The universal algebra generated by a power partial isometry

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Dedicated to Albrecht Böttcher on the occassion of his 60th birthday.

Abstract

A power partial isometry (PPI) is an element v of a C^*-algebra with the property that every power v^n is a partial isometry. The goal of this paper is to identify the universal C^*-algebra generated by a PPI with (a slight modification of) the algebra of the finite sections method for Toeplitz operators with continuous generating function, as first described by Albrecht Böttcher and Bernd Silbermann in [1].

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1 Introduction

Let $\mathcal{A}$ be a C^*-algebra. An element V of $\mathcal{A}$ is called a partial isometry if $vv^*v = v$. Simple examples show that a power of a partial isometry needs not to be a partial isometry again. One therefore calls v a *power partial isometry* (PPI) if every power of v is a partial isometry again.

Examples. (a) In a C^*-algebra with identity element e, every unitary element u (i.e. $u^*u = uu^* = e$) is a PPI. In particular, the function $u : t \mapsto t$ is a unitary element of the algebra $C(\mathbb{T})$ of the continuous functions on the complex unit circle $\mathbb{T}$, and the operator U of multiplication by the function u is a unitary operator on the Hilbert space $L^2(\mathbb{T})$ of the squared integrable functions on $\mathbb{T}$.
(b) In a C^*-algebra with identity element e, every isometry v (i.e. $v^*v = e$) and every co-isometry v (i.e. $vv^* = e$) is a PPI. In particular, the operators $V : (x_0, x_1, \ldots) \mapsto (0, x_0, x_1, \ldots)$ and $V^* : (x_0, x_1, x_2, \ldots) \mapsto (x_1, x_2, \ldots)$ of forward and backward shift, respectively, are PPIs on the Hilbert space $l^2(\mathbb{Z}^+)$ of the squared summable sequences on the non-negative integers.
(c) The matrix $V_n := (a_{ij})$ with $a_{i+1,i} = 1$ and $a_{ij} = 0$ if $i \neq j + 1$, considered as

an element of the algebra $\mathbb{C}^{n\times n}$ of the complex $n \times n$ matrices, is a PPI.

(*d*) If v_i is a PPI in a C^*-algebra $\mathcal{A}_i$ for every i in an index set I, then $(v_i)_{i\in I}$ is a PPI in the direct product $\prod_{i\in I} \mathcal{A}_i$. In particular, the operator (V, V^*), considered as an element of $L(l^2(\mathbb{Z}^+)) \times L(l^2(\mathbb{Z}^+))$, is a PPI.

Note that the PPI V_n in (*c*) and (V, V^*) in (*d*) are neither isometric nor co-isometric. ■

The goal of the present paper is to describe the universal C^*-algebra generated by a PPI. Recall that a C^*-algebra $\mathcal{A}$ generated by a PPI v is *universal* if, for every other C^*-algebra $\mathcal{B}$ generated by a PPI w, there is a *-homomorphism from $\mathcal{A}$ to $\mathcal{B}$ which sends v to w. The universal algebra generated by a unitary resp. isometric element is defined in an analogous way. The existence of a universal algebra generated by a PPI is basically a consequence of Example (*d*).

It follows from the Gelfand-Naimark theorem that the universal algebra generated by a unitary element is *-isomorphic to the algebra $C(\mathbb{T})$, generated by the unitary function u. Coburn [5] identified the universal algebra generated by an isometry as the Toeplitz algebra $\mathsf{T}(C)$ which is the smallest C^*-subalgebra of $L(l^2(\mathbb{Z}^+))$ which contains the isometry V, the shift operator. This algebra bears its name since it can be described as the smallest C^*-subalgebra of $L(l^2(\mathbb{Z}^+))$ which contains all Toeplitz operators $T(a)$ with generating function $a \in C(\mathbb{T})$. Recall that the Toeplitz operator with generating function $a \in L^1(\mathbb{T})$ is given the matrix $(a_{i-j})_{i,j=0}^{\infty}$ where a_k stands for the kth Fourier coefficient of a. This operator is bounded on $l^2(\mathbb{Z}^+)$ if and only if $a \in L^\infty(\mathbb{T})$ (see [2, 3]).

We will see that the universal algebra of a PPI is also related with Toeplitz operators, via the finite sections discretization with respect to the sequence of the projections $P_n : (x_0, x_1, \ldots) \mapsto (x_0, \ldots, x_{n-1}, 0, 0, \ldots)$ on $l^2(\mathbb{Z}^+)$. Write $\mathcal{F}$ for the set of all bounded sequences $(A_n)_{n\geq 1}$ of operators $A_n \in L(\operatorname{im} P_n)$ and $\mathcal{G}$ for the set of all sequences $(A_n) \in \mathcal{F}$ with $\|A_n\| \to 0$. Provided with entry-wise defined operations and the supremum norm, $\mathcal{F}$ becomes a C^*-algebra and $\mathcal{G}$ a closed ideal of $\mathcal{F}$. Since $L(\operatorname{im} P_n)$ is isomorphic to $\mathbb{C}^{n\times n}$, we can identify $\mathcal{F}$ with the direct product and $\mathcal{G}$ with the direct sum of the algebras $\mathbb{C}^{n\times n}$ for $n \geq 1$. Now consider the smallest C^*-subalgebra $\mathcal{S}(\mathsf{T}(C))$ of $\mathcal{F}$ which contains all sequences $(P_nT(a)P_n)$ with $a \in C(\mathbb{T})$ and its C^*-subalgebra $\mathcal{S}_{\geq 2}(\mathsf{T}(C))$ which is generated by the sequence (P_nVP_n) (note that V is the Toeplitz operator with generating function $t \mapsto t$). With these notations, the main result of the present paper can be formulated as follows.

Theorem 1 *The universal algebra generated by a PPI is *-isomorphic to the C^*-algebra $\mathcal{S}_{\geq 2}(\mathsf{T}(C))$ generated by the PPI (P_nVP_n).*

For a general account on C^*-algebras generated by partial isometries, with special emphasis on their relation to graph theory, see [4].

Before going into the details of the proof of Theorem 1, we provide some basic (and well known) facts on the algebras $\mathcal{S}(\mathsf{T}(C))$ and $\mathcal{S}_{\geq 2}(\mathsf{T}(C))$. Since the first entry of the sequence $(P_n V P_n)$ is zero, the first entry of every sequence in $\mathcal{S}_{\geq 2}(\mathsf{T}(C))$ is zero. So we can omit the first entry and consider the elements of $\mathcal{S}_{\geq 2}(\mathsf{T}(C))$ as sequences labeled by $n \geq 2$ (whence the notation). In fact this is the only difference between the algebras $\mathcal{S}(\mathsf{T}(C))$ and $\mathcal{S}_{\geq 2}(\mathsf{T}(C))$.

Proposition 2 *$\mathcal{S}_{\geq 2}(\mathsf{T}(C))$ consists of all sequences $(A_n)_{n\geq 2}$ where $(A_n)_{n\geq 1}$ is a sequence in $\mathcal{S}(\mathsf{T}(C))$.*

The sequences in $\mathcal{S}(\mathsf{T}(C))$ are completely described in the following theorem, where we let R_n denote the operator $(x_0, x_1, \ldots) \mapsto (x_{n-1}, \ldots, x_0, 0, 0, \ldots)$ on $l^2(\mathbb{Z}^+)$. Further we set $\tilde{a}(t) := a(t^{-1})$ for every function a on $\mathbb{T}$. This description was found by A. Böttcher and B. Silbermann and first published in their 1983 paper [1] on the convergence of the finite sections method for quarter plane Toeplitz operators (see also [6], Section 1.4.2).

Proposition 3 *The algebra $\mathcal{S}(\mathsf{T}(C))$ consists of all sequences $(A_n)_{n\geq 1}$ of the form*

$$(A_n) = (P_n T(a) P_n + P_n K P_n + R_n L R_n + G_n) \tag{1}$$

where $a \in C(\mathbb{T})$, K and L are compact operators, and $(G_n) \in \mathcal{G}$. The representation of a sequence $(A_n) \in \mathcal{S}(\mathsf{T}(C))$ in this form is unique.

Corollary 4 *$\mathcal{G}$ is a closed ideal of $\mathcal{S}(\mathsf{T}(C))$, and the quotient algebra $\mathcal{S}(\mathsf{T}(C))/\mathcal{G}$ is *-isomorphic to the C^*-algebra of all pairs*

$$(T(a) + K,\, T(\tilde{a}) + L) \in L(l^2(\mathbb{Z}^+)) \times L(l^2(\mathbb{Z}^+)) \tag{2}$$

with $a \in C(\mathbb{T})$ and K, L compact. In particular, the mapping which sends the sequence (1) *to the pair* (2) *is a *-homomorphism from $\mathcal{S}(\mathsf{T}(C))$ onto $\mathcal{S}(\mathsf{T}(C))/\mathcal{G}$ with kernel $\mathcal{G}$.*

It is not hard to see that the algebra of all pairs (2) is just the smallest C^*-subalgebra of $L(l^2(\mathbb{Z}^+)) \times L(l^2(\mathbb{Z}^+))$ that contains the PPI (V, V^*).

Corollary 5 *The set $\mathcal{J}$ of all pairs (K, L) of compact operators K, L is a closed ideal of $\mathcal{S}(\mathsf{T}(C))/\mathcal{G}$. The quotient algebra $(\mathcal{S}(\mathsf{T}(C))/\mathcal{G})/\mathcal{J}$ is *-isomorphic to $C(\mathbb{T})$. In particular, the mapping which sends the pair* (2) *to the function a is a *-homomorphism from $\mathcal{S}(\mathsf{T}(C))/\mathcal{G}$ onto $C(\mathbb{T})$ with kernel $\mathcal{J}$.*

Observe that all of the above examples (a) - (d) appear somewhere in the algebra $\mathcal{S}(\mathsf{T}(C))$ and its quotients.

2 Elementary properties of PPI

Our first goal is a condition ensuring that the product of two partial isometries is a partial isometry again.

Proposition 6 *Let u, v be partial isometries. Then uv is a partial isometry if and only if*

$$u^*uvv^* = vv^*u^*u, \tag{3}$$

i.e. if the initial projection u^*u of u and the range projection vv^* of v commute.

Proof. Condition (3) implies that

$$(uv)(uv)^*(uv) = uvv^*u^*uv = (uu^*u)(vv^*v) = uv;$$

hence, uv is a partial isometry. Conversely, if uv is a partial isometry, then a simple calculation gives

$$v^*(vv^*u^*u - u^*uvv^*)(u^*uvv^* - vv^*u^*u)v = 0.$$

With the C^*-axiom we conclude that $v^*(vv^*u^*u - u^*uvv^*) = 0$, hence $vv^*(vv^*u^*u - u^*uvv^*) = 0$, which finally gives

$$vv^*u^*u = vv^*u^*uvv^*.$$

The right-hand side of this equality is selfadjoint; so must be the left-hand side. Thus, $vv^*u^*u = (vv^*u^*u)^* = u^*uvv^*$, which is condition (3). ■

In particular, if v is a partial isometry, then v^2 is a partial isometry if and only if

$$v^*vvv^* = vv^*v^*v. \tag{4}$$

Proposition 7 *Let v be a partial isometry with property* (4) *(e.g. a PPI). Then*

$$e := v^*v + vv^* - v^*vvv^* = v^*v + vv^* - vv^*v^*v$$

is the identity element of the C^*-algebra generated by v. Moreover,

$$p := vv^* - vv^*v^*v = e - v^*v \quad \text{and} \quad \tilde{p} := v^*v - v^*vvv^* = e - vv^*$$

are mutually orthogonal projections (meaning that $p\tilde{p} = \tilde{p}p = 0$).

Proof. Condition (4) implies that e is selfadjoint. Further,

$$ve = vv^*v + vvv^* - vv^*vvv^* = v + vvv^* - vvv^* = v$$

and, similarly, $v^*e = v^*$. Taking adjoints it follows that $ev^* = v^*$ and $ev = v$, and e is the identity element. The remaining assertions are also easy to check. ■

We will often use the notation v^{*n} instead of $(v^*)^n$.

Proposition 8 *(a) If v is a PPI, then*

$$v^{*k}v^kv^nv^{*n} = v^nv^{*n}v^{*k}v^k \quad \textit{for all } k,\, n \geq 1. \tag{5}$$

(b) If v is a partial isometry and if (5) holds for $k = 1$ and for every $n \geq 1$, then v is a PPI.

Proof. Assertion (a) is a consequence of Proposition 6 (the partial isometry v^{n+k} is the product of the partial isometries v^k and v^n). Assertion (b) follows easily by induction. For $k = 1$, condition (5) reduces to

$$(v^*v)(v^nv^{*n}) = (v^nv^{*n})(v^*v).$$

Thus if v and v^n are partial isometries, then v^{n+1} is a partial isometry by Proposition 6. ∎

Lemma 9 *If v is a PPI, then $(v^nv^{*n})_{n\geq 0}$ and $(v^{*n}v^n)_{n\geq 0}$ are decreasing sequences of pairwise commuting projections.*

Proof. The PPI property implies that the v^nv^{*n} are projections and that

$$v^nv^{*n}n^{n+k}(v^*)^{n+k} = (v^nv^{*n}v^n)v^k(v^*)^{n+k} = v^nv^k(v^*)^{n+k} = n^{n+k}(v^*)^{n+k}$$

for $k,\, n \geq 0$. The assertions for the second sequence follow similarly. ∎

3 A distinguished ideal

Let $\mathcal{A}$ be a C^*-algebra generated by a PPI v. By $\text{alg}\,(v,\, v^*)$ we denote the smallest (symmetric, not necessarily closed) subalgebra of $\mathcal{A}$ which contains v and v^*. Further we write $\mathbb{N}_v$ for the set of all non-negative integers such that $pv^n\tilde{p} \neq 0$. From Proposition 7 we know that $0 \notin \mathbb{N}_v$. Finally, we set

$$\pi_n := pv^n\tilde{p}v^{*n}p \quad \text{and} \quad \tilde{\pi}_n := \tilde{p}v^{*n}pv^n\tilde{p}.$$

Proposition 10 *(a) The element $pv^n\tilde{p}$ is a partial isometry with initial projection $\tilde{\pi}_n$ and range projection π_n. Thus, the projections π_n and $\tilde{\pi}_n$ are Murray-von Neumann equivalent in $\mathcal{A}$, and they generate the same ideal of $\mathcal{A}$.*
(b) $\pi_m\pi_n = 0$ and $\tilde{\pi}_m\tilde{\pi}_n = 0$ whenever $m \neq n$.

Proof. (a) By definition,

$$\pi_n = pv^n\tilde{p}v^{*n}p = pv^n(e - vv^*)v^{*n}p = p(v^nv^{*n} - v^{n+1}(v^*)^{n+1})p.$$

Since $p = e - vv^*$ and v^nv^{*n} commute by Proposition 8,

$$\pi_n = p(v^nv^{*n} - v^{n+1}(v^*)^{n+1}) = (v^nv^{*n} - v^{n+1}(v^*)^{n+1})p.$$

Being a product of commuting projections (Lemma 9), π_n is itself a projection. Analogously, $\tilde{\pi}_n$ is a projection. Thus, $pv^n\tilde{p}$ is a partial isometry, and π_n and $\tilde{\pi}_n$ are Murray-von Neumann equivalent. Finally, the equality

$$\pi_n = \pi_n^2 = (pv^n\tilde{p}v^{*n}p)^2 = pv^n\tilde{\pi}_n v^{*n}p$$

shows that π_n belongs to the ideal generated by $\tilde{\pi}_n$. The reverse inclusion follows analogously. Assertion (b) is again a simple consequence of Lemma 9. ■

Let $\mathcal{C}_n$ denote the smallest closed ideal of $\mathcal{A}$ which contains the projection π_n (likewise, the projection $\tilde{\pi}_n$). We want to show that $\mathcal{C}_n$ is isomorphic to $\mathbb{C}^{(n+1)\times(n+1)}$ whenever $n \in \mathbb{N}_v$ (Proposition 17 below). For we need to establish a couple of facts on (finite) words in $\text{alg}\,(v,\, v^*)$.

Lemma 11 *Let a, b, c be non-negative integers. Then*

$$v^{*a}v^bv^{*c} = \begin{cases} (v^*)^{a-b+c} & \text{if } \min\{a,\, c\} \geq b, \\ v^{b-a}v^{*c} & \text{if } a \leq b \leq c, \\ v^{*a}v^{b-c} & \text{if } a \geq b \geq c \end{cases}$$

and

$$v^av^{*b}v^c = \begin{cases} v^{a-b+c} & \text{if } \min\{a,\, c\} \geq b, \\ v^a(v^*)^{b-c} & \text{if } a \geq b \geq c, \\ (v^*)^{b-a}v^c & \text{if } a \leq b \leq c. \end{cases}$$

Proof. Let $\min\{a,\, c\} \geq b$. Then

$$v^{*a}v^bv^{*c} = (v^*)^{a-b}v^{*b}v^bv^{*b}(v^*)^{c-b} = (v^*)^{a-b}v^{*b}(v^*)^{c-b} = (v^*)^{a-b+c},$$

where we used that v^{*b} is a partial isometry. If $a \leq b \leq c$, then

$$v^{*a}v^bv^{*c} = v^{*a}v^av^{b-a}(v^*)^{b-a}(v^*)^{c-b+a} = v^{b-a}(v^*)^{b-a}v^{*a}v^a(v^*)^{c-b+a}$$

by Proposition 8 (a). Thus,

$$v^{*a}v^bv^{*c} = v^{b-a}(v^*)^{b-a}v^{*a}v^av^{*a}(v^*)^{c-b} = v^{b-a}(v^*)^{b-a}v^{*a}(v^*)^{c-b} = v^{b-a}v^{*c}.$$

Similarly, $v^{*a}v^bv^{*c} = v^{*a}v^{b-c}$ if $a \geq b \geq c$. The second assertion of the lemma follows by taking adjoints. ■

Every word in $\text{alg}\,(v,\, v^*)$ is a product of powers v^n and v^{*m}. Every product $v^av^{*b}v^c$ and $v^{*a}v^bv^{*c}$ of three powers can be written as a product of at most two powers if one of the conditions

$$\min\{a,\, c\} \geq b \quad \text{or} \quad a \leq b \leq c \quad \text{or} \quad a \geq b \geq c \tag{6}$$

in Lemma 11 is satisfied. Since (6) is equivalent to $\max\{a,\, c\} \geq b$, such a product can not be written as a product of less than three powers by means of Lemma 11 if

$\max\{a, c\} < b$. Since it is not possible in a product $v^a v^{*b} v^c v^{*d}$ or $v^{*a} v^b v^{*c} v^d$ of four powers that $\max\{a, c\} < b$ *and* $\max\{b, d\} < c$, one can shorten every product of powers v^n and v^{*m} to a product of at most three powers. Summarizing we get the following lemma.

Lemma 12 *Every finite word in* $\text{alg}\,(v, v^*)$ *is of the form* $v^a v^{*b}$ *or* $v^{*b} v^a$ *with* $a, b \geq 0$ *or of the form* $v^a v^{*b} v^c$ *or* $v^{*a} v^b v^{*c}$ *with* $0 < \min\{a, c\} \leq \max\{a, c\} < b$.

Corollary 13 *Let* w *be a word in* $\text{alg}\,(v, v^*)$.
(a) *If* $pwp \neq 0$, *then* $w = v^a v^{*a}$ *for some* $a \geq 0$.
(b) *If* $\tilde{p} w \tilde{w} \neq 0$, *then* $w = v^{*a} v^a$ *with some* $a \geq 0$.

Proof. We only check assertion (a). By the preceding lemma, w is a product of at most three powers $v^a v^{*b} v^c$ or $v^{*a} v^b v^{*c}$. First let $w = v^a v^{*b} v^c$. Since $vp = pv^* = 0$, we conclude that $c = 0$ if $pwp \neq 0$. Writing

$$pwp = \begin{cases} pv^a v^{*a} (v^*)^{b-a} p = v^a v^{*a} p (v^*)^{b-a} p & \text{if} \quad a \leq b, \\ pv^{a-b} v^b v^{*b} p = pv^{a-b} p v^b v^{*b} & \text{if} \quad a \geq b, \end{cases}$$

we obtain by the same argument that $a = b$ if $pwp \neq 0$. Thus, $w = v^a v^{*a}$. The case when $w = v^{*a} v^b v^{*c}$ can be treated analogously. ∎

An element k of a C^*-algebra $\mathcal{A}$ is called an *element of algebraic rank one* if, for every $a \in \mathcal{A}$, there is a complex number α such that $kak = \alpha k$.

Proposition 14 *Let* $m, n \in \mathbb{N}_v$. *Then*
(a) π_n *is a projection of algebraic rank one in* $\mathcal{A}$.
(b) π_m *and* π_n *are Murray-von Neumann equivalent if and only if* $m = n$.

Analogous assertions hold for $\tilde{\pi}_n$ in place of π_n.

Proof. (a) Every element of $\mathcal{A}$ is a limit of linear combinations of words in v and v^*. It is thus sufficient to show that, for every word w, there is an $\alpha \in \mathbb{C}$ such that $\pi_n w \pi_n = \alpha \pi_n$. If $\pi_n w \pi_n = 0$, this holds with $\alpha = 0$. If $\pi_n w \pi_n = \pi_n pwp \pi_n \neq 0$, then $w = v^a v^{*a}$ for some $a \geq 0$ by Corollary 13. In this case,

$$\pi_n w \pi_n = \pi_n v^a v^{*a} \pi_n = p(v^n v^{*n} - v^{n+1} (v^*)^{n+1}) v^a v^{*a} (v^n v^{*n} - v^{n+1} (v^*)^{n+1}) p.$$

From Lemma 9 we infer that

$$(v^n v^{*n} - v^{n+1} (v^*)^{n+1}) v^a v^{*a} = \begin{cases} v^n v^{*n} - v^{n+1} (v^*)^{n+1} & \text{if} \quad a \leq n, \\ v^a v^{*a} - v^a v^{*a} = 0 & \text{if} \quad a \geq n+1. \end{cases}$$

Thus,

$$\pi_n w \pi_n = \begin{cases} p(v^n v^{*n} - v^{n+1} (v^*)^{n+1})^2 p = \pi_n & \text{if} \quad a \leq n, \\ 0 & \text{if} \quad a \geq n+1, \end{cases}$$

i.e. $\alpha = 1$ if $a \leq n$ and $\alpha = 0$ in all other cases.

(b) The projections π_m and π_n are Murray-von Neumann equivalent if and only if $\pi_m \mathcal{A} \pi_n \neq \{0\}$. So we have to show that $\pi_m \mathcal{A} \pi_n = \{0\}$ whenever $m \neq n$. Again it is sufficient to show that $\pi_m w \pi_n = 0$ for every word w.

Suppose there is a word w such that $\pi_m w \pi_n = \pi_m p w p \pi_n \neq 0$. Then $w = v^a v^{*a}$ for some $a \geq 0$ by Corollary 13. The terms in parentheses in

$$\pi_m w \pi_n = \pi_m v^a v^{*a} \pi_n = p(v^m v^{*m} - v^{m+1}(v^*)^{m+1})(v^a v^{*a})(v^n v^{*n} - v^{n+1}(v^*)^{n+1})p$$

commute by Lemma 9. Since

$$(v^m v^{*m} - v^{m+1}(v^*)^{m+1})(v^n v^{*n} - v^{n+1}(v^*)^{n+1}) = 0$$

for $m \neq n$ we conclude that $\pi_m w \pi_n = 0$, a contradiction. ∎

Lemma 15 (a) *If $a > n$ or $b > a$, then $v^b v^{*a} \pi_n = 0$.*
(b) *If $b \leq a \leq n$, then $v^b v^{*a} \pi_n = (v^*)^{a-b} \pi_n$.*

Proof. (a) One easily checks that $(v^*)^{n+1} p = 0$, which gives the first assertion. Let $b > a$. Then, since p commutes with $v^k v^{*k}$ and $vp = 0$,

$$\begin{aligned} v^b v^{*a} \pi_n &= v^{b-a} v^a v^{*a} (v^n v^{*n} - v^{n+1}(v^*)^{n+1}) p \\ &= v^{b-a} p v^a v^{*a} (v^n v^{*n} - v^{n+1}(v^*)^{n+1}) = 0. \end{aligned}$$

(b) Applying Lemma 11 to the terms in inner parentheses in

$$v^b v^{*a} \pi_n = ((v^b v^{*a} v^n) v^{*n} - (v^b v^{*a} v^{n+1})(v^*)^{n+1}) p,$$

one can simplify this expression to

$$((v^*)^{a-b} v^n v^{*n} - (v^*)^{a-b} v^{n+1})(v^*)^{n+1}) p = (v^*)^{a-b} \pi_n.$$ ∎

Corollary 16 (a) *If w is a word in v, v^*, then $w\pi_n \in \{0, \pi_n, v^*\pi_n, \ldots, v^{*n}\pi_n\}$.*
(b) *For every $w \in \mathcal{A}$, $w\pi_n$ is a linear combination of elements $v^{*i}\pi_n$ with $i \in \{0, 1, \ldots, n\}$.*
(c) *Every element of the ideal $\mathcal{C}_n$ generated by π_n is a linear combination of elements $v^{*i}\pi_n v^j$ with $i, j \in \{0, 1, \ldots n\}$.*

In particular, $\mathcal{C}_n$ is a finite-dimensional C^*-algebra. We are now in a position to describe this algebra exactly.

Proposition 17 (a) *For $n \in \mathbb{N}_v$, the algebra $\mathcal{C}_n$ is *-isomorphic to $\mathbb{C}^{(n+1)\times(n+1)}$.*
(b) *$\mathcal{C}_m \mathcal{C}_n = \{0\}$ whenever $m \neq n$.*

Proof. (a) The elements $e_{ij}^{(n)} := v^{*i}\pi_n v^j$ with $i,\, j \in \{0,\, 1,\, \ldots n\}$ span the algebra $\mathcal{C}_n$ by Corollary 16 (c). Thus, the assertion will follow once we have shown that these elements form a system of $(n+1) \times (n+1)$ matrix units in the sense that $(e_{ij}^{(n)})^* = e_{ji}^{(n)}$ and

$$e_{ij}^{(n)} e_{kl}^{(n)} = \delta_{jk} e_{il}^{(n)} \qquad \text{for all } i,\, j,\, k,\, l \in \{0,\, 1,\, \ldots n\}, \tag{7}$$

with δ_{jk} the standard Kronecker delta. The symmetry property is clear. To check (7), first let $j = k$. Then

$$e_{ij}^{(n)} e_{jl}^{(n)} = v^{*i}\pi_n(v^j v^{*j}\pi_n)v^l = v^{*i}\pi_n^2 v^l = e_{il}^{(n)}$$

by Lemma 15 (b). If $j > k$, then

$$e_{ij}^{(n)} e_{kl}^{(n)} = v^{*i}\pi_n(v^j v^{*k}\pi_n)v^l = 0$$

by Lemma 15 (a). Finally, if $j < k$, then

$$e_{ij}^{(n)} e_{kl}^{(n)} = v^{*i}(\pi_n v^j v^{*k})\pi_n v^l = v^{*i}(v^k v^{*j}\pi_n)^*\pi_n v^l = 0,$$

again by Lemma 15 (a). This proves (a). Assertion (b) follows from Proposition 14 (b). ∎

Given a PPI v, we let $\mathcal{G}_v$ stand for the smallest closed ideal which contains all projections π_n. If $\mathbb{N}_v$ is empty, then $\mathcal{G}_v$ is the zero ideal. Let $\mathbb{N}_v \neq \emptyset$. The ideal generated by a projection π_n with $n \in \mathbb{N}_v$ is isomorphic to $\mathbb{C}^{(n+1)\times(n+1)}$ by Proposition 17, and if $u,\, w$ are elements of $\mathcal{A}$ which belong to ideals generated by two different projections π_m and π_n, then $uw = 0$ by Proposition 14 (b). Hence, $\mathcal{G}_v$ is then isomorphic to the direct sum of all matrix algebras $\mathbb{C}^{(n+1)\times(n+1)}$ with $n \in \mathbb{N}_v$.

If $\mathcal{A}$ is the *universal* C^*-algebra generated by a PPI v, then $\mathbb{N}_v$ is the set of *all* positive integers. Indeed, the algebra $\mathcal{S}_{\geq 2}(\mathsf{T}(C))$ introduced in the introduction is generated by the PPI $v := (P_n V P_n)$, and $\mathbb{N}_v = \mathbb{N}$ in this concrete setting.

Corollary 18 *If $\mathcal{A}$ is the universal C^*-algebra generated by a PPI v, then $\mathbb{N}_v = \mathbb{N}$, and $\mathcal{G}_v$ is isomorphic to the ideal $\mathcal{G}_{\geq 2} := \mathcal{S}_{\geq 2}(\mathsf{T}(C)) \cap \mathcal{G}$.*

4 PPI with $\mathbb{N}_v = \emptyset$

Our next goal is to describe the C^*-algebra $\mathcal{A}$ which is generated by a PPI v with $\mathbb{N}_v = \emptyset$. This condition is evidently satisfied if one of the projections $p = e - v^*v$ and $\tilde{p} = e - vv^*$ is zero, in which cases the algebra generated by the PPI v is well known:

- If $p = 0$ and $\tilde{p} = 0$, then v is unitary, and $\mathcal{A}$ is *-isomorphic to $C(X)$ where $X \subseteq \mathbb{T}$ is the spectrum of v by the Gelfand-Naimark theorem.
- If $p = 0$ and $\tilde{p} \neq 0$, then v is a non-unitary isometry, $\mathcal{A}$ is *-isomorphic to the Toeplitz algebra $\mathsf{T}(C)$ by Coburn's theorem, and the isomorphism sends v to the forward shift V.
- If $p \neq 0$ and $\tilde{p} = 0$, then v is a non-unitary co-isometry, again $\mathcal{A}$ is *-isomorphic to the Toeplitz algebra $\mathsf{T}(C)$ by Coburn's theorem, and the isomorphism sends v to the backward shift V^*.

Thus the only interesting case is when $\mathbb{N}_v = \emptyset$, but $p \neq 0$ and $\tilde{p} \neq 0$. Let $\mathcal{C}$ and $\widetilde{\mathcal{C}}$ denote the smallest closed ideals of $\mathcal{A}$ which contain the projections p and $\tilde{p}$, respectively. For $i, j \geq 0$, set

$$f_{ij} := v^{*i} p v^j \quad \text{and} \quad \tilde{f}_{ij} := v^i \tilde{p} v^{*j}.$$

Lemma 19 *If v is a PPI with $\mathbb{N}_v = \emptyset$, then $(f_{ij})_{i,j \geq 0}$ is a (countable) system of matrix units, i.e. $f_{ij}^* = f_{ji}$ and*

$$f_{ij} f_{kl} = \delta_{jk} f_{il} \qquad \text{for all } i, j, k, l \geq 0. \tag{8}$$

If one of the f_{ij} is non-zero (e.g. if $f_{00} = p \neq 0$), then all f_{ij} are non-zero.

An analogous assertion holds for the family of the $\tilde{f}_{ij}$.

Proof. The symmetry condition is evident, and if $f_{ij} = 0$ then $f_{kl} = f_{ki} f_{ij} f_{jl} = 0$ for all k, l by (8). Property (8) on its hand will follow once we have shown that

$$p v^j v^{*k} p = \delta_{jk} p \qquad \text{for all } j, k \geq 0. \tag{9}$$

The assertion is evident if $j = k = 0$. If $j > 0$ and $k = 0$, then

$$\begin{aligned} p v^j p &= (e - v^* v) v^j (e - v^* v) \\ &= v^j - v^* v^{j+1} - v^{j-1}(v v^* v) + v^* v^j (v v^* v) \\ &= v^j - v^* v^{j+1} - v^{j-1} v + v^* v^j v = 0, \end{aligned}$$

and (9) holds. Analogously, (9) holds if $j = 0$ and $k > 0$. Finally, let $j, k > 0$. The assumption $\mathbb{N}_v = \emptyset$ ensures that

$$p v^{j-1} \tilde{p} = (e - v^* v) v^{j-1} (e - v v^*) = v^{j-1} - v^* v^j - v^j v^* + v^* v^{j+1} v^* = 0 \tag{10}$$

for all $j \geq 1$. Employing this identity we find

$$\begin{aligned} p v^j v^{*k} p &= (e - v^* v)\, v^j v^{*k} p \;=\; v^j v^{*k} p - (v^* v^{j+1} v^*)\,(v^*)^{k-1} p \\ &= v^j v^{*k} p - (v^{j-1} - v^* v^j - v^j v^*)(v^*)^{k-1} p \\ &= (e - v^* v) v^{j-1} (v^*)^{k-1} p. \end{aligned}$$

Thus, $p v^j v^{*k} p = p v^{j-1} (v^*)^{k-1} p$ for $j, k \geq 1$. Repeated application of this identity finally leads to one of the cases considered before. ■

Proposition 20 *Let* $\mathbb{N}_v = \emptyset$ *and* $p \neq 0$.

(a) *The ideal* $\mathcal{C}$ *of* $\mathcal{A}$ *generated by* p *coincides with the smallest closed subalgebra of* $\mathcal{A}$ *which contains all* f_{ij} *with* $i, j \geq 0$.

(b) $\mathcal{C}$ *is* **-isomorphic to the ideal of the compact operators on a separable infinite-dimensional Hilbert space.*

Analogous assertions hold for the projection $\tilde{p}$*, the algebra* $\widetilde{\mathcal{C}}$*, and the* $\tilde{f}_{ij}$.

Proof. For a moment, write $\mathcal{C}'$ for the smallest closed subalgebra of $\mathcal{A}$ which contains all f_{ij} with $i, j \geq 0$. The identities

$$f_{ij}v = v^{*i}pv^jv = f_{i,j+1}, \qquad vf_{0j} = vpv^j = v(e - v^*v)v^j = 0$$

and, for $i \geq 1$,

$$\begin{aligned} vf_{ij} &= vv^{*i}(e - v^*v)v^j \;=\; vv^{*i}v^j - (v(v^*)^{i+1}v)v^j \\ &= vv^{*i}v^j + ((v^*)^{i-1} - (v^*)^iv - vv^{*i})v^j \\ &= (v^*)^{i-1}v^j - (v^*)^ivv^j \;=\; (v^*)^{i-1}(e - v^*v)v^j \;=\; f_{i-1,j} \end{aligned}$$

(where we used the adjoint of (10)) and their adjoints show that $\mathcal{C}'$ is a closed ideal of $\mathcal{A}$. Since $p = f_{00}$ we conclude that $\mathcal{C} \subseteq \mathcal{C}'$. Conversely, we have $f_{ij} = v^{*i}pv^j \in \mathcal{C}$ for all $i, j \geq 0$ whence the reverse inclusion $\mathcal{C}' \subseteq \mathcal{C}$. This settles assertion (a).

For assertion (b), note that every C^*-algebra generated by a (countable) system of matrix units (in particular, the algebra $\mathcal{C}'$) is naturally *-isomorphic to the algebra of the compact operators on a separable infinite-dimensional Hilbert space (see, e.g., Corollary A.9 in Appendix A2 in [7]). ■

Lemma 21 *If* $\mathbb{N}_v = \emptyset$*, then* $\mathcal{C} \cap \widetilde{\mathcal{C}} = \{0\}$.

Proof. $\mathcal{C}$ and $\widetilde{\mathcal{C}}$ are closed ideals. Thus, $\mathcal{C} \cap \widetilde{\mathcal{C}} = \mathcal{C}\widetilde{\mathcal{C}}$, and we have to show that $f_{ij}\tilde{f}_{kl} = 0$ for all $i, j, k, l \geq 0$. Since

$$f_{ij}\tilde{f}_{kl} = (v^{*i}pv^j)\,(v^k\tilde{p}v^{*l}) = v^{*i}\,(pv^{j+k}\tilde{p})\,v^{*l},$$

this is a consequence of $\mathbb{N}_v = \emptyset$. ■

Remember that $p \neq 0$ and $\tilde{p} \neq 0$. From the preceding lemma we conclude that the mapping

$$\mathcal{A} \to \mathcal{A}/\mathcal{C} \times \mathcal{A}/\widetilde{\mathcal{C}}, \quad w \mapsto (w + \mathcal{C},\, w + \widetilde{\mathcal{C}})$$

is an injective *-homomorphism; thus $\mathcal{A}$ is *-isomorphic to the C^*-subalgebra of $\mathcal{A}/\mathcal{C} \times \mathcal{A}/\widetilde{\mathcal{C}}$ generated by $(v + \mathcal{C},\, v + \widetilde{\mathcal{C}})$. The element $v + \mathcal{C}$ is an isometry in $\mathcal{A}/\mathcal{C}$ (since $e - v^*v \in \mathcal{C}$), but it is not unitary (otherwise $e - vv^* \in \tilde{\mathcal{C}}$ would be a non-zero element of $\mathcal{C}$, in contradiction with Lemma 21). Analogously, $v + \widetilde{\mathcal{C}}$ is a non-unitary co-isometry. By Coburn's Theorem, there are *-isomorphisms

$\mu : \mathcal{A}/\mathcal{C} \to \mathsf{T}(C)$ and $\tilde{\mu} : \mathcal{A}/\tilde{\mathcal{C}} \to \mathsf{T}(C)$ which map $v + \mathcal{C} \mapsto V$ and $v + \tilde{\mathcal{C}} \mapsto V^*$, respectively. But then

$$\mu \times \tilde{\mu} : \mathcal{A}/\mathcal{C} \times \mathcal{A}/\widetilde{\mathcal{C}} \to \mathsf{T}(C) \times \mathsf{T}(C), \quad (a,\, \tilde{a}) \mapsto (\mu(a),\, \tilde{\mu}(\tilde{a}))$$

is a *-isomorphism which maps the C^*-subalgebra of $\mathcal{A}/\mathcal{C} \times \mathcal{A}/\widetilde{\mathcal{C}}$ generated by $(v+\mathcal{C},\, v+\widetilde{\mathcal{C}})$ to the C^*-subalgebra of $\mathsf{T}(C) \times \mathsf{T}(C)$ generated by the pair $(V,\, V^*)$. The latter algebra has been identified in Corollary 4. Summarizing we get:

Proposition 22 *Let the C^*-algebra $\mathcal{A}$ be generated by a PPI v with $\mathbb{N}_v = \emptyset$ and $p \neq 0$ and $\tilde{p} \neq 0$. Then $\mathcal{A}$ is *-isomorphic to the algebra $\mathcal{S}(\mathsf{T}(C))/\mathcal{G}$ (likewise, to $\mathcal{S}_{\geq 2}(\mathsf{T}(C))/\mathcal{G}_{\geq 2}$), and the isomorphism sends v to $(P_n V P_N)_{n \geq 1} + \mathcal{G}$ (likewise, to $(P_n V P_N)_{n \geq 2} + \mathcal{G}_{\geq 2}$).*

5 The general case

We are now going to finish the proof of Theorem 1. For we think of $\mathcal{A}$ as being faithfully represented as a C^*-algebra of bounded linear operators on a separable infinite-dimensional Hilbert space H (note that $\mathcal{A}$ is finitely generated, hence separable). As follows easily from (7), $z_n := \sum_{i=0}^n e_{ii}^{(n)}$ is the identity element of $\mathcal{C}_n$. So we can think of the z_n as orthogonal projections on H. Moreover, these projections are pairwise orthogonal by Proposition 17 (b). Thus, the operators $P_n := \sum_{i=1}^n z_n$ form an increasing sequence of orthogonal projections on H. Let $P \in L(H)$ denote the least upper bound of that sequence (which then is the limit of the P_n in the strong operator topology). Clearly, P is an orthogonal projection again (but note that P does not belong to $\mathcal{A}$ in general).

Lemma 23 (a) *Every z_n is a central projection of $\mathcal{A}$.*
(b) *P commutes with every element of $\mathcal{A}$.*

Proof. Assertion (b) is a consequence of (a). We show that

$$\begin{aligned} z_n &= \sum_{i=0}^n v^{*i} \pi_n v^i = \sum_{i=0}^n v^{*i} p (v^n v^{*n} - v^{n+1} (v^*)^{n+1}) v^i \\ &= \sum_{i=0}^n v^{*i} (e - v^* v)(v^n v^{*n} - v^{n+1} (v^*)^{n+1}) v^i \end{aligned}$$

commutes with v. Indeed,

$$\begin{aligned} v z_n &= v(e - v^* v)(v^n v^{*n} - v^{n+1} (v^*)^{n+1}) \\ &\quad + \sum_{i=1}^n v v^{*i} (e - v^* v)(v^n v^{*n} - v^{n+1} (v^*)^{n+1}) v^i \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} vv^{*i}(e - v^*v)v^n(v^{*n} - v(v^*)^{n+1})v^i \\
&= \sum_{i=1}^{n} (vv^{*i}v^n - v(v^*)^{i+1}v^{n+1})(v^{*n} - v(v^*)^{n+1})v^i \\
&= \sum_{i=1}^{n} ((v^*)^{i-1}v^n - (v^*)^i v^{n+1})(v^{*n} - v(v^*)^{n+1})v^i \quad \text{(by Lemma 11)} \\
&= \sum_{i=1}^{n} (v^*)^{i-1}(e - v^*v)(v^n v^{*n} - v^{n+1}(v^*)^{n+1})v^i \\
&= \sum_{i=0}^{n-1} v^{*i}(e - v^*v)(v^n v^{*n} - v^{n+1}(v^*)^{n+1})v^i v \\
&= \sum_{i=0}^{n} v^{*i}(e - v^*v)(v^n v^{*n} - v^{n+1}(v^*)^{n+1})v^i v \\
&\qquad -(v^*)^n(e - v^*v)(v^n v^{*n} - v^{n+1}(v^*)^{n+1})v^{n+1} \\
&= \sum_{i=0}^{n} v^{*i}(e - v^*v)(v^n v^{*n} - v^{n+1}(v^*)^{n+1})v^i v \; = \; z_n v
\end{aligned}
$$

again by Lemma 11. Thus, $vz_n = z_n v$. Since $z_n = z_n^*$, this implies that z_n also commutes with v^* and, hence, with every element of $\mathcal{A}$. ■

Consequently, $\mathcal{A} = P\mathcal{A}P \oplus (I - P)\mathcal{A}(I - P)$ where I stands for the identity operator on H. We consider the summands of this decomposition separately. The part $(I - P)\mathcal{A}(I - P)$ is generated by the PPI $v' := (I - P)v(I - P)$. Since

$$
\begin{aligned}
(I - P)pv^n\tilde{p}v^{*n}p(I - P) &= (I - P)\pi_n(I - P) \\
&= (I - P)z_n e_{00}^{(n)}\pi_n(I - P) = 0,
\end{aligned}
$$

we conclude that $\mathbb{N}_{v'} = \emptyset$. Thus, this part of $\mathcal{A}$ is described by Proposition 22.

The part $P\mathcal{A}P$ is generated by the PPI PvP. It follows from the definition of P that $\mathbb{N}_{PvP} = \mathbb{N}_v$ and that $\mathcal{G}_{PvP} = P\mathcal{G}_v P = \mathcal{G}_v$. We let $\prod_{n\in\mathbb{N}_v} \mathcal{C}_n$ stand for the direct product of the algebras $\mathcal{C}_n$ and consider the mapping

$$
P\mathcal{A}P \to \prod_{n\in\mathbb{N}_v} \mathcal{C}_n, \quad PAP \mapsto (z_n PAP z_n)_{n\in\mathbb{N}_v} = (z_n A z_n)_{n\in\mathbb{N}_v}. \tag{11}
$$

If $z_n A z_n = 0$ for every $n \in \mathbb{N}_v$, then

$$
PAP = \sum_{m,n\in\mathbb{N}_v} z_m PAP z_n = \sum_{n\in\mathbb{N}_v} z_n A z_n = 0.
$$

Thus, the mapping (11) is injective, and the algebra $P\mathcal{A}P$ is *-isomorphic to the C^*-subalgebra of $\prod_{n\in\mathbb{N}_v} \mathcal{C}_n$ generated by the sequence $(z_n v z_n)_{n\in\mathbb{N}_v}$. Further we infer from Proposition 17 (a) that $\mathcal{C}_n$ is isomorphic to $\mathbb{C}^{(n+1)\times(n+1)}$ if $n \in \mathbb{N}_v$. We are going to make the latter isomorphism explicit. For we note that

$$\begin{aligned} e_{ii}^{(n)} v e_{jj}^{(n)} &= v^{*i}\pi_n v^{i+1} v^{*j} \pi_n v^j \\ &= \begin{cases} v^{*i}\pi_n v^{i+1}(v^*)^{i+1}\pi_n v^{i+1} & \text{if} \quad i+1 = j, \\ 0 & \text{if} \quad i+1 \neq j \end{cases} \quad \text{(by Corollary 13)} \\ &= \begin{cases} v^{*i}\pi_n v^{i+1} & \text{if} \quad i+1 = j, \\ 0 & \text{if} \quad i+1 \neq j \end{cases} \quad \text{(by Lemma 15)} \\ &= \begin{cases} e_{i,i+1}^{(n)} & \text{if} \quad i+1 = j, \\ 0 & \text{if} \quad i+1 \neq j. \end{cases} \end{aligned}$$

We choose a unit vector $e_i^{(n)}$ in the range of $e_{ii}^{(n)}$ (recall Proposition 14 (a)), and let $f_i^{(n)}$ stand for the $n+1$-tuple $(0, \ldots, 0,\ 1,\ 0,\ \ldots,\ 0)$ with the 1 at the ith position. Then $(e_i^{(n)})_{i=0}^n$ forms an orthonormal basis of $\operatorname{im} z_n$, $(f_i^{(n)})_{i=0}^n$ forms an orthonormal basis of $\mathbb{C}^{n+1}$, the mapping $e_i^{(n)} \mapsto f_{n-i}^{(n)}$ extends to a linear bijection from $\operatorname{im} z_n$ onto $\mathbb{C}^{n+1}$, which finally induces a *-isomorphism ξ_n from $\mathcal{C}_n \cong L(\operatorname{im} z_n)$ onto $\mathbb{C}^{(n+1)\times(n+1)} \cong L(\mathbb{C}^{n+1})$. Then

$$\xi : \prod_{n\in\mathbb{N}_v} \mathcal{C}_n \to \prod_{n\in\mathbb{N}_v} \mathbb{C}^{(n+1)\times(n+1)}, \quad (A_n) \mapsto (\xi_n(A_n))$$

is a *-isomorphism which maps the C^*-subalgebra of $\prod_{n\in\mathbb{N}_v} \mathcal{C}_n$ generated by the sequence $(z_n v z_n)_{n\in\mathbb{N}_v}$ to the C^*-subalgebra of $\prod_{n\in\mathbb{N}_v} \mathbb{C}^{(n+1)\times(n+1)}$ generated by the sequence $(V_{n+1})_{n\in\mathbb{N}_v}$, where V_n is the matrix described in Example (c). Note that V_n is just the $n \times n$th finite section $P_n V P_n$ of the forward shift operator.

If now $\mathcal{A}$ is the universal algebra generated by a PPI v, then $\mathbb{N}_v = \mathbb{N}$, as we observed in Corollary 18. Thus, in this case, the algebra $P\mathcal{A}P$ is *-isomorphic to the smallest C^*-subalgebra of $\mathcal{F} = \prod_{n\geq 1} \mathbb{C}^{n\times n}$ generated by the sequence $(P_n V P_n)$, i.e. to the C^*-algebra $\mathcal{S}_{\geq 2}(\mathsf{T}(C))$.

It remains to explain what happens with the part $(I-P)\mathcal{A}(I-P)$ of $\mathcal{A}$. The point is that the quotient $P\mathcal{A}P/P\mathcal{G}_v P$ is generated by a PPI u for which $\mathbb{N}_u$ is empty. We have seen in Proposition 22 that both this quotient and the algebra $(I-P)\mathcal{A}(I-P)$ are canonically *-isomorphic to $\mathcal{S}_{\geq 2}(\mathsf{T}(C))/\mathcal{G}_{\geq 2}$. Thus, there is a *-homomorphism from $P\mathcal{A}P$ onto $(I-P)\mathcal{A}(I-P)$ which maps the generating PPI PvP of $P\mathcal{A}P$ to the generating PPI $(I-P)v(I-P)$ of $(I-P)\mathcal{A}(I-P)$. Hence, if $\mathcal{A}$ is the universal C^*-algebra generated by a PPI, then already $P\mathcal{A}P$ has the universal property, and $\mathcal{A} \cong P\mathcal{A}P \cong \mathcal{S}_{\geq 2}(\mathsf{T}(C))$. ∎

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