Maximal Regularity of the Time-Periodic Stokes Operator on Unbounded and Bounded Domains

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Abstract

We investigate the time-periodic Stokes equations with non-homogeneous divergence data in the whole space, the half space, bent half spaces and bounded domains. The solutions decompose into a well-studied stationary part and a purely periodic part, for which we establish L^p estimates. For the whole space and the half space case we use a reduction of the Stokes equations to $(n-1)$ heat equations. Perturbation and localisation methods yield the result on bent half spaces and bounded domains. A one-to-one correspondence between maximal regularity for the initial value problem and time periodic maximal regularity is proven, providing a short proof for the maximal regularity of the Stokes operator avoiding the notion of $\mathcal{R}$-boundedness. The results are applied to a quasilinear model governing the flow of nematic liquid crystals.

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1 Introduction

For $n \geq 2$ and a domain $\Omega \subset \mathbb{R}^n$ we consider the time-periodic Stokes system

$$\begin{cases} \partial_t u - \Delta u + \nabla p &= f & \text{in } \mathbb{R} \times \Omega, \\ \operatorname{div} u &= g & \text{in } \mathbb{R} \times \Omega, \\ u &= 0 & \text{on } \mathbb{R} \times \partial\Omega, \\ u(t+T, \cdot) &= u(t, \cdot). & \end{cases} \tag{1.1}$$

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Here, $T > 0$ is the length of one period, and the forcing term f and the function g are also assumed to fulfill the periodicity condition. We are interested in the following types of domains $\Omega \subset \mathbb{R}^n$:

- the *whole space* $\mathbb{R}^n$,
- the *(upper) half space* $\mathbb{R}^n_+ := \{x = (x', x_n) \in \mathbb{R}^n : x_n > 0\}$,
- a *bent half space* $\mathbb{R}^n_\omega := \{x = (x', x_n) \in \mathbb{R}^n : x_n > \omega(x')\}$, where the $\omega : \mathbb{R}^{n-1} \to \mathbb{R}$ is a Lipschitz function in $\mathrm{W}^{2,1}_{\mathrm{loc}}(\mathbb{R}^{n-1})$ such that the gradient $\nabla'\omega$ is bounded in $\mathbb{R}^{n-1}$, or
- a bounded domain with boundary of class $C^{1,1}$.

Our aim is to establish maximal L^p-L^q-regularity estimates for solutions to (1.1), where $p, q \in (1, \infty)$. For the Cauchy problem the maximal L^p regularity estimates have been studied by many researchers. For example, for the Stokes operator, such estimates are established by Solonnikov [27] using potential theory, by Giga and Sohr [18] using Dore and Venii's theory [9], and by Shibata and Shimizu [26] based on Weis' theorem [29]. As for maximal L^p regularity estimates in the case of nonzero divergence as in (1.1), the reader is referred to the results of Farwig and Sohr [13], Filonov and Shilkin [15] and Abels [1]. On the other hand, Kyed [20] recently established maximal L^p regularity estimates for the time-periodic problem on $\Omega = \mathbb{R}^n$. In this paper we generalize the previous results in the following points:

- the results of [1, 15] are extended to the time-periodic problem, and
- the result of [20] is extended to more general domains Ω.

In order to wove in the periodicity condition already on a functional analytic level, we consider the locally compact abelian group $G := \mathbb{T} \times \mathbb{R}^n := \mathbb{R}/T\mathbb{Z} \times \mathbb{R}^n$. We will fix the Haar measure μ defined on G by choosing

$$\int_G f \,\mathrm{d}\mu := \frac{1}{T}\int_0^T \int_{\mathbb{R}^n} f \,\mathrm{d}x\,\mathrm{d}t, \qquad f \in C_0(G).$$

Note that the topology and the differentiable structure on G is inherited from $\mathbb{R} \times \mathbb{R}^n$, see [20] for details. In particular, we may speak about function spaces like the space of compactly supported smooth functions $\mathrm{C}_0^\infty(G)$, the Schwartz-Bruhat space $\mathcal{S}(G)$ and the space of tempered distributions $\mathcal{S}'(G)$ [5, 28]. Introducing the time-periodic domains

$$\Omega_\mathbb{T} := \mathbb{T} \times \Omega$$

as open subsets of G, we can formulate (1.1) equivalently as

$$\begin{cases} \partial_t u - \Delta u + \nabla p &= f \qquad \text{in } \Omega_{\mathbb{T}}\,, \\ \operatorname{div} u &= g \qquad \text{in } \Omega_{\mathbb{T}}\,, \\ u &= 0 \qquad \text{on } \partial\Omega_{\mathbb{T}}\,, \end{cases} \tag{1.2}$$

where we have used the notation $\partial\Omega_{\mathbb{T}} := \mathbb{T} \times \partial\Omega$ for the spatial boundary of $\Omega_{\mathbb{T}}$. One main advantage of the notion of G is the possibility to introduce a Fourier transform $\mathcal{F}_G$, which constitutes a homeomorphism from $\mathcal{S}'(G)$ to $\mathcal{S}'(\widehat{G})$, where $\widehat{G}$ is the Pontryagin dual of G.

In analyzing the time-periodic problem it is useful to introduce a time averaging projection, as is discussed by Galdi and Kyed [17, 20]:

$$\mathcal{P}f := \frac{1}{T}\int_0^T f(t,\cdot)\,\mathrm{d}t = 1_{\mathbb{T}} * f, \qquad f \in C_0^\infty(\Omega_{\mathbb{T}}),$$

where $*$ denotes the convolution on the torus group $\mathbb{T}$ and $1_{\mathbb{T}}$ is the constant function 1. By Young's inequality we have $\|\mathcal{P}f\|_{\mathrm{L}^{\mathbf{p}}(\Omega_{\mathbb{T}})} \leq \|\mathcal{P}f\|_{\mathrm{L}^{\mathbf{p}}(\Omega_{\mathbb{T}})}$. Here, $\mathrm{L}^{\mathbf{p}}(\Omega_{\mathbb{T}})$ is the anisotropic Lebesgue space with exponents $\mathbf{p} = (p, q)$ in time and space, respectively; see Section 2 for details. Clearly, $\mathcal{P}f$ is independent of time and it holds $\mathcal{P}^2 = \mathcal{P}$. Therefore, $\mathcal{P} : \mathrm{L}^{\mathbf{p}}(\Omega_{\mathbb{T}}) \to \mathrm{L}^{\mathbf{p}}(\Omega_{\mathbb{T}})$ and $\mathcal{P}_\perp := \mathsf{id} - \mathcal{P}$ are continuous projections and thus induce a decomposition

$$\mathrm{L}^{\mathbf{p}}(\Omega_{\mathbb{T}}) = \mathrm{L}^q(\Omega) \oplus \mathrm{L}^{\mathbf{p}}_\perp(\Omega_{\mathbb{T}}),$$

where $\mathrm{L}^{\mathbf{p}}_\perp(\Omega_{\mathbb{T}}) := \mathcal{P}_\perp \mathrm{L}^{\mathbf{p}}(\Omega_{\mathbb{T}})$ and where we have used that $\mathcal{P}f$ is independent of time in order to identify $\mathcal{P}\mathrm{L}^{\mathbf{p}}(\Omega_{\mathbb{T}})$ with $\mathrm{L}^q(\Omega)$ in virtue of the isometry $\|\mathcal{P}f\|_{\mathrm{L}^q(\Omega)} = \|\mathcal{P}f\|_{\mathrm{L}^{\mathbf{p}}(\Omega_{\mathbb{T}})}$. It should be noted that estimating derivatives of $\mathcal{P}f$ and $\mathcal{P}_\perp f$ in a similar way, we obtain likewise a decomposition of higher order Sobolev spaces and of solenoidal spaces.

As in [20], we will solve problem (1.2) by decomposing it into a stationary problem and a time-dependent problem. That is, by applying $\mathcal{P}$ and $\mathcal{P}_\perp$ to (1.2), respectively, the Stokes problem is decomposed into to the following two problems.

(I) Stationary problem in $\mathrm{L}^q(\Omega)^n$:

$$\begin{cases} -\Delta u + \nabla p &= \mathcal{P}f \qquad \text{in } \Omega\,, \\ \operatorname{div} u &= \mathcal{P}g \qquad \text{in } \Omega\,, \\ u &= 0 \qquad \text{on } \partial\Omega\,. \end{cases} \tag{1.3}$$

The estimates for solutions in the L^q framework are well known; see e.g. [12, 14]. For the convenience of the reader the results are summarized in

Section 3. Our major assumption on $\mathcal{P}f$ here is $\mathcal{P}f \in \mathrm{L}^q(\Omega)^n$ for some $q \in (1,\infty)$. However, in unbounded domains, the condition $\mathcal{P}f \in \mathrm{L}^q(\Omega)^n$ is in general too weak to ensure that the solution u satisfies $\operatorname{div} u = \mathcal{P}g$ even when $\mathcal{P}g = 0$. Therefore, we often need to replace the divergence condition by $\nabla \operatorname{div} u = \nabla \mathcal{P}g$, that is,

$$\begin{cases} -\Delta u + \nabla p &= \mathcal{P}f \qquad \text{in } \Omega\,, \\ \nabla \operatorname{div} u &= \nabla \mathcal{P}g \qquad \text{in } \Omega\,, \\ u &= 0 \qquad \text{on } \partial\Omega\,. \end{cases} \tag{1.4}$$

The equation (1.4) can be solved under mild conditions on given data such as $\mathcal{P}f \in \mathrm{L}^q(\Omega)^n$ and $\mathcal{P}g \in \widehat{\mathrm{W}}^{1,q}(\Omega)$, $q \in (1,\infty)$. On the other hand, for example in the application to a Navier-Stokes type system as is discussed in Section 6, it is sometimes crucial to solve (1.3) exactly rather than (1.4). Therefore, additional conditions on $\mathcal{P}f$ and $\mathcal{P}g$ have to be imposed in these cases.

(II) Time-dependent problem in $\mathrm{L}^{\mathbf{p}}_{\perp}(\Omega_{\mathbb{T}})^n$:

$$\begin{cases} \partial_t u - \Delta u + \nabla p &= \mathcal{P}_{\perp} f \qquad \text{in } \Omega_{\mathbb{T}}\,, \\ \operatorname{div} u &= \mathcal{P}_{\perp} g \qquad \text{in } \Omega_{\mathbb{T}}\,, \\ u &= 0 \qquad \text{on } \partial\Omega\,. \end{cases} \tag{1.5}$$

The analysis of (1.5) is the main subject of this paper, and is discussed in Section 4. The main results are stated in three theorems depending on the type of domains: For the whole and the half space, this is Theorem 4.6, for bent half spaces Theorem 4.8, for bounded domains Theorem 4.11.

As for the time-dependent (periodic) problem (1.5), we can expect the existence of solutions which decay at spatial infinity under the mere assumption $\mathcal{P}_{\perp} f \in \mathrm{L}^{\mathbf{p}}_{\perp}(\Omega_{\mathbb{T}})^n$ even for unbounded domains. This remarkable feature also has an impact on the corresponding initial value problem: it allows us to give a fast and straightforward proof of the maximal L^p regularity for the Stokes operator on $\mathrm{L}^q(\Omega)$ *without* using the notion of $\mathcal{R}$-boundedness. We discuss this subject in Section 5. In fact, it follows directly from the abstract theory on the equivalence between maximal L^p regularity (for Cauchy problems) and time-periodic maximal L^p regularity, to which we contribute also in Section 5. This abstract theory was firstly established by Arendt and Bu [2] for generators of semigroups which are invertible. In Theorem 5.1 of this paper we extend their result to possibly non-invertible generators.

This paper is organized as follows: In Section 2, we introduce basic notation and definitions. We devote Section 3 to collecting the known results concerning problems (1.3) and (1.4). The treatment of problem (1.5) is carried out in Section 4. This section is further divided into Section 4.1, where we introduce a reduction of the Stokes system to $n-1$ heat equations, and Sections 4.2 – 4.4, where we treat the whole space, the half space, the bent half space and bounded domains, respectively. In Section 5 the relation between maximal L^p regularity for abstract Cauchy problems and time-periodic maximal L^p regularity is discussed. Finally, we give an application to the nonlinear simplified Ericksen-Leslie model governing the dynamics of nematic liquid crystal flows in Section 6.

2 Preliminaries

Throughout the paper, $T > 0$ denotes the time period and $\mathbb{T} := \mathbb{R}/T\mathbb{Z}$ the corresponding torus. The locally abelian group $G := \mathbb{T} \times \mathbb{R}^n$ is called *periodic whole space*. For a domain $\Omega \subset \mathbb{R}^n$ we introduce $\Omega_\mathbb{T} := \mathbb{T} \times \Omega$ and denote by $\mathrm{C}^m(\Omega_\mathbb{T})$, $k \in \mathbb{N}_0$, the space of m-times differentiable functions and by $\mathrm{C}_0^\infty(\Omega_\mathbb{T})$ the space of smooth and compactly supported functions in time and space. Furthermore, we introduce the notion $\mathrm{C}_0^\infty(\overline{\Omega_\mathbb{T}}) := \{u_{|\Omega_\mathbb{T}} \mid u \in \mathrm{C}_0^\infty(G)\}$. Here, the differentiable structure on $\Omega_\mathbb{T}$ is inherited from $\mathbb{R} \times \Omega$.

Let $\mathbf{p} := (p, q) \in (1, \infty)^2$. Then we introduce the anisotropic Lebesgue spaces $\mathrm{L}^\mathbf{p}(\Omega_\mathbb{T}) := \mathrm{L}^p(\mathbb{T}; \mathrm{L}^q(\Omega))$ with norm

$$\|u\|_{\mathrm{L}^\mathbf{p}(\Omega_\mathbb{T})} := \left(\frac{1}{T} \int_0^T \|u(t)\|_{\mathrm{L}^q(\Omega)}^p \, \mathrm{d}t \right)^{1/p} .$$

Since the topology and differentiable structure of G is inherited from $\mathbb{R}_t \times \mathbb{R}_x^n$ and since we deal with sufficiently smooth domains only, we can also define the anisotropic mixed-derivative Sobolev spaces

$$\begin{aligned} \mathrm{W}^{1,2,\mathbf{p}}(\Omega_\mathbb{T}) &:= \overline{\mathrm{C}_0^\infty(\overline{\Omega_\mathbb{T}})}^{\|\cdot\|_{\mathrm{W}^{1,2,\mathbf{p}}(\Omega_\mathbb{T})}}, \\ \|u\|_{\mathrm{W}^{1,2,\mathbf{p}}(\Omega_\mathbb{T})} &:= \sum_{|\alpha| \le 1, |\beta| \le 2} \|\partial_t^\alpha u\|_{\mathrm{L}^\mathbf{p}(\Omega_\mathbb{T})} + \|\partial_x^\beta u\|_{\mathrm{L}^\mathbf{p}(\Omega_\mathbb{T})} . \end{aligned}$$

Note that for domains Ω satisfying the segment condition we have

$$\mathrm{W}^{1,2,\mathbf{p}}(\Omega_\mathbb{T}) = \{f \in \mathrm{L}^\mathbf{p}(\Omega_\mathbb{T}) \mid \|f\|_{\mathrm{W}^{1,2,\mathbf{p}}(\Omega_\mathbb{T})} < \infty\},$$

where the derivatives which appear in the norm $\|f\|_{\mathrm{W}^{1,2,\mathbf{p}}(\Omega_\mathbb{T})}$ are to be understood in the sense of distributions. In the context of Stokes and

Navier-Stokes equations on unbounded domains, the concept of homogeneous (mixed-derivative) Sobolev spaces $\widehat{\mathrm{W}}^{0,1,\mathbf{p}}(\Omega_{\mathbb{T}})$ appears naturally. Such spaces are defined as

$$\widehat{\mathrm{W}}^{0,1,\mathbf{p}}(\Omega_{\mathbb{T}}) := \overline{\mathrm{C}_0^\infty(\overline{\Omega_{\mathbb{T}}})}^{\|\nabla\cdot\|_{\mathrm{L}^{\mathbf{p}}(\Omega_{\mathbb{T}})}} / \sim ,$$

where $\sim$ denotes the equivalence relation

$$u \sim v \text{ if and only if } \|\nabla u - \nabla v\|_{\mathrm{L}^{\mathbf{p}}(\Omega_{\mathbb{T}})} = 0.$$

The dual space of $\widehat{\mathrm{W}}^{0,1,\mathbf{p}'}(\Omega_{\mathbb{T}})$, where $\mathbf{p}' := (p', q')$ and p', q' are the respective Hölder conjugates, will be denoted by $\widehat{\mathrm{W}}^{0,-1,\mathbf{p}}(\Omega_{\mathbb{T}}) := [\widehat{\mathrm{W}}^{0,1,\mathbf{p}'}(\Omega_{\mathbb{T}})]^*$ and is endowed with the norm

$$\|g\|_{\widehat{\mathrm{W}}^{0,-1,\mathbf{p}}(\Omega_{\mathbb{T}})} := \sup_{0\neq\varphi\in C_0^\infty(\overline{\Omega_{\mathbb{T}}})} \frac{[g,\varphi]}{\|\nabla\varphi\|_{\mathrm{L}^{\mathbf{p}'}(\Omega_{\mathbb{T}})}}.$$

A subscript $\perp$ always denotes the projected part with respect to the complement projection $\mathcal{P}_\perp$ defined in Section 1. In all cases, we use corresponding notations for function spaces defined on Ω rather than $\Omega_{\mathbb{T}}$, i.e., for function spaces corresponding only to the space variable x. Moreover, we will denote by $\mathrm{L}^q_\sigma(\Omega)$ the closure of the solenoidal space $\mathrm{C}^\infty_{0,\sigma}(\Omega) := \{u \in \mathrm{C}_0^\infty(\Omega)^n \mid \operatorname{div} u = 0\}$ in the topology of $\mathrm{L}^q(\Omega)^n$. It is well known that $\mathrm{L}^q_\sigma(\Omega) = \{u \in \mathrm{L}^q(\Omega)^n \mid \operatorname{div} u = 0, n \cdot u_{|\partial\Omega} = 0\}$ for sufficiently smooth domains, see *e.g.* [16, Theorem III.2.3]. Furthermore, we introduce the space $\mathrm{L}^{\mathbf{p}}_\sigma(\Omega_{\mathbb{T}}) := \mathrm{L}^p(\mathbb{T}; \mathrm{L}^q_\sigma(\Omega))$.

Fourier variables of time-periodic functions will be denoted by $k \in \frac{2\pi}{T}\mathbb{Z}$, where $T > 0$ is the time period. Note that with this notation, k is not an integer in general. Moreover, we introduce the notation $\frac{2\pi}{T}\mathbb{Z}^* = \frac{2\pi}{T}\mathbb{Z} \setminus \{0\}$.

Concerning the boundary of the bent half spaces, we have to make certain regularity and smallness assumptions. Therefore, we give the following definition.

Definition 2.1. Let $K > 0$. We say that $\omega \in \mathrm{C}^{0,1}(\mathbb{R}^{n-1}) \cap \mathrm{W}^{2,1}_{\mathrm{loc}}(\mathbb{R}^{n-1})$ is of type K_1, or K_2, respectively, if $\|\nabla'\omega\|_\infty < K$ and if

(K_1) $n \geq 2$ and $\|\nabla'^2\omega\|_\infty < K$, or

(K_2) $n \geq 3$ and $\|\nabla'^2\omega\|_{\mathrm{L}^{n-1,\infty}(\mathbb{R}^{n-1})} < K$.

Remark 2.2. If $K > 0$ and $n \geq 3$, then the condition K_2 is fulfilled whenever $\||\cdot|\nabla'^2\omega\|_{\mathrm{L}^\infty(\mathbb{R}^{n-1})} < K$ or $\|\nabla'^2\omega\|_{\mathrm{L}^{n-1}(\mathbb{R}^{n-1})} < K$.

3 The Stationary Problem

The stationary problem (1.4) has been solved in [12, 14]. The results can be summarized as follows.

Proposition 3.1. *Let $n \geq 2$, $q \in (1,\infty)$ and let $\Omega = \mathbb{R}^n$ or $\Omega = \mathbb{R}^n_+$. Then for every $\mathcal{P}f \in \mathrm{L}^q(\Omega)^n$, $\mathcal{P}g \in \widehat{\mathrm{W}}^{1,q}(\Omega)$, there exists a unique solution $(u,p) \in \widehat{\mathrm{W}}^{2,q}(\Omega)^n \times \widehat{\mathrm{W}}^{1,q}(\Omega)$ to (1.4) satisfying*

$$\|\nabla^2 u\|_{\mathrm{L}^q(\Omega)} + \|\nabla p\|_{\mathrm{L}^q(\Omega)} \leq c\big(\|\mathcal{P}f\|_{\mathrm{L}^q(\Omega)} + \|\nabla \mathcal{P}g\|_{\mathrm{L}^q(\Omega)}\big) \tag{3.1}$$

with $c = c(n,q) > 0$.

Proposition 3.2. *Let $n \geq 3$, $q \in (1, n-1)$ and assume furthermore $\omega \in \mathrm{C}^{0,1}(\mathbb{R}^{n-1}) \cap \mathrm{W}^{2,1}_{\mathrm{loc}}(\mathbb{R}^{n-1})$ is such that for simplicity $\omega(0') = 0$. Then there exists a constant $K = K(n,q) > 0$ such that if ω is of type K_2, then for all $\mathcal{P}f \in \mathrm{L}^q(\mathbb{R}^n_\omega)^n$ and $\mathcal{P}g \in \widehat{\mathrm{W}}^{1,q}(\mathbb{R}^n_\omega)$ there exists a unique solution $(u,p) \in \widehat{\mathrm{W}}^{2,q}(\mathbb{R}^n_\omega)^n \times \widehat{\mathrm{W}}^{1,q}(\mathbb{R}^n_\omega)$ to (1.4) satisfying the estimate*

$$\|\nabla^2 u\|_{\mathrm{L}^q(\mathbb{R}^n_\omega)} + \|\nabla p\|_{\mathrm{L}^q(\mathbb{R}^n_\omega)} \leq c\big(\|\mathcal{P}f\|_{\mathrm{L}^q(\mathbb{R}^n_\omega)} + \|\nabla \mathcal{P}g\|_{\mathrm{L}^q(\mathbb{R}^n_\omega)}\big) \tag{3.2}$$

with a constant $c = c(n,q,\omega) > 0$.

Remark 3.3.

1. The uniqueness assertions in Proposition 3.1 and Proposition 3.2 are to be understood in the respective homogeneous Sobolev spaces, *i.e.*, only up to a constant for the pressure p and up to a linear polynomial $a + Ax$, where $a \in \mathbb{C}^n$ and $A \in \mathbb{C}^{n,n}$, for the velocity field u. However, in the half space and in bent half spaces, more information can be obtained due to the boundary condition. To be more precise, the velocity field u is

    - unique up to a linear term bx_n, where $b \in \mathbb{C}^n$, if $\Omega = \mathbb{R}^n_+$, or, more generally,
    - unique up to a linear term Ax, where $A \in \mathbb{C}^{n,n}$ and $A(x', \omega(x')) = 0$, if $\Omega = \mathbb{R}^n_\omega$. In particular, if ω is nonlinear, the velocity field u is unique. If however ω is a linear transformation, say, $\omega(x') = d'^T \cdot x'$ with $d' \in \mathbb{R}^{n-1}$, then u is unique up to a vector field of the form Ax where $A = a_n \otimes (-d'^T, 1)$ with a column vector $a_n \in \mathbb{C}^n$.

    Moreover, for any of the domains, we can employ the Sobolev embedding to solve (1.3) exactly if $q \in (1,n)$ by singling out a special divergence data with $\mathcal{P}g \in \mathrm{L}^{q^*}(\Omega)$ and a special solution with $\nabla u \in \mathrm{L}^{q^*}(\Omega)$, where $q^* := \frac{nq}{n-q}$ is the Sobolev index corresponding to q.

2. In [14], Proposition 3.2 is not stated with condition K_2, but with the conditions from Remark 2.2. However, revising the proof, a simple calculation as in (4.17) below shows that is suffices to assume K_2.

Note that we assume condition K_2 in Proposition 3.2. In particular we are restricted to $n \geq 3$ and $q \in (1, n-1)$. However, the problem

$$\left\{\begin{array}{rll} \lambda u - \Delta u + \nabla p &= \mathcal{P}f & \text{in } \mathbb{R}^n_\omega\,, \\ \operatorname{div} u &= \mathcal{P}g & \text{in } \mathbb{R}^n_\omega\,, \\ u &= 0 & \text{on } \partial\mathbb{R}^n_\omega\,. \end{array}\right. \tag{3.3}$$

has been solved by [12] for large resolvent parameters $|\lambda|$ under the assumption K_1, and therefore they could prove the following on bounded domains.

Proposition 3.4. *Let $n \geq 2$, $q \in (1, \infty)$ and let Ω be a bounded domain with a $\mathrm{C}^{1,1}$-boundary. Then for every $\mathcal{P}f \in \mathrm{L}^q(\Omega)^n$, $\mathcal{P}g \in \widehat{\mathrm{W}}^{1,q}(\Omega)$, there exists a unique solution $(u,p) \in \mathrm{W}^{2,q}(\Omega)^n \times \widehat{\mathrm{W}}^{1,q}(\Omega)$ to (1.3) satisfying*

$$\|u\|_{\mathrm{W}^{2,q}(\Omega)} + \|\nabla p\|_{\mathrm{L}^q(\Omega)} \leq c\big(\|\mathcal{P}f\|_{\mathrm{L}^q(\Omega)} + \|\nabla \mathcal{P}g\|_{\mathrm{L}^q(\Omega)}\big) \tag{3.4}$$

with $c = c(n,q,\Omega) > 0$.

4 The Time-Periodic Problem

4.1 Reduction of the Stokes System

In this section we recall the result of [25], which provides a reduction for the Stokes system based on the isomorphism to the space of solenoidal vector fields when the fluid domain is the whole space or has a graph boundary. Here, let us assume that $\Omega = \mathbb{R}^n$ or $\Omega = \mathbb{R}^n_+$ and introduce the linear operators $W : \mathrm{L}^q(\Omega)^n \to \mathrm{L}^q(\Omega)^{n-1}$ and $V = (V', V_n) : \mathrm{L}^q(\Omega)^{n-1} \to \mathrm{L}^q(\Omega)^n$ as

$$\begin{aligned} &Wu = u' + Su_n\,, \\ &V'w = w + SUS\cdot w\,,\ V_n w = -US\cdot w\,, \end{aligned} \tag{4.1}$$

with

$$\begin{aligned} &Sf = \nabla'(-\Delta')^{-\frac{1}{2}} f\,, \\ &Ug = (-\Delta')^{\frac{1}{2}} \int_{-\infty}^{x_n} e^{-(x_n-y_n)(-\Delta')^{\frac{1}{2}}} g(x_n, y_n)\,\mathrm{d}y_n\,, \end{aligned} \tag{4.2}$$

where we extend g by zero to the whole space in the case $\Omega = \mathbb{R}^n_+$. Since S is a singular integral operator it is classical that S is bounded in L^q for $q \in (1, \infty)$ and hence:

$$\|Sf\|_{\mathrm{L}^q(\mathbb{R}^{n-1})} \le C\|f\|_{\mathrm{L}^q(\mathbb{R}^{n-1})}. \tag{4.3}$$

It is also well known that the Poisson semigroup admits the estimate

$$\|Ug\|_{\mathrm{L}^q(\mathbb{R}_+;\mathrm{L}^q(\mathbb{R}^{n-1}))} \le C\|g\|_{\mathrm{L}^q(\mathbb{R}_+;\mathrm{L}^q(\mathbb{R}^{n-1}))}\,. \tag{4.4}$$

In particular, (4.3) and (4.4) imply the boundedness of W and V in $\mathrm{L}^q(\Omega)$. The key properties of W and V are stated as follows.

Lemma 4.1 ([25]). *Let $\Omega = \mathbb{R}^n$ or $\Omega = \mathbb{R}^n_+$. Then the following statements hold.*
(i) *The operator W satisfies*

$$\{\nabla p \in \mathrm{L}^q(\Omega)^n \mid p \in L^q_{loc}(\overline{\Omega})\,,\ \Delta p = 0 \text{ in } \Omega\} \subset \mathrm{Ker}_{L^q}(W)\,,$$

where $\mathrm{Ker}_{L^q}(W) = \{f \in L^q(\Omega)^n \mid Wf = 0\}$.
(ii) $\mathrm{Ran}_{L^q}(V) = L^q_\sigma(\Omega)$.
(iii) *$WV = I$ on $L^q(\Omega)^{n-1}$ and $VW = I$ on $L^q_\sigma(\Omega)$. In particular, the restriction $W|_{L^q_\sigma} : L^q_\sigma(\Omega) \to L^q(\Omega)^{n-1}$ is an isomorphism and its inverse is given by V. Moreover, for any $m \in \mathbb{N}$ the map $W|_{L^q_\sigma} : W^{m,q}_\sigma(\Omega) \to W^{m,q}(\Omega)^{n-1}$ is an isomorphism with its inverse V, and there are positive constants C and C' such that*

$$C'\|\nabla^m Wu\|_{L^q(\Omega)} \le \|\nabla^m u\|_{L^q(\Omega)} \le C\|\nabla^m Wu\|_{L^q(\Omega)} \tag{4.5}$$

for any $u \in W^{m,q}_\sigma(\Omega)$.

Remark 4.2. Although the properties of W in the Sobolev space $\mathrm{W}^{m,q}_\sigma(\Omega)$ and (4.5) are not explicitly stated in [25], these are easily obtained from the results in $\mathrm{L}^q_\sigma(\Omega)$ and the definitions in (4.1) and (4.2).

Moreover, it is clear that the results transfer to the time-dependent case, if we replace $\mathrm{L}^q(\Omega)$ by $\mathrm{L}^{\mathrm{p}}(\Omega_\mathbb{T})$ and similarly for Sobolev and solenoidal spaces.

4.2 The Whole Space and the Half Space

In this section we consider (1.5) in $\Omega_\mathbb{T} = G$ or $\Omega_\mathbb{T} = \mathbb{T} \times \mathbb{R}^n_+$. We start with two preparational lemmata, dealing with the problems

$$\begin{cases} \Delta u_g &= g \qquad \text{in } \Omega_\mathbb{T}\,, \\ \partial_n u_g &= 0 \qquad \text{on } \partial\Omega_\mathbb{T}\,, \end{cases} \tag{4.6}$$

and

$$\begin{cases} \partial_t w - \Delta w &= h \qquad \text{in } \Omega_{\mathbb{T}}\,, \\ \qquad\quad w &= 0 \qquad \text{on } \partial\Omega_{\mathbb{T}}\,. \end{cases} \tag{4.7}$$

It should be understood that the boundary conditions are omitted in the whole space case $\Omega_{\mathbb{T}} = G$.

Lemma 4.3. *Let $\mathbf{p} \in (1,\infty)^2$ and assume $g \in \widehat{\mathrm{W}}_{\perp}^{0,-1,\mathbf{p}}(\Omega_{\mathbb{T}})$. Then there is a unique $u_g \in \widehat{\mathrm{W}}_{\perp}^{0,1,\mathbf{p}}(\Omega_{\mathbb{T}})$ solving (4.6) in a weak sense, and there is $c > 0$ such that*

$$\|\nabla u_g\|_{L^{\mathbf{p}}(\Omega_{\mathbb{T}})} \leq c\|g\|_{\widehat{\mathrm{W}}^{0,-1,\mathbf{p}}(\Omega_{\mathbb{T}})}\,. \tag{4.8}$$

Moreover, if $m \in \{0,1\}$ and $g \in \mathrm{W}_{\perp}^{0,m,\mathbf{p}}(\Omega_{\mathbb{T}})$ in addition, then $\nabla u_g \in \mathrm{W}_{\perp}^{0,1+m,\mathbf{p}}(\Omega_{\mathbb{T}})$ and

$$\|\nabla^{2+m} u_g\|_{\mathrm{L}^{\mathbf{p}}(\Omega_{\mathbb{T}})} \leq C\|\nabla^m g\|_{\mathrm{L}^{\mathbf{p}}(\Omega_{\mathbb{T}})}\,. \tag{4.9}$$

Proof. By [12], there is for almost all times $t \in \mathbb{T}$ a unique solution $u_g(t,\cdot) \in \widehat{\mathrm{W}}^{1,q}(\mathbb{R}^n)$ in the weak sense to

$$\Delta u_g(t,\cdot) = g(t,\cdot) \qquad \text{in } \mathbb{R}^n\,,$$

with a corresponding *a priori* estimate. Integrating over time yields the result for $\Omega_{\mathbb{T}} = G$. In the half space case, the result follows easily via a reflection argument. The additional regularity follows by the uniqueness assertion when solving (4.6) with u_g and g replaced by $\partial_i u_g$ and $\partial_i g$, respectively. □

Lemma 4.4. *Let $\mathbf{p} \in (1,\infty)^2$ and $h \in \mathrm{L}_{\perp}^{\mathbf{p}}(\Omega_{\mathbb{T}})^n$. Then there exists a unique solution $w \in \mathrm{W}_{\perp}^{1,2,\mathbf{p}}(\Omega_{\mathbb{T}})^n$ to (4.7), and the following estimate holds:*

$$\|w, \partial_t w, \nabla^2 w\|_{\mathrm{L}^{\mathbf{p}}(\Omega_{\mathbb{T}})} \leq c\|h\|_{\mathrm{L}^{\mathbf{p}}(\Omega_{\mathbb{T}})}\,. \tag{4.10}$$

If additionaly $h \in \mathrm{L}_{\perp}^{\mathbf{s}}(\Omega_{\mathbb{T}})^n$ for some $\mathbf{s} \in (1,\infty)^2$, then $w \in \mathrm{W}_{\perp}^{1,2,\mathbf{s}}(\Omega_{\mathbb{T}})^n$.

Proof. Assume first $\Omega_{\mathbb{T}} = G$. Note that it suffices to assume $h \in \mathcal{P}_{\perp}\mathcal{S}(G)$. Therefore, an application of the Fourier transform yields the representation formula $w := \mathcal{F}_G^{-1} m \mathcal{F}_G h \in \mathcal{P}_{\perp}\mathcal{S}'(G)$, where

$$m(k,\xi) := \begin{cases} 0, & \text{if } k = 0, \\ \frac{1}{ik+|\xi|^2}, & k \in \frac{2\pi}{T}\mathbb{Z}^*. \end{cases}$$

In [20], the symbols m, $ik\cdot m$ and $(i\xi)^\alpha\cdot m$, $|\alpha|\leq 2$, have been shown to be $\mathrm{L}^{\mathbf{p}}$ multipliers for $\mathbf{p}=(p,p)$. However, since the prove in [20] rests on the Marcinkiewicz multiplier theorem, it follows by the work of Besov [4] that the general case $\mathbf{p}\in(1,\infty)^2$ is covered as well. Consequently $w\in \mathrm{W}^{1,2,\mathbf{p}}_{\perp}(G)$ and

$$\|w,\partial_t w,\nabla^2 w\|_{\mathrm{L}^{\mathbf{p}}(\mathbb{R}^n)}\leq c\|h\|_{\mathrm{L}^{\mathbf{p}}(G)}.$$

Since we have an explicit representation formula for w in terms of an everywhere defined multiplier, we obtain uniqueness on the level of tempered distributions. This observation also implies the additional regularity assertion.

In the case $\Omega_{\mathbb{T}}=\mathbb{T}\times\mathbb{R}^n_+$, the reflection principle immediately yields existence and the *a priori* estimate. For the uniqueness, let $w\in\mathrm{W}^{1,2,\mathbf{p}}_{\perp}(\Omega_{\mathbb{T}})$ be a solution to (4.7) with data $f=0$ and let $h\in\mathrm{L}^{\mathbf{p}'}_{\perp}(\Omega_{\mathbb{T}})$ be arbitrary. Then due to the existence part we find $v\in\mathrm{W}^{1,2,\mathbf{p}'}_{\perp}(\Omega_{\mathbb{T}})$ such that $\partial_t v-\Delta v=h$ and $v|_{\partial\mathbb{R}^n_+}=0$. Defining $\tilde v(t,x):=v(-t,x)$, we conclude

$$\langle w,h\rangle_{\mathrm{L}^2(\Omega_{\mathbb{T}})}=-\langle w,\partial_t\tilde v+\Delta\tilde v\rangle_{\mathrm{L}^2(\Omega_{\mathbb{T}})}=\langle\partial_t w-\Delta w,\tilde v\rangle_{\mathrm{L}^2(\Omega_{\mathbb{T}})}=0,$$

and hence $w=0$. The regularity assertion follows now by the reflection principle and the uniqueness in the whole space. The proof is complete. □

In order to formulate our main theorem of this section, we introduce the space

$$\begin{aligned}&Y^{\mathbf{p}}_{\perp}(\Omega_{\mathbb{T}}):=\{(f,g)\in\mathrm{L}^{\mathbf{p}}_{\perp}(\Omega_{\mathbb{T}})^n\times\widehat{\mathrm{W}}^{0,1,\mathbf{p}}_{\perp}(\Omega_{\mathbb{T}})\mid\partial_t g\in\widehat{\mathrm{W}}^{0,-1,\mathbf{p}}_{\perp}(\Omega_{\mathbb{T}})\},\\&\|(f,g)\|_{Y^{\mathbf{p}}_{\perp}(\Omega_{\mathbb{T}})}:=\|f\|_{\mathrm{L}^{\mathbf{p}}(\Omega_{\mathbb{T}})}+\|\nabla g\|_{\mathrm{L}^{\mathbf{p}}(\Omega_{\mathbb{T}})}+\|\partial_t g\|_{\widehat{\mathrm{W}}^{0,-1,\mathbf{p}}(\Omega_{\mathbb{T}})}.\end{aligned}\tag{4.11}$$

Remark 4.5. The assumption $\partial_t g\in\widehat{\mathrm{W}}^{0,-1,\mathbf{p}}_{\perp}(\Omega_{\mathbb{T}})$ and the Poincaré inequality with respect to the time variable imply that $g\in\widehat{\mathrm{W}}^{0,-1,\mathbf{p}}_{\perp}(\Omega_{\mathbb{T}})$. Hence, together with the condition $\nabla g\in\mathrm{L}^{\mathbf{p}}_{\perp}(\Omega_{\mathbb{T}})^n$ we have $g\in\mathrm{L}^{\mathbf{p}}_{\perp}(\Omega_{\mathbb{T}})$.

Theorem 4.6. *Let $\mathbf{p}\in(1,\infty)^2$ and $(\mathcal{P}_\perp f,\mathcal{P}_\perp g)\in Y^{\mathbf{p}}_{\perp}(\Omega_{\mathbb{T}})$. Then there exists a unique solution $(u,p)\in\mathrm{W}^{1,2,\mathbf{p}}_{\perp}(\Omega_{\mathbb{T}})^n\times\widehat{\mathrm{W}}^{0,1,\mathbf{p}}_{\perp}(\Omega_{\mathbb{T}})$ to (1.5) on $\Omega_{\mathbb{T}}$, and the following estimate holds.*

$$\|u,\partial_t u,\nabla^2 u,\nabla p\|_{\mathrm{L}^{\mathbf{p}}(\Omega_{\mathbb{T}})}\leq c\|(\mathcal{P}_\perp f,\mathcal{P}_\perp g)\|_{Y^{\mathbf{p}}_{\perp}(\Omega_{\mathbb{T}})}.\tag{4.12}$$

If additionally $(\mathcal{P}_\perp f,\mathcal{P}_\perp g)\in Y^{\mathbf{s}}_{\perp}(\Omega_{\mathbb{T}})^n$ for some $\mathbf{s}\in(1,\infty)^2$, then $(u,p)\in\mathrm{W}^{1,2,\mathbf{s}}_{\perp}(\Omega_{\mathbb{T}})^n\times\widehat{\mathrm{W}}^{0,1,\mathbf{s}}_{\perp}(\Omega_{\mathbb{T}})$.

Proof. Let u_g be the solution to (4.6) obtained in Lemma 4.3. From the assumptions on $\mathcal{P}_\perp g$ we can infer $\nabla' u_g \in \mathrm{W}^{1,2,\mathbf{p}}_\perp(\Omega_\mathbb{T})^{n-1}$, where we have written ∇' for the gradient with respect to the first $n-1$ space variables. Set $v = V\tilde{w}$. Here, $V : \mathrm{L}^{\mathbf{p}}(\Omega)^{n-1} \to \mathrm{L}^{\mathbf{p}}_\sigma(\Omega)$ is the isomorphism in Lemma 4.1 and $\tilde{w} := (w + \nabla' u_g)$ with w being the solution to (4.7) with right-hand side $h = W\mathbb{P}\mathcal{P}_\perp f - (\partial_t - \Delta)\nabla' u_g \in \mathrm{L}^{\mathbf{p}}(\Omega)^{n-1}$. Note that for $\Omega_\mathbb{T} = G$, $\tilde{w}$ is simply the solution to (4.7) with right-hand side $h = W\mathbb{P}\mathcal{P}_\perp f$, while in the half space case this is not true due to possibly non-trivial boundary values of $\nabla' u_g$.

Let ∇p_v be the pressure field defined as $\nabla p_v = \mathbb{Q}\Delta v$. Then, arguing as in [25], we can check that $(v, \nabla p_v)$ solves

$$\begin{cases} \partial_t v - \Delta v + \nabla p_v &= \mathbb{P}\mathcal{P}_\perp f \qquad \text{in } \Omega_\mathbb{T}\,, \\ \operatorname{div} v &= 0 \qquad \text{in } \Omega_\mathbb{T}\,, \\ v &= -\nabla u_g \qquad \text{on } \partial\Omega_\mathbb{T}\,. \end{cases} \tag{4.13}$$

For the convenience of the reader we give a sketch of the proof here.

Clearly $w \in \mathrm{W}^{1,2,\mathbf{p}}_\perp(\Omega_\mathbb{T})^{n-1}$ by Lemma 4.4, whence $\tilde{w} \in \mathrm{W}^{1,2,\mathbf{p}}_\perp(\Omega_\mathbb{T})^{n-1}$ and so $v = V\tilde{w} \in \mathrm{W}^{1,2,\mathbf{p}}_{\sigma,\perp}(\Omega_\mathbb{T})$ as well as $\tilde{w} = Wv$ by Lemma 4.1. Moreover, since w solves $\partial_t w - \Delta w = W\mathbb{P}\mathcal{P}_\perp f - (\partial_t - \Delta)\nabla' u_g$ in $\Omega_\mathbb{T}$ we see that v solves $\partial_t v - V\Delta W v = VW\mathbb{P}\mathcal{P}_\perp f$ in $\Omega_\mathbb{T}$, while

$$V\Delta W v = VW\Delta v = VW\mathbb{P}\Delta v,$$

for W commutes with Δ by its definition and $\mathbb{Q}\Delta v$ is the harmonic pressure and therefore $W\mathbb{Q}\Delta v = 0$ by Lemma 4.1. Since $VW = \mathsf{id}$ on $\mathrm{L}^{\mathbf{p}}_\sigma(\Omega)$ we finally observe that $\partial_t v - \Delta v + \nabla p_v = \mathbb{P}\mathcal{P}_\perp f$ in $\Omega_\mathbb{T}$ for $\nabla p_v = \mathbb{Q}\Delta v$. It is straightforward from the definition of V that $v' = \tilde{w}$ on $\partial\Omega$, which yields $v' = -\nabla' u_g$ on $\partial\Omega$ by the Dirichlet condition of w. Moreover, on the one hand $\partial_n u_g = 0$ on $\partial\Omega$, since u_g solves (4.6), and on the other hand $v_n = v \cdot n = 0$ on $\partial\Omega$ since $v \in \mathrm{L}^{\mathbf{p}}_\sigma(\Omega_\mathbb{T})$. Therefore $v = -\nabla u_g$ on $\partial\Omega$. That is, $(v, \nabla p_v)$ solves indeed (4.13).

Furthermore, we have the estimates $\|\nabla p_v\|_{\mathrm{L}^{\mathbf{p}}(\Omega_\mathbb{T})} \leq c\|\nabla^2 v\|_{\mathrm{L}^{\mathbf{p}}(\Omega_\mathbb{T})}$ and

$$\begin{aligned} c\|v, \partial_t v, \nabla^2 v\|_{\mathrm{L}^{\mathbf{p}}(\Omega_\mathbb{T})} &\leq c\|\tilde{w}, \partial_t \tilde{w}, \nabla^2 \tilde{w}\|_{\mathrm{L}^{\mathbf{p}}(\Omega_\mathbb{T})} \\ &\leq c\|W\mathbb{P}\mathcal{P}_\perp f - (\partial_t - \Delta)\nabla' u_g\|_{\mathrm{L}^{\mathbf{p}}(\Omega_\mathbb{T})} \\ &\leq v\big(\|\mathcal{P}_\perp f\|_{\mathrm{L}^{\mathbf{p}}(\Omega_\mathbb{T})} + \|\nabla' \mathcal{P}_\perp g\|_{\mathrm{L}^{\mathbf{p}}(\Omega_\mathbb{T})} + \|\partial_t \mathcal{P}_\perp g\|_{\widehat{\mathrm{W}}^{0,-1,\mathbf{p}}(\Omega_\mathbb{T})}\big)\,. \end{aligned}$$

Here we have used Lemma 4.1 and Lemma 4.4. Now we define $(u, \nabla p)$ as

$$\begin{aligned} u &= \nabla u_g + v\,, \\ \nabla p &= \nabla p_v + \mathbb{Q}\mathcal{P}_\perp f + \nabla\partial_t u_g - \nabla\mathcal{P}_\perp g\,, \end{aligned}$$

which satisfies (1.5) on $\Omega_{\mathbb{T}}$ with the *a priori* estimate (4.12).

Next we show the uniqueness. Let $(u,p) \in W_{\perp}^{1,2,\mathbf{p}}(\Omega_{\mathbb{T}})^n \times \widehat{\mathrm{W}}_{\perp}^{0,1,\mathbf{p}}(\Omega_{\mathbb{T}})$ be a solution with homogeneous data $(\mathcal{P}_{\perp}f, \mathcal{P}_{\perp}g) = (0,0)$. Then in particular $u \in \mathrm{L}^{\mathbf{p}}_{\sigma}(\Omega_{\mathbb{T}})$ and by applying the Helmholtz projection, we conclude that $Wu = W\mathbb{P}u \in W_{\perp}^{1,2,\mathbf{p}}(\Omega_{\mathbb{T}})^{n-1}$ solves (4.7) with homogeneous data. Hence, $Wu = 0$ by Lemma 4.4. Since $VW = \mathsf{id}$ on $\mathrm{L}^{\mathbf{p}}_{\sigma}(\Omega_{\mathbb{T}})$ by Lemma 4.1, it follows $u = VWu = 0$ and consequently also $p = 0$.

Similarly, the regularity assertion of Theorem 4.6 follows from the regularity assertion of Lemma 4.4. The proof is complete. □

4.3 The Bent Half Space

We consider bent periodic half spaces $G_\omega := \mathbb{T} \times \mathbb{R}^n_\omega$ that are merely small perturbations of the half space G_+, *i.e.*, if ω is close to the zero function in a certain sense. Given a Lipschitz continuous function $\omega : \mathbb{R}^{n-1} \to \mathbb{R}$, we define the transformation $\phi_\omega : G_\omega \to G_+$ defined via

$$\phi_\omega(t,x) := (t,\tilde{x}) := (t, x', x_n - \omega(x')).$$

For a function u defined on G_ω we introduce a function $\tilde{u}$ defined on G_+ by setting $\tilde{u}(t,\tilde{x}) := u(\phi_\omega^{-1}(t,\tilde{x}))$.

Proposition 4.7. *Let* $\mathbf{p} \in (1,\infty)^2$*,* $\omega \in \mathrm{C}^{0,1}(\mathbb{R}^{n-1}) \cap \mathrm{W}^{2,1}_{\mathrm{loc}}(\mathbb{R}^{n-1})$ *and* $K > 0$*. For* $m \in \{0,1,2\}$*, the mapping* $u \mapsto \tilde{u}$ *is an isomorphism between* $\mathrm{W}^{0,m,\mathbf{p}}(G_\omega)$ *and* $\mathrm{W}^{0,m,\mathbf{p}}(G_+)$ *as well as between* $\widehat{\mathrm{W}}^{0,1,\mathbf{p}}(G_\omega)$ *and* $\widehat{\mathrm{W}}^{0,1,\mathbf{p}}(G_+)$ *and between* $\widehat{\mathrm{W}}^{0,-1,\mathbf{p}}(G_\omega)$ *and* $\widehat{\mathrm{W}}^{0,-1,\mathbf{p}}(G_+)$*, if* ω *is either of type* K_1 *or of type* K_2 *when* $q \in (1, n-1)$*.*

Proof. It is readily seen that $\phi_\omega : G_\omega \to G_+$ is a bijection with Jacobian equal to 1. If we denote by $\tilde{\partial}_i$, $\tilde{\nabla}$ and similar expressions the corresponding differential operators with respect to the variable $\tilde{x} \in G_+$, then using $\partial_n\omega = 0$ we see

$$\begin{aligned} \partial_i u(x) &= (\tilde{\partial}_i - (\partial_i\omega)\tilde{\partial}_n)\tilde{u}(\tilde{x}), \\ \partial_i\partial_j u(x) &= (\tilde{\partial}_i\tilde{\partial}_j - (\partial_i\omega)\tilde{\partial}_j\tilde{\partial}_n - (\partial_j\omega)\tilde{\partial}_i\tilde{\partial}_n - (\partial_i\partial_j\omega)\tilde{\partial}_n \\ &\quad + (\partial_i\omega)(\partial_j\omega)\tilde{\partial}_n^2)\tilde{u}(\tilde{x}). \end{aligned} \tag{4.14}$$

Hence, there is $C = C(n) > 0$ such that

$$\begin{aligned} \|u\|_{\mathrm{L}^{\mathbf{p}}(G_\omega)} &= \|\tilde{u}\|_{\mathrm{L}^{\mathbf{p}}(G_+)}, \\ \|\nabla u\|_{\mathrm{L}^{\mathbf{p}}(G_\omega)} &\le C(1 + \|\nabla'\omega\|_\infty)\|\tilde{\nabla}\tilde{u}\|_{\mathrm{L}^{\mathbf{p}}(G_+)}, \end{aligned} \tag{4.15}$$

and

$$\begin{aligned}\|\nabla^2 u\|_{\mathrm{L}^{\mathbf{p}}(G_\omega)} \leq C(1+\|\nabla'\omega\|_\infty)^2\|\tilde\nabla^2\tilde u\|_{\mathrm{L}^{\mathbf{p}}(G_+)}\\ + C\|(\nabla'^2\omega)\tilde\partial_n\tilde u\|_{\mathrm{L}^{\mathbf{p}}(G_+)}\,.\end{aligned} \tag{4.16}$$

From (4.15) it follows immediately that the mapping $u \to \tilde u$ is an isomorphism between $\mathrm{W}^{0,m,\mathbf{p}}(G_\omega)$ and $\mathrm{W}^{0,m,\mathbf{p}}(G_+)$ for $m \in \{0,1\}$ as well as between $\widehat{\mathrm{W}}^{0,1,\mathbf{p}}(G_\omega)$ and $\widehat{\mathrm{W}}^{0,1,\mathbf{p}}(G_+)$.

In order to deal with the second derivatives, assume first that we are in the case K_2. Then the estimate of the term $\|(\nabla'^2\omega)\tilde\partial_n\tilde u\|_{\mathrm{L}^{\mathbf{p}}(G_+)}$ becomes non-trivial. Assume for now that $\partial_n\tilde u$ is smooth and compactly supported and write $\partial_n\tilde u(t,\cdot) = \varphi \in \mathrm{C}_0^\infty(\overline{\mathbb{R}^n_+})$ for a fixed but arbitrary $t \in \mathbb{T}$. By the generalized Sobolev embedding theorem there exists a constant $c > 0$, such that for all $\tilde x_n > 0$

$$\|\varphi(\cdot,\tilde x_n)\|_{\mathrm{L}^{s,q}(\mathbb{R}^{n-1})} \leq c\|\tilde\nabla'\varphi(\cdot,\tilde x_n)\|_{\mathrm{L}^{q}(\mathbb{R}^{n-1})},$$

where $s > q$ is defined via $\frac{1}{n-1} + \frac{1}{s} = \frac{1}{q}$. Therefore, we obtain by the generalized Hölder inequality

$$\begin{aligned}\|(\nabla'^2\omega)\varphi\|^q_{\mathrm{L}^q(\mathbb{R}^n_+)} &\leq c\int_0^\infty\int_{\mathbb{R}^{n-1}} |\nabla'^2\omega|^q|\varphi(\cdot,\tilde x_n)|^q \,\mathrm{d}x'\,\mathrm{d}\tilde x_n\\ &\leq c\|\nabla'^2\omega\|^q_{\mathrm{L}^{n-1,\infty}(\mathbb{R}^{n-1})}\int_0^\infty \|\varphi(\cdot,\tilde x_n)\|^q_{\mathrm{L}^{s,q}(\mathbb{R}^{n-1})}\,\mathrm{d}\tilde x_n\\ &\leq cK^q\|\tilde\nabla'\varphi\|^q_{\mathrm{L}^q(\mathbb{R}^n_+)}.\end{aligned} \tag{4.17}$$

Hence, integrating over time, we obtain

$$\|(\nabla'^2\omega)\partial_n\tilde u\|_{\mathrm{L}^{\mathbf{p}}(G_+)} \leq cK\|\tilde\nabla'\partial_n\tilde u\|_{\mathrm{L}^{\mathbf{p}}(G_+)} \leq cK\|\nabla^2\tilde u\|_{\mathrm{L}^{\mathbf{p}}(G_+)}. \tag{4.18}$$

If ω is of type K_1, we can immediately estimate

$$\|(\nabla'^2\omega)\partial_n\tilde u\|_{\mathrm{L}^{\mathbf{p}}(G_+)} \leq K\|\nabla\tilde u\|_{\mathrm{L}^{\mathbf{p}}(G_+)} \leq K\|\tilde u\|_{\mathrm{W}^{0,2,\mathbf{p}}(G_+)} \tag{4.19}$$

Collecting (4.15), (4.18) and (4.19), the mapping $u \to \tilde u$ is also an isomorphism between $W^{0,2,\mathbf{p}}(G_\omega)$ and $W^{0,2,\mathbf{p}}(G_+)$.

Moreover, let $F \in \widehat{\mathrm{W}}^{0,-1,\mathbf{p}}(G_\omega)$ and define $\tilde F$ via $[\tilde F,\tilde\varphi] := [F,\varphi]$ for all $\tilde\varphi \in \widehat{\mathrm{W}}^{0,1,\mathbf{p}}(G_+)$. Then using (4.15), we get

$$\begin{aligned}c^{-1}\|\tilde F\|_{\widehat{\mathrm{W}}^{-1,\mathbf{p}}(G_+)} &= c^{-1}\sup_{0\neq\tilde\varphi\in\widehat{\mathrm{W}}^{1,\mathbf{p}}(G_+)}\frac{|[\tilde F,\tilde\varphi]|}{\|\nabla\tilde\varphi\|_{\mathrm{L}^{\mathbf{p}}(G_+)}}\\ &\leq \sup_{0\neq\varphi\in\widehat{\mathrm{W}}^{1,\mathbf{p}}(G_\omega)}\frac{|[F,\varphi]|}{\|\nabla\varphi\|_{\mathrm{L}^{\mathbf{p}}(G_\omega)}} = \|F\|_{\widehat{\mathrm{W}}^{-1,\mathbf{p}}(G_\omega)},\end{aligned} \tag{4.20}$$

where $c = c(n,\omega) > 0$. □

We can now give the main theorem of this section, namely that there is a unique solution to the Stokes resolvent problem on the bent periodic half space, given that the bending is small in some sense. The space $Y^{\mathbf{p}}_{\perp}(G_\omega)$ has been introduced in (4.11).

Theorem 4.8. *Let* $\mathbf{p} \in (1,\infty)^2$ *and* $\omega \in \mathrm{C}^{0,1}(\mathbb{R}^{n-1}) \cap \mathrm{W}^{2,1}_{\mathrm{loc}}(\mathbb{R}^{n-1})$. *Then there is a constant* $K = K(n,q) > 0$ *with the following property:*
If

- ω *is either of type* K_1 *or of type* K_2 *when* $q \in (1, n-1)$, *and if*
- $(\mathcal{P}_\perp f, \mathcal{P}_\perp g) \in Y^{\mathbf{p}}_{\perp}(G_\omega)$,

then there exists a unique solution $(u,p) \in \mathrm{W}^{1,2,\mathbf{p}}_{\perp}(G_\omega)^n \times \widehat{\mathrm{W}}^{0,1,\mathbf{p}}_{\perp}(G_\omega)$ *to the problem* (1.5) *on* G_ω. *This solution satisfies the* a priori *estimate*

$$\|u, \partial_t u, \nabla^2 u, \nabla p\|_{\mathrm{L}^{\mathbf{p}}(G_\omega)} \leq c\|(\mathcal{P}_\perp f, \mathcal{P}_\perp g)\|_{Y^{\mathbf{p}}_{\perp}(G_\omega)}, \tag{4.21}$$

where $c = c(n, \mathbf{p}, \omega) > 0$ *is a constant. If additionally* $(\mathcal{P}_\perp f, \mathcal{P}_\perp g) \in Y^{\mathbf{s}}_{\perp}(G_\omega)$ *for some* $\mathbf{s} \in (1,\infty)^2$ *and if* $K \leq \min\{K(n,q), K(n,s)\}$, *then* $(u,p) \in \mathrm{W}^{1,2,\mathbf{s}}_{\perp}(G_\omega)^n \times \widehat{\mathrm{W}}^{0,1,\mathbf{s}}_{\perp}(G_\omega)$.

Proof. Our perturbation argument will be carried out on certain suitable Banach spaces. Namely, for $\Omega_{\mathbb{T}} = G_+$ and $\Omega_{\mathbb{T}} = G_\omega$ we introduce

$$X^{\mathbf{p}}_{\perp}(\Omega_{\mathbb{T}}) := \Big(W^{1,2,\mathbf{p}}_{\perp}(\Omega_{\mathbb{T}}) \cap W^{0,1,\mathbf{p}}_{0,\perp}(\Omega_{\mathbb{T}})\Big)^n \times \widehat{\mathrm{W}}^{0,1,\mathbf{p}}_{\perp}(\Omega_{\mathbb{T}}), \tag{4.22}$$

and equip them with the norms

$$\|u,p\|_{X^{\mathbf{p}}_{\perp}(\Omega_{\mathbb{T}})} := \|u, \partial_t u, \nabla^2 u, \nabla p\|_{\mathrm{L}^{\mathbf{p}}(\Omega_{\mathbb{T}})}.$$

Define operators $S_{\mathbf{p}} : X^{\mathbf{p}}_{\perp}(G_\omega) \to Y^{\mathbf{p}}_{\perp}(G_\omega)$ and $\tilde{S}_{\mathbf{p}} : \tilde{X}^{\mathbf{p}}_{\perp}(G_+) \to \tilde{Y}^{\mathbf{p}}_{\perp}(G_+)$ via

$$S_{\mathbf{p}}(u,p) := \begin{pmatrix} \partial_t u - \Delta u + \nabla p \\ -\operatorname{div} u \end{pmatrix}, \tag{4.23}$$

and a similar expression for $\tilde{S}_{\mathbf{p}}$. Observe that $\tilde{S}_q : \tilde{X}^{\mathbf{p}}_{\perp}(G_+) \to \tilde{Y}^{\mathbf{p}}_{\perp}(G_+)$ is an isomorphism due to Theorem 4.6. In virtue of (4.14) we obtain

$$S_{\mathbf{p}}(u,p) \circ \phi_\omega^{-1} = \tilde{S}_{\mathbf{p}}(\tilde{u},\tilde{p}) + \tilde{R}_{\mathbf{p}}(\tilde{u},\tilde{p}), \tag{4.24}$$

where the remainder $\tilde{R}_{\mathbf{p}}$ is given by

$$\tilde{R}_{\mathbf{p}}(\tilde{u},\tilde{p}) := \begin{pmatrix} -|\nabla'\omega|^2 \tilde{\partial}_n^2 \tilde{u} + 2(0,\nabla'\omega)\cdot \tilde{\nabla}\tilde{\partial}_n \tilde{u} + (\Delta'\omega)\tilde{\partial}_n \tilde{u} - (0,\nabla'\omega)\tilde{\partial}_n \tilde{p} \\ (0,\nabla'\omega)\cdot \tilde{\partial}_n \tilde{u} \end{pmatrix}.$$

It is our goal to show that $\tilde{R}_{\mathbf{p}} : \tilde{X}^{\mathbf{p}}_{\perp}(G_+) \to \tilde{Y}^{\mathbf{p}}_{\perp}(G_+)$ is relatively small with respect to $\tilde{S}_{\mathbf{p}}$ in the operator norm. Note that we can estimate the $\tilde{Y}^{\mathbf{p}}_{\perp}$-norm of $\tilde{R}_{\mathbf{p}}(\tilde{u},\tilde{p})$ by

$$\begin{aligned}\|\tilde{R}_{\mathbf{p}}(\tilde{u},\tilde{p})\|_{\tilde{Y}^{\mathbf{p}}_{\perp}(G_+)} \leq 4K(1+K)\|\tilde{\nabla}^2\tilde{u}\|_{\mathrm{L}^{\mathbf{p}}(G_+)} + K\|\tilde{\partial}_n\tilde{p}\|_{\mathrm{L}^{\mathbf{p}}(G_+)} \\ + \|({\Delta'}^2\omega)\tilde{\partial}_n\tilde{u}\|_{\mathrm{L}^{\mathbf{p}}(G_+)} + K\|\partial_t\tilde{\partial}_n\tilde{u}\|_{\widehat{\mathrm{W}}^{0,-1,\mathbf{p}}(G_+)}.\end{aligned} \tag{4.25}$$

The third term is estimated by $\|({\Delta'}^2\omega)\tilde{\partial}_n\tilde{u}\|_{\mathrm{L}^{\mathbf{p}}(G_+)} \leq cK\|\tilde{u}\|_{\mathrm{W}^{0,2,\mathbf{p}}(G_+)}$ in virtue of (4.18). Taking into account the trivial estimate

$$\|\partial_t\tilde{\partial}_n\tilde{u}\|_{\widehat{\mathrm{W}}^{0,-1,\mathbf{p}}(G_+)} \leq \|\partial_t\tilde{u}\|_{L^{\mathbf{p}}(G_+)}, \tag{4.26}$$

it finally follows from (4.25), (4.26) and Ehrling's lemma

$$\begin{aligned}\|\tilde{R}_{\mathbf{p}}(\tilde{u},\tilde{p})\|_{\tilde{Y}^{\mathbf{p}}_{\perp}(G_+)} &\leq CK(1+K)\|\tilde{u},\partial_t\tilde{u},\tilde{\nabla}^2\tilde{u},\tilde{\nabla}\tilde{p}\|_{\mathrm{L}^{\mathbf{p}}(G_+)} \\ &\leq CK(1+K)\|\tilde{S}_{\mathbf{p}}^{-1}\|\|\tilde{S}_{\mathbf{p}}(\tilde{u},\tilde{p})\|_{\tilde{Y}^{\mathbf{p}}_{\perp}(G_+)},\end{aligned}$$

where $C = C(n,\mathbf{p},\omega) > 0$. Hence, if we choose sufficiently small, then $\tilde{S}_{\mathbf{p}} + \tilde{R}_{\mathbf{p}} : \tilde{X}^{\mathbf{p}}_{\perp}(G_+) \to \tilde{Y}^{\mathbf{p}}_{\perp}(G_+)$ is an isomorphism and so is $S_{\mathbf{p}} : X^{\mathbf{p}}_{\perp}(G_\omega) \to Y^{\mathbf{p}}_{\perp}(G_\omega)$ by (4.24) and Proposition 4.7. In particular, we have the *a priori* estimate (4.21). As for the regularity assertion, it suffices to apply the above argument to the spaces $X^{\mathbf{p}}_{\perp} \cap X^{\mathbf{s}}_{\perp}$ and $Y^{\mathbf{p}}_{\perp} \cap Y^{\mathbf{s}}_{\perp}$ instead of $X^{\mathbf{p}}_{\perp}$ and $Y^{\mathbf{p}}_{\perp}$, which gives the existence of the solution (u_s,p_s) in $X^{\mathbf{p}}_{\perp} \cap X^{\mathbf{s}}_{\perp}$. Then the uniqueness in $X^{\mathbf{p}}_{\perp}$ implies $(u,p) = (u_s,p_s)$. The proof is complete. □

4.4 Bounded Domains

For the study of bounded domains $\Omega_{\mathbb{T}} = \mathbb{T} \times \Omega$, we consider again the operator $S_{\mathbf{p}} : X^{\mathbf{p}}_{\perp}(\Omega_{\mathbb{T}}) \to Y^{\mathbf{p}}_{\perp}(\Omega_{\mathbb{T}})$ given by (4.23), where $X^{\mathbf{p}}_{\perp}(\Omega_{\mathbb{T}})$ and $Y^{\mathbf{p}}_{\perp}(\Omega_{\mathbb{T}})$ are defined as in (4.22) and (4.11), respectively.

Lemma 4.9. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary of class $\mathrm{C}^{1,1}$ and $\mathbf{p} \in (1,\infty)^2$. The operator $S_{\mathbf{p}} : X^{\mathbf{p}}_{\perp}(\Omega_{\mathbb{T}}) \to Y^{\mathbf{p}}_{\perp}(\Omega_{\mathbb{T}})$ is injective and has a dense range. Moreover, there exists a constant $c = c(n,\mathbf{p},\Omega) > 0$ such that for $(u,p) \in X^{\mathbf{p}}_{\perp}(\Omega_{\mathbb{T}})$ and $(f,-g) := S_{\mathbf{p}}(u,p)$ it holds the estimate*

$$\begin{aligned}&\|u,\partial_t u,\nabla^2 u,\nabla p\|_{\mathrm{L}^{\mathbf{p}}(\Omega_{\mathbb{T}})} \\ &\qquad \leq c(\|(f,g)\|_{Y^{\mathbf{p}}_{\perp}(\Omega_{\mathbb{T}})} + \|u\|_{\mathrm{L}^{\mathbf{p}}(\Omega_{\mathbb{T}})} + \|\partial_t u\|_{[\mathrm{W}^{0,1,\mathbf{p}'}(\Omega_{\mathbb{T}})]'}).\end{aligned} \tag{4.27}$$

Proof. For $k \in \frac{2\pi}{T}\mathbb{Z}^*$, we denote by $\hat{f}(k)$ the k-th Fourier mode of a T-time-periodic function f with respect to the time variable, i.e.,

$$\hat{f}(k) = \frac{1}{T}\int_0^T f(t)e^{-ikt}\,\mathrm{d}t\,. \tag{4.28}$$

If $(u,p) \in X_\perp^{\mathbf{p}}(\Omega_\mathbb{T})$ satisfies $S_\mathbf{p}(u,p) = (0,0)$ then $(\hat{u}(k),\hat{p}(k)) \in \big(\mathrm{W}^{2,q}(\Omega) \cap \mathrm{W}_0^{1,q}(\Omega)\big)^n \times \widehat{\mathrm{W}}^{1,q}(\Omega)$ solves

$$\begin{cases} ik\hat{u}(k) - \Delta\hat{u}(k) + \nabla\hat{p}(k) &= \hat{f}(k) \qquad \text{in } \Omega\,, \\ \operatorname{div}\hat{u}(k) &= \hat{g}(k) \qquad \text{in } \Omega\,, \\ \hat{u}(k) &= 0 \qquad \text{on } \partial\Omega\,, \end{cases} \tag{4.29}$$

with $(\hat{f}(k),\hat{g}(k)) = (0,0)$. By [12, Theorem 1.2] we have $(\hat{u}(k),\hat{p}(k)) = (0,0)$. Since $k \in \frac{2\pi}{T}\mathbb{Z}^*$ was arbitrary and since $(\hat{u}(0),\hat{p}(0)) = (0,0)$ by the assumption $(u,p) \in X_\perp^{\mathbf{p}}(\Omega_\mathbb{T})$, it follows $(u,p) = (0,0)$. Therefore, $S_\mathbf{p}$ is injective.

Next we show that the range of $S_\mathbf{p}$ is dense in $Y_\perp^{\mathbf{p}}(\Omega_\mathbb{T})$. To this end introduce the notation

$$f_m(t) = F_m * f(t)\,, \tag{4.30}$$

where F_m is the m-th Fejér kernel with period T. By the vector valued Fejér theorem, see e.g. [3, Theorem 4.2.19], the trigonometric polynomials (f_m, g_m) converge to (f,g) in $Y_\perp^{\mathbf{p}}(\Omega_\mathbb{T})$ for any $(f,g) \in Y_\perp^{\mathbf{p}}(\Omega_\mathbb{T})$. On the other hand, it is known [12, Theorem 1.2] that (4.29) is uniquely solvable in $\big(\mathrm{W}^{2,q}(\Omega) \cap \mathrm{W}_0^{1,q}(\Omega)\big)^n \times \widehat{\mathrm{W}}^{1,q}(\Omega)$ for a given $(\hat{f}(k),\hat{g}(k)) \in \mathrm{L}^q(\Omega)^n \times \big(\widehat{\mathrm{W}}^{1,q}(\Omega) \cap \widehat{\mathrm{W}}^{-1,q}(\Omega)\big)$. Hence, trigonometric polynomials belong to the range of $S_\mathbf{p}$, which implies that the range of $S_\mathbf{p}$ is dense in $Y_\perp^{\mathbf{p}}(\Omega_\mathbb{T})$.

Finally let us prove (4.27). The proof follows a well-known localisation method. We choose finitely many balls $B_j \subset \mathbb{R}^n$, $j \in \{1,\dots,m\}$, where (after a possible rotation and translation that we will suppress in the following) each $j \in \{1\dots,m\}$ is of one of the two types:

- type $\mathbb{R}^n$: if $\overline{B}_j \subset \Omega$,
- type $\mathbb{R}^n_{\omega_j}$: if $\overline{B}_j \cap \Omega \neq \emptyset$.

Moreover, we choose corresponding smooth cut-off functions $\psi_j \in \mathrm{C}_0^\infty(\mathbb{R}^n)$ with $\operatorname{supp}\psi_j \subset B_j$ and $\sum_{j=1}^m \psi_j = 1$ in Ω. Note that we can choose the balls in such a way that the boundary graphs ω_j fulfill the regularity and smallness assumption in Theorem 4.8, see [12, 14] for details.

Since $(f,-g)=S_{\mathbf{p}}(u,p)$, we obtain for $j\in\{1,\dots,m\}$

$$\begin{aligned}\partial_t(\psi_j u)-\Delta(\psi_j u)+\nabla(\psi_j p)&=f_j,\\ \operatorname{div}(\psi_j u)&=g_j,\end{aligned}\tag{4.31}$$

where

$$\begin{aligned}f_j&:=\psi_j f-2(\nabla\psi_j)\nabla u-(\Delta\psi_j)u+(\nabla\psi_j)p,\\ g_j&:=\psi_j g+(\nabla\psi_j)\cdot u.\end{aligned}\tag{4.32}$$

Depending on whether $j\in\{1,\dots,m\}$ is of type $\mathbb{R}^n$ or $\mathbb{R}^n_{\omega_j}$, we interpret these equations as problems in G or G_{ω_j}, respectively.

Assume $j\in\{1,\dots,m\}$ is of type $\mathbb{R}^n_{\omega_j}$. We can apply Theorem 4.8 to problem (4.31) to obtain

$$\|(\psi_j u,\psi_j\partial_t u,\nabla^2(\psi_j u),\nabla(\psi_j p)\|_{\mathrm{L}^{\mathbf{p}}(G_{\omega_j})}\leq c\|(f_j,g_j)\|_{Y^{\mathbf{p}}_{\perp}(G_{\omega_j})}.$$

By the Poincaré inequality, the definition of f_j and g_j yields

$$\begin{aligned}\|f_j\|_{\mathrm{L}^{\mathbf{p}}(\Omega_{\mathbb{T}})}&\leq C(\psi_j)(\|f\|_{\mathrm{L}^{\mathbf{p}}(G_{\omega_j})}+\|u\|_{\mathrm{W}^{0,1,\mathbf{p}}(G_{\omega_j})}+\|p\|_{\mathrm{L}^{\mathbf{p}}(G_{\omega_j})}),\\ \|\nabla g_j\|_{\mathrm{L}^{\mathbf{p}}(\Omega_{\mathbb{T}})}&\leq C(\psi_j)(\|\nabla g\|_{\mathrm{L}^{\mathbf{p}}(G_{\omega_j})}+\|u\|_{\mathrm{W}^{0,1,\mathbf{p}}(G_{\omega_j})}).\end{aligned}$$

We still need to estimate the term $\|\partial_t g_j\|_{\widehat{\mathrm{W}}^{0,-1,\mathbf{p}}(G_{\omega_j})}$. Let $v\in\mathrm{C}^\infty_0(\overline{G_{\omega_j}})$ and define $v_0:=v-\frac{1}{|B|}\int_B v\,\mathrm{d}x$, where $B\subset\mathbb{R}^n$ is a ball containing $\operatorname{supp}\nabla\psi_j\cap\mathbb{R}^n_{\omega_j}$. As v_0 has vanishing mean in B, the Poincaré inequality yields constants $c_1,c_2>0$ such that

$$\begin{aligned}\|\nabla(\psi_j v_0)\|_{\mathrm{L}^{\mathbf{p}'}(\Omega_{\mathbb{T}})}&\leq c_1\|\nabla v\|_{\mathrm{L}^{\mathbf{p}'}(G_{\omega_j})},\\ \|(\nabla\psi_j)v_0\|_{\mathrm{W}^{0,1,\mathbf{p}'}(\Omega_{\mathbb{T}})}&\leq c_2\|\nabla v\|_{\mathrm{L}^{\mathbf{p}'}(G_{\omega_j})}.\end{aligned}$$

Note that

$$[\partial_t g_j,v]=-[\operatorname{div}(\psi_j u),\partial_t v]=-[\partial_t u,\nabla(\psi_j v_0)]+[\partial_t u,(\nabla\psi_j)v_0].$$

Therefore, we can calculate

$$\begin{aligned}\|\partial_t g_j\|_{\widehat{\mathrm{W}}^{0,-1,\mathbf{p}}(G_{\omega_j})}&\leq\sup_{0\neq v\in\mathrm{C}^\infty_0(\overline{G_{\omega_j}})}\frac{|[\partial_t g_j,v]|}{\|\nabla v\|_{\mathrm{L}^{\mathbf{p}'}(G_{\omega_j})}}\\ &\leq c_1\sup_{0\neq v\in\widehat{\mathrm{W}}^{0,1,\mathbf{p}'}(\Omega_{\mathbb{T}})}\frac{|[\partial_t u,\nabla v]|}{\|\nabla v\|_{\mathrm{L}^{\mathbf{p}'}(\Omega_{\mathbb{T}})}}+c_2\|\partial_t u\|_{[\mathrm{W}^{0,1,\mathbf{p}'}(\Omega)]'}\\ &=c_1\|\partial_t g\|_{\widehat{\mathrm{W}}^{0,-1,\mathbf{p}}(\Omega_{\mathbb{T}})}+c_2\|\partial_t u\|_{[\mathrm{W}^{0,1,\mathbf{p}'}(\Omega)]'}.\end{aligned}\tag{4.33}$$

Finally, if $j \in \{1, \dots, m\}$ is of type G, the same calculations can be performed using the periodic whole space result in Theorem 4.6 instead of the bent periodic half space result in Theorem 4.8. Summing up the finitely many inequalities obtained for $j \in \{1, \dots, m\}$ yields estimate (4.27) with the additional terms $\|p\|_{\mathrm{L}^{\mathbf{p}}(\Omega_{\mathbb{T}})}$ and $\|\nabla u\|_{\mathrm{L}^{\mathbf{p}}(\Omega_{\mathbb{T}})}$ on the right-hand side. However, since the pressure is defined only up to a constant, we can assume that the mean of p vanishes. In virtue of the Poincaré inequality and the existence of the Helmholtz projection we thus have

$$\|p\|_{\mathrm{L}^{\mathbf{p}}(\Omega_{\mathbb{T}})} \leq c\|\nabla p\|_{\mathrm{L}^{\mathbf{p}}(\Omega_{\mathbb{T}})} \leq c\|(f,g)\|_{Y^{\mathbf{p}}_{\perp}(\Omega_{\mathbb{T}})}. \tag{4.34}$$

Moreover, the term $\|\nabla u\|_{\mathrm{L}^{\mathbf{p}}(\Omega_{\mathbb{T}})}$ can be absorbed by a standard interpolation argument. The proof is complete. □

Next we show that the last three terms in (4.27) can be omitted for the case $\mathbf{p} \in [2,\infty)^2$. This will show that $S_{\mathbf{p}} : X^{\mathbf{p}}_{\perp}(\Omega_{\mathbb{T}}) \to Y^{\mathbf{p}}_{\perp}(\Omega_{\mathbb{T}})$ yields an isomorphism at least when $\mathbf{p} \in [2,\infty)^2$. The general case $\mathbf{p} \in (1,\infty)^2$ will be proved later in Theorem 4.11 by a duality argument.

Lemma 4.10. *Let $\mathbf{p} \in [2,\infty)^2$. Then, under the assumptions of Lemma 4.9 the following estimate holds.*

$$\|u, \partial_t u, \nabla^2 u, \nabla p\|_{\mathrm{L}^{\mathbf{p}}(\Omega_{\mathbb{T}})} \leq c\|(f,g)\|_{Y^{\mathbf{p}}_{\perp}(\Omega_{\mathbb{T}})}\,. \tag{4.35}$$

Proof. Firstly we consider the case $\mathbf{p} = (2,2)$. By [12, Theorem 1.2] the solution $(\hat{u}(k), \hat{p}(k)) \in \big(\mathrm{W}^{2,2}(\Omega) \cap \mathrm{W}^{1,2}_0(\Omega)\big)^n \times \widehat{\mathrm{W}}^{1,2}(\Omega)$ to (4.29) satisfies the estimate

$$\begin{aligned}
&|k|\|\hat{u}(k)\|_{\mathrm{L}^2(\Omega)} + \|\nabla^2\hat{u}(k)\|_{\mathrm{L}^2(\Omega)} + \|\nabla\hat{p}(k)\|_{\mathrm{L}^2(\Omega)} \\
&\quad \leq C\big(\|\hat{f}(k)\|_{\mathrm{L}^2(\Omega)} + \|\nabla\hat{g}(k)\|_{\mathrm{L}^2(\Omega)} + |k|\|\hat{g}(k)\|_{\widehat{\mathrm{W}}^{-1,2}(\Omega)}\big)\,.
\end{aligned}$$

By the Plancherel theorem we obtain (4.35). Next we consider the case $\mathbf{p} \in [2,\infty)^2$. First we note that by the result for $\mathbf{p} = (2,2)$ and the trivial embedding, it holds

$$\|\partial_t u\|_{[\mathrm{W}^{0,1,\mathbf{p}'}(\Omega_{\mathbb{T}})]'} \leq c\|\partial_t u\|_{\mathrm{L}^2(\Omega_{\mathbb{T}})} \leq c\|(f,\nabla g)\|_{Y^2_{\perp}(\Omega_{\mathbb{T}})} \leq c\|(f,\nabla g)\|_{Y^{\mathbf{p}}_{\perp}(\Omega_{\mathbb{T}})},$$

and thus we have from (4.27)

$$\|u, \partial_t u, \nabla^2 u, \nabla p\|_{\mathrm{L}^{\mathbf{p}}(\Omega_{\mathbb{T}})} \leq c\big(\|(f,\nabla g)\|_{Y^{\mathbf{p}}_{\perp}(\Omega_{\mathbb{T}})} + \|u\|_{\mathrm{L}^{\mathbf{p}}(\Omega_{\mathbb{T}})}\big)\,.$$

Now we will show (4.35) by a contradiction argument. If (4.35) does not hold then we would find sequences $(u_\ell, p_\ell) \subset X_\perp^{\mathbf{p}}(\Omega_\mathbb{T})$, $\ell \in \mathbb{N}$, such that

$$\begin{aligned} \|u_\ell, \partial_t u_\ell, \nabla^2 u_\ell, \nabla p_\ell\|_{\mathrm{L}^{\mathbf{p}}(\Omega_\mathbb{T})} &= 1, \qquad \text{for all } \ell \in \mathbb{N}, \\ \|(f_\ell, g_\ell)\|_{Y_\perp^{\mathbf{p}}(\Omega_\mathbb{T})} &\to 0 \qquad \text{as } \ell \to \infty, \end{aligned}$$

where $(f_\ell, -g_\ell) := S_{\mathbf{p}}(u_\ell, p_\ell)$. Suppressing the notion of subsequences, we thus have the weak convergence

$$u_\ell \rightharpoonup u \ \text{ in } \mathrm{W}_\perp^{1,2,\mathbf{p}}(\Omega_\mathbb{T})^n \cap \mathrm{W}_0^{0,1,\mathbf{p}}(\Omega_\mathbb{T}).$$

From (4.34) we immediately obtain $\|\nabla p\|_{\mathrm{L}^{\mathbf{p}}(\Omega_\mathbb{T})} \to 0$. Using the convergence of f_ℓ and g_ℓ we deduce that $(u, 0) \in X_\perp^{\mathbf{p}}(\Omega_\mathbb{T})$ solves

$$\begin{cases} \partial_t u - \Delta u &= 0 \qquad \text{in } \Omega_\mathbb{T}\,, \\ \nabla \operatorname{div} u &= 0 \qquad \text{in } \Omega_\mathbb{T}\,, \\ u &= 0 \qquad \text{on } \partial\Omega\,. \end{cases}$$

By the boundary condition we conclude that even $\operatorname{div} u = 0$ and consequently $S_{\mathbf{p}}(u, 0) = (0, 0)$. By Lemma 4.9, $S_{\mathbf{p}}$ is injective and consequently $u = 0$. Since we are on a bounded domain, the embedding $\mathrm{W}^{1,2,\mathbf{p}}(\Omega_\mathbb{T}) \hookrightarrow \mathrm{W}^{1,1,\mathbf{p}}(\Omega_\mathbb{T}) \hookrightarrow\hookrightarrow \mathrm{L}^{\mathbf{p}}(\Omega_\mathbb{T})$ is compact, which yields the contradiction

$$\begin{aligned} 1 &= \lim_{\ell \to \infty} \|u_\ell, \partial_t u_\ell, \nabla^2 u_\ell, \nabla p_\ell\|_{\mathrm{L}^{\mathbf{p}}(\Omega_\mathbb{T})} \\ &\le \lim_{\ell \to \infty} c(\|(f_\ell, g_\ell)\|_{Y_\perp^{\mathbf{p}}(\Omega_\mathbb{T})} + \|u_\ell\|_{\mathrm{L}^{\mathbf{p}}(\Omega_\mathbb{T})}) = 0. \end{aligned}$$

Consequently, estimate (4.35) has to hold. □

We are now in the position to state the main result for the case of bounded domains.

Theorem 4.11. *Let $\mathbf{p} \in (1, \infty)^2$ and $(\mathcal{P}_\perp f, \mathcal{P}_\perp g) \in Y_\perp^{\mathbf{p}}(\Omega_\mathbb{T})$. Then there exists a unique solution $(u, p) \in \mathrm{W}_\perp^{1,2,\mathbf{p}}(\Omega_\mathbb{T})^n \times \widehat{\mathrm{W}}_\perp^{0,1,\mathbf{p}}(\Omega_\mathbb{T})$ to (1.5) on $\Omega_\mathbb{T}$, and the following estimate holds.*

$$\|u, \partial_t u, \nabla^2 u, \nabla p\|_{\mathrm{L}^{\mathbf{p}}(\Omega_\mathbb{T})} \le c\|(\mathcal{P}_\perp f, \mathcal{P}_\perp g)\|_{Y_\perp^{\mathbf{p}}(\Omega_\mathbb{T})}. \tag{4.36}$$

If additionally $(\mathcal{P}_\perp f, \mathcal{P}_\perp g) \in Y_\perp^{\mathbf{s}}(\Omega_\mathbb{T})$ for some $\mathbf{s} \in (1, \infty)^2$, then $(u, p) \in W_\perp^{1,2,\mathbf{s}}(\Omega_\mathbb{T})^n \times \widehat{\mathrm{W}}_\perp^{0,1,\mathbf{s}}(\Omega_\mathbb{T})$.

Proof. Firstly we consider the case $\mathbf{p} \in [2,\infty)^2$. By Lemma 4.9, the operator $S_{\mathbf{p}} : X_{\perp}^{\mathbf{p}}(\Omega_{\mathbb{T}}) \to Y_{\perp}^{\mathbf{p}}(\Omega_{\mathbb{T}})$ is injective and possesses a dense range in $Y_{\perp}^{\mathbf{p}}(\Omega_{\mathbb{T}})$. Lemma 4.10 shows that the range is closed. Hence, $S_{\mathbf{p}}$ is even surjective and therefore an isomorphism. This completes the proof for $\mathbf{p} \in [2,\infty)^2$.

Next we consider the case $\mathbf{p} \in (1,2]^2$. It suffices to show the a priori estimate of the form (4.36), for the assertion is then shown by the same argument as in the case $\mathbf{p} \in [2,\infty)^2$. Let $(u,p) \in X_{\perp}^{\mathbf{p}}(\Omega_{\mathbb{T}})$ and set $(f,-g) = S_{\mathbf{p}}(u,p)$. By taking the truncation operator (4.30) if necessary, we may assume that all data are smooth in the time variable. We use a duality argument. For any $\varphi \in \mathrm{L}_{\perp}^{\mathbf{p}'}(\Omega_{\mathbb{T}})$ set $\tilde{\varphi}(t) = \varphi(-t)$ and set $(v,r) = S_{\mathbf{p}'}^{-1}(-\tilde{\varphi},0)$, which is well-defined since $\mathbf{p}' \in [2,\infty)^2$. Then $\tilde{v}(t) = v(-t)$ and $\tilde{r}(t) = r(-t)$ satisfy

$$\partial_t \tilde{v} + \Delta \tilde{v} - \nabla \tilde{r} = \varphi\,, \ \operatorname{div} \tilde{v} = 0 \text{ in } \Omega_{\mathbb{T}}\,,$$

and $\tilde{v}(t) = 0$ on $\partial\Omega$. By the integration by parts we have

$$\begin{aligned}
\langle \partial_t u, \varphi \rangle_{\mathrm{L}^2(\Omega_{\mathbb{T}})} &= \langle \partial_t u, \partial_t \tilde{v} + \Delta \tilde{v} - \nabla \tilde{r} \rangle_{\mathrm{L}^2(\Omega_{\mathbb{T}})} \\
&= \langle \partial_t u - \Delta u, \partial_t \tilde{v} \rangle_{\mathrm{L}^2(\Omega_{\mathbb{T}})} + \langle \partial_t g, \tilde{r} \rangle_{\mathrm{L}^2(\Omega_{\mathbb{T}})} \\
&= \langle f - \nabla p, \partial_t \tilde{v} \rangle_{\mathrm{L}^2(\Omega_{\mathbb{T}})} + \langle \partial_t g, \tilde{r} \rangle_{\mathrm{L}^2(\Omega_{\mathbb{T}})} \\
&= \langle f, \partial_t \tilde{v} \rangle_{\mathrm{L}^2(\Omega_{\mathbb{T}})} + \langle \partial_t g, \tilde{r} \rangle_{\mathrm{L}^2(\Omega_{\mathbb{T}})}
\end{aligned}$$

The result for the case $\mathbf{p}' \in (2,\infty)^2$ yields

$$\begin{aligned}
|\langle f, \partial_t \tilde{v} \rangle_{\mathrm{L}^2(\Omega_{\mathbb{T}})}| &\le \|f\|_{\mathrm{L}^{\mathbf{p}}(\Omega_{\mathbb{T}})} \|\partial_t \tilde{v}\|_{\mathrm{L}^{\mathbf{p}'}(\Omega_{\mathbb{T}})} \\
&\le c \|f\|_{\mathrm{L}^{\mathbf{p}}(\Omega_{\mathbb{T}})} \|\varphi\|_{\mathrm{L}^{\mathbf{p}'}(\Omega_{\mathbb{T}})}\,,
\end{aligned}$$

and

$$\begin{aligned}
|\langle \partial_t g, \tilde{r} \rangle_{\mathrm{L}^2(\Omega_{\mathbb{T}})}| &\le \|\partial_t g\|_{\widehat{\mathrm{W}}^{0,-1,\mathbf{p}}(\Omega_{\mathbb{T}})} \|\nabla \tilde{r}\|_{\mathrm{L}^{\mathbf{p}'}(\Omega_{\mathbb{T}})} \\
&\le c \|\partial_t g\|_{\widehat{\mathrm{W}}^{0,-1,\mathbf{p}}(\Omega_{\mathbb{T}})} \|\varphi\|_{\mathrm{L}^{\mathbf{p}'}(\Omega_{\mathbb{T}})}\,.
\end{aligned}$$

Hence we have

$$\|\partial_t u\|_{\mathrm{L}^{\mathbf{p}}(\Omega_{\mathbb{T}})} \le c(\|f\|_{\mathrm{L}^{\mathbf{p}}(\Omega_{\mathbb{T}})} + \|\partial_t g\|_{\widehat{\mathrm{W}}^{0,-1,\mathbf{p}}(\Omega_{\mathbb{T}})})\,. \tag{4.37}$$

Then we rewrite the equations $S_{\mathbf{p}}(u,p) = (f,-g)$ as

$$\begin{cases}
-\Delta u(t) + \nabla p(t) &= f(t) - \partial_t u(t) & \text{in } \Omega\,, \\
\operatorname{div} u(t) &= g(t) & \text{in } \Omega\,, \\
u(t) &= 0 & \text{on } \partial\Omega
\end{cases}$$

for each $t \in \mathbb{T}$. By [12, Theorem 1.2] again, we have

$$\|u(t), \nabla^2 u(t), \nabla p(t)\|_{\mathrm{L}^q(\Omega)} \leq c(\|f(t)\|_{\mathrm{L}^q(\Omega)} + \|\partial_t u(t)\|_{\mathrm{L}^q(\Omega)} + \|\nabla g(t)\|_{\mathrm{L}^q(\Omega)})$$

for each t, which implies (4.36) by taking the L^p norm in the time variable and taking into account estimate (4.37).

The general result for $\mathbf{p} \in (1,\infty)^2$ follows by interpolation of $\mathrm{L}^{\mathbf{s}}$ and $\mathrm{L}^{\mathbf{r}}$ with $\mathbf{s} = (s, 2)$ and $\mathbf{r} = (2, r)$ for suitable $s, r \in (1,\infty)$.

Since $\Omega_{\mathbb{T}}$ is bounded, the regularity assertion follows from the existence result for $\mathbf{p} = (p, q)$ and $\mathbf{s} = (s, t)$ and the uniqueness result for the exponent $\mathbf{r} = (r, r)$ with $r := \min\{p, q, s, t\}$. The proof is complete. □

5 The initial-value problem

In this section, we demonstrate the impact of the time-periodic problem on the initial value problem by giving a short and direct argument which shows that the Stokes operator A_q admits maximal L^p regularity on $\mathrm{L}^q_\sigma(\Omega)$ for $\Omega = \mathbb{R}^n$, $\Omega = \mathbb{R}^n_+$ and sufficiently smooth bounded domains $\Omega \subset \mathbb{R}^n$. Recall that a generator $-A$ of a C_0 semigroup on a Banach space X is said to admit *maximal* L^p *regularity* on $(0, T)$, if for every $f \in \mathrm{L}^p(0, T; X)$ the unique solution to the abstract Cauchy problem $\partial_t u + Au = f$, $u(0) = 0$ satisfies $u \in \mathrm{L}^p(0, T; D(A)) \cap \mathrm{W}^{1,p}(0, T; X) := \mathbb{E}_T(A)$. Weis' theorem [29] states that maximal L^p regularity is equivalent to $\mathcal{R}$-boundedness of the resolvent family $\{t(it + A)^{-1} \mid t \in \mathbb{R}\}$. This characterization has been used extensively to show maximal regularity of various differential operators, in particular for the Stokes operator $A_q : D(A_q) \subset \mathrm{L}^q_\sigma(\Omega) \to \mathrm{L}^q_\sigma(\Omega)$, where $A_q := -\mathbb{P}\Delta$ and $D(A_q) := \mathrm{W}^{2,q}(\Omega)^n \cap \mathrm{W}^{1,q}_0(\Omega)^n \cap \mathrm{L}^q_\sigma(\Omega)$.

We shall show that the notion of $\mathcal{R}$-boundedness can be avoided completely in the case of the Stokes operator, by extending the abstract result of Arendt and Bu [2]. For convenience let us introduce the notion of the abstract maximal L^pregularity for the time-periodic problem as follows. We say that A admits *time-periodic maximal* L^p *regularity* on $(0, T)$, if for every $f \in \mathrm{L}^p(0, T; X)$ with $\int_0^T f \,\mathrm{d}t = 0$ the abstract time-periodic problem

$$\partial_t u + Au = f \qquad \text{in } (0, T)\,,\ u(0) = u(T) \tag{5.1}$$

is uniquely solvable in $\mathrm{L}^p(0, T; D(A)/\mathrm{Ker}\, A) \cap \mathrm{W}^{1,p}(0, T; X)$. We note that, in the above definition, the condition $\int_0^T f \,\mathrm{d}t = 0$ is added in order to cover the case when A is not invertible.

Recall the notation $i\frac{2\pi}{T}\mathbb{Z}^* := i\frac{2\pi}{T}\mathbb{Z} \setminus \{0\}$. Then we have the following abstract result on the equivalence between maximal L^p regularity and time-periodic maximal L^p regularity.

Theorem 5.1. *Let $-A$ be a generator of C_0-semigroup on a Banach space X. Then the following assertions are equivalent.*
(i) *The operator A admits maximal L^p regularity on $(0, T)$ for all $T > 0$.*
(ii) *For any $T > 0$ satisfying $i\frac{2\pi}{T}\mathbb{Z}^* \cap \sigma(A) = \emptyset$ the operator A admits time-periodic maximal L^p regularity on $(0, T)$.*
(iii) *The operator A admits time-periodic maximal L^p regularity on $(0, T_1)$ for some $T_1 > 0$.*

Remark 5.2. The above equivalence is proved in [2, Theorem 5.1] in the case of an invertible generator $-A$. Theorem 5.1 of the present paper seems to be new for generators which are not necessarily invertible. This case is important for applications since elliptic operators such as the Laplace operator and the Stokes operators are not invertible in general if the domain is unbounded.

Proof of Theorem 5.1. We may assume that $-A$ generates a C_0-analytic semigroup on X. In fact, it is well-known that maximal L^p regularity implies the analyticity of the semigroup, see e.g. [8]. On the other hand, if A admits time-periodic maximal L^p regularity, then $\{ik(ik + A)^{-1} \mid k \in \frac{2\pi}{T}\mathbb{Z}^*\}$ is easily seen to be a Fourier multiplier and is hence bounded, yielding again the analyticity of the semigroup. The assertion (ii) $\to$ (iii) is easy, for $-A$ is the generator of C_0-analytic semigroup, and thus, $i\frac{2\pi}{T_1}\mathbb{Z}^* \cap \sigma(A) = \emptyset$ holds for sufficiently small $T_1 > 0$. The result (iii) $\to$ (i) follows from [21, Remark 3.2], where it is applied to the Dirichlet Laplacian. For the convenience of the reader, we give a proof here. Recall that for analytic semi-groups e^{-tA} on a Banach space X, we have the following characterization of the trace space at $t = 0$:

$$(X, D(A))_{1-1/p,p} = \big\{x \in X \mid Ae^{-tA}x \in \mathrm{L}^p(0, T; X)\big\} = \big\{u(0) \mid u \in \mathbb{E}_T(A)\big\},$$

see [24, Corollary 1.14, Proposition 6.2]. Let now $f \in \mathrm{L}^p(0, T_1; X)$ and write $f = \mathcal{P}f + \mathcal{P}_\perp f \in X \oplus \mathrm{L}^p_\perp(0, T_1; X)$. By assumption, there is a solution $v \in \mathbb{E}_{T_1}(A)$ on $(0, T_1)$ to $\partial_t v + Av = \mathcal{P}_\perp f$. The characterization of the trace space shows $e^{-tA}v(0) \in \mathbb{E}_{T_1}(A)$. Define $w \in \mathrm{L}^p(0, T_1; X)$ via $w(t) := \int_0^t e^{-(t-s)A}\mathcal{P}f \, \mathrm{d}s$. Since $\mathcal{P}f \in X$ does not depend on time, it follows that $w \in \mathbb{E}_{T_1}(A)$. Thus,

$$u := v + w - e^{-tA}v(0) \in \mathbb{E}_{T_1}(A)$$

is a solution to $\partial_t u + Au = f$, $u(0) = 0$, on $(0, T_1)$. Therefore, A admits maximal L^p regularity on $(0, T_1)$, and thus, by [8], on $(0, T)$ for any $T > 0$.

Finally we prove (i) → (ii). Let $T > 0$ be such that $i\frac{2\pi}{T}\mathbb{Z}^* \cap \sigma(A) = \emptyset$. We note that there is $\lambda > 0$ such that $A + \lambda + i\mu$ is invertible on X for any $\mu \in \mathbb{R}$. In particular $1 \in \rho(e^{-T(A+\lambda)})$, see e.g. [10, Corollary IV.3.12]. Therefore [2, Theorem 5.1] implies that $A+\lambda$ admits time-periodic maximal L^p regularity on $(0, T)$. Let $f \in \mathrm{L}^p(0, T; X)$ with $\int_0^T f \,\mathrm{d}t = 0$. By the choice of T, for any $k \in \frac{2\pi}{T}\mathbb{Z}^*$, there exist unique solutions $\hat{u}(k), \hat{v}(k) \in D(A)$ to the problems

$$\big(ik + A\big)\,\hat{u}(k) = \hat{f}(k)\,,\ \big(ik + A + \lambda\big)\,\hat{v}(k) = \hat{f}(k)\,,$$

respectively. Here, the Fourier modes $\hat{f}(k)$ are defined as in (4.28), and we will set $\hat{u}(0) := \hat{v}(0) := 0$. Then the Fourier inverses u and v are well-defined at least as X-valued tempered distributions. By the time-periodic maximal L^p regularity of $A + \lambda$ we immediately obtain $v \in \mathbb{E}_T(A)$. Hence, $v_m := F_m * v \to v$ in $\mathbb{E}_T(A)$ by [3, Theorem 4.2.19], where F_m is the m-th Fejér kernel with period T. Using this fact, we shall show that also $u_m := F_m * u$ converges in $\mathbb{E}_T(A)$. For this purpose we observe that

$$\begin{aligned} u_m(t) &= v_m(t) + F_m * (u - v)(t) \\ &= v_m + \frac{1}{m+1}\sum_{j=1}^{m}\sum_{k\in M_j}\big((ik + A)^{-1} - (ik + A + \lambda)^{-1}\big)\hat{f}(k)e^{ikt} \\ &= v_m(t) - \frac{\lambda}{m+1}\sum_{j=1}^{m}\sum_{k\in M_j}(ik + A + \lambda)^{-1}(ik + A)^{-1}\hat{f}(k)e^{ikt}\,, \end{aligned}$$

where $M_j := \{k \in \frac{2\pi}{T}\mathbb{Z}^* \mid |k| \le \frac{2\pi j}{T}\}$. Since $-A$ generates an analytic semigroup, we have the estimate

$$\begin{aligned} \|(ik + A + \lambda)^{-1}(ik + A)^{-1}\hat{f}(k)\|_X &\le \frac{C}{|\lambda + ik| \cdot |k|}\|\hat{f}(k)\|_X \\ &\le \frac{C}{k^2}\|f\|_{L^p(0,T;X)}\,,\ k \in \frac{2\pi}{T}\mathbb{Z}^*\,. \end{aligned}$$

Therefore, the Fourier series of $u-v$ converges in $BC([0,T];X)$, and its limit is $u-v$. Consequently, the same is true for its arithmetic mean $F_m*(u-v)$. In conclusion, $u_m = v_m + F_m * (u - v) \to u$ in $BC([0, T]; X)$ as $m \to \infty$ Since the problem

$$\partial_t w + (A + \lambda)w = \lambda u \text{ in } (0, T)\,,\ w(0) = w(T)$$

is uniquely solvable in $\mathbb{E}_T(A)$, and since each Fourier mode of $w-(u-v)$ vanishes, we finally obtain $u := v + w \in \mathbb{E}_T(A)$. Thus, the function u is a solution to (5.1) as desired. If $u' \in \mathbb{E}_T(A)$ is another solution to (5.1), then $\hat{v}(k) := \hat{u}(k) - \hat{u}'(k)$ solves $(ik + A)\hat{v}(k) = 0$ for all $k \in \frac{2\pi}{T}\mathbb{Z}$. Hence $\hat{v}(0) \in \operatorname{Ker} A$ and $\hat{v}(k) = 0$ for $k \neq 0$ due to $i\frac{2\pi}{T}\mathbb{Z}^* \cap \sigma(A) = \emptyset$. This shows $u = u'$ in $\mathrm{L}^p(0,T; D(A)/\operatorname{Ker} A)$. The proof is complete. □

In view of the preceeding sections, Theorem 5.1 implies the maximal L^p regularity of the Stokes operator on the whole space, the half space and on sufficiently smooth bounded domains. We emphasize again that while the result itself is well-known, it is the simplicity of its proof that is striking.

Corollary 5.3. *Let $p, q \in (1,\infty)$, $T > 0$ and assume that $\Omega = \mathbb{R}^n$, $\Omega = \mathbb{R}^n_+$ or $\Omega \subset \mathbb{R}^n$ is a bounded domain with a $\mathrm{C}^{1,1}$-smooth boundary. Then the Stokes operator A_q on $\mathrm{L}^q_\sigma(\Omega)$ admits maximal L^p regularity on $(0,T)$.*

Proof. It is well-known that $-A_q$ generates an analytic semi-group. By Theorem 4.6 and Theorem 4.11, respectively, A_q admits time-periodic maximal L^p regularity. Hence, Theorem 5.1 applies. □

6 Nematic Liquid Crystal Flow

In this section, we apply the linear theory to a time-periodic nonlinear model. Given an exterior force $f = (f_u, f_d) \in \mathrm{L}^{\mathbf{p}}(\Omega_{\mathbb{T}})^{2n}$, consider the time-periodic problem

$$\begin{cases} \partial_t u - \nu\Delta u + u\cdot\nabla u + \nabla p = f_u - \kappa \operatorname{div}([\nabla d]^T[\nabla d])\,, & \text{in } \Omega_{\mathbb{T}}\,, \\ \partial_t d - \sigma\Delta d + u\cdot\nabla d = f_d + \sigma|\nabla d|^2(d + d_0)\,, & \text{in } \Omega_{\mathbb{T}}\,, \\ \operatorname{div} u = 0\,, & \text{in } \Omega_{\mathbb{T}}\,, \\ (u,d) = 0\,, & \text{on } \partial\Omega_{\mathbb{T}}\,, \end{cases} \tag{LCD}$$

where $\nu, \sigma, \kappa > 0$, $d_0 \in \mathbb{R}^n$ with $|d_0| = 1$. The domain Ω is assumed to be the whole space $\mathbb{R}^n$, the half space $\mathbb{R}^n_+$ or a bounded domain of class $C^{1,1}$. System (LCD) is a modified version of the so-called *simplified Ericksen-Leslie model* describing a nematic liquid crystal flow. Here, the function u denotes the velocity of the flow, p the pressure and d the deviation of the macroscopic molecular orientation d_0. The constants ν, σ and κ represent viscosity, the competition between kinetic energy and potential energy and the microscopic elastic relaxation time for the molecular orientation field, respectively. Note that one usually includes the condition that the molecular orientation is a vector field of constant norm 1. Since we will allow for general, small time-periodic forces $f \in \mathrm{L}^{\mathbf{p}}(\Omega_{\mathbb{T}})^{2n}$, one cannot expect such a condition to be fulfilled in our case. However, due to the smallness of the forcing terms, it is always guaranteed that the solution stays in a neighbourhood of d_0. The model itself bases on the continuum theory of liquid crystals developed by Ericksen and Leslie, see for example the survey article [11], and has been considered for the first time by [22] and [23]. In [19], the simplified Ericksen-Leslie model was treated on bounded domains using quasilinear theory based upon the maximal L^p-regularity of the Stokes operator. In a similar manner, with the linear theory developed in the previous sections, we shall prove the following result in the time-periodic case.

Theorem 6.1. *Let $n \geq 2$ and suppose that $\Omega \subset \mathbb{R}^n$ is a bounded domain of class $\mathrm{C}^{1,1}$. Suppose $T > 0$ and let $\mathbf{p} = (p, q) \in (1, \infty)^2$ satisfy $2/p + n/q < 1$. Then there is $\varepsilon > 0$ such that for all $f = (f_u, f_d) \in \mathrm{L}^{\mathbf{p}}(\Omega_{\mathbb{T}})^{2n}$ with $\|f\|_{\mathrm{L}^{\mathbf{p}}(\Omega_{\mathbb{T}})} < \varepsilon$ the problem (LCD) admits a solution*

$$(u, d, p) \in \mathrm{W}^{1,2,\mathbf{p}}(\Omega_{\mathbb{T}})^{2n} \times \mathrm{W}^{0,1,\mathbf{p}}(\Omega_{\mathbb{T}}).$$

Moreover, the molecular orientation $\frac{d+d_0}{|d+d_0|}$ is well-defined on $\Omega_{\mathbb{T}}$.

A particular problem arises in the case of the unbounded domains, where we will be restricted to dimensions $n \geq 4$ due to the regularity loss in the steady-state part. In order to deal with unbounded domains at all, we use again the projection $\mathcal{P}$ to split $\mathrm{L}^{\mathbf{p}}(\Omega_{\mathbb{T}}) = \mathrm{L}^q(\Omega) \oplus \mathrm{L}^{\mathbf{p}}_{\perp}(\Omega_{\mathbb{T}})$. Then, we introduce for $q \in (1, \infty)$ and $r \in (1, n/2)$ the domains

$$\begin{aligned}
D(A_{r,q}) &:= \widehat{\mathrm{W}}^{2,q}(\Omega)^n \cap \widehat{\mathrm{W}}^{2,r}(\Omega)^n \cap \widehat{\mathrm{W}}_0^{1,\frac{nr}{n-r}}(\Omega)^n \cap \mathrm{L}_\sigma^{\frac{nr}{n-2r}}(\Omega)\,,\\
D(\Delta_{r,q}) &:= \widehat{\mathrm{W}}^{2,q}(\Omega)^n \cap \widehat{\mathrm{W}}^{2,r}(\Omega)^n \cap \widehat{\mathrm{W}}_0^{1,\frac{nr}{n-r}}(\Omega)^n \cap \mathrm{L}^{\frac{nr}{n-2r}}(\Omega)^n\,,\\
D^{r,q}(\Omega) &:= D(A_{r,q}) \times D(\Delta_{r,q})\,.
\end{aligned}$$

Moreover, for $\mathbf{s}, \mathbf{p} \in (1, \infty)^2$ we introduce the intersection spaces

$$\begin{aligned}
\mathrm{L}^{\mathbf{s},\mathbf{p}}(\Omega_{\mathbb{T}}) &:= \mathrm{L}^{\mathbf{s}}(\Omega_{\mathbb{T}}) \cap \mathrm{L}^{\mathbf{p}}(\Omega_{\mathbb{T}}),\\
\mathrm{W}^{1,2,\mathbf{s},\mathbf{p}}(\Omega_{\mathbb{T}}) &:= \mathrm{W}^{1,2,\mathbf{s}}(\Omega_{\mathbb{T}}) \cap \mathrm{W}^{1,2,\mathbf{p}}(\Omega_{\mathbb{T}}).
\end{aligned}$$

Theorem 6.2. *Let $T > 0$ and let $\mathbf{s} = (s, r)$, $\mathbf{p} = (p, q)$ satisfy $s, r \in (1, \infty)$, $p, q \in (1, \infty)$. Assume furthermore $r \leq n/3$ and $2/p + n/q < 1$. Then there is $\varepsilon > 0$ such that for all $f = (f_u, f_d) \in \mathrm{L}^{\mathbf{s},\mathbf{p}}(\Omega_{\mathbb{T}})^{2n}$ with $\|f\|_{\mathrm{L}^{\mathbf{s},\mathbf{p}}(\Omega_{\mathbb{T}})} < \varepsilon$ the problem (LCD) admits a solution $(u, d, p) \in \mathrm{L}^1_{\mathrm{loc}}(\Omega_{\mathbb{T}})$ with $\nabla p \in \mathrm{L}^{\mathbf{s},\mathbf{p}}(\Omega_{\mathbb{T}})^n$ and*

$$\begin{pmatrix} u \\ d \end{pmatrix} = \mathcal{P} \begin{pmatrix} u \\ d \end{pmatrix} + \mathcal{P}_\perp \begin{pmatrix} u \\ d \end{pmatrix} \in \left[D^{r,q}(\Omega) \oplus \mathrm{W}^{1,2,\mathbf{s},\mathbf{p}}_\perp(\Omega_{\mathbb{T}})^{2n} \right].$$

Moreover, the molecular orientation $\frac{d+d_0}{|d+d_0|}$ is well-defined on $\Omega_\mathbb{T}$.

We need some preparations for the proof. We will use the notation

$$\mathbb{E} := D^{r,q}(\Omega) \oplus \left\{ \begin{array}{c} \left[\mathrm{W}_\perp^{1,2,\mathbf{s},\mathbf{p}}(\Omega_\mathbb{T})^n \cap \mathrm{W}_0^{0,1,\mathbf{p}}(\Omega_\mathbb{T})^n \cap \mathrm{L}_\sigma^{\mathbf{p}}(\Omega_\mathbb{T})\right] \\ \times \left[\mathrm{W}_\perp^{1,2,\mathbf{s},\mathbf{p}}(\Omega_\mathbb{T})^n \cap \mathrm{W}_0^{0,1,\mathbf{p}}(\Omega_\mathbb{T})^n\right] \end{array} \right\}.$$

Moreover, let us write $z := (u, d) \in \mathbb{E}$ and

$$L := \begin{pmatrix} \nu A_{r,q} & 0 \\ 0 & -\sigma\Delta_{r,q} \end{pmatrix}.$$

By Proposition 3.1 and Theorem 4.6 (or Proposition 3.4 and Theorem 4.11 in the case of a bounded domain) there is a unique solution $\bar{z} = (\bar{u}, \bar{d}) \in \mathbb{E}$ to the problem

$$\partial_t \bar{z} + L\bar{z} = \begin{pmatrix} \mathbb{P}f_u \\ f_d \end{pmatrix} \tag{6.1}$$

with $\|\bar{z}\|_\mathbb{E} \leq C_M \|f\|_{\mathrm{L}^{\mathbf{s},\mathbf{p}}(\Omega_\mathbb{T})} < C_M\varepsilon$. Consider now $F : \mathbb{E} \to \mathrm{L}^{\mathbf{s},\mathbf{p}}(\Omega_\mathbb{T})^{2n}$ defined via

$$F(z) := \begin{pmatrix} \mathbb{P}(u \cdot \nabla u + \kappa \operatorname{div}([\nabla d]^T[\nabla d])) \\ u \cdot \nabla d - \sigma|\nabla d|^2(d + d_0) \end{pmatrix}.$$

We shall show that indeed $F(\mathbb{E}) \subset \mathrm{L}^{\mathbf{s},\mathbf{p}}(\Omega_\mathbb{T})^{2n}$.

Lemma 6.3. *Under the assumptions of Theorem 6.2 let $z = \mathcal{P}z + \mathcal{P}_\perp z \in \mathbb{E}$. Then for all $\alpha \in [\frac{nr}{n-2r}, \infty]$ and $\beta \in [\frac{nr}{n-r}, \infty]$ we have*

$$\|\mathcal{P}z\|_{\mathrm{L}^\alpha(\Omega)} + \|\nabla\mathcal{P}z\|_{\mathrm{L}^\beta(\Omega)} \leq c\|\mathcal{P}z\|_{D^{r,q}(\Omega)}, \tag{6.2}$$

$$\|\mathcal{P}_\perp z\|_{\mathrm{W}^{0,1,\infty}(\Omega_\mathbb{T})} \leq c\|\mathcal{P}_\perp z\|_{\mathrm{W}^{1,2,\mathbf{p}}(\Omega_\mathbb{T})}. \tag{6.3}$$

Proof. Recall the estimate for $\Omega = \mathbb{R}^n$ and $\Omega = \mathbb{R}^n_+$

$$\|v\|_{\mathrm{L}^\infty(\Omega)} \leq c(\|\nabla v\|_{\mathrm{L}^q(\Omega)} + \|v\|_{\mathrm{L}^\gamma(\Omega)}), \tag{6.4}$$

which holds true if $\gamma \in (1, \infty)$ and $q \in (n, \infty)$, see [16, II.9.7]. Setting $v := \nabla\mathcal{P}z$, $\beta := \frac{nr}{n-r}$ and using interpolation, we obtain

$$\|\nabla\mathcal{P}z\|_{\mathrm{L}^\alpha(\Omega)} \leq c\|\mathcal{P}z\|_{D^{r,q}(\Omega)}, \qquad \alpha \in (\frac{nr}{n-r}, \infty]. \tag{6.5}$$

Another application of (6.4) with $v := \mathcal{P}z$ and $\gamma := \frac{nr}{n-2r}$ yields the full estimate (6.2). Estimate (6.3) is well known, see *e.g.* [7, Lemma 4.4]. □

Lemma 6.4. *Under the assumptions of Theorem 6.2 it holds the inclusion* $F(\mathbb{E}) \subset \mathrm{L}^{\mathbf{s},\mathbf{p}}(\Omega_{\mathbb{T}})^{2n}$ *and there is* $C_L > 0$ *such that for* $z \in \mathbb{E}$ *with* $\|z\|_{\mathbb{E}} < \rho$, $\rho \in (0,1]$ *we have*

$$\|F(z)\|_{\mathrm{L}^{\mathbf{s},\mathbf{p}}(\Omega_{\mathbb{T}})} \leq C_L \rho^2. \tag{6.6}$$

Proof. We shall show a more precise statement. Assume $z_i = (u_i, d_i) \in \mathbb{E}$, $i \in \{1,2\}$. Let us write $v_i := \mathcal{P} u_i$ and $w_i := \mathcal{P}_\perp u_i$ for convenience. Then

$$\begin{aligned}
\|\mathcal{P}u_1 \cdot \nabla \mathcal{P}u_2\|_r &\leq \|\mathcal{P}u_1\|_{\frac{nr}{n-2r}} \|\nabla \mathcal{P}u_2\|_{\frac{n}{2}} \leq c\|z_1\|_{\mathbb{E}}\|z_2\|_{\mathbb{E}},\\
\|\mathcal{P}u_1 \cdot \nabla \mathcal{P}u_2\|_q &\leq \|\mathcal{P}u_1\|_\infty \|\nabla \mathcal{P}u_2\|_q \leq c\|z_1\|_{\mathbb{E}}\|z_2\|_{\mathbb{E}},\\
\|\mathcal{P}_\perp u_1 \cdot \nabla \mathcal{P}u_2\|_{\mathbf{s},\mathbf{p}} &\leq \|\mathcal{P}_\perp u_1\|_{\mathbf{s},\mathbf{p}} \|\nabla \mathcal{P}u_2\|_\infty \leq c\|z_1\|_{\mathbb{E}}\|z_2\|_{\mathbb{E}},\\
\|\mathcal{P}u_1 \cdot \nabla \mathcal{P}_\perp u_2\|_{\mathbf{s},\mathbf{p}} &\leq \|\mathcal{P}u_1\|_\infty \|\nabla \mathcal{P}_\perp u_2\|_{\mathbf{s},\mathbf{p}} \leq c\|z_1\|_{\mathbb{E}}\|z_2\|_{\mathbb{E}},\\
\|\mathcal{P}_\perp u_1 \cdot \nabla \mathcal{P}_\perp u_2\|_{\mathbf{s},\mathbf{p}} &\leq \|\mathcal{P}_\perp u_1\|_\infty \|\nabla \mathcal{P}_\perp u_2\|_{\mathbf{s},\mathbf{p}} \leq c\|z_1\|_{\mathbb{E}}\|z_2\|_{\mathbb{E}}.
\end{aligned} \tag{6.7}$$

This shows $u \cdot \nabla u \in \mathrm{L}^{\mathbf{s},\mathbf{p}}(\Omega_{\mathbb{T}})^n$, and similarly $u \cdot \nabla d \in \mathrm{L}^{\mathbf{s},\mathbf{p}}(\Omega_{\mathbb{T}})^n$. For the term $|\nabla d|^2(d + d_0)$ we notice $2q > 2r > \frac{nr}{n-r}$, and hence with (6.2)

$$\begin{aligned}
&\||\nabla d_1|^2(d_2 + d_0)\|_{\mathbf{s},\mathbf{p}}\\
&\leq (\|\nabla \mathcal{P} d_1\|_{2r} + \|\nabla \mathcal{P} d_1\|_{2q} + \|\nabla \mathcal{P}_\perp d_1\|_{2\mathbf{s},2\mathbf{p}})^2(\|z_2\|_\infty + 1)\\
&\leq c(\|\mathcal{P} z_1\|_{D^{r,q}(\Omega)} + \|\nabla \mathcal{P}_\perp d_1\|_{\mathbf{s},\mathbf{p}} + \|\nabla \mathcal{P}_\perp d_1\|_\infty)^2(\|z_2\|_\infty + 1),\\
&\leq c\|z_1\|_{\mathbb{E}}^2(\|z_2\|_{\mathbb{E}} + 1)
\end{aligned}$$

Finally,

$$\begin{aligned}
\|\mathrm{div}\,([\nabla d_1]^T[\nabla d_2])\|_{\mathrm{L}^{\mathbf{s},\mathbf{p}}(\Omega_{\mathbb{T}})} &\leq \|\nabla d_1\|_{\mathrm{L}^\infty(\Omega_{\mathbb{T}})}\|\nabla^2 d_2\|_{\mathrm{L}^{\mathbf{s},\mathbf{p}}(\Omega_{\mathbb{T}})}\\
&\quad + \|\nabla d_2\|_{\mathrm{L}^\infty(\Omega_{\mathbb{T}})}\|\nabla^2 d_1\|_{\mathrm{L}^{\mathbf{s},\mathbf{p}}(\Omega_{\mathbb{T}})}\\
&\leq c\|z_1\|_{\mathbb{E}}\|z_2\|_{\mathbb{E}}
\end{aligned}$$

Since all estimates are at least quadradic, this concludes the proof. □

We are now in the position to prove Theorems 6.1 and 6.2.

Proof of Theorems 6.1 and 6.2. Let us concentrate on Theorem 6.2, since the proof of Theorem 6.1 is similar and in fact easier. We will apply the contraction mapping principle on the set $B_\rho \subset \mathbb{E}$,

$$B_\rho := \{z \in \mathbb{E} : \|z - \bar{z}\|_{\mathbb{E}} \leq \rho\},$$

to the mapping $S : \mathbb{E} \to \mathbb{E}$ which assigns to $y \in \mathbb{E}$ the unique solution $z = Sy \in \mathbb{E}$ to the problem

$$\partial_t z + Lz = \begin{pmatrix} \mathbb{P} f_u \\ f_d \end{pmatrix} - F(y).$$

By the linear theory, $\|z - \bar{z}\|_{\mathbb{E}} \le C_M \|F(z)\|_{\mathrm{L}^{\mathbf{s},\mathbf{p}}(\Omega_{\mathbb{T}})}$. Due to estimate (6.6) it holds $S(B_\rho) \subset B_\rho$ for $\rho \in (0, [C_M C_L]^{-1})$.

Similarly, for $y_i = (u_i, d_i) \in B_\rho$, $i \in \{1, 2\}$, the term $\|Sy_1 - Sy_2\|_{\mathbb{E}}$ can be estimated by four summands. With (6.7) we can calculate

$$\begin{aligned}\|u_1 \cdot \nabla u_1 - u_2 \cdot \nabla u_2\|_{\mathrm{L}^{\mathbf{s},\mathbf{p}}(\Omega_{\mathbb{T}})} &\le \|u_1 \cdot \nabla(u_1 - u_2) + (u_1 - u_2) \cdot \nabla u_2\|_{\mathrm{L}^{\mathbf{s},\mathbf{p}}(\Omega_{\mathbb{T}})} \\ &\le c(\|y_1\|_{\mathbb{E}} + \|y_2\|_{\mathbb{E}})\|y_1 - y_2\|_{\mathbb{E}} \le c(\rho + \varepsilon)\|y_1 - y_2\|_{\mathbb{E}},\end{aligned}$$

and

$$\begin{aligned}&\||\nabla d_1|^2 d_1 - |\nabla d_2|^2 d_2 + (|\nabla d_1|^2 - |\nabla d_2|^2) d_0\|_{\mathrm{L}^{\mathbf{s},\mathbf{p}}(\Omega_{\mathbb{T}})} \\ &\quad = \||\nabla d_1|^2 (d_1 - d_2) + (|\nabla d_1|^2 - |\nabla d_2|^2)(d_2 + d_0)\|_{\mathrm{L}^{\mathbf{s},\mathbf{p}}(\Omega_{\mathbb{T}})} \\ &\quad \le c(\|y_1\|_{\mathbb{E}}^2 \|y_1 - y_2\|_{\mathbb{E}} + \|y_1 - y_2\|_{\mathbb{E}}(\|y_1\|_{\mathbb{E}} + \|y_2\|_{\mathbb{E}})(\|y_2\|_{\mathbb{E}} + 1)) \\ &\quad \le c(\rho + \varepsilon)(\rho + \varepsilon + 1)\|y_1 - y_2\|_{\mathbb{E}},\end{aligned}$$

as well as

$$\begin{aligned}&\|\mathrm{div}\,([\nabla d_1]^T [\nabla d_1]) - \mathrm{div}\,([\nabla d_2]^T [\nabla d_2])\|_{\mathrm{L}^{\mathbf{s},\mathbf{p}}(\Omega_{\mathbb{T}})} \\ &\qquad \le c(\|y_1\|_{\mathbb{E}} + \|y_2\|_{\mathbb{E}})\|y_1 - y_2\|_{\mathbb{E}} \le c(\rho + \varepsilon)\|y_1 - y_2\|_{\mathbb{E}}.\end{aligned}$$

This yields $\|Sy_1 - Sy_2\|_{\mathbb{E}} \le \frac{1}{2}\|y_1 - y_2\|_{\mathbb{E}}$ for sufficiently small $\rho, \varepsilon > 0$. Therefore, the contraction mapping principle yields a unique fixed point $z = (u, d) \in B_\rho$ such that $S(z) = z$.

If we additionally assume $\rho, \varepsilon > 0$ to be sufficiently small such that

$$\|d\|_\infty \le c\|z\|_{\mathbb{E}} \le c(\rho + \varepsilon) < 1,$$

then also $d + d_0 \neq 0$ on $\Omega_{\mathbb{T}}$. □

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