A Method for Obtaining Time-Periodic L^p Estimates

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We introduce a method for showing *a priori* L^p estimates for time-periodic, linear, partial differential equations set in a variety of domains such as the whole space, the half space and bounded domains. The method is generic and can be applied to a wide range of problems. We demonstrate it on the heat equation. The main idea is to replace the time axis with a torus in order to reformulate the problem on a locally compact abelian group and to employ Fourier analysis on this group. As a by-product, maximal L^p regularity for the corresponding initial-value problem follows *without* the notion of $\mathcal{R}$-boundedness. Moreover, we introduce the concept of a time-periodic fundamental solution.

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1 Introduction

The purpose of this paper is to introduce a generic method that establishes L^p estimates for time-periodic solutions to a large class of linear, partial differential equations. The method works particularly well for parabolic problems. To emphasize the strength and simplicity of the method in the parabolic case, we demonstrate it on the time-periodic heat equation:

$$\begin{cases} \partial_t u - \Delta u = f & \text{in } \mathbb{R} \times \Omega, \\ u = 0 & \text{on } \mathbb{R} \times \partial\Omega, \\ u(t+T, x) = u(t, x), \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ is a domain, $u : \mathbb{R} \times \Omega \to \mathbb{R}$ the unknown function dependent on a time variable t and a spatial variable x with $(t, x) \in \mathbb{R} \times \Omega$, $T > 0$ a fixed positive time-period, and $f : \mathbb{R} \times \Omega \to \mathbb{R}$ the data, which is also assumed to be T-time-periodic. The method to be introduced enables us to identify a Banach space X^p of T-time-periodic functions vanishing on the boundary $\partial\Omega$ with the property that the differential operator $\partial_t - \Delta$ maps X^p homeomorphically onto $L^p\big((0, T) \times \Omega\big)$ for any $p \in (1, \infty)$. As a consequence, we obtain the L^p estimate

$$\|u\|_{X^p} \leq c\|f\|_{L^p((0,T)\times\Omega)} \tag{1.2}$$

for a solution to (1.1). Observe that $L^p\big((0, T) \times \Omega\big)$ is the "natural" L^p space for T-time-periodic data, whence X^p can be described as the maximal regularity space in the L^p setting for the T-time-periodic problem (1.1). The Banach space X^p will be characterized as a Sobolev-type space. A proper identification of X^p and in particular the L^p estimate (1.2) is crucial in many applications; not only the analysis of corresponding non-linear problems, but also the investigation of Hopf bifurcations and other phenomena.

We shall first treat the whole-space case $\Omega = \mathbb{R}^n$, then the half-space case $\Omega = \mathbb{R}^n_+$ and finally the case of a sufficiently smooth bounded domain. In the first case we establish a direct representation formula for the solution u in terms of Fourier multipliers. The estimate (1.1) is then shown using classical tools from harmonic analysis. In the half space case the reflection principle applies. For the case of bounded domains we employ localization techniques. One may recognize these steps as the standard procedure for analyzing elliptic problems. In fact, we consider it a novelty of our method that it enables us to treat time-periodic parabolic problems with the same approach and tools used for the corresponding elliptic problem. This is by no means a trivial approach. In fact, the vast literature on time-periodic problems is almost solely based on the idea of treating first the corresponding initial value problem and then subsequently showing, employing for example a Poincaré map, existence of at least one initial value that produces a time-periodic solution. In comparison, our method is much more direct. Moreover, no maximal regularity results for the initial-value problem are needed. In fact, we shall show that maximal L^p regularity for the corresponding initial-value problem follows effortlessly and *without* employing the notion of $\mathcal{R}$-boundedness from our method. This suggests that in the investigation of classical Cauchy problems, one should treat the time-periodic problem before the initial-value problem.

We start by briefly explaining the method in the whole-space case $\Omega = \mathbb{R}^n$. The main idea is to reformulate the time-periodic problem as a PDE on the locally compact abelian group $G := \mathbb{T} \times \mathbb{R}^n$, where $\mathbb{T}$ denotes the torus $\mathbb{R}/T\mathbb{Z}$. As both the data f and the solution u are T-periodic in the t variable, they can naturally be interpreted as functions on G. Moreover, a differentiable structure on G is canonically inherited from $\mathbb{R} \times \mathbb{R}^n$ via the quotient mapping $\pi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{T} \times \mathbb{R}^n$ in such a way that (1.1) can be equivalently reformulated as a partial differential equation

$$\partial_t u - \Delta u = f \quad \text{in } G \tag{1.3}$$

in a setting of functions $u : G \to \mathbb{R}$ and $f : G \to \mathbb{R}$. Note that the periodicity conditions become superfluous in this setting. The main advantage of the G setting, however, is the availability of the Fourier transform $\mathscr{F}_G$ in combination with the space of tempered distributions $\mathscr{S}'(G)$, the dual of the Schwartz-Bruhat space $\mathscr{S}(G)$. With these tools at our disposal, we derive from (1.3) the representation formula

$$u = \mathscr{F}_G^{-1}\left[\frac{1}{-ik + |\xi|^2}\,\mathscr{F}_G\big[f\big]\right] \tag{1.4}$$

for data $f \in \mathscr{S}(G)$. Here, (k, ξ) denote points in the dual group $\widehat{G} := \frac{2\pi}{T}\mathbb{Z} \times \mathbb{R}^n$. The representation formula is key to the L^p estimates. We further let $\delta_{\mathbb{Z}}$ denote the Dirac mass on $\frac{2\pi}{T}\mathbb{Z}$ and split the right-hand side in (1.4) into two parts:

$$u = \mathscr{F}_{\mathbb{R}^n}^{-1}\left[\frac{1}{|\xi|^2}\,\mathscr{F}_{\mathbb{R}^n}\big[1_{\mathbb{T}} *_{\mathbb{T}} f\big]\right] + \mathscr{F}_G^{-1}\left[\frac{\big(1-\delta_{\mathbb{Z}}(k)\big)}{-ik + |\xi|^2}\,\mathscr{F}_G\big[f\big]\right] =: u_1 + u_2, \tag{1.5}$$

where $1_{\mathbb{T}} *_{\mathbb{T}} f$ denotes convolution on the torus $\mathbb{T}$ with the constant function 1. The technical advantage of decomposing u as in (1.5) is immediately clear. The first part u_1 is namely expressed in terms of a classical Fourier multiplier in the $\mathbb{R}^n$ setting for which we can use standard tools from harmonic analysis to show L^p estimates. In the simple case of the heat equation under investigation here, we recognize the Fourier multiplier to be the symbol of the Laplacian. The second part u_2 is expressed in terms of a Fourier multiplier with *no* singularities. As a consequence, we are able to establish very strong L^p estimates for u_2. In contrast to the $\mathbb{R}^n$ case, however, there is no comprehensive theory available in the abstract G setting to establish such estimates via Fourier multipliers. To overcome this challenge, we employ a transference principle for group multipliers, see Theorem 5.1. The transference principle allows us to "transfer" the multiplier into an $\mathbb{R}^{n+1}$ setting and then use the classical tools. The principle was originally established by de Leeuw [6] and later generalized by Edwards and Gaudry in [7]. Combining the estimates of u_1 and u_2, we obtain the desired estimate (1.2) of u and an identification of the space X^p.

Another advantage of the reformulation (1.3) in the group setting is the ability to express the solution u as a convolution of a "fundamental solution" with the data

f. One can derive directly from the representation (1.4) that $u = \mathit{\Gamma} *_G f$, where $\mathit{\Gamma} = \mathscr{F}_G^{-1}\big[\frac{1}{-ik+|\xi|^2}\big]$ and $*_G$ denotes convolution on the group G. In fact, from (1.5) we see that $\mathit{\Gamma} = 1_{\mathbb{T}} \otimes \mathit{\Gamma}^{\mathrm{L}} + \mathit{\Gamma}^{\perp}$, where $\mathit{\Gamma}^{\mathrm{L}}$ denotes the well-known fundamental solution to the Laplace equation and $\mathit{\Gamma}^{\perp} = \mathscr{F}_G^{-1}\Bigg[\frac{\big(1-\delta_{\mathbb{Z}}(k)\big)}{-ik+|\xi|^2}\Bigg]$. Provided we can obtain pointwise estimates of $\mathit{\Gamma}^{\perp}$, this observation can be used to extract pointwise information from u. In particular, the asymptotic structure of u as $|x| \to \infty$ can be analyzed in this way. In Section 6 such pointwise estimates of $\mathit{\Gamma}^{\perp}$ are established.

2 Statement of the main result

Although our aim is to introduce a generic method, we shall nevertheless state as our main theorem the result obtained by applying the method to the specific case of the time-periodic heat equation (1.1). The details of the method are then revealed in the proof of the theorem.

We state the main theorem in a context of classical Sobolev spaces of time-periodic functions. Let $\Omega \subset \mathbb{R}^n$ be a domain. We first introduce the space

$$C^\infty_{0,\mathrm{per}}(\mathbb{R}\times\Omega) := \{f \in C^\infty(\mathbb{R}\times\Omega) \mid f(t+T,x) = f(t,x) \ \wedge \ f \in C^\infty_0\big([0,T]\times\Omega\big)\},$$

of smooth time-period functions with compact support in the spatial variable. Clearly,

$$\|f\|_p := \left(\frac{1}{T}\int_0^T \int_\Omega |f(t,x)|^p \,\mathrm{d}x\mathrm{d}t\right)^{\frac{1}{p}},$$
$$\|f\|_{1,2,p} := \left(\sum_{|\alpha|\le 1} \|\partial_t^\alpha f\|_p^p + \sum_{|\beta|\le 2} \|\partial_x^\beta f\|_p^p\right)^{\frac{1}{p}}$$

are norms on $C^\infty_{0,\mathrm{per}}(\mathbb{R}\times\Omega)$, and we can thus define the Lebesgue and (anisotropic) Sobolev spaces

$$L^p_{\mathrm{per}}(\mathbb{R}\times\Omega) := \overline{C^\infty_{0,\mathrm{per}}(\mathbb{R}\times\Omega)}^{\|\cdot\|_p}, \quad W^{1,2,p}_{\mathrm{per}}\big(\mathbb{R}\times\Omega\big) := \overline{C^\infty_{0,\mathrm{per}}(\mathbb{R}\times\overline{\Omega})}^{\|\cdot\|_{1,2,p}}$$

of time-periodic functions. Note that for domains Ω satisfying the segment condition we have

$$W^{1,2,p}_{\mathrm{per}}(\mathbb{R}\times\Omega) = \{f \in L^p_{\mathrm{per}}(\mathbb{R}\times\Omega) \mid \|f\|_{1,2,p} < \infty\},$$

where the derivatives which appear in the norm $\|f\|_{1,2,p}$ are to be understood in the sense of distributions. We introduce the operators

$$\mathcal{P}, \mathcal{P}_\perp : C^\infty_{0,\mathrm{per}}(\mathbb{R}\times\Omega) \to C^\infty_{0,\mathrm{per}}(\mathbb{R}\times\Omega), \quad \mathcal{P}f := \frac{1}{T}\int_0^T f(t,x)\,\mathrm{d}t, \quad \mathcal{P}_\perp := \mathrm{Id} - \mathcal{P},$$

which are clearly complementary projections. By continuity, $\mathcal{P}$ and $\mathcal{P}_\perp$ extend to bounded operators on $L^p_{\mathrm{per}}(\mathbb{R}\times\Omega)$ and $W^{1,2,p}_{\mathrm{per}}\big(\mathbb{R}\times\Omega\big)$. Of course, $\mathcal{P}f$ is independent of the time variable $t \in \mathbb{R}$ and can be considered as a function in the space variable $x \in \Omega$ only. Finally we denote by $\dot{W}^{2,p}(\Omega)$ the classical homogeneous Sobolev spaces with semi-norm $|\cdot|_{2,p}$. We can now formulate the main theorem:

Theorem 2.1. *Assume that either* $\Omega = \mathbb{R}^n$, $\Omega = \mathbb{R}^n_+$, *or* $\Omega \subset \mathbb{R}^n$ *is a bounded domain with a* $C^{1,1}$*-smooth boundary. Let* $p \in (1,\infty)$. *For any* $f \in L^p_{\mathrm{per}}(\mathbb{R}\times\Omega)$ *there is a solution* u *to* (1.1) *with*

$$u(t,x) = u_s(x) + u_p(t,x) \in \dot{W}^{2,p}(\Omega) \oplus \mathcal{P}_\perp W^{1,2,p}_{\mathrm{per}}\big(\mathbb{R}\times\Omega\big) \tag{2.1}$$

satisfying

$$|u_s|_{2,p} \leq c_1 \, \|\mathcal{P}f\|_p, \tag{2.2}$$

$$\|u_p\|_{1,2,p} \leq c_2 \, \|\mathcal{P}_\perp f\|_p, \tag{2.3}$$

where $c_1 = c_1(n,p,\Omega) > 0$ *and* $c_2 = c_2(n,p,\Omega,T) > 0$. *If* $v = v_s + v_p$ *is another solution with* $v_s \in \dot{W}^{2,r_1}(\Omega)$ *and* $v_p \in \mathcal{P}_\perp W^{1,2,r_2}_{\mathrm{per}}\big(\mathbb{R}\times\Omega\big)$, $r_1, r_2 \in (1,\infty)$, *then* $u_s - v_s$ *is a polynomial of order 1 and* $u_p = v_p$.

Remark 2.2. Observe that the constant c_1 in (2.2) is independent of the time period T, whereas the constant c_2 in (2.3) depends on T and in fact, as will become clear in the proof, tends to infinity as $T \to \infty$. If one is not interested in the dependency of the constants on T, it may be more convenient to combine (2.2)–(2.3) into the estimate

$$|u_s|_{2,p} + \|u_p\|_{1,2,p} \leq c\|f\|_p, \tag{2.4}$$

where $c = c(n,p,\Omega,T) > 0$.

Remark 2.3. The statement in Theorem 2.1 concerning uniqueness can be improved considerably for the purely periodic part u_p of the solution. In the whole-space case $\Omega = \mathbb{R}^n$, for example, it will follow directly from the proof that if u and v are two solutions to (1.1) with $\mathcal{P}_\perp u, \mathcal{P}_\perp v \in \mathscr{S}'_{per}(\mathbb{R}\times\mathbb{R}^n)$, then $\mathcal{P}_\perp u = \mathcal{P}_\perp v$. Here, $\mathscr{S}'_{per}(\mathbb{R}\times\mathbb{R}^n)$ denotes the space of tempered distributions that are T-periodic in the t variable.

Remark 2.4. The Lebesgue space $L^p_{\mathrm{per}}(\mathbb{R}\times\Omega)$ can be identified with $L^p_{\mathrm{per}}(\mathbb{R}; L^p(\Omega))$. It is natural to ask whether Theorem 2.1 can be extended to mixed-norm spaces of the form $L^p_{\mathrm{per}}(\mathbb{R}; L^q(\Omega))$. The proof of (2.2)-(2.3) in this paper relies on the Marcinkiewicz multiplier theorem, which requires $p = q$. In order to extend the result to the case $p \neq q$, one can use a result of Besov [4] on a mixed-norm Littlewood-Paley theorem, upon which the classical theorems of Fourier multipliers are based.

Observe that the projections $\mathcal{P}$ and $\mathcal{P}_\perp$ decompose a time-periodic solution u into a stationary part $u_s = \mathcal{P}u$ and a *purely periodic* part $u_p = \mathcal{P}_\perp u$. From Theorem 2.1 we learn that the two parts must be treated separately in order to identify the Banach space of "maximal regularity" for problem (1.1), that is, a Banach space $X^p(\mathbb{R}\times\Omega)$ of T-time-periodic functions vanishing on the boundary $\mathbb{R}\times\partial\Omega$ with the property that $\partial_t - \Delta : X^p(\mathbb{R}\times\Omega) \to L^p_{\mathrm{per}}(\mathbb{R}\times\Omega)$ is a homeomorphism. As a corollary to Theorem 2.1 we directly obtain:

Corollary 2.5. *With the same assumptions as in Theorem 2.1, let*

$$X^p_\perp(\mathbb{R}\times\Omega) := \{u \in \mathcal{P}_\perp W^{1,2,p}_{\mathrm{per}}\big(\mathbb{R}\times\Omega\big) \mid u_{|\partial\Omega} = 0\}.$$

Then

$$\partial_t - \Delta : \ X^p_\perp(\mathbb{R}\times\Omega) \to \mathcal{P}_\perp L^p_{\mathrm{per}}(\mathbb{R}\times\Omega)$$

is a homeomorphism.

Remark 2.6. In Corollary 2.5 we only identify the maximal regularity space $X^p_\perp(\mathbb{R}\times\Omega)$ for the purely periodic part $\mathcal{P}_\perp u$ of a solution to (1.1). In order to identify the full maximal regularity space of the problem, we would need to also address the stationary part $\mathcal{P}u$. More specifically, we would need to identify a Banach space $X^p_s(\Omega)$ of functions that vanish at the boundary $\partial\Omega$ with the property that $-\Delta : X^p_s(\Omega) \to L^p(\Omega)$ is a homeomorphism. Since $L^p_{\mathrm{per}}(\mathbb{R}\times\Omega) = L^p(\Omega) \oplus \mathcal{P}_\perp L^p_{\mathrm{per}}(\mathbb{R}\times\Omega)$, we could then conclude that

$$\partial_t - \Delta : \ X^p_s(\Omega) \oplus X^p_\perp(\mathbb{R}\times\Omega) \to L^p_{\mathrm{per}}(\mathbb{R}\times\Omega)$$

is a homeomorphism. Since the identification of $X^p_s(\Omega)$ is a well-known problem in the context of elliptic equations and completely decoupled from the time-periodic nature of (1.1), we shall not address it here. Of course, in certain cases, for example when Ω is bounded, the homogeneous Sobolev space $\big(\dot{W}^{2,p}(\Omega), |\cdot|_{2,p}\big)$ is a Banach space, in which case the identification of $X^p_s(\Omega)$ follows directly from Corollary 2.5. Note that Corollary 2.5 makes it possible to construct hybrid L^p-type Banach spaces of maximal regularity for (1.1). If for example $X^p_s(\Omega)$ and $Y^p_s(\Omega)$ are Banach spaces such that $-\Delta : X^p_s(\Omega) \to Y^p_s(\Omega)$ is a homeomorphism, then

$$\partial_t - \Delta : \ X^p_s(\Omega) \oplus X^p_\perp(\mathbb{R}\times\Omega) \to Y^p_s(\Omega) \oplus \mathcal{P}_\perp L^p_{\mathrm{per}}(\mathbb{R}\times\Omega)$$

is a homeomorphism. Such constructions may be useful in applications where for example weak Lebesgue or Besov spaces are better suited to capture the properties of solutions to the elliptic problem satisfied by $\mathcal{P}u$, in our case $-\Delta\mathcal{P}u = \mathcal{P}f$.

3 Impact on the initial-value problem

A remarkable consequence of Corollary 2.5 concerns initial value problems. Recall that a generator $-A$ of a C_0 semi-group on a Banach space X is said to admit *maximal L^p regularity* on $(0,T)$, if for every $f \in L^p(0,T;X)$ the unique solution to the abstract Cauchy problem $\partial_t u + Au = f$, $u(0) = 0$ satisfies $u \in L^p(0,T;D(A)) \cap W^{1,p}(0,T;X) =: \mathbb{E}_T(A)$. It is well-known that maximal L^p regularity is equivalent to $\mathcal{R}$-boundedness of the resolvent family $\{t(it+A)^{-1} \mid t \in \mathbb{R}\}$, see [11]. This characterization has been used extensively to show maximal regularity for various differential operators, in particular for the Dirichlet Laplacian. Using Corollary 2.5, we are able to establish maximal L^p regularity for the Dirichlet Laplacian while completely circumventing the notion of $\mathcal{R}$-boundedness.

Recall that the Dirichlet Laplacian $\Delta_p : D(\Delta_p) \subset L^p(\Omega) \to L^p(\Omega)$ with domain $D(\Delta_p) := W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)$, defined via $\Delta_p u := \Delta u$ for $u \in D(\Delta_p)$, is a generator of an analytic C_0 semi-group $e^{t\Delta_p}$.

Theorem 3.1. *Let $p \in (1,\infty)$, $T > 0$ and assume that $\Omega = \mathbb{R}^n$, $\Omega = \mathbb{R}^n_+$ or Ω is a bounded domain with a $C^{1,1}$-smooth boundary. The Dirichlet Laplacian $-\Delta_p$ admits maximal L^p regularity on $(0,T)$.*

Proof. Recall for an analytic semi-group e^{-tA} on a Banach space X the following characterization of the trace space at $t = 0$:

$$(X, D(A))_{1-1/p,p} = \{x \in X \mid Ae^{-tA}x \in L^p(0,T;X)\} = \{u(0) \mid u \in \mathbb{E}_T(A)\}, \tag{3.1}$$

see for example [9, Corollary 1.14, Proposition 6.2]. Let $f \in L^p(0,T;L^p(\Omega))$. Canonically, we can interpret f as a periodic function in $L^p_{\text{per}}(\mathbb{R}\times\Omega)$ and write $f = \mathcal{P}f + \mathcal{P}_\perp f \in L^p(\Omega) \oplus \mathcal{P}_\perp L^p_{\text{per}}(\mathbb{R}\times\Omega)$. Corollary 2.5 gives a solution $v \in \mathbb{E}_T(\Delta_p)$ on $(0,T)$ to $\partial_t v - \Delta_p v = \mathcal{P}_\perp f$. Since $\mathrm{e}^{t\Delta_p}$ is analytic, also $\mathrm{e}^{t\Delta_p}v(0) \in \mathbb{E}_T(\Delta_p)$ by (3.1). Furthermore, we define $w(t) := \int_0^t \mathrm{e}^{(t-s)\Delta_p}\mathcal{P}f\,\mathrm{d}s$. Since $\mathcal{P}f \in L^p(\Omega)$ does not depend on time, it is easy to verify that $w \in \mathbb{E}_T(\Delta_p)$. Thus, a solution $u \in \mathbb{E}_T(\Delta_p)$ to $u_t - \Delta_p u = f$, $u(0) = 0$ is given by $u := v + w - \mathrm{e}^{t\Delta_p}v(0)$. □

Remark 3.2. The argument above is based on one given by Arendt and Bu in [3]. Observe that the proof of Theorem 3.1 yields maximal regularity for any generator $-A$ of an analytic semi-group for which a result corresponding to Corollary 2.5 can be established.

4 Preliminaries

In the following, we let G denote the group

$$G := \mathbb{T}\times\mathbb{R}^n := \mathbb{R}/T\mathbb{Z}\times\mathbb{R}^n \tag{4.1}$$

with addition as the group operation. We shall reformulate (1.1) and the main theorem in a setting of functions defined on G. For this purpose, we must first introduce a topology and an appropriate differentiable structure on G. Both will be inherited from $\mathbb{R}\times\mathbb{R}^n$. More precisely, we equip G with the quotient topology induced by the canonical quotient mapping

$$\pi : \mathbb{R}\times\mathbb{R}^n \to \mathbb{T}\times\mathbb{R}^n, \quad \pi(t,x) := ([t],x). \tag{4.2}$$

Equipped with the quotient topology, G becomes a locally compact abelian group. The restriction $\Pi := \pi_{|[0,T)\times\mathbb{R}^n}$ is used to identify G with the domain $[0,T)\times\mathbb{R}^n$; Π is clearly a (continuous) bijection. Via Π, one can identify the Haar measure $\mathrm{d}g$ on G as the product of the Lebesgue measure on $[0,T)$ and the Lebesgue measure on $\mathbb{R}^n$. The Haar measure is unique up to a normalization factor, which we choose such that

$$\int_G u(g)\,\mathrm{d}g = \frac{1}{T}\int_0^T\int_{\mathbb{R}^n} u\circ\Pi(t,x)\,\mathrm{d}x\mathrm{d}t.$$

For the sake of convenience, we will omit the Π in integrals of G-defined functions with respect to $\mathrm{d}x\mathrm{d}t$. Next, we define by

$$C^\infty(G) := \{u : G \to \mathbb{R} \mid u\circ\pi \in C^\infty(\mathbb{R}\times\mathbb{R}^n)\} \tag{4.3}$$

the space of smooth functions on G. For $u \in C^\infty(G)$ we define derivatives

$$\forall(\alpha,\beta) \in \mathbb{N}_0^n \times \mathbb{N}_0 : \quad \partial_t^\beta \partial_x^\alpha u := \left[\partial_t^\beta \partial_x^\alpha (u \circ \pi)\right] \circ \Pi^{-1}. \tag{4.4}$$

It is easy to verify for $u \in C^\infty(G)$ that also $\partial_t^\beta \partial_x^\alpha u \in C^\infty(G)$. We further introduce the subspace

$$C_0^\infty(G) := \{u \in C^\infty(G) \mid \operatorname{supp} u \text{ is compact}\}$$

of compactly supported smooth functions. If $\Omega \subset \mathbb{R}^n$ is a domain, then the spaces $C^\infty(\mathbb{T} \times \Omega)$ and $C_0^\infty(\mathbb{T} \times \Omega)$ are defined analogously.

With a differentiable structure defined on G, we can introduce the space of tempered distributions on G. For this purpose, we first recall the Schwartz-Bruhat space of generalized Schwartz functions; see for example [5]. More precisely, we define

$$\mathscr{S}(G) := \{u \in C^\infty(G) \mid \forall(\alpha,\beta,\gamma) \in \mathbb{N}_0^n \times \mathbb{N}_0^n \times \mathbb{N}_0 : \ \rho_{\alpha,\beta,\gamma}(u) < \infty\},$$
$$\rho_{\alpha,\beta,\gamma}(u) := \sup_{(t,x)\in G} |x^\alpha \partial_x^\beta \partial_t^\gamma u(t,x)|.$$

Equipped with the semi-norm topology of the family $\left\{\rho_{\alpha,\beta,\gamma} \mid (\alpha,\beta,\gamma) \in \mathbb{N}_0^n \times \mathbb{N}_0^n \times \mathbb{N}_0\right\}$, $\mathscr{S}(G)$ becomes a topological vector space. The dual space $\mathscr{S}'(G)$ equipped with the weak* topology is referred to as the space of tempered distributions on G. For a tempered distribution $u \in \mathscr{S}'(G)$, distributional derivatives $\partial_t^\beta \partial_x^\alpha u \in \mathscr{S}'(G)$ are defined by duality as in the classical case.

We shall also introduce tempered distributions on G's dual group $\widehat{G}$. We associate each $(k,\xi) \in \frac{2\pi}{T}\mathbb{Z} \times \mathbb{R}^n$ with the character $\chi : G \to \mathbb{C}$, $\chi(t,x) := \mathrm{e}^{ix\cdot\xi + ikt}$ on G. In particular, we stress the fact that k is not an integer, but $k \in \frac{2\pi}{T}\mathbb{Z}$. It is standard to verify that all characters are of this form, and we can thus identify $\widehat{G} = \frac{2\pi}{T}\mathbb{Z} \times \mathbb{R}^n$. By default, $\widehat{G}$ is equipped with the compact-open topology, which in this case coincides with the product of the discrete topology on $\frac{2\pi}{T}\mathbb{Z}$ and the Euclidean topology on $\mathbb{R}^n$. The Haar measure on $\widehat{G}$ is simply the product of the counting measure on $\frac{2\pi}{T}\mathbb{Z}$ and the Lebesgue measure on $\mathbb{R}^n$. By

$$C^\infty(\widehat{G}) := \{w \in C(\widehat{G}) \mid \forall k \in \frac{2\pi}{T}\mathbb{Z} : \ w(k,\cdot) \in C^\infty(\mathbb{R}^n)\}$$

the space of smooth functions on $\widehat{G}$ is introduced. The Schwartz-Bruhat space on the dual group $\widehat{G}$ is defined by

$$\mathscr{S}(\widehat{G}) := \{w \in C^\infty(\widehat{G}) \mid \forall(\alpha,\beta,\gamma) \in \mathbb{N}_0^n \times \mathbb{N}_0^n \times \mathbb{N}_0 : \ \hat{\rho}_{\alpha,\beta,\gamma}(w) < \infty\},$$
$$\hat{\rho}_{\alpha,\beta,\gamma}(w) := \sup_{(k,\xi)\in\widehat{G}} |\xi^\alpha \partial_\xi^\beta k^\gamma w(k,\xi)|,$$

and equipped with the canonical semi-norm topology.

We denote by $\mathscr{F}_G$ the Fourier transform associated to the locally compact abelian group G. It is explicitly given by

$$\mathscr{F}_G : L^1(G) \to C(\widehat{G}), \quad \mathscr{F}_G(u)(k,\xi) := \frac{1}{T}\int_0^T \int_{\mathbb{R}^n} u(t,x)\, \mathrm{e}^{-ix\cdot\xi - ikt}\, \mathrm{d}x\mathrm{d}t.$$

The inverse Fourier transform is formally defined by

$$\mathscr{F}_G^{-1} : L^1(\widehat{G}) \to C(G), \quad \mathscr{F}_G^{-1}(w)(t,x) := \sum_{k\in\frac{2\pi}{T}\mathbb{Z}} \int_{\mathbb{R}^n} w(k,\xi)\, \mathrm{e}^{ix\cdot\xi + ikt}\, \mathrm{d}\xi.$$

It is standard to verify that $\mathscr{F}_G : \mathscr{S}(G) \to \mathscr{S}(\widehat{G})$ is a homeomorphism with $\mathscr{F}_G^{-1}$ as the actual inverse, provided the Lebesgue measure $\mathrm{d}\xi$ is normalized appropriately. By duality, $\mathscr{F}_G$ extends to a homeomorphism $\mathscr{S}'(G) \to \mathscr{S}'(\widehat{G})$.

Let $\Omega \subset \mathbb{R}^n$ be a domain. In analogy to the spaces $L^p_{\mathrm{per}}(\mathbb{R}\times\Omega)$ and $W^{1,2,p}_{\mathrm{per}}(\mathbb{R}\times\Omega)$, we define by

$$L^p(\mathbb{T}\times\Omega) := \overline{C_0^\infty(\mathbb{T}\times\Omega)}^{\|\cdot\|_p}, \quad W^{1,2,p}(\mathbb{T}\times\Omega) := \overline{C_0^\infty(\mathbb{T}\times\overline{\Omega})}^{\|\cdot\|_{1,2,p}}$$

the corresponding Lebesgue and Sobolev spaces, where $\|\cdot\|_p$ is the L^p-norm with respect to the Haar measure $\mathrm{d}g$. Again, we have the equality

$$W^{1,2,p}(\mathbb{T}\times\Omega) = \{f \in L^p(\mathbb{T}\times\Omega) \mid \|f\|_{1,2,p} < \infty\}$$

if the domain Ω satisfies the segment condition.

As in Section 2, we define the time-averaging projection $\mathcal{P}$ and its complement $\mathcal{P}_\perp$. Observe that $\mathcal{P}$ can also be expressed as the convolution over the torus $\mathbb{T}$ with the constant function $1_\mathbb{T}$, that is, $\mathcal{P}f = 1_\mathbb{T} *_\mathbb{T} f$. Hence, by Young's inequality $\mathcal{P}$ is continuous both in $L^p(\mathbb{T}\times\Omega)$ and $W^{1,2,p}(\mathbb{T}\times\Omega)$. Therefore, we may define the Banach space

$$X^p_\perp(\mathbb{T}\times\Omega) := \{u \in \mathcal{P}_\perp W^{1,2,p}(\mathbb{T}\times\Omega) \mid u_{|\mathbb{T}\times\partial\Omega} = 0\}.$$

5 Proof of the main result

With the terminology introduced in the previous section, we are now able to disclose the details of our method and in doing so prove the main theorem.

We start by recalling a theorem on transference of Fourier multipliers, which enables us to "transfer" multipliers from one group setting into another. The theorem is originally due to de Leeuw [6], who established the transference principle between the torus group and $\mathbb{R}$. The more general version below is due to Edwards and Gaudry [7, Theorem B.2.1].

Theorem 5.1. *Let G and H be locally compact abelian groups. Moreover, let $\Phi : \widehat{G} \to \widehat{H}$ be a continuous homomorphism and $p \in [1,\infty]$. Assume that $m \in L^\infty(\widehat{H};\mathbb{C})$ is a continuous L^p-multiplier, that is, there is a constant $B > 0$ such that*

$$\forall f \in L^2(H)\cap L^p(H) : \ \|\mathscr{F}_H^{-1}[m\cdot\mathscr{F}_H(f)]\|_p \le B\|f\|_p.$$

Then $m \circ \Phi \in L^\infty(\widehat{G};\mathbb{C})$ *is also an* L^p*-multiplier with*

$$\forall f \in L^2(G) \cap L^p(G): \; \|\mathscr{F}_G^{-1}\big[m \circ \Phi \cdot \mathscr{F}_G(f)\big]\|_p \leq B\|f\|_p.$$

5.1 The Whole Space

With the theorem above on transference of Fourier multipliers, we are able to establish the essential part of the main theorem in the whole-space case.

Lemma 5.2. *Let* $p \in (1,\infty)$. *For any* $f \in \mathcal{P}_\perp L^p(G)$ *there exists a solution* $u \in \mathcal{P}_\perp W^{1,2,p}(G)$ *to*

$$\partial_t u - \Delta u = f \quad \text{in } G \tag{5.1}$$

that satisfies

$$\|u\|_{1,2,p} \leq c\,\|f\|_p, \tag{5.2}$$

where $c = c(n,p,T) > 0$. *If* $v \in \mathcal{P}_\perp \mathscr{S}'(G)$ *is another solution, then* $u = v$.

Proof. It clearly suffices to assume $f \in \mathcal{P}_\perp \mathscr{S}(G)$. Observe that $f = \mathcal{P}_\perp f$. Recalling that $\mathscr{F}_G\big[\mathcal{P}_\perp f\big] = (1-\delta_\mathbb{Z})\mathscr{F}_G\big[f\big]$, we can apply the Fourier transform $\mathscr{F}_G$ in (5.1) to deduce that

$$u := \mathscr{F}_G^{-1}\left[\frac{1-\delta_\mathbb{Z}(k)}{-ik+|\xi|^2}\,\mathscr{F}_G\big[f\big]\right] \tag{5.3}$$

is a solution. Since the Fourier multiplier

$$M : \widehat{G} \to \mathbb{C}, \quad M(k,\xi) := \frac{1-\delta_\mathbb{Z}(k)}{-ik+|\xi|^2}$$

is bounded, that is, $M \in L^\infty(\widehat{G})$, it is clear that u given by the formula above is well-defined as an element in $\mathscr{S}'(G)$. To analyze u further, we wish to apply Theorem 5.1. For this purpose, let χ be a "cut-off" function with

$$\chi \in C^\infty(\mathbb{R};\mathbb{R}), \quad \chi(\eta) = 0 \text{ for } |\eta| \leq \frac{\pi}{T}, \quad \chi(\eta) = 1 \text{ for } |\eta| \geq \frac{2\pi}{T}.$$

We then define

$$m : \mathbb{R}\times\mathbb{R}^n \to \mathbb{C}, \quad m(\eta,\xi) := \frac{\chi(\eta)}{-i\eta+|\xi|^2}. \tag{5.4}$$

We let $H := \mathbb{R}\times\mathbb{R}^n$ and consider H to be a locally compact group in the canonical way. The dual group $\widehat{H}$ can then also be identified with $\mathbb{R}\times\mathbb{R}^n$ and we can thus consider m as mapping $m : \widehat{H} \to \mathbb{C}$. In order to employ Theorem 5.1, we define $\Phi : \widehat{G} \to \widehat{H}$, $\Phi(k,\xi) := (k,\xi)$. Clearly, Φ is a continuous homomorphism. Moreover, $M = m \circ \Phi$. Consequently, if we can show that m is a continuous $L^p(H)$-multiplier, we may conclude from Theorem 5.1 that M is an $L^p(G)$-multiplier. Since the only zero of the denominator $-i\eta+|\xi|^2$ in definition (5.4) of m is $(\eta,\xi)=(0,0)$, and since the numerator $\chi(\eta)$ in (5.4) vanishes in a neighborhood of $(0,0)$, we see that m is continuous; in fact m is smooth. We shall now apply Marcinkiewicz's multiplier theorem, see for example [8, Corollary 5.2.5] or [10, Chapter IV, §6], to show that m is an $L^p(\mathbb{R}\times\mathbb{R}^n)$-multiplier. Note that classical harmonic analysis can be employed at this point since m is a Fourier multiplier in the Euclidean $\mathbb{R}\times\mathbb{R}^n$ setting. We must verify that

$$\sup_{\epsilon\in\{0,1\}^{n+1}}\ \sup_{(\eta,\xi)\in\mathbb{R}\times\mathbb{R}^n}\left|\xi_1^{\epsilon_1}\cdots\xi_n^{\epsilon_n}\eta^{\epsilon_{n+1}}\partial_1^{\epsilon_1}\cdots\partial_n^{\epsilon_n}\partial_\eta^{\epsilon_{n+1}}m(\eta,\xi)\right|\leq c. \tag{5.5}$$

Since m is smooth, (5.5) follows if we can show that all functions of type

$$(\eta,\xi)\to\xi_1^{\epsilon_1}\cdots\xi_n^{\epsilon_n}\eta^{\epsilon_{n+1}}\partial_1^{\epsilon_1}\cdots\partial_n^{\epsilon_n}\partial_\eta^{\epsilon_{n+1}}m(\eta,\xi)$$

stay bounded as $|(\eta,\xi)|\to\infty$. Since m is a rational function with non-vanishing denominator away from $(0,0)$, this is easy to verify. Consequently, we conclude (5.5) and by Marcinkiewicz's multiplier theorem that m is an $L^p(H)$-multiplier. As mentioned above, it follows that M is an $L^p(G)$-multiplier and thus the estimate $\|u\|_p\leq c\|f\|_p$ holds. Note that the neighbourhood in which m is vanishing becomes small as $T\to\infty$, and hence also the corresponding bound in (5.5) grows for large periods T. We can repeat the argument above for $\partial_t u$ and $\partial_x^\alpha u$, $|\alpha|\leq 2$ and obtain the estimates $\|\partial_t u\|_p\leq c(n,p,T)\|f\|_p$ and $\|\partial_x^\alpha u\|_p\leq c(n,p,T)\|f\|_p$. We have thus shown (5.2). It is clear from the representation formula (5.3) that $\mathcal{P}_\perp u=u$, whence we have $u\in\mathcal{P}_\perp W^{1,2,p}(G)$.

It remains to show uniqueness. Assume that $v\in\mathcal{P}_\perp\mathscr{S}'(G)$ is another solution. Applying the Fourier transform $\mathscr{F}_G$, it then follows $(-ik+|\xi|^2)\mathscr{F}_G\big[u-v\big]=0$ and thus $\operatorname{supp}\mathscr{F}_G\big[u-v\big]\subset\{(0,0)\}$. However, since $0=\mathscr{F}_G\big[\mathcal{P}(u-v)\big]=\delta_{\mathbb{Z}}\cdot\mathscr{F}_G\big[u-v\big]$, we must have $(0,0)\notin\operatorname{supp}\mathscr{F}_G\big[u-v\big]$. We conclude $\operatorname{supp}\mathscr{F}_G\big[u-v\big]=\emptyset$ and consequently $u=v$. □

5.2 The Half Space

In this section we consider the half-space case $\mathbb{T}\times\mathbb{R}^n_+$.

Lemma 5.3. *Let $p\in(1,\infty)$. For any $f\in\mathcal{P}_\perp L^p(\mathbb{T}\times\mathbb{R}^n_+)$ there exists a unique solution $u\in\mathcal{P}_\perp W^{1,2,p}(\mathbb{T}\times\mathbb{R}^n_+)$ to*

$$\begin{cases}\partial_t u-\Delta u=f & \text{in } \mathbb{T}\times\mathbb{R}^n_+\,,\\ \qquad\quad u=0 & \text{on } \mathbb{T}\times\partial\mathbb{R}^n_+\,,\end{cases} \tag{5.6}$$

and there is a constant $c=c(n,p,T)>0$ such that the following estimate holds:

$$\|u\|_{1,2,p}\leq c\|f\|_p\,. \tag{5.7}$$

If additionally $f\in\mathcal{P}_\perp L^s(\mathbb{T}\times\mathbb{R}^n_+)$ for some $s\in(1,\infty)$, then also $u\in\mathcal{P}_\perp W^{1,2,s}(\mathbb{T}\times\mathbb{R}^n_+)$.

Proof. The existence of the solution $u \in \mathcal{P}_\perp W^{1,2,p}(\mathbb{T}\times\mathbb{R}^n_+)$ to (5.6) satisfying (5.7) follows easily from the reflection principle in combination with Lemma 5.2. For the uniqueness, let $u \in \mathcal{P}_\perp W^{1,2,p}(\mathbb{T}\times\mathbb{R}^n_+)$ be a solution to (5.6) with data $f = 0$ and let $h \in \mathcal{P}_\perp L^{p'}(\mathbb{T}\times\mathbb{R}^n_+)$. Then by the above we find $v \in \mathcal{P}_\perp W^{1,2,p'}(\mathbb{T}\times\mathbb{R}^n_+)$ such that $\partial_t v - \Delta v = h$ and $v|_{\mathbb{T}\times\partial\mathbb{R}^n_+} = 0$. Defining $w(t,x) := v(-t,x)$, we conclude

$$\int_{\mathbb{T}}\int_{\mathbb{R}^n_+} uh \,\mathrm{d}x\,\mathrm{d}t = -\int_{\mathbb{T}}\int_{\mathbb{R}^n_+} u(\partial_t w + \Delta w)\,\mathrm{d}x\,\mathrm{d}t = \int_{\mathbb{T}}\int_{\mathbb{R}^n_+} (\partial_t u - \Delta u)w \,\mathrm{d}x\,\mathrm{d}t = 0.$$

Since h is arbitrary, it follows $u = 0$. The regularity assertion follows by the reflection principle and the uniqueness in $\mathcal{P}_\perp\mathscr{S}'(G)$ stated in Lemma 5.2. □

5.3 The Bent Half Space

In this section we consider bent half spaces $\mathbb{T}\times\mathbb{R}^n_\omega$, where $\mathbb{R}^n_\omega$ is merely a small perturbation of the half space $\mathbb{R}^n_+$. More specifically, we consider a Lipschitz continuous function $\omega : \mathbb{R}^{n-1} \to \mathbb{R}$, close to the zero function in a certain sense, and define $\mathbb{R}^n_\omega := \{(x', x_n) \in \mathbb{R}^n \mid x_n > \omega(x')\}$.

We introduce the transformation $\phi_\omega : \mathbb{R}^n_\omega \to \mathbb{R}^n_+$ via $\phi_\omega(x) := \tilde{x} := (x', x_n - \omega(x'))$. For a function u defined on $\mathbb{T}\times\mathbb{R}^n_\omega$ we set $\Phi[u](t,\tilde{x}) := \tilde{u}(t,\tilde{x}) := u(t, \phi_\omega^{-1}(\tilde{x}))$, where $(t,\tilde{x}) \in \mathbb{T}\times\mathbb{R}^n_+$. It should be understood that $\|u\|_p$ and $\|u\|_{1,2,p}$ denote norms over the space-time domain $\mathbb{T}\times\mathbb{R}^n_\omega$, while $\|\tilde{u}\|_p$ and $\|\tilde{u}\|_{1,2,p}$ denote norms over the space-time domain $\mathbb{T}\times\mathbb{R}^n_+$.

Proposition 5.4. *Let $p \in (1,\infty)$ and assume that $\omega \in C^{0,1}(\mathbb{R}^{n-1}) \cap W^{2,1}_{\mathrm{loc}}(\mathbb{R}^{n-1})$ is such that $\|\nabla'\omega\|_\infty, \|\nabla'^2\omega\|_\infty < \infty$. Then $\Phi : \mathcal{P}_\perp L^p(\mathbb{T}\times\mathbb{R}^n_\omega) \to \mathcal{P}_\perp L^p(\mathbb{T}\times\mathbb{R}^n_+)$ and $\Phi : \mathcal{P}_\perp W^{1,2,p}(\mathbb{T}\times\mathbb{R}^n_\omega) \to \mathcal{P}_\perp W^{1,2,p}(\mathbb{T}\times\mathbb{R}^n_+)$ yield homeomorphisms.*

Proof. It is readily seen that $\phi_\omega : \mathbb{R}^n_\omega \to \mathbb{R}^n_+$ is a bijection with Jacobian equal to 1. We denote by $\tilde{\partial}_i$, $\tilde{\nabla}$ the corresponding differential operators with respect to the variable $\tilde{x} \in \mathbb{R}^n_+$. Formally setting $\partial_n\omega = 0$, we see

$$\begin{aligned} \partial_i u(t,x) &= (\tilde{\partial}_i - (\partial_i\omega)\tilde{\partial}_n)\tilde{u}(t,\tilde{x}), \\ \partial_i\partial_j u(t,x) &= \big[\tilde{\partial}_i\tilde{\partial}_j - (\partial_i\omega)\tilde{\partial}_j\tilde{\partial}_n - (\partial_j\omega)\tilde{\partial}_i\tilde{\partial}_n - (\partial_i\partial_j\omega)\tilde{\partial}_n + (\partial_i\omega)(\partial_j\omega)\tilde{\partial}_n^2\big]\tilde{u}(t,\tilde{x}). \end{aligned} \tag{5.8}$$

Hence, there is $C = C(n) > 0$ such that

$$\begin{aligned} \|u\|_p &= \|\tilde{u}\|_p\,, \\ \|\nabla u\|_p &\le C\big(1 + \|\nabla'\omega\|_\infty\big)\|\tilde{\nabla}\tilde{u}\|_p\,, \\ \|\nabla^2 u\|_p &\le C\big[\big(1 + \|\nabla'\omega\|_\infty\big)^2\|\tilde{\nabla}^2\tilde{u}\|_p + \|\nabla'^2\omega\|_\infty\|\tilde{\partial}_n\tilde{u}\|_p\big], \end{aligned} \tag{5.9}$$

which proves the claim. □

Lemma 5.5. *Let* $p \in (1,\infty)$ *and* $\omega \in C^{0,1}(\mathbb{R}^{n-1}) \cap W^{2,1}_{\mathrm{loc}}(\mathbb{R}^{n-1})$*. Then there is a constant* $K = K(n,p) > 0$ *with the following property: If* $\|\nabla'\omega\|_\infty, \|\nabla'^2\omega\|_\infty < K$*, then for any* $f \in \mathcal{P}_\perp L^p(\mathbb{T}\times\mathbb{R}^n_\omega)$ *there exists a unique solution* $u \in \mathcal{P}_\perp W^{1,2,p}(\mathbb{T}\times\mathbb{R}^n_\omega)$ *to*

$$\begin{cases} \partial_t u - \Delta u = f & \text{in } \mathbb{T}\times\mathbb{R}^n_\omega\,, \\ u = 0 & \text{on } \mathbb{T}\times\partial\mathbb{R}^n_\omega\,, \end{cases} \tag{5.10}$$

and there is a constant $c = c(n,p,\omega,T) > 0$ *such that the following estimate holds:*

$$\|u\|_{1,2,p} \le c\|f\|_p. \tag{5.11}$$

If additionally $\|\nabla'\omega\|_\infty, \|\nabla'^2\omega\|_\infty \le \min\{K(n,p),K(n,s)\}$ *and* $f \in \mathcal{P}_\perp L^s(\mathbb{T}\times\mathbb{R}^n_\omega)$ *for some* $s \in (1,\infty)$*, then* $u \in \mathcal{P}_\perp W^{1,2,s}(\mathbb{T}\times\mathbb{R}^n_\omega)$*.*

Proof. Observe that $\partial_t - \tilde\Delta : X^p_\perp(\mathbb{T}\times\mathbb{R}^n_+) \to \mathcal{P}_\perp L^p(\mathbb{T}\times\mathbb{R}^n_+)$ is an isomorphism due to Lemma 5.3. In virtue of (5.8) we obtain

$$(\partial_t - \Delta)u\circ\phi_\omega^{-1} = (\partial_t - \tilde\Delta + \tilde R)\tilde u, \tag{5.12}$$

where $\tilde R : X^p_\perp(\mathbb{T}\times\mathbb{R}^n_+) \to \mathcal{P}_\perp L^p(\mathbb{T}\times\mathbb{R}^n_+)$ is given by

$$\tilde R\tilde u := -|\nabla'\omega|^2\tilde\partial_n^2\tilde u + 2(\nabla'\omega,0)\cdot\tilde\nabla\tilde\partial_n\tilde u + (\Delta'\omega)\tilde\partial_n\tilde u.$$

We have

$$\|\tilde R\tilde u\|_p \le 4K(1+K)\|\tilde u\|_{1,2,p} \le 4K(1+K)\|\|(\partial_t-\tilde\Delta)^{-1}\|\|\|(\partial_t-\tilde\Delta)\tilde u\|_p.$$

Thus for sufficiently small $K>0$, $\partial_t - \tilde\Delta + \tilde R$ is an isomorphism and so is $\partial_t - \Delta$ in virtue of (5.12) and Proposition 5.4. In particular, we have the *a priori* estimate (5.11). The regularity assertion follows if we consider intersection spaces, *e.g.* $X^p_\perp \cap X^s_\perp$ instead of $X^p_\perp$. □

5.4 Bounded Domains

The key lemma for bounded domains $\Omega \subset \mathbb{R}^n$ with a boundary of class $C^{1,1}$ reads as follows.

Lemma 5.6. *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded domain with boundary of class* $C^{1,1}$ *and let* $p \in (1,\infty)$*. The operator* $\partial_t - \Delta : X^p_\perp(\mathbb{T}\times\Omega) \to \mathcal{P}_\perp L^p(\mathbb{T}\times\Omega)$ *is injective and has a dense range. Moreover, there exists a constant* $c = c(n,p,\Omega,T) > 0$ *such that for all* $u \in X^p_\perp(\mathbb{T}\times\Omega)$ *we have the estimate*

$$\|u\|_{1,2,p} \le c\big(\|(\partial_t-\Delta)u\|_p + \|u\|_p\big). \tag{5.13}$$

Proof. Let us consider for $k \in \frac{2\pi}{T}\mathbb{Z}\setminus\{0\}$ the Helmholtz equation

$$\begin{cases} ikv - \Delta v = h & \text{in } \Omega\,, \\ v = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{5.14}$$

Standard elliptic theory yields for every $h \in L^p(\Omega)$ a unique solution $v \in W^{2,p}(\Omega)$ to (5.14). If $u \in X^p_\perp(\mathbb{T}\times\Omega)$ satisfies $(\partial_t - \Delta)u = 0$, then $\mathscr{F}_\mathbb{T} u(k,\cdot) \in W^{2,p}(\Omega)$ solves (5.14) with a homogeneous right-hand side, where $\mathscr{F}_\mathbb{T}$ denotes the Fourier transform on the torus. Consequently $\mathscr{F}_\mathbb{T} u(k,\cdot) = 0$. Since $k \in \frac{2\pi}{T}\mathbb{Z}\setminus\{0\}$ was arbitrary and $\mathscr{F}_\mathbb{T} u(0,\cdot) = 0$ by the assumption $\mathcal{P}u = 0$, it follows $u = 0$. Therefore, $\partial_t - \Delta$ is injective.

Next we show that the range of $\partial_t - \Delta$ is dense in $\mathcal{P}_\perp L^p(\mathbb{T}\times\Omega)$. Since trigonometric polynomials with coefficients in E are dense in $L^p(\mathbb{T};E)$ for any Banach space E, see for example [2, Theorem 4.2.19], it suffices to find $u \in X^p_\perp(\mathbb{T}\times\Omega)$ satisfying $(\partial_t - \Delta)u = e^{ikt}h$ for $h \in L^p(\Omega)$. Solving (5.14), we directly find that such u is given by $u := e^{ikt}v$.

Finally let us prove (5.13). The proof follows a well-known localization method. We choose finitely many balls $B_j \subset \mathbb{R}^n$, $j \in \{1,\dots,m\}$ covering Ω, where (after a possible rotation and translation, which we will suppress in the following) each $j \in \{1\dots,m\}$ is of one of the two types:

- type $\mathbb{R}^n$: if $\overline{B}_j \subset \Omega$,
- type $\mathbb{R}^n_{\omega_j}$: if $\overline{B}_j \cap \partial\Omega \neq \emptyset$.

In the latter case, $\omega_j : \mathbb{R}^{n-1} \to \mathbb{R}$ denote Lipschitz functions with $\overline{B}_j \cap \partial\Omega \subset \operatorname{graph}\omega_j$ in the respective local coordinates. Note that if we choose the balls sufficiently small, the functions ω_j meet the regularity and smallness assumption in Lemma 5.5 due to the boundary regularity of Ω. Moreover, we choose corresponding smooth cut-off functions $\psi_j \in C_0^\infty(\mathbb{R}^n)$ satisfying $\operatorname{supp}\psi_j \subset B_j$ and $\sum_{j=1}^m \psi_j = 1$ in Ω.

Let us write $f = (\partial_t - \Delta)u$. We obtain for $j \in \{1,\dots,m\}$

$$\partial_t(\psi_j u) - \Delta(\psi_j u) = f_j, \tag{5.15}$$

where $f_j := \psi_j f - 2(\nabla\psi_j)\nabla u - (\Delta\psi_j)u$. Depending on whether $j \in \{1,\dots,m\}$ is of type $\mathbb{R}^n$ or $\mathbb{R}^n_{\omega_j}$, we interpret (5.15) as a problem in $\mathbb{T}\times\mathbb{R}^n$ or $\mathbb{T}\times\mathbb{R}^n_{\omega_j}$, and obtain

$$\|\psi_j u\|_{1,2,p} \leq \|f_j\|_p \leq C(\psi_j)(\|f\|_p + \|\nabla u\|_p + \|u\|_p)$$

in virtue of Lemma 5.2 and 5.5, respectively. Summing up the finitely many inequalities obtained for $j \in \{1,\dots,m\}$ and absorbing the term $\|\nabla u\|_p$ on the right-hand side by a standard interpolation argument, we conclude the estimate (5.13). □

Next we show that the last term in (5.13) can be omitted.

Lemma 5.7. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary of class $C^{1,1}$ and let $p \in (1,\infty)$. There exists a constant $c = c(n,p,\Omega,T) > 0$ such that for all $u \in X^p_\perp(\mathbb{T}\times\Omega)$ the following estimate holds:*

$$\|u\|_{1,2,p} \leq c\|(\partial_t - \Delta)u\|_p\,. \tag{5.16}$$

Proof. We will show (5.16) by a contradiction argument. If (5.16) does not hold, then we find a sequence $(u_m)_{m\in\mathbb{N}} \subset X^p_\perp(\mathbb{T}\times\Omega)$ such that $\|u_m\|_{1,2,p} = 1$ for all $m \in \mathbb{N}$ and

$\|(\partial_t - \Delta)u_m\|_p \to 0$ as $m \to \infty$. Suppressing the notion of subsequences, we thus have the weak convergence $u_m \rightharpoonup u$ in $X^p_\perp(\mathbb{T}\times\Omega)$, and u solves

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{in } \mathbb{T}\times\Omega\,, \\ u = 0 & \text{on } \mathbb{T}\times\partial\Omega\,. \end{cases}$$

By Lemma 5.6 we conclude $u = 0$. Since the domain Ω is bounded, the embedding $W^{1,2,p}(\mathbb{T}\times\Omega) \hookrightarrow W^{1,1,p}(\mathbb{T}\times\Omega) \hookrightarrow\hookrightarrow L^p(\mathbb{T}\times\Omega)$ is compact, whence $\|u_m\|_p \to 0$ as $m \to \infty$. This yields the contradiction

$$1 = \lim_{m\to\infty} \|u_m\|_{1,2,p} \leq \lim_{m\to\infty} c\big(\|(\partial_t - \Delta)u_m\|_p + \|u_m\|_p\big) = 0.$$

Therefore, estimate (5.16) has to hold. □

Lemma 5.8. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary of class $C^{1,1}$ and let $p \in (1,\infty)$. For any $f \in \mathcal{P}_\perp L^p(\mathbb{T}\times\Omega)$ there exists a unique solution $u \in \mathcal{P}_\perp W^{1,2,p}(\mathbb{T}\times\Omega)$ to*

$$\begin{cases} \partial_t u - \Delta u = f & \text{in } \mathbb{T}\times\Omega\,, \\ u = 0 & \text{on } \mathbb{T}\times\partial\Omega\,, \end{cases} \tag{5.17}$$

and there is a constant $c = c(n,p,\Omega,T) > 0$ such that the following estimate holds:

$$\|u\|_{1,2,p} \leq c\|f\|_p\,. \tag{5.18}$$

If additionally $f \in \mathcal{P}_\perp L^s(\mathbb{T}\times\Omega)$ for some $s \in (1,\infty)$, then also $u \in \mathcal{P}_\perp W^{1,2,s}(\mathbb{T}\times\Omega)$.

Proof. The operator $\partial_t - \Delta : X^p_\perp(\mathbb{T}\times\Omega) \to \mathcal{P}_\perp L^p(\mathbb{T}\times\Omega)$ is injective and has a dense range by Lemma 5.6. By Lemma 5.7, the range is also closed. Hence, $\partial_t - \Delta$ is an isomorphism, which gives the unique solvability of (5.17) as well as the estimate (5.18). The regularity assertion follows immediately from the unique solvability of (5.17) in $W^{1,2,\min\{s,p\}}(\mathbb{T}\times\Omega)$. □

5.5 Proof of the main theorem

Proof of Theorem 2.1. Existence of a solution $u_s \in \dot{W}^{2,p}(\Omega)$ to

$$\begin{cases} -\Delta u_s = \mathcal{P}f & \text{in } \Omega, \\ u_s = 0 & \text{on } \partial\Omega \end{cases} \tag{5.19}$$

that satisfies (2.2) is well-known from standard theory on elliptic partial differential equations. Via the canonical quotient map, the spaces $C^\infty_{0,\mathrm{per}}(\mathbb{R}\times\Omega)$ and $C^\infty_0(\mathbb{T}\times\Omega)$ are isometrically isomorphic in the norms $\|\cdot\|_p$ and $\|\cdot\|_{1,2,p}$. By construction, also the Sobolev spaces $W^{1,2,p}_{\mathrm{per}}(\mathbb{R}\times\Omega)$ and $W^{1,2,p}(\mathbb{T}\times\Omega)$ are isometrically isomorphic. Hence

Lemma 5.2 in the case $\Omega = \mathbb{R}^n$, Lemma 5.3 in the case $\Omega = \mathbb{R}^n_+$, and Lemma 5.8 in the case of a bounded domain, provide a solution $u_p \in \mathcal{P}_\perp W^{1,2,p}_{\mathrm{per}}(\mathbb{R} \times \Omega)$ to

$$\begin{cases} \partial_t u_p - \Delta u_p = \mathcal{P}_\perp f & \text{in } \mathbb{R} \times \Omega, \\ u_p = 0 & \text{on } \mathbb{R} \times \partial\Omega, \\ u_p(t+T, x) = u_p(t,x) \end{cases} \tag{5.20}$$

that satisfies (2.3). Setting $u := u_s + u_p$, we thus obtain the desired solution to (1.1). Assume $v = v_s + v_p$ is another solution to (1.1) with $v_s \in \dot{W}^{2,r_1}(\Omega)$ and $v_p \in \mathcal{P}_\perp W^{1,2,r_2}_{\mathrm{per}}\big(\mathbb{R} \times \Omega\big)$. Since u_p and v_p both solve (5.20), the uniqueness statements of Lemma 5.2, Lemma 5.3 or Lemma 5.8 yield $u_p = v_p$. Similarly, both u_s and v_s solve (5.19), whence $u_s - v_s$ is a polynomial of order 1 by standard elliptic theory. □

6 Fundamental Solution

The introduction of distributions on the group G enables us to define in a natural manner a fundamental solution to (1.3) as a solution $\varGamma \in \mathscr{S}'(G)$ to

$$\partial_t \varGamma - \Delta \varGamma = \delta_G, \tag{6.1}$$

where δ_G denotes the Dirac delta distribution on G. Employing the Fourier transform $\mathscr{F}_G$ and the decomposing as in (1.5), we immediately obtain the expression

$$\varGamma = 1_{\mathbb{T}} \otimes \varGamma^{\mathrm{L}} + \varGamma^\perp, \tag{6.2}$$

where $1_{\mathbb{T}}$ is the constant function 1 on the torus $\mathbb{T}$, $\varGamma^{\mathrm{L}}$ denotes the well-known fundamental solution to the Laplace equation

$$\varGamma^{\mathrm{L}} : \mathbb{R}^n \setminus \{0\} \to \mathbb{C}, \quad \varGamma^{\mathrm{L}} := \begin{cases} -\dfrac{1}{2\pi} \log|x| & (n = 2), \\ \dfrac{1}{(n-2)\omega_n} |x|^{2-n} & (n > 2) \end{cases}$$

and

$$\varGamma^\perp \in \mathscr{S}'(G), \quad \varGamma^\perp := \mathscr{F}_G^{-1}\left[\frac{\big(1 - \delta_{\mathbb{Z}}(k)\big)}{-ik + |\xi|^2}\right]. \tag{6.3}$$

At the outset, $\varGamma^\perp$ is defined as a tempered distribution. The following theorem asserts that $\varGamma^\perp$ has a realization which can be estimated pointwise.

Theorem 6.1. *The distribution $\varGamma^\perp \in \mathscr{S}'(G)$ defined in (6.3) satisfies for $r \in [2, \infty)$:*

$$\|\varGamma^\perp(\cdot, x)\|_{L^r(\mathbb{T})} \leq c|x|^{-(n-2+\frac{2(r-1)}{r})} \,\mathrm{e}^{-\frac{1}{2}\sqrt{\frac{\pi}{T}}|x|}. \tag{6.4}$$

Moreover

$$\sup_{t \in \mathbb{T}} |\varGamma^\perp(t,x)| \leq c|x|^{-n} \,\mathrm{e}^{-\frac{1}{2}\sqrt{\frac{\pi}{T}}|x|}. \tag{6.5}$$

Proof. The Fourier transform

$$\mathscr{F}_{\mathbb{T}}\big[\Gamma^{\perp}\big] = \mathscr{F}^{-1}_{\mathbb{R}^n}\left[\frac{\big(1-\delta_{\mathbb{Z}}(k)\big)}{-ik+|\xi|^2}\right] \in \mathscr{S}'\Big(\frac{2\pi}{T}\mathbb{Z}\times\mathbb{R}^n\Big)$$

is the function $\mathscr{F}_{\mathbb{T}}\big[\Gamma^{\perp}\big] : \frac{2\pi}{T}\mathbb{Z}\times\big(\mathbb{R}^n\setminus\{0\}\big)\to\mathbb{C}$ given by

$$\mathscr{F}_{\mathbb{T}}\big[\Gamma^{\perp}\big](k,x) = \big(1-\delta_{\mathbb{Z}}(k)\big)\,\frac{i}{4}\left(\frac{\sqrt{-ik}}{2\pi|x|}\right)^{\frac{n-2}{2}} H^{(1)}_{\frac{n}{2}-1}\Big(\sqrt{-ik}\cdot|x|\Big), \tag{6.6}$$

where $H^{(1)}_{\nu}$ denotes the Hankel function of the first kind, and $\sqrt{z}$ the square root of z with *positive* imaginary part. One may verify the identity (6.6) either directly or by recalling that for all $k\in\frac{2\pi}{T}\mathbb{Z}\setminus\{0\}$ both expressions are fundamental solutions in $\mathscr{S}'(\mathbb{R}^n)$ to the Helmholtz equation. It is well known that $H^{(1)}_{\nu}(z):\mathbb{C}\setminus\{0\}\to\mathbb{C}$ is analytic and behaves asymptotically as $H^{(1)}_{\nu}(z)\sim|z|^{-\frac{1}{2}}\,\mathrm{e}^{-\operatorname{Im}z}$ as $|z|\to\infty$; see for example [1, 9.2.3]. Provided $\nu\in\mathbb{R}_+$, one has $H^{(1)}_{\nu}(z)\sim|z|^{-\nu}$ as $|z|\to 0$; see [1, 9.1.9]. Combining (6.6) with the asymptotic behaviour of $H^{(1)}_{\nu}$ in both 0 and ∞, we let $r'=\frac{r}{r-1}$ and estimate in the case $n>2$:

$$\begin{aligned}
\|\mathscr{F}_{\mathbb{T}}\big[\Gamma^{\perp}\big](\cdot,x)\|^{r'}_{\ell^{r'}(\frac{2\pi}{T}\mathbb{Z})} &\leq c\sum_{k\in\frac{2\pi}{T}\mathbb{Z}\setminus\{0\}}\left(|k|^{\frac{n-2}{4}}|x|^{\frac{2-n}{2}}\Big|H^{(1)}_{\frac{n}{2}-1}\big(\sqrt{-ik}\cdot|x|\big)\Big|\right)^{r'}\\
&\leq c\left(\sum_{0<|k|<\frac{1}{2}|x|^{-2}}|x|^{r'(2-n)}+\sum_{\frac{1}{2}|x|^{-2}\leq|k|}|k|^{\frac{n-3}{4}r'}|x|^{\frac{1-n}{2}r'}\,\mathrm{e}^{-\frac{r'}{\sqrt{2}}|k|^{\frac{1}{2}}|x|}\right)\\
&\leq c\left(|x|^{r'(2-n)-2}\chi_{\left[0,\sqrt{\frac{T}{4\pi}}\right)}(|x|)+\sum_{0<|k|}|k|^{\frac{n-3}{4}}|x|^{\frac{1-n}{2}r'}\,\mathrm{e}^{-\frac{r'}{\sqrt{2}}|k|^{\frac{1}{2}}|x|}\chi_{\left(\sqrt{\frac{T}{4\pi}},\infty\right)}(|x|)\right).
\end{aligned}$$

An elementary computation yields $\sum_{j=1}^{\infty} j^{\frac{n-3}{4}}q^{j^{\frac{1}{2}}}\leq cq$ for $0\leq q\leq\alpha<1$, which employed above implies

$$\begin{aligned}
&\|\mathscr{F}_{\mathbb{T}}\big[\Gamma^{\perp}\big](\cdot,x)\|^{r'}_{\ell^{r'}(\frac{2\pi}{T}\mathbb{Z})}\\
&\qquad\leq c\left(|x|^{r'(2-n)-2}\chi_{\left[0,\sqrt{\frac{T}{4\pi}}\right)}(|x|)+|x|^{\frac{1-n}{2}r'}\,\mathrm{e}^{-r'\sqrt{\frac{\pi}{T}}|x|}\,\chi_{\left(\sqrt{\frac{T}{4\pi}},\infty\right)}(|x|)\right)\\
&\qquad\leq c|x|^{r'(2-n)-2}\,\mathrm{e}^{-r'\frac{1}{2}\sqrt{\frac{\pi}{T}}|x|}.
\end{aligned}$$

By Hausdorff-Young's inequality, we thus obtain in the case $n>2$:

$$\|\Gamma^{\perp}(\cdot,x)\|_{L^r(\mathbb{T})}\leq c\|\mathscr{F}_{\mathbb{T}}\big[\Gamma^{\perp}\big](\cdot,x)\|_{\ell^{r'}(\frac{2\pi}{T}\mathbb{Z})}\leq c|x|^{-(n-2+\frac{2(r-1)}{r})}\,\mathrm{e}^{-\frac{1}{2}\sqrt{\frac{\pi}{T}}|x|}.$$

To obtain a similar estimate in the case $n = 2$, we first recall that $H_0^{(1)}(z) \sim \log|z|$ as $|z| \to 0$ (see [1, 9.1.8]) and estimate:

$$
\begin{aligned}
\sum_{0<|k|<\frac{1}{2}|x|^{-2}} & \left(|k|^{\frac{n-2}{4}}|x|^{\frac{2-n}{2}}\left|H_0^{(1)}\big(\sqrt{-ik}\cdot|x|\big)\right|\right)^{r'} \\
&\leq c \sum_{0<|k|<\frac{1}{2}|x|^{-2}} \left|\log\big(|k|^{\frac{1}{2}}|x|\big)\right|^{r'} \\
&\leq c \int_0^{\frac{T}{4\pi}|x|^{-2}} \left|\log\big(t^{\frac{1}{2}}|x|\big)\right|^{r'} \mathrm{d}t \cdot \chi_{\left[0,\sqrt{\frac{T}{4\pi}}\right)}(|x|) \\
&\leq c|x|^{-2} \int_0^{\sqrt{\frac{T}{4\pi}}} |\log s|^{r'} s\,\mathrm{d}s \cdot \chi_{\left[0,\sqrt{\frac{T}{4\pi}}\right)}(|x|) \leq c|x|^{-2} \cdot \chi_{\left[0,\sqrt{\frac{T}{4\pi}}\right)}(|x|).
\end{aligned}
$$

We can now proceed just as in the case $n > 2$ and conclude (6.1) also in the case $n = 2$. The pointwise estimate (6.5) follows by the same argument with $r = \infty$ and $r' = 1$. □

Remark 6.2. As an immediate, but non-trivial, consequence of Theorem 6.1 and the structure (6.2) of the fundamental solution $\varGamma$, we can identify the leading term in the asymptotic expansion as $|x| \to \infty$ of a solution to (1.1) in the case $\Omega = \mathbb{R}^n$. If f is sufficiently regular, say $f \in C^\infty_{0,\mathrm{per}}(\mathbb{R} \times \mathbb{R}^n)$, a solution u to (1.1) that satisfies $\lim_{|x|\to\infty} u(t,x) = 0$ coincides with the convolution of $\varGamma$ with f:

$$
\begin{aligned}
u = \varGamma * f &= (1_{\mathbb{T}} \otimes \varGamma^{\mathrm{L}}) * f + \varGamma^{\perp} * f \\
&= \varGamma^{\mathrm{L}} *_{\mathbb{R}^n} \big(1_{\mathbb{T}} *_{\mathbb{T}} f\big) + \varGamma^{\perp} * f.
\end{aligned}
$$

An asymptotic expansion of the convolution of $\varGamma^{\mathrm{L}}$ with $1_{\mathbb{T}} *_{\mathbb{T}} f \in C_0^\infty(\mathbb{R}^n)$ is well-known:

$$
\varGamma^{\mathrm{L}} *_{\mathbb{R}^n} \big(1_{\mathbb{T}} *_{\mathbb{T}} f\big)(x) = \varGamma^{\mathrm{L}}(x) \cdot \int_{\mathbb{R}^n} \big(1_{\mathbb{T}} *_{\mathbb{T}} f\big)(y)\,\mathrm{d}y + O(|x|^{1-n}). \tag{6.7}
$$

From (6.4) we deduce for sufficiently large values of $|x|$ that

$$
|\varGamma^{\perp} * f(t,x)| \leq \int_{\mathbb{R}^n} \|\varGamma^{\perp}(\cdot, x-y)\|_{L^2(\mathbb{T})} \|f(\cdot,y)\|_{L^2(\mathbb{T})}\,\mathrm{d}y \leq c|x|^{-(n-1)}\,\mathrm{e}^{-\frac{1}{4}\sqrt{\frac{\pi}{T}}|x|}
$$

and observe that the asymptotic expansion of $\varGamma^{\perp} * f$ adds only terms of higher order to (6.7). More precisely, we find that

$$
u = \varGamma * f = \varGamma^{\mathrm{L}}(x) \cdot \left(\int_{\mathbb{R}^n} \frac{1}{T} \int_0^T f(t,y)\,\mathrm{d}t\mathrm{d}y \right) + O(|x|^{1-n}),
$$

which means that a solution to the time-periodic heat equation has the *same* behaviour at spatial infinity as a solution to the stationary Poisson equation. We emphasize that this insight follows naturally from our method, but is by no means clear in the more traditional approaches to time-periodic equations.

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