Extension-restriction theorems for algebras of approximation sequences

Steffen Roch

Abstract

The C^*-algebra $\mathcal{S}(\mathsf{T}(C))$ of the finite sections discretization for Toeplitz operators with continuous generating function is fairly well understood. Since its description in [3], this algebra had served both as a source of inspiration and as an archetypal example of an algebra generated by an discretization procedure. The latter is no accident: it turns out that, after suitable extension by compact sequences and suitable fractal restriction, *every* separable C^*-algebra of approximation sequences has the same structure as $\mathcal{S}(\mathsf{T}(C))$. We explain what this statement means and give a proof.

Keywords: finite sections discretization, Toeplitz operators, fractal restriction, compact extension, Silbermann algebras

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1 Introduction

Let H be a Hilbert space and $\mathcal{P} = (P_n)_{n\geq 1}$ a filtration on H, i.e. a sequence of orthogonal projections of finite rank that converges strongly to the identity operator on H. By $\mathcal{F}^{\mathcal{P}}$ we denote the set of all bounded sequences $(A_n)_{n\geq 1}$ of operators $A_n \in L(\operatorname{im} P_n)$, and by $\mathcal{G}^{\mathcal{P}}$ the set of all sequences $(A_n) \in \mathcal{F}^{\mathcal{P}}$ with $\|A_n\| \to 0$. Provided with the operations

$$(A_n) + (B_n) := (A_n + B_n), \quad (A_n)(B_n) := (A_n B_n), \quad (A_n)^* := (A_n^*)$$

and the norm $\|(A_n)\| := \sup \|A_n P_n\|$, $\mathcal{F}^{\mathcal{P}}$ becomes a unital C^*-algebra and $\mathcal{G}^{\mathcal{P}}$ a closed ideal of $\mathcal{F}^{\mathcal{P}}$. The importance of the quotient algebra $\mathcal{F}^{\mathcal{P}}/\mathcal{G}^{\mathcal{P}}$ in numerical analysis stems from the fact that a coset $(A_n) + \mathcal{G}^{\mathcal{P}}$ is invertible in $\mathcal{F}^{\mathcal{P}}/\mathcal{G}^{\mathcal{P}}$ if and only if the A_n are invertible for all sufficiently large n and if the norms of the inverses are uniformly bounded, which is equivalent to saying that (A_n) is a stable sequence. Note in that connection that

$$\|(A_n) + \mathcal{G}^{\mathcal{P}}\|_{\mathcal{F}^{\mathcal{P}}/\mathcal{G}^{\mathcal{P}}} = \limsup \|A_n\|. \tag{1}$$

With every non-empty subset A of $L(H)$, we associate the smallest closed and symmetric subalgebra $\mathcal{S}^{\mathcal{P}}(\mathsf{A})$ of $\mathcal{F}^{\mathcal{P}}$ that contains all sequences $(P_nAP_n)_{n\geq 1}$ with $A \in \mathsf{A}$. The algebra $\mathcal{S}(\mathsf{T}(C))$ of the finite sections discretization for Toeplitz operators with continuous generating function arises exactly in this way. Here, H is the Hilbert space $l^2(\mathbb{Z}^+)$, P_n is the projection which sends the sequence $(x_0, x_1, \ldots)$ to $(x_0, \ldots, x_{n-1}, 0, 0, \ldots)$, and A is the C^*-algebra $\mathsf{T}(C)$ generated by all Toeplitz operators $T(a)$ with a a continuous function on the complex unit circle $\mathbb{T}$. Recall that $T(a)$ is given by the matrix representation $(a_{i-j})_{i,j\geq 0}$ with respect to the standard basis of $l^2(\mathbb{Z}^+)$, where a_k denotes the kth Fourier coefficient of a. (We agree to omit the superscript $\mathcal{P}$ when the filtration is specified in this way.)

The sequences in $\mathcal{S}(\mathsf{T}(C))$ are completely described in the following theorem, discovered by A. Böttcher and B. Silbermann and first published in their 1983 paper [3] on the convergence of the finite sections method for quarter plane Toeplitz operators (see also [4] and [6], Section 1.4.2). Here R_n stands for the operator $(x_0, x_1, \ldots) \mapsto (x_{n-1}, \ldots, x_0, 0, 0, \ldots)$ on $l^2(\mathbb{Z}^+)$. It is not hard to see that for each sequence $\mathbf{A} = (A_n) \in \mathcal{S}(\mathsf{T}(C))$, the strong limits $W(\mathbf{A}) := \text{s-lim}\, A_nP_n$ and $\widetilde{W}(\mathbf{A}) := \text{s-lim}\, R_nA_nR_nP_n$ exist and that W and $\widetilde{W}$ are unital *-homomorphisms from $\mathcal{S}(\mathsf{T}(C))$ to $L(l^2(\mathbb{Z}^+))$ (actually, to $\mathsf{T}(C)$).

Theorem 1 (*a*) *The algebra $\mathcal{S}(\mathsf{T}(C))$ consists of all sequences $(A_n)_{n\geq 1}$ of the form*

$$(A_n) = (P_nT(a)P_n + P_nKP_n + R_nLR_n + G_n) \tag{2}$$

where $a \in C(\mathbb{T})$, K and L are compact operators on $l^2(\mathbb{Z}^+)$, and $(G_n) \in \mathcal{G}$. The representation of a sequence $(A_n) \in \mathcal{S}(\mathsf{T}(C))$ in this form is unique.

(*b*) *For every sequence $\mathbf{A} \in \mathcal{S}(\mathsf{T}(C))$, the coset $\mathbf{A}+\mathcal{G}$ is invertible in the quotient algebra $\mathcal{S}(\mathsf{T}(C))/\mathcal{G}$ (equivalently, in $\mathcal{F}/\mathcal{G}$) if and only if the operators $W(\mathbf{A})$ and $\widetilde{W}(\mathbf{A})$ are invertible.*

Corollary 2 *The quotient algebra $\mathcal{S}(\mathsf{T}(C))/\mathcal{G}$ is *-isomorphic to the C^*-algebra of all pairs*

$$(T(a) + K,\, T(\tilde{a}) + L) \in L(l^2(\mathbb{Z}^+)) \times L(l^2(\mathbb{Z}^+)) \tag{3}$$

with $a \in C(\mathbb{T})$ and K, L compact. In particular, the mapping which sends the sequence (2) *to the pair* (3) *is a *-homomorphism from $\mathcal{S}(\mathsf{T}(C))$ onto $\mathcal{S}(\mathsf{T}(C))/\mathcal{G}$ with kernel $\mathcal{G}$.*

Now we can give a first idea of what the statement in the abstract that "an algebra has the same structure as $\mathcal{S}(\mathsf{T}(C))$" means. It is not hard to check that the set $\mathcal{J}$ of all sequences $(P_nKP_n + R_nLR_n + G_n)$ with K, L compact and $(G_n) \in \mathcal{G}$ forms a closed two-sided ideal of $\mathcal{S}(\mathsf{T}(C))$. By Corollary 2, the quotient $\mathcal{J}/\mathcal{G}$ is naturally isomorphic to the product $K(l^2(\mathbb{Z}^+)) \times K(l^2(\mathbb{Z}^+))$. Thus, this quotient has two natural irreducible representations $(K, L) \mapsto K$ and

$(K, L) \mapsto L$ which extend uniquely to irreducible representations of $\mathcal{S}(\mathsf{T}(C))/\mathcal{G}$. These extensions coincide (up to unitary equivalence) with the quotients $W^{\mathcal{G}}$ and $\widetilde{W}^{\mathcal{G}}$ of the homomorphisms W and $\widetilde{W}$ by $\mathcal{G}$ (note that $\mathcal{G}$ lies in the kernel of both W and $\widetilde{W}$), and these quotients have the property described in assertion (b) of Theorem 1.

To fix the latter property formally, we introduce the following notions. Given a unital C^*-algebra $\mathcal{A}$ and a family $\mathcal{W}$ of unital *-homomorphisms from $\mathcal{A}$ to certain unital C^*-algebras, we say that $\mathcal{W}$ is *strongly spectral* for $\mathcal{A}$ if an element $a \in \mathcal{A}$ is invertible if and only if $W(a)$ is invertible for every $W \in \mathcal{W}$. We say that $\mathcal{W}$ is *spectral* for $\mathcal{A}$ if an element $a \in \mathcal{A}$ is invertible if and only if $W(a)$ is invertible for every $W \in \mathcal{W}$ and if $\sup_{W \in \mathcal{W}} \|(W(A))^{-1}\| < \infty$. Clearly, both notions coincide if $\mathcal{W}$ is a finite family.

We recall further that a C^*-algebra $\mathcal{A}$ is called *elementary* if it is isomorphic to the algebra of the compact operators on some Hilbert space, and that it is a *dual* algebra if it is isomorphic to a direct sum of elementary algebras. If $\mathcal{J}$ a closed ideal of $\mathcal{A}$ which is elementary (respective dual) when considered as a C^*-algebra, then we call $\mathcal{J}$ an *elementary* (respective a *dual*) ideal of $\mathcal{A}$. It is easy to see that every dual ideal $\mathcal{J}$ is generated (as a C^*-algebra) by its elementary ideals, $\mathcal{K}_t$ with $t \in T$, say. Since every closed ideal of $\mathcal{J}$ is also a closed ideal of $\mathcal{A}$, $\mathcal{J}$ can be identified with the smallest closed ideal of $\mathcal{A}$ which contains all elementary ideals ideals $\mathcal{K}_t$. See [2] for an overview on dual algebras.

Now we can make the statement that a C^*-subalgebra $\mathcal{A}$ of $\mathcal{F}$ has the "same structure as $\mathcal{S}(\mathsf{T}(C))$" more precise: It means that $\mathcal{A}$ contains $\mathcal{G}$, that the quotient $\mathcal{A}/\mathcal{G}$ contains an ideal $\mathcal{J}$ which is dual, and hat the extensions of the unitary representations of $\mathcal{J}/\mathcal{G}$ to $\mathcal{A}/\mathcal{G}$ form a spectral family for $\mathcal{A}/\mathcal{G}$.

The motivation for that paper came from the observation made again and again over the years since Silbermann's paper [17] that many concrete algebras of approximation sequences have "the same structure as $\mathcal{S}(\mathsf{T}(C))$". This observation appeared (at least to the author) as a big miracle, since neither were the discretized operators very close to Toeplitz operators, nor had the used discretization procedures something in common with the fairly simple idea of taking finite sections. To get an impression, here is a very incomplete list of papers from different fields where this phenomenon occurs: [3, 5, 7, 8, 10]. The explanation of that fact proposed in the present paper is that at least every *separable* subalgebra of $\mathcal{F}$ has "the same structure as $\mathcal{S}(\mathsf{T}(C))$", after *suitable extension by compact sequences and suitable fractal restriction.*

The paper is organized as follows. The phrases "fractal restriction" and "extension by compact sequences" are explained in the following two sections, followed by a section which studies "fractal algebras of compact sequences". The goal of these sections is to introduce the language and to provide some facts for later reference without proofs. The heart of the paper is Section 5 where two versions of extension-restriction theorems are derived.

2 Fractal restriction

The idea behind the notion of a fractal algebra comes from a remarkable property of the algebra $\mathcal{S}(\mathsf{T}(C))/\mathcal{G}$: the structure of this algebra is determined by two representations W and $\widetilde{W}$, and these representations are defined by certain strong limits. A consequence of this "limit form" is that the operators $W(\mathbf{A})$ and $\widetilde{W}(\mathbf{A})$ can be determined from each subsequence of the sequence $\mathbf{A} \in \mathcal{S}(\mathsf{T}(C))$. This observation implies that whenever a *subsequence* of a sequence $\mathbf{A} \in \mathcal{S}(\mathsf{T}(C))$ is stable, then the operators $W(\mathbf{A})$ and $\widetilde{W}(\mathbf{A})$ are already invertible and, hence, the *full* sequence $\mathbf{A}$ is stable by Theorem 1.

One can state this observation in a slightly different way: every sequence in $\mathcal{S}(\mathsf{T}(C))$ can be rediscovered from each of its (infinite) subsequences up to a sequence tending to zero in the norm. In that sense, the essential information on a sequence in $\mathcal{S}(\mathsf{T}(C))$ is stored in each of its subsequences. Subalgebras of $\mathcal{F}$ with this property were called *fractal* in [16] in order to emphasize exactly this self-similarity aspect. We will recall some basic properties of fractal algebras that will be needed in what follows and start with the official definition of a fractal algebra. (Note that this definition also makes sense in a more general context when $\mathcal{F}$ is the direct product and $\mathcal{G}$ the direct sum of a sequence of C^*-algebras.)

Let $\eta : \mathbb{N} \to \mathbb{N}$ be a strictly increasing sequence. By $\mathcal{F}_\eta$ we denote the set of all subsequences $(A_{\eta(n)})$ of sequences (A_n) in $\mathcal{F}$. One can make $\mathcal{F}_\eta$ to a C^*-algebra in a natural way. The mapping $R_\eta : \mathcal{F} \to \mathcal{F}_\eta, \ (A_n) \mapsto (A_{\eta(n)})$ is called the *restriction* of $\mathcal{F}$ onto $\mathcal{F}_\eta$. For every subset $\mathcal{S}$ of $\mathcal{F}$, we abbreviate $R_\eta \mathcal{S}$ by $\mathcal{S}_\eta$. It is easy to see that $\mathcal{G}_\eta$ coincides with the ideal of the sequences in $\mathcal{F}_\eta$ which tend to zero in the norm.

Let $\mathcal{A}$ be a C^*-subalgebra of $\mathcal{F}$. A *-homomorphism W from $\mathcal{A}$ into a C^*-algebra $\mathcal{B}$ is called *fractal* if, for every strictly increasing sequence $\eta : \mathbb{N} \to \mathbb{N}$, there is a mapping $W_\eta : \mathcal{A}_\eta \to \mathcal{B}$ such that $W = W_\eta R_\eta|_{\mathcal{A}}$. Thus, the image of a sequence in $\mathcal{A}$ under a fractal homomorphism can be reconstructed from each of its (infinite) subsequences. It is not hard to check that W_η is a *-homomorphism again. The homomorphisms W and $\widetilde{W}$ defined in Section 1 are archetypal examples of a fractal homomorphism.

Definition 3 *(a) A C^*-subalgebra $\mathcal{A}$ of $\mathcal{F}$ is called* fractal *if the canonical homomorphism $\pi : \mathcal{A} \to \mathcal{A}/(\mathcal{A} \cap \mathcal{G}), \ \mathbf{A} \mapsto \mathbf{A} + (\mathcal{A} \cap \mathcal{G})$ is fractal.*
(b) A sequence $\mathbf{A} \in \mathcal{F}$ is called fractal *if the smallest C^*-subalgebra of $\mathcal{F}$ which contains the sequence $\mathbf{A}$ and the identity sequence is fractal.*

Here are some equivalent characterizations of fractal algebras.

Theorem 4 *(a) A C^*-subalgebra $\mathcal{A}$ of $\mathcal{F}$ is fractal if and only if the implication*

$$R_\eta(\mathbf{A}) \in \mathcal{G}_\eta \ \Rightarrow \ \mathbf{A} \in \mathcal{G} \tag{4}$$

holds for every sequence $\mathbf{A} \in \mathcal{A}$ *and every strictly increasing sequence* η.

(*b*) *If* $\mathcal{A}$ *is a fractal* C^**-subalgebra of* $\mathcal{F}$, *then* $\mathcal{A}_\eta \cap \mathcal{G}_\eta = (\mathcal{A} \cap \mathcal{G})_\eta$ *for every strictly increasing sequence* η.

(*c*) *A* C^**-subalgebra* $\mathcal{A}$ *of* $\mathcal{F}$ *is fractal if and only if the algebra* $\mathcal{A} + \mathcal{G}$ *is fractal.*

In many instances, the following theorem offers a comfortable way to check the fractality of a specific subalgebra of $\mathcal{F}$ (for example, that of $\mathcal{S}(\mathsf{T}(C))$), where the homomorphism W appearing in the theorem is the product of fractal homomorphisms W and $\widetilde{W}$).

Theorem 5 *A unital* C^**-subalgebra* $\mathcal{A}$ *of* $\mathcal{F}$ *is fractal if and only if there is a unital and fractal* **-homomorphism* W *from* $\mathcal{A}$ *into a unital* C^**-algebra* $\mathcal{B}$ *such that, for every sequence* $\mathbf{A} \in \mathcal{A}$, *the coset* $\mathbf{A} + (\mathcal{A} \cap \mathcal{G})$ *is invertible in* $\mathcal{A}/(\mathcal{A} \cap \mathcal{G})$ *if and only if* $W(\mathbf{A})$ *is invertible in* $\mathcal{B}$.

The following results from [16] give a first impression of the power of fractality.

Proposition 6 *Let* $\mathcal{A}$ *be a unital fractal* C^**-subalgebra of* $\mathcal{F}$ *and* $\mathbf{A} = (A_n) \in \mathcal{A}$. *Then,*

(*a*) *the sequence* $\mathbf{A}$ *is stable if and only if it possesses a stable subsequence.*

(*b*) *the limit* $\lim_{n\to\infty} \|A_n\|$ *exists and is equal to* $\|\mathbf{A} + \mathcal{G}\|$.

Thus, the upper limit in (1) is in fact a limit if the sequence (A_n) belongs to a fractal algebra. A similar improvement can be observed for the convergence of certain spectral quantities. We will need the following notions.

Let $(M_n)_{n\in\mathbb{N}}$ be a sequence of non-empty compact subsets of the complex plane. The *upper limit* $\limsup M_n$ (also called the *partial limiting set*) resp. the *lower limit* $\liminf M_n$ (or the *uniform limiting set*) of the sequence (M_n) consists of all points $x \in \mathbb{C}$ which are a partial limit resp. the limit of a sequence (m_n) of points $m_n \in M_n$. The upper and lower limit of a sequence (M_n) coincide if and only if this sequence converges with respect to the Hausdorff metric

$$h(L, M) := \max\{\max_{l\in L} \operatorname{dist}(l, M), \max_{m\in M} \operatorname{dist}(m, L)\}.$$

Recall in this connection that non-empty compact subsets of $\mathbb{C}$ form a complete metric space with respect to the Hausdorff distance and that the relatively compact subsets of this space are precisely its bounded subsets.

Assertion (*a*) of the following result is the analog of the limsup-formula (1) for norms; assertion (*b*) its improvement in the presence of fractality.

Proposition 7 (*a*) *If* $(A_n) \in \mathcal{F}$ *is a normal sequence, then*

$$\limsup \sigma(A_n) = \sigma_{\mathcal{F}/\mathcal{G}}((A_n) + \mathcal{G}).$$

(b) *If* (A_n) *is a normal sequence in a unital and fractal* C^**-subalgebra of* $\mathcal{F}$*, then*

$$\limsup \sigma(A_n) = \liminf \sigma(A_n). \tag{5}$$

(c) *A normal sequence* $(A_n) \in \mathcal{F}$ *is fractal if and only if* (5) *holds.*

Similar results hold for other spectral quantities, for example for the sequences of the condition numbers, the sets of the singular values, the ϵ-pseudospectra, and the numerical ranges of the A_n (see Chapter 3 in [6]). These results indicate that, given an (in general non-fractal) sequence in $\mathcal{F}$, it is of vital importance to single out (one of) its fractal subsequences. That this is indeed possible is a consequence of the following *fractal restriction theorem* first proved in [11]. The proof given there was based on Proposition 7 (c) and is rather involved. A much simpler proof employs the converse of Proposition 6 (b), which on its hand follows easily from Theorem 4 (a).

Theorem 8 *Let* $\mathcal{A}$ *be a separable* C^**-subalgebra of* $\mathcal{F}$*. Then there is a strictly increasing sequence* $\eta : \mathbb{N} \to \mathbb{N}$ *such that the restricted algebra* $\mathcal{A}_\eta$ *is fractal. In particular, every sequence in* $\mathcal{F}$ *possesses a fractal subsequence.*

One cannot expect that Theorem 8 holds for arbitrary C^*-subalgebras of $\mathcal{F}$; for example it is certainly not true for the algebra l^∞. On the other hand, non-separable fractal algebras exist: the finite sections algebra for Toeplitz operators with *piecewise continuous* generating function can serve as an example.

3 Compactness and essential fractality

The ideal $\mathcal{J}$ of the finite sections algebra $\mathcal{S}(\mathsf{T}(C))$ is evidently related with compact operators. We will make this relation precise by introducing an ideal $\mathcal{K}$ of $\mathcal{F}$ consisting of sequences of compact type. The role of this ideal in numerical analysis is comparable with the role of the ideal of the compact operators in operator theory. Throughout this section we assume that $\mathcal{F}$ is a direct product of matrix algebras $\mathbb{C}^{\delta(n) \times \delta(n)}$ with dimension function $\delta : \mathbb{N} \to \mathbb{N}$ tending to infinity.

Definition 9 *Let* $\mathcal{K}$ *denote the smallest closed ideal of* $\mathcal{F}$ *which contains all sequences* $(K_n) \in \mathcal{F}$ *with* $\operatorname{rank} K_n \leq 1$ *for every* n*. We refer to the elements of* $\mathcal{K}$ *as* compact sequences *and call a sequence in* $\mathcal{F}$ *a* Fredholm sequence *if it is invertible modulo the ideal* $\mathcal{K}$*.*

Thus, a sequence $(A_n) \in \mathcal{F}$ belongs to $\mathcal{K}$ if and only if, for every $\varepsilon > 0$, there is a sequence $(K_n) \in \mathcal{F}$ such that

$$\sup_n \|A_n - K_n\| < \varepsilon \quad \text{and} \quad \sup_n \operatorname{rank} K_n < \infty. \tag{6}$$

It is a simple consequence of this fact that $\mathcal{G} \subseteq \mathcal{K}$.

We say that a sequence $\mathbf{A} \in \mathcal{F}$ has *finite essential rank* if it is the sum of a sequence in $\mathcal{G}$ and a sequence $(K_n) \in \mathcal{F}$ with $\sup_n \operatorname{rank} K_n < \infty$. If $\mathbf{A}$ is of finite essential rank, then there is a smallest integer $r \geq 0$ such that $\mathbf{A}$ can be written as $(G_n) + (K_n)$ with $(G_n) \in \mathcal{G}$ and $\sup_n \operatorname{rank} K_n = r$. We call this integer the *essential rank* of $\mathbf{A}$ and write $\operatorname{ess\,rank} \mathbf{A} = r$. If $\mathbf{A}$ is not of finite essential rank, then we put $\operatorname{ess\,rank} \mathbf{A} = \infty$. Thus, the sequences of essential rank 0 are just the sequences in $\mathcal{G}$.

In our running example, the intersection $\mathcal{S}(\mathsf{T}(C)) \cap \mathcal{K}$ is just the ideal $\mathcal{J}$, and the essential rank of the sequence $(P_n K P_n + R_n L R_n + G_n)$ equals $\operatorname{rank} K + \operatorname{rank} L$.

Both the compactness and the Fredholm property of a sequence $(A_n) \in \mathcal{F}$ can be characterized in terms of the asymptotic behavior of the singular values of the A_n; see Sections 4.2 and 5.1 in [13] for the following results. We denote the decreasingly ordered singular values of an $n \times n$ matrix A by

$$\|A\| = \Sigma_1(A) \geq \Sigma_2(A) \geq \ldots \geq \Sigma_n(A) \geq 0 \tag{7}$$

and set $\sigma_k(A) := \Sigma_{n-k+1}(A)$.

Theorem 10 *The following conditions are equivalent for a sequence $(K_n) \in \mathcal{F}$:*
(a) $\lim_{k\to\infty} \sup_{n \geq k} \Sigma_k(K_n) = 0$*;*
(b) $\lim_{k\to\infty} \limsup_{n\to\infty} \Sigma_k(K_n) = 0$*;*
(c) (K_n) is compact.

Corollary 11 *(a) A sequence $(K_n) \in \mathcal{F}$ is of essential rank r if and only if*

$$\limsup_{n\to\infty} \Sigma_r(K_n) > 0 \quad \textit{and} \quad \lim_{n\to\infty} \Sigma_{r+1}(K_n) = 0.$$

(b) If $(K_n) \in \mathcal{K}$, then $\lim_{n\to\infty} \sigma_k(K_n) = 0$ for every k.

Theorem 12 *The following conditions are equivalent for a sequence $(A_n) \in \mathcal{F}$:*
(a) (A_n) is a Fredholm sequence.
(b) There are sequences $(B_n) \in \mathcal{F}$ and $(J_n) \in \mathcal{K}$ with $\sup_n \operatorname{rank} J_n < \infty$ such that $B_n A_n = I_n + J_n$ for all $n \in \mathbb{N}$.
(c) There is a $k \in \mathbb{N}$ such that $\liminf_{n\to\infty} \sigma_{k+1}(A_n) > 0$.

The smallest non-negative integer k which satisfies condition (c) in the previous theorem is called the *α-number* $\alpha(\mathbf{A})$ of the Fredholm sequence $\mathbf{A} = (A_n)$. It corresponds to the kernel dimension of a Fredholm operator. Equivalently, $\alpha(\mathbf{A})$ is the smallest non-negative integer k for which there exist a sequence $(B_n) \in \mathcal{F}$ and a sequence $(J_n) \in \mathcal{K}$ of essential rank k such that $B_n A_n^* A_n = I_n + J_n$ for all $n \in \mathbb{N}$.

Next we extend the concept of fractality and define fractality with respect to the ideal of the compact sequences. For details see [15].

Definition 13 *A C^*-subalgebra $\mathcal{A}$ of $\mathcal{F}$ is said to be* essentially fractal (*or* $\mathcal{K}$-fractal) *if the canonical homomorphism* $\pi^{\mathcal{K}} : \mathcal{A} \to \mathcal{A}/(\mathcal{A} \cap \mathcal{K})$ *is fractal and if* $(\mathcal{A} \cap \mathcal{K})_\eta = \mathcal{A}_\eta \cap \mathcal{K}_\eta$ *for each strictly increasing sequence* $\eta : \mathbb{N} \to \mathbb{N}$.

In the same way one defines $\mathcal{J}$-fractality with respect to an arbitrary closed ideal $\mathcal{J}$ of $\mathcal{F}$. If $\mathcal{J} = \mathcal{G}$, then the second condition in Definition 13 is automatically satisfied by Theorem 4; thus, $\mathcal{G}$-fractality coincides with fractality in the sense of Definition 3.

The following result shows that essential fractality implies what one expects.

Theorem 14 *Let $\mathcal{A}$ be an essentially fractal and unital C^*-subalgebra of $\mathcal{F}$. A sequence in $\mathcal{A}$ is compact (Fredholm) and only if one of its subsequences is compact (Fredholm), respectively.*

Corollary 15 *Let $\mathcal{A}$ be a fractal C^*-subalgebra of $\mathcal{F}$. If $(\mathcal{A} \cap \mathcal{K})_\eta = \mathcal{A}_\eta \cap \mathcal{K}_\eta$ for each strictly increasing sequence $\eta : \mathbb{N} \to \mathbb{N}$, then $\mathcal{A}$ is essentially fractal.*

Essential fractality has striking consequences for the behavior of the smallest singular values of a Fredholm sequence.

Theorem 16 *Let $\mathcal{A}$ be an essentially fractal and unital C^*-subalgebra of $\mathcal{F}$. A sequence $(A_n) \in \mathcal{A}$ is Fredholm if and only if there is a $k \in \mathbb{N}$ such that*

$$\limsup_{n\to\infty} \sigma_k(A_n) > 0. \tag{8}$$

Consequently, if a sequence (A_n) in an essentially fractal and unital C^*-subalgebra of $\mathcal{F}$ fails to be Fredholm, then

$$\lim_{n\to\infty} \sigma_k(A_n) = 0 \qquad \text{for each } k \in \mathbb{N}. \tag{9}$$

We call a sequence with that property *not normally solvable*, in analogy with the corresponding notion from operator theory.

Corollary 17 *Let $\mathcal{A}$ be an essentially fractal and unital C^*-subalgebra of $\mathcal{F}$. Then a sequence in $\mathcal{A}$ is either Fredholm or not normally solvable.*

Example 18 The finite sections algebra $\mathcal{S}(\mathsf{T}(C))$ for Toeplitz operators is essentially fractal, as follows easily from the description of $\mathcal{S}(\mathsf{T}(C))$ in Theorem 1 in combination with Corollary 15. The finite sections algebra $\mathcal{S}(\mathsf{BDO}(\mathbb{N}))$ for band-dominated operators examined in [13] is an example of an algebra which is essentially fractal but not fractal. Finally, the sequence (A_n) where

$$A_n := \begin{cases} \text{diag}\,(0,\, 0,\, \ldots 0,\, 1) & \text{if } n \text{ is even} \\ \text{diag}\,(0,\, 1,\, \ldots 1,\, 1) & \text{if } n \text{ is odd,} \end{cases}$$

is fractal, but not essentially fractal: its subsequence (A_{2n}) is compact, whereas (A_{2n+1}) is Fredholm. ■

As for fractality, there is an *essential fractal restriction theorem* (see [15]). Its proof is based on the fact that, for every separable C^*-subalgebra $\mathcal{A}$ of $\mathcal{F}$, there is a sequence η such that not only the sequence of the norms $(\|A_{\eta(n)}\|)_{n\geq 1} = (\Sigma_1(A_{\eta(n)}))_{n\geq 1}$ converges for every $(A_n) \in \mathcal{A}$ (which was basic in the proof of the fractal restriction theorem), but also *every* sequence $(\Sigma_k(A_{\eta(n)}))_{n\geq 1}$ of the singular values, for every $k \in \mathbb{N}$.

Theorem 19 *Let $\mathcal{A}$ be a separable C^*-subalgebra of $\mathcal{F}$. Then there is a strictly increasing sequence $\eta : \mathbb{N} \to \mathbb{N}$ such that the restricted algebra $\mathcal{A}_\eta$ is essentially fractal.*

4 Fractal algebras of compact sequences

Compact sequences in fractal algebras behave particularly well. To state these results, we need some more notions.

A non-zero element k of a C^*-algebra $\mathcal{A}$ is said to be of *algebraic rank one* if, for each $a \in \mathcal{A}$, there is a complex number μ such that $kak = \mu k$. We let $\mathcal{C}(\mathcal{A})$ stand for the smallest closed subalgebra of $\mathcal{A}$ which contains all elements of algebraic rank one. If such elements do not exist, we set $\mathcal{C}(\mathcal{A}) = \{0\}$. In any case, $\mathcal{C}(\mathcal{A})$ is a closed ideal of $\mathcal{A}$, the elements of which we call *compact*. The following theorem summarizes some well-known equivalent descriptions of $\mathcal{C}(\mathcal{A})$ (see Theorem 1.4.5 in [1], and recall the notion of a dual algebra from the introduction).

Theorem 20 *Let $\mathcal{A}$ be a unital C^*-algebra and $\mathcal{J}$ a closed ideal of $\mathcal{A}$. The following assertions are equivalent:*

(a) $\mathcal{J} = \mathcal{C}(\mathcal{J})$.

(b) $\mathcal{J}$ is a dual algebra.

(c) The spectrum of every self-adjoint element of $\mathcal{J}$ is at most countable and has 0 as only possible accumulation point.

Every dual ideal of a C^*-algebra comes with an associated lifting theorem (see [17] for a first version of that theorem and [6] for a proof).

Theorem 21 (Lifting theorem for dual ideals) *Let $\mathcal{A}$ be a unital C^*-algebra. For every element t of a set T, let $\mathcal{J}_t$ be an elementary ideal of $\mathcal{A}$ such that $\mathcal{J}_s\mathcal{J}_t = \{0\}$ whenever $s \neq t$, and let $W_t : \mathcal{A} \to L(H_t)$ denote the irreducible representation of $\mathcal{A}$ which extends the (unique up to unitary equivalence) irreducible representation of $\mathcal{J}_t$. Let further $\mathcal{J}$ stand for the smallest closed ideal of $\mathcal{A}$ which contains all ideals $\mathcal{J}_t$.*

(a) An element $a \in \mathcal{A}$ is invertible if and only if the coset $a + \mathcal{J}$ is invertible in $\mathcal{A}/\mathcal{J}$ and if every operator $W_t(a)$ is invertible in $L(H_t)$.

(b) The separation property holds, i.e. $W_s(\mathcal{J}_t) = \{0\}$ whenever $s \neq t$.

(c) *If* $j \in \mathcal{J}$*, then* $W_t(j)$ *is compact for every* $t \in T$*.*

(d) *If* $a + \mathcal{J}$ *is invertible in* $\mathcal{A}/\mathcal{J}$*, then all operators* $W_t(a)$ *are Fredholm and all but a finite number of them is invertible.*

A basic observation of [12] is that a compact sequence in a fractal algebra has the spectral property of Theorem 20 (c). The following result is thus an immediate consequence of that theorem. It implies in particular that every unital and fractal C^*-subalgebra of $\mathcal{F}$ which contains non-trivial compact sequences is subject to the lifting theorem.

Corollary 22 *Let* $\mathcal{A}$ *be a unital and fractal* C^**-subalgebra of* $\mathcal{F}$ *which contains the ideal* $\mathcal{G}$*. Then the ideal* $(\mathcal{A} \cap \mathcal{K})/\mathcal{G}$ *of* $\mathcal{A}/\mathcal{G}$ *is a dual algebra.*

5 Restriction-extension theorems

5.1 Weights of elementary algebras of sequences

A projection in a C^*-algebra is a self-adjoint element p with $p^2 = p$. A closed ideal $\mathcal{J}$ of a C^*-algebra $\mathcal{A}$ *lifts projections*, if every projection in $\mathcal{A}/\mathcal{J}$ contains a representative which is a projection in $\mathcal{A}$. Closed ideals of C^*-algebras do not lift projections in general (take $\mathcal{A} = C([0, 1])$ and $\mathcal{J} = \{f \in \mathcal{A} : f(0) = f(1) = 0\}$). The following proposition states that elementary ideals of $\mathcal{F}/\mathcal{G}$ lift projections. More general, every dual ideal of a C^*-algebra owns the projection lifting property.

Proposition 23 *Let* $\mathcal{J}$ *be an elementary* C^**-subalgebra of* $\mathcal{F}/\mathcal{G}$*.*

(a) *Every projection* $p \in \mathcal{J}$ *lifts to a sequence* $(\Pi_n) \in \mathcal{F}$ *of orthogonal projections, i.e.,* $(\Pi_n) + \mathcal{G} = p$*.*

(b) *If* p *and* q *are rank one projections in* $\mathcal{J}$ *which lift to projections* (Π_n^p) *and* (Π_n^q) *in* $\mathcal{F}$*, respectively, then* $\dim \operatorname{im} \Pi_n^p = \dim \operatorname{im} \Pi_n^q$ *for all sufficiently large* n*.*

Thus, *for large* n, the numbers $\dim \operatorname{im} \Pi_n^p$ are uniquely determined by the algebra $\mathcal{J}$; they do neither depend on the rank one projection p nor on the choice of its lifting. For a precise formulation of that property, define an equivalence relation $\sim$ on the set of all sequences of non-negative integers by calling two sequences (α_n), (β_n) *equivalent* if $\alpha_n = \beta_n$ for all sufficiently large n. Then Proposition 23 states that the equivalence class which contains the sequence $(\dim \operatorname{im} \Pi_n^p)_{n \geq 1}$ is uniquely determined by the algebra $\mathcal{J}$. We denote this equivalence class by $\alpha^{\mathcal{J}}$ and call it the *weight* of the elementary algebra $\mathcal{J}$. We say that $\mathcal{J}$ is *of positive weight* if $\alpha^{\mathcal{J}}$ contains a sequence of positive numbers, and $\mathcal{J}$ is *of weight one* if $\alpha^{\mathcal{J}}$ contains the constant sequence of ones. Note that the weight is bounded if $\mathcal{J}$ is in $\mathcal{K}/\mathcal{G}$; in this case (Π_n^p) is a compact sequence of projections and therefore of finite essential rank.

5.2 Silbermann pairs and $\mathcal{J}$-Fredholm sequences

Next we are going to examine the consequences of the lifting theorem in the context of Silbermann pairs.

A *Silbermann pair* $(\mathcal{A}, \mathcal{J})$ consists of a unital C^*-subalgebra $\mathcal{A}$ of $\mathcal{F}$ and a closed ideal $\mathcal{J}$ of $\mathcal{A}$ which contains $\mathcal{G}$ properly and which is contained in $\mathcal{K}$, and for which $\mathcal{J}/\mathcal{G}$ is a dual algebra. Every sequence in $\mathcal{A}$ which is invertible modulo $\mathcal{J}$ is called an *$\mathcal{J}$-Fredholm* sequence. Note that every $\mathcal{J}$-Fredholm sequence is a Fredholm sequence in sense of Definition 9 (but a Fredholm sequence in $\mathcal{A}$ need not be $\mathcal{J}$-Fredholm because $\mathcal{J}$ may be properly contained in $\mathcal{A} \cap \mathcal{K}$).

Under the hypotheses of Corollary 22, $(\mathcal{A}, \mathcal{A} \cap \mathcal{K})$ is a Silbermann pair, and a sequence in $\mathcal{A}$ is $(\mathcal{A} \cap \mathcal{K})$-Fredholm if and only if it is Fredholm. The study of Silbermann pairs (in the special case when $\mathcal{J}/\mathcal{G}$ is an elementary subalgebra of $\mathcal{K}/\mathcal{G}$) was initiated by Silbermann in [18].

Let $(\mathcal{A}, \mathcal{J})$ be a Silbermann pair. Being dual by definition, the algebra $\mathcal{J}/\mathcal{G}$ is the direct sum of a family $(I_t)_{t \in T}$ of elementary algebras with associated bijective representations $W_t : I_t \to K(H_t)$. These representations extent to irreducible representations $\mathcal{A} \to L(H_t)$ which we denote by W_t again. In this context, the Lifting theorem 21 specifies as follows.

Theorem 24 *Let $(\mathcal{A}, \mathcal{J})$ be a Silbermann pair.*

(a) A sequence $\mathbf{A} \in \mathcal{A}$ is stable if and only if it is $\mathcal{J}$-Fredholm and if the operators $W_t(\mathbf{A})$ are invertible for each $t \in T$.

(b) $W_s(I_t) = \{0\}$ whenever $s \neq t$.

(c) If $\mathbf{J} \in \mathcal{J}$, then $W_t(\mathbf{J})$ is a compact operator for every $t \in T$.

(d) If $\mathbf{A} \in \mathcal{A}$ is $\mathcal{J}$-Fredholm, then all operators $W_t(\mathbf{A})$ are Fredholm and all but a finite number of them is invertible.

For every elementary ideal I_t, let (α_n^t) be a representative of the weight α^{I_t}. Assertion (d) of Theorem 24 implies that for every $\mathcal{J}$-Fredholm sequence $\mathbf{A} \in \mathcal{A}$, the sum

$$\alpha_n(\mathbf{A}) := \sum_{t \in T} \alpha_n^t \, \dim \ker W_t(\mathbf{A}) \tag{10}$$

is finite. This definition obviously depends on the choice of the representatives of the weights. But the equivalence class of the sequence $(\alpha_n(\mathbf{A}))$ modulo $\sim$ is uniquely determined, since only a finite number of items in the sum (10) is not zero.

A basic phenomenon of a $\mathcal{J}$-Fredholm sequence (A_n) is the following splitting property of the (increasingly ordered) singular values $\sigma_k(A_n)$ of A_n.

Theorem 25 *Let $(\mathcal{A}, \mathcal{J})$ be a Silbermann pair and $\mathbf{A} = (A_n) \in \mathcal{A}$ a $\mathcal{J}$-Fredholm sequence. Then $\mathbf{A}$ is a Fredholm sequence, and*

$$\lim_{n\to\infty} \sigma_{\alpha_n(\mathbf{A})}(A_n) = 0 \quad \textit{whereas} \quad \liminf_{n\to\infty} \sigma_{\alpha_n(\mathbf{A})+1}(A_n) > 0. \tag{11}$$

The proof makes use of results on lifting of families of mutually orthogonal projections and on generalized (or Moore-Penrose) invertibility. For details see [12].

Theorem 25 has some remarkable consequences. First note that the number

$$\alpha(\mathbf{A}) := \limsup_{n\to\infty} \alpha_n(\mathbf{A}) \tag{12}$$

is well defined and finite for every $\mathcal{J}$-Fredholm sequence $\mathbf{A} \in \mathcal{A}$. Since $(\alpha_n(\mathbf{A}))$ is a sequence of non-negative integers, it has a constant subsequence the entries of which are equal to $\alpha(\mathbf{A})$. Together with (11), this shows that

$$\liminf_{n\to\infty} \sigma_{\alpha(\mathbf{A})}(A_n) = 0 \quad \text{and} \quad \liminf_{n\to\infty} \sigma_{\alpha(\mathbf{A})+1}(A_n) > 0. \tag{13}$$

Corollary 26 *Let $(\mathcal{A}, \mathcal{J})$ be a Silbermann pair and $\mathbf{A} \in \mathcal{A}$ a $\mathcal{J}$-Fredholm sequence. Then $\mathbf{A}$ is a Fredholm sequence, and its α-number is given by* (12)*.*

Many subalgebras of $\mathcal{F}$ which arise from concrete approximation methods have the property that every rank one projection in $\mathcal{J}/\mathcal{G}$ lifts to a sequence of projections of rank one (equivalently, that every elementary algebra I_t has weight one). In this case we call $(\mathcal{A}, \mathcal{J})$ a Silbermann pair of *local weight one*. For Silbermann pairs with this property, Theorem 25 and its Corollary 26 specify as follows.

Corollary 27 *Let $(\mathcal{A}, \mathcal{J})$ be a Silbermann pair of local weight one and $\mathbf{A} = (A_n) \in \mathcal{A}$ a $\mathcal{J}$-Fredholm sequence. Then*

$$\alpha(\mathbf{A}) = \sum_{t\in T} \dim\ker W_t(\mathbf{A}), \tag{14}$$

and the sequence $\mathbf{A}$ has the $\alpha(\mathbf{A})$-splitting property, i.e., the number of the singular values of A_n which tend to zero is $\alpha(\mathbf{A})$.

5.3 Spectral Silbermann pairs

A Silbermann pair $(\mathcal{A}, \mathcal{J})$ is called *spectral* or *strongly spectral* if the family $\{W_t\}_{t\in T}$ of the lifting homomorphisms of $(\mathcal{A}, \mathcal{J})$ is spectral or strongly spectral for the algebra $\mathcal{A}/\mathcal{G}$, respectively. For strongly spectral Silbermann pairs, the assertions of the lifting theorem can be completed as follows.

Theorem 28 *Let $(\mathcal{A}, \mathcal{J})$ be a strongly spectral Silbermann pair and $\mathbf{A} \in \mathcal{A}$. Then*

(a) $\mathbf{A}$ is stable if and only if all operators $W_t(\mathbf{A})$ are invertible;

(b) $\|\mathbf{A} + \mathcal{G}\|_{\mathcal{F}/\mathcal{G}} = \max_{t\in T} \|W_t(\mathbf{A})\|$.

(c) $\mathbf{A}$ is $\mathcal{J}$-Fredholm if and only if all operators $W_t(\mathbf{A})$ are Fredholm and if there are only finitely many of them which are not invertible;

(d) $\mathbf{A} \in \mathcal{J}$ if and only if all operators $W_t(\mathbf{A})$ are compact and if, for each $\varepsilon > 0$, there are only finitely many of them with $\|W_t(\mathbf{A})\| > \varepsilon$.

Proof. Assertion (a) is a re-formulation of the strong spectral condition. Assertion (b) is a consequence of (a) and of general properties of strongly spectral families of homomorphisms; see Theorem 5.39 in [6] (where strongly spectral families were called *sufficient*).

(c) The 'only if' part follows from the Lifting theorem 24 (d). Conversely, let $\mathbf{A} \in \mathcal{A}$ be a sequence for which all operators $W_t(\mathbf{A})$ are Fredholm and for which there is a finite subset T_0 of T which consists of all t such that $W_t(\mathbf{A})$ is not invertible. Then all operators $W_t(\mathbf{A}^*\mathbf{A})$ are Fredholm, and they are invertible if $t \notin T_0$. Let $t \in T_0$. Then $W_t(\mathbf{A}^*\mathbf{A})$ is a Fredholm operator of index 0. Hence, there is a compact operator K_t such that $W_t(\mathbf{A}^*\mathbf{A}) + K_t$ is invertible. Choose a sequence $\mathbf{K}_t \in \mathcal{J}$ with $W_t(\mathbf{K}_t) = K_t$ and $W_s(\mathbf{K}_t) = 0$ for $s \neq t$ (which is possible by the separation property in Theorem 24). Then the sequence $\mathbf{K} := \sum_{t \in T_0} \mathbf{K}_t$ belongs to the ideal $\mathcal{J}$, and all operators $W_t(\mathbf{A}^*\mathbf{A} + \mathbf{K})$ are invertible. By assertion (a), the sequence $\mathbf{A}^*\mathbf{A} + \mathbf{K}$ is stable. Similarly, one finds a sequence $\mathbf{L} \in \mathcal{J}$ such that $\mathbf{A}\mathbf{A}^* + \mathbf{L}$ is stable. Thus, the sequence $\mathbf{A}$ is invertible modulo $\mathcal{J}$, whence its $\mathcal{J}$-Fredholm property.

(d) The 'only if' part follows again from the Lifting theorem 24 (c). For the 'if' part, let $\mathbf{K} \in \mathcal{A}$ be a sequence such that, for every $\varepsilon > 0$, there are only finitely many $t \in T$ with $\|W_t(\mathbf{A})\| > \varepsilon$. For $n \in \mathbb{N}$, let T_n stand for the (finite) subset of T which collects all t with $\|W_t(\mathbf{K})\| > 1/n$. For each $t \in T_n$, choose a sequence $\mathbf{K}^t \in \mathcal{J}$ with $W_t(\mathbf{K}^t) = W_t(\mathbf{K})$ and $W_s(\mathbf{K}^t) = 0$ for $s \neq t$ (employ the separation property in Theorem 24 again), and set $\mathbf{K}_n := \sum_{t \in T_n} \mathbf{K}^t$. Then $W_t(\mathbf{K} - \mathbf{K}_n) = 0$ for $t \in T_n$ and $W_t(\mathbf{K}_n) = 0$ for $t \notin T_n$. Hence, $\sup_{t \in T} \|W_t(\mathbf{K} - \mathbf{K}_n)\| \leq 1/n$ for every $n \in \mathbb{N}$. By Theorem 28 (b), the supremum in this estimate coincides with $\|\mathbf{K} - \mathbf{K}_n + \mathcal{G}\|_{\mathcal{F}/\mathcal{G}}$. Thus, $\mathbf{K}$ is the norm limit of a sequence in $\mathcal{J}$, whence $\mathbf{K} \in \mathcal{J}$. ■

An example for a strongly spectral Silbermann pair is $(\mathcal{S}(\mathsf{T}(C)), \mathcal{J})$, with $\mathcal{J}$ specified as in the introduction. A sequence in $\mathcal{S}(\mathsf{T}(C))$ is Fredholm if and only its strong limit is a Fredholm operator (note that $T(a)$ and $T(\tilde{a})$ are Fredholm operators only simultaneously). Equivalently, the sequence $\mathbf{A} := (P_n T(a) P_n + P_n K P_n + R_n L R_n + G_n)$ with $a \in C(\mathbb{T})$, K, L compact and $(G_n) \in \mathcal{G}$ is Fredholm if and only if $T(a)$ is a Fredholm operator. In this case,

$$\alpha(\mathbf{A}) = \dim \ker(T(a) + K) + \dim \ker(T(\tilde{a}) + L). \tag{15}$$

In particular, if $K = L = 0$ and if a is not the zero function, then

$$\begin{aligned} \alpha(\mathbf{A}) &= \dim \ker T(a) + \dim \ker T(\tilde{a}) \\ &= \max\{\dim \ker T(a), \, \dim \ker T(\tilde{a})\}. \end{aligned}$$

The second equality holds by a theorem of Simonenko and Coburn which states that one of the quantities $\dim \ker T(a)$ and $\dim \ker T(\tilde{a})$ is zero for each non-zero Toeplitz operator.

5.4 Silbermann algebras

We call a unital fractal C^*-subalgebra $\mathcal{A}$ of $\mathcal{F}$ a *Silbermann algebra* if $(\mathcal{A},\, \mathcal{A}\cap\mathcal{K})$ is a spectral Silbermann pair; a Silbermann algebra $\mathcal{A}$ is called *strong* if the pair $(\mathcal{A},\, \mathcal{A}\cap\mathcal{K})$ is strongly spectral. The following is an immediate consequence of Theorem 28.

Theorem 29 (*a*) *Let $\mathcal{A}$ be a Silbermann algebra. A sequence $\mathbf{A}\in\mathcal{A}$ is compact if and only if all operators $W_t(\mathbf{A})$ are compact and if, for each $\varepsilon > 0$, only a finite number of them has a norm greater than ε.*

Let $\mathcal{A}$ be a strong Silbermann algebra. A sequence $\mathbf{A}\in\mathcal{A}$ is

(*b*) *Fredholm if and only if all operators $W_t(\mathbf{A})$ are Fredholm and if only a finite number of them is not invertible;*

(*c*) *stable if and only if all operators $W_t(\mathbf{A})$ are invertible.*

The goal of this section is fractality and essential fractality properties of Silbermann algebras, which we prepare by two technical results. The first one describes the fractal algebras among the elementary subalgebras of $\mathcal{K}$.

Proposition 30 *Let $\mathcal{J}$ be a C^*-subalgebra of $\mathcal{K}$ which contains $\mathcal{G}$ properly and for which the quotient algebra $\mathcal{J}/\mathcal{G}$ is elementary. Then $\mathcal{J}$ is fractal if and only if $\mathcal{J}/\mathcal{G}$ is of positive weight.*

Proof. First let $\mathcal{J}/\mathcal{G}$ be of positive weight, and let $\eta : \mathbb{N}\to\mathbb{N}$ be strictly increasing. The mapping $W_\eta : \mathcal{J}\to\mathcal{F}_\eta/\mathcal{G}_\eta,\ \mathbf{J}\mapsto R_\eta\mathbf{J}+\mathcal{G}_\eta$ is a *-homomorphism which has the ideal $\mathcal{G}$ in its kernel. Since $\mathcal{J}/\mathcal{G}$ is elementary (thus, simple), either $\ker W_\eta = \mathcal{G}$ or $\ker W_\eta = \mathcal{J}$. The latter is impossible: the positivity of the weight implies that there is a sequence in $\mathcal{J}$ which consists of non-zero projections, and no restriction of that sequence can tend to zero in the norm. Thus, $\ker W_\eta = \mathcal{G}$, and the quotient homomorphism $W_\eta^\pi : \mathcal{J}/\mathcal{G}\to\mathcal{J}_\eta/\mathcal{G}_\eta$ which sends $\mathbf{J}+\mathcal{G}$ to $W_\eta(\mathbf{J})$ is a *-isomorphism between these algebras. Define

$$\pi_\eta : \mathcal{J}_\eta\to\mathcal{J}/\mathcal{G},\quad R_\eta\mathbf{J}\mapsto (W_\eta^\pi)^{-1}(R_\eta\mathbf{J}+\mathcal{G}_\eta).$$

Then $\pi_\eta R_\eta$ is the canonical homomorphism from $\mathcal{J}$ onto $\mathcal{J}/\mathcal{G}$, whence the fractality of $\mathcal{J}$.

Let now $\mathcal{J}$ be a fractal algebra and assume that $\mathcal{J}/\mathcal{G}$ is not of positive weight. Then the weight (α_n) of $\mathcal{J}/\mathcal{G}$ contains infinitely many zeros, say $\alpha_{\eta(n)} = 0$ for a certain strictly increasing sequence η. Thus, every algebraic rank one projection p in $\mathcal{J}/\mathcal{G}$ lifts to a sequence (Π_n) of projections with $\Pi_{\eta(n)} = 0$ for all n. Then $(\Pi_n)\in\mathcal{G}$ by Theorem 4, i.e., $p = 0$. This is impossible, since $\mathcal{J}$ contains $\mathcal{G}$ properly. ∎

The following lemma is immediate from the definition of a fractal homomorphism.

Lemma 31 *Let $\mathcal{A}$ be a C^*-subalgebra of $\mathcal{F}$, $\mathcal{J}$ a closed ideal of $\mathcal{A}$ with $\mathcal{G} \subseteq \mathcal{J}$, $W : \mathcal{J} \to L(H)$ an irreducible representation of $\mathcal{J}$ with $\mathcal{G} \subseteq \ker W$, and $W' : \mathcal{A} \to L(H)$ the (unique) irreducible extension of W. If W is fractal, then W' is fractal.*

We say that a Silbermann pair $(\mathcal{A}, \mathcal{J})$ has *positive* or *constant local weight* if every elementary component of $\mathcal{J}/\mathcal{G}$ has a positive or constant weight, respectively. We call a Silbermann algebra $\mathcal{A}$ of *positive* or *constant local weight* if the Silbermann pair $(\mathcal{A}, \mathcal{A} \cap \mathcal{K})$ has the corresponding property. If $\mathcal{A}$ is a Silbermann algebra of constant local weight, then the numbers $\alpha_n(\mathbf{A})$ in (10) are independent of n.

Theorem 32 *Let $(\mathcal{A}, \mathcal{J})$ be a strongly spectral Silbermann pair of positive local weight. Then*
(a) $\mathcal{A}$ is a fractal algebra,
(b) $\mathcal{J} = \mathcal{A} \cap \mathcal{K}$.

Proof. (a) Let W_t be the lifting of an elementary component $\mathcal{J}_t$ of $\mathcal{J}$. We use the notation W_t also for the extensions of W_t to irreducible representations of $\mathcal{J}$ and $\mathcal{A}$. Since $(\mathcal{A}, \mathcal{J})$ is a strongly spectral Silbermann pair, the family $\{W_t\}_{t \in T}$ of the lifting homomorphisms of $\mathcal{J}/\mathcal{G}$ is strongly spectral for $\mathcal{A}/\mathcal{G}$. Moreover, all homomorphisms W_t are fractal. Indeed, since every elementary component of $\mathcal{J}$ is fractal by Proposition 30 and $\mathcal{G} \subseteq \ker W_t$, the representations W_t of $\mathcal{J}_t$ are fractal. Lemma 31 implies that the W_t are also fractal as representations of the algebra $\mathcal{A}$. Now the fractality of $\mathcal{A}$ follows from Theorem 5.
(b) $\mathcal{A}$ is fractal by assertion (a), so $(\mathcal{A} \cap \mathcal{K})/\mathcal{G}$ is a dual algebra by Corollary 22. Let $\{W_t\}_{t \in T}$ refer to the family of the lifting homomorphisms associated with that algebra. Since $\mathcal{J}/\mathcal{G}$ is a closed ideal of $(\mathcal{A} \cap \mathcal{K})/\mathcal{G}$, the lifting homomorphisms of $\mathcal{J}/\mathcal{G}$ form a subset $\{W_t\}_{t \in S}$ of $\{W_t\}_{t \in T}$, with S a non-empty subset of T.

Let $\mathbf{A} \in \mathcal{A}$ be invertible modulo $\mathcal{A} \cap \mathcal{K}$. By Theorem 28 (c), all operators $W_t(\mathbf{A})$ are Fredholm, and only a finite number of them is not invertible. Hence, all operators $W_t(\mathbf{A})$ with $t \in S$ are Fredholm, and all but finitely many of them are invertible. Again by By Theorem 28 (c), the sequence $\mathbf{A}$ is invertible modulo $\mathcal{J}$. Hence, invertibility modulo $\mathcal{A} \cap \mathcal{K}$ is equivalent to invertibility modulo $\mathcal{J}$ for sequences in $\mathcal{A}$, which implies $\mathcal{A} \cap \mathcal{K} = \mathcal{J}$. ■

Corollary 33 *If $(\mathcal{A}, \mathcal{J})$ is a strongly spectral Silbermann pair of positive local weight, then $\mathcal{A}$ is a strong Silbermann algebra.*

Corollary 34 *Let $\mathcal{J}$ be a C^*-subalgebra of $\mathcal{F}$ with the following properties:*
(a) $\mathcal{G} \subset \mathcal{J} \subset \mathcal{K}$,
(b) $\mathcal{J}/\mathcal{G}$ is a dual algebra,
(c) $\mathcal{J}$ is of positive local weight, and
(d) at least one elementary components of $\mathcal{J}/\mathcal{G}$ has infinite dimension.

Then the minimal unitization $\mathbb{C}\mathbf{I} + \mathcal{J}$ *of* $\mathcal{J}$ *is fractal.*

Indeed, by Theorem 32, we have to show that $(\mathbb{C}\mathbf{I} + \mathcal{J},\ \mathcal{J})$ is a strongly spectral Silbermann pair. But this follows easily from the lifting theorem ■

Next we turn our attention to relations between a strong Silbermann algebra $\mathcal{A}$ and its restriction $R_\eta \mathcal{A} = \mathcal{A}_\eta$.

Theorem 35 *Let* $\mathcal{A}$ *be a unital and fractal* C^**-subalgebra of* $\mathcal{F}$ *which contains the ideal* $\mathcal{G}$*, and let* $\eta : \mathbb{N} \to \mathbb{N}$ *be strictly increasing. Then the following assertions are equivalent*:

(a) $\mathcal{A}$ *is a strong Silbermann algebra.*

(b) $(\mathcal{A}_\eta,\ (\mathcal{A} \cap \mathcal{K})_\eta)$ *is a strongly spectral Silbermann pair of positive local weight.*

Proof. First we show that if $\mathcal{A}$ is fractal, then $(\mathcal{A}_\eta,\ (\mathcal{A} \cap \mathcal{K})_\eta)$ is a Silbermann pair of positive local weight. If $\mathcal{A}$ is fractal, its restriction $\mathcal{A}_\eta$ is fractal. Then, by Corollary 22, $(\mathcal{A}_\eta \cap \mathcal{K}_\eta)/\mathcal{G}_\eta$ is a dual algebra. Being a closed ideal of a dual algebra, the algebra $(\mathcal{A} \cap \mathcal{K})_\eta$ is dual itself, and this algebra consists of compact sequences in $\mathcal{F}_\eta$ only. Hence, $(\mathcal{A}_\eta,\ (\mathcal{A} \cap \mathcal{K})_\eta)$ is a Silbermann pair.

Let p be a minimal projection in $(\mathcal{A} \cap \mathcal{K})_\eta)/\mathcal{G}_\eta$, and let $(P_{\eta(n)})$ be a lifting of p to a sequence of projections. Since $\mathcal{A}_\eta$ is fractal, this sequence is fractal. Theorem 4 (a) then implies that $P_{\eta(n)} \neq 0$ for all sufficiently large n. Hence, the weight of the elementary component containing p is positive.

Next we show that for every fractal algebra $\mathcal{A}$ and every strictly increasing sequence η, there is a natural *-isomorphism

$$\xi_\eta : \mathcal{A}/\mathcal{G} \to \mathcal{A}_\eta/\mathcal{G}_\eta, \quad \mathbf{A} + \mathcal{G} \mapsto (R_\eta \mathbf{A}) + \mathcal{G}_\eta.$$

Indeed, ξ_η is well defined since $\mathbf{A} - \mathbf{B} \in \mathcal{G}$ implies that $(R_\eta \mathbf{A}) - (R_\eta \mathbf{B}) \in \mathcal{G}_\eta$, and ξ_η has a trivial kernel since $R_\eta \mathbf{A} \in \mathcal{G}_\eta$ implies $\mathbf{A} \in \mathcal{G}$ via Theorem 4. The isomorphism ξ_η maps $(\mathcal{A} \cap \mathcal{K})/\mathcal{G}$ onto $(\mathcal{A} \cap \mathcal{K})_\eta/\mathcal{G}_\eta$. Hence, both ideals are canonically isomorphic, which implies that if W is one of the lifting homomorphisms of the Silbermann pair $(\mathcal{A},\ \mathcal{A} \cap \mathcal{K})$, then $W\xi_\eta^{-1}$ is a lifting homomorphism of the pair $(\mathcal{A}_\eta, (\mathcal{A} \cap \mathcal{K})_\eta)$ and, conversely, if W_η is a lifting homomorphism of $(\mathcal{A}_\eta, (\mathcal{A} \cap \mathcal{K})_\eta)$, then $W_\eta \xi_\eta$ is a lifting homomorphism of $(\mathcal{A},\ \mathcal{A} \cap \mathcal{K})$.

Now it is easy to see that the strongly spectral hypotheses in (a) and (b) imply each other. For example, we verify that $(b) \Rightarrow (a)$; the reverse implication follows similarly. Let the Silbermann pair $(\mathcal{A}_\eta,\ (\mathcal{A} \cap \mathcal{K})_\eta)$ be strongly spectral, and let $\{W_t\}_{t \in T}$ refer to the family of the lifting homomorphisms of the Silbermann pair $(\mathcal{A},\ \mathcal{A} \cap \mathcal{K})$. Let $\mathbf{A} \in \mathcal{A}$ be a sequence for which all operators $W_t(\mathbf{A} + \mathcal{G})$ are invertible. Then all operators $W_t \xi_\eta^{-1} \xi_\eta(\mathbf{A} + \mathcal{G})$ with $t \in T$ are invertible. Since the $W_t \xi_\eta^{-1}$ run through the lifting homomorphisms of the pair $(\mathcal{A}_\eta,\ (\mathcal{A} \cap \mathcal{K})_\eta)$, and since this pair is strongly spectral, the restricted coset $\xi_\eta(\mathbf{A} + \mathcal{G})$ is invertible in $\mathcal{A}_\eta/\mathcal{G}_\eta$. Then $\mathbf{A} + \mathcal{G}$ is invertible in $\mathcal{A}/\mathcal{G}$, since ξ_η is an isomorphism. ■

Corollary 36 *Restrictions of strong Silbermann algebras are strong Silbermann algebras.*

Proof. If $\mathcal{A}$ is a strong Silbermann algebra and η a strictly increasing sequence, then $(\mathcal{A}_\eta, (\mathcal{A}\cap\mathcal{K})_\eta)$ is a strongly spectral Silbermann pair, and every elementary component of $(\mathcal{A}\cap\mathcal{K})_\eta)/\mathcal{G}_\eta$ has positive weight by Theorem 35. By Corollary 33, $\mathcal{A}_\eta$ is a strong Silbermann algebra. ■

Corollary 37 *Strict Silbermann algebras are essentially fractal.*

Indeed, the canonical homomorphism $\mathcal{A} \to \mathcal{A}/(\mathcal{A}\cap\mathcal{K})$ is fractal since $\mathcal{A}$ is fractal and $\mathcal{G}$ is in the kernel of that homomorphism. Further, $(\mathcal{A}\cap\mathcal{K})_\eta = \mathcal{A}_\eta \cap \mathcal{K}_\eta$ by Theorem 32 (b). ■

Corollary 38 *Let* $\mathbf{A}$ *be a Fredholm sequence in a strong Silbermann algebra of constant local weight. Then every restriction* $\mathbf{A}_\eta$ *of* $\mathbf{A}$ *is Fredholm, and the sequences* $\mathbf{A}$ *and* $\mathbf{A}_\eta$ *have the same* α*-number.*

5.5 Forcing the spectral property

Again, let $\mathcal{F} := \mathcal{F}^\delta$ be an algebra of matrix sequences with dimension function δ and $\mathcal{G} := \mathcal{G}^\delta$ the associated ideal of zero sequences. We say that a C^*-subalgebra $\mathcal{A}^{ext}$ of $\mathcal{F}$ is an *extension* of a C^*-subalgebra $\mathcal{A}$ of $\mathcal{F}$ *by compact sequences* if there is a subset $\mathcal{K}'$ of the ideal $\mathcal{K}$ of the compact sequences in $\mathcal{F}$ such that $\mathcal{A}^{ext}$ is the smallest C^*-subalgebra of $\mathcal{F}$ which contains $\mathcal{A}$ and $\mathcal{K}'$. The goal of this section is to prove the following result, where extension by compact sequences and fractal restriction are used to force the spectral property.

Theorem 39 *Let* $\mathcal{A}$ *be a unital separable* C^**-subalgebra of* $\mathcal{F}$*. Then there are an extension* $\mathcal{A}^{ext}$ *of* $\mathcal{A}$ *by compact sequences and a strictly increasing sequence* η *such that the restriction* $\mathcal{A}^{ext}_\eta$ *is a Silbermann algebra.*

In other words, after extending $\mathcal{A}$ by a suitable set of compact sequences and then passing to a suitable restriction, we arrive at a spectral Silbermann pair $(\mathcal{A}^{ext}_\eta, \mathcal{A}^{ext}_\eta \cap \mathcal{K}_\eta)$.

Proof. Let $\mathcal{A}_0$ be a countable dense subset of $\mathcal{A}$ which contains the identity sequence. The set $\mathcal{A}_0^*\mathcal{A}_0$ is still countable and dense in $\mathcal{A}$.

For each sequence $\mathbf{A} = (A_n)$ in $\mathcal{A}_0$, we write

$$A_n^* A_n = E_n^* \text{diag}\,(\lambda_1(A_n), \ldots, \lambda_{\delta(n)}(A_n)) E_n \tag{16}$$

with a unitary matrix E_n and increasingly ordered eigenvalues $0 \le \lambda_1(A_n) \le \ldots \le \lambda_{\delta(n)}(A_n)$ of $A_n^* A_n$. For $l, r \in \mathbb{N}$, let $K_{l,r,n}$ be the $\delta(n) \times \delta(n)$-matrix which is zero if $\max\{l, r\} > \delta(n)$ and which has a 1 at the lrth entry and zeros at all other entries if $\max\{l, r\} \le \delta(n)$. The sequence $\mathbf{K}^{\mathbf{A},l,r}$ with entries

$K_n^{\mathbf{A},l,r} := E_n^* K_{l,r,n} E_n$ (note that E_n depends on $\mathbf{A}$) is a sequence of matrices of rank at most 1, hence compact. Let $\mathcal{A}^{ext}$ stand for the smallest C^*-subalgebra of $\mathcal{F}$ which contains the algebra $\mathcal{A}$, the ideal $\mathcal{G}$, and all sequences $\mathbf{K}^{\mathbf{A},l,r}$ with $\mathbf{A} \in \mathcal{A}_0$ and $l, r \in \mathbb{N}$. This algebra is still separable. Hence, by Theorems 8 and 19, there is a strictly increasing sequence η such that the restriction $\mathcal{A}^{ext}_\eta$ is fractal and essentially fractal. We claim that $\mathcal{A}^{ext}_\eta$ is a Silbermann algebra. Note that the sequences $\mathbf{K}^{\mathbf{A},r} := \sum_{l=1}^r \mathbf{K}^{\mathbf{A},l,l}$ with entries

$$K_n^{\mathbf{A},r} := E_n^* \mathrm{diag}\,(1, \ldots, 1, 0, \ldots, 0) E_n \tag{17}$$

where r ones followed by $\delta(n) - r$ zeros belong to $\mathcal{A}^{ext}$.

To simplify notation, we will assume that η is the identity mapping (otherwise replace δ by $\delta \circ \eta$ in what follows). Let $\mathcal{J} := \mathcal{A}^{ext} \cap \mathcal{K}$. Since $\mathcal{A}$ is a C^*-subalgebra and $\mathcal{J}$ is a closed ideal of $\mathcal{A}^{ext}$, the algebraic sum $\mathcal{A}+\mathcal{J}$ is a C^*-subalgebra of $\mathcal{A}^{ext}$. This subalgebra contains $\mathcal{A}$, $\mathcal{G}$ and all sequences $\mathbf{K}^{\mathbf{A},l,r}$. Thus, $\mathcal{A}^{ext} = \mathcal{A} + \mathcal{J}$.

Since $\mathcal{A}^{ext}$ is fractal, the ideal $\mathcal{J}/\mathcal{G}$ is dual by Corollary 22. Let $(I_t)_{t \in T}$ denote the set of its elementary components and let $W_t : I_t \to L(H_t)$ stand for the irreducible representation associated with I_t. As before, we will denote an irreducible representation of I_t and its irreducible extensions to $\mathcal{A}^{ext}/\mathcal{G}$ and $\mathcal{A}^{ext}$ by the same symbol.

Let $\mathbf{A} \in \mathcal{A}_0$. We claim that the coset $\mathbf{K}^{\mathbf{A},1,1} + \mathcal{G} = \mathbf{K}^{\mathbf{A},1} + \mathcal{G}$ is an algebraic rank one projection in $(\mathcal{A}^{ext} \cap \mathcal{K})/\mathcal{G}$. Indeed, the entries $K_n^{\mathbf{A},1}$ are projection matrices of rank one. Hence, for every positive sequence $(B_n^* B_n) \in \mathcal{A}^{ext}$, there is a sequence (β_n) of complex numbers such that

$$K_n^{\mathbf{A},1} B_n^* B_n K_n^{\mathbf{A},1} = \beta_n K_n^{\mathbf{A},1} \quad \text{for every } n \in \mathbb{N}.$$

The sequence $(\beta_n K_n^{\mathbf{A},1})_{n \in \mathbb{N}}$ is fractal, and β_n is the largest singular value of $\beta_n K_n^{\mathbf{A},1}$. By Proposition 6 (b), the sequence (β_n) is convergent. Since every sequence in $\mathcal{A}^{ext}$ is a linear combination of four positive sequences, we conclude that, for every sequence $\mathbf{C} = (C_n) \in \mathcal{A}^{ext}$, there is a convergent sequence (γ_n) of complex numbers such that

$$K_n^{\mathbf{A},1} C_n K_n^{\mathbf{A},1} = \gamma_n K_n^{\mathbf{A},1} \quad \text{for every } n \in \mathbb{N}.$$

Put $\gamma := \lim_{n \to \infty} \gamma_n$. Then $\mathbf{K}^{\mathbf{A},1} \mathbf{C} \mathbf{K}^{\mathbf{A},1} - \gamma \mathbf{K}^{\mathbf{A},1} \in \mathcal{G}$, which proves the claim.

Since the elementary components of $\mathcal{J}/\mathcal{G}$ are generated by algebraic rank one projections, there is a $t(\mathbf{A}) \in T$ such that $\mathbf{K}^{\mathbf{A},1} + \mathcal{G} \in I_{t(\mathbf{A})}$. Since $I_{t(\mathbf{A})}$ is an ideal, the equality

$$\mathbf{K}^{\mathbf{A},l,r} = \mathbf{K}^{\mathbf{A},l,1} \mathbf{K}^{\mathbf{A},1,1} \mathbf{K}^{\mathbf{A},1,r}$$

implies that $\mathbf{K}^{\mathbf{A},l,r} + \mathcal{G} \in I_{t(\mathbf{A})}$ for every pair $l, r \in \mathbb{N}$. In particular, all cosets $\mathbf{K}^{\mathbf{A},r} + \mathcal{G}$ belong to $I_{t(\mathbf{A})}$. Since the cosets $\mathbf{K}^{\mathbf{A},l,l} + \mathcal{G}$ are linearly independent algebraic rank one projections and the representation $W_{t(\mathbf{A})}$ is irreducible, the operators $W_{t(\mathbf{A})}(\mathbf{K}^{\mathbf{A},l,l})$ form a linearly independent set of projection operators of rank one. Hence, the Hilbert space $H_{t(\mathbf{A})}$ has infinite dimension.

Since $\mathcal{A}_0$ is dense in $\mathcal{A}$, the sequences $\mathbf{A}+\mathbf{K}$ with $\mathbf{A} \in \mathcal{A}_0$ and $\mathbf{K} \in \mathcal{A}^{ext} \cap \mathcal{K}$ form a dense subset $\mathcal{A}_0^{ext}$ of $\mathcal{A}^{ext}$. Let $\mathbf{B} := \mathbf{A}+\mathbf{K}$ be a sequence of this form, for which $\mathbf{B}^*\mathbf{B}$ is not a Fredholm sequence (equivalently, $\mathbf{B}^*\mathbf{B}$ is not a $\mathcal{J}$-Fredholm sequence, since $\mathcal{J}$ contains all compact sequences in $\mathcal{A}^{ext}$). Then $\mathbf{A}^*\mathbf{A} = (A_n^*A_n)$ is not a Fredholm sequence, hence

$$\lim_{n\to\infty} \lambda_r(A_n) = 0 \quad \text{for every } r \in \mathbb{N} \tag{18}$$

by (9) (recall (16) and remember that $\mathcal{A}^{ext}$ is essentially fractal after restriction). From (16) – (18) we conclude that $\mathbf{A}^*\mathbf{A}\mathbf{K}^{\mathbf{A},r} \in \mathcal{G}$ for every $r \in \mathbb{N}$, hence

$$W_{t(\mathbf{A})}(\mathbf{A}^*\mathbf{A})W_{t(\mathbf{A})}(\mathbf{K}^{\mathbf{A},r}) = 0 \quad \text{for every } r \in \mathbb{N}. \tag{19}$$

Since $(W_{t(\mathbf{A})}(\mathbf{K}^{\mathbf{A},r}))_{r\geq 1}$ is an increasing sequence of orthogonal projections on $H_{t(\mathbf{A})}$, this sequence converges strongly, and its limit, P, is the orthogonal projection from $H_{t(\mathbf{A})}$ onto the closure of the linear span of the union of the ranges of the $W_{t(\mathbf{A})}(\mathbf{K}^{\mathbf{A},r})$ (see, for example, Theorem 4.1.2 in [9]). So we conclude from (19) that $W_{t(\mathbf{A})}(\mathbf{A}^*\mathbf{A})P = 0$. Thus, and by Theorem 24 (c),

$$W_{t(\mathbf{A})}(\mathbf{B}^*\mathbf{B})P = W_{t(\mathbf{A})}(\mathbf{A}^*\mathbf{A})P + W_{t(\mathbf{A})}(\mathbf{A}^*\mathbf{K}+\mathbf{K}^*\mathbf{A}+\mathbf{K}^*\mathbf{K})P$$

is a compact operator. Then $W_{t(\mathbf{A})}(\mathbf{B}^*\mathbf{B})$ cannot be invertible: otherwise, the projection P were compact, but the range of P has infinite dimension, which follows by the same arguments as the infinite dimensionality of $H_{t(\mathbf{A})}$.

Thus, whenever $\mathbf{B} \in \mathcal{A}_0^{ext}$ and $\mathbf{B}^*\mathbf{B}$ is not a Fredholm sequence, then one of the operators $W_t(\mathbf{B}^*\mathbf{B})$ is not invertible. Conversely, if all operators $W_t(\mathbf{B}^*\mathbf{B})$ with $t \in T$ are invertible, then $\mathbf{B}^*\mathbf{B}$ is a Fredholm sequence. By Theorem 24 (e) this implies that, whenever all operators $W_t(\mathbf{B}^*\mathbf{B})$ with $t \in T$ are invertible, then the sequence $\mathbf{B}^*\mathbf{B}$ is a stable. This fact holds for all sequences $\mathbf{B}$ in the dense subset $\mathcal{A}_0^{ext}$ of $\mathcal{A}^{ext}$, from which it is easy to conclude that the family $(W_t)_{t\in T}$ is spectral for $\mathcal{A}^{ext}$. ■

It is not clear if one can make $\mathcal{A}^{ext}$ to a *strong* Silbermann algebra by a suitable restriction. The point is that the implication, obtained at the end of the previous proof, is verified only for sequences in a dense subset of $\mathcal{A}^{ext}$. Another open question is if there is a version of Theorem 39 which works without restriction if one already starts with a fractal and essentially fractal separable subalgebra $\mathcal{A}$ of $\mathcal{F}$.

5.6 Forcing local rank one

Our final goal is forcing the local rank one property by a suitable extension and restriction. The following result on restrictions of Silbermann pairs has been already shown as part of the proof of Theorem 35.

Proposition 40 *Let $(\mathcal{A}, \mathcal{J})$ be a Silbermann pair (in $\mathcal{F}$) and $\eta : \mathbb{N} \to \mathbb{N}$ strictly increasing. Then $(\mathcal{A}_\eta, \mathcal{J}_\eta)$ is a Silbermann pair (in $\mathcal{F}_\eta$).*

Let $\mathcal{J}/\mathcal{G} \cong \oplus_{t \in T} K(H_t)$ be the representation of the dual algebra $\mathcal{J}/\mathcal{G}$ as a direct sum of elementary components. Since these components are simple, every homomorphic image of $K(H_t)$ is either $\{0\}$ or isomorphic to $K(H_t)$. Given a strictly increasing η, let T_η denote the set of all $t \in T$ which are *not* mapped to the zero ideal under the homomorphism

$$(A_n) + \mathcal{G} \mapsto (A_{\eta(n)}) + \mathcal{G}_\eta. \tag{20}$$

The following lemma says that the weights of $\mathcal{J}/\mathcal{G}$ behave naturally under the mapping (20), provided that $\mathcal{J}$ is fractal. Recall from Corollary 34 that the latter condition holds if all elementary components of $\mathcal{J}/\mathcal{G}$ have positive weight and if at least one of them has infinite dimension.

Lemma 41 *Let $\mathcal{J}$ be a fractal algebra and $t \in T_\eta$. Let $\mathcal{J}_{t,\eta}$ be the image of the elementary component $\mathcal{J}_t \cong K(H_t)$ of $\mathcal{J}/\mathcal{G}$ under the homomorphism* (20)*. Then $\mathcal{J}_{t,\eta} \cong K(H_t)$ is elementary, and the weight of $\mathcal{J}_{t,\eta}$ is the restriction α_η of the weight α of $\mathcal{J}_t$.*

Proof. Let $(A_{\eta(n)}) + \mathcal{G}_\eta$ be a projection of algebraic rank one in $\mathcal{J}_{t,\eta}$. The fractality of $\mathcal{J}$ ensures that the quotient homomorphism (20) is an isomorphism from $\mathcal{J}/\mathcal{G}$ to $\mathcal{J}_\eta/\mathcal{G}_\eta$ and, thus, from $\mathcal{J}_t$ to $\mathcal{J}_{t,\eta}$. Hence, the pre-image $(A_n)+\mathcal{G} \in \mathcal{J}_t$ of $(A_{\eta(n)}) + \mathcal{G}_\eta$ is uniquely determined, and it is a projection of algebraic rank one. It is clear that if (Π_n) is a projection which lifts $(A_n) + \mathcal{G}$, then $(\Pi_{\eta(n)})$ is a projection which lifts $(A_{\eta(n)}) + \mathcal{G}_\eta$. This is the assertion. ∎

In the next result we will see how to force local constant weight (recall that the latter means that every elementary component of $\mathcal{J}/\mathcal{G}$ has constant weight).

Theorem 42 *Let $(\mathcal{A}, \mathcal{J})$ be a Silbermann pair and assume that $\mathcal{J}$ is separable and fractal. Then there is a strictly increasing sequence η such that the Silbermann pair $(\mathcal{A}_\eta, \mathcal{J}_\eta)$ has constant local weight.*

Proof. Since $\mathcal{J}$ is separable, the number of the elementary components of $\mathcal{J}/\mathcal{G}$ is at most countable. We enumerate these components by $\mathcal{J}^{(1)}, \mathcal{J}^{(2)}, \ldots$ and denote the weight of $\mathcal{J}^{(i)}$ by $\alpha^{(i)}$. Every weight $\alpha^{(i)}$ is bounded and has, thus, a constant subsequence. This is used in the following construction.

Starting with the identity mapping η_0 on $\mathbb{N}$, there is a subsequence η_1 of η_0 such that the restriction $\alpha^{(1)}_{\eta_1}$ is a positive constant. We continue in this way and get, for every $k \geq 1$, a subsequence η_k of η_{k-1} such that the restriction $\alpha^{(k)}_{\eta_k}$ is a positive constant. If the number of elementary components of $\mathcal{J}/\mathcal{G}$ is finite, we let η be the last of the sequences η_k obtained in this way; otherwise we set $\eta(n) := \eta_n(n)$ for $n \in \mathbb{N}$. In each case, every restriction $\alpha^{(i)}_\eta$ is a positive constant.

Then $(\mathcal{A}_\eta,\, \mathcal{J}_\eta)$ is a Silbermann pair by Proposition 40, and the local weights of that pair are just the η-restrictions of the weights of $(\mathcal{A},\, \mathcal{J})$ by Lemma 41. Thus, $(\mathcal{A}_\eta,\, \mathcal{J}_\eta)$ is a Silbermann pair with constant local weight. ■

Let now $\mathcal{A}$ be a separable Silbermann algebra with constant local weight. As before, we denote the elementary components of $(\mathcal{A} \cap \mathcal{K})/\mathcal{G}$ by $\mathcal{J}^{(1)}$, $\mathcal{J}^{(2)}$, ... and write $\alpha^{(i)}$ for the weight of $\mathcal{J}^{(i)}$, which now can be thought as a non-negative integer. For every i, let $p_i \in \mathcal{J}^{(i)}$ be a minimal (hence, algebraic rank one) projection. Given p_i, choose a projection $(\Pi_n^{(i)})_{n\ge1} \in \mathcal{A}$ which lifts p_i and which is specified such that $\dim \operatorname{im} \Pi_n^{(i)} = \alpha^{(i)}$ for all n, and write $\Pi_n^{(i)}$ as a sum $\sum_{k=1}^{\alpha^{(i)}} \Pi_n^{(i,k)}$ of projections with $\dim \operatorname{im} \Pi_n^{(i,k)} = 1$ (for example, choose unitary matrices $E_n^{(i)}$ such that

$$\Pi_n^{(i)} = E_n^{(i)} \operatorname{diag}(1,\, \ldots,\, 1,\, 0,\, \ldots,\, 0)(E_n^{(i)})^*,$$

with $\alpha^{(i)}$ ones followed by $\delta(n) - \alpha^{(i)}$ zeros; then set

$$\Pi_n^{(i,k)} := E_n^{(i)} \operatorname{diag}(0,\, \ldots,\, 0,\, 1,\, 0,\, \ldots,\, 0)(E_n^{(i)})^*,$$

with the 1 at the kth position). $(\Pi_n^{(i,k)})_{n\ge1}$ is a sequence of rank one projections in $\mathcal{F}$. Let $\tilde{\mathcal{A}}$ be the smallest closed subalgebra of $\mathcal{F}$ which contains the algebra $\mathcal{A}$ and all sequences $(\Pi_n^{(i,k)})_{n\ge1}$. Since $\tilde{\mathcal{A}}$ is again a separable C^*-algebra, the fractal restriction theorem applies. Let η be a strictly increasing sequence such that $\tilde{\mathcal{A}}_\eta$ is fractal.

Theorem 43 *Let $\mathcal{A}$ be a separable Silbermann algebra. Then the algebra $\tilde{\mathcal{A}}_\eta$ obtained in this way is a Silbermann algebras of local weight one.*

Proof. For brevity, we write $\mathcal{A}$ and $\tilde{\mathcal{A}}$ for $\mathcal{A}_\eta$ and $\tilde{\mathcal{A}}_\eta$, respectively, and set $\mathcal{J} := \mathcal{A} \cap \mathcal{K}$ and $\tilde{\mathcal{J}} := \tilde{\mathcal{A}} \cap \mathcal{K}$. Then $\tilde{\mathcal{A}}$ is both a fractal algebra and an extension of $\mathcal{A}$ by compact sequences. In particular,

$$\tilde{\mathcal{A}}/\tilde{\mathcal{J}} = (\mathcal{A} + \tilde{\mathcal{J}})/\tilde{\mathcal{J}} \cong \mathcal{A}/(\mathcal{A} \cap \tilde{\mathcal{J}}) = \mathcal{A}/(\mathcal{A} \cap \mathcal{K}) = \mathcal{A}/\mathcal{J}.$$

We are going to relate the irreducible representations of $\mathcal{A}$, associated with the elementary components of $\mathcal{J}/\mathcal{G}$, with the irreducible representations of $\tilde{\mathcal{A}}$, associated with the elementary components of $\tilde{\mathcal{J}}/\mathcal{G}$.

Let $W : \mathcal{J} \to L(K)$ be an irreducible representation of $\mathcal{J}$ associated with an elementary component of $\mathcal{J}/\mathcal{G}$. Let $x \in K$ by a cyclic vector and extend W to an irreducible representation $\tilde{W}$ of $\mathcal{A}$ by setting $\tilde{W}(a)W(j)x := W(aj)x$ for $a \in \mathcal{A}$ and $j \in \mathcal{J}$.

Since $\mathcal{G}$ is in the kernel of W, the quotient of $\tilde{W}$ by $\mathcal{G}$ is well defined; we denote it by $\tilde{W}$ again. Then there are an irreducible representation $\pi : \tilde{\mathcal{J}} \to L(\tilde{H})$, a closed subspace H of $\tilde{H}$ which is invariant with respect to $\pi(\mathcal{J})$, and a unitary operator $U : H \to K$ such that

$$W(j) = U\pi(j)|_H U^* \quad \text{for } j \in \mathcal{J}. \tag{21}$$

Set $U^*x =: y \in H$. Then $y \neq 0$; hence y is cyclic for π, and we can extend π to an irreducible representation $\tilde{\pi} : \tilde{\mathcal{A}} \to L(\tilde{H})$ by

$$\tilde{\pi}(a)\pi(j)y := \pi(aj)y \quad \text{for } a \in \tilde{\mathcal{A}},\ j \in \tilde{\mathcal{J}}.$$

From (21) we conclude that $W(aj)x = U\pi(aj)|_H U^*x$ for $j \in \mathcal{J}$ and $a \in \mathcal{A}$; hence,

$$\begin{aligned} \tilde{W}(a)W(j)x &= W(aj)x \\ &= U\pi(aj)|_H U^*x \\ &= U\tilde{\pi}(a)\pi(j)|_H y \\ &= U\tilde{\pi}(a)|_H U^*U\pi(j)U^*x \\ &= U\tilde{\pi}(a)|_H U^*W(j)x. \end{aligned}$$

Since the vectors $W(j)x$ lie dense in K, we conclude that H is an invariant subspace for $\tilde{\pi}(\mathcal{A})$ and $\tilde{W}(a) = U\tilde{\pi}(a)|_H U^*$ for $a \in \mathcal{A}$. Note that every $\tilde{\pi}$ obtained in this way is an irreducible representation of $\tilde{\mathcal{J}}$ with $\mathcal{G}$ in its kernel; so it is associated with an elementary component of $\tilde{\mathcal{J}}/\mathcal{G}$.

Let $(I_t)_{t\in T}$ be the elementary components of $\mathcal{J}/\mathcal{G}$ and $W_t : \mathcal{A} \to L(K_t)$ the lifting homomorphism associated with I_t. For every $t \in T$, we construct an extension of W_t to an irreducible representation of $\tilde{\mathcal{A}}$ as above. In general, this extension will not be unique. For a given irreducible representation $\tilde{\pi}$ of $\tilde{\mathcal{A}}$ coming from an irreducible representation of $\tilde{\mathcal{J}}/\mathcal{G}$, we let $T_{\tilde{\pi}}$ denote the set of all $t \in T$ for which the extension of W_t defined as above leads to $\tilde{\pi}$, i.e., for which there are a closed subspace H_t of $\tilde{H}$ invariant with respect to $\tilde{\pi}(\mathcal{A})$ and a unitary operator $U_t : H_t \to K_t$ such that

$$W_t(a) = U_t\tilde{\pi}(a)|_{H_t}U_t^* \quad \text{for } a \in \mathcal{A}. \tag{22}$$

We claim that the Hilbert spaces H_t with $t \in T_{\tilde{\pi}}$ are pairwise orthogonal. Let $s,\, t \in T_{\tilde{\pi}}$ and $s \neq t$. Inserting $a \in I_s$ into (22) we get $0 = W_t(a) = U_t\tilde{\pi}(a)|_{H_t}U_t^*$ whence $\tilde{\pi}(I_s)|_{H_t} = 0$. On the other hand, let $0 \neq x_s \in K_s$. Since $W_s : I_s \to L(K_s)$ is irreducible, x_s is algebraically cyclic, i.e. $W_s(I_s)x_s = K_s$. With $y_s := U_s x_s$ we obtain

$$H_s = U_s^*K_s = U_s^*W_s(I_s)x_s = \tilde{\pi}(I_s)U_s x_s = \tilde{\pi}(I_s)y_s.$$

Let now $z_s \in H_s$ and $z_t \in H_t$. By the previous line, there is a $j_s \in I_s$ such that $z_s = \tilde{\pi}(j_s)y_s$. Then

$$\langle z_s,\, z_t\rangle = \langle \tilde{\pi}(j_s)y_s\, z_t\rangle = \langle y_s,\, \tilde{\pi}(j_s)^* z_t\rangle = \langle y_s,\, \tilde{\pi}(j_s^*)z_t\rangle = 0$$

because of $\tilde{\pi}(I_s)|_{H_t} = 0$ as observed above. This proves the claim.

Now we can finish the proof of the theorem. Let $\tilde{\mathbf{A}} \in \tilde{\mathcal{A}}$ be a sequence such that all operators $\tilde{\pi}(\tilde{\mathbf{A}})$ are invertible and the norms of their inverses are uniformly bounded by a constant C, where $\tilde{\pi}$ runs through the irreducible representations of $\tilde{\mathcal{A}}$ coming from the elementary components of $\tilde{\mathcal{J}}/\mathcal{G}$. Write $\tilde{\mathbf{A}}$ as $\mathbf{A} + \tilde{\mathbf{J}}$ where $\mathbf{A} \in \mathcal{A}$ and $\tilde{\mathbf{J}} \in \tilde{\mathcal{J}}$ (recall that $\tilde{\mathcal{A}}$ is an extension of $\mathcal{A}$ by compact sequences). Since $\tilde{\mathcal{J}}$ is dual, there are only finitely many π with $\|\pi(\tilde{\mathbf{J}})\| \geq 1/(2C)$. Thus, all operators $\tilde{\pi}(\mathbf{A})$ are Fredholm, only a finite number of them is not invertible, and the norms of their inverses are uniformly bounded.

From (22) we then conclude that all operators $W_t(\mathbf{A})$ are Fredholm, that only a finite number of them is not invertible, and that the norms of their inverses are uniformly bounded. Then all operators $W_t(\mathbf{A}^*\mathbf{A})$ are Fredholm with index 0, only a finite number of them is not invertible, and the norms of their inverses are uniformly bounded. Thus, there is a sequence $\mathbf{J} \in \mathcal{J}$ such that all operators $W_t(\mathbf{A}^*\mathbf{A} + \mathbf{J})$ are invertible and the norms of their inverses is uniformly bounded; the (finite) sum $\mathbf{J}$ of the (compact) orthogonal projections onto the kernels of $W_t(\mathbf{A}^*\mathbf{A})$ will do the job. Because $\mathcal{A}$ is a Silbermann algebra, the sequence $\mathbf{A}^*\mathbf{A} + \mathbf{J}$ is invertible. In particular, $\mathbf{A}^*\mathbf{A}$ is a Fredholm sequence. Analogously one shows that $\mathbf{A}\mathbf{A}^*$ is a Fredholm sequence. But then $\mathbf{A}$ itself is a Fredholm sequence in $\mathcal{A}$, which implies that $\mathbf{A} + \tilde{\mathbf{J}} = \tilde{\mathbf{A}}$ is a Fredholm sequence in $\tilde{\mathcal{A}}$, hence invertible modulo $\tilde{\mathcal{J}}$. By the lifting theorem, applied to the Silbermann pair $(\tilde{\mathcal{A}}, \tilde{\mathcal{J}})$, the sequence $\tilde{\mathbf{A}}$ is stable. Thus, $\tilde{\mathcal{A}}$ is a Silbermann algebra. ■

If one starts with a strong Silbermann algebra $\mathcal{A}$, then this construction yields a strong Silbermann algebra $\tilde{\mathcal{A}}_\eta$ of local weight one.

5.7 A few examples

Block Toeplitz operators. As in the introduction, we consider Toeplitz operators $T(a)$ on $l^2(\mathbb{Z}^+)$ and their finite sections with respect to the filtration (P_n), but now the Toeplitz operators are generated by matrix-valued continuous functions $a : \mathbb{T} \to \mathbb{C}^{N \times N}$. We denote the related algebra of the (full) finite sections discretization for these operators by $\mathcal{S}(\mathsf{T}(C^{N \times N}))$. It is not hard to derive the analogues of Theorem 1 and Corollary 2 in this setting, with the sequences $(R_n L R_n)$ replaced by the sequences $(R_n L_i R_n$ when $n = kN + i$ with compact operators L_i, and with the homomorphism $\widetilde{W}$ replaced by the family of the N homomorphisms

$$\widetilde{W_i}(\mathbf{A}) := \text{s-lim}_{n \to \infty} R_{nN+i} A_{nN+i} R_{nN+i} P_{nN+i}$$

with $i \in \{0, 1, \ldots, N-1\}$. In particular, *every restriction* $\mathcal{S}_{\eta_i}(\mathsf{T}(C^{N \times N}))$ *with* $\eta_i(n) := nN + i$ *has the same structure as* $\mathcal{S}(\mathsf{T}(C))$.

One-sided Almost Mathieu operators. These are the operators

$$H_{\alpha, \lambda, \theta} := V_{-1} + V_1 + aI : l^2(\mathbb{Z}^+) \to l^2(\mathbb{Z}^+)$$

where V_1 and V_{-1} stand for the forward and backward shift operator, respectively, and where $a(n) := \lambda \cos 2\pi(n\alpha + \theta)$ with real parameters α, λ and θ. Further we write AP_α for the smallest closed subalgebra of $L(l^2(\mathbb{Z}^+))$ which contains all operators $H_{\alpha,\lambda,\theta}$ with arbitrary λ and θ (but with fixed α) and all compact operators. The algebra of the finite sections discretization of AP_α with respect to the same filtration (P_n) as before is denoted by $\mathcal{S}(\mathsf{AP}_\alpha)$.

We only deal with the non-periodic case when $\alpha \in (0, 1)$ is irrational. Then we write α as a continued fraction with nth approximant p_n/q_n such that

$$|\alpha - p_n/q_n| < q_n^{-2}.$$

It has been shown in [10, Theorem 5.1] that the restriction $\mathcal{S}_\eta(\mathsf{AP}_\alpha)$ with $\eta(n) := q_n$ is again an algebra with (exactly) the same structure as $\mathcal{S}(\mathsf{T}(C))$.

Operators in Cuntz algebras. For $N \geq 2$, let O_N denote the smallest C^*-subalgebra of $L(l^2(\mathbb{Z}^+))$ which contains the operators

$$S_i : (x_k)_{k\geq 0} \mapsto (y_k)_{k\geq 0} \quad \text{with} \quad y_k := \begin{cases} x_r & \text{if } k = rN + i \\ 0 & \text{else} \end{cases} \tag{23}$$

with $i = 0, \ldots, N-1$. These operators are isometries, and they satisfy the Cuntz axiom

$$S_0 S_0^* + \ldots + S_{N-1} S_{N-1}^* = I. \tag{24}$$

Since the (abstract) Cuntz algebra $\mathcal{O}_N$ is simple, $\mathcal{O}_N$ is *-isomorphic to the (concrete = represented) algebra O_N. We use the same filtration (P_n) as before and consider the smallest closed subalgebra $\mathcal{S}(\mathsf{O}_N)$ of $\mathcal{F} = \mathcal{F}^{\mathcal{P}}$ which contains all finite sections sequences $(P_n A P_n)$ with $A \in \mathsf{O}_N$. It turns out that the algebra $\mathcal{S}(\mathsf{O}_N)$ fails to be fractal, and it is a main result of [14] that $\eta(n) := N^n$ is a right choice to make the restricted algebra $\mathcal{S}_\eta(\mathsf{O}_N)$ fractal.

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Author's address:

Steffen Roch, Technische Universität Darmstadt, Fachbereich Mathematik, Schlossgartenstrasse 7, 64289 Darmstadt, Germany.
E-mail: roch@mathematik.tu-darmstadt.de