On the continuity of the solutions to the Navier-Stokes equations with initial data in critical Besov spaces

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Abstract

It is well-known that there exists a unique local-in-time strong solution u of the initial-boundary value problem for the Navier-Stokes sytem in a three-dimensional smooth bounded domain when the initial velocity u_0 belongs to critical Besov spaces. A typical space is $B = B^{-1+3/q}_{q,s}$ with $3 < q < \infty$, $2 < s < \infty$ satisfying $2/s+3/q \leq 1$ or $B = \overset{\circ}{B}{}^{-1+3/q}_{q,\infty}$. In this paper we show that the solution u is continuous in time up to initial time with values in B. Moreover, the solution map $u_0 \mapsto u$ is locally Lipschitz from B to $C([0,T];B)$. This implies that in the range $3 < q < \infty$, $2 < s \leq \infty$ with $3/q + 2/s \leq 1$ the problem is well-posed which is in strong contrast to norm inflation phenomena for $B^{-1}_{\infty,s}$, $1 \leq s < \infty$.

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1 Introduction

We consider the initial-boundary value problem of the Navier-Stokes equations in a bounded domain $\Omega \subset \mathbb{R}^3$ with $C^{2,1}$ boundary $\partial\Omega$,

$$u_t - \Delta u + u \cdot \nabla u + \nabla p = f, \quad \operatorname{div} u = 0 \quad \text{in} \quad (0,T) \times \Omega \tag{1.1}$$
$$u\big|_{\partial\Omega} = 0, \quad u(0) = u_0$$

where $T \in (0,\infty]$. We are interested in local-in-time strong solutions in a Bochner space $L^s(0,T;L^q(\Omega))$ or, more generally, a weighted Bochner space with weight in time,

$$L^s_\alpha(L^q) := L^s_\alpha(0,T;L^q(\Omega)) = \left\{v \text{ measurable in } (0,T) \times \Omega : \|v\|_{L^s_\alpha(L^q)} < \infty\right\}$$

with

$$\|v\|_{L^s_\alpha(L^q)} := \left(\int_0^T \left(\tau^\alpha \|v(\tau)\|_q\right)^s \mathrm{d}\tau\right)^{1/s}$$

where $\alpha \geq 0$ and $1 \leq s < \infty$; for $s = \infty$ the standard modification for the norm $\|\cdot\|_{L^\infty_\alpha(L^q)}$ is to be used. By definition $L^s_0(L^q) = L^s(L^q) = L^s(0,T;L^q(\Omega))$.

There is a large literature on the existence of a local-in-time strong solution under various regularity condition on the initial data and the external force f [2], [13], [14], [15], [17], [18], [21], [25], [26], [27]. The first contribution in this direction seems to be the work of Kiselev and Ladyzhenskaya [19]. Since then the condition on initial data and the external force f has been weakened, in other words, u_0 can be taken in a larger space.

In the scale of Besov spaces it is shown in [11], [12] that a necessary and sufficient condition to get $L^s(L^q)$-strong solutions is that the initial data u_0 belongs to a solenoidal Besov space $\mathbb{B}_{q,s}^{-1+3/q}(\Omega)$ provided that $s = s_q$ where $2/s_q + 3/q = 1$ ($3 < q < \infty$). In this case, the so-called Serrin class $L^s(L^q)$ allows to prove regularity and uniqueness of weak solutions of the Navier-Stokes system. See also [7] for a review.

The existence of strong solutions is extended for s larger than s_q by introducing a weighted Bochner space. In fact, in [8] a local-in-time strong solution in $L^s_\alpha(L^q)$ is constructed if the initial data belongs to $\mathbb{B}_{q,s}^{-1+3/q}$ for $3 < q < \infty$, $s_q \le s < \infty$ with $2/s_q + 3/q = 1$ and $2/s + 3/q = 1 - 2\alpha$. In [9], this result is extended to the case $s = \infty$ by replacing $\mathbb{B}_{q,s}^{-1+3/q}$ by $\overset{\circ}{\mathbb{B}}{}_{q,\infty}^{-1+3/q}$ which is obtained as a continuous interpolation space. In [8, 9] u_0 is assumed to belong also to the space L^2_σ to compare with weak solutions. However, just for existence of a strong solution, this additional L^2_σ assumption is unnecessary to get an $L^s_\alpha(L^q)$-solution. The explanation of the Besov spaces will be given in the Appendix for the reader's convenience.

In this paper, we shall prove that $L^s_\alpha(L^q)$-solutions are indeed in $C\big([0,T]; \mathbb{B}_{q,s}^{-1+3/q}\big)$ for initial data $u_0 \in \mathbb{B}_{q,s}^{-1+3/q}$ when $s_q \le s < \infty$, or in $C\big([0,T]; \overset{\circ}{\mathbb{B}}{}_{q,\infty}^{-1+3/q}\big)$ for $u_0 \in \overset{\circ}{\mathbb{B}}{}_{q,\infty}^{-1+3/q}$ when $s = \infty$, see Theorems 1.1 and 1.2 below, respectively. Moreover, we will show in Theorems 1.3 ($s_q \le s < \infty$) and 1.4 ($s = \infty$) that they are globally well-posed for small initial data. Theorems 1.1 and 1.2 are in strong contrast to the so-called *norm inflation phenomenon* in limiting – homogeneous or inhomogeneous – Besov spaces for the corresponding Cauchy problem on $\mathbb{R}^n$, $n \ge 2$. Bourgain and Pavlovič [4] construct for any $\delta > 0$ mild solutions with initial values u_0 in the Schwartz class such that $\|u_0\|_{\dot{B}^{-1}_{\infty,\infty}} \le \delta$, but $\|u(t)\|_{\dot{B}^{-1}_{\infty,\infty}} > 1/\delta$ for some $0 < t < \delta$. Note that on the one hand, $\dot{B}^{-1}_{\infty,\infty}$ is the largest scale-invariant Banach space of tempered distributions, see Meyer [24]. On the other hand, $BMO^{-1} \subset \dot{B}^{-1}_{\infty,\infty}$ is the largest scale-invariant space for which global well-posedness for small initial data in BMO^{-1} has been proved so far, cf. Koch-Tataru [20]. Yoneda [32] clarifies the approach in [4] and extends the result to $\dot{B}^{-1}_{\infty,s}$, $s > 2$, to be more precise, to a space V satisfying $\dot{B}^{-1}_{\infty,2} \subset V \subset \dot{B}^{-1}_{\infty,s}$. Wang [31] proves this norm inflation phenomenon even for all $1 \le s < \infty$. Finally, Cheskidov and Shvidkoy [5] consider weak solutions of Leray-Hopf type such that $\limsup_{t\to 0} \|u(t) - u_0\|_{B^{-1}_{\infty,\infty}} \ge \delta_0$ for some $\delta_0 > 0$ independent of u_0. Since the inhomogeneous Besov space $B^{-1+3/q}_{q,\infty}$, $1 < q < \infty$, is continuously embedded into $B^{-1}_{\infty,\infty}$ on $\mathbb{R}^3$, this result also yields the ill-posedness of weak solutions at $t = 0$ measured in the space $B^{-1+3/q}_{q,\infty}$. This negative result underlines the importance of using the continuous interpolation space $\overset{\circ}{\mathbb{B}}{}_{q,\infty}^{-1+3/q}$ rather than $\mathbb{B}_{q,\infty}^{-1+3/q}$ in Theorem 1.2

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with $C^{2,1}$ boundary. Let $0 < T \le \infty$, $2 < s < \infty$, $3 < q < \infty$, and $0 \le \alpha < 1/2$ satisfy $2/s + 3/q = 1 - 2\alpha$. Moreover, let u be an $L^s_\alpha(L^q)$-strong solution with initial data $u_0 \in \mathbb{B}_{q,s}^{-1+3/q}$ and $f = \operatorname{div} F$ satisfying*

$F \in L^{s/2}_{2\alpha}\left(0,T;L^{q/2}(\Omega)\right)$. *Then*

$$u \in C\big([0,T];\mathbb{B}^{-1+3/q}_{q,s}\big). \tag{1.2}$$

Theorem 1.2 ($s=\infty$)**.** *Let Ω, T, q be as in Theorem 1.1, and let $2\alpha = 1-3/q$. Then an $L^\infty_\alpha(L^q)$-strong solution u with initial data $u_0 \in \overset{\circ}{\mathbb{B}}{}^{-1+3/q}_{q,\infty}$ and $f = \operatorname{div} F$ satisfying $F \in L^\infty_{2\alpha}\left([0,T];L^{q/2}(\Omega)\right)$ and $\|F\|_{L^\infty_{2\alpha}(0,t;L^{q/2})} \to 0$ as $t \downarrow 0$ satisfies*

$$u \in C\left([0,T];\overset{\circ}{\mathbb{B}}{}^{-1+3/q}_{q,\infty}\right). \tag{1.3}$$

We further observe the continuity of solutions with respect to initial data and external forces.

Theorem 1.3. *Under the assumptions of Theorem 1.1, let v be an $L^s_\alpha(L^q)$-strong solution with initial data $v_0 \in \mathbb{B}^{-1+3/q}_{q,s}$ and external force $G \in L^{s/2}_{2\alpha}\left(0,T;L^{q/2}(\Omega)\right)$. Then there are constants ε_* and C depending only on Ω such that if $T_0 \le T$ is taken so that $\|u\|_{L^s_\alpha(0,T_0;L^q)} \le \varepsilon_*$, $\|v\|_{L^s_\alpha(0,T_0;L^q)} \le \varepsilon_*$, then for all $t \in (0,T_0)$*

$$\|(u-v)(t)\|_{\mathbb{B}^{-1+3/q}_{q,s}} \le C\left(\|u_0-v_0\|_{\mathbb{B}^{-1+3/q}_{q,s}} + \|F-G\|_{L^{s/2}_{2\alpha}\left(0,T_0;L^{q/2}(\Omega)\right)}\right). \tag{1.4}$$

Theorem 1.4 ($s=\infty$)**.** *Under the assumptions of Theorem 1.2, let v be an $L^\infty_\alpha(L^q)$-strong solution with initial data $v_0 \in \overset{\circ}{\mathbb{B}}{}^{-1+3/q}_{q,\infty}$ and external force $G \in L^\infty_{2\alpha}\big(0,T;L^{q/2}(\Omega)\big)$ such that $\|G\|_{L^\infty_{2\alpha}(0,t;L^{q/2})} \to 0$ as $t \downarrow 0$. Then there are constants ε_* and C depending only on Ω such that if $T_0 \le T$ is taken so that $\|u\|_{L^\infty_\alpha(0,T_0;L^q)} \le \varepsilon_*$, $\|v\|_{L^\infty_\alpha(0,T_0;L^q)} \le \varepsilon_*$, then*

$$\|(u-v)(t)\|_{\mathbb{B}^{-1+3/q}_{q,\infty}} \le C\big(\|u_0-v_0\|_{\mathbb{B}^{-1+3/q}_{q,\infty}} + \|F-G\|_{L^\infty_{2\alpha}(0,T_0;L^{q/2})}\big), \quad t \in (0,T_0). \tag{1.5}$$

To show Theorem 1.1 and 1.2, we shall use a semigroup characterization of Besov spaces. We recall the Helmholtz projection $\mathbb{P}_q : L^q(\Omega) \to L^q_\sigma(\Omega)$ and the Stokes operator $A = A_q = -\mathbb{P}_q\Delta$ in $L^q_\sigma(\Omega)$, the closure of $C^\infty_{c,\sigma}(\Omega)$ in $L^q(\Omega)$; here $C^\infty_{c,\sigma}(\Omega)$ denotes the space of smooth solenoidal vector fields with compact support. The semigroup generated by $-A_q$ is denoted by e^{-tA_q} and defines the solution operator $u_0 \mapsto u(t)$ for the Stokes equations in case that $f=0$. Then

$$u_0 \in \mathbb{B}^{-1+3/q}_{q,s} \quad \text{iff} \quad \int_0^T \left(\tau^\alpha \left\|e^{-\tau A}u_0\right\|_q\right)^s \mathrm{d}\tau < \infty$$

with the usual modification if $s=\infty$; for more details see the Appendix in Sect. 5. The results on continuity and well-posedness hold for a (mild) solution $u \in L^s_\alpha(L^q)$ of the corresponding integral equation

$$u(t) = e^{-tA}u_0 - \int_0^t e^{-(t-\tau)A}\left(\mathbb{P}\operatorname{div}(u\otimes u) - \mathbb{P}\operatorname{div}F\right)(\tau)\,\mathrm{d}\tau. \tag{1.6}$$

In Sect. 2 we prepare abstract lemmata for Theorems 1.1–1.4 to be proved in Sect. 4. The essential technical estimates will be performed in Sect. 3. In the Appendix the abstract interpolation spaces introduced in Sect. 2 are identified with solenoidal Besov spaces.

Note that $L^s_\alpha(L^q)$-strong solutions in [9] are defined as the subset of classical weak solutions of Leray-Hopf type in which $u \in L^s_\alpha(L^q)$. Finally, recall that related results can be found in articles by Amann [2] and Haak and Kunstmann [16] as special cases of a more general abstract theory using interpolation-extrapolation scales of Banach spaces; see Remark 4.1 for more details.

2 Abstract spaces

Let X be a Banach space equipped with the norm $\|\cdot\|_X$, and let $-A$ denote the generator of a C_0-analytic semigroup e^{-tA} in X. Assume that $\{z \in \mathbb{C} : \operatorname{Re} z \geq 0\}$ is included in the resolvent set of A. Then $A^{-1} : X \to \mathcal{D}(A)$ is bounded and $A : \mathcal{D}(A) \to X$ is an isometry when $\mathcal{D}(A)$ is equipped with the graph norm $\|A \cdot\|_X$. Moreover, the semigroup e^{-tA} decays exponentially in time, i.e., $\left\|e^{-tA}\right\|_{\mathrm{op}(X)} \leq C_0 e^{-\nu t}$ with some positive constants C_0 and ν; here $\|\cdot\|_{\mathrm{op}(X)}$ denotes the operator norm on X.

Under these assumptions we define the extrapolation space $Z = X_{-1}$ with norm $\|z\|_Z = \|A^{-1}z\|_X$ as the completion $\overline{(X, \|\cdot\|_{-1})}$. Then A_{-1}, defined as the closure of A in X_{-1}, is the unique continuous extension of the isometry $A : \mathcal{D}(A) \to X$ and yields an isometry $A_{-1} : X = \mathcal{D}(A_{-1}) \subset X_{-1} \to X_{-1}$. The semigroup operators e^{-tA} possess continuous extensions to X_{-1} defining an exponentially decaying analytic semigroup with infinitesimal generator A_{-1}, see Proposition 2.1 below. For simplicity we will denote this semigroup by e^{-tA} again. For details we refer to [1, Chapter V], [6, Chapter II.5]. If X is reflexive, then Z is isomorphic to $\big(\mathcal{D}(A')\big)'$, see [6, Chapter II, Exercise 5.9(4)].

Hence, with an abuse of notation, we will write

$$A : X \to Z = AX = \big(\mathcal{D}(A')\big)'$$

defining the isometry $\|Ax\|_Z = \|x\|_X$ for $x \in X$.

For $\alpha \geq 0$, $1 \leq s \leq \infty$, $0 < T < \infty$ and $f \in Z$ we define a "norm" by

$$|f|_{s,\alpha,T} := \begin{cases} \left(\displaystyle\int_0^T \tau^{\alpha s} \left\|e^{-\tau A} f\right\|_X^s \, \mathrm{d}\tau \right)^{1/s} & \text{when } \ s < \infty, \\ \sup_{0<\tau<T} \tau^{\alpha} \left\|e^{-\tau A} f\right\|_X & \text{when } \ s = \infty. \end{cases}$$

The space of all $f \in Z$ having finite norm $|f|_{s,\alpha,T}$ is denoted by $X_{s,\alpha,T}$. By definition, the embedding $X_{s,\alpha,T}$ to Z is continuous.

Writing the norm of f in $X_{s,\alpha,T}$ in the form

$$|f|_{s,\alpha,T} = \left(\int_0^T \tau^{(\alpha+1/s)s} \left\|A e^{-\tau A} f\right\|_Z^s \frac{\mathrm{d}\tau}{\tau} \right)^{1/s} \tag{2.1}$$

we conclude from real interpolation theory applied to the spaces Z and $X = \mathcal{D}(A_{-1})$, see e.g. [23, Proposition 6.2], that this norm is equivalent to the norm on the space $(Z, X)_{1-\alpha-1/s,s}$. Since the semigroup e^{-tA} is assumed to decay exponentially, $T \in (0, \infty]$ can be chosen arbitrarily and the usual additional term $\|f\|_Z$ on the right-hand side of (2.1) can be omitted. In the limit case where $s = \infty$, [23, Exercise 6.1.1 (1)] implies that $X_{\infty,\alpha,T} = (Z, X)_{1-\alpha,\infty}$. Thus, for fixed $\theta = 1 - \alpha - \frac{1}{s} \in (0, 1)$, *i.e.*, $\alpha = \alpha(s) = 1 - \theta - \frac{1}{s} \in [0, 1-\theta]$, we get the scale of interpolation spaces $(Z, X)_{\theta,s}$ for $\frac{1}{1-\theta} =: s_1 \leq s \leq \infty$ and with continuous embeddings

$$\begin{aligned} X &\subset X_{s_1,\alpha(s_1),T} = (Z, X)_{\theta,s_1} \subset X_{s,\alpha(s),T} = (Z, X)_{\theta,s} \\ &\subset X_{\infty,\alpha(\infty),T} = (Z, X)_{\theta,\infty} \subset Z. \end{aligned} \tag{2.2}$$

Proposition 2.1. *(i) For $t > 0$ and $u_0 \in Z$ we have that $e^{-tA}u_0 \in Z$ such that*

$$\|e^{-tA}u_0\|_Z \le \|e^{-tA}\|_{\mathrm{op}(X)}\|u_0\|_Z.$$

Moreover, e^{-tA} extends to a bounded linear operator from Z to X. To be more precise, there exists a constant $c > 0$ independent of t and $u_0 \in Z$ such that

$$\|e^{-tA}u_0\|_X \le ct^{-1}\|u_0\|_Z, \quad t > 0.$$

(ii) The space X is continuously embedded into $X_{s,\alpha,T}$ for all $\alpha \ge 0$, $1 \le s \le \infty$, $0 < T \le \infty$.

(iii) The norms $|\cdot|_{s,\alpha,T}$, $0 < T \le \infty$, are equivalent to each other.

Proof. (i) By analyticity we observe that for $u_0 \in Z$ and for $t > 0$

$$\begin{aligned}\|e^{-tA}u_0\|_Z &= \|A_{-1}^{-1}e^{-tA}u_0\|_X = \|e^{-tA}A_{-1}^{-1}u_0\|_X \\ &\le \|e^{-tA}\|_{\mathrm{op}(X)}\|A^{-1}u_0\|_X \le \|e^{-tA}\|_{\mathrm{op}(X)}\|u_0\|_Z.\end{aligned}$$

If $u_0 = A_{-1}x \in Z$ with $x \in X$, then

$$\|e^{-tA}u_0\|_X = \|A_{-1}e^{-tA}x\|_X \le ct^{-1}\|x\|_X = ct^{-1}\|u_0\|_Z,$$

with some constant c independent of f and t.

(ii), (iii) are standard consequences of real interpolation theory and have been mentioned already above. Here we use the fact that e^{-tA} decays exponentially. □

In view of Proposition 2.1 we often suppress the T-dependence of $X_{s,\alpha,T}$ and assume that $0 < T < \infty$. Besides the spaces $X_{\infty,\alpha,T}$ we also need the closed subspace $\overset{\circ}{X}_{\infty,\alpha}$ defined by

$$\overset{\circ}{X}_{\infty,\alpha} = \left\{ f \in X_{\infty,\alpha} : \sup_{0<\tau<T} \tau^{\alpha} \left\|e^{-\tau A}f\right\|_X \to 0 \quad \text{as} \quad T \to 0 \right\}.$$

Note that by [23, Exercise 6.1.1 (1)] $\overset{\circ}{X}_{\infty,\alpha}$ coincides with the continuous interpolation space $(Z, X)^0_{1-\alpha,\infty}$. Thus obviously $X \subset \overset{\circ}{X}_{\infty,\alpha} \subset X_{\infty,\alpha} \subset Z$.

3 Estimates of Continuity

The first continuity result considers the homogeneous part $e^{-tA}u_0$ in (1.6) and can be proved by general interpolation theory since by (2.1), (2.2) $X_{s,\alpha} = A\,(X, \mathcal{D}(A))_{1-\alpha-\frac{1}{s},s} = (Z, X)_{1-\alpha-\frac{1}{s},s}$. However, we present a direct proof for completeness.

Proposition 3.1. *Let $s \in [1, \infty]$ and $\alpha \ge 0$. Assume that $u_0 \in X_{s,\alpha}$.*

(i) For $t \in (0, T]$ the estimate

$$\left|e^{-tA}u_0\right|_{s,\alpha,T} \le C_T|u_0|_{s,\alpha,T}$$

holds with the constant $C_T = \sup_{t\in(0,T)} \|e^{-tA}\|_{\mathrm{op}(X)}$.

(ii) $e^{-tA}u_0 \in C\big([0,\infty); X_{s,\alpha}\big)$ *if* $u_0 \in X_{s,\alpha}$ *and* $s < \infty$.

(iii) $e^{-tA}u_0 \in C\big([0,\infty); \overset{\circ}{X}_{\infty,\alpha}\big)$ *if* $u_0 \in \overset{\circ}{X}_{\infty,\alpha}$.

(iv) For $u_0 \in X_{\infty,\alpha}$*, continuity holds except at* $t = 0$*, i.e.,* $e^{-tA} \in C\big((0,\infty); \overset{\circ}{X}_{\infty,\alpha}\big)$*. Moreover,* $e^{-tA}u_0 \overset{*}{\rightharpoonup} u_0$ *as* $t \to 0$ *in* $X_{\infty,\alpha}$*; for the latter result* X *is assumed to be reflexive.*

To prove Proposition 3.1, we use the strong continuity of the semigroup e^{-tA} on X and on $\mathcal{D}(A)$ near $t = 0$.

Lemma 3.2. *(i)* $\left\|\left(e^{-tA} - \mathrm{I}\right) f\right\|_X \le c_{\mu,T} t^\mu \left\|A^\mu f\right\|_X$ *for* $\mu \in (0,1]$*,* $t \in (0,T)$ *and* $f \in \mathcal{D}(A^\mu)$ *with a constant* $c_{\mu,T} > 0$ *independent of* f *and* $t > 0$*.*

(ii) $\left\|\left(e^{-tA} - \mathrm{I}\right) e^{-\tau A} f\right\|_X \le c_{\mu,T} \left(\dfrac{t}{\tau}\right)^\mu \|f\|_X$ *for* $\mu \in (0,1]$*,* $t \in (0,T)$ *and* $f \in X$ *with* $c_{\mu,T}$ *independent of* t*,* τ *and* f*.*

Proof of Lemma 3.2. (i) By the fundamental theorem of calculus,

$$e^{-tA}f - f = -\int_0^t Ae^{-\tau A}f \; \mathrm{d}\tau = \int_0^t A^{1-\mu}e^{-\tau A}A^\mu f \; \mathrm{d}\tau.$$

Since $\left\|A^\lambda e^{-tA}\right\|_{\mathrm{op}(X)} \le c_\lambda \tau^{-\lambda}$ $(\lambda > 0)$ by analyticity, we observe that

$$\left\|e^{-tA}f - f\right\|_X \le c_{1-\mu} \int_0^t \frac{\mathrm{d}\tau}{\tau^{1-\mu}} \left\|A^\mu f\right\|_X = c'_\mu t^\mu \left\|A^\mu f\right\|_X.$$

(ii) This follows from (i) since $\left\|A^\mu e^{-\tau A}\right\|_{\mathrm{op}(X)} \le c_\mu \tau^{-\mu}$. □

Proof of Proposition 3.1 (i) This estimate is easy; for example, for $s < \infty$ we have

$$\left|e^{-tA}u_0\right|^s_{s,\alpha,T} = \int_0^T \tau^{\alpha s} \left\|e^{-(\tau+t)A}u_0\right\|^s_X \; \mathrm{d}\tau \le C_T^s |u_0|^s_{s,\alpha,T}.$$

(ii) Let $t_0, t \ge 0$. Then

$$\left|e^{-tA}u_0 - e^{-t_0A}u_0\right|^s_{s,\alpha,T} = \int_0^T \tau^{\alpha s} \left\|\left(e^{-tA} - e^{-t_0A}\right) e^{-\tau A}u_0\right\|^s_X \; \mathrm{d}\tau$$

converges to 0 as $t \to t_0$ by Lebesgue's Theorem on Dominated Convergence since the integrand is uniformly estimated from above by an integrable function in $(0,T)$ and converges to 0 in the pointwise sense. This proves the continuity of $e^{-tA}u_0$ in $[0,\infty)$ with values in $X_{s,\alpha}$.

(iii) Let $t, t_0 \ge 0$. We take $\delta \in (0,T)$ and divide the supremum into two parts:

$$\left|e^{-tA}u_0 - e^{-t_0A}u_0\right|_{\infty,\alpha,T} \le \left(\sup_{\delta<\tau<T} + \sup_{0<\tau<\delta}\right) \tau^\alpha \left\|\left(e^{-tA} - e^{-t_0A}\right) e^{-\tau A}u_0\right\|_X =: J_1 + J_2.$$

Similarly to the case $s < \infty$, we observe that $J_1 \to 0$ as $t \to t_0$. The second term is estimated as

$$J_2 \le 2C_0 \sup_{0<\tau<\delta} \tau^\alpha \left\| e^{-\tau A} u_0 \right\|_X .$$

If $u_0 \in \overset{\circ}{X}_{\infty,\alpha}$, the right-hand side (which is independent of t, t_0) tends to zero as $\delta \to 0$. Thus we conclude the continuity of $e^{-tA}u_0$ up to $t = 0$ with values in $\overset{\circ}{X}_{\infty,\alpha}$.

(iv) If $u_0 \in X_{\infty,\alpha}$, the function $e^{-tA}u_0$ may not be continuous at $t = 0$ with values in $X_{\infty,\alpha}$. However, since $e^{-tA}u_0 \in X$ by Proposition 2.1 for $t > 0$ and $X \subset \overset{\circ}{X}_{s,\alpha}$, the assertion $e^{-tA}u_0 \in C\big((0,\infty); \overset{\circ}{X}_{s,\alpha}\big)$ holds.

For the analysis at $t = 0$ note that we consider $X_{\infty,\alpha} = (Z, X)_{1-\alpha,\infty}$ as the dual space of $(Z', X')_{1-\alpha,1} = (X', \mathcal{D}(A'))_{\alpha,1}$ which is equipped with the norm $\int_0^T \tau^{-\alpha} \|A' e^{-\tau A'} \varphi\|_{X'} \, \mathrm{d}\tau$ for $\varphi \in (X', \mathcal{D}(A'))_{\alpha,1}$. Given φ we get that

$$\begin{aligned} |\langle e^{-tA}u_0 - u_0, \varphi\rangle| &= |\langle u_0, e^{-tA}\varphi - \varphi\rangle| \\ &\le \|u_0\|_{X_{\infty,\alpha}} \|e^{-tA}\varphi - \varphi\|_{(X',\mathcal{D}(A'))_{\alpha,1}}. \end{aligned}$$

To show that the latter term converges to 0 as $t \to 0$ we note that part (ii) of this proposition holds also for negative α as is easily seen. □

To estimate nonlinear terms as on the right hand side of (1.6), we consider for $\mu > 0$ the integral operator

$$(Nf)(t) = \int_0^t A^\mu e^{-(t-\tau)A} f(\tau) \, \mathrm{d}\tau \tag{3.1}$$

for $f \in L^{s_1}_{\alpha_1}(0, T; Y)$. Here Y is another Banach space containing X and e^{-tA} can be extended to Y having a regularizing estimate

$$\left\| e^{-tA} a \right\|_X \le c_T t^{-\eta} \|a\|_Y, \quad a \in Y, \quad t \in (0, T) \tag{3.2}$$

for some $\eta > 0$ with c_T independent of a.

We recall the weighted Hardy-Littlewood-Sobolev inequality [28], [29].

Lemma 3.3. *Assume that $\lambda \in (0, 1)$ satisfies the scale balance of exponents $1/s_1 + \lambda + \alpha_1 - \alpha_2 = 1 + 1/s_2$ under the restrictions of exponents $1 < s_1 \le s_2 < \infty$, $\alpha_2 \le \alpha_1$ and $-1/s_1 < \alpha_1 < 1 - 1/s_1$, $-1/s_2 < \alpha_2 < 1 - 1/s_2$. Then the integral operator*

$$(I_\lambda f)(t) = \int_{\mathbb{R}} |t - \tau|^{-\lambda} f(\tau) \, \mathrm{d}\tau$$

is bounded from $L^{s_1}_{\alpha_1}(\mathbb{R})$ to $L^{s_2}_{\alpha_2}(\mathbb{R})$.

By $\left\| A^\mu e^{-tA} \right\|_{\mathrm{op}(X)} \le C t^{-\mu}$ and (3.2)

$$\|(Nf)(t)\|_X \le C \int_0^t (t - \tau)^{-\mu-\eta} \|f(\tau)\|_Y \, \mathrm{d}\tau, \tag{3.3}$$

so that Lemma 3.3 yields the following:

Proposition 3.4. *Assume that $\lambda = \mu + \eta \in (0,1)$ for positive μ, η as in (3.1), (3.2). Then N defined by* (3.1) *is a bounded operator from $L^{s_1}_{\alpha_1}(0,T;Y)$ to $L^{s_2}_{\alpha_2}(0,T;X)$. Here the exponents are taken as in Lemma 3.3.*

We claim that $Nf(\cdot)$ belongs to $C([0,T];X_{s_2,\alpha_2,T})$.

Theorem 3.5. *Assume that $\lambda = \mu+\eta \in (0,1)$ for positive μ,η as in (3.1), (3.2) satisfies the scale balance $1/s_1 + \lambda + \alpha_1 - \alpha_2 = 1 + 1/s_2$ for exponents $1 < s_1 \le s_2 < \infty$, $\alpha_2 \le \alpha_1$ where $0 \le \alpha_1 < 1 - 1/s_1$, $-1/s_2 < \alpha_2 < 1 - 1/s_2$. If $f \in L^{s_1}_{\alpha_1}(0,T;Y)$, then*

$$|Nf(t)|_{s_2,\alpha_2,T} \le C\|f\|_{L^{s_1}_{\alpha_1}(0,t;Y)}, \quad t \in [0,T]. \tag{3.4}$$

Moreover,

$$Nf \in C([0,T];X_{s_2,\alpha_2,T}).$$

Proof. By definition we get from (3.3) that

$$\begin{aligned}
|Nf(t)|_{s_2,\alpha_2,T} &= \left(\int_0^T \tau^{\alpha_2 s_2} \left\|e^{-\tau A}(Nf)(t)\right\|_X^{s_2} \,\mathrm{d}\tau\right)^{1/s_2} \\
&\le C\left(\left\|\int_0^t (t+\tau-\rho)^{-\mu-\eta}\,\|f(\rho)\|_Y \,\mathrm{d}\rho\right\|^{s_2}_{L^{s_2}_{\alpha_2}(0,T)}\right)^{1/s_2} \\
&= C\left(\int_0^T \left(\tau^{\alpha_2}\int_{\mathbb{R}} |t+\tau-\rho|^{-\lambda}\|(f\chi)(\rho)\|_Y \,\mathrm{d}\rho\right)^{s_2} \mathrm{d}\tau\right)^{1/s_2}
\end{aligned}$$

with $\chi = \chi_{(0,t)}$, the characteristic function of the interval $(0,t)$. Using the change of variables $\tau' = \tau + t$ and that $0 \le \tau' - t \le \tau'$ Lemma 3.3 implies that

$$\begin{aligned}
|Nf(t)|_{s_2,\alpha_2,T} &\le C\left(\int_t^{t+T}\left((\tau'-t)^{\alpha_2}\int_{\mathbb{R}}|\tau'-\rho|^{-\lambda}\|(f\chi)(\rho)\|_Y\,\mathrm{d}\rho\right)^{s_2}\mathrm{d}\tau'\right)^{1/s_2} \\
&\le C\big\|I_\lambda(\|f\chi\|_Y)\big\|_{L^{s_2}_{\alpha_2}(t,t+T)} \\
&\le C\big\|\,\|f\chi\|_Y\big\|_{L^{s_1}_{\alpha_1}} = C\|f\|_{L^{s_1}_{\alpha_1}(0,t;Y)}.
\end{aligned}$$

The proof of continuity is based on the previous estimates. By definition for $t_1 \ge t_2 \ge 0$, we observe that

$$\begin{aligned}
&(Nf)(t_1) - (Nf)(t_2) \\
&= \int_{t_2}^{t_1} A^\mu e^{-(t_1-\rho)A} f(\rho)\,\mathrm{d}\rho + \int_0^{t_2}\left(A^\mu e^{-(t_1-\rho)A} - A^\mu e^{-(t_2-\rho)A}\right) f(\rho)\,\mathrm{d}\rho \\
&=: I_1 + I_2
\end{aligned}$$

The first term is easy to estimate. Replacing f by $f\chi_{(t_2,t_1)}$ and rewriting I_1 as an integral for $f\chi_{(t_2,t_1)}(\rho)$ with $\rho \in (0,t_1)$, (3.4) proves that

$$\begin{aligned}
|I_1|_{s_2,\alpha_2,T} &\le C\left\|\int_0^{t_1}(t_1+\tau-\rho)^{-\mu-\eta}\|f(\rho)\chi_{(t_2,t_1)}(\rho)\|_Y\,\mathrm{d}\rho\right\|_{L^{s_2}_{\alpha_2}(0,T)} \\
&\le C\|f\|_{L^{s_1}_{\alpha_1}(t_2,t_1;Y)} \to 0
\end{aligned}$$

as $t_1 - t_2 \to 0$. The integral I_2 is divided into two parts:

$$|I_2|_{s_2,\alpha_2,T}^{s_2} = \int_0^T \tau^{\alpha_2 s_2} \left\|e^{-\tau A} I_2\right\|_X^{s_2} \,\mathrm{d}\tau = \left(\int_0^\delta + \int_\delta^T\right) \tau^{\alpha_2 s_2} \left\|e^{-\tau A} I_2\right\|_X^{s_2} \,\mathrm{d}\tau.$$

The first part is estimated as follows:

$$\begin{aligned} C \int_0^\delta \tau^{\alpha_2 s_2} & \left\| \int_0^{t_2} A^\mu e^{-(t_2+\tau-\rho)A} f(\rho) \,\mathrm{d}\rho \right\|_X^{s_2} \mathrm{d}\tau \\ & \le C \int_0^\delta \tau^{\alpha_2 s_2} \Big(\int_0^{t_2} (t_2+\tau-\rho)^{-\lambda} \|f(\rho)\|_Y \,\mathrm{d}\rho \Big)^{s_2} \mathrm{d}\tau. \end{aligned}$$

Replacing δ by T, we conclude - as for the estimate of $|I_1|_{s_2,\alpha_2,T}$ - from Lemma 3.3 that the right-hand double integral is bounded by $C\|f\|^{s_2}_{L^{s_1}_{\alpha_1}(0,t_2;Y)}$. Hence, as a function of δ, the right-hand side converges to 0 as $\delta \to 0$, uniformly in $0 \le t_2 \le t_1 \le T$.

To estimate the integral over (δ, T) in $|I_2|^{s_2}_{s_2,\alpha_2,T}$ we observe that

$$\int_\delta^T \tau^{\alpha_2 s_2} \left\|e^{-\tau A} I_2\right\|_X^{s_2} \,\mathrm{d}\tau = \int_\delta^T \tau^{\alpha_2 s_2} \varphi(\tau, t_1, t_2) \,\mathrm{d}\tau$$

where by Lemma 3.2 (ii) for any $\nu_1 \in (0,1)$

$$\begin{aligned} \varphi(\tau, t_1, t_2) &= \left\| \int_0^{t_2} \left(e^{-(t_1-t_2)A} - \mathrm{I}\right) e^{-\tau A} A^\mu e^{-(t_2-\rho)A} f(\rho) \,\mathrm{d}\rho \right\|_X^{s_2} \\ &\le C \left| \frac{t_2 - t_1}{\tau} \right|^{\nu_1 s_2} \left(\int_0^{t_2} \left\|e^{-\tau A/2} A^\mu e^{-(t_2-\rho)A} f(\rho)\right\|_X \,\mathrm{d}\rho \right)^{s_2}. \end{aligned}$$

Thus

$$\begin{aligned} \int_\delta^T & \tau^{\alpha_2 s_2} \varphi(\tau, t_1, t_2) \,\mathrm{d}\tau \\ & \le C \left| \frac{t_2 - t_1}{\delta} \right|^{\nu_1 s_2} \int_0^T \left(\int_0^{t_2} (t_2+\tau-\rho)^{-\lambda} \|f(\rho)\|_Y \,\mathrm{d}\rho \right)^{s_2} \mathrm{d}\tau \\ & \le C \left| \frac{t_2 - t_1}{\delta} \right|^{\nu_1 s_2} \|f\|^{s_2}_{L^{s_1}_{\alpha_1}(0,t_2;Y)} \end{aligned}$$

converges to 0 as $t_2 - t_1 \to 0$ for fixed $\delta > 0$.

Now the proof of continuity in the case of finite s_2 is complete. □

Next we handle the case $X_{\infty,\alpha}$.

Theorem 3.6. *Assume that $\lambda = \mu + \eta \in (0,1)$ for positive μ, η as in (3.1), (3.2), and that $0 \le \alpha_2 = \lambda + \alpha_1 - 1$, $0 < \alpha_1 < 1$. Let $f \in L^\infty_{\alpha_1}(0,T;Y)$ and $\|f\|_{L^\infty_{\alpha_1}(t)} := \|f\|_{L^\infty_{\alpha_1}(0,t;Y)}$ for $0 \le t \le T$. Assume that*

$$\|f\|_{L^\infty_{\alpha_1}(t)} \to 0 \quad \text{as} \quad t \to 0. \tag{3.5}$$

(i) For $t \in (0,T)$

$$\|Nf(t)\|_{X_{\infty,\alpha_2,T}} \le C\|f\|_{L^\infty_{\alpha_1}(t)}$$

Particularly, $Nf(t) \to 0$ as $t \to 0$ and $Nf(t) \in \overset{\circ}{X}_{\infty,\alpha_2,T}$.

(ii) $Nf \in C\big([0,T], \overset{\circ}{X}_{\infty,\alpha_2}\big)$.

Proof. (i) We first observe, by (3.2) and the analyticity of e^{-tA}, that for $0 \le \tau < T$

$$\tau^{\alpha_2} \left\| e^{-\tau A} Nf(t) \right\|_X \le C\tau^{\alpha_2} \int_0^t (t+\tau-\rho)^{-\lambda} \rho^{-\alpha_1} \,\mathrm{d}\rho \, \|f\|_{L^\infty_{\alpha_1}(t)}.$$

Thus, for $t \le \tau < T$,

$$\begin{aligned}\sup_{t\le\tau<T} \tau^{\alpha_2} \left\| e^{-\tau A} Nf(t) \right\|_X &\le C \sup_{t\le\tau<T} \tau^{\alpha_2} \int_0^\tau (\tau-\rho)^{-\lambda} \rho^{-\alpha_1} \,\mathrm{d}\rho \, \|f\|_{L^\infty_{\alpha_1}(t)} \\ &\le CB\|f\|_{L^\infty_{\alpha_1}(t)}\end{aligned}$$

by the scale balance, where $B = B(1-\lambda, 1-\alpha_1)$ is the Beta function. For $\tau \le t$ we have

$$\begin{aligned}\sup_{0<\tau<t} \tau^{\alpha_2} \left\| e^{-\tau A} Nf(t) \right\|_X &\le C \sup_{0<\tau<t} \tau^{\alpha_2} \int_0^t (t-\rho)^{-\lambda} \rho^{-\alpha_1} \,\mathrm{d}\rho \, \|f\|_{L^\infty_{\alpha_1}(t)} \\ &= CB \sup_{0<\tau<t} \tau^{\alpha_2} t^{-\alpha_2} \|f\|_{L^\infty_{\alpha_1}(t)} \\ &= CB\|f\|_{L^\infty_{\alpha_1}(t)}.\end{aligned} \tag{3.6}$$

Hence, under the assumption (3.5),

$$\|Nf(t)\|_{X_{\infty,\alpha_2}} \le CB\|f\|_{L^\infty_{\alpha_1}(t)} \to 0 \quad \text{as} \quad t \to 0.$$

For fixed $t > 0$, a modification of (3.6) also yields for $0 < \tau < \tau_0 < t$ the estimate

$$\sup_{0<\tau<\tau_0} \tau^{\alpha_2} \left\| e^{-\tau A} Nf(t) \right\|_X \le C(t) \sup_{0<\tau<\tau_0} \tau^{\alpha_2} \cdot \|f\|_{L^\infty_{\alpha_1}(t)},$$

i.e., $Nf(t) \in \overset{\circ}{X}_{\infty,\alpha_2}$.

(ii) It remains to prove the continuity for $t \ge \delta > 0$ for arbitrary $\delta > 0$. By definition for $t_1 \ge t_2 \ge \delta > 0$, we observe that

$$\begin{aligned}&(Nf)(t_1) - (Nf)(t_2) \\ &\quad= \int_{t_2}^{t_1} A^\mu e^{-(t_1-\rho)A} f(\rho) \,\mathrm{d}\rho + \int_0^{t_2} \left(e^{-(t_1-t_2)A} - \mathrm{I}\right) A^\mu e^{-(t_2-\rho)A} f(\rho) \,\mathrm{d}\rho \\ &\quad=: I_1 + I_2.\end{aligned}$$

The term I_1 is easy to treat. Due to the boundedness of the operator family $e^{-\tau A}$, $0 \le \tau \le \delta/2$, on X it suffices to consider $\|I_1\|_X$ directly. If $0 < \tau < \delta/2$,

$$\begin{aligned}\|I_1\|_X &\le c \int_{t_2}^{t_1} (t_1-\rho)^{-\lambda} \rho^{-\alpha_1} \,\mathrm{d}\rho \, \|f\|_{L^\infty_{\alpha_1}(T)} \\ &\le c_\delta \int_{t_2}^{t_1} (t_1-\rho)^{-\lambda} \,\mathrm{d}\rho \, \|f\|_{L^\infty_{\alpha_1}(T)} \\ &\le C_\delta \|f\|_{L^\infty_{\alpha_1}(T)} (t_1-t_2)^{1-\lambda}.\end{aligned}$$

Thus

$$\limsup_{\substack{t_1-t_2\to 0\\ t_1,t_2\ge\delta}} \sup_{0<\tau<\delta/2} \tau^{\alpha_2} \left\| e^{-\tau A} I_1 \right\|_X = 0.$$

For the estimate of I_2 we consider $\|I_2\|_X$ directly. By Lemma 3.2 (ii) and with $0 < \mu < 1-\lambda$

$$\begin{aligned}\|I_2\|_X &\le c\int_0^{t_2}\left(\frac{t_1-t_2}{t_2-\rho}\right)^{\mu}(t_2-\rho)^{-\lambda}\|f(\rho)\|_Y\,\mathrm{d}\rho\\ &\le c(t_1-t_2)^{\mu}\int_0^{t_2}(t_2-\rho)^{-\lambda-\mu}\rho^{-\alpha_1}\,\mathrm{d}\rho\,\|f\|_{L^\infty_{\alpha_1}(T)}\\ &\le c_\delta(t_1-t_2)^{\mu}\|f\|_{L^\infty_{\alpha_1}(T)}\end{aligned}$$

since $t_2 \ge \delta > 0$. We thus conclude that

$$\limsup_{\substack{t_1-t_2\to 0\\ t_1,t_2\ge\delta}}\ \sup_{0<\tau<\delta/2}\left\|\tau^{\alpha_2}e^{-\tau A}I_2\right\|_X = 0.$$

Now the assertion $Nf \in C\big((0,T];X_{\infty,\alpha_2}\big)$ is proved. □

4 Proofs of Main Theorems

We shall prove Theorem 1.1 and Theorem 1.2 based on the abstract results given in the previous section.

Proof of Theorem 1.1. We first note that $u_0 \in \mathbb{B}^{-1+3/q}_{q,s}$ is equivalent to $u_0 \in X_{s,\alpha}$ if $X = L^q_\sigma(\Omega)$ and A is taken as the Stokes operator, where $2/s + 3/q = 1-2\alpha$. The $L^s_\alpha(L^q)$-strong solution u satisfies the integral equation

$$\begin{aligned}u(t) &= e^{-tA}u_0 - \int_0^t e^{-(t-\rho)A}\,\mathbb{P}\nabla\cdot\big((u\otimes u)(\rho) - F(\rho)\big)\,\mathrm{d}\rho\\ &= e^{-tA}u_0 - \int_0^t A^{1/2}e^{-(t-\rho)A}\,A^{-1/2}\mathbb{P}\nabla\cdot\big((u\otimes u)(\rho)-F(\rho)\big)\,\mathrm{d}\rho \qquad (4.1)\end{aligned}$$

where $A^{-1/2}\mathbb{P}\nabla$ is bounded in any $L^r(\Omega)$-space, $1 < r < \infty$ (see Giga-Miyakawa [15] and [26]). We observe from the assumptions $u \in L^s_\sigma(L^q)$ and $F \in L^{s/2}_{2\alpha}(L^{q/2})$ that

$$f := A^{-1/2}\mathbb{P}\nabla\cdot\big((u\otimes u) - F\big) \in L^{s/2}_{2\alpha}\big(0,T;L^{q/2}_\sigma(\Omega)\big).$$

We take $Y = L^{q/2}_\sigma(\Omega)$ and $X = L^q_\sigma(\Omega)$ and rewrite (4.1) as

$$u(t) = e^{-tA}u_0 + Nf(t)$$

with $\mu = 1/2$. By Proposition 3.1, $e^{-tA}u_0 \in C\left([0,\infty),X_{s,\alpha}\right)$. Since, see [15],

$$\left\|e^{-tA}v\right\|_X \le C_T t^{-\eta}\|v\|_Y$$

with $\eta = 3/2q$, the operator N satisfies the assumptions (3.1), (3.3) with $\mu = 1/2, \eta = 3/2q$. Thus Theorem 3.5 implies that $Nf \in C\left([0,\infty),X_{s,\alpha}\right)$. The proof is complete. □

Proof of Theorem 1.2. We note the condition $u_0 \in \overset{\circ}{\mathbb{B}}{}^{-1+3/q}_{q,\infty}$ is equivalent to say that $u_0 \in \overset{\circ}{X}_{\infty,\alpha}$ with $3/q = 1 - 2\alpha$. We recall the construction of the solution of (4.1) by the iteration

$$
\begin{aligned}
u_1 &= e^{-tA} u_0, \\
u_{m+1}(t) &= e^{-tA} u_0 - \int_0^t A^{1/2} e^{-(t-\rho)A}\, A^{-1/2} \mathbb{P} \nabla \cdot (u_m \otimes u_m - F)\, \mathrm{d}\rho \quad (m \geq 1).
\end{aligned}
$$

If $u_0 \in \overset{\circ}{X}_{\infty,\alpha}$, by Theorem 3.6 we see that $\|u_{m+1}\|_{L^\infty_\alpha(0,T;X)} \to 0$ as $T \to 0$ and the limit solution u has the same property $\|u\|_{L^\infty_\alpha(0,T,X)} \to 0$ as $T \to 0$.

We now consider (4.1) and apply Proposition 3.1 (ii) and Theorem 3.6 to get the desired continuity. □

Proof of Theorem 1.3. Let u, v be $L^s_\alpha(L^q)$-strong solutions of (1.1) with data $f = \operatorname{div} F, u_0$ and $g = \operatorname{div} G, v_0$. Being mild solutions of (4.1) the difference $w = u - v$ solves the integral equation

$$
w(t) = e^{-tA} w_0 - \int_0^t A^{1/2} e^{-(t-\rho)A}\, A^{-1/2} \mathbb{P} \nabla \cdot (w \otimes u + v \otimes w - (F - G))\, \mathrm{d}\rho. \tag{4.2}
$$

By the *a priori* estimates of Proposition 3.1 and Theorem 3.5 w satisfies the estimate

$$
\|w\|_{X_{s,\alpha,T}} \leq C\big(\|u_0 - v_0\|_{s,\alpha,T} + \|F - G\|_{X_{s/2,2\alpha,T}} + \|w\|_{X_{s,\alpha,T}}(\|u\|_{X_{s,\alpha,T}} + \|v\|_{X_{s,\alpha,T}})\big).
$$

Choosing T sufficiently small, the term involving $\|w\|_{X_{s,\alpha,T}}$ on the right-hand side can be absorbed, thus proving the estimate (1.4). □

Proof of Theorem 1.4. The proof is similar to the proof of Theorem 1.3 using Theorem 3.6 instead of Theorem 3.5. □

Remark 4.1. *(i) Our continuity results (Theorem 1.1 and Theorem 1.2) have strong overlap with results in [16, Remark 4.19, Theorem 4.20]. Their external force f is allowed to be of the form $f = f_0 + \operatorname{div} F$, where both f and F are t-dependent but with values in $L^2(\Omega)$. If there are no external forces, our results are contained in their results. They obtained such results as applications of a heavy, technical machinery whereas our approach is more direct and simple. For example, their basic space X in which the equation* (1.6) *is considered is a Besov space introduced as a real interpolation space of homogeneous versions of $\mathcal{D}(A^{1/2})$ and $\mathcal{D}(A^{-1/2})$ while in our approach* (1.6) *is mainly analyzed in a classical t-weighted $L^q_\sigma(\Omega)$ space.*

(ii) The results of Amann [2] cannot be compared with ours. In Sect. 5 he considers more regular solutions with initial value in $\overset{\circ}{\mathbb{B}}{}^{-1+3/q}_{q,\infty}$ but with forces in weighted C^0 spaces so that solutions are classically regular for $t > 0$, see [2, Theorem 6.1]. Although a weaker force is discussed in Remark 7.3, his space is not Besov type but H^s type when he considers the problem in a domain.

5 Appendix: Besov Spaces

For $1 < q < \infty$, $1 \le r \le \infty$ and $t \in \mathbb{R}$ let $B^t_{q,r}(\mathbb{R}^3)$ denote the usual Besov spaces, see [30, 2.3.1], and define for the bounded domain $\Omega \subset \mathbb{R}^3$ the space $B^t_{q,r}(\Omega)$ by restriction of elements in $B^t_{q,r}(\mathbb{R}^3)$ in the sense of distributions to Ω; the norm of $u \in B^t_{q,r}(\Omega)$ is defined by $\|u\|_{B^t_{q,r}(\Omega)} = \inf\big\{\|v\|_{B^t_{q,r}(\mathbb{R}^3)} : v \in B^t_{q,r}(\mathbb{R}^3), v_{|\Omega} = u\big\}$. Concerning Besov spaces on Ω with vanishing trace - if possible -, the definition is modified as follows: Considering only vector fields rather than scalar-valued functions and the range $t \in [-2, 2]$ we follow Amann [2], [3] and define

$$\mathbf{B}^t_{q,r}(\Omega) = \begin{cases} \{u \in B^t_{q,r}(\Omega)^3;\ u_{|\partial\Omega} = 0\}, & 1/q < t \le 2, \\ \{u \in B^{1/q}_{q,r}(\mathbb{R}^3)^3;\ \mathrm{supp}(u) \subset \overline{\Omega}\}, & 1/q = t, \\ B^t_{q,r}(\Omega)^3, & 0 \le t < 1/q, \\ \big(\mathbf{B}^{-t}_{q',r'}(\Omega)\big)'\ \ (1 < r \le \infty), & -2 \le t < 0. \end{cases} \tag{5.1}$$

For spaces of solenoidal vector fields on Ω let

$$\mathbb{B}^t_{q,r}(\Omega) = \begin{cases} \mathbf{B}^t_{q,r}(\Omega) \cap L^q_\sigma(\Omega), & 0 < t \le 2, \\ \mathrm{cl}\big(C^\infty_{c,\sigma}(\Omega)\big) \text{ in } \mathbf{B}^0_{q,r}(\Omega), & t = 0, \\ \big(\mathbb{B}^{-t}_{q',r'}(\Omega)\big)'\ \ (1 < r \le \infty), & -2 \le t < 0, \end{cases} \tag{5.2}$$

where "cl" denotes the closure. Note that $u \in \mathbb{B}^t_{q,r}(\Omega)$ with $\frac{1}{q} < t \le 2$ vanishes on $\partial\Omega$ by (5.1), but that only the normal component of u vanishes on $\partial\Omega$ when $0 < t \le \frac{1}{q}$ since $u \in L^q_\sigma(\Omega)$.

Moreover, we need the spaces (little Nikol'skii spaces)

$$\overset{\circ}{\mathbb{B}}{}^t_{q,\infty}(\Omega) := \mathrm{cl}\big(\mathbf{H}^t_q(\Omega) \cap L^q_\sigma(\Omega)\big) \text{ in } \mathbb{B}^t_{q,\infty}(\Omega),$$

where $\mathbf{H}^t_q(\Omega)$ is a Bessel potential space defined by restriction of the usual Bessel potential space $H^t_q(\mathbb{R}^3)^3$ to vector fields on Ω (and vanishing on $\partial\Omega$ as in (5.1)), cf. [3, pp. 3-4]. Using the notation $(\cdot,\cdot)_{\theta,r}$, $1 \le r < \infty$, of real interpolation, and $(\cdot,\cdot)^0_{\theta,\infty}$ for the continuous interpolation functor, Theorem 3.4 in [2] states that for $0 < \theta < 1$

$$(L^q_\sigma(\Omega), \mathcal{D}(A_q))_{\theta,r} = \mathbb{B}^{2\theta}_{q,r}(\Omega), \tag{5.3}$$

$$(L^q_\sigma(\Omega), \mathcal{D}(A_q))^0_{\theta,\infty} = \overset{\circ}{\mathbb{B}}{}^{2\theta}_{q,\infty}(\Omega). \tag{5.4}$$

Note that $\mathcal{D}(A_q)$ is equipped with its graph norm, and that for a bounded domain this graph norm can be simplified to $\|A_q \cdot\|_q$. As is well-known ([23, Proposition 6.2, Exercise 6.1.1 (1)], equivalent norms on the spaces $(L^q_\sigma(\Omega), \mathcal{D}(A_q))_{\theta,r}$, $1 \le r \le \infty$, are given by

$$\|u\|_{\mathbb{B}^{2\theta}_{q,r}} \sim \begin{cases} \Big(\displaystyle\int_0^T \big(\tau^{1-\theta}\|A_q e^{-\tau A_q} u\|_q\big)^r \frac{\mathrm{d}\tau}{\tau}\Big)^{1/r} & \text{if } 1 \le r < \infty, \\ \sup_{(0,T)} \tau^{1-\theta}\|A_q e^{-\tau A_q} u\|_q & \text{if } r = \infty, \end{cases}$$

where $T \in (0, \infty)$ can be chosen arbitrarily. The space $\overset{\circ}{\mathbb{B}}{}^{2\theta}_{q,\infty}(\Omega)$ is equipped with the norm of $\mathbb{B}^{2\theta}_{q,\infty}(\Omega)$ but elements $u \in \overset{\circ}{\mathbb{B}}{}^{2\theta}_{q,\infty}(\Omega)$ enjoy the further property that

$$\lim_{\tau \to 0} \tau^{1-\theta}\|A_q e^{-\tau A_q} u\|_q = 0. \tag{5.5}$$

To find similar representations for negative exponents of regularity as well recall that for $-1 < \theta < 0$ and $1 < r < \infty$ by (5.2), (5.3)

$$\big(L^q_\sigma(\Omega), \mathcal{D}(A_{q'})'\big)_{-\theta,r} = \Big(\big(L^{q'}_\sigma(\Omega), \mathcal{D}(A_{q'})\big)_{-\theta,r'}\Big)' = \big(\mathbb{B}^{-2\theta}_{q',r'}(\Omega)\big)' = \mathbb{B}^{2\theta}_{q,r}(\Omega).$$

For the cases $r = 1$ and $r = \infty$ recall from Sect. 2 that A_q is an isomorphism from $\mathcal{D}(A_q)$ to $L^q_\sigma(\Omega)$ and also from $L^q_\sigma(\Omega)$ to $\mathcal{D}(A_{q'})'$. Hence, for $1 \le r \le \infty$ and $-1 < \theta < 0$

$$\big(\mathcal{D}(A_{q'})', L^q_\sigma(\Omega)\big)_{1+\theta,r} = A\Big(\big(L^q_\sigma(\Omega), \mathcal{D}(A_q)\big)_{1+\theta,r}\Big), \tag{5.6}$$

with a similar result for the continuous interpolation functor $(\cdot,\cdot)^0_{\theta,\infty}$. Then we get the characterizations (here $-1 < \theta < 0$):

$$\big(\mathcal{D}(A_{q'})', L^q_\sigma(\Omega)\big)_{1+\theta,r} = \mathbb{B}^{2\theta}_{q,r}(\Omega), \quad 1 \le r < \infty, \tag{5.7}$$

$$\big(\mathcal{D}(A_{q'})', L^q_\sigma(\Omega)\big)_{1+\theta,\infty} = \mathbb{B}^{2\theta}_{q,\infty}(\Omega) \cong \mathbf{B}^{2\theta}_{q,\infty}(\Omega)/\big(\mathbb{B}^{-2\theta}_{q',1}(\Omega)\big)^{\perp}, \tag{5.8}$$

$$\big(\mathcal{D}(A_{q'})', L^q_\sigma(\Omega)\big)^0_{1+\theta,\infty} = \overset{\circ}{\mathbb{B}}{}^{2\theta}_{q,\infty}(\Omega) = \mathrm{cl}\big(\mathbf{H}^2_q(\Omega)\big) \text{ in } \big(\mathbb{B}^{-2\theta}_{q',1}(\Omega)\big)'. \tag{5.9}$$

Actually, (5.7) for $r = 1$ and (5.9) follow from [2, Theorem 3.4], [3, p. 4], for all $-1 < \theta < 0$; the space $\overset{\circ}{\mathbb{B}}{}^{2\theta}_{q,\infty}(\Omega)$ also coincides with the closure $\mathrm{cl}\big(L^q_\sigma(\Omega)\big)$ in $\mathbb{B}^{2\theta}_{q,\infty}(\Omega)$. To prove (5.8) we use (5.3), the duality theorem of real interpolation to get that

$$\big(\mathcal{D}(A_{q'})', L^q_\sigma(\Omega)\big)_{1+\theta,\infty} = \Big(\big(L^{q'}_\sigma(\Omega), \mathcal{D}(A_{q'})\big)_{-\theta,1}\Big)' = \big(\mathbb{B}^{-2\theta}_{q',1}(\Omega)\big)'$$

and the definition from (5.2). The second part of (5.8) is based on the isomorphism

$$\mathbb{B}^{2\theta}_{q,\infty}(\Omega) = \big(\mathbb{B}^{-2\theta}_{q',1}(\Omega)\big)' \cong \mathbf{B}^{2\theta}_{q,\infty}(\Omega)/(\mathbb{B}^{-2\theta}_{q',1}(\Omega))^{\perp},$$

see also [2, Remark 3.6] and its proof; here $\mathbf{B}^{2\theta}_{q,\infty}(\Omega) = \big(\mathbf{B}^{-2\theta}_{q',1}(\Omega)\big)'$ by [30, Theorems 4.3.2, 4.8.1], since in our application $-2\theta = 1 - \frac{3}{q} = 2\alpha > \frac{1}{q'} - 1$ and $-2\theta - \frac{1}{q'} = -\frac{2}{q} \notin \mathbb{Z}$.

Thus for any $1 \le r \le \infty$ and $-1 < \theta < 0$, by (5.6), (5.7), (5.8) and (5.3), $\big(\mathcal{D}(A_{q'})', L^q_\sigma(\Omega)\big)_{1+\theta,r} = A(\mathbb{B}^{2+2\theta}_{q,r}(\Omega)) = \mathbb{B}^{2\theta}_{q,r}(\Omega)$ with equivalent norm

$$\|u\|_{A(\mathbb{B}^{2+2\theta}_{q,r})} \sim \begin{cases} \Big(\displaystyle\int_0^T \big(\tau^{-\theta}\|e^{-\tau A_q}u\|_q\big)^r \frac{\mathrm{d}\tau}{\tau}\Big)^{1/r} & \text{if } 1 \le r < \infty, \\ \sup_{\tau\in(0,T)} \tau^{-\theta}\|e^{-\tau A_q}u\|_q & \text{if } r = \infty. \end{cases} \tag{5.10}$$

This result was used in [12] when $\frac{2}{r} + \frac{3}{q} = 1$, $\theta = 0$, $2 < r < \infty$. For the continuous interpolation space $\big(\mathcal{D}(A_{q'})', L^q_\sigma(\Omega)\big)^0_{1+\theta,\infty} = \overset{\circ}{\mathbb{B}}{}^{2\theta}_{q,\infty}(\Omega)$ we have the norm defined in (5.10), with the additional property that

$$\lim_{\tau\to 0} \tau^{-\theta}\|e^{-\tau A_q}u\|_q = 0.$$

Summarizing the previous arguments we get the following theorem.

Theorem 5.1. *Choose any $T \in (0, \infty)$.*

(i) Let $2 < s < \infty$, $3 < q < \infty$ and $0 < \alpha < \frac{1}{2}$ such that $\frac{2}{s} + \frac{3}{q} = 1 - 2\alpha$. Then the real interpolation space $\big(\mathcal{D}(A_{q'})', L^q_\sigma(\Omega)\big)_{1-\alpha,s}$ coincides with the Besov space $\mathbb{B}^{-1+3/q}_{q,s}(\Omega)$ and has the equivalent norm $\big(\int_0^T (\tau^\alpha \|e^{-\tau A_q} u\|_q)^s \, \mathrm{d}\tau\big)^{1/s}$.

(ii) If $3 < q < \infty$ and $0 < \alpha < \frac{1}{2}$ such that $\frac{3}{q} = 1 - 2\alpha$, the real interpolation space $\big(\mathcal{D}(A_{q'})', L^q_\sigma(\Omega)\big)_{1-\alpha,\infty}$ coincides with the space of Besov-type $\mathbb{B}^{-1+3/q}_{q,\infty}(\Omega)$ and has the equivalent norm $\sup_{\tau \in (0,T)} \tau^\alpha \|e^{-\tau A_q} u\|_q$.

(iii) The interpolation space $\big(\mathcal{D}(A_{q'})', L^q_\sigma(\Omega)\big)^0_{1-\alpha,\infty}$ equals the Besov space $\mathring{\mathbb{B}}^{-1+3/q}_{q,\infty}(\Omega)$, equipped with the norm of $\mathbb{B}^{-1+3/q}_{q,\infty}(\Omega)$ such that the property $\lim_{\tau \to 0} \tau^\alpha \|e^{-\tau A_q} u\|_q = 0$ additionally holds for $u \in \mathring{\mathbb{B}}^{-1+3/q}_{q,\infty}(\Omega)$.

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