac{1}{E}\pi^\star \mathrm{A}^\alpha \wedge d\nu^\alpha.$$

Note that $\mathfrak{g} \cong \mathbb{R}$ alone does not imply that G is abelian. $G = \mathbb{S}^1 \rtimes \mathbb{Z}_2$ with $(r,g) \cdot (r',e) = (r - r', g)$ for $r, r' \in \mathbb{S}^1$ and $g \neq e \in \mathbb{Z}_2$, is a simple counterexample, where $\mathrm{Ad}((0,g)) = -\mathrm{id}_{\mathfrak{g}}$, and thus ν in Theorem 5.2 would not be invariant and global for this Lie group G.

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