Beyond fractality: piecewise fractal and quasifractal algebras

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Abstract

Fractality is a property of algebras of approximation sequences with several useful consequences: for example, if (A_n) is a sequence in a fractal algebra, then the pseudospectra of the A_n converge in the Hausdorff metric. The fractality of a *separable* algebra of approximation sequences can always be forced by a suitable restriction This observation leads to the question to describe the possible fractal restrictions of a given algebra. In this connection we define two classes of algebras beyond the class of fractal algebras (piecewise fractal and quasifractal algebras), give examples for algebras with these properties, and present some first results on the structure of quasifractal algebras (being continuous fields over the set of their fractal restrictions).

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1 Introduction

Fractality is a special property of algebras of approximation sequences which typically arise as follows.

Let H be a Hilbert space and $\mathcal{P} = (P_n)_{n \ge 1}$ a filtration on H, i.e., a sequence of orthogonal projections of finite rank that converges strongly to the identity operator on H. Let $\mathcal{F}^\mathcal{P}$ denote the set of all bounded sequences $(A_n)_{n\ge1}$ of operators $A_n \in L(\operatorname{im} P_n)$ and $\mathcal{G}^\mathcal{P}$ the set of all sequences $(A_n) \in \mathcal{F}^\mathcal{P}$ with $\|A_n\| \to 0$. Provided with the operations

$$(A_n) + (B_n) := (A_n + B_n), \quad (A_n)(B_n) := (A_nB_n), \quad (A_n)^* := (A_n^*) \qquad (1)$$

and the norm $\|(A_n)\| := \sup \|A_nP_n\|$, $\mathcal{F}^\mathcal{P}$ becomes a unital C^*-algebra and $\mathcal{G}^\mathcal{P}$ a closed ideal of $\mathcal{F}^\mathcal{P}$. The importance of the quotient algebra $\mathcal{F}^\mathcal{P}/\mathcal{G}^\mathcal{P}$ in numerical analysis stems from the fact that a coset $(A_n) + \mathcal{G}^\mathcal{P}$ is invertible in $\mathcal{F}^\mathcal{P}/\mathcal{G}^\mathcal{P}$ if and only if the A_n are invertible for all sufficiently large n and if the norms of the inverses are uniformly bounded, which is equivalent to saying that (A_n) is a stable sequence.

With every non-empty subset A of $L(H)$, we associate the smallest C^*-subalgebra $\mathcal{S}^{\mathcal{P}}(\mathsf{A})$ of $\mathcal{F}^{\mathcal{P}}$ that contains all sequences $(P_nAP_n)_{n\geq 1}$ with $A \in \mathsf{A}$. Algebras of this form are the prototypes of algebras of approximation sequences mentioned above.

To make this concrete, consider the algebra $\mathcal{S}(\mathsf{T}(C))$ of the finite sections discretization (FSD) for Toeplitz operators with continuous generating function. Here, H is the Hilbert space $l^2(\mathbb{Z}^+)$, P_n is the projection on H sending $(x_0, x_1, \ldots)$ to $(x_0, \ldots, x_{n-1}, 0, 0, \ldots)$ (we agree to omit the superscript $\mathcal{P}$ when the filtration is specified in this way), and A is the C^*-algebra $\mathsf{T}(C)$ generated by all Toeplitz operators $T(a)$ with a a continuous function on the complex unit circle $\mathbb{T}$. Recall that $T(a)$ is given by the matrix representation $(a_{i-j})_{i,j\geq 0}$ with respect to the standard basis of $l^2(\mathbb{Z}^+)$, where

$$a_k := \frac{1}{2\pi}\int_0^{2\pi} a(e^{is})e^{-iks}\,ds, \quad k \in \mathbb{Z}, \tag{2}$$

denotes the kth Fourier coefficient of a. It is well known that the algebra $\mathsf{T}(C)$ has a nice description, as follows.

Theorem 1 $\mathsf{T}(C) = \{T(a) + K : a \in C(\mathbb{T})$ *and* $K \in K(l^2(\mathbb{Z}^+))\}$.

Here, $K(l^2(\mathbb{Z}^+))$ is the ideal of the compact operators on $l^2(\mathbb{Z}^+)$.

Similarly, the sequences in the algebra $\mathcal{S}(\mathsf{T}(C))$ are completely characterized in the following theorem by Böttcher and Silbermann [2] (see also [3], [4, Section 1.4.2] and the pioneering paper [10]). Therein R_n stands for the operator $(x_0, x_1, \ldots) \mapsto (x_{n-1}, \ldots, x_0, 0, 0, \ldots)$ on $l^2(\mathbb{Z}^+)$. It is not hard to see that for each sequence $\mathbf{A} = (A_n) \in \mathcal{S}(\mathsf{T}(C))$, the strong limits $W(\mathbf{A}) := \text{s-lim}\, A_nP_n$ and $\widetilde{W}(\mathbf{A}) := \text{s-lim}\, R_nA_nR_nP_n$ exist and that W and $\widetilde{W}$ are unital *-homomorphisms from $\mathcal{S}(\mathsf{T}(C))$ to $L(l^2(\mathbb{Z}^+))$ (actually, to $\mathsf{T}(C)$).

Theorem 2 *(a) The algebra $\mathcal{S}(\mathsf{T}(C))$ consists of all sequences $(A_n)_{n\geq 1}$ of the form*

$$(A_n) = (P_nT(a)P_n + P_nKP_n + R_nLR_n + G_n) \tag{3}$$

where $a \in C(\mathbb{T})$, K and L are compact operators on $l^2(\mathbb{Z}^+)$, and $(G_n) \in \mathcal{G}$. The representation of a sequence $(A_n) \in \mathcal{S}(\mathsf{T}(C))$ in this form is unique.

(b) For every sequence $\mathbf{A} \in \mathcal{S}(\mathsf{T}(C))$, the coset $\mathbf{A}+\mathcal{G}$ is invertible in the quotient algebra $\mathcal{S}(\mathsf{T}(C))/\mathcal{G}$ (equivalently, $\mathbf{A}+\mathcal{G}$ is invertible in $\mathcal{F}/\mathcal{G}$ or, again equivalently, $\mathbf{A}$ is stable) if and only if the operators $W(\mathbf{A})$ and $\widetilde{W}(\mathbf{A})$ are invertible.

The algebra $\mathcal{S}(\mathsf{T}(C))$ of the FSD of the Toeplitz operators gives a first example of a fractal algebra. The idea behind the notion of a fractal algebra comes from a remarkable property of the algebra $\mathcal{S}(\mathsf{T}(C))/\mathcal{G}$: the structure of this algebra is determined by two representations W and $\widetilde{W}$. These representations are defined by certain strong limits, hence, the operators $W(\mathbf{A})$ and $\widetilde{W}(\mathbf{A})$ can be determined from each subsequence of the sequence $\mathbf{A} \in \mathcal{S}(\mathsf{T}(C))$. This observation implies that whenever a *subsequence* of a sequence $\mathbf{A} \in \mathcal{S}(\mathsf{T}(C))$ is stable, then the operators $W(\mathbf{A})$ and $\widetilde{W}(\mathbf{A})$ are already invertible and, hence, the *full* sequence $\mathbf{A}$ is stable by Theorem 2.

One can state this observation in a slightly different way: every sequence in $\mathcal{S}(\mathsf{T}(C))$ can be rediscovered from each of its (infinite) subsequences up to a sequence tending to zero in the norm. In that sense, the essential information on a sequence in $\mathcal{S}(\mathsf{T}(C))$ is stored in each of its subsequences. Subalgebras of $\mathcal{F}$ with this property were called *fractal* in [9] (see also [6]) in order to emphasize this self-similarity aspect. We will recall some basic properties of fractal algebras that will be needed in what follows and start with the official definition of a fractal algebra. We will state this definition in the slightly more general context where $\mathcal{C} = (\mathcal{C}_n)_{n \in \mathbb{N}}$ is a sequence of unital C^*-algebras and $\mathcal{F}^{\mathcal{C}}$ is the set of all bounded sequences (A_n) with $A_n \in \mathcal{C}_n$. With the operations as in (1) and with the supremum norm, $\mathcal{F}^{\mathcal{C}}$ becomes a unital C^*-algebra and the set $\mathcal{G}^{\mathcal{C}}$ of all sequences in $\mathcal{F}^{\mathcal{C}}$ tending to zero in the norm forms a closed ideal of $\mathcal{F}^{\mathcal{C}}$. Again, we will often simply write $\mathcal{F}$ and $\mathcal{G}$ in place of $\mathcal{F}^{\mathcal{C}}$ and $\mathcal{G}^{\mathcal{C}}$. Note in that connection that

$$\|(A_n) + \mathcal{G}\|_{\mathcal{F}/\mathcal{G}} = \limsup_{n \to \infty} \|A_n\|_{\mathcal{C}_n} \tag{4}$$

for every sequence $(A_n) \in \mathcal{F}$.

The sequences in $\mathcal{G}$ are often called *zero sequences*. Thus, $(G_n) \in \mathcal{F}$ is a zero sequence if $\lim_{n \to \infty} \|G_n\| = 0$. We call a sequence $(G_n) \in \mathcal{F}$ a *partial zero sequence* if $\liminf_{n \to \infty} \|G_n\| = 0$. The perhaps simplest way to define fractal algebras is the following (which is equivalent to the original definition in [9]).

Definition 3 *A C^*-subalgebra of $\mathcal{F}$ is called* fractal *if every partial zero sequence in $\mathcal{A}$ is a zero sequence.*

The fractality of the algebra $\mathcal{S}(\mathsf{T}(C))$ can be seen as follows. Suppose $\mathbf{A} := (P_n T(a) P_n + P_n K P_n + R_n L R_n + G_n)$ is a partial zero sequence in $\mathcal{S}(\mathsf{T}(C))$. Then, necessarily, $W(\mathbf{A}) = T(a) + K = 0$ and $\widetilde{W}(\mathbf{A}) = T(\tilde{a}) + L = 0$ with $\tilde{a}(t) := a(t^{-1})$. Hence, $\mathbf{A} \in \mathcal{G}$.

Here are some facts which illustrate the importance of the notion of fractality.

(F1) For a sequence (A_n) in a fractal subalgebra $\mathcal{A}$ of $\mathcal{F}$, the sets of the singular values (the pseudospectra, the numerical ranges) of the A_n converge with respect to the Hausdorff metric to the set of the singular values (the pseudospectra, the numerical ranges) of the coset $(A_n) + \mathcal{G}$ (see [4], Chapter 3).

(F2) The ideal of the compact sequences in a fractal algebra has a nice structure: it is a dual subalgebra of $\mathcal{A}/\mathcal{G}$ as shown in [7] (see the part before Corollary 25 for the definition of a compact sequence and the result).

(F3) If (A_n) is a sequence in a fractal algebra, then $\lim_{n\to\infty} \|A_n\|$ exists (compare this fact with (4) which holds for an arbitrary sequence in $\mathcal{F}$).

Property (F3) is crucial for the present paper. It follows easily from the definition of a fractal algebra and, conversely, the existence of $\lim_{n\to\infty} \|A_n\|$ for every sequence (A_n) in $\mathcal{A}$ implies that $\mathcal{A}$ is fractal.

It is certainly not true that every subalgebra of $\mathcal{F}$ is fractal ($\mathcal{F}$ itself is not fractal), but it is a remarkable consequence of (F3) that *every separable C^*-subalgebra of $\mathcal{F}$ has a fractal restriction.* To state this precisely, we need some more notation. Let $\eta : \mathbb{N} \to \mathbb{N}$ be a strictly increasing sequence. By $\mathcal{F}_\eta$ we denote the set of all subsequences $(A_{\eta(n)})$ of sequences (A_n) in $\mathcal{F}$. One can make $\mathcal{F}_\eta$ to a C^*-algebra in a natural way. The mapping $R_\eta : \mathcal{F} \to \mathcal{F}_\eta,\ (A_n) \mapsto (A_{\eta(n)})$ is called the *restriction* of $\mathcal{F}$ onto $\mathcal{F}_\eta$. For every subset $\mathcal{S}$ of $\mathcal{F}$, we abbreviate $R_\eta\mathcal{S}$ by $\mathcal{S}_\eta$. It is easy to see that $\mathcal{G}_\eta$ coincides with the ideal of the sequences in $\mathcal{F}_\eta$ which tend to zero in the norm. Since the strictly increasing sequences $\eta : \mathbb{N} \to \mathbb{N}$ are in one-to-one correspondence to the infinite subsets $\mathbb{M} := \eta(\mathbb{N})$ of $\mathbb{N}$, we will also use the notation $\mathcal{A}|_\mathbb{M}$ in place of $R_\eta\mathcal{A} = \mathcal{A}_\eta$. With these notations, we can formulate the following result of [6] (a shorter proof is in [8]).

Theorem 4 (Fractal restriction theorem) *If $\mathcal{A}$ is a separable C^*-subalgebra of $\mathcal{F}$, then there is a strictly increasing sequence $\eta : \mathbb{N} \to \mathbb{N}$ such that the restricted algebra $\mathcal{A}_\eta$ is fractal.*

One cannot expect that Theorem 4 holds for arbitrary C^*-subalgebras of $\mathcal{F}$; for example it is certainly not true for the algebra $\mathcal{F}$. On the other hand, non-separable fractal algebras exist: the algebra of the FSD for Toeplitz operators with *piecewise continuous* generating function can serve as an example.

The goal of this paper is to present some first steps into the world beyond fractal algebras. Repeated use of the fractal restriction theorem will lead us to the fractal exhaustion theorem, which then will give rise to single out two classes of non-fractal algebras, the piecewise fractal and the quasifractal algebras. For both classes, we present typical examples and study some properties. For piecewise fractal algebras, this will be quite simple: they are just constituted by a finite number of fractal algebras, and (F1) – (F3) hold for each of the finite restrictions separately. For quasifractal algebras, it is our first goal to get an overview of the possible fractal restrictions. In particular, we will define a topology on the set of all (equivalence classes of) fractal restrictions which makes this set to a compact Hausdorff space. Then we show that every quasifractal algebra can be considered as a continuous field of C^*-algebras over this space.

2 Fractal exhaustion of C^*-subalgebras of $\mathcal{F}$

The restriction process in Theorem 4 can be iterated to yield a complete decomposition of a separable subalgebra of $\mathcal{F}$ into fractal restrictions.

Theorem 5 (Fractal exhaustion theorem) *Let $\mathcal{A}$ be a separable C^*-subalgebra of $\mathcal{F}$. Then there exist a (finite or infinite) number of infinite subsets $\mathbb{M}_1, \mathbb{M}_2, \ldots$ of $\mathbb{N}$ with*

$$\mathbb{M}_i \cap \mathbb{M}_j = \emptyset \quad \text{for } i \neq j \quad \text{and} \quad \cup_i \mathbb{M}_i = \mathbb{N}$$

such that every restriction $\mathcal{A}|_{\mathbb{M}_i}$ is a fractal subalgebra of $\mathcal{F}|_{\mathbb{M}_i}$.

Proof. With Theorem 4, we find an infinite subset $\mathbb{M}_1$ of $\mathbb{N}$ such that $\mathcal{A}|_{\mathbb{M}_1}$ is fractal. Without loss of generality we may assume that $1 \in \mathbb{M}_1$ (otherwise we include 1 into $\mathbb{M}_1$). If $\mathbb{N} \setminus \mathbb{M}_1$ is a finite set, we include these finitely many points into $\mathbb{M}_1$. The algebra $\mathcal{A}|_{\mathbb{M}_1}$ is still fractal, and we are done.

If $\mathbb{N} \setminus \mathbb{M}_1$ is an infinite set, we apply Theorem 4 to the restriction $\mathcal{A}|_{\mathbb{N}\setminus\mathbb{M}_1}$ and get an infinite subset $\mathbb{M}_2$ of $\mathbb{N}\setminus\mathbb{M}_1$ such that $\mathcal{A}|_{\mathbb{M}_2}$ is fractal. Without loss we may assume that the smallest number in $\mathbb{N} \setminus \mathbb{M}_1$ belongs to $\mathbb{M}_2$. If now $\mathbb{N} \setminus (\mathbb{M}_1 \cup \mathbb{M}_2)$ is finite, we include these finitely many points into $\mathbb{M}_2$ and are done.

If $\mathbb{N} \setminus (\mathbb{M}_1 \cup \mathbb{M}_2)$ is infinite, we proceed in this way and obtain a finite (in case one of the sets $\mathbb{N} \setminus (\mathbb{M}_1 \cup \ldots \cup \mathbb{M}_k)$ is finite) or infinite sequence $\mathcal{A}|_{\mathbb{M}_1}, \mathcal{A}|_{\mathbb{M}_2}, \ldots$ of fractal restrictions of $\mathcal{A}$. It follows from our construction that the $\mathbb{M}_i$ are pairwise disjoint, and the inclusion of the smallest number of $\mathbb{N} \setminus (\mathbb{M}_1 \cup \ldots \cup \mathbb{M}_k)$ into $\mathbb{M}_{k+1}$ guaranties that $k \in \mathbb{M}_1 \cup \ldots \cup \mathbb{M}_k$, which gives the exhausting property. ■

If the number of restrictions in Theorem 5 is infinite, then the relation between the algebra $\mathcal{A}$ and its restrictions may be quite loose. For example, there could be a sequence $\mathbf{A}$ in $\mathcal{A}$ such that every restriction $\mathbf{A}|_{\mathbb{M}_k}$ tends to zero in the norm, but $\mathbf{A}$ does not belong to $\mathcal{G}$ (consider a sequence the restriction of which to $\mathbb{M}_k$ is $(P_1, 0, \ldots)$ for every k). This cannot happen if the number of restrictions is finite, which leads to the following definition.

Definition 6 *A C^*-subalgebra $\mathcal{A}$ of $\mathcal{F}$ is called* piecewise fractal *if there are finitely many infinite subsets $\mathbb{M}_1, \ldots, \mathbb{M}_k$ of $\mathbb{N}$ with*

$$\mathbb{M}_i \cap \mathbb{M}_j = \emptyset \quad \text{for } i \neq j \quad \text{and} \quad \cup_{i=1}^k \mathbb{M}_i = \mathbb{N} \tag{5}$$

such that every restriction $\mathcal{A}|_{\mathbb{M}_i}$ is a fractal subalgebra of $\mathcal{F}|_{\mathbb{M}_i}$.

A typical example of a piecewise fractal algebra (in fact, a close relative of the algebra of the FSD for Toeplitz operators) will be examined in the following section. It is clear that, in piecewise fractal algebras, properties (F1) – (F3) hold separately on each of the finitely many fractal restrictions.

It turns out that several important properties of a sequence $\mathbf{A}$ in $\mathcal{F}$ can be expressed in terms of the family of *all* fractal restrictions of $\mathbf{A}$. To explain this observation, we introduce a class of subalgebras of $\mathcal{F}$ which is still small enough to own a useful fractality property, but which is also large enough to cover all separable subalgebras of $\mathcal{F}$.

Definition 7 *A C^*-subalgebra $\mathcal{A}$ of $\mathcal{F}$ is called* quasifractal *if every restriction of $\mathcal{A}$ has a fractal restriction.*

Lemma 8 (a) *Piecewise fractal C^*-subalgebras of $\mathcal{F}$ are quasifractal.*
(b) *Separable C^*-subalgebras of $\mathcal{F}$ are quasifractal.*

Proof. Let $\mathcal{A}$ be piecewise fractal and let $\mathbb{M}$ be an infinite subset of $\mathbb{N}$. Then there is an infinite subset $\mathbb{M}_i$ of $\mathbb{N}$ as in (5) such that the intersection $\mathbb{M} \cap \mathbb{M}_i =: \mathbb{K}$ is infinite. Then $\mathbb{K}$ defines a fractal restriction of $\mathcal{A}|_{\mathbb{M}}$, which proves (a). Assertion (b) is a direct consequence of the fractal restriction theorem. ■

Proposition 9 *Let $\mathcal{A}$ be a quasifractal C^*-subalgebra of $\mathcal{F}$. Then a sequence $\mathbf{A} \in \mathcal{A}$ is a zero sequence (is stable) if and only if every fractal restriction of $\mathbf{A}$ goes to zero (is stable, respectively).*

Proof. If $\mathbf{A}$ is a zero sequence, then every restriction of $\mathbf{A}$ goes to zero as well. If $\mathbf{A} = (A_n)$ is not in $\mathcal{G}$, there are a restriction η and a positive constant C such that $\|A_{\eta(n)}\| \geq C$ for all $n \in \mathbb{N}$. Due to the quasifractality of $\mathcal{A}$, there is a fractal restriction μ of η. The restricted sequence $(A_{\mu(n)})$ does not tend to zero. The argument for stability is similar. ■

In particular, this result holds when $\mathbf{A}$ is a sequence in $\mathcal{F}$ and $\mathcal{A}$ is the smallest C^*-subalgebra of $\mathcal{F}$ which contains $\mathbf{A}$. Since $\mathcal{A}$ is separable, it is quasifractal. Note also that the fractal exhaustion theorem (Theorem 5) holds for general quasifractal algebras (in place of separable algebras) as well.

3 The FSD for block Toeplitz operators

We are now going to extend the results cited in the introduction to the FSD for Toeplitz operators with matrix-valued generating functions, which will provide us which an archetypal example of a piecewise fractal algebra.

3.1 Block Toeplitz operators

Throughout this section, N denotes a fixed positive integer. For a C^*-subalgebra B of $L^\infty(\mathbb{T})$, we write $B^{N \times N}$ for the C^*-algebra of all $N \times N$-matrices with entries in B. The elements of $B^{N \times N}$ are considered as functions on $\mathbb{T}$ with values in $\mathbb{C}^{N \times N}$.

Let $a \in L^\infty(\mathbb{T})^{N\times N}$. The kth Fourier coefficient a_k of a is given as in (2). We define the Toeplitz operator $T(a)$ and the Hankel operator $H(a)$ with generating function a via their matrix representations $(a_{i-j})_{i,j\geq 0}$ and $(a_{i+j+1})_{i,j\geq 0}$ with respect to the standard basis of $l^2(\mathbb{Z}^+)$ in verbatim the same way as for $N = 1$, having in mind that in the present setting, the a_k are $N \times N$-matrices. To emphasize the latter fact, $T(a)$ and $H(a)$ are usually referred to as *block Toeplitz* and *block Hankel operators*. For B as above, we write $\mathsf{T}(B^{N\times N})$ for the smallest closed subalgebra of $L(l^2(\mathbb{Z}^+))$ which contains all Toeplitz operators $T(a)$ with $a \in B^{N\times N}$.

Every Toeplitz operator $T(a)$ generated by a (scalar-valued) function $a \in L^\infty(\mathbb{T})$ can also be viewed as an $N \times N$-block Toeplitz operator generated by a certain function $a_{\langle N\rangle} \in L^\infty(\mathbb{T})^{N\times N}$. In particular, if a is a trigonometric polynomial a, then $a_{\langle N\rangle}$ has only finitely many non-vanishing Fourier coefficients and is, hence, a function in $C^{N\times N} := C(\mathbb{T})^{N\times N}$. Since the trigonometric polynomials are dense in $C(\mathbb{T})$ we obtain:

Proposition 10 $\mathsf{T}(B) \subseteq \mathsf{T}(B^{N\times N})$ *for* $B = C(\mathbb{T}), L^\infty(\mathbb{T})$.

This inclusion holds for other function classes as well, e.g., for $B = PC$, the algebra of the piecewise continuous functions. We will not need these results in the present paper.

The analogue of Theorem 1 reads as follows.

Theorem 11 $\mathsf{T}(C^{N\times N}) = \{T(a) + K : a \in C(\mathbb{T})^{N\times N}$ *and* $K \in K(l^2(\mathbb{Z}^+))\}$.

Proof. As in case $N = 1$ one can show that the right hand side is a C^*-subalgebra of $L(l^2(\mathbb{Z}^+))$. Since this algebra contains all Toeplitz operators $T(a)$ with $a \in C(\mathbb{T})^{N\times N}$, the inclusion $\subseteq$ follows. For the reverse inclusion, we have to show that $K(l^2(\mathbb{Z}^+)) \subseteq \mathsf{T}(C^{N\times N})$. This follows from $K(l^2(\mathbb{Z}^+)) \subseteq \mathsf{T}(C)$ by Theorem 1 and $\mathsf{T}(C) \subseteq \mathsf{T}(C^{N\times N})$ by Proposition 10. ■

3.2 An adapted FSD for block Toeplitz operators

Let the filtration $\mathcal{P} = (P_n)$ and the reflection operators R_n on $l^2(\mathbb{Z}^+)$ be as in Theorem 2, and let $a \in C(\mathbb{T})^{N\times N}$. In contrast to the case $N = 1$, where every finite section $P_nT(a)P_n$ is a finite Toeplitz matrix again, the block Toeplitz structure of the $P_nT(a)P_n$ gets lost when $N > 1$ and n is not divisible by N. It is therefore only natural to consider the *adapted* or restricted sequence $(P_{nN}T(a)P_{nN})_{n\geq 1}$ instead of the full sequence $(P_nT(a)P_n)_{n\geq 1}$ of all finite sections of $T(a)$.

Accordingly, we set $\mathcal{P}_N := (P_{nN})_{n\geq 1}$ and let $\mathcal{S}_{N\mathbb{N}}(\mathsf{T}(C^{N\times N}))$ stand for the smallest closed subalgebra of $\mathcal{F}^{\mathcal{P}_N}$ which contains all sequences $(P_{nN}T(a)P_{nN})_{n\geq 1}$ with $a \in C^{N\times N}$. The algebra $\mathcal{S}(\mathsf{T}(C^{N\times N}))$ of the *full* FSD for block Toeplitz operators, which is generated by the sequences $(P_nT(a)P_n)_{n\geq 1}$ will be the subject of the following section.

A common basis both for the adapted and the full FSD is provided by the following lemma.

Lemma 12 *Let $0 \le i < N$. The strong limits*

$$W(\mathbf{A}) := \text{s-lim}_{n\to\infty} A_n P_n, \quad \widetilde{W}_i(\mathbf{A}) := \text{s-lim}_{n\to\infty} R_{nN+i} A_{nN+i} R_{nN+i}$$

exist for every sequence $\mathbf{A} = (A_n) \in \mathcal{S}(\mathsf{T}(C^{N\times N}))$. In particular, if $\mathbf{A} = (P_n T(a) P_n)$ with $a \in C(\mathbb{T})^{N\times N}$, then $W(\mathbf{A}) = T(a)$ and $\widetilde{W}_i(\mathbf{A}) = T(\widetilde{a}_i)$ with

$$\widetilde{a}_i(t) := \begin{cases} R_N\, a(t^{-1})\, R_N & \text{if } i = 0, \\ \begin{pmatrix} R_i & 0 \\ 0 & tR_{N-i} \end{pmatrix} a(t^{-1}) \begin{pmatrix} R_i & 0 \\ 0 & t^{-1}R_{N-i} \end{pmatrix} & \text{if } i > 0. \end{cases} \tag{6}$$

The operators R_k in (6) are understood as $k \times k$ matrices.

Proof. The existence of the strong limits is either evident or follows from (6), which on its hand rests on the equality

$$R_{nN+i} T(a) R_{nN+i} = P_{nN+i} T(\widetilde{a}_i) P_{nN+i}, \tag{7}$$

holding for general $a \in L^\infty(\mathbb{T})^{N\times N}$. Note that it is *clear* that (7) holds with a *certain* function $\widetilde{a}_i$. The concrete form of these functions, as shown in (6), follows by straightforward, but somewhat tedious, calculations showing that the kth Fourier coefficient of $\widetilde{a}_i$ coincides with the kth Fourier coefficient of the function on the right hand side of the equality (6). ∎

Theorem 13 (a) *The algebra $\mathcal{S}_{N\mathbb{N}}(\mathsf{T}(C^{N\times N}))$ of the adapted FSD coincides with the set of all sequences*

$$(P_{nN} T(a) P_{nN} + P_{nN} K P_{nN} + R_{nN} L R_{nN} + G_{nN})_{n\ge 1} \tag{8}$$

where $a \in C(\mathbb{T})^{N\times N}$, K, $L \in K(l^2(\mathbb{Z}^+))$, and $(G_n) \in \mathcal{G}^{\mathcal{P}}$.
(b) *The sequence (8) is stable if and only if the operators $T(a)+K$ and $T(\widetilde{a}_0)+L$ are invertible.*

Proof. Let $\mathcal{S}$ denote the set of all sequences (8). Proceeding as in the proof of Theorem 2 (a) and using Lemma 12, which we need here for $i = 0$ only, we obtain that $\mathcal{S}$ is a C^*-subalgebra of $\mathcal{F}$ and that W and $\widetilde{W}_0$ are *-homomorphisms on $\mathcal{S}$. Since $\mathcal{S}$ contains all sequences $(P_{nN} T(a) P_{nN})$ with $a \in C(\mathbb{T})^{N\times N}$, we conclude that $\mathcal{S}_{N\mathbb{N}}(\mathsf{T}(C^{N\times N})) \subseteq \mathcal{S}$.

For the reverse inclusion we have to show that all sequences $(P_{nN} K P_{nN} + R_{nN} L R_{nN} + G_{nN})_{n\ge 1}$ with K, $L \in K(l^2(\mathbb{Z}^+))$ and $(G_n) \in \mathcal{G}$ belong to the algebra $\mathcal{S}_{N\mathbb{N}}(\mathsf{T}(C^{N\times N}))$. From Theorem 2 (a) and Lemma 10 we know that

$$(P_n K P_n + R_n L R_n + G_n)_{n\ge 1} \in \mathcal{S}(\mathsf{T}(C)) \subseteq \mathcal{S}(\mathsf{T}(C^{N\times N})),$$

hence the restriction of that sequence to $N\mathbb{N}$ belongs to $\mathcal{S}_{N\mathbb{N}}(\mathsf{T}(C^{N\times N}))$. This settles the proof of (a). Assertion (b) follows as in the proof of Theorem 2 (b). ■

As a by-product we obtain that the algebra $\mathcal{S}_{N\mathbb{N}}(\mathsf{T}(C^{N\times N}))$ of the adapted FSD can also be characterized as the smallest closed subalgebra of $\mathcal{F}^{\mathcal{P}_N}$ which contains all sequences $(P_{nN}AP_{nN})_{n\geq 1}$ with $A \in \mathsf{T}(C^{N\times N})$.

3.3 The full FSD for block Toeplitz operators

Now we turn our attention to the algebra $\mathcal{S}(\mathsf{T}(C^{N\times N}))$ of the full FSD for block Toeplitz operators. In analogy with Theorems 2 (b) and 13, we will derive a complete description of that algebra. For that goal, we define the remainder function $\kappa : \mathbb{N} \to \{0, 1, \ldots, N-1\}$ such that N divides $n - \kappa(n)$.

Theorem 14 (a) *The algebra* $\mathcal{S}(\mathsf{T}(C^{N\times N}))$ *of the full FSD coincides with the set of all sequences*

$$(P_nT(a)P_n + P_nKP_n + R_nL_{\kappa(n)}R_n + G_n)_{n\geq 1} \tag{9}$$

where $a \in C(\mathbb{T})^{N\times N}$, $K, L_0, L_1, \ldots, L_{N-1} \in K(l^2(\mathbb{Z}^+))$, *and* $(G_n) \in \mathcal{G}^{\mathcal{P}}$.
(b) *The sequence* (9) *is stable if and only if the operators* $T(a)+K$ *and* $T(\widetilde{a}_i)+L_i$ *are invertible for every* $0 \leq i < N$.

Proof. (a) Let again $\mathcal{S}$ denote the set of all sequences (9). The inclusion $\mathcal{S}(\mathsf{T}(C^{N\times N})) \subseteq \mathcal{S}$ follows as in the proof of Theorem 13, using now Lemma 12 in its general form. The more interesting part of the proof is the reverse inclusion $\mathcal{S} \subseteq \mathcal{S}(\mathsf{T}(C^{N\times N}))$.

The sequences $(P_nT(a)P_n)$ with $a \in C(\mathbb{T})^{N\times N}$ belong to $\mathcal{S}(\mathsf{T}(C^{N\times N}))$ by definition. From Proposition 10 we conclude that $\mathcal{S}(\mathsf{T}(C)) \subseteq \mathcal{S}(\mathsf{T}(C^{N\times N}))$; hence, the sequences (P_nKP_n) with $K \in K(l^2(\mathbb{Z}^+))$ and the sequences in $\mathcal{G}^{\mathcal{P}}$ belong to $\mathcal{S}(\mathsf{T}(C^{N\times N}))$ by Theorem 2 (a). It remains to show that the sequences

$$(0, \ldots, 0, R_jLR_j, 0, \ldots, 0, R_{j+N}LR_{j+N}, 0, \ldots)$$

(starting with a block of $j-1$ zeros; all subsequent blocks of zeros have length $N-1$) belong to $\mathcal{S}(\mathsf{T}(C^{N\times N}))$ for every $1 \leq j \leq N$ and $L \in K(l^2(\mathbb{Z}^+))$. Since the algebra $\mathcal{S}(\mathsf{T}(C^{N\times N}))$ is closed, it is sufficient to show that all sequences

$$(0, \ldots, 0, R_jP_kLP_kR_j, 0, \ldots, 0, R_{j+N}P_kLP_kR_{j+N}, 0, \ldots)$$

with $k \in \mathbb{N}$ belong to $\mathcal{S}(\mathsf{T}(C^{N\times N}))$. This sequence is the product of the sequence $(R_nP_kLP_kR_n)_{n\geq 1}$, which is in $\mathcal{S}(\mathsf{T}(C))$ by Theorem 2 (a) and hence also in $\mathcal{S}(\mathsf{T}(C^{N\times N}))$, with the sequence

$$(0, \ldots, 0, R_jP_kR_j, 0, \ldots, 0, R_{j+N}P_kR_{j+N}, 0, \ldots). \tag{10}$$

So it remains to show that these sequences are in $\mathcal{S}(\mathsf{T}(C^{N\times N}))$ for every $k \in \mathbb{N}$. This task can be further reduced to showing that the sequence

$$(0, \ldots, 0, R_j P_1 R_j, 0, \ldots, 0, R_{j+N} P_1 R_{j+N}, 0, \ldots) \tag{11}$$

is in $\mathcal{S}(\mathsf{T}(C^{N\times N}))$ for every $1 \le j \le N$. Indeed, with the shift operators $V_{\pm 1}$ defined on $L^2(\mathbb{Z}^+)$ by

$$V_1 : (x_0, x_1, \ldots) \mapsto (0, x_0, x_1, \ldots), \quad V_{-1} : (x_0, x_1, \ldots) \mapsto (x_1, x_2, \ldots),$$

we have $(P_n V_{\pm 1} P_n)_{n\ge 1} \in \mathcal{S}(\mathsf{T}(C))$ and

$$P_n V_{-1} P_n \cdot R_n P_1 R_n \cdot P_n V_1 P_n = R_n (P_2 - P_1) R_n.$$

Thus if the sequence (11) is in $\mathcal{S}(\mathsf{T}(C^{N\times N}))$ then the sequence

$$(0, \ldots, 0, R_j(P_2 - P_1)R_j, 0, \ldots, 0, R_{j+N}(P_2 - P_1)R_{j+N}, 0, \ldots)$$

obtained by multiplying (11) by $(P_n V_{-1} P_n)$ from the left and by $(P_n V_1 P_n)$ from the right, is in $\mathcal{S}(\mathsf{T}(C^{N\times N}))$; hence, the sequence (10) is in $\mathcal{S}(\mathsf{T}(C^{N\times N}))$ when $k = 2$. Repeating this argument we get the assertion for general k.

So we are left with verifying that (11) is in $\mathcal{S}(\mathsf{T}(C^{N\times N}))$. For $1 \le j \le N$, let $B_j = (b_{kl})_{k,l=1}^N$ and $D_j = (d_{kl})_{k,l=1}^N$ be the $N \times N$-matrices with $b_{j1} = d_{jj} = 1$ and with all other entries being zero and set

$$A_j := \begin{pmatrix} 0 & B_j & 0 & 0 & \\ 0 & 0 & B_j & 0 & \\ 0 & 0 & 0 & B_j & \\ & & & & \ddots \end{pmatrix}, \quad C_j := \begin{pmatrix} C_j & 0 & 0 & \\ 0 & C_j & 0 & \\ 0 & 0 & C_j & \\ & & & \ddots \end{pmatrix}.$$

These are block Toeplitz operators in $\mathsf{T}(C^{N\times N})$; hence the sequences $(P_n A_j P_n)$ and $(P_n C_j P_n)$ belong to $\mathcal{S}(\mathsf{T}(C^{N\times N}))$. A straightforward computation gives

$$\begin{aligned} & P_n C_j P_n - P_n A_j P_n A_j^* P_n \\ &= \begin{cases} \operatorname{diag}(0, \ldots, 0) & \text{if } 1 \le n < j, \\ \operatorname{diag}(0, \ldots, 0, 1, 0, \ldots, 0) & \text{if } j \le n \le N, \\ \operatorname{diag}(0, \ldots, 0) & \text{if } N+1 \le n < N+j, \\ \operatorname{diag}(0, \ldots, 0, 1, 0, \ldots, 0) & \text{if } N+j \le n \le 2N, \\ \operatorname{diag}(0, \ldots, 0) & \text{if } 2N+1 \le n < 2N+j, \\ \operatorname{diag}(0, \ldots, 0, 1, 0, \ldots, 0) & \text{if } 2N+j \le n \le 3N \end{cases} \end{aligned}$$

and so on, with the ones standing at the jth, $(N+j)$th and $(2N+j)$th position in lines 2, 4 and 6, respectively. For $j = N$ we conclude that the sequence $(E_n^N)_{n=1}^\infty$ with

$$E_n^N := \begin{cases} \operatorname{diag}(0, \ldots, 0, 1) & \text{if } n = kN, \\ \operatorname{diag}(0, \ldots, 0) & \text{else} \end{cases}$$

belongs to $\mathcal{S}(\mathsf{T}(C^{N\times N}))$. Similarly, for $j = N-1$ and $j = N-2$, the sequences (E_n^j) with

$$E_n^{N-1} := \begin{cases} \operatorname{diag}(0,\, \ldots,\, 0,\, 1) & \text{if } n = kN-1, \\ \operatorname{diag}(0,\, \ldots,\, 0,\, 1,\, 0) & \text{if } n = kN, \\ \operatorname{diag}(0,\, \ldots,\, 0) & \text{else} \end{cases}$$

and

$$E_n^{N-2} := \begin{cases} \operatorname{diag}(0,\, \ldots,\, 0,\, 1) & \text{if } n = kN-2, \\ \operatorname{diag}(0,\, \ldots,\, 0,\, 1,\, 0) & \text{if } n = kN-1, \\ \operatorname{diag}(0,\, \ldots,\, 0,\, 1,\, 0,\, 0) & \text{if } n = kN, \\ \operatorname{diag}(0,\, \ldots,\, 0) & \text{else} \end{cases}$$

are elements of $\mathcal{S}(\mathsf{T}(C^{N\times N}))$. Employing a shift argument as before we conclude that with (E_n^N) also the sequence $(E_n^{N,1})_{n=1}^\infty$ with

$$E_n^{N,1} := \begin{cases} \operatorname{diag}(0,\, \ldots,\, 0,\, 1,\, 0) & \text{if } n = kN, \\ \operatorname{diag}(0,\, \ldots,\, 0) & \text{else} \end{cases}$$

and hence the sequence $(F_n^{N-1}) := (E_n^{N-1}) - (E_n^{N,1})$ with

$$F_n^{N-1} := \begin{cases} \operatorname{diag}(0,\, \ldots,\, 0,\, 1) & \text{if } n = kN-1, \\ \operatorname{diag}(0,\, \ldots,\, 0) & \text{else} \end{cases}$$

belongs to $\mathcal{S}(\mathsf{T}(C^{N\times N}))$. Similarly, with (E_n^N) and (F_n^{N-1}), also the shifted sequences $(E_n^{N,2})$ and $(F_n^{N-1,1})$ with

$$E_n^{N,2} := \begin{cases} \operatorname{diag}(0,\, \ldots,\, 0,\, 1,\, 0,\, 0) & \text{if } n = kN, \\ \operatorname{diag}(0,\, \ldots,\, 0) & \text{else} \end{cases}$$

and

$$F_n^{N-1,1} := \begin{cases} \operatorname{diag}(0,\, \ldots,\, 0,\, 1,\, 0) & \text{if } n = kN-1, \\ \operatorname{diag}(0,\, \ldots,\, 0) & \text{else} \end{cases}$$

belong to $\mathcal{S}(\mathsf{T}(C^{N\times N}))$. Then also the sequence $(F_n^{N-2}) := (E_n^{N-2}) - (E_n^{N,2}) - (F_n^{N-1,1})$ with

$$F_n^{N-2} := \begin{cases} \operatorname{diag}(0,\, \ldots,\, 0,\, 1) & \text{if } n = kN-2, \\ \operatorname{diag}(0,\, \ldots,\, 0) & \text{else} \end{cases}$$

lie in $\mathcal{S}(\mathsf{T}(C^{N\times N}))$. So we have found that the sequences $(F_n^N) := (E_n^N)$, (F_n^{N-1}) and (F_n^{N-2}), i.e., the sequences (11) with $j = N$, $j = N-1$ and $j = N-2$, are in $\mathcal{S}(\mathsf{T}(C^{N\times N}))$. Continuing in this way, we get the assertion for general j. This finishes the proof of assertion (a); assertion (b) follows again as in the proof of Theorem 2 (b). ■

Corollary 15 *The algebra $\mathcal{S}(\mathsf{T}(C^{N\times N}))/\mathcal{G}^{\mathcal{P}}$ is *-isomorphic to the C^*-algebra of all $(N+1)$-tuples $(W(\mathbf{A}),\, \widetilde{W_0}(\mathbf{A}),\, \ldots,\, \widetilde{W}_{N-1}(\mathbf{A}))$ with $\mathbf{A} \in \mathcal{S}(\mathsf{T}(C^{N\times N}))$.*

4 Quasifractal algebras

4.1 An example

We start with a concrete example of a quasifractal algebra which we will obtain by a discretization of continuous functions of Toeplitz operators. Let $X = [0, 1]$ (or another compact metric, hence separable, space) and (ξ_n) a dense sequence in X. Let $\mathcal{S}(X, \mathsf{T}(C))$ stand for the smallest C^*-subalgebra of $\mathcal{F}$ which contains all sequences $(P_n A(\xi_n) P_n)$ where $A : X \to \mathsf{T}(C)$ is a continuous function. If we apply this discretization to a constant function $A : X \to \mathsf{T}(C)$, we just get the usual FSD for A. In this sense, $\mathcal{S}(\mathsf{T}(C)) \subseteq \mathcal{S}(X, \mathsf{T}(C))$.

Theorem 16 *The algebra $\mathcal{S}(X, \mathsf{T}(C))$ is quasifractal.*

Proof. Consider an arbitrary restriction of $\mathcal{S}(X, \mathsf{T}(C))$ given by a strictly increasing sequence η. By compactness, the sequence $(\xi_{\eta(n)})$ has a convergent subsequence $(\xi_{\mu(n)})$ with limit $\mu^* \in X$. Let $A : X \to \mathsf{T}(C)$ be continuous. Then $\|A(\mu(n)) - A(\mu^*)\| \to 0$. Hence, the sequence $(P_{\mu(n)} A(\mu(n)) P_{\mu(n)})$ differs from the sequence $(P_{\mu(n)} A(\mu^*) P_{\mu(n)}) \in \mathcal{S}(\mathsf{T}(C))_\mu$ by a zero sequence. This shows that $\mathcal{S}(X, \mathsf{T}(C))_\mu = \mathcal{S}(\mathsf{T}(C))_\mu$. Since $\mathcal{S}(\mathsf{T}(C))$ is fractal, this implies the fractality of the restriction $\mathcal{S}(X, \mathsf{T}(C))_\mu$. Since η was arbitrary, the algebra $\mathcal{S}(X, \mathsf{T}(C))$ is quasifractal. ∎

4.2 The fractal variety of an algebra

Let $\mathcal{C}$ be a sequence of unital C^*-algebras and $\mathcal{A}$ be a C^*-subalgebra of the algebra $\mathcal{F}^{\mathcal{C}}$. By $\mathrm{fr}\,\mathcal{A}$ we denote the set of all infinite subsets $\mathbb{M}$ of $\mathbb{N}$ such that the restriction $\mathcal{A}|_{\mathbb{M}}$ is fractal. We say that $\mathbb{M}_1, \mathbb{M}_2 \in \mathrm{fr}\,\mathcal{A}$ are *equivalent* if $\mathbb{M}_1 \cup \mathbb{M}_2 \in \mathrm{fr}\,\mathcal{A}$. This relation is reflexive and symmetric. The following lemma implies that it is also transitive and, hence, an equivalence relation.

Lemma 17 *If $\mathbb{M}_1, \mathbb{M}_2 \in \mathrm{fr}\,\mathcal{A}$ and $\mathbb{M}_1 \cap \mathbb{M}_2$ is infinite, then $\mathbb{M}_1 \cup \mathbb{M}_2 \in \mathrm{fr}\,\mathcal{A}$.*

Proof. Let $\mathbb{M}$ be an infinite subset of $\mathbb{M}_1 \cup \mathbb{M}_2$, and let $\mathbf{A} \in \mathcal{A}$ be a sequence for which $\mathbf{A}|_{\mathbb{M}}$ is a zero sequence. We show that then $\mathbf{A}|_{\mathbb{M}_1 \cup \mathbb{M}_2}$ is a zero sequence, whence the fractality of $\mathcal{A}|_{\mathbb{M}_1 \cup \mathbb{M}_2}$ by definition.

One of the sets $\mathbb{M} \cap \mathbb{M}_1$, $\mathbb{M} \cap \mathbb{M}_2$ is infinite; say $\mathbb{M} \cap \mathbb{M}_1$. Then $\mathbf{A}|_{\mathbb{M} \cap \mathbb{M}_1}$ is a zero subsequence of $\mathbf{A}|_{\mathbb{M}_1} \in \mathcal{A}|_{\mathbb{M}_1}$. Since $\mathcal{A}|_{\mathbb{M}_1}$ is fractal, $\mathbf{A}|_{\mathbb{M}_1}$ is a zero sequence. But then $\mathbf{A}|_{\mathbb{M}_1 \cap \mathbb{M}_2}$ is a zero subsequence of $\mathbf{A}|_{\mathbb{M}_2} \in \mathcal{A}|_{\mathbb{M}_2}$. Since $\mathcal{A}|_{\mathbb{M}_2}$ is fractal, we conclude that $\mathbf{A}|_{\mathbb{M}_2}$ is a zero sequence. Thus, $\mathbf{A}|_{\mathbb{M}_1 \cup \mathbb{M}_2}$ is a zero sequence. ∎

We write $\mathbb{M}_1 \sim \mathbb{M}_2$ if $\mathbb{M}_1, \mathbb{M}_2 \in \mathrm{fr}\,\mathcal{A}$ are equivalent, denote the set of all equivalence classes of the relation $\sim$ by $(\mathrm{fr}\,\mathcal{A})^\sim$, and call $(\mathrm{fr}\,\mathcal{A})^\sim$ the *fractal variety* of $\mathcal{A}$. If $\mathcal{A}$ is fractal, then $(\mathrm{fr}\,\mathcal{A})^\sim$ is a singleton, consisting of the equivalence class of $\mathbb{N}$.

Our goal is to define a topology on $(\mathrm{fr}\,\mathcal{A})^\sim$ which makes $(\mathrm{fr}\,\mathcal{A})^\sim$ to a compact Hausdorff space. For $\mathcal{A}$ as above, let $\mathcal{L}(\mathcal{A})$ denote the smallest closed complex subalgebra of $l^\infty := l^\infty(\mathbb{N})$ which contains all sequences $(\|A_n\|)$ where (A_n) is a sequence in $\mathcal{A}$. Clearly, $\mathcal{L}(\mathcal{A})$ is a commutative C^*-algebra, and $\mathcal{L}(\mathcal{A})$ is unital if $\mathcal{A}$ is unital.

For a C^*-subalgebra $\mathcal{L}$ of l^∞, we let $\mathrm{cr}\,\mathcal{L}$ stand for the set of all infinite subsets $\mathbb{M}$ of $\mathbb{N}$ such that all sequences in the restriction $\mathcal{L}|_\mathbb{M}$ converge. The algebra $\mathcal{L}$ is called *quasiconvergent* if every infinite subset of $\mathbb{N}$ has an infinite subset in $\mathrm{cr}\,\mathcal{L}$.

Proposition 18 *If $\mathcal{A}$ is a C^*-subalgebra of $\mathcal{F}$, then* $\mathrm{fr}\,\mathcal{A} = \mathrm{cr}\,\mathcal{L}(\mathcal{A})$.

Proof. If $\mathbb{M} \in \mathrm{fr}\,\mathcal{A}$, then the sequence $(\|A_n\|)_{n\in\mathbb{M}}$ converges for every sequence $(A_n) \in \mathcal{A}$ by Fact (F3) in the introduction; hence, $\mathbb{M} \in \mathrm{cr}\,\mathcal{L}(\mathcal{A})$. Conversely, let $\mathbb{M} \in \mathrm{cr}\,\mathcal{L}(\mathcal{A})$, and let $(A_n)_{n\in\mathbb{M}}$ be a partial zero sequence in $\mathcal{A}|_\mathbb{M}$. Then the sequence (A_n) is in $\mathcal{L}(\mathcal{A})$; hence the sequence $(\|A_n\|)_{n\in\mathbb{M}}$ converges. The limit of this sequence is necessarily equal to 0; hence $(A_n)_{n\in\mathbb{M}}$ is a zero sequence, and $\mathcal{A}|_\mathbb{M}$ is fractal by definition. ∎

Corollary 19 *A C^*-subalgebra $\mathcal{A}$ of $\mathcal{F}$ is quasifractal if and only if the associated C^*-subalgebra $\mathcal{L}(\mathcal{A})$ of l^∞ is quasiconvergent.*

4.3 Quasiconvergent algebras

Let c and c_0 denote the algebras of the convergent sequences and of the zero sequences on $\mathbb{N}$, respectively. The restrictions of l^∞, c and c_0 to an infinite subset $\mathbb{M}$ of $\mathbb{N}$ can be identified with $l^\infty(\mathbb{M})$, $c(\mathbb{M})$ and $c_0(\mathbb{M})$.

Let $\mathcal{L}$ be a C^*-subalgebra of l^∞ and $\mathbb{M}$ be an infinite subset $\mathbb{M}$ of $\mathbb{N}$. The restriction $\mathcal{L}|_\mathbb{M}$ is called *non-degenerated* if $\mathcal{L}|_\mathbb{M}$ is not contained in $c_0(\mathbb{M})$. The algebra $\mathcal{L}$ is called *non-degenerated* if no restriction if no restriction of $\mathcal{L}$ to an infinite subset of $\mathbb{N}$ is degenerated. Every unital algebra $\mathcal{L}$ is non-degenerated.

For every $\mathbb{M} \in \mathrm{cr}\,\mathcal{L}$, the mapping

$$\varphi_\mathbb{M} : \mathcal{L} \to \mathbb{C}, \quad a \mapsto \lim(a|_\mathbb{M}) \tag{12}$$

is a continuous linear functional on $\mathcal{L}$ which is a character if $\mathbb{M}$ is non-degenerated. Since $\mathcal{L} \cap c_0$ is in the kernel of the mapping (12), the quotient mapping

$$\varphi_\mathbb{M} : \mathcal{L}/(\mathcal{L} \cap c_0) \to \mathbb{C}, \quad a + (\mathcal{L} \cap c_0) \mapsto \lim(a|_\mathbb{M}) \tag{13}$$

is well defined. This mapping is a character of $\mathcal{L}/(\mathcal{L}\cap c_0)$ if $\mathbb{M}$ is non-degenerated.

Proposition 20 *Let $\mathcal{L}$ be a unital and quasiconvergent C^*-subalgebra of l^∞. Then the set $\{\varphi_\mathbb{M} : \mathbb{M} \in \mathrm{cr}\,\mathcal{L}\}$ is strictly spectral for $\mathcal{L}/(\mathcal{L} \cap c_0)$, i.e., if $b \in \mathcal{L}/(\mathcal{L} \cap c_0)$ and $\varphi_\mathbb{M}(b)$ is invertible for all $\mathbb{M} \in \mathrm{cr}\,\mathcal{L}$, then b is invertible.*

Proof. Suppose that $a + (\mathcal{L} \cap c_0)$ is not invertible in $\mathcal{L}/(\mathcal{L} \cap c_0)$. Then $a + c_0$ is not invertible in $\mathcal{L}/c_0$; hence, a is a partial zero sequence. Let $\mathbb{M}'$ be an infinite subset of $\mathbb{N}$ such that $a|_{\mathbb{M}'} \to 0$. Since $\mathcal{L}$ is quasiconvergent, there is an infinite subset $\mathbb{M}$ of $\mathbb{M}'$ which belongs to $\operatorname{cr} \mathcal{L}$. The character associated with $\mathbb{M}$ satisfies $\varphi_{\mathbb{M}}(a) = 0$. Conversely, if $a \in \mathcal{L}$ and $\varphi_{\mathbb{M}}(a) \neq 0$ for all $\mathbb{M} \in \operatorname{cr} \mathcal{L}$, then $a + (\mathcal{L} \cap c_0)$ is invertible in $\mathcal{L}/(\mathcal{L} \cap c_0)$. This is the strict spectral property. ■

To conclude that $\{\varphi_{\mathbb{M}} : \mathbb{M} \in \operatorname{cr} \mathcal{L}\}$ is all of the maximal ideal space $\operatorname{Max}(\mathcal{L}/(\mathcal{L} \cap c_0))$ we need a further property of $\mathcal{L}$: separability.

Proposition 21 *Let $\mathcal{L}$ be a unital, separable and quasiconvergent C^*-subalgebra of l^∞. Then $\{\varphi_{\mathbb{M}} : \mathbb{M} \in \operatorname{cr} \mathcal{L}\} = \operatorname{Max}(\mathcal{L}/(\mathcal{L} \cap c_0))$.*

Proof. The assertion is a consequence of an observation by Nistor and Prudhon: since $\mathcal{L}/(\mathcal{L} \cap c_0)$ is separable, every strictly spectral family for $\mathcal{L}/(\mathcal{L} \cap c_0)$ is exhaustive (see [5] for the terminology and a proof). A short direct proof of the proposition goes as follows. Let φ be a character of $\mathcal{L}/(\mathcal{L} \cap c_0)$. We extend φ to a character on $\mathcal{L}$ by $\varphi : a \mapsto \varphi(a + (\mathcal{L} \cap c_0))$. Since $\mathcal{L}$ is separable, the kernel of φ is separable. Let $(j_n)_{n \in \mathbb{N}}$ be a sequence which is dense in $\ker \varphi$. Then the element

$$j := \sum_{j=1}^{\infty} \frac{1}{2^n} \frac{j_n^* j_n}{\|j_n\|^2}$$

belongs to $\ker \varphi$, implying that $j + c_0$ is not invertible in $\mathcal{L}/c_0$. By Proposition 20, there is a set $\mathbb{M} \in \operatorname{cr} \mathcal{L}$ such that $\varphi_{\mathbb{M}}(j) = 0$. Since characters are positive, we conclude that $\varphi_{\mathbb{M}}(j_n^* j_n) = 0$, hence $\varphi_{\mathbb{M}}(j_n) = 0$ for all $n \in \mathbb{N}$. The continuity of $\varphi_{\mathbb{M}}$ and the density of (j_n) in $\ker \varphi$ imply that $\varphi_{\mathbb{M}}$ vanishes on $\ker \varphi$. Thus, the characters φ and $\varphi_{\mathbb{M}}$ coincide. ■

The make the equality established in the previous proposition to a bijection between (cosets of) $\operatorname{cr} \mathcal{L}$ and $\operatorname{Max}(\mathcal{L}/(\mathcal{L} \cap c_0))$, we need to understand which sets $\mathbb{M} \in \operatorname{cr} \mathcal{L}$ generate the same character $\varphi_{\mathbb{M}}$. Proceeding similarly as in the previous section, we call $\mathbb{M}_1, \mathbb{M}_2 \in \operatorname{cr} \mathcal{L}$ *equivalent* if $\mathbb{M}_1 \cup \mathbb{M}_2 \in \operatorname{cr} \mathcal{L}$. The so-defined relation $\sim$ is an equivalence relation, and $\mathbb{M}_1 \sim \mathbb{M}_2$ if and only if $\varphi_{\mathbb{M}_1} = \varphi_{\mathbb{M}_2}$. We denote the equivalence class of $\mathbb{M} \in \operatorname{cr} \mathcal{L}$ by $\mathbb{M}^\sim$ and write $(\operatorname{cr} \mathcal{L})^\sim$ for the set of all equivalence classes. Then, by construction, the mapping

$$(\operatorname{cr} \mathcal{L})^\sim \to \{\varphi_{\mathbb{M}} : \mathbb{M} \in \operatorname{cr} \mathcal{L}\}, \quad \mathbb{M}^\sim \mapsto \varphi_{\mathbb{M}}$$

is a (well defined) bijection. Combining this observation with the result of Proposition 21 we obtain:

Corollary 22 *Let $\mathcal{L}$ be a unital, separable and quasiconvergent C^*-subalgebra of l^∞. Then $\mathbb{M}^\sim \mapsto \varphi_{\mathbb{M}}$ is a bijection from $(\operatorname{cr} \mathcal{L})^\sim$ onto $\operatorname{Max}(\mathcal{L}/(\mathcal{L} \cap c_0))$.*

4.4 Quasifractal algebras as continuous fields

Recall from Proposition 18 that $\mathrm{fr}\,\mathcal{A} = \mathrm{cr}\,\mathcal{L}(\mathcal{A})$ for every C^*-subalgebra $\mathcal{A}$ of $\mathcal{F}$. If $\mathcal{A}$ is quasifractal, then $\mathcal{L}(\mathcal{A})$ is quasiconvergent by Corollary 19, and the relations $\sim$ on $\mathrm{fr}\,\mathcal{A}$ and $\mathrm{cr}\,\mathcal{L}(\mathcal{A})$ are compatible in the sense that $(\mathrm{fr}\,\mathcal{A})^\sim = (\mathrm{cr}\,\mathcal{L}(\mathcal{A}))^\sim$. Thus, if $\mathcal{A}$ is unital and quasifractal and $\mathcal{L}(\mathcal{A})$ is separable, then there is a (well defined) bijection

$$(\mathrm{fr}\,\mathcal{A})^\sim \to \mathrm{Max}\,(\mathcal{L}(\mathcal{A})/(\mathcal{L}(\mathcal{A}) \cap c_0)), \quad \mathbb{M}^\sim \mapsto \varphi_\mathbb{M}. \tag{14}$$

We employ this bijection to transfer the Gelfand topology of $\mathrm{Max}\,(\mathcal{L}(\mathcal{A})/(\mathcal{L}(\mathcal{A}) \cap c_0))$ onto $(\mathrm{fr}\,\mathcal{A})^\sim$, thus making the latter to a compact Hausdorff space.

We claim that the algebra $\mathcal{A}/(\mathcal{A} \cap \mathcal{G})$ is *-isomorphic to a continuous field of C^*-algebras over the base space $(\mathrm{fr}\,\mathcal{A})^\sim$ in the following sense[1].

Definition 23 *Let X be a compact Hausdorff space and let $\mathcal{B}$ be the direct product of a family $\{\mathcal{B}_x\}_{x\in X}$ of C^*-algebras, labeled by X. A* continuous field of C^*-algebras over X *is a C^*-subalgebra $\mathcal{C}$ of $\mathcal{B}$ with the following properties:*

(a) $\mathcal{C}$ is maximal, i.e., $\mathcal{B}_x = \{c(x) : c \in \mathcal{C}\}$ for every $x \in X$,

(b) the function $X \to \mathbb{C}$, $x \mapsto \|c(x)\|$ is continuous for every $c \in \mathcal{C}$.

The algebras $\mathcal{B}_x$ are called the fibers of $\mathcal{A}$, and X is the base space.

Set $X = (\mathrm{fr}\,\mathcal{A})^\sim$, for $\mathbb{M} \in \mathrm{fr}\,\mathcal{A}$ define $\mathcal{B}_\mathbb{M}$ as $\mathcal{A}|_\mathbb{M}/(\mathcal{A}|_\mathbb{M} \cap \mathcal{G}|_\mathbb{M})$ (note that these algebras depend on the equivalence class $\mathbb{M}^\sim$ of $\mathbb{M}$ only), and let $\mathcal{B}$ be the direct product of the family $\{\mathcal{B}_\mathbb{M}\}_{\mathbb{M}\in\mathrm{fr}\,\mathcal{A}}$. Every sequence $\mathbf{A} \in \mathcal{A}$ determines a function in $\mathcal{B}$ via

$$\mathbb{M} \mapsto \mathbf{A}|_\mathbb{M} + (\mathcal{A}|_\mathbb{M} \cap \mathcal{G}|_\mathbb{M}) \in \mathcal{A}|_\mathbb{M}/(\mathcal{A}|_\mathbb{M} \cap \mathcal{G}|_\mathbb{M}). \tag{15}$$

Let $\mathcal{C}$ be the set of all functions (15) with $\mathbf{A} \in \mathcal{A}$.

Theorem 24 *Let $\mathcal{A}$ be a unital and quasifractal C^*-subalgebra of $\mathcal{F}$ for which $\mathcal{L}(\mathcal{A})$ is separable. Then*

(a) $\mathcal{C}$ is a continuous field of C^*-algebras over $(\mathrm{fr}\,\mathcal{A})^\sim$,

(b) the mapping which sends $\mathbf{A} + (\mathcal{A} \cap \mathcal{G})$ to the function (15) is a *-isomorphism from $\mathcal{A}/(\mathcal{A} \cap \mathcal{G})$ onto $\mathcal{C}$.

Proof. (a) Evidently, $\mathcal{C}$ is maximal. Let $\mathbf{A} = (A_n) \in \mathcal{A}$. Then

$$\|\mathbf{A}|_\mathbb{M} + (\mathcal{A}|_\mathbb{M} \cap \mathcal{G}|_\mathbb{M})\| = \lim_{n\in\mathbb{M}} \|A_n\| = \varphi_\mathbb{M}(\mathbf{a} + (\mathcal{L}(\mathcal{A}) \cap c_0)),$$

where $\mathbf{a} := (\|A_n\|) \in \mathcal{L}$. Since $\mathcal{M} \mapsto \varphi_\mathbb{M}(\mathbf{a} + (\mathcal{L}(\mathcal{A}) \cap c_0))$ is a continuous function, it follows that condition (b) of Definition 23 is also satisfied.

[1]Note that one usually adds a third condition to the definition of a continuous field, namely that $\mathcal{C}$ is a $C(X)$-algebra.

(b) It is evident that this mapping is a surjective *-homomorphism. If $\mathbf{A} \in \mathcal{A}$ and $\mathbf{A}|_{\mathbb{M}} \in \mathcal{G}|_{\mathbb{M}}$ for every $\mathbb{M} \in \operatorname{fr} \mathcal{A}$, then $\mathbf{A} \in \mathcal{G}$ by Proposition 9. Thus, the mapping in assertion (b) is also injective. ■

To state our last result, we need some more notation. For $\mathcal{F}$ as in the setting of the FSD for Toeplitz operators, let $\mathcal{K}$ denote the smallest closed ideal of $\mathcal{F}$ which contains all sequences (K_n) with $\sup \operatorname{rank} K_n < \infty$. The sequences in $\mathcal{K}$ are called *compact*. Further, a C^*-algebra is called *elementary* if it is *-isomorphic to an algebra $K(H)$, the compact operators on a certain Hilbert space H, and a C^*-algebra is called *dual* if it is *-isomorphic to a direct sum of elementary algebras. See [1] for more on dual algebras.

For example, the compact sequences in $\mathcal{S}(\mathsf{T}(C))$ are just the sequences

$$(A_n) = (P_n K P_n + R_n L R_n + G_n)$$

where K and L are compact operators on $l^2(\mathbb{Z}^+)$ and $(G_n) \in \mathcal{G}$, and the algebra $(\mathcal{S}(\mathsf{T}(C)) \cap \mathcal{K})/\mathcal{G}$ is isomorphic to the algebra of all pairs (K, L), hence, to the direct sum of two copies of $K(l^2(\mathbb{Z}^+))$.

A basic observation in [7] states that whenever $\mathcal{A}$ is fractal, then $(\mathcal{A} \cap \mathcal{K})/\mathcal{G}$ is a dual algebra. Combining this observation with Theorem 24 we obtain

Corollary 25 *Let $\mathcal{A}$ be a unital and quasifractal C^*-subalgebra of $\mathcal{F}$ for which $\mathcal{L}(\mathcal{A})$ is separable. Then $\mathcal{A}/(\mathcal{A} \cap \mathcal{K})$ is *-isomorphic to a continuous field of dual algebras over* $(\operatorname{fr} \mathcal{A})^\sim$.

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