ASYMPTOTIC BEHAVIOR FOR THE QUASI-GEOSTROPHIC EQUATIONS WITH FRACTIONAL DISSIPATION IN $\mathbb{R}^2$

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Abstract. In this paper the Surface Quasi-Geostrophic Equations (QGE) with fractional dissipation in $\mathbb{R}^2$ are considered. Our aim is to study the long-time behavior of QGE in the subcritical case. To this end we investigate the global well-posedness and global attractor for QGE in $H^s(\mathbb{R}^2)$ via commutator estimates for nonlinear terms, a new iterative technique for estimates of higer order derivatives and with the help of a nonlocal damping term. Besides, by using the fractional Lieb-Thirring inequality, estimates of the finite Hausdorff and fractal dimensions of the global attractor are found.

Contents

1. Introduction

We consider the asymptotic behavior of solutions of the following 2D (surface) quasi-geostrophic equations (QGE) with fractional dissipation in $\mathbb{R}^2$:

$$\begin{cases} \theta_t + u \cdot \nabla\theta + \kappa(-\Delta)^\alpha\theta = F(x,\theta) \\ \theta(x,0) = \theta^0 \end{cases} \tag{1.1}$$

Key words. Global attractor; Fractional dissipation; Quasi-geostrophic equations; Nonlocal damping; Unbounded domain; Finite fractal dimension.

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where $F(x,\theta)$ is a given function and $\kappa>0$ is the viscosity. The nonlocal operator $(-\Delta)^\alpha$, $1/2<\alpha\le 1$, is defined through the Fourier transform

$$\widehat{(-\Delta)^\alpha} g(\xi)=|\xi|^{2\alpha}\widehat{g}(\xi), \tag{1.2}$$

where $\widehat{g}$ is the Fourier transform of g (see Sect. 2 below). For notational convenience, we write $\Lambda=(-\Delta)^{1/2}$. In (1.1) $\theta=\theta(x,t)$ represents the potential temperature. The velocity $u=(u_1,u_2)$ is incompressible and determined from θ by a stream function ψ via the relations

$$u=(u_1,u_2)=\big(-\frac{\partial\psi}{\partial x_2},\frac{\partial\psi}{\partial x_1}\big) \qquad \text{and} \qquad (-\Delta)^{\frac{1}{2}}\psi=-\theta. \tag{1.3}$$

The equality relating u to θ in (1.3) can be rewritten in terms of Riesz transforms

$$u=(u_1,u_2)=(\partial_{x_2}\Lambda^{-1}\theta,-\partial_{x_1}\Lambda^{-1}\theta)=\mathcal{R}^\perp\theta, \tag{1.4}$$

where $\mathcal{R}^\perp=(-\mathcal{R}_2,\mathcal{R}_1)$ and $\mathcal{R}_j, j=1,2$ denote the Riesz transforms defined by

$$\widehat{\mathcal{R}_j f(\xi)}=-i\frac{\xi_j}{|\xi|}\widehat{f}(\xi).$$

There is an extensive literature on the 2D quasi-geostrophic equations (1.1), (1.3), since the system is regarded as an important model in geophysical fluid dynamics, especially for atmospheric and oceanic fluid flow in case of small Rossby and Ekman numbers (see [36]). Actually, the global well-posedness and the large time behavior for the system (1.1), (1.3) without external force (*i.e.* $f=0$) and damping have been intensively investigated due to both their mathematical importance and their potential for applications in meteorology and oceanography (see [7, 12, 14, 16, 17, 18, 26, 46, 47] and references therein). For example, Berselli considered the long time behavior of solutions of the 2D QGE with periodic boundary condition in [6]. Existence and uniqueness of solutions were proved by Constantin-Wu [15] and Wu [45]. After that, by developing a generalized maximum principle, Ju proved the existence of the H^s–global attractor for the 2D QGE with periodic boundary condition in [25].

The interesting problem involving the long-time behavior a of dynamical system can be described naturally in terms of attractors of the corresponding semigroup (see [2, 24, 41] and references therein). In bounded domains, people are interested in finding the existence of the attractor for a large class of equations such as reaction-diffusion equations, nonlinear wave equations, two-dimensional QGE and Navier-Stokes systems, etc. Besides, under some natural assumptions it has been proven that for all equations mentioned above, the attractor has a finite Hausdorff and fractal dimension (see [2, 24, 41, 43]).

In recent years, more and more articles refer to the case of unbounded spatial domains or the whole space. It is known that the behavior of solutions for the above equations becomes much more complicated, mainly because compact embeddings and the Poincaré inequality do not hold; actually, these tools are very important to assure the existence of bounded absorbing sets when one focuses on bounded domains with smooth boundary or a spatial domain with periodic boundary condition. The above mentioned evolution equations of mathematical physics on unbounded domains are usually treated under some additional assumptions on the external force such that the equations have the property of damping (see [39, 42]). We would like to mention that Abergel [1], Marín-Rubio, Real [34] and Rosa [38] studied the 2D Navier-Stokes equations on a strip in $\mathbb{R}^2$ and arbitrary domains of $\mathbb{R}^2$ satisfying the Poincaré inequality, respectively. Efendiev-Zelik [20] considered nonlinear reaction-diffusion systems (with damping) in unbounded domains and obtained the attractors for the system in weighted Sobolev spaces. The global attractor for the 2D QGE in $\mathbb{R}^2$ with damping has proved by Wang-Tang [44] in the framework of L^p. However, the estimate of Hausdorff and fractal dimensions of the global attractor in L^p is unsolved. We mention that in [20] as well as in [44] the system contains the globally acting damping term λu (with $\lambda>0$) on the left-hand side of the equation; this term is implicitly contained in the bounded domain case due to the Poincaré inequality.

The main purpose of this paper is to study the existence of the H^s–global attractor for the QGE in $\mathbb{R}^2$ and find estimates of Hausdorff and fractal dimensions of the attractor. For simplicity, we consider the external force $F(x,\upsilon)$ in the following form:

$$F(x,\upsilon) = g_1(x)f_1(\upsilon) + f_2(\upsilon) + g_2(x), \tag{1.5}$$

where $f_1(\theta), g_1(x)$ and $g_2(x)$ are given functions, and $f_2(\upsilon)$ is a nonlocally acting functional given by

$$f_2(\upsilon) = \rho * \upsilon \text{ with } \rho = \rho(x) \in L^1(\mathbb{R}^2); \tag{1.6}$$

here the $*$–product denotes convolution on $\mathbb{R}^2$. Assuming that $\hat{\rho}(\xi)$ is negative and converges to 0 as $|\xi| \to \infty$ (or even has compact support in $\mathbb{R}^2$) the term f_2 yields in Fourier space the damping term $-\hat{\rho}\hat{\upsilon}$ instead of $\lambda\hat{\upsilon}$ on the left-hand side of the equation. Since $\hat{\rho}(\xi) \to 0$ as $|\xi| \to \infty$, the term f_2 defines a much weaker damping than the classical term $\lambda\upsilon$ as in [44]. Concerning $g_1(x)f_1(\upsilon)$ we assume the dissipativity condition $g_1(x)f_1(\upsilon)\upsilon \le 0$ on $\mathbb{R}^2 \times \mathbb{R}$.

There are new challenges to consider the asymptotic behavior for (1.1), (1.3) and (1.5), (1.6), particularly, concerning the existence of global attractors and the finite dimensionality of the attractor. The most difficult issue is the absence of standard compact Sobolev embeddings as mentioned before. To overcome this difficulty we use a suitable cut-off function to decompose the whole space $\mathbb{R}^2$ into a bounded ball and its complement; then the asymptotic compactness of the semigroup follows from the compact Sobolev embedding on the ball and estimates in its complement. However, when proving the same kind of estimates both in H^s and L^p, the dissipative term $(-\Delta)^\alpha$, $1/2 < \alpha \le 1$, and the nonlinear term $u \cdot \nabla\theta$ give much more trouble than for reaction-diffusion systems. Moreover, the existence of an absorbing set in $H^s(\mathbb{R}^2)$ cannot be obtained immediately by the uniform Gronwall lemma, since we study the global attractor in a higher order Sobolev space $H^s(\mathbb{R}^2)$, $s > 2(\alpha-1)$, whereas the dissipative term $(-\Delta)^\alpha$, $1/2 < \alpha \le 1$, only supplies H^α regularity. It is necessary to obtain the absorbing set by an iterative technique (boot-strapping argument) and commutator estimates involving the term $u \cdot \nabla\theta$.

Besides, it is notable that the absence of a damping term $\lambda\theta$ and the explicit dependence of the external force $f_1 = f_1(\theta)$ on the temperature lead to new difficulties, especially, to get estimates of $\|\Lambda^s f_1(\theta)\|_{L^2}$. In this situation, for instance, we are faced with the problem of proving existence and uniqueness of solutions and uniform bounded estimates of them, see Sect. 4 and the Appendix (Sect. 8) for detailed proofs. For the analysis of $f_1(\theta)$ we use Littlewood-Paley theory to get new estimates for $\|\Lambda^s f_1(\theta)\|_{L^2}$, see Sect. 3. Finally, we also would like to estimate the Hausdorff and fractal dimensions of the global attractor of the fractional dissipative QGE. Thanks to the fractional Lieb-Thirring inequality obtained in [32], our idea is feasible if we can prove that the semigroup is uniformly differentiable on the attractor by controlling the nonlinear term.

This article is organized as follows. In Sect. 2, we present some notation and recall the theory of global attractors for infinite dimensional dissipative dynamical systems and several preliminary results which will be used frequently. In Theorems 2.6 and 2.7 we state the global well-posedness of the 2D QGE and the main result of this paper, respectively. To get the main result, we introduce some Littlewood-Paley theory and prove a crucial estimate for the external force $f_1(\theta)$ which is necessary to get global existence and uniform estimates for solutions of (1.1), (1.3) in Sect. 3. In Sect. 4 and Sect. 5, we present *a priori* estimates which will yield existence of absorbing sets in $H^s(\mathbb{R}^2)$ and prove smallness of the $H^s(\mathbb{R}^2)$-norm on the complement of a bounded ball. In Sect. 6, we first prove the asymptotic compactness of the solution semigroup and then, by combining results from previous sections, deduce the existence of the global attractor. We conclude the proof of Theorem 2.7 by proving the finite dimensionality of the attractor in Sect. 7. Last but not least, global existence and uniqueness of solutions for (1.1), (1.3) are proved in Sect. 8.

2. Preliminaries and Main Results

2.1. **Notation and function spaces.** We first recall some notation and basic results from harmonic analysis. For any $p \in [1,\infty]$, $L^p(\mathbb{R}^d) = L^p$ denotes the space of the pth-power integrable functions on $\mathbb{R}^d$ and $\|\cdot\|_{L^p}$ denotes the norm of L^p. For the duality product (or scalar product) of L^p with $L^{p'}$, $p' = \frac{p}{p-1}$, we simply write $(\cdot,\cdot)$. Let $\mathcal{S}(\mathbb{R}^d) = \mathcal{S}$ be the Schwartz class of rapidly decreasing smooth functions. For $f \in \mathcal{S}(\mathbb{R}^d)$, the Fourier transform of f is defined by

$$\mathcal{F}f(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-i\xi\cdot x}\,\mathrm{d}x; \tag{2.1}$$

the inverse Fourier transform is denoted by $\mathcal{F}^{-1}$. Given $s \in \mathbb{R}$ and $1 \le p \le \infty$, the inhomogeneous Bessel potential space $H^{s,p}(\mathbb{R}^d)$ is the set of all $u \in \mathcal{S}'(\mathbb{R}^d) = \mathcal{S}'$, the set of tempered distributions, such that

$$\|u\|_{H^{s,p}} := \|\mathcal{F}^{-1}[(1+|\xi|^2)^{s/2}\widehat{u}]\|_{L^p} < \infty.$$

$H^{s,p}$ is a Banach space for $s \in \mathbb{R}$, $1 \le p \le \infty$, and even reflexive for $1 < p < \infty$. For $s \in \mathbb{R}$, the homogeneous Sobolev space $\dot{H}^{s,p}(\mathbb{R}^d)$ is defined as the space of all tempered distributions modulo polynomials u in $\mathcal{S}'(\mathbb{R}^d)/\mathcal{P}$ such that

$$\|u\|_{\dot{H}^{s,p}} := \|\mathcal{F}^{-1}(|\xi|^s\widehat{u})\|_{L^p} < \infty;$$

here $\mathcal{P}$ denotes the set of polynomials on $\mathbb{R}^d$. In what follows, we write $H^{s,p} = H^{s,p}(\mathbb{R}^d), \dot{H}^{s,p} = \dot{H}^{s,p}(\mathbb{R}^d)$ and $H^s = H^{s,2}, \dot{H}^s = \dot{H}^{s,2}$ for brevity.

2.2. **Preliminary results.** We first recall the basic concepts and results about global attractors and asymptotic compactness of a semigroup. See [24, 28, 37, 41] for some basic properties.

Definition 2.1. Let M be a complete metric space. A one-parameter family $\{S(t)\}_{t\ge 0}$ of maps $S(t) : M \to M, t \ge 0$, is called a C^0- or *continuous semigroup* if it satisfies:

(1) $S(0)$ is the identity map on M,

(2) $S(t+s) = S(t)S(s)$ for all $t, s \ge 0$,

(3) for each $x \in M$ the function $S(t)x$ is continuous in $t \ge 0$.

Let $\{S(t)\}_{t\ge 0}$ be a C^0-semigroup in a complete metric space M. A subset B_0 of M is called an *absorbing set* in M if, for any bounded subset B of M, there exists some $t_1 \ge 0$ such that $S(t)B \subset B_0$, for all $t \ge t_1$. A subset $\mathcal{A}$ of M is called a *global attractor* for the semigroup if $\mathcal{A}$ enjoys the following properties:

(1) $\mathcal{A}$ is an invariant compact set, *i.e.*, $S(t)\mathcal{A} = \mathcal{A}$ for any $t \ge 0$,

(2) $\mathcal{A}$ attracts all bounded sets of M. In other words, for any bounded subset B of M,

$$d(S(t)B, \mathcal{A}) \to 0 \text{ as } t \to \infty,$$

where $d(B, A) = \sup_{x\in B}\inf_{y\in A} d(x,y)$ is the semidistance of two sets B and A.

A C^0-semigroup $\{S(t)\}_{t\ge 0}$ is said to be *asymptotically compact* in M if

$$\{S(t_n)u_n\}_{n=1}^{\infty} \text{ has a convergent subsequence in } M$$

for any bounded sequence $\{u_n\}_{n=1}^{\infty}$ in M and any sequence $\{t_n\}_{n=1}^{\infty}$ in $(0,\infty)$ such that $t_n \to \infty$.

Proposition 2.2. *Assume that M is a complete metric space and let $\{S(t)\}_{t\ge 0}$ be a C^0-semigroup on M. If $\{S(t)\}_{t\ge 0}$ has a bounded absorbing set and is asymptotically compact in M, then $\{S(t)\}_{t\ge 0}$ possesses a global attractor.*

Next, we review the Uniform Gronwall Lemma and some commutator and product estimates, which are used frequently in this article.

Lemma 2.3 (Uniform Gronwall Lemma)**.** *Let g, h and y be non-negative locally integrable functions on $]t_0, +\infty[$ such that*

$$\frac{dy(t)}{dt} \leq g(t)y(t) + h(t), \qquad \forall t \ \geq t_0,$$

and

$$\int_t^{t+r} g(s)\,\mathrm{d}s \leq a_1, \qquad \int_t^{t+r} h(s)\,\mathrm{d}s \leq a_2, \qquad \int_t^{t+r} y(s)\,\mathrm{d}s \leq a_3, \qquad \forall t \ \geq t_0,$$

where $r > 0$ and a_1, a_2, a_3 are non-negative constants. Then

$$y(t+r) \leq \left(\frac{a_3}{r} + a_2\right) e^{a_1}, \quad \forall t \ \geq t_0.$$

Lemma 2.4 (Commutator and product estimates, [27])**.** *Suppose that $s > 0$ and $p \in]1, +\infty[$. If $f, g \in \mathcal{S}(\mathbb{R}^d)$, then*

$$\|\Lambda^s(fg) - f\Lambda^s g\|_{L^p} \leq C(\|\nabla f\|_{L^{p_1}} \|g\|_{\dot{H}^{s-1,p_2}} + \|f\|_{\dot{H}^{s,p_3}} \|g\|_{L^{p_4}})$$

and

$$\|\Lambda^s(fg)\|_{L^p} \leq C(\|f\|_{L^{p_1}} \|g\|_{\dot{H}^{s,p_2}} + \|f\|_{\dot{H}^{s,p_3}} \|g\|_{L^{p_4}}) \tag{2.2}$$

with $p_1, p_2, p_3, p_4 \in]1, +\infty[$ such that

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.$$

We note that (2.2) even holds when $p = 1$ and $p_1, p_2, p_3, p_4 \in]1, +\infty]$, see [23]. The limit case $p = \infty$ in (2.2) is found in [8].

Lemma 2.5 (Positivity Lemma, [17, 18, 25])**.** *Let $0 \leq \alpha \leq 1$ and $\theta \in L^q(\mathbb{R}^2), \Lambda^{2\alpha}\theta \in L^q(\mathbb{R}^2)$ where $q \geq 2$. Then*

$$\int_{\mathbb{R}^2} |\theta|^{q-2}\theta\Lambda^{2\alpha}\theta\,\mathrm{d}x \geq \frac{2}{q}\int_{\mathbb{R}^2} \left(\Lambda^\alpha|\theta|^{q/2}\right)^2 \mathrm{d}x \geq 0.$$

Finally, we note that by (1.4) and the theory of singular integrals we have that for any $p \in]1, \infty[$ there exists a constant $c(p)$ such that (see [40])

$$\|u\|_{L^p} \leq c(p)\|\theta\|_{L^p}. \tag{2.3}$$

The L^2–inner product on $\mathbb{R}^2$, $\int_{\mathbb{R}^2} f\,g\,\mathrm{d}x$, will also be denoted by (f, g).

2.3. **Main results.** We now state the result about the existence of global strong solutions of 2D quasi-geostrophic equations (1.1).

Theorem 2.6. *Let $\alpha \in]\frac{1}{2}, 1]$, $\kappa > 0$ and $\theta^0 \in H^s(\mathbb{R}^2)$ with $s > 2(1-\alpha)$. Suppose further that*

$$\rho \in L^1(\mathbb{R}^2), \quad g_1 \in H^s(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2), \quad g_2 \in H^{s-\alpha}(\mathbb{R}^2) \cap L^2(\mathbb{R}^2) \cap L^{q_0}(\mathbb{R}^2)$$

for some $q_0 > \frac{2}{2\alpha-1}$, $f_1 \in \mathcal{C}^\infty(\mathbb{R})$ satisfying

$$f_1(0) = f_1'(0) = 0, \quad |f_1'(x)| \leq K_1, |f_1''(x)| \leq K_2 \ \text{ for any } x \in \mathbb{R} \text{ and for some } \ K_1, K_2 \geq 0.$$

Then for any $T > 0$, there is unique solution θ of (1.1), (1.3) and (1.5), (1.6) such that

$$\theta \in \mathcal{C}([0, T); H^s(\mathbb{R}^2)) \cap L^2(0, T; H^{s+\alpha}(\mathbb{R}^2)).$$

By Theorem 2.6 the proof of which is given in the Appendix (Sect. 8) we define the nonlinear operator semigroup $\{S(t)\}_{t\geq 0}$ in H^s as

$$S(t) : H^s(\mathbb{R}^2) \longrightarrow H^s(\mathbb{R}^2), \quad \theta^0 \mapsto S(t)\theta^0 = \theta(t),$$

which is generated by the solution of (1.1) with initial data $\theta^0 \in H^s(\mathbb{R}^2)$. Our main results on the attractor and its dimensions read as follows:

Theorem 2.7. *Let the conditions in Theorem 2.6 hold. Assume further that*

(1) $g_1(x) \geq 0$ *for almost all* $x \in \mathbb{R}^2$ *and* $f_1(\upsilon)\upsilon \leq 0$,

(2) $\hat{\rho}(\xi) \leq 0, \ \hat{\rho}(0) < 0$.

Then the semigroup $S(t)_{t\geq 0}$ *generated by the system* (1.1), (1.3) *and* (1.5), (1.6) *has a global attractor* $\mathcal{A}$ *in* $H^s(\mathbb{R}^2)$ *which attracts any bounded subset in* $H^s(\mathbb{R}^2)$.

Moreover, if either $K_1\|g_1\|_{L^\infty}$ *is sufficiently small or* $f_1' \leq 0$, *then the global attractor* $\mathcal{A}$ *has finite Hausdorff and fractal dimensions.*

Remark 2.8. (1) If $g_2 = 0$ in Theorem 2.6, then all solutions θ tend to 0 in $L^2(\mathbb{R}^2)$ as $t \to \infty$ so that $\mathcal{A} = \emptyset$, see the exponential estimate (4.9). To avoid this trivial case, we have in mind a function $g_2 \neq 0$. The crucial consequence of condition (1) in Theorem 2.7 is the fact that $g_1(x)f_1(\theta)\theta \leq 0$ everywhere. Of course, this condition can be reformulated as $g_1(x) \leq 0$ and $f_1(\upsilon)\upsilon \geq 0$. As can be seen from the proofs of Proposition 4.1 and Proposition 5.1 condition (2) on $\hat{\rho}$ can be weakened to the assumption that $\hat{\rho}(\xi) \leq \kappa|\xi|^{2\alpha} - \beta$ on $\mathbb{R}^2$ with any $\beta > 0$.

(2) To prove the finite dimensionality of the attractor $\mathcal{A}$ a condition on f_1' is needed, see the proof of (7.27) below. Actually, it suffices to guarantee that $\hat{\kappa}_0 - f_1'g_1$ is uniformly bounded from below by a positive constant where $\hat{\kappa}_0 > 0$ depends on κ and ρ.

(3) It is common to add a damping term when considering the long-time behavior of evolutionary equations in unbounded domains or the whole space, since the damping term can lead to solutions of the homogeneous equation of (1.1) with exponential decay; this property replaces the role of the Poincaré inequality as in the case of bounded domains (see [11, 20]). Condition (1.6) together with the asumption (2) in Theorem 2.7 plays the role of the damping term which guarantees some decay of the solution θ, but it is much weaker since we restrict the damping property of the external force to a non-localized term of convolution type which in Fourier space acts a multiplication by a decaying function, maybe even of compact support. Nevertheless, by Fourier analysis, we get a similar exponential decay property of the homogeneous solution. A typical example is given by ρ such that $\hat{\rho}(\xi) = e^{-|\xi|^2/\varepsilon^2}$ for small $\varepsilon > 0$; it leads to a damping (in Fourier space) concentrated in the ball of radius ε. However, our approach yields a new challenge to get uniformly bounded estimates of θ in $H^s(\mathbb{R}^2)$.

(4) Obviously, the result of Theorem 2.7 is also formally true without the condition (1.6) when we consider the fractional dissipative 2D QGE on some arbitrary open domain Ω such that the Poincaré inequality holds:

$$\|\phi\|_{L^2}^2 \leq \frac{1}{\lambda_1}\|\Lambda^\alpha \phi\|_{L^2}^2, \quad \forall \phi \in \dot{H}^\alpha(\Omega),$$

for some absolute constant $\lambda_1 > 0$.

(5) In [25] the author considered the asymptotic behavior for the quasi-geostrophic equations in the case of a spatial domain with periodic boundary condition. Here we generalize this result to the case of unbounded domains (the whole space) by proving the asymptotic compactness of the semigroup instead of its compactness. In [25, Sect. 6], the author remarked that one can obtain that the global attractor of the 2D QGE with $\alpha = 1$ has finite dimensions. As far as we know, there is no reference to estimates of Hausdorff and fractal dimensions for the 2D QGE on unbounded domains. In this paper we prove that the finiteness of the dimensions of the global attractor in H^s for all $\alpha \in (\frac{1}{2}, 1]$ can be achieved via the (fractional) Lieb-Thirring inequality, see Sect. 7.

3. Key Inequalities in Besov Spaces

To prove global existence and asymptotic behavior of solutions of QGE, we need estimates of $\Lambda^s f(\theta)$ of the external force $f(\theta)$ in terms of θ. For convenience, we introduce the Littlewood-Paley theory, see, *e.g.*, [3, 10]. Choose functions $\chi, \varphi \in \mathcal{S}(\mathbb{R}^d)$ supported in $\mathfrak{B} = \{\xi \in \mathbb{R}^d : |\xi| \leq 4/3\}$ and

$\mathfrak{C} = \{\xi \in \mathbb{R}^d : 3/4 \le |\xi| \le 8/3\}$, respectively, such that

$$\sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1 \quad \forall \xi \in \mathbb{R}^d\backslash\{0\}, \qquad \chi(\xi) + \sum_{j\ge 0} \varphi(2^{-j}\xi) = 1 \quad \forall \xi \in \mathbb{R}^d.$$

Let $\check{\varphi} = \mathcal{F}^{-1}\varphi$ and $\check{\chi} = \mathcal{F}^{-1}\chi$. The *nonhomogeneous dyadic blocks* Δ_j and the low-frequency cut-off operators S_j are defined by

$$\Delta_j u = 0 \ \text{ if } \ j \le -2, \quad \Delta_{-1} u(x) = \chi(D)u(x) = \int_{\mathbb{R}^d} \check{\chi}(y)u(x-y)\,\mathrm{d}y,$$
$$\Delta_j u(x) = \varphi(2^{-j}D)u(x) = 2^{jd}\int_{\mathbb{R}^d} \check{\varphi}(2^j y)u(x-y)\,\mathrm{d}y \quad \forall j \ge 0,$$

where $u \in \mathcal{S}'$. The *homogeneous dyadic blocks* $\dot{\Delta}_j$ are defined for all $j \in \mathbb{Z}$ by

$$\dot{\Delta}_j u(x) = \varphi(2^{-j}D)u(x) = 2^{jd}\int_{\mathbb{R}^d} \check{\varphi}(2^j y)u(x-y)\,\mathrm{d}y.$$

Given $s \in \mathbb{R}$ and $p, q \in [1,\infty]$, the *inhomogeneous Besov space* $B^s_{p,q}(\mathbb{R}^d)$ is the set of tempered distributions u such that

$$\|u\|_{B^s_{p,q}} := \big(2^{js}\|\Delta_j u\|_{L^p}\big)_{\ell^q(\mathbb{Z})} < \infty.$$

To define homogeneous Besov spaces, we first denote by $\mathcal{S}'_h$ the set of tempered distributions u such that the low frequency condition

$$\lim_{\lambda\to\infty} \|\phi(\lambda D)u\|_{L^\infty} = 0 \ \text{ for any } \phi \in \mathcal{D}(\mathbb{R}^d)$$

is satisfied, see [3, Definition 1.26]. Let $s \in \mathbb{R}$, $p, q \in [1,\infty]$. The *homogeneous Besov space* $\dot{B}^s_{p,q}(\mathbb{R}^d)$ is the set of tempered distributions $u \in \mathcal{S}'_h$ such that

$$\|u\|_{\dot{B}^s_{p,q}} := \big(2^{js}\|\dot{\Delta}_j u\|_{L^p}\big)_{\ell^q(\mathbb{Z})} < \infty.$$

We point out that $B^s_{p,q}(\mathbb{R}^d) = \dot{B}^s_{p,q}(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ and $\|u\|_{B^s_{p,q}} \approx \|u\|_{\dot{B}^s_{p,q}} + \|u\|_{L^p}$ when $s > 0$. Next, we recall the explicit definition of the fractional Laplacian $(-\Delta)^s$ when $s \in (0,1)$,

$$(-\Delta)^s u(x) = C(d,s)\, P.V. \int_{\mathbb{R}^d} \frac{u(x)-u(y)}{|x-y|^{d+s}}\,\mathrm{d}y, \tag{3.1}$$

where $u \in \mathcal{S}$, see [19, Sect. 3]; here $P.V.$ means the limit "in the principle value sense", and $C(d,s) > 0$ is a constant depending on d and s.

Proposition 3.1. *(1) Suppose that $f \in \mathcal{C}^1(\mathbb{R})$ satisfies $f(0) = 0$ and $\|f'\|_{L^\infty} \le K < \infty$. Then for any $u \in \dot{B}^s_{p,1}(\mathbb{R}^d), p \in [1,\infty]$, we have for any $0 < s < 1$*

$$\|\Lambda^s (f\circ u)\|_{L^p} \le C\|u\|_{\dot{B}^s_{p,1}}, \tag{3.2}$$

where $C = C(K,d,s)$ is a positive constant.

(2) Let $f \in \mathcal{C}^2(\mathbb{R})$, $f'(0) = 0$ and $\|f''\|_{L^\infty} \le K < \infty$. Then, with $p \in [1,\infty]$, for any $u, v \in L^{p_1}(\mathbb{R}^d) \cap \dot{H}^{s,p_2}(\mathbb{R}^d) \cap \dot{B}^s_{p_3,1}(\mathbb{R}^d) \cap L^{p_4}(\mathbb{R}^d)$ and any $0 < s < 1$ we have

$$\|\Lambda^s(f\circ u - f\circ v)\|_{L^p} \le C\left(\sup_{w\in\overline{[u,v]}} \|w\|_{L^{p_1}} \|u-v\|_{\dot{H}^{s,p_2}} + \sup_{w\in\overline{[u,v]}} \|w\|_{\dot{B}^s_{p_3,1}} \|u-v\|_{L^{p_4}}\right), \tag{3.3}$$

where $C = C(K,d,s)$ is a positive constant, $\overline{[u,v]}$ denotes the closed line segment $\{u + \sigma(v-u) : \sigma \in [0,1]\}$ and $p_1, p_2, p_3, p_4 \in]1,\infty]$ satisfy $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}$.

Proof. By (3.1) and the assumption on f, we have

$$|\Lambda^s (f \circ u)(x)| \leq KC(d,s) \int_{\mathbb{R}^d} \frac{|u(x+y)-u(x)|}{|y|^{d+s}} \,\mathrm{d}y,$$

and Minkowski's inequality implies that

$$\|\Lambda^s (f \circ u)\|_{L^p} \leq KC(d,s) \int_{\mathbb{R}^d} \frac{1}{|y|^{d+s}} \|u(\cdot + y) - u\|_{L^p} \,\mathrm{d}y. \tag{3.4}$$

By [3, Theorem 2.36] the latter term is equivalent to $\|u\|_{\dot{B}^s_{p,1}}$.

We note that if we ignore the restriction $u \in \mathcal{S}'_h$ the homogeneous space Besov $\dot{B}^s_{p,1}$ is a space of functions modulo constants (see [3, p. 120]; in this case, [5, Theorem 6.3.1] proves the same equivalence. Thus, the inequality (3.2) is established.

As to (3.3), we get the desired result by writing $f \circ u - f \circ v = \int_0^1 f'(v + \sigma(u-v))(u-v)\,\mathrm{d}\sigma$, and combine Lemma 2.4 and (3.2). Moreover, we use the embedding $\dot{B}^s_{p_3,1} \hookrightarrow \dot{H}^{s,p_3}$. □

By the relationship of homogeneous and inhomogeneous Besov spaces, it is easy to get the following corollary.

Corollary 3.2. *Suppose that $f \in \mathcal{C}^1(\mathbb{R})$ satisfies $f(0) = 0$ and $\|f'\|_{L^\infty} \leq K < \infty$. Then for any $u \in B^s_{2,1}(\mathbb{R}^d)$ and $0 < s < 1$, we have*

$$\|f \circ u\|_{H^s} \leq C\|u\|_{B^s_{2,1}}, \tag{3.5}$$

where $C = C(K,d,s)$ is a positive constant.

For composition estimates of $f \circ u$ and any positive s, we refer to the following proposition.

Proposition 3.3 (Theorem 2.87 and Corollary 2.91 in [3]). *Let $f \in \mathcal{C}^\infty(\mathbb{R})$ vanish at 0, let $s > 0$, and $p, r \in [1,\infty]$. If u belongs to $B^s_{p,r} \cap L^\infty$, then so does $f \circ u$ and*

$$\|f \circ u\|_{B^s_{p,r}} \leq C\left(s, f', \|u\|_{L^\infty}\right) \|u\|_{B^s_{p,r}}. \tag{3.6}$$

If furthermore $f'(0) = 0$, then for $u, v \in B^s_{p,r} \cap L^\infty$ the function $f \circ u - f \circ v$ belongs to $B^s_{p,r} \cap L^\infty$ and

$$\|f \circ u - f \circ v\|_{B^s_{p,r}} \leq C\left(\|u-v\|_{B^s_{p,r}} \sup_{w \in \overline{[u,v]}} \|w\|_{L^\infty} + \|u-v\|_{L^\infty} \sup_{w \in \overline{[u,v]}} \|w\|_{B^s_{p,r}}\right),$$

where C depends on f'', $\|u\|_{L^\infty}$ and $\|v\|_{L^\infty}$.

4. Uniform Bounded Estimates

In the following, we denote by C a positive constant, which is independent of time t and of the initial data θ^0. The constant C may vary from line to line. We begin with uniform estimates of the solutions in L^2 and L^{q_0}. Here and in the following we choose q_0 related to $s > 2(1-\alpha)$ such that

$$2\alpha - 1 > \frac{2}{q_0} > 1 - s. \tag{4.1}$$

Proposition 4.1 (Existence of an absorbing ball in L^2 and L^{q_0}). *Let the conditions of Theorem 2.7 hold. Then there exist C and for any bounded ball $B \subset L^2(\mathbb{R}^2) \cap L^{q_0}(\mathbb{R}^2)$ there exists $T_1 = T_1(B) > 0$ such that for all initial values $\theta^0 \in B$ and $t \geq T_1$ the solution $\theta(t) = S(t)\theta^0$ satisfies*

$$\|\theta(t)\|_{L^2} \leq C, \tag{4.2}$$

$$\int_t^{t+1} \|\Lambda^\alpha \theta(\tau)\|^2_{L^2} \,\mathrm{d}\tau \leq C, \tag{4.3}$$

$$\|\theta(t)\|_{L^{q_0}} \leq C. \tag{4.4}$$

Proof. We first note that $\theta^0 \in H^s(\mathbb{R}^2), s > 2(1-\alpha)$, with the above choice of q_0, see (4.1), implies $\theta^0 \in L^2(\mathbb{R}^2) \cap L^{q_0}(\mathbb{R}^2)$ for some $q_0 > \frac{2}{2\alpha-1}$. Multiplying $(1.1)_1$ with $|\theta|^{p-2}\theta$ with $2 \le p \le q_0$ and integrating over $\mathbb{R}^2$, we obtain

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta\|_{L^p}^p + \kappa\int_{\mathbb{R}^2}\Lambda^{2\alpha}\theta|\theta|^{p-2}\theta\,\mathrm{d}x = -\int_{\mathbb{R}^2} u\cdot\nabla\theta|\theta|^{p-2}\theta\,\mathrm{d}x + \int_{\mathbb{R}^2}F(x,\theta)|\theta|^{p-2}\theta\,\mathrm{d}x. \tag{4.5}$$

Due to the incompressibility of u and an integration by parts, the first term on the right hand side of (4.5) vanishes. From (1.5) and (4.5) with $p=2$ and the assumption on f_1, g_1, f_2 in Theorem 2.7, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta\|_{L^2}^2 + \kappa\|\Lambda^\alpha\theta\|_{L^2}^2 \le \int_{\mathbb{R}^2}(\rho*\theta\cdot\theta + g_2(x)\theta)\,\mathrm{d}x. \tag{4.6}$$

Using Plancherel's Theorem, it follows that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}|\widehat{\theta}|^2\,\mathrm{d}\xi + \kappa\int_{\mathbb{R}^2}|\xi|^{2\alpha}|\widehat{\theta}|^2\,\mathrm{d}\xi &\le \int_{\mathbb{R}^2}\left(\hat{\rho}(\xi)|\hat{\theta}(\xi)|^2 + \hat{g}_2(\xi)\bar{\hat{\theta}}\right)\mathrm{d}\xi\\
&= \int_{|\xi|\le\sigma}\hat{\rho}(\xi)|\hat{\theta}(\xi)|^2\,\mathrm{d}\xi + \int_{|\xi|\ge\sigma}\hat{\rho}(\xi)|\hat{\theta}(\xi)|^2\,\mathrm{d}\xi + \int_{\mathbb{R}^2}\hat{g}_2(\xi)\bar{\hat{\theta}}\,\mathrm{d}\xi\\
&\le -|\hat{\rho}_\sigma|\int_{|\xi|\le\sigma}|\hat{\theta}(\xi)|^2\,\mathrm{d}\xi + \int_{\mathbb{R}^2}\hat{g}_2(\xi)\bar{\hat{\theta}}\,\mathrm{d}\xi,\end{aligned} \tag{4.7}$$

where $\sigma > 0$ is chosen such that $\hat{\rho}_\sigma = \max\{\hat{\rho}(\xi) : \xi \in \mathbb{R}^2, |\xi| \le \sigma\} < 0$. Besides, it is obvious that

$$\kappa\int_{\mathbb{R}^2}|\xi|^{2\alpha}|\widehat{\theta}|^2\,\mathrm{d}\xi \ge \kappa\sigma^{2\alpha}\int_{|\xi|\ge\sigma}|\hat{\theta}(\xi)|^2\,\mathrm{d}\xi.$$

Combining the above inequalities we get, with $\kappa_0 = \min\{\kappa\sigma^{2\alpha}, |\hat{\rho}_\sigma|\} > 0$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}|\widehat{\theta}|^2\mathrm{d}\xi + \kappa_0\int_{\mathbb{R}^2}|\widehat{\theta}|^2\,\mathrm{d}\xi \le \int_{\mathbb{R}^2}\hat{g}_2(\xi)\bar{\hat{\theta}}\,\mathrm{d}\xi \le \frac{1}{2\kappa_0}\int_{\mathbb{R}^2}|\hat{g}_2(\xi)|^2\mathrm{d}\xi + \frac{\kappa_0}{2}\int_{\mathbb{R}^2}|\hat{\theta}|^2\mathrm{d}\xi. \tag{4.8}$$

Absorbing the last term from the right-hand side, multiplying by $e^{\kappa_0 t}$ and integrating on $(0,t)$, we find that

$$\|\theta(t)\|_{L^2}^2 \le e^{-\kappa_0 t}\|\theta^0\|_{L^2}^2 + \frac{1}{\kappa_0^2}\|g_2\|_{L^2}^2 \quad (\text{for } t > 0) \tag{4.9}$$

$$\le \frac{2}{\kappa_0^2}\|g_2\|_{L^2}^2 \quad \text{for } t > t_1^*, \tag{4.10}$$

where $t \ge t_1^* := \frac{1}{\kappa_0}\ln\left(\frac{\kappa_0^2\|\theta^0\|_{L^2}^2}{\|g_2\|_{L^2}^2}\right)$. By this result, the assumption on ρ and (4.6), we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta\|_{L^2}^2 + \kappa\|\Lambda^\alpha\theta\|_{L^2}^2 \le \|g_2\|_{L^2}\|\theta\|_{L^2} \le C_0(g_2,\kappa_0) \;\text{ for any } t \ge t_1^*. \tag{4.11}$$

Integrating (4.11) over $[t, t+1]$, $t \ge t_1^*$, and using (4.9), we have

$$\kappa\int_t^{t+1}\|\Lambda^\alpha\theta\|_{L^2}^2\,\mathrm{d}\tau \le \frac{1}{\kappa_0^2}\|g_2\|_{L^2}^2 + C_0. \tag{4.12}$$

It remains to prove (4.4). Since $g_1(x)f_1(\theta)\theta \le 0$ and $\left|\int_{\mathbb{R}^2} g_2|\theta|^{q_0-2}\theta\,\mathrm{d}x\right| \le \|g_2\|_{L^{q_0}}\|\theta\|_{L^{q_0}}^{q_0-1}$, we deduce from (4.5) and Lemma 2.5 that

$$\frac{1}{q_0}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta\|_{L^{q_0}}^{q_0} + \frac{2\kappa}{q_0}\|\Lambda^\alpha|\theta|^{\frac{q_0}{2}}\|_{L^2}^2 \le \|\rho\|_{L^1}\|\theta\|_{L^{q_0}}^{q_0} + \|g_2\|_{L^{q_0}}\|\theta\|_{L^{q_0}}^{q_0-1} \tag{4.13}$$

and consequently that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\theta\|_{L^{q_0}} \le \|\rho\|_{L^1}\|\theta\|_{L^{q_0}} + \|g_2\|_{L^{q_0}}. \tag{4.14}$$

If $\alpha=1$, then $H^\alpha=H^1\hookrightarrow L^{q_0}$, and we get by (4.9), (4.12) that for any $t\geq t_1^*$

$$\int_t^{t+1}\|\theta\|_{L^{q_0}}\,\mathrm{d}y\leq\int_t^{t+1}\|\theta\|_{L^2}\,\mathrm{d}y+\int_t^{t+1}\|\Lambda^\alpha\theta\|_{L^2}\,\mathrm{d}y\leq C. \tag{4.15}$$

Combining (4.14)–(4.15) and the fact that $g_2\in L^{q_0}(\mathbb{R}^2)$, the uniform Gronwall lemma implies that

$$\|\theta(t)\|_{L^{q_0}}\leq C,\qquad\forall\ t\geq t_1^*+1,$$

and we finish the proof of the case $\alpha=1$ by choosing $T_1=t_1^*+1$.

If $1/2<\alpha<1$ and $q_i:=\frac{2}{(1-\alpha)^i}$, $i\in\mathbb{N}$, there exists i_0 such that $q_{i_0}\leq q_0<q_{i_0+1}$. By the conditions on θ^0 and g_2, the inequalities (4.13) and (4.14) hold true for q_0 replaced by $q_i, i=1,\dots,i_0$. Since $H^\alpha\hookrightarrow L^{\frac{2}{1-\alpha}}$, by (4.15), one has for $i=1$ and any $t\geq t_1^*$

$$\int_t^{t+1}\|\theta\|_{L^{q_1}}\,\mathrm{d}y\leq\int_t^{t+1}\|\theta\|_{L^2}\,\mathrm{d}y+\int_t^{t+1}\|\Lambda^\alpha\theta\|_{L^2}\,\mathrm{d}y\leq C. \tag{4.16}$$

Thus by (4.16) and (4.13), (4.14) with q_1, we get with the Uniform Gronwall Lemma

$$\|\theta(t)\|_{L^{q_1}}+\int_t^{t+1}\|\Lambda^\alpha\,|\theta(\tau)|^{\frac{q_1}{2}}\,\|_{L^2}^2\,\mathrm{d}\tau\leq C,\quad\forall\ t\geq t_1^*+1,$$

which together with the embedding estimate $\|\theta\|_{L^{q_2}}\leq c\|\Lambda^\alpha\,|\theta(\tau)|^{\frac{q_1}{2}}\,\|_{L^2}^{1/q_1}$ implies that

$$\int_t^{t+1}\|\theta\|_{L^{q_2}}\,\mathrm{d}y\leq C,\quad\forall\ t\geq t_1^*+1.$$

By iteration, we find that

$$\|\theta(t)\|_{L^{q_{i_0}}}+\int_t^{t+1}\|\Lambda^\alpha\,|\theta(y)|^{\frac{q_{i_0}}{2}}\,\|_{L^2}^2\,\mathrm{d}y\leq C,\quad\forall\ t\geq t_1^*+i_0,$$

and in a final step with q_0 instead of q_{i_0+1},

$$\int_t^{t+1}\|\theta\|_{L^{q_0}}\,\mathrm{d}y\leq C,\qquad\forall\ t\geq t_1^*+i_0.$$

Finally, by using (4.14), one gets that $\|\theta(t)\|_{L^{q_0}}\leq C$ for all $t\geq t_1^*+i_0+1$, finishing the proof in the case $\alpha<1$ by choosing $T_1=t_1^*+i_0+1$. □

Remark 4.2. Two further results can immediately be obtained from (4.9), (4.12) in Proposition 4.1. The estimates (4.17) and (4.18) below are equivalent up to a change of constants C, C_μ and T_1.

(1) By (4.16) we have

$$\int_t^{t+1}\|\theta(\tau)\|_{H^\alpha}^2\,\mathrm{d}\tau\leq C,\qquad\forall\ t\geq T_1. \tag{4.17}$$

(2) Using (4.17) for $(t_1,t_1+1),\dots,(t_1+n,t_1+n+1)$ with arbitrary $n\in\mathbb{N}$ an easy argument by finite geometric sums implies that

$$\int_{t_1}^t e^{-\mu(t-\tau)}\|\theta\|_{H^\alpha}^2\,\mathrm{d}\tau\leq C_\mu,\quad\forall\ t\geq t_1+1\geq T_1+1\text{ and }\forall\,\mu>0. \tag{4.18}$$

On the other hand, exploiting (4.18) for $t=t_1+1$ and all $t_1\geq T_1$, we get (4.17).

Proposition 4.3 (Existence of an absorbing ball in H^s)**.** *If the conditions of Theorem 2.7 hold, then for each bounded ball $B\subset H^s(\mathbb{R}^2)$ there exists a $T_2=T_2(B)$ such that for all $\theta^0\in B$*

$$\|\theta(t)\|_{H^s}\leq C,\quad\forall\ t\geq T_2, \tag{4.19}$$

$$\int_t^{t+1}\|\Lambda^{s+\alpha}\theta(\tau)\|_{L^2}^2\,d\tau\leq C,\quad\forall\ t\geq T_2. \tag{4.20}$$

Proof. Taking the inner product of $(1.1)_1$ with $\Lambda^{2s}\theta$ in L^2, we find that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda^s\theta\|_{L^2}^2+\kappa\|\Lambda^{s+\alpha}\theta\|_{L^2}^2=-\int_{\mathbb{R}^2}(u\cdot\nabla\theta)\,\Lambda^{2s}\theta\,\mathrm{d}x+\int_{\mathbb{R}^2}F(x,\theta)\Lambda^{2s}\theta\,\mathrm{d}x. \tag{4.21}$$

Case 1. $2(1-\alpha)<s<1+2\eta_0$ **with some** $0<2\eta_0<\min\big\{s,2\alpha-1,2\big(\alpha-\frac{2}{q_0}\big)\big\}$**.**

It is easy to see from (4.1) that $s+\alpha-\eta_0>1$. Concerning the integral containing the term $f_2\Lambda^{2s}\theta$ we note that $\int(\rho*\theta)\Lambda^{2s}\theta\,\mathrm{d}x=\int(\rho*\Lambda^s\theta)\Lambda^s\theta\,\mathrm{d}x$. Consequently, Hölder's and Young's inequalities imply that

$$\begin{aligned}&\big|\big(F(x,\theta),\Lambda^{2s}\theta\big)\big| \\ &\quad\le\frac{1}{2}\left(\|\Lambda^{s-2\eta_0}\left(f_1(\theta)g_1\right)\|_{L^2}^2+\|\Lambda^{s+2\eta_0}\theta\|_{L^2}^2\right)+\|\rho\|_{L^1}\|\Lambda^s\theta\|_{L^2}^2+\big(C\|\Lambda^{s-\alpha}g_2\|_{L^2}^2+\frac{\kappa}{8}\|\Lambda^{s+\alpha}\theta\|_{L^2}^2\big).\end{aligned} \tag{4.22}$$

By Lemma 2.4, Proposition 3.1, the condition of f_1,g_1 and s,q_0, and then Young's inequality, we have

$$\begin{aligned}\|\Lambda^{s-2\eta_0}\left(f_1(\theta)g_1\right)\|_{L^2}^2&\le C\|f_1(\theta)\|_{L^{\frac{1}{\eta_0}}}^2\|g_1\|_{\dot H^{s-2\eta_0,\frac{2}{1-2\eta_0}}}^2+\|f_1(\theta)\|_{\dot H^{s-2\eta_0,\frac{2}{1-\alpha+\eta_0}}}^2\|g_1\|_{L^{\frac{2}{\alpha-\eta_0}}}^2\\ &\le C\left(\|\theta\|_{L^{\frac{1}{\eta_0}}}^2+\|\theta\|_{\dot B^{s-2\eta_0}_{\frac{2}{1-\alpha+\eta_0},1}}^2\right)\|g_1\|_{H^s}^2\\ &\le C\|\theta\|_{H^{s+\alpha-\eta_0}}^2\|g_1\|_{H^s}^2\\ &\le C\|\theta\|_{H^s}^2+\frac{\kappa}{8}\|\theta\|_{H^{s+\alpha}}^2,\end{aligned} \tag{4.23}$$

$$\|\Lambda^{s+2\eta_0}\theta\|_{L^2}^2\le C\|\theta\|_{H^s}^2+\frac{\kappa}{8}\|\theta\|_{H^{s+\alpha}}^2, \tag{4.24}$$

where we employed the embeddings $H^s\hookrightarrow\dot H^{s-2\eta_0,\frac{2}{1-2\eta_0}}\hookrightarrow L^{\frac{2}{\alpha-\eta_0}}$, then $H^{s+\alpha-\eta_0}\hookrightarrow\dot B^{s-2\eta_0}_{\frac{2}{1-\alpha+\eta_0},1}$ and interpolation arguments. Similar to the estimates of [45, pp. 1165-1166], using the exponent $\beta:=\frac{1}{2}+\frac{1}{q_0}<\alpha$, (2.3) applied also to $\Lambda^{s+1-\beta}u$, and (4.4), we obtain for $t\ge T_1$ that

$$\begin{aligned}-\big(u\cdot\nabla\theta,\Lambda^{2s}\theta\big)&\le\|\Lambda^{s+1-\beta}(u\theta)\|_{L^2}\|\Lambda^{s+\beta}\theta\|_{L^2}\\ &\le C\left(\|u\|_{L^{q_0}}\|\Lambda^{s+1-\beta}\theta\|_{L^{\frac{2q_0}{q_0-2}}}+\|\theta\|_{L^{q_0}}\|\Lambda^{s+1-\beta}u\|_{L^{\frac{2q_0}{q_0-2}}}\right)\|\Lambda^{s+\beta}\theta\|_{L^2}\\ &\le C\|\theta\|_{L^{q_0}}\|\theta\|_{\dot H^{s-\beta+2(\frac{1}{2}+\frac{1}{q_0})}}\|\Lambda^{s+\beta}\theta\|_{L^2}\\ &\le C\|\theta\|_{H^s}^2+\frac{\kappa}{8}\|\Lambda^{s+\alpha}\theta\|_{L^2}^2.\end{aligned} \tag{4.25}$$

Summarizing (4.21)-(4.25), we arrive at the estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda^s\theta\|_{L^2}^2+\kappa\|\Lambda^{s+\alpha}\theta\|_L^2\le C\|\theta\|_{H^s}^2+C\|\Lambda^{s-\alpha}g_2\|_{L^2}^2,\quad\forall\ t\ge T_1. \tag{4.26}$$

If $s\le\alpha$, then $H^\alpha\hookrightarrow H^s$, and (4.17), (4.26) and the uniform Gronwall lemma lead to (4.19) and (4.20).

If $s>\alpha$, the above proof of (4.21)–(4.25) with a sufficiently small $\eta_0>0$ yields a similar estimate for $s=\alpha$, *i.e.*,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda^\alpha\theta\|_{L^2}^2+\kappa\|\Lambda^{2\alpha}\theta\|_{L^2}\le C\|\theta\|_{H^\alpha}^2+C\|g_2\|_{L^2}^2,\quad\forall\ t\ge T_1. \tag{4.27}$$

Then by (4.2), (4.17), (4.27), and the uniform Gronwall lemma, we have for $s^{(1)}=\alpha$

$$\|\theta(t)\|_{H^{s^{(1)}}}\le C,\quad\forall\ t\ge T_1+1, \tag{4.28}$$

$$\int_t^{t+1}\|\theta\|_{H^{s^{(1)}+\alpha}}\,\mathrm{d}\tau\le C,\quad\forall\ t\ge T_1+1. \tag{4.29}$$

Next we iterate with $s^{(n+1)} = s^{(n)} + \alpha,\ n \geq 1,\ n = 1, \dots, n_0$, such that $s^{(n_0)} \leq s < s^{(n_0+1)}$. From (4.27) with α replaced by 2α (but $\Lambda^{2\alpha}$ by $\Lambda^{3\alpha}$) and (4.28), (4.29) we get for $t \geq T_1 + 2$

$$\|\theta(t)\|_{H^{s^{(2)}}} \leq C, \quad \forall\, t \geq T_1 + 2,$$

$$\int_t^{t+1} \|\theta\|_{H^{s^{(2)}+\alpha}} \,\mathrm{d}\tau \leq C, \quad \forall\, t \geq T_1 + 2.$$

Therefore, by the uniform Gronwall lemma and a bootstrapping argument, there exists $T_2 \geq T_1 > 0$ such that the result of this proposition holds for $0 < s < 1 + 2\eta_0$.

Case 2. $s \geq 1 + 2\eta_0$**.**

In view of Case 1 the estimates (4.19), (4.20) hold for $1 < s < 1 + 2\eta_0$. Moreover, by Sobolev's embedding, we obtain the uniform estimate

$$\|\theta(t)\|_{L^\infty} \leq C \quad \forall\, t \geq \widehat{T}_2, \tag{4.30}$$

for some $\widehat{T}_2 > T_1$. By (3.6) with $B^s_{2,2} = H^s$, (4.30) and the product estimate (2.2), the inequality (4.23) for the external force $f_1(\theta)$ is replaced by

$$\begin{aligned} \left|\big(f_1(\theta)g_1, \Lambda^{2s}\theta\big)\right| &\leq \|\Lambda^s\left(f_1(\theta)g_1\right)\|_{L^2}\|\Lambda^s\theta\|_{L^2} \\ &\leq C\big(\|\Lambda^s f_1(\theta)\|_{L^2}\|g_1\|_{L^\infty} + \|f_1(\theta)\|_{L^\infty}\|\Lambda^s g_1\|_{L^2}\big)\|\Lambda^s\theta\|_{L^2} \\ &\leq C\|g_1\|_{H^s}\|\theta\|^2_{H^s} \quad \forall\, t \geq \widehat{T}_2. \end{aligned} \tag{4.31}$$

Thus, we obtain an inequality similar to (4.26). Now we choose $1 + \eta_0 + \alpha \leq s^{(1)} < 1 + \eta_0 + 2\alpha$ and $s^{(n+1)} = s^{(n)} + \alpha,\ n \geq 1$. By using the same method as above, we get the desired result by choosing $T_2 \geq \widehat{T}_2$. □

Remark 4.4. Using the elementary argument used in Remark 4.2 the estimate (4.20) implies that

$$\int_{t_1}^t e^{-\mu(t-\tau)}\|\theta\|^2_{H^{s+\alpha}} \,\mathrm{d}\tau \leq C_\mu \quad \forall\, t \geq t_1 \geq T_2 \text{ and } \forall\, \mu > 0. \tag{4.32}$$

5. Uniform Smallness Estimates

We begin with this section by choosing a cut-off function $\chi : \mathbb{R}^+ \longrightarrow [0, 1]$ such that

$$\chi(r) = 0 \quad \text{for} \quad 0 \leq r \leq 1 \quad \text{and} \quad \chi(r) = 1 \quad \text{for} \quad r \geq 2,$$

and define $\chi_k = \chi\left(\frac{|\cdot|}{k}\right)$, $k > 0$. It is easy to see that $\mathrm{supp}\chi_k \subset \{x \in \mathbb{R}^2 : |x| \geq k\}$; moreover, for any multi-index $\gamma \in \mathbb{N}^2, |\gamma| > 0$, we get that

$$|D^\gamma \chi_k(x)| \leq Ck^{-|\gamma|} \quad \text{for} \quad |x| \in [k, 2k], \qquad D^\gamma\chi_k(x) = 0 \quad \text{for} \quad |x| \in [k, 2k]^c.$$

Clearly, we have

$$\|\chi_k\|_{\dot{H}^\sigma} \leq Ck^{-(\sigma-1)} \quad \text{for} \quad \sigma > 1 \quad \text{and} \quad \|\chi_k\|_{\dot{H}^{\varrho,q}} \leq Ck^{-(\varrho-2/q)} \quad \text{for} \quad \varrho q > 2, 1 < q < \infty. \tag{5.1}$$

By using the cut-off functions χ_k, we first prove that the L^2– and H^s–norm of solutions are arbitrary small uniformly on the exterior domains $\mathbb{R}^2\backslash\Omega_k$, where $\Omega_k = \{x \in \mathbb{R}^2 : |x| \leq k\}$ for $k > 0$. As in Sect. 4 let $B \subset H^s$ denote a bounded ball of initial values θ^0 of (1.1).

Proposition 5.1. *Let the conditions of Theorem 2.7 hold. Then, for any* $\varepsilon > 0$*, there exist* $T_3 = T_3(\varepsilon, B) > 0$ *and* $K_1 = K_1(\varepsilon, B) > 0$ *such that any solution* $\theta = S(\cdot)\theta^0$*,* $\theta^0 \in B$*, of* (1.1) *satisfies*

$$\int_{\mathbb{R}^2} \chi_k^2|\theta(t)|^2 \,\mathrm{d}x \leq \varepsilon, \quad \forall\, t \geq T_3 \quad \textit{and} \quad k \geq K_1, \tag{5.2}$$

$$\int_t^{t+1}\int_{\mathbb{R}^2} \chi_k^2\left|\Lambda^\alpha\theta\right|^2 \,\mathrm{d}x\,\mathrm{d}\tau \leq \varepsilon, \quad \forall\, t \geq T_3 \quad \textit{and} \quad k \geq K_1. \tag{5.3}$$

Proof. Multiplying $(1.1)_1$ with $\chi_k^2\theta$ and integrating over $\mathbb{R}^2$, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}|\chi_k\theta|^2\,\mathrm{d}x+\kappa\int_{\mathbb{R}^2}(\Lambda^{2\alpha}\theta)\chi_k^2\theta\,\mathrm{d}x=\int_{\mathbb{R}^2}\big(F(x,\theta)-u\cdot\nabla\theta\big)\chi_k^2\theta\,\mathrm{d}x. \tag{5.4}$$

Since

$$\begin{aligned}\int_{\mathbb{R}^2}(\Lambda^{2\alpha}\theta)\chi_k^2\theta\,\mathrm{d}x&=\int_{\mathbb{R}^2}\Lambda^\alpha\theta\cdot\Lambda^\alpha(\chi_k^2\theta)\,\mathrm{d}x\\&=\int_{\mathbb{R}^2}\chi_k\Lambda^\alpha\theta\cdot\Lambda^\alpha(\chi_k\theta)\,\mathrm{d}x-\int\Lambda^\alpha\theta\cdot[\chi_k,\Lambda^\alpha](\chi_k\theta)\,\mathrm{d}x\\&=\int_{\mathbb{R}^2}|\Lambda^\alpha(\chi_k\theta)|^2\,\mathrm{d}x+\int[\chi_k,\Lambda^\alpha]\theta\cdot\Lambda^\alpha(\chi_k\theta)\,\mathrm{d}x-\int\Lambda^\alpha\theta\cdot[\chi_k,\Lambda^\alpha](\chi_k\theta)\,\mathrm{d}x,\end{aligned}$$

(5.4) implies that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}|\chi_k\theta|^2\,\mathrm{d}x+\kappa\int_{\mathbb{R}^2}|\Lambda^\alpha(\chi_k\theta)|^2\,\mathrm{d}x=&-\int_{\mathbb{R}^2}(u\cdot\nabla\theta)\chi_k^2\theta\,\mathrm{d}x+\int_{\mathbb{R}^2}F(x,\theta)\chi_k^2\theta\,\mathrm{d}x-\\&-\kappa\int[\chi_k,\Lambda^\alpha]\theta\cdot\Lambda^\alpha(\chi_k\theta)\,\mathrm{d}x+\kappa\int\Lambda^\alpha\theta\cdot[\chi_k,\Lambda^\alpha](\chi_k\theta)\,\mathrm{d}x.\end{aligned} \tag{5.5}$$

Let us consider each term on the right hand side of (5.5) as follows. Since $\nabla\cdot u=0$, using integration by parts and Hölder's inequality, we have

$$\begin{aligned}\left|\int_{\mathbb{R}^2}(u\cdot\nabla\theta)\chi_k^2\theta\,\mathrm{d}x\right|&=\left|-\int_{\mathbb{R}^2}(u\cdot\nabla\chi_k)\chi_k|\theta|^2\,\mathrm{d}x\right|\\&\le\frac{C}{k}\|u\|_{L^{q_0}}\|\theta\|^2_{L^{\frac{2q_0}{q_0-1}}}\\&\le\frac{C}{k}\|\theta\|_{L^{q_0}}\|\theta\|^2_{H^\alpha},\end{aligned} \tag{5.6}$$

where we used (2.3) and the embedding $H^\alpha\hookrightarrow L^{\frac{2q_0}{q_0-1}}$. For the second term, we see that

$$\begin{aligned}\int_{\mathbb{R}^2}F(x,\theta)\chi_k^2\theta\,\mathrm{d}x&\le\int_{\mathbb{R}^2}\big((\rho*\theta)\chi_k^2\theta+g_2(x)\chi_k^2\theta\big)\,\mathrm{d}x\\&=\int_{\mathbb{R}^2}\rho*(\chi_k\theta)\cdot\chi_k\theta\,\mathrm{d}x+\int_{B_k^c}[\chi_k,\rho*]\theta\cdot\chi_k\theta\,\mathrm{d}x+\int_{\mathbb{R}^2}g_2(x)\chi_k^2\theta\,\mathrm{d}x\\&=:J_1+J_2+J_3.\end{aligned} \tag{5.7}$$

The integral J_1 is used together with the second term on the left-hand side of (5.5) as in (4.7)-(4.8) to yield a positive left-hand side integral of $|\chi_k\theta|^2$, see (5.13) below. In the second integrand J_2 let us analyze the commutator term $[\chi_k,\rho*]\theta(x)$ in the pointwise sense. We get for $|x|\ge k$

$$\begin{aligned}[\chi_k,\rho*]\theta(x)&=\int_{\mathbb{R}^2}\rho(x-y)\big(\chi_k(x)-\chi_k(y)\big)\,\theta(y)\,\mathrm{d}y\\&=\int_{B_R(x)\cup B_R^c(x)}\rho(x-y)\big(\chi_k(x)-\chi_k(y)\big)\,\theta(y)\,\mathrm{d}y,\end{aligned}$$

where the radius of the ball $B_R(x)$ will be chosen suitably. For $y\in B_R(x)$ we have $|\chi_k(x)-\chi_k(y)|\le\frac{cR}{k}$ using an estimate of $\nabla\chi_k$. Hence

$$\left|\int_{B_R(x)}\rho(x-y)(\chi_k(x)-\chi_k(y))\,\theta(y)\,\mathrm{d}y\right|\le\frac{cR}{k}\int_{\mathbb{R}^2}|\rho(x-y)|\,|\theta(y)|\,\mathrm{d}y.$$

Moreover, for $y \in B_R^c(x)$ we use the estimate

$$\left| \int_{B_R^c(x)} \rho(x-y)(\chi_k(x) - \chi_k(y))\, \theta(y)\, \mathrm{d}y \right| \le 2(|\rho|\chi_{R/2}) * |\theta|(x).$$

Given any $\epsilon > 0$, we find R such that $\|\rho\, \chi_{R/2}\|_1 \le \epsilon$. Thus, we have

$$\begin{aligned} J_2 &\le \left(2\|(|\rho|\chi_{R/2}) * |\theta|\|_{L^2} + \frac{cR}{k}\||\rho| * |\theta|\|_{L^2} \right) \|\chi_k \theta\|_{L^2} \\ &\le \left(2\epsilon + \frac{CR}{k}\|\rho\|_{L^1} \right) \|\theta\|_{L^2}^2. \end{aligned} \tag{5.8}$$

Finally, the term J_3 will be considered (5.12).

For the third term on the right hand side of (5.5) we apply Lemma 2.4 to $[\chi_k, \Lambda^\alpha] = [\chi_k - 1, \Lambda^\alpha]$ and (5.1) to get

$$\begin{aligned} \|[\chi_k, \Lambda^\alpha]\theta\|_{L^2} &\le C\left(\|\nabla\chi_k\|_{L^4}\|\theta\|_{\dot H^{\alpha-1,4}} + \|\chi_k\|_{\dot H^{\alpha,4}}\|\theta\|_{L^4}\right) \\ &\le C\left(\|\nabla\chi_k\|_{L^4}\|\theta\|_{\dot H^{\alpha-1/2}} + \|\chi_k\|_{\dot H^{\alpha,4}}\|\theta\|_{H^\alpha}\right) \\ &\le C\left(k^{-1/2} + k^{-(\alpha-1/2)}\right)\|\theta\|_{H^\alpha} \end{aligned} \tag{5.9}$$

so that

$$\begin{aligned} \left|\kappa \int [\chi_k, \Lambda^\alpha]\theta \cdot \Lambda^\alpha(\chi_k\theta)\, \mathrm{d}x\right| &\le C\|\Lambda^\alpha(\chi_k\theta)\|_{L^2}\|[\chi_k, \Lambda^\alpha]\theta\|_{L^2} \\ &\le Ck^{-(2\alpha-1)}\|\theta\|_{H^\alpha}^2 + \frac{\kappa}{2}\|\Lambda^\alpha(\chi_k\theta)\|_{L^2}^2. \end{aligned} \tag{5.10}$$

Concerning the forth term on the right hand side of (5.5) we use the decomposition $[\chi_k, \Lambda^\alpha](\chi_k\theta) = [\chi_k, \Lambda^\alpha]\theta - [\chi_k, \Lambda^\alpha]((1-\chi_k)\theta)$ and estimate the term arising from $[\chi_k, \Lambda^\alpha]\theta \cdot \Lambda^\alpha\theta$ as in (5.10) to get the same bound (even without the term $\frac{\kappa}{2}\|\Lambda^\alpha(\chi_k\theta)\|_{L^2}^2$). For the remaining term with $[\chi_k, \Lambda^\alpha]((1-\chi_k)\theta)$ Lemma 2.4 and (5.1) imply that

$$\begin{aligned} \|[\chi_k, \Lambda^\alpha]((1-\chi_k)\theta)\|_{L^2} &\le C\left(\|\nabla\chi_k\|_{L^4}\|(1-\chi_k)\theta\|_{\dot H^{\alpha-1,4}} + \|\chi_k\|_{\dot H^{\alpha,4}}\|(1-\chi_k)\theta\|_{L^4}\right) \\ &\le C\|\nabla\chi_k\|_{L^4}\left(\|1-\chi_k\|_{L^6}\|\theta\|_{\dot H^{\alpha-1/2,3}} + \|\theta\|_{L^{\frac{2q_0}{q_0-1}}}\|\chi_k\|_{\dot H^{\alpha-1/2,2q_0}}\right) \\ &\quad + C\|\chi_k\|_{\dot H^{\alpha,4}}\|\theta\|_{H^\alpha} \\ &\le C\left(k^{-1/6} + k^{-(\alpha-1/2)}\right)\|\theta\|_{H^\alpha}, \end{aligned} \tag{5.11}$$

where we used the embedding $\dot H^{\alpha-1/2} \hookrightarrow \dot H^{\alpha-1,4}$, the estimate $\|1-\chi_k\|_{L^6} \le ck^{1/3}$ and $q_0 > 1/\alpha$.

By (5.5)-(5.11), we get with $\beta_0 = \min\left(1, \alpha - \frac{1}{2}, \frac{1}{6}\right)$ that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2} |\chi_k\theta|^2\, \mathrm{d}x + \kappa\int_{\mathbb{R}^2} |\Lambda^\alpha(\chi_k\theta)|^2\, \mathrm{d}x \\ &\le \frac{C}{k}\|\theta\|_{L^{q_0}}\|\theta\|_{H^\alpha}^2 + Ck^{-\beta_0}\|\theta\|_{H^\alpha}^2 \\ &+ \int_{\mathbb{R}^2} \rho * (\chi_k\theta) \cdot \chi_k\theta\, \mathrm{d}x + \left(4\epsilon + \frac{CR}{k}\|\rho\|_{L^1}\right)\|\theta\|_{L^2}^2 + \int_{\mathbb{R}^2} g_2(x)\chi_k^2\theta\, \mathrm{d}x. \end{aligned}$$

Applying Plancherel's Theorem we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}|\widehat{\chi_k\theta}|^2\,\mathrm{d}\xi+\kappa\int_{\mathbb{R}^2}|\xi|^{2\alpha}\left|\widehat{\chi_k\theta}\right|^2\,\mathrm{d}\xi\\
&\le Ck^{-\beta_0}(1+\|\theta\|_{L^{q_0}})\|\theta\|_{H^\alpha}^2+\int_{\mathbb{R}^2}\hat\rho\left|\widehat{\chi_k\theta}\right|^2\,\mathrm{d}\xi\\
&+\left(4\epsilon+\frac{CR}{k}\|\rho\|_{L^1}\right)\|\theta\|_{L^2}^2+C\left(\int_{\mathbb{R}^2}|\chi_k g_2(x)|^2\,\mathrm{d}x\right)^{\frac12}\left(\int_{\mathbb{R}^2}|\widehat{\chi_k\theta}|^2\,\mathrm{d}\xi\right)^{\frac12}. \qquad (5.12)
\end{aligned}$$

By the method as in (4.7)–(4.8), we find after using Young's inequality that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}|\widehat{\chi_k\theta}|^2\,\mathrm{d}\xi+\frac{\kappa_0}{2}\int_{\mathbb{R}^2}\left|\widehat{\chi_k\theta}\right|^2\,\mathrm{d}\xi\\
&\quad\le Ck^{-\beta_0}(1+\|\theta\|_{L^{q_0}})\|\theta\|_{H^\alpha}^2+\left(4\epsilon+\frac{CR}{k}\|\rho\|_{L^1}\right)\|\theta\|_{L^2}^2+C\|\chi_k g_2\|_{L^2}^2. \qquad (5.13)
\end{aligned}$$

Then an integration of (5.13) with the weight $e^{+\kappa t}$, (4.2) and (4.4) yield for $T_1+1\le t_1\le t$

$$\begin{aligned}
\int_{\mathbb{R}^2}|\widehat{\chi_k\theta(t)}|^2\,\mathrm{d}\xi&\le e^{-\kappa_0(t-t_1)}\int_{\mathbb{R}^2}|\chi_k\theta(t_1)|^2\,\mathrm{d}x+C\int_{t_1}^t e^{-\kappa_0(t-\tau)}\|\chi_k g_2\|_{L^2}^2\,\mathrm{d}\tau\\
&\quad+Ck^{-\beta_0}\int_{t_1}^t e^{-\kappa_0(t-\tau)}\|\theta\|_{H^\alpha}^2\,\mathrm{d}\tau+\left(\epsilon C+\frac{CR}{k}\|\rho\|_{L^1}\right). \qquad (5.14)
\end{aligned}$$

Let us estimate each term on the right hand side of (5.14). By (4.2), we have

$$e^{-\kappa_0(t-t_1)}\int_{\mathbb{R}^2}|\chi_k\theta(t_1)|^2\,\mathrm{d}x\le\frac{\varepsilon}{4}, \qquad (5.15)$$

for $t\ge T_3(\varepsilon,B)>t_1=T_1$. Secondly, since $g_2\in L^2(\mathbb{R}^2)$, we deduce that

$$C\int_{t_1}^t e^{-\kappa_0(t-\tau)}\|\chi_k g_2\|_{L^2}^2\,\mathrm{d}\tau\le C\int_{|x|\ge k}|g_2(x)|^2\,\mathrm{d}x\le\frac{\varepsilon}{4} \qquad (5.16)$$

for $k\ge k_1(\varepsilon,B)$. Finally, by (4.18) again, one has

$$Ck^{-\beta_0}\int_{t_1}^t e^{-\kappa_0(t-\tau)}\|\theta\|_{H^\alpha}^2\,\mathrm{d}\tau\le\frac{\varepsilon}{4}, \qquad (5.17)$$

for $k\ge k_2(\varepsilon,B)$. For the last term, we can choose $\epsilon\le\varepsilon/(8C)$ (with a corresponding R_ϵ as above). Then, we find $k_3(\varepsilon,B)$ such that $\frac{CR_\epsilon}{k}\|\rho\|_{L^1}\le\frac{\varepsilon}{8}$ for $k\ge k_3$. Defining $K_1=K_1(\varepsilon,B)=\max\{k_1,k_2,k_3\}$, by (5.14)–(5.17), we get (5.2).

Combining the inequalities (5.12)–(5.17) and (5.2), we have

$$\int_t^{t+1}\int_{\mathbb{R}^2}|\Lambda^\alpha(\chi_k\theta)|^2\,\mathrm{d}x\,\mathrm{d}\tau\le\varepsilon,\quad\forall\ t\ge T_3\ \text{ and }\ k\ge K_1.$$

To complete the proof note that $\Lambda^\alpha(\chi_k\theta)=\chi_k\Lambda^\alpha\theta+[\Lambda^\alpha,\chi_k]\theta$. Hence (5.9) yields the estimate

$$\|\Lambda^\alpha(\chi_k\theta)\|_{L^2}^2\le 2\left\|\chi_k^2\left(\Lambda^\alpha\theta\right)\right\|_{L^2}^2+Ck^{-\beta_0}\|\theta\|_{H^\alpha}^2.$$

Redefining ε and K_1 (5.2) and (5.3) are proved. □

Remark 5.2. For any $\varepsilon>0$ there exist $\widehat T_3=\widehat T_3(\varepsilon,B)>0$ and $\widehat K_1=\widehat K_1(\varepsilon,B)>0$ such that

$$\int_{t_1}^t e^{-\mu(t-\tau)}\int_{\mathbb{R}^2}|\chi_k\Lambda^\alpha\theta|^2\,\mathrm{d}x\,\mathrm{d}\tau\le\varepsilon,\qquad\forall\ t\ge t_1+1\ge\widehat T_3,k\ge\widehat K_1\text{ and }\forall\,\mu>0. \qquad (5.18)$$

For the proof we exploit (5.3) and the elementary argument as in Remark 4.2.

Proposition 5.3. *Let the conditions of Theorem 2.7 hold. Then, for every $0 < s < \alpha$ and any $\varepsilon > 0$, there exist $T_4 = T_4(\varepsilon, B) > 0$ and $K_2 = K_2(\varepsilon, B) > 0$ such that any solution $\theta = S(\cdot)\theta^0$, $\theta^0 \in B$, of (1.1) satisfies*

$$\int_t^{t+1} \int_{\mathbb{R}^2} \chi_k \left|\Lambda^s \theta\right|^2 \,\mathrm{d}x \,\mathrm{d}\tau \le \varepsilon, \quad \forall\ t \ge T_4 \quad \textit{and} \quad k \ge K_2, \tag{5.19}$$

$$\int_{t_1}^{t} e^{-\mu(t-\tau)} \int_{\mathbb{R}^2} \chi_k \left|\Lambda^s \theta\right|^2 dx\, d\tau \le \varepsilon, \quad \forall\ t \ge t_1 + 1 \ge T_4, k \ge K_2 \ \textit{and for}\ \mu > 0. \tag{5.20}$$

Proof. For $s < \alpha$, by using Lemma 2.4 and (5.1), (5.2), we have

$$\begin{aligned}
&\int_{\mathbb{R}^2} \chi_k \left|\Lambda^s \theta\right|^2 \,\mathrm{d}x \\
&\quad = \int_{\mathbb{R}^2} \theta [\Lambda^s, \chi_k] \Lambda^s \theta \,\mathrm{d}x + \int_{\mathbb{R}^2} \theta \chi_k \Lambda^{2s} \theta \,\mathrm{d}x \\
&\quad \le \|\theta\|_{L^2} \|[\Lambda^s, \chi_k]\Lambda^s\theta\|_{L^2} + \|\chi_k\theta\|_{L^2} \|\Lambda^{2s}\theta\|_{L^2} \\
&\quad \le C\|\theta\|_{L^2} \left(\|\nabla\chi_k\|_{L^{\frac{4}{2-s}}} \|\Lambda^s\theta\|_{\dot{H}^{s-1,\frac{4}{s}}} + \|\chi_k\|_{\dot{H}^{s,\frac{4}{s}}} \|\Lambda^s\theta\|_{L^{\frac{4}{2-s}}} \right) + \|\chi_k\theta\|_{L^2} \|\theta\|_{H^{s+\alpha}} \\
&\quad \le Ck^{-s/2} \|\theta\|_{L^2} \|\theta\|_{\dot{H}^{\frac{3s}{2}}} + \|\chi_k\theta\|_{L^2} \|\theta\|_{H^{s+\alpha}} \\
&\quad \le Ck^{-s/2} \|\theta\|_{L^2} \|\theta\|_{H^{s+\alpha}} + \|\chi_k\theta\|_{L^2} \|\theta\|_{H^{s+\alpha}}
\end{aligned}$$

Thus, applying Propositions 4.1, 4.3 and 5.1, there exists T_4 such that for $t \ge T_4 > \max\{T_1, T_2, T_3\}$

$$\int_t^{t+1} \int_{\mathbb{R}^2} \chi_k \left|\Lambda^s \theta\right|^2 \,\mathrm{d}x \,\mathrm{d}\tau \le C \int_t^{t+1} \|\theta\|_{H^{s+\alpha}} \left(k^{-s/2} + \|\chi_k\theta\|_{L^2}\right) \mathrm{d}\tau \le \varepsilon, \quad k \ge k_4(\varepsilon, B).$$

Choosing $K_2 = k_4(\varepsilon, B)$ we finish the proof of (5.19).

Then (5.20) is an easy consequence of (5.19). Here T_4 and K_2 depend on μ. □

Proposition 5.4. *Let the conditions of Theorem 2.7 hold. Then, for every $s > 2(1-\alpha)$ and any $\varepsilon > 0$ sufficiently small, there exist $T_5 = T_5(\varepsilon, B) > 0$ and $K_3 = K_3(\varepsilon, B) > 0$ such that any solution $\theta = S(\cdot)\theta^0$, $\theta^0 \in B$, of (1.1) satisfies*

$$\int_{\mathbb{R}^2} |\chi_k \Lambda^s \theta(t)|^2 \,\mathrm{d}x \le \varepsilon, \quad \forall t \ge T_5 \quad \textit{and} \quad k \ge K_3. \tag{5.21}$$

Proof. We divide the proof into several steps.

Step 1. Result for $0 < s < 2\alpha - 1 - \frac{2}{q_0}$.

Applying the operator Λ^s to $(1.1)_1$ and multiplying the resulting equation by $\chi_k \Lambda^s \theta$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^2} \chi_k \left|\Lambda^s\theta\right|^2 \,\mathrm{d}x + \kappa \left(\Lambda^{s+2\alpha}\theta, \chi_k\Lambda^s\theta\right) = \left(\Lambda^s F, \chi_k \Lambda^s\theta\right) - \left(\Lambda^s(u\cdot\nabla\theta), \chi_k\Lambda^s\theta\right). \tag{5.22}$$

We first note that

$$\left(\Lambda^{s+2\alpha}\theta, \chi_k\Lambda^s\theta\right) = \left(\Lambda^{s+\alpha}\theta, [\Lambda^\alpha, \chi_k]\ \Lambda^s\theta\right) + \left(\Lambda^{s+\alpha}\theta, \chi_k\Lambda^{s+\alpha}\theta\right).$$

Inserting this identity into (5.22), it follows that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^2} \chi_k \left|\Lambda^s\theta\right|^2 \,\mathrm{d}x + \kappa \int_{\mathbb{R}^2} \chi_k \left|\Lambda^{s+\alpha}\theta\right|^2 \,\mathrm{d}x \\
&= \left(\Lambda^s\left(f_1(\theta)g_1\right) + \rho * \Lambda^s\theta, \chi_k\Lambda^s\theta\right) + \left(g_2, \Lambda^s(\chi_k\Lambda^s\theta)\right) - \kappa\left(\Lambda^{s+\alpha}\theta, [\Lambda^\alpha, \chi_k]\,\Lambda^s\theta\right) - \left(\Lambda^s(u\cdot\nabla\theta), \chi_k\Lambda^s\theta\right) \\
&=: (I_1 + I_\rho) + I_2 + I_3 + I_4.
\end{aligned} \tag{5.23}$$

Since $s < \alpha$, we choose $0 < \delta = 2\eta_0 < \min\{\alpha - s, s, 2\alpha - 1, 2(\alpha - \frac{2}{q_0})\}$ (which is smaller than the exponent $2\eta_0$ in the proof of (4.22), (4.23)) and rewrite the first term as

$$I_1 = \left(\Lambda^{s-\delta}\left(f_1(\theta)g_1\right), [\Lambda^\delta, \chi_k]\ \Lambda^s\theta\right) + \left(\Lambda^{s-\delta}\left(f_1(\theta)g_1\right), \chi_k\Lambda^{s+\delta}\theta\right). \tag{5.24}$$

In view of (4.23)

$$\|\Lambda^{s-\delta}\left(f_1(\theta)g_1\right)\|_{L^2} \le C\|\theta\|_{H^{s+\alpha}}\|g_1\|_{H^s}. \tag{5.25}$$

On the other hand, by Lemma 2.4 we have for any $\eta_0 < \delta' \le \min\{\alpha + 1/2, s + \alpha - 1 + \eta_0\}$

$$\begin{aligned}\left\|[\Lambda^{\delta'}, \chi_k]\ \Lambda^s\theta\right\|_{L^2} &\le C\|\nabla\chi_k\|_{L^4}\|\Lambda^s\theta\|_{\dot H^{\delta'-1,4}} + C\|\chi_k\|_{\dot H^{\delta', \frac{2}{\eta_0}}}\|\Lambda^s\theta\|_{L^{\frac{2}{1-\eta_0}}} \\ &\le C\left(k^{-1/2} + k^{-(\delta'-\eta_0)}\right)\|\theta\|_{H^{s+\alpha}}.\end{aligned} \tag{5.26}$$

Therefore, we obtain (with $\delta' = 2\eta_0$ in (5.26))

$$\left|\left(\Lambda^{s-\delta}\left(f_1(\theta)g_1\right), [\Lambda^\delta, \chi_k]\ \Lambda^s\theta\right)\right| \le C\left(k^{-1/2} + k^{-\eta_0}\right)\|\theta\|^2_{H^{s+\alpha}}\|g_1\|_{H^s} \tag{5.27}$$

$$\left|\left(\Lambda^{s-\delta}\left(f_1(\theta)g_1\right), \chi_k\Lambda^{s+\delta}\theta)\right)\right| \le C\|g_1\|_{H^s}\|\theta\|_{H^{s+\alpha}}\left(\int_{\mathbb{R}^2}\chi_k\left|\Lambda^{s+\delta}\theta\right|^2\,\mathrm{d}x\right)^{\frac{1}{2}}. \tag{5.28}$$

Moreover,

$$|I_\rho| = |\,|(\rho * \Lambda^s\theta, \chi_k\Lambda^s\theta)| \le \|\rho\|_{L^1}\|\theta\|_{H^s}\left(\int_{\mathbb{R}^2}\chi_k\left|\Lambda^s\theta\right|^2\,\mathrm{d}x\right)^{\frac{1}{2}}.$$

For I_3 we exploit (5.9) with θ replaced by $\Lambda^s\theta$ to get the bound

$$\begin{aligned}|I_3| &\le \|[\Lambda^\alpha, \chi_k]\ \Lambda^s\theta\|_{L^2}\|\theta\|_{H^{s+\alpha}} \\ &\le C\big(k^{-1/2} + k^{-(\alpha-1/2)}\big)\|\theta\|^2_{H^{s+\alpha}}.\end{aligned} \tag{5.29}$$

The term I_2 is written in the form $\big(g_2, \Lambda^s(\chi_k\Lambda^s\theta)\big) = \big(g_2, [\Lambda^s, \chi_k]\,\Lambda^s\theta)\big) + \big(\chi_k g_2, \Lambda^{2s}\theta\big)$ where we use (5.26) with $\delta' := s$ for the first term to get that

$$|(g_2, [\Lambda^s, \chi_k]\,\Lambda^s\theta)| \le \|g_2\|_{L^2}(k^{-\frac{1}{2}} + k^{-(s-\eta_0)})\|\theta\|_{H^{s+\alpha}} \tag{5.30}$$

$$|(\chi_k g_2, \Lambda^{2s}\theta)| \le \|\chi_k g_2\|_{L^2}\|\theta\|_{H^{s+\alpha}} \tag{5.31}$$

To estimate the crucial term I_4 on the right hand side of (5.23), recall that $s < 2\alpha - 1 - \frac{2}{q_0} < \alpha$. Then we rewrite the term as

$$\begin{aligned}(\Lambda^s(u\cdot\nabla\theta), \chi_k\Lambda^s\theta) &= \big(\Lambda^{2s-\alpha}(u\cdot\nabla\theta), [\Lambda^{\alpha-s}, \chi_k]\ \Lambda^s\theta\big) + \big(\Lambda^{2s-\alpha}(u\cdot\nabla\theta), \chi_k\Lambda^\alpha\theta)\big) \\ &=: I_{41} + I_{42}.\end{aligned}$$

By Lemma 2.4, (2.3) and the incompressibility of u, we have

$$\|\Lambda^{2s-\alpha}(u\cdot\nabla\theta)\|_{L^2} \le \|\Lambda^{2s-\alpha+1}(u\theta)\|_{L^2} \le C\|\theta\|_{L^{q_0}}\|\theta\|_{H^{2s-\alpha+1, \frac{2q_0}{q_0-2}}} \le C\|\theta\|_{L^{q_0}}\|\theta\|_{H^{s+\alpha}}, \tag{5.32}$$

where we used that $H^{s+\alpha} \hookrightarrow H^{2s-\alpha+1, \frac{2q_0}{q_0-2}}$. Hence, by (5.32),

$$|I_{42}| \le C\|\theta\|_{L^{q_0}}\|\theta\|_{H^{s+\alpha}}\left(\int_{\mathbb{R}^2}\left|\chi_k\Lambda^\alpha\theta\right|^2\,\mathrm{d}x\right)^{\frac{1}{2}}. \tag{5.33}$$

As to I_{41}, we apply Lemma 2.4, (2.3), (5.1) and Sobolev embeddings to get

$$\begin{aligned}\|[\Lambda^{\alpha-s}, \chi_k]\ \Lambda^s\theta\|_{L^2} &\le C\|\nabla\chi_k\|_{L^4}\|\Lambda^s\theta\|_{\dot H^{\alpha-s-1,4}} + \|\chi_k\|_{\dot H^{\alpha-s, \frac{2}{1-\alpha+2/q_0}}}\|\Lambda^s\theta\|_{L^{\frac{2}{\alpha-2/q_0}}} \\ &\le C\big(k^{-1/2} + k^{-(2\alpha-1-\frac{2}{q_0}-s)}\big)\|\theta\|_{H^{s+\alpha}};\end{aligned}$$

so that together with (5.32)

$$|I_{41}| \le C\big(k^{-1/2} + k^{-(2\alpha-1-\frac{2}{q_0}-s)}\big)\|\theta\|_{L^{q_0}}\|\theta\|_{H^{s+\alpha}}^2. \tag{5.34}$$

Let $\delta_0 > 0$ be a lower bound for all exponents of $\frac{1}{k}$ appearing in (5.26)–(5.34). Combining (5.23)–(5.34), and the estimate (4.4) for $\|\theta\|_{L^{q_0}}$, we obtain for all $t \ge \max\{T_2, T_3, T_4\}$ (with $\delta = 2\eta_0$)

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}\chi_k\,|\Lambda^s\theta|^2\,\mathrm{d}x + \kappa\int_{\mathbb{R}^2}\chi_k\,\big|\Lambda^{s+\alpha}\theta\big|^2\,\mathrm{d}x\\
&\qquad\le Ck^{-\delta_0}\|\theta\|_{H^{s+\alpha}}^2\big(1+\|g_1\|_{H^s}\big) + C\big(k^{-\delta_0}\|g_2\|_{L^2} + \|\chi_k g_2\|_{L^2}\big)\|\theta\|_{H^{s+\alpha}}\\
&\qquad\quad + C\|g_1\|_{H^s}\|\theta\|_{H^{s+\alpha}}\left(\int_{\mathbb{R}^2}\chi_k\,\big|\Lambda^{s+\delta}\theta\big|^2\,\mathrm{d}x\right)^{\frac12} + C\|\theta\|_{H^{s+\alpha}}\left(\int_{\mathbb{R}^2}|\chi_k\Lambda^\alpha\theta|^2\,\mathrm{d}x\right)^{\frac12}\\
&\qquad\quad + \|\rho\|_{L^1}\|\theta\|_{H^s}\left(\int_{\mathbb{R}^2}\chi_k\,|\Lambda^s\theta|^2\,\mathrm{d}x\right)^{\frac12}.
\end{aligned}$$

Next we add the term $\kappa_0\int_{\mathbb{R}^2}\chi_k\,|\Lambda^s\theta|^2\,\mathrm{d}x$ on both sides of the above inequality, multiply by $e^{\kappa_0 t}$ and integrate on (t_1, t). Moreover, using a bound C for $\|g_1\|_{H^s}$, $\|g_2\|_{H^{s-\alpha}}$, $\|\rho\|_{L^1}$ and, by (4.19), for $\|\theta(\cdot)\|_{H^s}$ on (T_2, ∞), we get for $t \ge t_1 \ge \max\{T_2, T_3, T_4\}$ that

$$\begin{aligned}
&\int_{\mathbb{R}^2}\chi_k\,|\Lambda^s\theta(t)|^2\,\mathrm{d}x + \kappa\int_{t_1}^t e^{-\kappa_0(t-\tau)}\int_{\mathbb{R}^2}\chi_k\,\big|\Lambda^{s+\alpha}\theta\big|^2\,\mathrm{d}x\,\mathrm{d}\tau\\
&\quad\le e^{-\kappa_0(t-t_1)}\int_{\mathbb{R}^2}\chi_k\,|\Lambda^s\theta(t_1)|^2\,\mathrm{d}x\\
&\qquad + C\big(k^{-\delta_0} + \|\chi_k g_2\|_{L^2}\big)\int_{t_1}^t e^{-\kappa_0(t-\tau)}\big(\|\theta(\tau)\|_{H^{s+\alpha}} + \|\theta(\tau)\|_{H^{s+\alpha}}^2\big)\,\mathrm{d}\tau\\
&\qquad + C\int_{t_1}^t e^{-\kappa_0(t-\tau)}\|\theta\|_{H^{s+\alpha}}\left(\left(\int_{\mathbb{R}^2}\chi_k\,\big|\Lambda^{s+\delta}\theta\big|^2\,\mathrm{d}x\right)^{\frac12} + \left(\int_{\mathbb{R}^2}|\chi_k\Lambda^\alpha\theta|^2\,\mathrm{d}x\right)^{\frac12}\right)\mathrm{d}\tau\\
&\qquad + (\kappa_0 + C)\int_{t_1}^t e^{-\kappa_0(t-\tau)}\left(\int_{\mathbb{R}^2}\chi_k\,|\Lambda^s\theta|^2\,\mathrm{d}x + \left(\int_{\mathbb{R}^2}\chi_k\,|\Lambda^s\theta|^2\,\mathrm{d}x\right)^{\frac12}\right)\mathrm{d}\tau\\
&\quad =: \Upsilon_1 + \Upsilon_2 + (\Upsilon_{31} + \Upsilon_{32}) + (\Upsilon_{41} + \Upsilon_{42}).
\end{aligned}\tag{5.35}$$

Let us estimate the terms Υ_i. Firstly, by (4.19), we have

$$\Upsilon_1 \le e^{-\kappa_0(t-t_1)}\|\Lambda^s\theta(t_1)\|_{L^2}^2 \le \frac{\epsilon}{8}\quad\text{for } t \ge T_{51} > t_1,$$

where $\epsilon > 0$ is an arbitrary small constant. Secondly, by (4.32), the assumption $g_2 \in L^2(\mathbb{R}^2)$, and an additional application of Hölder's inequality to the term $\int_{t_1}^t e^{-\kappa_0(t-\tau)}\|\theta\|_{H^{s+\alpha}}^1\,\mathrm{d}\tau$, we deduce that

$$\Upsilon_2 \le C\big(k^{-\delta_0} + \|\chi_k g_2\|_{L^2}\big) \le \frac{\epsilon}{8}\quad\text{for}\ \ k \ge k_4(\epsilon, B).$$

Concerning $\Upsilon_{31}, \Upsilon_{32}, \Upsilon_{41}, \Upsilon_{42}$ note that $s+\delta < \alpha$. Then by Hölder's inequality applied to the integral over τ and (4.32), (5.18), (5.20), we deduce that

$$\Upsilon_{31} + \Upsilon_{32} + \Upsilon_{41} + \Upsilon_{42} \le \frac{\epsilon}{8}\quad\text{for}\ \ t \ge T_{52} > t_1,\quad k \ge k_5(\epsilon, B).$$

Choosing $T_5 = \max\{T_{51}, T_{52}\}$ and $K_3' = \max\{K_2, k_4, k_5\}$ and summarizing the estimates of Υ_i we finish the proof of Step 1 with $\epsilon = \varepsilon$.

Next, we shall deal in several steps with the general case when $s \geq 2\alpha - 1 - \frac{2}{q_0}$ and define $s^{(n)} = n\big(2\alpha - 1 - \frac{2}{q_0}\big)$. Since $H^s \hookrightarrow H^{s^{(1)}} \hookrightarrow H^{s^{(1)}-\eta}$ for any $0 < \eta < s^{(1)}$, we have the result from Step 1 with s replaced by $s^{(1)} - \eta$, *i.e.*,

$$\int_{\mathbb{R}^2} \chi_k \left|\Lambda^{s^{(1)}-\eta}\theta(t)\right|^2 \mathrm{d}x \leq \epsilon, \quad \forall t \geq T_5 \text{ and } k \geq K_3'. \tag{5.36}$$

$$\int_{t_1}^{t} e^{-\kappa_0(t-\tau)} \int_{\mathbb{R}^2} \chi_k \left|\Lambda^{s^{(1)}+\alpha-\eta}\theta\right|^2 \mathrm{d}x\, \mathrm{d}\tau \leq \epsilon, \quad \forall t \geq T_{51} > t_1 \geq T_5, \quad k \geq K_3'. \tag{5.37}$$

Step 2. Result for $s^{(1)} \leq s < s^{(2)}$**.**

For the first term $(\Lambda^s (f_1(\theta)g_1), \chi_k \Lambda^s \theta)$, we again write I_1 in the form (5.24) with $\delta = 2\eta_0$ and $0 < 2\eta_0 < \min\big\{s^{(1)} + \alpha - s, s, 2\big(\alpha - \frac{2}{q_0}\big)\big\}$. We need to estimate $\Lambda^{s-\delta}(f_1(\theta)g_1)$. If $s - \delta < 1$, then it is easy to get (5.25) (similar to (4.23)). If $s - \delta > 1$, we obtain (5.25) by an estimate similar to (4.31). Moreover, by $s + \delta < s^{(1)} + \alpha$ and (5.37), we see that $\int_{t_1}^t e^{-\kappa_0(t-\tau)} \int_{\mathbb{R}^2} \chi_k \left|\Lambda^{s+\delta}\theta\right|^2 \mathrm{d}x\, \mathrm{d}\tau$ can be arbitrary small.

In Step 2, we may have $s \geq \alpha$. In this case we use an estimate for I_2 in (5.23) with $s \geq \alpha$ similar to I_3 in (5.29), namely

$$\begin{aligned} |I_2| &= \left| \left(\Lambda^{s-\alpha} g_2, [\Lambda^\alpha, \chi_k]\, \Lambda^s \theta\right) + \left(\Lambda^{s-\alpha} g_2, \chi_k \Lambda^{s+\alpha}\theta\right) \right| \\ &\leq C\left(k^{-1/2} + k^{-(\alpha-1/2)}\right) \|\theta\|_{H^{s+\alpha}} \|g_2\|_{H^{s-\alpha}} + C\int_{\mathbb{R}^2} \chi_k \left|\Lambda^{s-\alpha} g_2\right|^2 \mathrm{d}x + \frac{\kappa}{8}\int_{\mathbb{R}^2} \chi_k \left|\Lambda^{s+\alpha}\theta\right|^2 \mathrm{d}x. \end{aligned}$$

Since $g_2 \in H^{s-\alpha}$, we deduce that

$$C\int_{\mathbb{R}^2} \chi_k \left|\Lambda^{s-\alpha} g_2\right|^2 \mathrm{d}x \leq C\int_{|x|\geq k} \left|\Lambda^{s-\alpha} g_2\right|^2 \mathrm{d}x\, \mathrm{d}\tau \to 0 \quad \text{as} \quad k \to \infty. \tag{5.38}$$

Concerning I_4 we choose $0 < \eta < \min\{\frac{1}{q_0}, s^{(2)} - s\}$, and write $-I_4$ in the form (*cf.* Step 1)

$$\begin{aligned} (\Lambda^s(u\cdot\nabla\theta), \chi_k \Lambda^s\theta) &= \left(\Lambda^{2s-s^{(1)}-\alpha+\eta}(u\cdot\nabla\theta), \left[\Lambda^{s^{(1)}+\alpha-s-\eta}, \chi_k\right]\Lambda^s\theta \,+\, \chi_k \Lambda^{s^{(1)}+\alpha-\eta}\theta\right) \\ &=: I_{41}' + I_{42}'. \end{aligned} \tag{5.39}$$

As before, by Lemma 2.4, (2.3) and the incompressibility of u, we have

$$\|\Lambda^{2s-s^{(1)}-\alpha+\eta}(u\cdot\nabla\theta)\|_{L^2} \leq C\|\theta\|_{L^{q_0}} \|\theta\|_{H^{2s-s^{(1)}-\alpha+\eta+1, \frac{2q_0}{q_0-2}}} \leq C\|\theta\|_{L^{q_0}}\|\theta\|_{H^{s+\alpha}},$$

where we used $\eta < s^{(2)} - s$ so that $H^{s+\alpha} \hookrightarrow H^{2s-s^{(1)}-\alpha+\eta+1, \frac{2q_0}{q_0-2}}$. Consequently,

$$\left|I_{42}'\right| \leq C\|\theta\|_{L^{q_0}}\|\theta\|_{H^{s+\alpha}} \left(\int_{\mathbb{R}^2} \chi_k \left|\Lambda^{s^{(1)}+\alpha-\eta}\theta\right|^2 \mathrm{d}x\right)^{\frac{1}{2}}.$$

As to I_{41}', using Lemma 2.4 and (2.3), then (5.1) and Sobolev embeddings, we get

$$\begin{aligned} &\left\|\left[\Lambda^{s^{(1)}+\alpha-s-\eta}, \chi_k\right]\Lambda^s\theta\right\|_{L^2} \\ &\quad\leq C\|\nabla\chi_k\|_{L^4}\|\Lambda^s\theta\|_{\dot{H}^{s^{(1)}+\alpha-s-\eta-1,4}} + \|\chi_k\|_{\dot{H}^{s^{(1)}+\alpha-s-\eta, \frac{2}{1-\alpha+2/q_0-\eta}}} \|\Lambda^s\theta\|_{L^{\frac{2}{\alpha-2/q_0+\eta}}} \\ &\quad\leq C\left(k^{-1/2} + k^{-(s^{(2)}-s)}\right)\|\theta\|_{H^{s+\alpha}}, \end{aligned}$$

where we exploited $H^{s+\alpha} \hookrightarrow \dot{H}^{s+\alpha} \hookrightarrow \dot{H}^{s^{(1)}+\alpha-\eta-1,4}$ (note that $s^{(1)} + \alpha - \eta - 1$ can be negative) and $H^{s+\alpha} \hookrightarrow \dot{H}^{s, \frac{2}{\alpha-2/q_0+\eta}}$. Hence

$$\left|I_{41}'\right| \leq C\left(k^{-1/2} + k^{-(s^{(2)}-s)}\right)\|\theta\|_{L^{q_0}}\|\theta\|_{H^{s+\alpha}}^2.$$

Proceeding as in Step 1 and using (4.4) the terms Υ_2 and Υ_3 (related to I'_{42}) in (5.35) are replaced – with an appropriate $\delta_1 > 0$ – by

$$\Upsilon'_2 = \begin{cases} Ck^{-\delta_1}\displaystyle\int_{t_1}^{t} e^{-\kappa_1(t-\tau)}\big(\|\theta(\tau)\|_{H^{s+\alpha}} + \|\theta(\tau)\|^2_{H^{s+\alpha}}\big)\,\mathrm{d}\tau + C\int_{\mathbb{R}^2}\chi_k\left|\Lambda^{s-\alpha}g_2\right|^2 \mathrm{d}x, & \text{if } s \geq \alpha,\\ C\big(k^{-\delta_0} + \|\chi_k g_2\|_{L^2}\big)\displaystyle\int_{t_1}^{t} e^{-\kappa_0(t-\tau)}\big(\|\theta(\tau)\|_{H^{s+\alpha}} + \|\theta(\tau)\|^2_{H^{s+\alpha}}\big)\,\mathrm{d}\tau, & \text{if } s < \alpha \end{cases}$$

$$\Upsilon'_3 = C\int_{t_1}^{t} e^{-\kappa_0(t-\tau)}\|\theta\|_{H^{s+\alpha}}\left[\left(\int_{\mathbb{R}^2}\chi_k\left|\Lambda^{s+\delta}\theta\right|^2 \mathrm{d}x\right)^{\frac{1}{2}} + \left(\int_{\mathbb{R}^2}\chi_k\left|\Lambda^{s^{(1)}+\alpha-\eta}\theta\right|^2 \mathrm{d}x\right)^{\frac{1}{2}}\right]\mathrm{d}\tau.$$

Applying Hölder's inequality,(4.32), (5.37) and (5.38), Υ_2 and Υ_3 are bounded by

$$\Upsilon'_2 \leq Ck^{-\delta_1} \leq \frac{\epsilon}{8}, \text{ for } k \geq k_6(\epsilon),$$
$$\Upsilon'_3 \leq C\epsilon^{\frac{1}{2}} \leq \frac{\epsilon^{\frac{1}{4}}}{8} \quad \forall\, t \geq T_{52} > t_1 \geq T_5 \text{ and } k \geq k_6(\epsilon)$$

for ϵ sufficiently small.

Therefore, choosing T_5 as defined before and $K''_3 = \max\{k_6, K'_3\}$, we finish the proof of Step 2 with $\epsilon^{\frac{1}{4}} = \varepsilon$.

Step 3. Result for $s^{(2)} \leq s < s^{(3)}$.

As before, we have from Step 2 the same results as (5.36)–(5.37) with $s^{(1)}$ replaced by $s^{(2)}$, *i.e.*,

$$\int_{\mathbb{R}^2}\chi_k\left|\Lambda^{s^{(2)}-\eta}\theta(t)\right|^2 \mathrm{d}x \leq \epsilon^{\frac{1}{4}}, \quad \forall t \geq T_5 \text{ and } k \geq K''_3$$
$$\int_{t_1}^{t} e^{-\kappa_0(t-\tau)}\int_{\mathbb{R}^2}\chi_k\left|\Lambda^{s^{(2)}+\alpha-\eta}\theta\right|^2 \mathrm{d}x\,\mathrm{d}\tau \leq \epsilon^{\frac{1}{4}}, \quad \forall t \geq T_{52} > t_1 \geq T_5, \quad k \geq K''_3.$$

Therefore, as in Step 2, we deal with the third case $s^{(2)} \leq s < s^{(3)}$. In particular, we have with $\delta_2 > 0$

$$\Upsilon''_2 \leq Ck^{-\delta_2} \leq \frac{\epsilon}{8} \text{ for } k \geq k_7(\epsilon, B), \quad \Upsilon''_3 \leq C\epsilon^{\frac{1}{8}} \leq \frac{\epsilon^{\frac{1}{16}}}{8},$$

and get the result with an adequate K'''_3 and $\epsilon^{\frac{1}{16}} = \varepsilon$.

Step 4. End of the proof.

For any $s > 0$ in Theorem 2.6, there exists $n_0 \in \mathbb{N}$ such that $s^{(n_0-1)} \leq s < s^{(n_0)}$. We repeat the above proof $(n_0 - 1)$-times and get the result with suitable K_3 and T_5 and with $\epsilon^{\left(\frac{1}{4}\right)^{n_0-1}} = \varepsilon$.

Therefore, the proposition is proved. □

6. Global Attractor in H^s

In this section, we prove the existence of the global attractor in H^s. In view of Proposition 2.2 and Sect. 4, it is sufficient to prove the asymptotic compactness of the semigroup. As in Sect. 4 let $B \subset H^s(\mathbb{R}^2)$ denote any bounded ball of initial values θ^0 for (1.1)

Proposition 6.1. *Let the assumptions of Theorem 2.7 hold. Then the solutions $\theta = S(\cdot)\theta^0$, $\theta \in B$, of* (1.1) *satisfy*

$$\int_t^{t+1}\|\Lambda^{s-\alpha}(\partial_t\theta)\|^2_{L^2}\,\mathrm{d}\tau \leq C, \quad \forall\, t \geq T_2 + 1, \quad \textit{if } s > \alpha, \tag{6.1}$$
$$\int_t^{t+1}\|\partial_t\theta\|^2_{L^2}\,\mathrm{d}\tau \leq C, \quad \forall\, t \geq T_2 + 1. \tag{6.2}$$

Proof. For $s>\alpha$, taking the inner product of $(1.1)_1$ with $\Lambda^{2(s-\alpha)}\partial_t\theta$ in L^2, we find that

$$\|\Lambda^{s-\alpha}(\partial_t\theta)\|_{L^2}^2+\frac{\kappa}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda^s\theta\|_{L^2}^2=-\int_{\mathbb{R}^2}(u\cdot\nabla\theta)\,\Lambda^{2(s-\alpha)}\partial_t\theta\,\mathrm{d}x+\int_{\mathbb{R}^2}F(x,\theta)\Lambda^{2(s-\alpha)}\partial_t\theta\,\mathrm{d}x$$
$$=:I_1+I_2. \tag{6.3}$$

By Hölder's and Young's inequalities, Lemma 2.4 and (1.4), (2.3), I_1 is bounded by

$$\begin{aligned}I_1&\le\|\Lambda^{s-\alpha}(u\cdot\nabla\theta)\|_{L^2}\|\Lambda^{s-\alpha}(\partial_t\theta)\|_{L^2}\\&\le C\|\Lambda^{s-\alpha+1}(u\theta)\|_{L^2}^2+\frac14\|\Lambda^{s-\alpha}(\partial_t\theta)\|_{L^2}^2\\&\le C\|\theta\|_{L^{q_0}}^2\|\Lambda^{s-\alpha+1}\theta\|^2_{L^{\frac{2q_0}{q_0-2}}}+\frac14\|\Lambda^{s-\alpha}(\partial_t\theta)\|_{L^2}^2\\&\le C\|\theta\|_{L^{q_0}}^2\|\theta\|_{H^{s+\alpha}}^2+\frac14\|\Lambda^{s-\alpha}(\partial_t\theta)\|_{L^2}^2,\end{aligned}\tag{6.4}$$

where we used $H^{s+\alpha}\hookrightarrow H^{s-\alpha+1,\frac{2q_0}{q_0-2}}$. For I_2, applying (3.2) and the trivial estimate $|f_1(\theta)|\le K|\theta|$, we have for $\alpha<s<1+\alpha$

$$\begin{aligned}I_2&\le\|\Lambda^{s-\alpha}F(x,\theta)\|_{L^2}\|\Lambda^{s-\alpha}(\partial_t\theta)\|_{L^2}\\&\le C\left(\|\Lambda^{s-\alpha}(f_1(\theta)g_1)\|_{L^2}+\|\rho*\Lambda^{s-\alpha}\theta\|_{L^2}+\|\Lambda^{s-\alpha}g_2\|_{L^2}\right)\|\Lambda^{s-\alpha}(\partial_t\theta)\|_{L^2}\\&\le C\left(\|f_1(\theta)\|_{L^{\frac{2}{\alpha-\sigma}}}\|g_1\|_{\dot H^{s-\alpha,\frac{2}{1-\alpha+\sigma}}}+\|f_1(\theta)\|_{\dot H^{s-\alpha,\frac{2}{1-\alpha+\sigma}}}\|g_1\|_{L^{\frac{2}{\alpha-\sigma}}}\right)\|\Lambda^{s-\alpha}(\partial_t\theta)\|_{L^2}\\&\quad+\left(\|\rho\|_{L^1}\|\theta\|_{H^{s-\alpha}}+\|\Lambda^{s-\alpha}g_2\|_{L^2}\right)\|\Lambda^{s-\alpha}(\partial_t\theta)\|_{L^2}\\&\le C\left(\|\theta\|_{\dot H^{1-\alpha+\sigma}}\|g_1\|_{\dot H^{s-\sigma}}+\|\theta\|_{\dot B^{s-\sigma}_{2,1}}\|g_1\|_{\dot H^{1-\alpha+\sigma}}\right)\|\Lambda^{s-\alpha}(\partial_t\theta)\|_{L^2}\\&\quad+\left(\|\rho\|_{L^1}\|\theta\|_{H^{s-\alpha}}+\|\Lambda^{s-\alpha}g_2\|_{L^2}\right)\|\Lambda^{s-\alpha}(\partial_t\theta)\|_{L^2}\\&\le C\|\theta\|_{H^s}^2\left(\|g_1\|_{H^s}^2+\|\rho\|_{L^1}^2\right)+C\|\Lambda^{s-\alpha}g_2\|_{L^2}^2+\frac14\|\Lambda^{s-\alpha}(\partial_t\theta)\|_{L^2}^2,\end{aligned}\tag{6.5}$$

where $0<\sigma<\min\{2\alpha-1,s+\alpha-1\}$.

For $s\ge1+\alpha$, we also know that $\theta\in L^\infty$, see (4.30). Then, by (3.6) in Proposition 3.3, we deduce the same inequality as (6.5) for all $t\ge\widehat{T}_5$. Inserting (6.4) and (6.5) into (6.3) and integrating over $[t,t+1],t\ge T_2$, we obtain in view of (4.4), $g_1\in H^s$, and $\rho\in L^1$ that

$$\int_t^{t+1}\|\Lambda^{s-\alpha}(\partial_t\theta)\|_{L^2}^2\,\mathrm{d}\tau+\kappa\|\Lambda^s\theta(t+1)\|_{L^2}^2\le\kappa\|\Lambda^s\theta(t)\|_{L^2}^2+C\int_t^{t+1}\|\theta\|_{H^{s+\alpha}}^2\,\mathrm{d}\tau. \tag{6.6}$$

Now (4.20) yields the estimate (6.1).

As to (6.2), we multiply (1.1) by $\partial_t\theta$ and get

$$\begin{aligned}&\|\partial_t\theta\|_{L^2}^2+\frac{\kappa}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda^\alpha\theta\|_{L^2}^2\\&\quad=-\int_{\mathbb{R}^2}(u\cdot\nabla\theta)\,\partial_t\theta\,\mathrm{d}x+\int_{\mathbb{R}^2}F(x,\theta)\partial_t\theta\,\mathrm{d}x\\&\quad\le C\left(\|u\cdot\nabla\theta\|_{L^2}^2+\|F(x,\theta)\|_{L^2}^2\right)+\frac12\|\partial_t\theta\|_{L^2}^2\\&\quad\le C\left(\|\theta\|_{L^{q_0}}^2\|\nabla\theta\|^2_{L^{\frac{2q_0}{q_0-2}}}+\|f_1(\theta)g_1\|_{L^2}^2+\|\rho\|_{L^1}^2\|\theta\|_{L^2}^2+\|g_2\|_{L^2}^2\right)+\frac12\|\partial_t\theta\|_{L^2}^2\\&\quad\le C\left(\|\theta\|_{L^{q_0}}^2\|\theta\|_{H^{2\alpha}}^2+\|\theta\|_{H^{2\alpha}}^2\|g_1\|_{L^2}^2+\|\rho\|_{L^1}^2\|\theta\|_{L^2}^2+\|g_2\|_{L^2}^2\right)+\frac12\|\partial_t\theta\|_{L^2}^2,\end{aligned}\tag{6.7}$$

since $H^{2\alpha} \hookrightarrow H^{1,\frac{2q_0}{q_0-2}} \hookrightarrow L^\infty$ and $\|f'\|_{L^\infty} \le K$. Next, we claim that

$$\int_t^{t+1} \|\theta\|_{H^{2\alpha}}^2 \,\mathrm{d}\tau \le C, \quad \forall\, t \ge T_1 + 1. \tag{6.8}$$

In fact, we recall (4.27) in Proposition 4.3 which holds even when $s < \alpha$ and $g_1 \in L^\infty$ only; for the proof the crucial term $\int f_1(\theta) g_1(x)\, \Lambda^{2\alpha}\theta$ is estimated by $C\|\theta\|_{L^2}^2 \|g_1\|_{L^\infty}^2 + \frac{\kappa}{8}\|\Lambda^{2\alpha}\theta\|_{L^2}^2$. Then use (4.3) and the uniform Gronwall Lemma to get a uniform bound of $\|\Lambda^\alpha \theta(t)\|_{L^2}$ for all $t \ge T_1 + 1$. Now an integration of (4.27) yields (6.8). Combining the latter estimates with (6.7), we obtain (6.2). □

Remark 6.2. Combining (4.2) and a uniform bound for $\|\Lambda^\alpha\theta(t)\|_{L^2}$ as above, we see that

$$\|\theta(t)\|_{H^\alpha} \le C, \quad \forall\, t \ge T_1 + 1. \tag{6.9}$$

Proposition 6.3. *Under the assumptions of Theorem 2.7, the solutions $\theta = S(\cdot)\theta^0$, $\theta^0 \in B$, of* (1.1) *satisfy*

$$\|\partial_t \theta(t)\|_{L^2}^2 \le C, \quad \forall\, t \ge T_2 + 2, \tag{6.10}$$

$$\|\Lambda^{s-\alpha}(\partial_t\theta)(t)\|_{L^2}^2 \le C, \quad \forall\, t \ge T_2 + 3, \text{ if } s > \alpha. \tag{6.11}$$

Proof. By differentiating (1.1) in time and writing $\omega = \partial_t\theta$, we have

$$\partial_t \omega + u\cdot\nabla\omega + \kappa(-\Delta)^\alpha \omega = -u_t\cdot\nabla\theta + f_1'(\theta) g_1(x)\omega + \rho * \omega. \tag{6.12}$$

Taking the inner product of (6.12) with ω and using $|f_1'(\theta)| \le K$ and $g_1 \in L^\infty$, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\|\omega\|_{L^2}^2 + \kappa\|\Lambda^\alpha\omega\|_{L^2}^2 \le -\int_{\mathbb{R}^2} (u_t\cdot\nabla\theta)\,\omega\,\mathrm{d}x + C\|\omega\|_{L^2}^2. \tag{6.13}$$

Since $2\alpha - 1 > \frac{2}{q_0} > 1 - s$, we first choose ϵ_0 small enough such that

$$0 < \epsilon_0 < \frac{1}{2}\min\Big\{2\alpha - 1 - \frac{2}{q_0},\ \alpha,\ s - 2(1-\alpha)\Big\}; \tag{6.14}$$

then we have

$$2 < \frac{2}{2\alpha - 1 - 2\epsilon_0} < q_0, \quad 2(1-\alpha) + 2\epsilon_0 < s,$$

and hence $0 < \widetilde{s} < \frac{1}{2}\min\{1, s\}$ with $\widetilde{s} := 1 - \alpha + \epsilon_0$. Using Hölder's, Young's and interpolation inequalities, the first term on the right hand side of (6.13) is bounded by

$$\begin{aligned}
\left|\int_{\mathbb{R}^2} (u_t\cdot\nabla\theta)\,\omega\,\mathrm{d}x\right| &\le \|\Lambda^{-\alpha+\epsilon_0}(u_t\cdot\nabla\theta)\|_{L^2}\|\Lambda^{\alpha-\epsilon_0}\omega\|_{L^2} \\
&\le \frac{1}{2}\|\Lambda^{1-\alpha+\epsilon_0}(u_t\theta)\|_{L^2}^2 + \frac{1}{2}\|\Lambda^{\alpha-\epsilon_0}\omega\|_{L^2}^2 \\
&\le \frac{1}{2}\|\Lambda^{\tilde{s}}(u_t\theta)\|_{L^2}^2 + C\|\omega\|_{L^2}^2 + \frac{\kappa}{4}\|\Lambda^\alpha\omega\|_{L^2}^2.
\end{aligned} \tag{6.15}$$

Concerning the term $\|\Lambda^{\tilde{s}}(u_t\theta)\|_{L^2}$ we apply Lemma 2.4 and (1.4), (2.3) to get that

$$\begin{aligned}
\|\Lambda^{\tilde{s}}(u_t\theta)\|_{L^2}^2 &\le C\left(\|\Lambda^{\tilde{s}}u_t\|_{L^{\frac{1}{1-\alpha+\epsilon_0}}}^2 \|\theta\|_{L^{\frac{2}{2\alpha-1-2\epsilon_0}}}^2 + \|u_t\|_{L^{\frac{2}{1-\alpha+\epsilon_0}}}^2 \|\Lambda^{\tilde{s}}\theta\|_{L^{\frac{2}{\alpha-\epsilon_0}}}^2\right) \\
&\le C\left(\|\Lambda^{\alpha-\epsilon_0}\omega\|_{L^2}^2\|\theta\|_{L^2\cap L^{q_0}}^2 + \|\Lambda^{\alpha-\epsilon_0}\omega\|_{L^2}^2\|\Lambda^{2\widetilde{s}}\theta\|_{L^2}^2\right) \\
&\le C\|\Lambda^{\alpha-\epsilon_0}\omega\|_{L^2}^2\left(\|\theta\|_{L^2\cap L^{q_0}}^2 + \|\Lambda^{2\widetilde{s}}\theta\|_{L^2}^2\right),
\end{aligned} \tag{6.16}$$

where we used

$$\dot{H}^{\alpha-\epsilon_0} \hookrightarrow \dot{H}^{\tilde{s},\frac{1}{1-\alpha+\epsilon_0}}, \quad \dot{H}^{\alpha-\epsilon_0} \hookrightarrow L^{\frac{2}{1-\alpha+\epsilon_0}} \quad\text{and}\quad \dot{H}^{2\widetilde{s}} \hookrightarrow \dot{H}^{\tilde{s},\frac{2}{\alpha-\epsilon_0}}.$$

Thus, by (4.2), (4.4), (4.19), and an interpolation inequality applied to $\|\Lambda^{\alpha-\epsilon_0}\omega\|_{L^2}$,

$$\|\Lambda^{\tilde{s}}(u_t\theta)\|_{L^2}^2 \le C\|\omega\|_{L^2}^2 + \frac{\kappa}{2}\|\Lambda^\alpha\omega\|_{L^2}^2. \tag{6.17}$$

Combining (6.13)–(6.17), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\omega\|_{L^2}^2 + \kappa\|\Lambda^\alpha\omega\|_{L^2}^2 \le C\|\omega\|_{L^2}^2, \quad \forall t \ge T_2 + 1. \tag{6.18}$$

Finally, we use (6.2) in Proposition 6.1 and the uniform Gronwall Lemma to get (6.10).

In the following let $s > \alpha$. To prove (6.11) we take the inner product of (6.12) with $\Lambda^{2(s-\alpha)}\omega$ and obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda^{s-\alpha}\omega\|_{L^2}^2 + \kappa\|\Lambda^s\omega\|_{L^2}^2 = \big(f_1'(\theta)g_1(x)\omega + \rho*\omega - u\cdot\nabla\omega - u_t\cdot\nabla\theta,\, \Lambda^{2(s-\alpha)}\omega\big). \tag{6.19}$$

We have

$$\big|\big(f_1'(\theta)g_1(x)\omega,\, \Lambda^{2(s-\alpha)}\omega\big)\big| \le \frac{1}{2}\big\|\Lambda^{s-\alpha}\left(f_1'(\theta)g_1\omega\right)\big\|_{L^2}^2 + \frac{1}{2}\big\|\Lambda^{s-\alpha}\omega\big\|_{L^2}^2 \tag{6.20}$$

where for some $0 < \frac{2}{q_0} < \eta < 2\alpha - 1$

$$\big\|\Lambda^{s-\alpha}\left(f_1'(\theta)g_1\omega\right)\big\|_{L^2}^2 \le C\left\|f_1'(\theta)g_1\right\|_{L^{q_0}}^2 \|\omega\|^2_{H^{s-\alpha,\frac{2q_0}{q_0-2}}} + C\|\omega\|^2_{L^{\frac{2}{\alpha-\eta}}} \left\|f_1'(\theta)g_1\right\|^2_{\dot{H}^{s-\alpha,\frac{2}{1-\alpha+\eta}}}. \tag{6.21}$$

If $\alpha < s < 1+\alpha$, we use (3.2) and the embeddings $\dot{H}^{s-\eta+2/q_0} \hookrightarrow \dot{H}^{s-\alpha,\frac{2}{1-\alpha+\eta-2/q_0}}$ and $H^s \hookrightarrow \dot{B}^{s-\eta+2/q_0}_{2,1} \hookrightarrow \dot{B}^{s-a}_{\frac{2}{1-\alpha+\eta-2/q_0},1}$, to show that the most right hand side terms of (6.21) is bounded by

$$\begin{aligned}
\left\|f_1'(\theta)g_1\right\|_{\dot{H}^{s-\alpha,\frac{2}{1-\alpha+\eta}}} &\le C\left(\|f_1'(\theta)\|_{L^{q_0}}\|g_1\|_{\dot{H}^{s-\alpha,\frac{2}{1-\alpha+\eta-2/q_0}}} + \|f_1'(\theta)\|_{\dot{H}^{s-\alpha,\frac{2}{1-\alpha+\eta-2/q_0}}}\|g_1\|_{L^{q_0}}\right) \\
&\le C\left(\|\theta\|_{L^{q_0}}\|g_1\|_{\dot{H}^{s-\eta+\frac{2}{q_0}}} + \|\theta\|_{\dot{B}_{2,1}^{s-\eta+\frac{2}{q_0}}}\|g_1\|_{L^{q_0}}\right) \\
&\le C\|\theta\|_{H^s}\|g_1\|_{H^s}.
\end{aligned}$$

For $s \ge 1+\alpha$, due to the embeddings $\dot{H}^{s-\eta} \hookrightarrow \dot{H}^{s-\alpha,\frac{2}{1-\alpha+\eta}}$ and $\dot{H}^s \hookrightarrow \dot{H}^{s-\eta,\frac{2}{1-\eta}}$, we get with (2.2) and (3.6) that

$$\begin{aligned}
\left\|f_1'(\theta)g_1\right\|_{\dot{H}^{s-\alpha,\frac{2}{1-\alpha+\eta}}} &\le \|f_1'(\theta)\|_{\dot{H}^{s-\eta}}\|g_1\|_{L^\infty} + \|g_1\|_{\dot{H}^{s-\eta}}\|f_1'(\theta)\|_{L^\infty} \\
&\le C\|g_1\|_{H^s}\|\theta\|_{H^s} \quad \text{for all} \quad t \ge \widehat{T}_5,
\end{aligned}$$

i.e., an estimate of $f_1'(\theta)g_1$ as above. Moreover,

$$\left\|f_1'(\theta)g_1\right\|_{L^{q_0}} \le C\left\|\theta\right\|_{L^{q_0}}\left\|g_1\right\|_{L^\infty} \le C\left\|\theta\right\|_{L^{q_0}}.$$

Inserting the above inequalities into (6.21), we get with (4.4), (4.19) and an interpolation estimate for ω that

$$\begin{aligned}
\big\|\Lambda^{s-\alpha}\left(f_1'(\theta)g_1\omega\right)\big\|_{L^2}^2 &\le C\left\|\theta\right\|_{L^{q_0}}^2 \|\omega\|^2_{H^{s-\alpha,\frac{2q_0}{q_0-2}}} + C\|\omega\|^2_{L^{\frac{2}{\alpha-\eta}}}\|\theta\|_{H^s}^2 \\
&\le \frac{\kappa}{4}\|\Lambda^s\omega\|_{L^2}^2 + C\|\omega\|_{L^2}^2.
\end{aligned} \tag{6.22}$$

Furthermore, it is easy to see that

$$\big(\rho*\omega,\, \Lambda^{2(s-\alpha)}\omega\big) \le \|\rho\|_{L^1}\big\|\Lambda^{s-\alpha}\omega\big\|_{L^2}^2.$$

For the nonlinear terms involving $u_t\cdot\nabla\theta$ and $u\cdot\nabla\omega$ on the right hand side of (6.19) we consider two cases; actually, it suffices to consider $u_t\cdot\nabla\theta$ only since the other term satisfies similar estimates.

Case 1. $s > 2\alpha - 1$.
By Hölder's and Young's inequalities we get that

$$\begin{aligned}\left|\int_{\mathbb{R}^2} (u_t \cdot \nabla\theta)\,\Lambda^{2(s-\alpha)}\omega\,\mathrm{d}x\right| &\le \|\Lambda^{s-2\alpha}\,(u_t\cdot\nabla\theta)\,\|_{L^2}\|\Lambda^s\omega\|_{L^2}\\ &\le C\|\Lambda^{s-2\alpha+1}\,(u_t\theta)\,\|_{L^2}^2 + \frac{\kappa}{4}\|\Lambda^s\omega\|_{L^2}^2. \end{aligned}\tag{6.23}$$

As to $\|\Lambda^{s-2\alpha+1}\,(u_t\theta)\,\|_{L^2}$, we apply Lemma 2.4 and (1.4), (2.3) to get with the exponents ϵ_0, $\tilde{s}$ that

$$\begin{aligned}\|\Lambda^{s-2\alpha+1}\,(u_t\theta)\,\|_{L^2}^2 &\le C\left(\|\Lambda^{s-2\alpha+1}u_t\|^2_{L^{\frac{2q_0}{q_0-2}}}\|\theta\|^2_{L^{q_0}} + \|u_t\|^2_{L^{\frac{2}{2\alpha-1-2\epsilon_0}}}\|\Lambda^{s-2\alpha+1}\theta\|^2_{L^{\frac{1}{1-\alpha+\epsilon_0}}}\right)\\ &\le C\left(\|\Lambda^{s-2\alpha+1+\frac{2}{q_0}}\omega\|_{L^2}^2\|\theta\|^2_{L^{q_0}} + \|\omega\|^2_{L^{\frac{2}{2\alpha-1-2\epsilon_0}}}\|\theta\|^2_{H^s}\right)\\ &\le C\left(\|\Lambda^{s-2\alpha+1+\frac{2}{q_0}}\omega\|_{L^2}^2 + \|\Lambda^{2\tilde{s}}\omega\|_{L^2}^2\right). \end{aligned}\tag{6.24}$$

Here we used the embeddings $\dot{H}^{2\tilde{s}} \hookrightarrow L^{\frac{2}{2\alpha-1-2\epsilon_0}}$, $H^s \hookrightarrow H^{s-2\alpha+1,\frac{1}{1-\alpha+\epsilon_0}}$ and (4.4), (4.19). Since $s-\alpha < s-2\alpha+1+\frac{2}{q_0} < s$ and $0 < 2\tilde{s} < s$, by interpolation and Young's inequalities, (6.19)–(6.24) lead to the estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda^{s-\alpha}\omega\|_{L^2}^2 + \kappa\|\Lambda^s\omega\|_{L^2}^2 \le C\left(\|\Lambda^{s-\alpha}\omega\|_{L^2}^2 + \|\omega\|_{L^2}^2\right), \quad t \ge T_2$$

By using (6.1) and (6.2) in Proposition 6.1 and the Uniform Gronwall Lemma, we obtain the result (6.11) for $s > 2\alpha - 1$.

Case 2. $s \le 2\alpha - 1$.
In this case, we have $s-\alpha < 2(s-\alpha)+1 \le s$. Since $\|\theta\|_{L^\infty} \le c\|\theta\|_{H^{2\alpha}}$, we see that

$$\begin{aligned}\left|\int_{\mathbb{R}^2} (u_t \cdot \nabla\theta)\,\Lambda^{2(s-\alpha)}\omega\,\mathrm{d}x\right| &\le \|\Lambda^{-1}\,(u_t\cdot\nabla\theta)\,\|_{L^2}\|\Lambda^{2s-2\alpha+1}\omega\|_{L^2}\\ &\le C\|u_t\theta\|_{L^2}\|\Lambda^{2s-2\alpha+1}\omega\|_{L^2}\\ &\le C\|u_t\|_{L^2}\|\theta\|_{L^\infty}\|\Lambda^{2s-2\alpha+1}\omega\|_{L^2}\\ &\le C\|\omega\|_{L^2}^2\|\theta\|^2_{H^{2\alpha}} + C\|\Lambda^{s-\alpha}\omega\|_{L^2}^2 + \frac{\kappa}{2}\|\Lambda^s\omega\|_{L^2}^2. \end{aligned}\tag{6.25}$$

Then, by (6.10) and (6.25), (6.19) yields the estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda^{s-\alpha}\omega\|_{L^2}^2 + \kappa\|\Lambda^s\omega\|_{L^2}^2 \le C\left(\|\Lambda^{s-\alpha}\omega\|_{L^2}^2 + \|\omega\|_{L^2}^2 + \|\theta\|^2_{H^{2\alpha}}\right), \quad \forall\, t \ge T_2 + 2.$$

Finally, by (6.1), (6.8) and the Uniform Gronwall Lemma, we get the result (6.11) for $s \le 2\alpha - 1$.

Now the proof of this proposition is completed. □

Proof of Theorem 2.7 (existence of the attractor). By Proposition 4.3 $\{S(t)\}_{t\ge0}$ has a bounded absorbing set in H^s; we denote it by B_0. In the following, we check the asymptotic compactness of the semigroup, *i.e.*, we have to show that

$$\text{for any } \ \{\theta_n^0\}_{n\ge1} \subset B_0 \text{ and } t_n \to \infty, \ \ \{\theta_n(t_n)\}_{n\ge1} \text{ is precompact in } H^s,$$

where $\theta_n(t_n) = S(t_n)\theta_n^0$.

First, we consider two arbitrary solutions θ_1, θ_2 of problem (1.1) with initial data $\theta_1^0, \theta_2^0 \in B_0$, respectively, and their difference $\delta\theta := \theta_1 - \theta_2$ in $H^{\max(s,\alpha)}$. For the proof we consider two cases, namely $s \le \alpha$ and $s > \alpha$.

Case 1. $0 < s \le \alpha$

By (1.4), we have corresponding velocities u_1 and u_2 and set $\delta u = u_1 - u_2$. From $(1.1)_1$, one has

$$\partial_t\delta\theta + u_1\cdot\nabla\delta\theta + \kappa(-\Delta)^\alpha\delta\theta = -\delta u\cdot\nabla\theta_2 + F(x,\theta_1) - F(x,\theta_2). \tag{6.26}$$

Taking the inner product of (6.26) with $\delta\theta$, we obtain

$$\kappa\|\Lambda^\alpha\delta\theta\|_{L^2}^2 \le -\left(\partial_t\delta\theta,\delta\theta\right) - \left(\delta u\cdot\nabla\theta_2,\delta\theta\right) + C\|\delta\theta(t)\|_{L^2}^2 \tag{6.27}$$

since $(u_1\cdot\nabla\delta\theta,\delta\theta)=0$ and due to the properties of g_1, g_2, ρ and f_1. We deal with each term on the right hand side of (6.27). For the first term, by Hölder's inequality and (6.10) in Proposition 6.3, we have for $t\ge T_2+2$

$$\left|\left(\partial_t\delta\theta,\delta\theta\right)\right| \le \|\partial_t\delta\theta(t)\|_{L^2}\|\delta\theta(t)\|_{L^2} \le C\|\delta\theta(t)\|_{L^2}. \tag{6.28}$$

Applying Hölder's and Young's inequalities, we see that

$$\begin{aligned}\left|\left(\delta u\cdot\nabla\theta_2,\delta\theta\right)\right| &\le \|\Lambda^{-\alpha+\epsilon_0}\left(\delta u\cdot\nabla\theta_2\right)\|_{L^2}\|\Lambda^{\alpha-\epsilon_0}\delta\theta(t)\|_{L^2}\\ &\le C\|\Lambda^{1-\alpha+\epsilon_0}\left(\delta u\,\theta_2\right)\|_{L^2}^2 + C\|\delta\theta(t)\|_{L^2}^2 + \frac{\kappa}{4}\|\Lambda^\alpha\delta\theta(t)\|_{L^2}^2.\end{aligned}$$

Recalling the exponents ϵ_0 and $\tilde{s}=1-\alpha+\epsilon_0$ from the proof of Proposition 6.3, we get as in (6.16)

$$\begin{aligned}\|\Lambda^{\tilde{s}}(\delta u\,\theta_2)\|_{L^2}^2 &\le C\|\Lambda^{\alpha-\epsilon_0}\delta\theta\|_{L^2}^2\left(\|\theta_2\|_{L^2\cap L^{q_0}}^2 + \|\Lambda^{2\widetilde{s}}\theta_2\|_{L^2}^2\right)\\ &\le C\|\Lambda^{\alpha-\epsilon_0}\delta\theta\|_{L^2}^2\|\theta_2\|_{H^s}^2\\ &\le C\|\delta\theta\|_{L^2}^2 + \frac{\kappa}{4}\|\Lambda^\alpha\delta\theta\|_{L^2}^2,\end{aligned}$$

where $2\widetilde{s}<s$ and $H^s\hookrightarrow L^{q_0}$. Thus the second term on the right hand side of (6.27) is bounded by

$$\left|\left(\delta u\cdot\nabla\theta_2,\delta\theta\right)\right| \le C\|\delta\theta\|_{L^2}^2 + \frac{\kappa}{2}\|\Lambda^\alpha\delta\theta\|_{L^2}^2. \tag{6.29}$$

Inserting (6.28), (6.29) into (6.27) and using (6.9), we obtain

$$\|\delta\theta(t)\|_{H^\alpha}^2 \le C\|\delta\theta(t)\|_{L^2}^2 + C\|\delta\theta(t)\|_{L^2} \quad \text{for any } t\ge T_4. \tag{6.30}$$

Now, we choose $K=\max\{K_1,K_3\}$ where K_1, K_3 are defined in Propositions 5.1 and 5.4, let $\Omega_K=\{x\in\mathbb{R}^2: |x|\le K\}$ and assume $t_n>t_m>T_5$. Applying (6.30) to the solutions θ_m and $\theta_n(\cdot+t_n-t_m)$ with initial values θ_m^0 and $\theta_n(t_n-t_m)=S(t_n-t_m)\theta_n^0$, respectively, evaluated at $t=t_m$, it follows that

$$\begin{aligned}\|\theta_n(t_n)-\theta_m(t_m)\|_{H^\alpha}^2 &\le C\left(\|\theta_n(t_n)-\theta_m(t_m)\|_{L^2(\Omega_K)}^2 + \|\theta_n(t_n)-\theta_m(t_m)\|_{L^2(\Omega_K^c)}^2\right)\\ &\quad + C\left(\|\theta_n(t_n)-\theta_m(t_m)\|_{L^2(\Omega_K)} + \|\theta_n(t_n)-\theta_m(t_m)\|_{L^2(\Omega_K^c)}\right).\end{aligned} \tag{6.31}$$

Since $H^s(\Omega_K)\hookrightarrow L^2(\Omega_K)$ is compact, (a subsequence of) $\{\theta_n(t_n)\}_{n\ge1}$ is a Cauchy sequence in $L^2(\Omega_K)$. On the other hand, in view of (5.2) in Proposition 5.1, for any $\varepsilon>0$, there exists $N_\varepsilon\in\mathbb{N}$ such that for all $m,n>N_\varepsilon$, by (6.31)

$$\|\theta_n(t_n)-\theta_m(t_m)\|_{H^\alpha}^2 \le \varepsilon. \tag{6.32}$$

Therefore, the semigroup $S(t)$ is asymptotically compact in H^α, and it is easy to that $S(t)$ is also asymptotically compact in H^s for $s\le\alpha$.

Case 2. $s>\alpha$

To this end, taking the inner product of $\Lambda^{2(s-\alpha)}\delta\theta$ with (6.26), we obtain

$$\kappa\|\Lambda^s\delta\theta\|_{L^2}^2 = -\left(\partial_t\delta\theta+\delta u\cdot\nabla\theta_2+u_1\cdot\nabla\delta\theta-F(x,\theta_1)+F(x,\theta_2),\,\Lambda^{2(s-\alpha)}\delta\theta\right). \tag{6.33}$$

For the first term on the right hand side of (6.33), we get from (6.11) that for $t\ge T_2+3$

$$\left|\left(\partial_t\delta\theta,\Lambda^{2(s-\alpha)}\delta\theta\right)\right| \le \|\Lambda^{s-\alpha}\left(\partial_t\delta\theta\right)(t)\|_{L^2}\|\Lambda^{s-\alpha}\delta\theta(t)\|_{L^2} \le C\|\Lambda^{s-\alpha}\delta\theta(t)\|_{L^2}. \tag{6.34}$$

Since $s > \alpha$ implies $s > 2\alpha - 1$, similar to (6.23)–(6.24), we have

$$\begin{aligned}
\left|\left(\delta u\cdot\nabla\theta_2, \Lambda^{2(s-\alpha)}\delta\theta\right)\right| &\le \|\Lambda^{s-2\alpha}\left(\delta u\cdot\nabla\theta_2\right)\|_{L^2}\|\Lambda^s\delta\theta\|_{L^2}\\
&\le C\left(\|\Lambda^{s-2\alpha+1}\delta u\|_{L^{\frac{2q_0}{q_0-2}}}\|\theta_2\|_{L^{q_0}} + \|\delta u\|_{L^{\frac{2}{2\alpha-1-2\epsilon_0}}}\|\Lambda^{s-2\alpha+1}\theta_2\|_{L^{\frac{1}{1-\alpha+\epsilon_0}}}\right)\|\Lambda^s\delta\theta\|_{L^2}\\
&\le C\left(\|\Lambda^s\delta\theta\|_{L^2}^{\eta}\|\Lambda^{s-\alpha}\delta\theta\|_{L^2}^{1-\eta}\|\theta_2\|_{L^{q_0}} + \|\Lambda^s\delta\theta\|_{L^2}^{\frac{2\tilde{s}}{s}}\|\delta\theta\|_{L^2}^{1-\frac{2\tilde{s}}{s}}\|\theta_2\|_{H^s}\right)\|\Lambda^s\delta\theta\|_{L^2}\\
&\le C\|\Lambda^{s-\alpha}\delta\theta\|_{L^2}^2\|\theta_2\|_{L^{q_0}}^{\frac{2}{1-\eta}} + C\|\delta\theta\|_{L^2}^2\|\theta_2\|_{H^s}^{\frac{2s}{s-2\tilde{s}}} + \frac{\kappa}{4}\|\Lambda^s\delta\theta\|_{L^2}^2, \qquad (6.35)
\end{aligned}$$

and, by analogy,

$$\left|\left(u_1\cdot\nabla\delta\theta, \Lambda^{2(s-\alpha)}\delta\theta\right)\right| \le C\|\Lambda^{s-\alpha}\delta\theta\|_{L^2}^2\|\theta_1\|_{L^{q_0}}^{\frac{2}{1-\eta}} + C\|\delta\theta\|_{L^2}^2\|\theta_1\|_{H^s}^{\frac{2s}{s-2\tilde{s}}} + \frac{\kappa}{4}\|\Lambda^s\delta\theta\|_{L^2}^2. \qquad (6.36)$$

Finally, by (3.3) in Proposition 3.1 and similar to (6.21)–(6.22), we have

$$\begin{aligned}
\left|\left(F(x,\theta_1) - F(x,\theta_2), \Lambda^{2(s-\alpha)}\delta\theta\right)\right| &= \left|\left(g_1[f_1(\theta_1) - f_1(\theta_2)] + \rho * \delta\theta, \Lambda^{2(s-\alpha)}\delta\theta\right)\right|\\
&\le \|\Lambda^{s-\alpha}\left(g_1[f_1(\theta_1) - f_1(\theta_2)]\right)\|_{L^2}\|\Lambda^{s-\alpha}\delta\theta\|_{L^2} + C\|\Lambda^{s-\alpha}\delta\theta\|_{L^2}^2\\
&\le \frac{\kappa}{4}\|\Lambda^s\delta\theta\|_{L^2}^2 + C\|\delta\theta\|_{L^2}^2. \qquad (6.37)
\end{aligned}$$

Therefore, inserting (6.34)–(6.37) into (6.33), we have

$$\begin{aligned}
\|\Lambda^s\delta\theta\|_{L^2}^2 \le C\|\Lambda^{s-\alpha}\delta\theta\|_{L^2} &+ C\|\Lambda^{s-\alpha}\delta\theta\|_{L^2}^2\left(\|\theta_1\|_{L^{q_0}}^{\frac{2}{1-\eta}} + \|\theta_2\|_{L^{q_0}}^{\frac{2}{1-\eta}}\right)\\
&+ C\|\delta\theta\|_{L^2}^2\left(1 + \|\theta_1\|_{H^s}^{\frac{2s}{s-\tilde{s}}} + \|\theta_2\|_{H^s}^{\frac{2s}{s-\tilde{s}}}\right).
\end{aligned}$$

By (4.4) and (4.19), one has for $t \ge T_4$

$$\|\delta\theta(t)\|_{H^s}^2 \le C\|\delta\theta(t)\|_{H^{s-\alpha}} + C\|\delta\theta(t)\|_{H^{s-\alpha}}^2 + C\|\delta\theta(t)\|_{L^2},$$

and, by $s > \alpha$ and Young's inequality, we conclude that

$$\|\delta\theta(t)\|_{H^s}^2 \le C\|\delta\theta(t)\|_{L^2}^2 + C\|\delta\theta(t)\|_{L^2}^{\frac{2\alpha}{s+\alpha}} \text{ for any } t \ge T_4. \qquad (6.38)$$

Thus, similarly to (6.31) and (6.32), we prove that $S(t)$ is asymptotically compact in H^s for $s > \alpha$.

Now the existence of a global attractor $\mathcal{A}$ in H^s is proved. □

7. Dimension of the Attractor

In this section, we pay attention to the finite dimensionality of the global attractor $\mathcal{A}$ obtained in Section 6. The general theory developed by Constantin, Foias and Temam [13] cannot be applied since the solution operator for the linearized flowin the whole space $\mathbb{R}^2$ is not compact. Therefore, we use the method introduced by Ghidaglia and Temam in [21], see also Temam [41, Chapter V, 3.3-3.4]. To this aim, we use the Hausdorff and fractal dimensions of a compact set $\mathscr{C} \subset \mathscr{X}$, denoted by $d_H^{\mathscr{X}}(\mathscr{C})$ and $d_f^{\mathscr{X}}(\mathscr{C})$, respectively (see [41] for detail), where $\mathscr{X}$ is a metric space. It is clear that $d_H^{\mathscr{X}}(\mathscr{C}) \le d_f^{\mathscr{X}}(\mathscr{C})$.

Lemma 7.1. *Let X, Y be metric spaces, $\mathscr{C} \subset \mathscr{X}$ be a compact subset, and $\mathcal{L} : \mathscr{C} \mapsto Y$ be μ-Hölder continuous on $\mathscr{C}$. Then*

$$d_f^Y(\mathcal{L}(\mathscr{C})) \le \frac{1}{\mu}\, d_f^X(\mathscr{C}),$$

$$d_H^Y(\mathcal{L}(\mathscr{C})) \le \frac{1}{\mu}\, d_H^X(\mathscr{C}).$$

In particular, the fractal dimension does not increase under a Lipschitz continuous mapping.

Lemma 7.1 for d_f is found as Lemma 1.3 in [33]; the case d_H is proved similarly. Next, we prove the following proposition which is crucial to get the dimension estimates of the attractor.

Proposition 7.2. *Let $\theta_1^0, \theta_2^0 \in \mathcal{A}$, and let $\theta_1(t)$ and $\theta_2(t)$ be the solutions to (1.1) with initial data θ_1^0 and θ_2^0, respectively. Then there exists $C > 0$ such that for every $t > 0$*

$$\|\theta_1(t) - \theta_2(t)\|_{L^2}^2 \le e^{Ct} \|\theta_1^0 - \theta_2^0\|_{L^2}^2. \tag{7.1}$$

Proof. Define $\delta\theta = \theta_1 - \theta_2$. Then, similarly to (6.26), we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\delta\theta\|_{L^2}^2 + \kappa\|\Lambda^\alpha \delta\theta\|_{L^2}^2 = -\int_{\mathbb{R}^2} \delta u \cdot \nabla\theta_2 \, \delta\theta \, \mathrm{d}x + \int_{\mathbb{R}^2} \left[(f_1(\theta_1) - f_1(\theta_2))\, g_1(x) + \rho * \delta\theta \right] \delta\theta \, \mathrm{d}x.$$

In view of (6.29) and the conditions of f_1, g_1 and ρ, it is easy to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\delta\theta\|_{L^2}^2 + \kappa\|\Lambda^\alpha \delta\theta\|_{L^2}^2 \le C\|\delta\theta\|_{L^2}^2.$$

Thus the result of this proposition follows from Gronwall's Lemma. □

Since $S(t)\mathcal{A} = \mathcal{A}$, by using (6.30) and (6.38), we have for $\theta_1, \theta_2 \in \mathcal{A}$

$$\|\theta_1(t) - \theta_2(t)\|_{H^s}^2 \le C\|\theta_1(t) - \theta_2(t)\|_{L^2}^2 + C\|\theta_1(t) - \theta_2(t)\|_{L^2}^{\frac{2\alpha}{\max\{s,\alpha\}+\alpha}}, \tag{7.2}$$

and by (7.1), we conclude that

$$\|\theta_1(t) - \theta_2(t)\|_{H^s}^2 \le Ce^{Ct}\Big(\|\theta_1^0 - \theta_2^0\|_{L^2}^2 + \|\theta_1^0 - \theta_2^0\|_{L^2}^{\frac{2\alpha}{\max\{s,\alpha\}+\alpha}} \Big).$$

This means that $S(t)$ is a Hölder continuous mapping from $X = L^2(\mathbb{R}^2)$ to $Y = H^s(\mathbb{R}^2)$. Applying Lemma 7.1 and $S(t)\mathcal{A} = \mathcal{A}$ again, we see that

$$d_f^{H^s}(\mathcal{A}) \le C(s,\alpha)\, d_f^{L^2}(\mathcal{A}) \quad \text{and} \quad d_H^{H^s}(\mathcal{A}) \le C(s,\alpha)\, d_H^{L^2}(\mathcal{A}). \tag{7.3}$$

So, instead of estimating the dimension of $\mathcal{A}$ in the space $H^s(\mathbb{R}^2)$, we estimate below its dimension in the simpler space $L^2(\mathbb{R}^2)$, and write $d_H(\mathcal{A})$ and $d_f(\mathcal{A})$ to denote the dimensions of $\mathcal{A}$ in $L^2(\mathbb{R}^2)$ for simplicity.

To this end we start with some necessary preparation. For given $\theta^0 \in H^s(\mathbb{R}^2)$ and corresponding solution $\theta(t) = S(t)\theta^0$, $t \ge 0$, of (1.1), we see that the linearized flow $\mathbb{\theta}$ around θ satisfies the equation

$$\begin{cases} \mathbb{\theta}_t + u \cdot \nabla\mathbb{\theta} + \kappa(-\Delta)^\alpha\mathbb{\theta} &= -\mathbb{u} \cdot \nabla\theta + f_1'(\theta) g_1(x)\mathbb{\theta} + \rho * \mathbb{\theta} \\ \mathbb{\theta}(x,0) &= \zeta \end{cases} \tag{7.4}$$

where $\mathbb{u} = \mathcal{R}^\perp\mathbb{\theta}$. As for the nonlinear problem, one can show that for given $\zeta \in L^2(\mathbb{R}^2)$, there exists a unique solution $\mathbb{\theta}$ of (7.4) such that

$$\mathbb{\theta} \in L^\infty(0,T; L^2(\mathbb{R}^2)) \cap L^2(0,T; H^\alpha(\mathbb{R}^2)) \ \text{ for any } T > 0.$$

Let $L(t;\theta^0)$ denote the corresponding (linear) solution operator to (7.4) for fixed $\theta(t) = S(t)\theta^0$, *i.e.*,

$$L(t;\theta^0) : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2), \quad \zeta \mapsto L(t;\theta^0)\zeta = \mathbb{\theta}(t).$$

We will prove that $L(t;\theta^0)$ is bounded (see Proposition 7.6 below) and that $\{S(t)\}_{t\ge 0}$ is uniformly differentiable on $\mathcal{A}$ in the following sense:

$$\lim_{\varepsilon\to 0} \sup_{\substack{\theta^0, \widetilde{\theta}^0 \in \mathcal{A} \\ 0 < \|\theta^0 - \widetilde{\theta}^0\|_{L^2} \le \varepsilon}} \frac{\|S(t)\theta^0 - S(t)\widetilde{\theta}^0 - L(t;\theta^0)(\theta^0 - \widetilde{\theta}^0)\|_{L^2}}{\|\theta^0 - \widetilde{\theta}^0\|_{L^2}} = 0, \tag{7.5}$$

see Proposition 7.7 below.

We write (7.4) as

$$\mathbb{\theta}_t = \mathscr{F}'(\theta)\mathbb{\theta} \equiv -u \cdot \nabla\mathbb{\theta} - \kappa(-\Delta)^\alpha\mathbb{\theta} - \mathbb{u} \cdot \nabla\theta + f_1'(\theta) g_1(x)\mathbb{\theta} + \rho * \mathbb{\theta}, \tag{7.6}$$

where $\mathscr{F}'(\theta)$ is the Fréchet derivative of the nonlinear operator $\mathscr{F}$ at θ defined by the formulation of QGE as the fixed point problem $\theta_t = \mathscr{F}(\theta)$. Then the numbers $q_m, m \in \mathbb{N}$, are defined by

$$q_m = \limsup_{t\to\infty} \sup_{\theta^0\in\mathcal{A}} \sup_{\substack{\zeta_i\in L^2 \\ \|\zeta_i\|_{L^2}\le 1 \\ i=1,\dots,m}} \frac{1}{t}\int_0^t \mathrm{Tr}\big[\mathscr{F}'(S(\tau)\theta^0)\circ Q_m(\tau)\big]\,\mathrm{d}\tau, \tag{7.7}$$

where $Q_m(\tau) = Q_m(\tau;\theta^0,\zeta_1,\dots,\zeta_m)$ is the orthogonal projector in $L^2(\mathbb{R}^2)$ onto the subspace spanned by $L(\tau;\theta^0)\zeta_1,\dots,L(\tau;\theta^0)\zeta_m$. The trace (denoted by Tr) of $F'(S(\tau)\theta^0)\circ Q_m(\tau)$ in (7.7) is defined for *a.a.* τ, see [41, Chapter V, 2.3]. If $q_m < 0$ for some $m \in \mathbb{N}$, then the global attractor has finite Hausdorff and fractal dimensions, respectively, estimated by

$$d_H(\mathcal{A}) \le m, \tag{7.8}$$

$$d_f(\mathcal{A}) \le m\left(1 + \max_{1\le j\le m-1}\frac{(q_j)_+}{|q_m|}\right), \tag{7.9}$$

see [41, p. 291].

Before going further, we recall some important results which are necessary in our proof.

Lemma 7.3 (Lieb-Thirring inequality, see [4, 29])**.** *Let $\varphi_1,\dots,\varphi_N \in H^1(\mathbb{R}^d)$ be an orthonormal family of vectors in $L^2(\mathbb{R}^d)$. Then for*

$$\rho_\varphi(x) := \sum_{i=1}^N |\varphi_i(x)|^2$$

the following estimate holds:

$$\int_{\mathbb{R}^d} \rho_\varphi^{1+2/d}\,\mathrm{d}x \le C_d \sum_{i=1}^N \int_{\mathbb{R}^d} |\nabla\varphi_i|^2\,\mathrm{d}x, \tag{7.10}$$

where C_d depends only on d.

Lemma 7.4 (Fractional Lieb-Thirring inequality, see inequality (9) in [32])**.** *For all $d \ge 1$ and $s > 0$, there exists a constant $C > 0$ depending only on d and s such that for all $N \in \mathbb{N}$ and for every L^2-normalized and anti-symmetric function $\Psi \in H^s(\mathbb{R}^{dN})$,* i.e., *$\|\Psi\|_{L^2(\mathbb{R}^{dN})} = 1$ and*

$$\Psi(x_1,\dots,x_i,\dots,x_j,\dots,x_N) = -\Psi(x_1,\dots,x_j,\dots,x_i,\dots,x_N), \forall\ i \ne j,$$

there holds

$$\left(\Psi, \sum_{i=1}^N (-\Delta_i)^s\Psi\right) \ge C\int_{\mathbb{R}^d} \rho_\Psi(x)^{1+2s/d}\,\mathrm{d}x, \tag{7.11}$$

where $\rho_\Psi(x)$ is defined by

$$\rho_\Psi(x) := \sum_{j=1}^N \int_{\mathbb{R}^{d(N-1)}} |\Psi(x_1,\dots,x_{j-1},x,x_{j+1},\dots,x_N)|^2 \prod_{i\ne j}\mathrm{d}x_i.$$

Remark 7.5. ([30]) Let $\varphi_1,\dots,\varphi_N \in H^s(\mathbb{R}^d)$ be an orthonormal family in $L^2(\mathbb{R}^d)$ and

$$\Psi(x_1,\dots,x_N) = (N!)^{-1/2}\det\{\varphi_i(x_j)\}_{i,j=1}^N.$$

Then Ψ is an L^2-normalized and anti-symmetric function, it holds

$$\rho_\Psi(x) = \sum_{i=1}^N |\varphi_i(x)|^2 =: \rho_\varphi(x), \qquad \left(\Psi, \sum_{i=1}^N (-\Delta_i)^s\Psi\right) = \sum_{i=1}^N \int_{\mathbb{R}^d} |\Lambda^s\varphi_i|^2\,\mathrm{d}x,$$

and the results in (7.11) imply that

$$\int_{\mathbb{R}^d} \rho_\varphi^{1+2s/d}\,\mathrm{d}x \le C_{d,s} \sum_{i=1}^N \int_{\mathbb{R}^d} |\Lambda^s \varphi_i|^2\,\mathrm{d}x. \tag{7.12}$$

This result is reduced to (7.10) when $s = 1$.

Proposition 7.6. *Let θ be the unique solution of* (1.1) *with the initial data $\theta^0 \in H^s(\mathbb{R}^2), s > 2(1-\alpha)$. Then for every $\zeta \in L^2(\mathbb{R}^2)$, the linear problem* (7.4) *has a unique solution Θ such that*

$$\Theta \in L^\infty(0,T;L^2(\mathbb{R}^2)) \cap L^2(0,T;H^\alpha(\mathbb{R}^2)) \text{ for any } T > 0.$$

Proof. We only prove the *a priori* estimates of the solution; the other assertions are achieved by standard methods. Taking the inner product of $(7.4)_1$ with Θ in L^2 and using the conditions on f_1, g_1 and ρ, it follows that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Theta\|_{L^2}^2 + \kappa\|\Lambda^\alpha\Theta\|_{L^2}^2 &= -\int_{\mathbb{R}^2} \mathbb{u}\cdot\nabla\theta\,\Theta\,\mathrm{d}x + \int_{\mathbb{R}^2} f_1'(\theta)g_1(x)|\Theta|^2\,\mathrm{d}x + \int_{\mathbb{R}^2} \rho*\Theta\,\Theta\,\mathrm{d}x \\ &\le -\int_{\mathbb{R}^2} \mathbb{u}\cdot\nabla\theta\,\Theta\,\mathrm{d}x + C\|\Theta\|_{L^2}^2. \end{aligned}\tag{7.13}$$

The trilinear term $-\int_{\mathbb{R}^2} \mathbb{u}\cdot\nabla\theta\ \omega\,\mathrm{d}x$ is controlled as the term $\int_{\mathbb{R}^2} u_t\cdot\nabla\theta\ \omega\,\mathrm{d}x$, see (6.15), (6.16), and admits the bound

$$-\int_{\mathbb{R}^2} \mathbb{u}\cdot\nabla\theta\ \Theta\,\mathrm{d}x \le \frac{\kappa}{2}\|\Lambda^\alpha\Theta\|_{L^2}^2 + C\|\Theta\|_{L^2}^2. \tag{7.14}$$

Inserting (7.14) into (7.13), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Theta\|_{L^2}^2 + \kappa\|\Lambda^\alpha\Theta\|_{L^2}^2 \le C\|\Theta\|_{L^2}^2. \tag{7.15}$$

The Gronwall inequality leads to $\Theta \in L^\infty(0,T;L^2(\mathbb{R}^2)) \cap L^2(0,T;H^\alpha(\mathbb{R}^2))$ for any $T > 0$. Finally, since (7.4) is a homogeneous linear equation, the same *a priori* estimate yields uniqueness. ☐

Proposition 7.7. *$\{S(t)\}_{t\ge0}$ is uniformly differentiable on $\mathcal{A}$ in the sense of* (7.5).

Proof. Let $\theta(t)$ and $\widetilde{\theta}(t)$ denote the solution of (1.1) with initial data θ^0 and $\widetilde{\theta}^0$ in $\mathcal{A}$, respectively, *i.e.*, $\theta(t)$ and $\widetilde{\theta}(t)$ satisfy

$$\begin{aligned}&\theta_t + u\cdot\nabla\theta + \kappa(-\Delta)^\alpha\theta = F(x,\theta) \quad \text{where } u = \mathcal{R}^\perp\theta,\\ &\widetilde{\theta}_t + \widetilde{u}\cdot\nabla\widetilde{\theta} + \kappa(-\Delta)^\alpha\widetilde{\theta} = F(x,\widetilde{\theta}) \quad \text{where } \widetilde{u} = \mathcal{R}^\perp\widetilde{\theta}.\end{aligned}$$

We denote by $\delta\theta = \theta - \widetilde{\theta}$, $\delta u = u - \widetilde{u}$, and consider $\Theta(t) = L(t,\theta^0)(\theta^0 - \widetilde{\theta}^0)$, the solution of (7.4) with initial value $\Theta(x,0) = \theta^0 - \widetilde{\theta}^0$. Then, after some elementary algebra, the equation satisfied by $\vartheta = \delta\theta - \Theta = \theta - \widetilde{\theta} - \Theta$ (and $\mathbb{U} = \mathcal{R}^\perp\vartheta$) reads as

$$\vartheta_t + u\cdot\nabla\vartheta + \kappa(-\Delta)^\alpha\vartheta + \mathbb{U}\cdot\nabla\theta + \delta u\cdot\nabla\delta\theta = g_1(x)\left(f_1(\theta) - f_1(\widetilde{\theta}) - f'(\theta)\Theta\right) + \rho*\vartheta. \tag{7.16}$$

Taking the inner product with ϑ we get

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\vartheta\|_{L^2}^2 + \kappa\|\Lambda^\alpha\vartheta\|_{L^2}^2 &= -\int_{\mathbb{R}^2}(\mathbb{U}\cdot\nabla\theta)\vartheta\,\mathrm{d}x - \int_{\mathbb{R}^2}(\delta u\cdot\nabla\delta\theta)\vartheta\,\mathrm{d}x \\ &\quad + \int_{\mathbb{R}^2}\Big(g_1(x)\big(f_1(\theta) - f_1(\widetilde{\theta}) - f_1'(\theta)\Theta\big)\,\vartheta + \rho*\vartheta\,\vartheta\Big)\,\mathrm{d}x.\end{aligned}\tag{7.17}$$

Similarly to (7.14) and by the uniform boundedness of $\theta, \tilde\theta \in H^s(\mathbb{R}^2)$ on $(0,\infty)$, the first term on the right hand side of (7.17) is bounded by

$$-\int_{\mathbb{R}^2} \mathbb{U}\cdot\nabla\theta\ \vartheta\,\mathrm{d}x \le \frac{\kappa}{4}\|\Lambda^\alpha\vartheta\|_{L^2}^2 + C\|\vartheta\|_{L^2}^2. \tag{7.18}$$

For the second term we start with the estimate $\left|\int_{\mathbb{R}^2}(\delta u\cdot\nabla\delta\theta)\vartheta\,\mathrm{d}x\right| \le C\|\Lambda^{\tilde s}(\delta u\,\delta\theta)\|_{L^2}^2 + C\|\Lambda^{\alpha-\epsilon_0}\vartheta\|_{L^2}^2$ where $\tilde s = 1-\alpha+\epsilon_0$, *cf.* the proof of Proposition 6.3, and estimate $\Lambda^{\tilde s}(\delta u\,\delta\theta)$ as in $(6.16)_1$ as follows:

$$\begin{aligned}
\|\Lambda^{\tilde s}(\delta u\,\delta\theta)\|_{L^2}^2 &\le C\left(\|\Lambda^{\tilde s}\delta u\|^2_{L^{\frac{1}{1-\alpha+\epsilon_0}}}\|\delta\theta\|^2_{L^{\frac{2}{2\alpha-1-2\epsilon_0}}} + \|\delta u\|^2_{L^{\frac{2}{1-\alpha+\epsilon_0}}}\|\Lambda^{\tilde s}\delta\theta\|^2_{L^{\frac{2}{\alpha-\epsilon_0}}}\right)\\
&\le C\|\Lambda^{\alpha-\epsilon_0}\delta u\|_{L^2}^2\|\delta\theta\|^2_{\dot H^{2\tilde s}}\\
&\le C\|\Lambda^{\alpha-\epsilon_0}\delta\theta\|_{L^2}^2\|\delta\theta\|_{L^2}^{2\gamma}\|\delta\theta\|_{H^s}^{2(1-\gamma)}
\end{aligned}$$

where $\gamma = \frac{s-2\tilde s}{s}\in(0,1)$. Hence, with $\sigma = \frac{\alpha-\epsilon_0}{\alpha}$, we obtain that

$$-\int_{\mathbb{R}^2}(\delta u\cdot\nabla\delta\theta)\vartheta\,\mathrm{d}x \le C\|\Lambda^\alpha\delta\theta\|_{L^2}^{2\sigma}\|\delta\theta\|_{L^2}^{2(1-\sigma+\gamma)} + \frac{\kappa}{4}\|\Lambda^\alpha\vartheta\|_{L^2}^2 + C\|\vartheta\|_{L^2}^2. \tag{7.19}$$

As to the last term on the right hand side of (7.17), the assumptions on f_1 imply the elementary estimate $|f_1(\theta) - f_1(\widetilde\theta) - f_1'(\theta)\mathbb{\theta}| \le c|\delta\theta|^2 + c|\vartheta|$. Using the embeddings $\dot H^{1/2}(\mathbb{R}^2)\hookrightarrow L^4(\mathbb{R}^2)$, $\dot H^{1/4}(\mathbb{R}^2)\hookrightarrow L^{8/3}(\mathbb{R}^2)$, and interpolation, the integral over $|\delta\theta|^2|\vartheta|$ can be estimated as

$$\begin{aligned}
\int_{\mathbb{R}^2}|\delta\theta|^2|\vartheta|\,\mathrm{d}x &\le C\|\delta\theta\|_{L^{8/3}}^2\|\vartheta\|_{L^4}\\
&\le C\|\Lambda^{1/2}\vartheta\|_{L^2}^2 + C\|\Lambda^{1/4}\delta\theta\|_{L^2}^4\\
&\le \frac{\kappa}{4}\|\Lambda^\alpha\vartheta\|_{L^2}^2 + C\|\vartheta\|_{L^2}^2 + C\|\Lambda^\alpha\delta\theta\|_{L^2}^{1/\alpha}\|\delta\theta\|_{L^2}^{4-1/\alpha}.
\end{aligned}$$

Consequently, the most right hand side term of (7.17) is controlled by

$$\int_{\mathbb{R}^2}\big(g_1\big(f_1(\theta) - f_1(\widetilde\theta) - f_1'(\theta)\mathbb{\theta}\big)\,\vartheta + \rho*\vartheta\,\vartheta\big)\,\mathrm{d}x \le \frac{\kappa}{4}\|\Lambda^\alpha\vartheta\|_{L^2}^2 + C\|\vartheta\|_{L^2}^2 + C\|\Lambda^\alpha\delta\theta\|_{L^2}^{\frac{1}{\alpha}}\|\delta\theta\|_{L^2}^{4-\frac{1}{\alpha}}. \tag{7.20}$$

Combining (7.17)–(7.20), we are led to the estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\vartheta\|_{L^2}^2 + \kappa\|\Lambda^\alpha\vartheta\|_{L^2}^2 \le C\|\vartheta\|_{L^2}^2 + C\|\Lambda^\alpha\delta\theta\|_{L^2}^{2\sigma}\|\delta\theta\|_{L^2}^{2(1-\sigma+\gamma)} + C\|\Lambda^\alpha\delta\theta\|_{L^2}^{\frac{1}{\alpha}}\|\delta\theta\|_{L^2}^{4-\frac{1}{\alpha}}.$$

Note that the last terms on the right-hand side are of the type $C\|\Lambda^\alpha\delta\theta\|_{L^2}^{2\sigma_j}\|\delta\theta\|_{L^2}^{2(1-\sigma_j+\gamma_j)}$, $j=1,2$, with $\sigma_1=\sigma, \gamma_1=\gamma$ and $\sigma_2=\frac{1}{2\alpha}, \gamma_2=1$. Then Gronwall's inequality with $\vartheta(0)=0$ gives

$$\begin{aligned}
\|\vartheta(t)\|_{L^2}^2 &\le Ce^{Ct}\sum_{j=1}^2\int_0^t\|\Lambda^\alpha\delta\theta\|_{L^2}^{2\sigma_j}\|\delta\theta\|_{L^2}^{2(1-\sigma_j+\gamma_j)}\,\mathrm{d}\tau\\
&\le Ce^{Ct}\sum_{j=1}^2\|\delta\theta\|_{L^\infty(0,t;L^2)}^{2\gamma_j}\int_0^t\|\Lambda^\alpha\delta\theta\|_{L^2}^{2\sigma_j}\|\delta\theta\|_{L^2}^{2(1-\sigma_j)}\,\mathrm{d}\tau\\
&\le Ce^{Ct}\sum_{j=1}^2\|\delta\theta\|_{L^\infty(0,t;L^2)}^{2\gamma_j}\left(\int_0^t\|\Lambda^\alpha\delta\theta\|_{L^2}^2\,\mathrm{d}\tau\right)^{\sigma_j}\left(\int_0^t\|\delta\theta\|_{L^2}^2\,\mathrm{d}\tau\right)^{1-\sigma_j}.
\end{aligned} \tag{7.21}$$

To control the norms of $\delta\theta$ in (7.21) we consider the equation

$$\partial_t\delta\theta + (u\cdot\nabla)\delta\theta + \kappa(-\Delta)^\alpha\delta\theta = -\delta u\cdot\nabla\widetilde\theta + F(x,\theta) - F(x,\widetilde\theta),$$

cf. (6.26), which admits the estimate

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\delta\theta\|_{L^2}^2+\kappa\|\Lambda^\alpha\delta\theta\|_{L^2}^2=-\big(\delta u\cdot\nabla\widetilde{\theta},\delta\theta\big)+C\|\delta\theta\|_{L^2}^2\le\frac{\kappa}{2}\|\Lambda^\alpha\delta\theta\|_{L^2}^2+C\|\delta\theta\|_{L^2}^2,$$

cf. (7.14) and also (6.26), (6.27). Absorbing the first term on the right hand side, Gronwall's inequality yields the exponential estimate

$$\|\delta\theta(t)\|_{L^2}^2+\kappa\int_0^t\|\Lambda^\alpha\delta\theta\|_{L^2}^2\,\mathrm{d}\tau\le e^{Ct}\|\delta\theta(0)\|_{L^2}^2. \tag{7.22}$$

Applying (7.22) to (7.21) we see that

$$\|\vartheta(t)\|_{L^2}^2\le\sum_{j=1}^2 Ct^{1-\sigma_j}e^{C_jt}\|\delta\theta(0)\|_{L^2}^{2(\gamma_j+1)},$$

i.e., $\|\vartheta(t)\|_{L^2}\le C(t)\big(\|\delta\theta(0)\|_{L^2}^{\gamma_1+1}+\|\delta\theta(0)\|_{L^2}^{\gamma_2+1}\big)$. Hence, for every $t>0$,

$$\frac{\|\theta(t)-\widetilde{\theta}(t)-\Theta(t)\|_{L^2}}{\|\theta^0-\widetilde{\theta}^0\|_{L^2}}\le C(t)\big(\|\theta^0-\widetilde{\theta}^0\|_{L^2}^{\gamma_1}+\|\theta^0-\widetilde{\theta}^0\|_{L^2}^{\gamma_2}\big)\to 0\quad\text{as}\quad\|\theta^0-\widetilde{\theta}^0\|_{L^2}\to 0.$$

Now the proof of this proposition is finished. □

Proposition 7.8. *Let θ be the unique solution of* (1.1) *with initial data $\theta^0\in H^s(\mathbb{R}^2)$, $s>2(1-\alpha)$. Then there exists a constant $\kappa'>0$ such that*

$$\frac{1}{t}\int_0^t\|\theta\|_{H^\alpha}^2\,\mathrm{d}\tau\le\frac{C}{\kappa'}\left(1+\frac{1}{t}\right)\left(\|\theta^0\|_{L^2}^2+1\right),\quad t>0, \tag{7.23}$$

$$\frac{1}{t}\int_0^t\|\theta\|_{H^{s+\alpha}}^2\,\mathrm{d}\tau\le\frac{1}{\kappa' t}\|\theta^0\|_{H^s}^2+\frac{C}{\kappa'}\left(1+\frac{1}{t}\right)\left(\|\theta^0\|_{L^2}^2+1\right),\quad t>T_1. \tag{7.24}$$

Proof. We recall that an argument as in (4.7)-(4.11) yields a constant $\kappa'>0$ and the estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\theta\|_{L^2}^2+\kappa'\|\theta\|_{H^\alpha}^2\le C\left(\|\theta^0\|_{L^2}^2+1\right) \tag{7.25}$$

based on which (7.23) is obvious. As to (7.24), in view of (4.26), one has

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda^s\theta\|_{L^2}^2+2\kappa\|\Lambda^{s+\alpha}\theta\|_{L^2}^2&\le C\|\theta\|_{H^s}^2+C\big(\|\theta\|_{L^{q_0}}+\|u\|_{L^{q_0}}\big)\|\Lambda^{s+\beta}\theta\|_{L^2}^2+C\|\Lambda^{s-\alpha}g_2\|_{L^2}^2\\&\le\kappa\|\Lambda^{s+\alpha}\theta\|_{L^2}^2++C\Big(1+\|\theta\|_{L^{q_0}}^{\frac{s+\alpha}{\alpha-\beta}}\Big)\|\theta\|_{L^2}^2+C\|\Lambda^{s-\alpha}g_2\|_{L^2}^2.\end{aligned} \tag{7.26}$$

Absorbing the term $\kappa\|\Lambda^{s+\alpha}\theta\|_{L^2}^2$ and using (7.23), the result is proved. □

Proof of Theorem 2.7 (dimension of the attractor). In order to estimate the numbers q_m in (7.7), fix $\theta^0\in\mathcal{A}$ and let $\theta(t)=S(t)\theta^0$ as well as $\Theta_j(t)=L(t;\theta^0)\zeta_j$, where $\zeta_1,\dots,\zeta_m\in L^2(\mathbb{R}^2)$. For fixed $t>0$ let $\{\varphi_1(t),\dots,\varphi_m(t)\}$ be an orthonormal basis in $L^2(\mathbb{R}^2)$ for $\operatorname{span}\{\Theta_1(t),\dots,\Theta_m(t)\}$. Since $\Theta_i(t)\in H^\alpha(\mathbb{R}^2)$ (for at least *a.a.* t), we can assume that $\varphi_j(t)\in H^\alpha(\mathbb{R}^2)$. Then we get with the orthogonal projection $Q_m(\tau)=\sum_j\big(\cdot,\varphi_j(\tau)\big)\varphi_j(\tau)$

$$\begin{aligned}&\operatorname{Tr}\big[\mathscr{F}'(\theta(\tau))\circ Q_m(\tau)\big]\\&\quad=\sum_{j=1}^m\big(\mathscr{F}'(\theta(\tau))\varphi_j,\varphi_j\big)\\&\quad=\sum_{j=1}^m\Big(-u\cdot\nabla\varphi_j-\frac{\kappa}{2}(-\Delta)^\alpha\varphi_j-\mathfrak{v}_{\varphi_j}\cdot\nabla\theta+f_1'(\theta)g_1(x)\varphi_j-\frac{\kappa}{2}(-\Delta)^\alpha\varphi_j+\rho*\varphi_j,\varphi_j\Big),\end{aligned}$$

where $\mathbb{u}_{\varphi_j} = \mathcal{R}^\perp \varphi_j$. In view of (4.7) and (4.8), one has

$$\Big(-\frac{\kappa}{2}(-\Delta)^\alpha \varphi_j + \rho * \varphi_j, \varphi_j\Big) \le -\widehat{\kappa}_0 \|\varphi_j\|_{L^2}^2 \quad \text{with} \quad \widehat{\kappa}_0 := \min\Big\{\frac{\kappa\sigma^{2\alpha}}{2}, |\hat{\rho}_\sigma|\Big\} > 0.$$

Let by assumption $K_1\|g_1\|_{L^\infty}$ be so small that $\widetilde{\kappa}_0 := \widehat{\kappa}_0 - K_1\|g_1\|_{L^\infty} > 0$. Thus, we have

$$\operatorname{Tr}\big[\mathscr{F}'(\theta(\tau)) \circ Q_m(\tau)\big] \le \sum_{j=1}^m \Big(-\frac{\kappa}{2}\|\Lambda^\alpha \varphi_j\|_{L^2}^2 - \widetilde{\kappa}_0 \|\varphi_j\|_{L^2}^2 - \big(\mathbb{u}_{\varphi_j} \cdot \nabla\theta, \varphi_j\big)\Big), \tag{7.27}$$

Under the alternative assumption in Theorem 2.7 that $f_1' \le 0$ and $g_1 \ge 0$ the integral $\big(f_1'(\theta) g_1 \varphi_j, \varphi_j\big)$ is nonpositive so that we may choose $\widetilde{\kappa}_0 = \widehat{\kappa}_0$ without any smallness assumption on $K_1\|g_1\|_{L^\infty}$.

Now, we estimate the last term on the right hand side of (7.27).

$$\begin{aligned}
\left|\sum_{j=1}^m \big(\mathbb{u}_{\varphi_j} \cdot \nabla\theta, \varphi_j\big)\right| &= \left|\int_{\mathbb{R}^2} \Big(\sum_{j=1}^m \mathcal{R}^\perp \varphi_j \, \varphi_j\Big) \cdot \nabla\theta \, \mathrm{d}x\right| \\
&\le \int_{\mathbb{R}^2} |\nabla\theta| \Big(\sum_{j=1}^m |\mathcal{R}\varphi_j|^2\Big)^{1/2} \Big(\sum_{j=1}^m |\varphi_j|^2\Big)^{1/2} \mathrm{d}x \\
&\le c\|\Lambda\theta\|_{L^{\frac{1+\alpha}{\alpha}}} \left\|\Big(\sum_{j=1}^m |\mathcal{R}\varphi_j|^2\Big)^{1/2}\right\|_{L^{2(1+\alpha)}} \left\|\Big(\sum_{j=1}^m |\varphi_j|^2\Big)^{1/2}\right\|_{L^{2(1+\alpha)}}.
\end{aligned}$$

Since the Riesz operator $\mathcal{R} = (\mathcal{R}_1, \mathcal{R}_2)$ is bounded on $L^q(\mathbb{R}^2)$ for every $1 < q < \infty$, the corresponding vector-valued estimate (*cf.* [22, Theorem 5.5.1]) implies that

$$\left\|\Big(\sum_{j=1}^m |\mathcal{R}\varphi_j|^2\Big)^{1/2}\right\|_{L^{2(1+\alpha)}} \le C\left\|\Big(\sum_{j=1}^m |\varphi_j|^2\Big)^{1/2}\right\|_{L^{2(1+\alpha)}} = C\|\rho_\varphi\|_{L^{1+\alpha}}^{1/2}$$

where $\rho_\varphi(x) = \sum_{j=1}^m |\varphi_j(x)|^2$. Hence the above estimates imply that

$$\left|\sum_{j=1}^m \big(\mathbb{u}_{\varphi_j} \cdot \nabla\theta, \varphi_j\big)\right| \le c\|\Lambda\theta\|_{L^{\frac{1+\alpha}{\alpha}}} \|\rho_\varphi\|_{L^{1+\alpha}}. \tag{7.28}$$

In the following, we differ between the cases $\alpha = 1$ and $1/2 < \alpha < 1$.

Case 1. $\alpha = 1$.

Since $\{\varphi_j\}$ is orthonormal in $L^2(\mathbb{R}^2)$, the Lieb-Thirring inequality (Lemma 7.3) implies that

$$\|\rho_\varphi\|_{L^2}^2 = \int_{\mathbb{R}^2} \rho_\varphi^2 \, \mathrm{d}x \le C_2 \sum_{j=1}^m \|\nabla\varphi_j\|_{L^2}^2 \tag{7.29}$$

with C_2 as in (7.10). Insert (7.29) into (7.28) to find by Young's inequality that

$$\left|\sum_{j=1}^m \big(\mathbb{u}_{\varphi_j} \cdot \nabla\theta, \varphi_j\big)\right| \le C_0\|\Lambda\theta\|_{L^2}^2 + \frac{\kappa}{4}\sum_{j=1}^m \|\nabla\varphi_j\|_{L^2}^2. \tag{7.30}$$

Hence (7.27) gives

$$\begin{aligned}\mathrm{Tr}\big[\mathscr{F}'(\theta(\tau))\circ Q_m(\tau)\big] &\le \sum_{j=1}^{m}\Big(-\frac{\kappa}{4}\|\nabla\varphi_j\|_{L^2}^2-\widetilde{\kappa}_0\|\varphi_j\|_{L^2}^2\Big)+C_0\|\Lambda\theta\|_{L^2}^2\\ &\le -\sum_{j=1}^{m}\widetilde{\kappa}_0\|\varphi_j\|_{L^2}^2+C_0\|\theta\|_{H^1}^2\\ &= -\widetilde{\kappa}_0 m+C_0\|\theta\|_{H^1}^2. \end{aligned}\tag{7.31}$$

Thanks to (7.23) with $\alpha=1$ the energy dissipation flux,

$$\epsilon=\limsup_{t\to\infty}\sup_{\theta^0\in\mathcal{A}}\frac{1}{t}\int_0^t\|S(\tau)\theta^0\|_{H^1}^2\,\mathrm{d}\tau\le\frac{C}{\kappa}\big(\|\theta^0\|_{L^2}^2+1\big)$$

is finite. Then, from (7.31), we find for q_m in (7.7) that

$$q_m\le-\widetilde{\kappa}_0 m+C_0\epsilon,\quad\forall m\in\mathbb{N}.\tag{7.32}$$

Therefore, if $m'\in\mathbb{N}$ is defined by

$$m'-1\le\frac{C_0\epsilon}{\widetilde{\kappa}_0}<m',$$

then $q_{m'}<0$, so that due to (7.8)

$$\dim_H(\mathcal{A})\le m'\le 1+\frac{C_0\epsilon}{\widetilde{\kappa}_0}.$$

Moreover, if $m''\in\mathbb{N}$ is defined by

$$m''-1\le\frac{2C_0\epsilon}{\widetilde{\kappa}_0}<m'',$$

then by [41, Lemma VI.2.2] $q_{m''}<0$ and $\frac{(q_j)_+}{|q_m|}\le 1$ for all $j=1,\dots,m''$, so that by (7.9)

$$\dim_f(\mathcal{A})\le 2m''\le 2+\frac{4C_0\epsilon}{\widetilde{\kappa}_0}.$$

Case 2. $1/2<\alpha<1$.

We begin with (7.28) and the Fractional Lieb-Thirring inequality (7.12) to get that

$$\|\rho_\varphi\|_{L^{1+\alpha}}^{1+\alpha}=\int_{\mathbb{R}^n}\rho_\varphi^{1+\alpha}\,\mathrm{d}x\le C_{2,\alpha}\sum_{j=1}^{m}|\Lambda^\alpha\varphi_i|^2\,\mathrm{d}x.\tag{7.33}$$

Thus, it follows that

$$\begin{aligned}\left|\sum_{j=1}^{m}\big(\mathbb{u}_{\varphi_j}\cdot\nabla\theta,\varphi_j\big)\right| &\le\|\Lambda\theta\|_{L^{\frac{1+\alpha}{\alpha}}}\left(C_{2,\alpha}\sum_{j=1}^{m}\|\Lambda^\alpha\varphi_j\|_{L^2}^2\right)^{\frac{1}{1+\alpha}}\\ &\le C_0'\|\Lambda\theta\|_{L^{\frac{1+\alpha}{\alpha}}}^{\frac{1+\alpha}{\alpha}}+\frac{\kappa}{4}\sum_{j=1}^{m}\|\Lambda^\alpha\varphi_j\|_{L^2}^2.\end{aligned}\tag{7.34}$$

Next, we deal with the term $\|\Lambda\theta\|_{L^{\frac{1+\alpha}{\alpha}}}$. Since $1/2<\alpha<1$ and $s>2(1-\alpha)$, it is easy to see that $2-2\alpha<\frac{2}{1+\alpha}<2-\alpha<s+\alpha$ so that $\dot{H}^{\frac{2}{1+\alpha}}\hookrightarrow\dot{H}^{1,\frac{1+\alpha}{\alpha}}$. Then an interpolation inequality and the fact $\theta\in L^\infty(0,\infty;H^s(\mathbb{R}^2))$ which follows from the invariance of $S(\cdot)$ on $\mathcal{A}\subset H^s$, imply that

$$\|\Lambda\theta\|_{L^{\frac{1+\alpha}{\alpha}}}\le C\|\Lambda^{\frac{2}{1+\alpha}}\theta\|_{L^2}\le C\|\Lambda^{2-2\alpha}\theta\|_{L^2}^{\frac{1-\alpha}{1+\alpha}}\|\Lambda^{2-\alpha}\theta\|_{L^2}^{\frac{2\alpha}{1+\alpha}}\le C\|\theta\|_{H^s}^{\frac{1-\alpha}{1+\alpha}}\|\theta\|_{H^{s+\alpha}}^{\frac{2\alpha}{1+\alpha}}$$

and consequently that

$$\|\Lambda\theta\|_{L^{\frac{1+\alpha}{\alpha}}}^{\frac{1+\alpha}{\alpha}} \leq C'\|\theta\|_{H^{s+\alpha}}^2 \tag{7.35}$$

for some absolute constant $C' > 0$. Combining (7.34) and (7.35), we have

$$\left|\sum_{j=1}^{m}\left(\mathfrak{v}_{\varphi_j}\cdot\nabla\theta,\varphi_j\right)\right| \leq C_0''\|\theta\|_{H^{s+\alpha}}^2 + \frac{\kappa}{4}\sum_{j=1}^{m}\|\Lambda^\alpha\varphi_j\|_{L^2}^2.$$

Together with (7.27) we conclude that

$$\begin{aligned}\operatorname{Tr}\left[\mathscr{F}'(\theta(\tau))\circ Q_m(\tau)\right] &\leq \sum_{j=1}^{m}\left(-\frac{\kappa}{4}\|\Lambda^\alpha\varphi_j\|_{L^2}^2 - \widetilde{\kappa}_0\|\varphi_j\|_{L^2}^2\right) + C_0''\|\theta\|_{H^{s+\alpha}}^2\\ &\leq -\widetilde{\kappa}_0 m + C_0''\|\theta\|_{H^{s+\alpha}}^2.\end{aligned}$$

We define the modified energy dissipation flux

$$\epsilon' = \limsup_{t\to\infty}\sup_{\theta^0\in\mathcal{A}}\frac{1}{t}\int_0^t\|S(t)\theta^0\|_{H^{s+\alpha}}^2\mathrm{d}\tau,$$

which is finite due to (7.24). For the remaining part of the proof, one proceeds as in the case $\alpha = 1$.

Now the proof of the finite dimensionality of the global attractor $\mathcal{A}$ and of Theorem 2.7 is completed. □

8. Appendix

For the sake of completeness of this paper, we show global existence and uniqueness of solutions for system (1.1), (1.3) and (1.5), (1.6).

Theorem 8.1. *Let $\alpha \in]\frac{1}{2}, 1]$, $\kappa > 0$ and $\theta^0 \in H^s(\mathbb{R}^2)$ with $s > 2(1-\alpha)$. Suppose further that*

$$\rho \in L^1(\mathbb{R}^2), g_1 \in H^s(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2), g_2 \in H^{s-\alpha}(\mathbb{R}^2)\cap L^2(\mathbb{R}^2)\cap L^{q_0}(\mathbb{R}^2)$$

for some $q_0 > \frac{2}{2\alpha-1}$, and that $f_1 \in \mathcal{C}^\infty(\mathbb{R})$ satisfies

$$f_1(0) = f_1'(0) = 0, \quad \|f_1'\|_{L^\infty} \leq K_1 < \infty, \ \|f_1''\|_{L^\infty} \leq K_2 < \infty.$$

Then for any $T > 0$, there is unique solution θ of (1.1), (1.3) and (1.5), (1.6) such that

$$\theta \in \mathcal{C}([0,T); H^s(\mathbb{R}^2)) \cap L^2(0,T; H^{s+\alpha}(\mathbb{R}^2)).$$

Proof. ***Step 1.*** We use a smoothing method to construct a sequence of approximate solutions as follows. For $n \geq 1$, let J_n be the spectral cut-off defined by

$$\widehat{J_n f} = 1_{B_n}\hat{f} \quad \text{ with } B_n = \{\xi\in\mathbb{R}^2 : |\xi_1| \leq n, |\xi_2| \leq n\}.$$

Note that J_n commutes with the differential operators Λ^δ for any $\delta > 0$, $\Delta, \nabla, \operatorname{div}$ and with Riesz operators.

Consider the following ODE in the space $L_n^2 := \{f \in L^2(\mathbb{R}^2) : \operatorname{supp}\hat{f} \subset B_n\}$:

$$\begin{aligned}\partial_t\theta = -J_n\operatorname{div}(J_n u J_n\theta) - \kappa(-\Delta)^\alpha J_n\theta + J_n F(x, J_n\theta),\\ J_n u = \mathcal{R}^\perp J_n\theta, \quad \theta(x,0) = J_n\theta^0.\end{aligned} \tag{8.1}$$

From the Picard-Lindelöf theorem, we get a unique maximal solution θ_n in $\mathcal{C}^1([0,T_n^*); L_n^2)$ for some time interval $[0, T_n^*)$. Since $J_n^2 = J_n$, we see that θ_n and $J_n\theta_n$ are solutions with the same initial data. By uniqueness, we have $J_n\theta_n = \theta_n$ (and thus $J_n u_n = u_n$). Therefore,

$$\begin{aligned}\partial_t\theta_n + J_n\operatorname{div}(u_n\theta_n) + \kappa(-\Delta)^\alpha\theta_n = J_n F(x,\theta_n),\\ u_n = \mathcal{R}^\perp\theta_n, \quad \theta_n(x,0) = J_n\theta^0.\end{aligned} \tag{8.2}$$

As J_n is an orthogonal projector for the L^2 inner product, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta_n\|_{L^2}^2+\kappa\|\Lambda^\alpha\theta_n\|_{L^2}^2\le\big(K_1\|g_1\|_{L^\infty}+\|\rho\|_{L^1}\big)\|\theta_n\|_{L^2}^2+\|g_2\|_{L^2}\|\theta_n\|_{L^2}. \tag{8.3}$$

Then Gronwall's Lemma shows that

$$\|\theta_n\|_{L^2}\le C_1(t):=e^{t(K_1\|g_1\|_{L^\infty}+\|\rho\|_{L^1})}\left(\|\theta^0\|_{L^2}+t\|g_2\|_{L^2}\right).$$

This implies that θ_n remains bounded in L^2_n for finite time, whence $T_n^*=+\infty$. Similarly, we use Lemma 2.5 to get

$$\frac{1}{q_0}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta_n\|_{L^{q_0}}^{q_0}\le\left(K_1\|g_1\|_{L^\infty}+\|\rho\|_{L^1}\right)\|\theta_n\|_{L^{q_0}}^{q_0}+\|g_2\|_{L^{q_0}}\|\theta_n\|_{L^{q_0}}^{q_0-1}$$

so that by Gronwall's Lemma

$$\|\theta_n\|_{L^{q_0}}\le C_2(t):=e^{t\left(K_1\|g_1\|_{L^\infty}+\|\rho\|_{L^1}\right)}\left(\|\theta^0\|_{L^{q_0}}+t\|g_2\|_{L^{q_0}}\right). \tag{8.4}$$

Now, multiplying $(8.2)_1$ by $\Lambda^{2s}\theta_n$, we have in view of (4.25) and (4.31)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda^s\theta_n\|_{L^2}^2+\kappa\|\Lambda^{s+\alpha}\theta_n\|_{L^2}^2&\le C\|\theta_n\|_{H^s}^2+C\big(\|\theta_n\|_{L^{q_0}}+\|u_n\|_{L^{q_0}}\big)\|\Lambda^{s+\beta}\theta_n\|_{L^2}^2+C\|\Lambda^{s-\alpha}g_2\|_{L^2}^2\\&\le C\big(1+C_2^{\frac{\alpha}{\alpha-\beta}}(t)\big)\|\Lambda^s\theta_n\|_{L^2}^2+C_1^2(t)+C\|\Lambda^{s-\alpha}g_2\|_{L^2}^2+\frac{\kappa}{2}\|\Lambda^{s+\alpha}\theta_n\|_{L^2}^2.\end{aligned}$$

After absorbing the last term of the above inequality, we use Gronwall's Lemma and (8.4) to get with continuous functions $C_3(t)$ and $C_4(t)$ which are independent of $n\in\mathbb{N}$ that

$$\|\Lambda^s\theta_n(t)\|_{L^2}^2+\kappa\int_0^t\|\Lambda^{s+\alpha}\theta_n\|_{L^2}^2\mathrm{d}\tau=e^{C_3(t)}\big(\|\theta^0\|_{H^s}^2+Ct\|\Lambda^{s-\alpha}g_2\|_{L^2}^2+C_4(t)\big).$$

Therefore, for each $T>0$, the sequence of approximate solutions $\{\theta_n\}$ is uniformly bounded with respect to $n\in\mathbb{N}$ on $[0,T]$; to be more precise,

$$\begin{aligned}&\{\theta_n\}\subset L^\infty(0,T;H^s(\mathbb{R})^2)\cap L^2(0,T;H^{s+\alpha}(\mathbb{R})^2)\quad\text{is bounded}\\&\{\theta_n\}\subset L^\infty(0,T;L^{q_0}(\mathbb{R}^2))\quad\text{is bounded.}\end{aligned} \tag{8.5}$$

Step 2. We prove that $\{\theta_n\}$ is a Cauchy sequence in the space

$$L^\infty(0,T_*;H^{s'}(\mathbb{R}^2))\cap L^2(0,T_*;H^{s'+\alpha}(\mathbb{R}^2))\quad\text{with }2(1-\alpha)<s'<\min\{1,s\}$$

where $T_*<T$ is to be determined later. Set for $m,n\in\mathbb{N}$ with $m>n$

$$\theta_{m,n}=\theta_m-\theta_n,\ u_{m,n}=u_m-u_n,\ \text{ and }\ J_{m,n}=J_m-J_n,$$

and note that $\|J_{m,n}\Lambda^{-\delta}v\|_{L^2}\le\frac{c}{n^\delta}\|v\|_{L^2}$ for any $\delta>0$ and $v\in L^2(\mathbb{R}^2)$. For $\theta_{m,n}$ we find that

$$\begin{aligned}\partial_t\theta_{m,n}+J_m(u_m\cdot\nabla\theta_{m,n})+\kappa\Lambda^\alpha\theta_{m,n}&=J_m\big(F(x,\theta_m)-F(x,\theta_n)-u_{m,n}\cdot\nabla\theta_n\big)\\&\quad+J_{m,n}\big(F(x,\theta_n)-u_n\cdot\nabla\theta_n\big),\\\theta_{m,n}(x,0)&=J_{m,n}\theta^0.\end{aligned}$$

Testing this equation with $\Lambda^{2s'}\theta_{m,n}$ we get that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Lambda^{s'}\theta_{m,n}\|_{L^2}^2+\kappa\|\Lambda^{s'+\alpha}\theta_{m,n}\|_{L^2}^2\\&\quad=\Big(J_m\big(F(x,\theta_n)-F(x,\theta_n)-u_{m,n}\cdot\nabla\theta_n-u_m\cdot\nabla\theta_{m,n}\big),\Lambda^{2s'}\theta_{m,n}\Big)\\&\qquad+\Big(J_{m,n}\big(F(x,\theta_n)-u_n\cdot\nabla\theta_n\big),\Lambda^{2s'}\theta_{m,n}\Big).\end{aligned} \tag{8.6}$$

Since

$$\left|\left(J_m\left(F(x,\theta_m)-F(x,\theta_n)\right),\, \Lambda^{2s'}\theta_{m,n}\right)\right| \le C\|\Lambda^{s'}(F(x,\theta_m)-F(x,\theta_n))\|_{L^2}^2 + C\|\Lambda^{s'}\theta_{m,n}\|_{L^2}^2,$$

we apply (3.3) in Proposition 3.1 and Lemma 2.4 to get with

$$p_1 = \frac{2}{1-s'},\quad p_2 = \frac{2}{s'},\quad p_3 = q_0,\quad p_4 = \frac{2q_0}{q_0-2}$$

for the first right-hand side term the estimate

$$\begin{aligned}
&\|\Lambda^{s'}(F(x,\theta_m)-F(x,\theta_n))\|_{L^2}^2 \\
&\quad\le \|\Lambda^{s'}\left(g_1[f_1(\theta_m)-f_1(\theta_n)]\right)\|_{L^2}^2 + \|\Lambda^{s'}(\rho*\theta_{m,n})\|_{L^2}^2 \\
&\quad\le C\|g_1\|_{L^\infty}^2\|f_1(\theta_m)-f_1(\theta_n)\|_{\dot H^{s'}}^2 + C\|f_1(\theta_m)-f_1(\theta_n)\|_{L^\infty}^2\|g_1\|_{\dot H^{s'}}^2 + C\|\theta_{m,n}\|_{H^{s'}}^2 \\
&\quad\le C\left(\|\theta_{m,n}\|_{L^{p_1}}^2 \sup_{\theta\in\overline{[\theta_m,\theta_n]}}\|\theta\|_{\dot B^{s'}_{p_2,1}}^2 + \sup_{\theta\in\overline{[\theta_m,\theta_n]}}\|\theta\|_{L^{p_3}}^2\|\theta_{m,n}\|_{\dot H^{s',p_4}}^2\right) \\
&\qquad + C\|\theta_{m,n}\|_{L^\infty}^2\|g_1\|_{H^s}^2 + C\|\theta_{m,n}\|_{H^{s'}}^2 \\
&\quad\le C\|\theta_{m,n}\|_{H^{s'}}^2\left(\|\theta_m\|_{H^{s'+1-\frac{2}{p_2}+\eta}}^2 + \|\theta_n\|_{H^{s'+1-\frac{2}{p_2}+\eta}}^2\right) \\
&\qquad + C\left(\|\theta_m\|_{L^{q_0}}^2 + \|\theta_n\|_{L^{q_0}}^2\right)\|\theta_{m,n}\|_{H^{s'-\frac{2}{p_4}+1}}^2 + C\|\theta_{m,n}\|_{H^{s'+\alpha-\delta}}^2
\end{aligned}\tag{8.7}$$

for some $0<\eta<\alpha+\frac{2}{p_2}-1,\ 0<\delta<s'+\alpha-1<\alpha$. Similar to the estimate of (4.22)–(4.23), we choose an η_0 small enough such that $2(1-\alpha)<s'<1+2\eta_0$ where $0<2\eta_0<\min\left\{s',\alpha,2\left(\alpha-\frac{2}{q_0}\right)\right\}$ and hence $s'+\alpha-\eta_0>1$. Using (4.22)–(4.23) we get that

$$\begin{aligned}
\left|\left(J_{m,n}F(x,\theta_n),\, \Lambda^{2s'}\theta_{m,n}\right)\right| &\le \|J_{m,n}\Lambda^{s'-s}\Lambda^{s-2\eta_0}(g_1f(\theta_n))\|_{L^2}\|\Lambda^{s'+2\eta_0}\theta_{m,n}\|_{L^2} \\
&\quad + \left(\|J_{m,n}\Lambda^{s'-\alpha}\rho*\theta_n\|_{L^2} + \|J_{m,n}\Lambda^{s'-\alpha}g_2\|_{L^2}\right)\left\|\Lambda^{s'+\alpha}\theta_{m,n}\right\|_{L^2} \\
&\le \frac{C}{n^{2(s-s')}}\|\theta_n\|_{H^{s+\alpha-\eta_0}}^2 + \|\theta_{m,n}\|_{L^2}^2 + C\|J_{m,n}\Lambda^{s'-\alpha}g_2\|_{L^2}^2 \\
&\quad + C\|J_{m,n}\Lambda^{s'-\alpha}\theta_n\|_{L^2}^2 + \frac{\kappa}{8}\left\|\Lambda^{s'+\alpha}\theta_{m,n}\right\|_{L^2}^2.
\end{aligned}\tag{8.8}$$

Furthermore, note that with some $\beta_0>0$,

$$\|J_{m,n}\Lambda^{s'-\alpha}g_2\|_{L^2}^2 \le \left\{\begin{array}{ll} \dfrac{C}{n^{2(\alpha-s')}}\|g_2\|_{L^2}^2, & \text{if } s'<\alpha \\ \dfrac{C}{n^{2(s-s')}}\|g_2\|_{H^{s-\alpha}}^2, & \text{if } s'\ge\alpha \end{array}\right\} \le \frac{C}{n^{\beta_0}}.$$

By analogy, and in view of (8.5), $\|J_{m,n}\Lambda^{s'-\alpha}\theta_n\|_{L^2}^2 \le \frac{C}{n^{\beta_0}}\|\theta_n\|_{H^s}^2$. Hence (8.8) can be reduced to

$$\left|\left(J_{m,n}F(x,\theta_n),\, \Lambda^{2s'}\theta_{m,n}\right)\right| \le \frac{C}{n^{\beta_0}}\left(1+\|\theta_n\|_{H^{s+\alpha}}^2\right) + C\|\theta_{m,n}\|_{L^2}^2 + \frac{\kappa}{8}\left\|\Lambda^{s'+\alpha}\theta_{m,n}\right\|_{L^2}^2.$$

The remaining terms in (8.6) are estimated as

$$\begin{aligned}
\left|\left(J_m(u_{m,n}\cdot\nabla\theta_n),\,\Lambda^{2s'}\theta_{m,n}\right)\right| &\le C\left\|\Lambda^{s'-\alpha+1}(u_{m,n}\theta_n)\right\|_{L^2}^2+\frac{\kappa}{8}\left\|\Lambda^{s'+\alpha}\theta_{m,n}\right\|_{L^2}^2 \quad \text{with}\\
\left\|\Lambda^{s'-\alpha+1}(u_{m,n}\theta_n)\right\|_{L^2}^2 &\le C\left\|u_{m,n}\right\|_{L^{p_1}}^2\left\|\theta_n\right\|_{\dot H^{s'-\alpha+1,p_2}}^2+C\|\theta_n\|_{L^{p_3}}^2\left\|u_{m,n}\right\|_{\dot H^{s'-\alpha+1,p_4}}^2\\
&\le C\|\theta_{m,n}\|_{H^{s'}}^2\|\theta_n\|_{H^{\widehat{s}}}^2+C\|\theta_n\|_{L^{p_3}}^2\|\theta_{m,n}\|_{H^{\widetilde{s}}}^2,
\end{aligned}$$

$$\begin{aligned}
\left|\left(J_m(u_m\cdot\nabla\theta_{m,n}),\,\Lambda^{2s'}\theta_{m,n}\right)\right| &\le C\left\|\Lambda^{s'-\alpha+1}(u_m\theta_{m,n})\right\|_{L^2}^2+\frac{\kappa}{8}\left\|\Lambda^{s'+\alpha}\theta_{m,n}\right\|_{L^2}^2 \quad \text{with}\\
\left\|\Lambda^{s'-\alpha+1}(u_m\theta_{m,n})\right\|_{L^2} &\le C\|\theta_{m,n}\|_{H^{s'}}^2\|\theta_m\|_{H^{\widehat{s}}}^2+C\|\theta_m\|_{L^{p_3}}^2\|\theta_{m,n}\|_{H^{\widetilde{s}}}^2,
\end{aligned}$$

where $\widehat{s}=s'-\alpha+2-\frac{2}{p_2}$, $\widetilde{s}=s'-\alpha+2-\frac{2}{p_4}$. By analogy,

$$\begin{aligned}
\left|\left(J_{m,n}(u_n\cdot\nabla\theta_n),\,\Lambda^{2s'}\theta_{m,n}\right)\right| &\le C\left\|J_{m,n}\Lambda^{s'-\alpha+1}(u_n\theta_n)\right\|_{L^2}^2+\frac{\kappa}{8}\left\|\Lambda^{s'+\alpha}\theta_{m,n}\right\|_{L^2}^2 \quad \text{with}\\
\left\|J_{m,n}\Lambda^{s'-\alpha+1}(u_n\theta_n)\right\|_{L^2}^2 &\le \frac{1}{n^{2(s-s')}}\left\|\Lambda^{s-\alpha+1}(u_n\theta_n)\right\|_{L^2}^2\\
&\le \frac{C}{n^{2(s-s')}}\left(\|\theta_n\|_{L^{p_3}}^2\|u_n\|_{\dot H^{s-\alpha+1,p_4}}^2+C\|u_n\|_{L^{p_3}}^2\|\theta_n\|_{\dot H^{s-\alpha+1,p_4}}^2\right)\\
&\le \frac{C}{n^{2(s-s')}}\|\theta_n\|_{L^{q_0}}^2\|\theta_n\|_{H^{s-\alpha+2-\frac{2}{p_4}}}^2\\
&\le \frac{C}{n^{2(s-s')}}\|\theta_n\|_{L^{q_0}}^2\|\theta_n\|_{H^{s+\alpha}}^2.
\end{aligned}$$

Moreover, we also have

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta_{m,n}\|_{L^2}^2+\kappa\|\Lambda^\alpha\theta_{m,n}\|_{L^2}^2 &= \left(J_m(F(x,\theta_m)-F(x,\theta_n)-u_{m,n}\cdot\nabla\theta_n),\theta_{m,n}\right)\\
&\quad+\left(J_{m,n}(F(x,\theta_n)-u_n\cdot\nabla\theta_n),\,\theta_{m,n}\right),
\end{aligned} \tag{8.9}$$

and the first two terms on the right hand side are estimated as follows:

$$\begin{aligned}
\left|\left(J_m(F(x,\theta_m)-F(x,\theta_n)),\theta_{m,n}\right)\right| &\le C\|\theta_{m,n}\|_{L^2}^2\\
\left|\left(J_m(u_{m,n}\cdot\nabla\theta_n),\theta_{m,n}\right)\right| &\le C\|\Lambda^{1-\alpha}(u_{m,n}\theta_n)\|_{L^2}^2+\frac{\kappa}{8}\|\Lambda^\alpha\theta_{m,n}\|_{L^2}^2 \quad \text{with}\\
\|\Lambda^{1-\alpha}(u_{m,n}\theta_n)\|_{L^2}^2 &\le C\left\|u_{m,n}\right\|_{p_1}^2\left\|\theta_n\right\|_{\dot H^{1-\alpha,p_2}}^2+C\|\theta_n\|_{L^{p_3}}^2\left\|u_{m,n}\right\|_{\dot H^{1-\alpha,p_4}}^2\\
&\le C\left\|\theta_{m,n}\right\|_{H^{s'}}^2\left\|\theta_n\right\|_{H^{\widehat{\alpha}}}^2+C\|\theta_n\|_{L^{q_0}}^2\|\theta_{m,n}\|_{H^{\widetilde{\alpha}}}^2,
\end{aligned}$$

where $\widehat{\alpha}=2-\alpha-\frac{2}{p_2}$ and $\widetilde{\alpha}=2-\alpha-\frac{2}{p_4}$. Furthermore,

$$\begin{aligned}
\left|\left(J_{m,n}F(x,\theta_n),\,\theta_{m,n}\right)\right| &\le \left\|J_{m,n}\Lambda^{-\alpha}\left(f_1(\theta_n)g_1+\rho*\theta_n+g_2\right)\right\|_{L^2}^2+\frac{\kappa}{8}\left\|\Lambda^\alpha\theta_{m,n}\right\|_{L^2}^2\\
&\le \frac{C}{n^{2\alpha}}\left(\|f_1(\theta_n)g_1\|_{L^2}^2+\|\theta_n\|_{L^2}^2+\|g_2\|_{L^2}^2\right)+\frac{\kappa}{8}\left\|\Lambda^\alpha\theta_{m,n}\right\|_{L^2}^2\\
&\le \frac{C}{n^{2\alpha}}\left(1+\|\theta_n\|_{H^\alpha}^2\right)+\frac{\kappa}{8}\|\Lambda^\alpha\theta_{m,n}\|_{L^2}^2,
\end{aligned}$$

$$\begin{aligned}
\left|\left(J_{m,n}(u_n\cdot\nabla\theta_n),\,\theta_{m,n}\right)\right| &\le C\left\|J_{m,n}\Lambda^{1-\alpha-s}\Lambda^s\left(u_n\theta_n\right)\right\|_{L^2}^2+\frac{\kappa}{8}\left\|\Lambda^\alpha\theta_{m,n}\right\|_{L^2}^2\\
&\le \frac{C}{n^{2(s+\alpha-1)}}\|\theta_n\|_{L^{p_3}}^2\|\Lambda^s\theta_n\|_{L^{p_4}}^2+\frac{\kappa}{8}\left\|\Lambda^\alpha\theta_{m,n}\right\|_{L^2}^2\\
&\le \frac{C}{n^{2(s+\alpha-1)}}\|\theta_n\|_{L^{q_0}}^2\|\theta_n\|_{H^{s+\alpha}}^2+\frac{\kappa}{8}\|\Lambda^\alpha\theta_{m,n}\|_{L^2}^2.
\end{aligned}$$

Next we estimate the energy term

$$D_{m,n}(t) := \|\theta_{m,n}\|^2_{L^\infty_t H^{s'}} + \|\theta_{m,n}\|^2_{L^2_t H^{s'+\alpha}}$$

which for $0 < t < 1$ is obviously bounded by a multiple of the modified energy term $\widetilde{D}_{m,n}(t) := \|\theta_{m,n}\|^2_{L^\infty_t H^{s'}} + \|\Lambda^\alpha \theta_{m,n}\|^2_{L^2_t H^{s'}}$; here $L^\infty_t H^{s'} = L^\infty(0,t;H^{s'}(\mathbb{R}^2))$ *etc.* Applying the assertion (8.5) on uniform boundedness for the sequence of solutions $\{\theta_n\}$ on $(0,T)$, integrating (8.6) as well as (8.9) over $[0,t]$ with $t < T \le 1$ and using the previous estimates we obtain for $D_{m,n}(t)$ that

$$\begin{aligned} D_{m,n} \le CtD_{m,n} + C\int_0^t \Big(\|\theta_{m,n}\|^2_{H^{\widetilde{s}}} + \|\theta_{m,n}\|^2_{H^{s'+\alpha-\delta}} + \|\theta_{m,n}\|^2_{H^{\widetilde{\alpha}}} + \|\theta_{m,n}\|^2_{H^{s'-\frac{2}{p_4}+1}}\Big)\,\mathrm{d}\tau + \|J_{m,n}\theta^0\|^2_{H^{s'}} \\ + C\int_0^t \|\theta_{m,n}\|^2_{H^{s'}}\Big(\|\theta_n\|^2_{H^{\widehat{s}}} + \|\theta_m\|^2_{H^{\widehat{s}}} + \|\theta_n\|^2_{H^{\widehat{\alpha}}} + \|\theta_m\|^2_{H^{s'+1-\frac{2}{p_2}+\eta}} + \|\theta_n\|^2_{H^{s'+1-\frac{2}{p_2}+\eta}}\Big)\,\mathrm{d}\tau \\ + \frac{C}{n^{2(s+\alpha-1)}} + \Big(\frac{1_{(0,\infty)}(\alpha-s')}{n^{2(\alpha-s')}} + \frac{C}{n^{2(s-s')}} + \frac{C}{n^{2\alpha}}\Big)(1+t). \end{aligned} \tag{8.10}$$

Because $s_0 := \max\{s' - \frac{2}{p_4} + 1, s' + 1 - \frac{2}{p_2} + \eta, \widehat{s}, \widetilde{s}, \widehat{\alpha}, \widetilde{\alpha}\} < s' + \alpha$ and all powers of n in the last line of (8.10) are bounded from below by some (modified) $\beta_0 > 0$, we rewrite (8.10) in view of (8.5) applied to $\theta_{m,n} = \theta_m - \theta_n$ in the short form

$$\begin{aligned} D_{m,n}(t) \le CtD_{m,n}(t) + C\int_0^t \Big(\|\theta_{m,n}\|^2_{H^{s_0}} + \|\theta_{m,n}\|^2_{H^{s'}}\big(\|\theta_m\|^2_{H^{s_0}} + \|\theta_n\|^2_{H^{s_0}}\big)\Big)\,\mathrm{d}\tau \\ + \|J_{m,n}\theta^0\|^2_{H^{s'}} + \frac{C}{n^{\beta_0}}(1+t). \end{aligned} \tag{8.11}$$

Using the interpolation inequality $\|\theta\|_{H^{s_0}} \le \|\theta\|^{\eta_0}_{H^{s'+\alpha}} \|\theta\|^{1-\eta_0}_{H^{s'}}$ where $0 < \eta_0 = \frac{s_0 - s'}{\alpha} < 1$ we get that

$$\int_0^t \|\theta_{m,n}\|^2_{H^{s_0}}\,\mathrm{d}\tau \le \Big(\int_0^t \|\theta_{m,n}\|^2_{H^{s'+\alpha}}\,\mathrm{d}\tau\Big)^{\eta_0}\Big(\int_0^t \|\theta_{m,n}\|^2_{H^{s'}}\,\mathrm{d}\tau\Big)^{1-\eta_0} \le t^{1-\eta_0}\, D_{m,n}(t).$$

By analogy, applying (8.5) to θ_m, θ_n, we see that

$$\int_0^t \|\theta_{m,n}\|^2_{H^{s'}}\big(\|\theta_m\|^2_{H^{s_0}} + \|\theta_n\|^2_{H^{s_0}}\big)\,\mathrm{d}\tau \le D_{m,n}(t)\, Ct^{1-\eta_0}.$$

Moreover, $\|J_{m,n}\theta^0\|_{H^{s'}} \le cn^{-(s-s')}\|\theta^0\|_{H^s}$.

Thus, (8.11) leads to the simpler estimate

$$D_{m,n}(t) \le \frac{C(1+t)}{n^{\beta_0}} + C(t + t^{1-\eta})D_{m,n}(t) \tag{8.12}$$

We choose $T_* \le 1$ such that $C(T_* + T_*^{1-\eta}) \le \frac{1}{2}$ and get the estimate $D_{m,n}(t) \le Cn^{-\beta}$ on $[0,T_*)$ for all $m > n$. Hence $\{\theta_n\}$ is a Cauchy sequence in $L^\infty(0,T_*;H^{s'}(\mathbb{R})^2) \cap L^2(0,T_*;H^{s'+\alpha}(\mathbb{R})^2)$.

Step 3. From (8.1), we find that the limit θ of $\{\theta_n\}$ in $L^\infty(0,T_*;H^{s'}(\mathbb{R}^2)) \cap L^2(0,T_*;H^{s'+\alpha}(\mathbb{R}^2))$ is a solution of the system (1.1) with the initial data $\theta^0 \in H^s(\mathbb{R}^2)$, and that, by a weak convergence argument, even

$$\theta \in L^\infty(0,T_*;H^s(\mathbb{R}^2)) \cap L^2(0,T_*;H^{s+\alpha}(\mathbb{R}^2)). \tag{8.13}$$

Due to the uniform boundedness of the sequence $\{\theta_n\}$ on any interval $[0,T)$, the local solution can be extended to a global one. As the proof of uniqueness is based on estimates similar to those for $\theta_{m,n}$, it will be omitted.

It remains to prove the continuity of the solution in time. In fact, it is easily seen from the equation (1.1) that $\partial_t\theta \in L^2(0,T;H^{s-\alpha}(\mathbb{R}^2) + L^2(\mathbb{R}^2))$ and hence $\theta \in \mathcal{C}([0,T];H^{s-\alpha}(\mathbb{R}^2) + L^2(\mathbb{R}^2))$. Since $\theta \in L^\infty(0,T_*;H^s(\mathbb{R}^2))$, a classical density and reflexivity argument ([31, Lemma 8.1, p. 275]) implies that even $\theta \in \mathcal{C}([0,T);H^s(\Omega))$. Now the proof of this theorem is completed. □

Acknowledgement

C. Qian is partly supported by NSFC 11501517 and 11671364, the Natural Science Foundation of Zhejiang Province (LQ16A010001) and China Scholarship Council (CSC) NO. 201608330096 to visit the Technische Universität Darmstadt.

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