Convergence of Geometric Subdivision Schemes

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Abstract

Non-linear $\mathbb{R}^d$-valued curve subdivision has a high potential of generating limit curves sensitive to the geometry of initial points. A natural condition characterizing geometric subdivision schemes is the commutation of the refinement rules with similarities. In this paper, we introduce this class of geometric subdivision schemes and address the question of convergence. We prove that uniform decay of the edge lengths is necessary and uniform summability thereof is sufficient for convergence. For a special subclass the necessary condition is also sufficient and thus fully characterizes convergence.

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1 Introduction

For univariate linear subdivision schemes there exists a general framework concerning convergence and smoothness of the limit, see [8] for a survey. When dealing with geometric subdivision, newly generated points on the finer level depend on points of the coarser level in a non-linear way. Even in a simplified setting regarding only uniform and stationary subdivision (the same non-linear rules are used in each part of the sequence and every refinement step) the question of convergence to a continuous limit curve and smoothness thereof is a complicated one. This is why analysis of such schemes is typically specialized to a concrete algorithm, e.g. see the schemes analyzed in [5, 6, 15, 17, 13], or a certain regularity of the limit is only motivated by numerical experiments, like the κ-scheme in [1]. However, a general framework for manifold-valued subdivision is developed in [9, 16, 11, 12] focusing on adaption of linear schemes to non-linear geometry. For non-linear schemes in a linear space, there are at least two contributions aiming at general results, namely [7] and [10].

In the first one, [7], Dyn and Hormann present necessary and sufficient conditions for convergence based on piecewise linear parameterizations as well as sufficient conditions for continuously varying tangents by a purely geometric approach. During their analysis, they focus on the special class of interpolatory schemes in the plane. In this setting, they show that the sequence of largest edge lengths has to be a null sequence for all convergent schemes, while summability of maximal edge lengths implies convergence.

In the second one, [10], a broad class of subdivision schemes, called GLUE-schemes, is introduced. These schemes have to be geometric (i.e., commute with similarities), local (i.e., new points depend only on a fixed number of old points), uniform (i.e., the same rules are applied everywhere) and equilinear (i.e., linear polygons are mapped to linear polygons with half spacing), which is abbreviated in the acronym GLUE. Further, an analysis for Hölder regularity up to $C^{1,\alpha}$ (and for a subclass even $C^{2,\alpha}$) using a newly introduced quantity named relative distortion is presented. Many schemes belong to this class. For instance [1, 6, 13, 17] and especially [5] fit into this framework. For the last one, an application of the presented theory yields $C^{1,1}$-smoothness for all initial data with relative distortion smaller than 0.08. However, the equilinear condition is quite restrictive. For example, the treatment of corner cutting schemes with arbitrary ratio [4] is not possible except for Chaikin's algorithm [3].

In this paper, we investigate convergence of a broader class than GLUE-schemes and will see that the sequence of maximal edge lengths has to be a null sequence for all convergent schemes, while summability is a sufficient condition. For a special subclass the property of being a null sequence is even sufficient and therefore fully characterizes convergence.

This paper can be seen as a continuation of both, [7] and [10]: On one hand, we take the class of GLUE-schemes and skip equilinearity to obtain a broader set of schemes called GLU-schemes. In particular, dropping equilinearity is a substantial generalization as it allows to treat some uncovered schemes as GLU-schemes, like Example 2.3. On the other hand, we follow the structure of the part of [7] about convergence and prove analogous results in the setting of GLU-schemes. This includes going from the plane to arbitrary space dimensions. Also the restrictive choice of interpolatory rules is replaced by a next general property, namely continuity of one refinement rule at the origin. This allows to choose a whole variety of functions for both refinement rules, what is at the same time the reason why proofs have to be significantly generalized.

The paper is organized as follows: In Section 2 after introducing some notations needed to formulate the setting, the class of GLU-schemes is presented and a non-interpolatory and non-equilinear example is given. The question of convergence is addressed in Section 3 by stating a necessary and proving a sufficient condition while another example shows that in general the necessary one is not enough to guarantee convergence. The Section is concluded by introducing a subclass of GLU-scheme for which a characterization of convergence holds. Some more technical proofs are collected in Section 4 while a conclusion of this paper is given in Section 5.

2 Setup

Because the analysis presented here can be regarded as a generalization of [10], we adapt the setting and repeat the notation used there. Our analysis applies not only to the plane as in [7] but to arbitrary dimensions. For this, let us denote Euclidean d-space by $\mathbb{E} := \mathbb{R}^d$ with space dimension $d \in \mathbb{N}$ fixed. We will investigate subdivision of finite sequences of points in $\mathbb{E}$, called *chains*. To this end, we denote the space of chains with $N \in \mathbb{N}$ points by $\mathbb{E}^N$. When starting with a short initial chain, further subdivision could be undefined due to a collaps of number of points, see Definition 2.1. This results in a need of chains with sufficiently many points, say $\mathbb{E}^{\geq n} := \bigcup_{N \geq n} \mathbb{E}^N$ for the set of chains with at least n points. Chains are always denoted by upper case bold face letters, e.g. $\mathbf{P} \in \mathbb{E}^N$, and the corresponding standard lower case letters, tagged with a subscript, are used for the points, e.g. $p_i \in \mathbb{E}$, where indices always start from 0. Points are understood as row vectors, which will be seperated by semicolons, while columns of points yield chains. The *length* of a chain $\mathbf{P} = [p_0; \ldots; p_{N-1}] \in \mathbb{E}^{\geq n}$ is the number of its points and is denoted by $\#\mathbf{P} := N$. We can extract subchains out of longer chains $\mathbf{P} \in \mathbb{E}^{\geq n}$ by means of truncation operators,

$$T_i^n \mathbf{P} := [p_i; \ldots; p_{i+n-1}] \in \mathbb{E}^n, \quad 0 \leq i \leq \#\mathbf{P} - n.$$

Two more terminologies are needed to formulate the class of subdivision schemes we are interested in.

A *similarity*, denoted by $S = (\varrho, Q, s) : \mathbb{E} \to \mathbb{E}$, is characterized by a scalar *scaling factor* $\varrho > 0$, an orthogonal *transformation matrix* $Q \in \mathbb{R}^{d \times d}$ and a *shift vector* $s \in \mathbb{E}$ acting on points by $S(p) = \varrho p Q + s$. Application of S to a chain $\mathbf{P}$ is understood pointwise, i.e. $p'_i = S(p_i)$ where p'_i is the i-th point of $S(P)$. The group of similarities in $\mathbb{E}$ is denoted by $\mathcal{S}(\mathbb{E})$.

An important chain will be the *constant null chain*

$$\mathbf{n}_N := [n; \ldots; n] \in \mathbb{E}^N$$

formed by repeating the point $n := 0 \in \mathbb{E}$.

Definition 2.1. *Given $m \in \mathbb{N}$, let $n := 2m + 1$. The map $\mathbf{G} : \mathbb{E}^{\geq n} \to \mathbb{E}^{\geq n}$ defines a* **g**eometric, **l**ocal, **u**niform subdivision scheme (or briefly GLU-scheme) in $\mathbb{E}$ *with spread n if $\#\mathbf{G}(\mathbf{P}) = 2\#\mathbf{P} - n + 1$ and if it satisfies the following properties:*

(G) $\mathbf{G}$ *commutes with similarities, i.e,* $\mathbf{G} \circ S = S \circ \mathbf{G}, \quad S \in \mathcal{S}(\mathbb{E})$.

(L) *The points p'_{2i} and p'_{2i+1} of the chain $\mathbf{P}' := \mathbf{G}(\mathbf{P})$ depend only on finitely many points, namely $p_i, \ldots, p_{i+m}$.*

(U) *There exist functions $g_0, g_1 : \mathbb{E}^{m+1} \to \mathbb{E}$ independent of i such that*

$$p'_{2i+\lambda} = g_\lambda(p_i, \ldots, p_{i+m}), \quad \lambda \in \{0, 1\}, \ 0 \leq i < \#\mathbf{P} - m,$$

while one of these functions, say g_0, is continuous in $\mathbf{n}_{m+1}$.

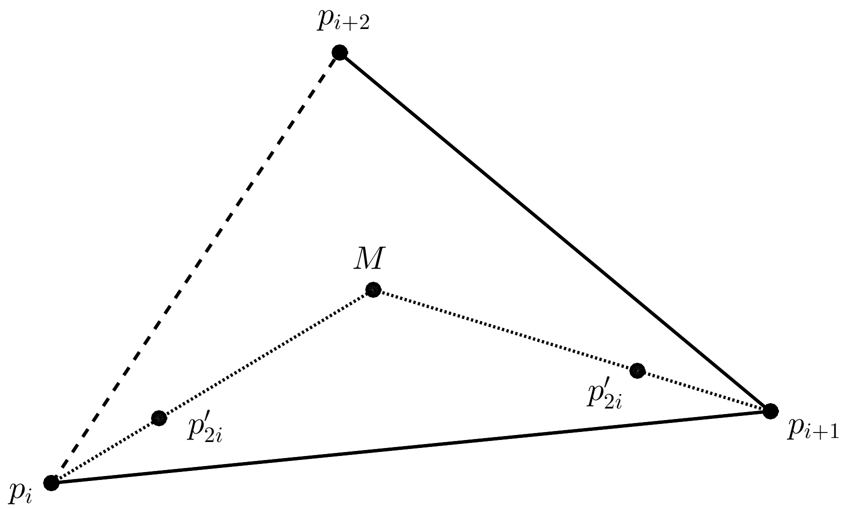

Figure 1: Definition of the scheme 2.3.

Remark 2.2. The following analysis only uses the property of commutation with scaling and translation. Hence, commutation with rotations is not needed and the class of schemes for which the convergence analysis applies is even larger. But for consistency to [10] and because commutation with rotations is anyway a natural property for geometric subdivision, we stay with this definition. We also note that continuity in $\mathbf{n}_{m+1}$ is only demanded for g_0. Besides commutation with similarities, no further conditions are imposed on g_1.

The class of GLUE-schemes considered in [10] differs in two aspects:

First, both refinement rules g_0 and g_1 of a GLUE-scheme have to be continuously differentiable in another special chain, while this property is replaced here by continuity only of g_0 in $\mathbf{n}_{m+1}$.

Second, the equilinear condition (E) is obviously dropped, what is a significant generalization. For example, de Rham's scheme with arbitrary cutting ratio [4] or generalized Lane-Riesenfeld algorithms[1] [1] fail to meet condition (E). Thus, they are no GLUE-schemes, but GLU-schemes as long as continuity of g_0 in $\mathbf{n}_{m+1}$ is satisfied.

The class of schemes addressed in [7], namely interpolatory planar subdivision, is also a subset of GLU-schemes, if g_1 fulfills condition (G): When choosing $d = 2$ and g_0 to be interpolatory, this function is continuous in $\mathbf{n}_{m+1}$ and commutes with similarities. As long as g_1 commutes, too, these schemes are GLU-schemes. For example, both newly introduced schemes in [7] are GLU-schemes.

[1]Some refinement rules allow to view generalized Lane-Riesenfeld algorithms as GLUE-schemes, but for the whole variety of rules, condition (E) is violated.

Example 2.3. We introduce a 1-parameter family of GLU-schemes that we will analyze in Section 3. Because these schemes use three points p_i, p_{i+1}, p_{i+2} to generate the new ones, it is $m = 2$. Let $M = M(T_i^3\mathbf{p})$ be the incenter of the triangle formed by the three points, see Figure 1. It is given by

$$M = \frac{\|p_{i+2} - p_{i+1}\|p_i + \|p_{i+2} - p_i\|p_{i+1} + \|p_{i+1} - p_i\|p_{i+2}}{\|p_{i+2} - p_{i+1}\| + \|p_{i+2} - p_i\| + \|p_{i+1} - p_i\|},$$

where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{E}$. For a given parameter $\alpha \in [0, 1]$ we define the refinement rules as

$$p'_{2i+\lambda} = g_\lambda(p_i, p_{i+1}, p_{i+2}) := \alpha p_{i+\lambda} + (1 - \alpha)M, \qquad \lambda \in \{0, 1\}.$$

Clearly by construction, this scheme fulfills condition (L). To varify property (G) let $S = (\varrho, Q, s)$ be a similarity and observe using $\|S(q) - S(p)\| = \varrho\|(q - p)Q\| = \varrho\|q - p\|$ for $q, p \in \mathbb{E}$,

$$\begin{aligned} M(S(T_i^3\mathbf{p})) &= \frac{\varrho\Big(\|p_{i+2} - p_{i+1}\|Sp_i + \|p_{i+2} - p_i\|Sp_{i+1} + \|p_{i+1} - p_i\|Sp_{i+2}\Big)}{\varrho\|p_{i+2} - p_{i+1}\| + \varrho\|p_{i+2} - p_i\| + \varrho\|p_{i+1} - p_i\|} \\ &= \varrho\frac{\|p_{i+2} - p_{i+1}\|p_i + \|p_{i+2} - p_i\|p_{i+1} + \|p_{i+1} - p_i\|p_{i+2}}{\|p_{i+2} - p_{i+1}\| + \|p_{i+2} - p_i\| + \|p_{i+1} - p_i\|}Q + s \\ &= S(M(T_i^3\mathbf{p})). \end{aligned}$$

So the refinement rules also commute with similarities,

$$\begin{aligned} g_\lambda(S(p_i), S(p_{i+1}), S(p_{i+2})) &= \alpha S(p_{i+\lambda}) + (1 - \alpha)M(S(T_i^3\mathbf{p})) \\ &= \varrho\Big(\alpha p_{i+\lambda} + (1 - \alpha)M(T_i^3\mathbf{p})\Big)Q + s \\ &= S(g_\lambda(p_i, p_{i+1}, p_{i+2})), \end{aligned}$$

and thus condition (G) holds. The check of condition (U) is performed after the characterization of continuity given in Lemma 2.5 below. To formulate this result some more notation and the reproduction of constant chains is needed.

With the Euclidean norm on $\mathbb{E}$ denoted by $\|\cdot\|$, we can equip the spaces of chains with a norm $|\cdot|_0$ defined by

$$|\mathbf{P}|_0 := \max_{i=0,\ldots,\#\mathbf{P}-1} \|p_i\|$$

for $\mathbf{P} \in \mathbb{E}^{\geq n}$, recalling that indices always starts from 0 and that $\#\mathbf{P}$ is the number of points in $\mathbf{P}$. Based on this, we will also use a semi-norm

$$|\mathbf{P}|_1 := \max_{i=1,\ldots,\#\mathbf{P}-1} \|p_i - p_{i-1}\|$$

on $\mathbf{P} \in \mathbb{E}^{\geq n}$, which is just the maximal edge length occuring in $\mathbf{P}$. Repeated subdivision of the *initial chain* $\mathbf{P} \in \mathbb{E}^{\geq n}$ yields the chains $\mathbf{P}^\ell := \mathbf{G}^\ell(\mathbf{P})$, $\ell \in \mathbb{N}$. Throughout, the points of $\mathbf{P}^\ell$ are denoted by p_i^ℓ.

Lemma 2.4. *A GLU-scheme* $\mathbf{G}$ *reproduces constant chains, i.e. for all* $s \in \mathbb{E}$

$$g_\lambda(s, \ldots, s) = s, \qquad \lambda \in \{0, 1\}.$$

Proof. Property (G) implies commutation of the functions g_0 and g_1 according to $g_\lambda(S(\mathbf{p})) = S(g_\lambda(\mathbf{p}))$, $S \in \mathcal{S}(\mathbb{E})$. The similarity $S := (1/2, \mathrm{Id}, s/2)$ satisfies $S(s) = s$. Hence, $g_\lambda(s, \ldots, s) = S(g_\lambda(s, \ldots, s)) = g_\lambda(s, \ldots, s)/2 + s/2$, and by solving for $g_\lambda(s, \ldots, s)$ showing the reproduction

$$g_\lambda(s, \ldots, s) = s.$$

□

The reason why we demand continuity of g_0 in $\mathbf{n}_{m+1}$ is given in the following Lemma. Essentially, it guarantees that bounded data remain bounded under g_0. Additionally, continuity and Lipschitz continuity coincides for GLU-schemes.

Lemma 2.5. *For a GLU-scheme* $\mathbf{G}$*, the following statements are equivalent:*

i) $g_0 : \mathbb{E}^{m+1} \to \mathbb{E}$ *is continuous in* $\mathbf{n}_{m+1}$.

ii) g_0 *is bounded, i.e. there exists* $C > 0$ *such that* $\|g_0(\mathbf{p})\| \leq C$ *for all* $\mathbf{p} \in B_1$.

iii) g_0 *is Lipschitz continuous in* $\mathbf{n}_{m+1}$*, i.e. there exists* $C > 0$ *such that* $\|g_0(\mathbf{p})\| \leq Cr$ *for all* $r > 0$ *and* $\mathbf{p} \in B_r$.

Here, $B_r := \{\mathbf{p} \in \mathbb{E}^{m+1} : |\mathbf{p}|_0 \leq r\}$ *is the closed* $|\cdot|_0$*-ball in* $\mathbb{E}^{m+1}$ *with radius* r *centered at the origin.*

Proof. i) $\Rightarrow$ *ii)*: By continuity of g_0, there exists $\delta > 0$ with

$$\|g_0(\mathbf{p})\| = \|g_0(\mathbf{p}) - n\| = \|g_0(\mathbf{p}) - g_0(\mathbf{n}_{m+1})\| \leq 1$$

for $\mathbf{p} \in B_\delta$, where we also used the reproduction of constants, $g_0(\mathbf{n}_{m+1}) = n$. Let $\mathbf{q} \in B_1$ and hence $\delta\mathbf{q} \in B_\delta$. Then we have by commutation with scaling

$$\|g_0(\mathbf{q})\| = \frac{1}{\delta}\|g_0(\delta\mathbf{q})\| \leq \frac{1}{\delta} =: C, \qquad \mathbf{q} \in B_1.$$

ii) $\Rightarrow$ *iii)*: For all $\mathbf{p} \in B_r$ we have $\mathbf{p}/r \in B_1$. Using scaling commutation again

$$\|g_0(\mathbf{p})\| = r\|g_0(\mathbf{p}/r)\| \leq Cr.$$

iii) $\Rightarrow$ *i)*: We have to show that for all $\epsilon > 0$ there exists $\delta > 0$ with $\|g_0(\mathbf{p})\| \leq \epsilon$ for all $\mathbf{p} \in B_\delta$. Given $\epsilon > 0$, we choose $\delta := \epsilon/C$ and use *iii)* with $r = \delta$ and find

$$\|g_0(\mathbf{p})\| \leq C\delta = \epsilon \qquad \text{for all } \mathbf{p} \in B_\delta.$$

□

In Example 2.3, the newly generated points lie in the convex hull of the old ones. Especially for input data $\mathbf{p} \in B_1$,

$$\|g_0(\mathbf{p})\| \leq \max_{i=0,\ldots,2} \|p_i\| = |\mathbf{p}|_0 \leq 1,$$

showing that g_0 is bounded. By Lemma 2.5 g_0 is continuous in $\mathbf{n}_3$ and the scheme fulfills property (U).

3 Convergence

In this section, we will see that summability of $|\mathbf{G}^\ell(\mathbf{P})|_1$ is a sufficient condition for a GLU-scheme $\mathbf{G}$ to be convergent. This result is used to analyze Example 2.3. Further, the property of $|\mathbf{G}^\ell(\mathbf{P})|_1$ being a null sequence is necessary for convergent GLU-schemes. With an additional restriction, this condition is also sufficient, but in general it does not guarantee convergence since we have a divergent example of such a GLU-scheme. For the notion of convergence of a refined chain, we use a standard definition based on uniform convergence of points at dyadic values.

Definition 3.1. *A GLU-scheme* $\mathbf{G}$ *is called* convergent at $\mathbf{P} \in \mathbb{E}^{\geq n}$ *if there exists a continuous* limit curve $\Phi[\mathbf{P}] : \mathbb{R} \to \mathbb{E}$ *such that*

$$\lim_{\ell\to\infty} \max_{i=0,\ldots,\#\mathbf{P}-1} \|\Phi[\mathbf{P}](2^{-\ell}i) - p_i^\ell\| = 0.$$

$\mathbf{G}$ *is called* (everywhere) convergent, *if it is convergent at all* $\mathbf{P} \in \mathbb{E}^{\geq n}$.

As stated in [8], this definition is closely related to convergence of a special sequence of functions in the L_∞-norm: For each $\mathbf{P}^\ell$ we define a piecewise linear continuous function $f_\ell : \mathbb{R} \to \mathbb{E}$ with vertices at $f_\ell(2^{-\ell}i) := p_i^\ell$ for $i = 0, \ldots, \#\mathbf{P}^\ell - 1$ and repetition at the ends, i.e. $f_\ell(2^{-\ell}i) := p_0^\ell$ for $i < 0$ and $f_\ell(2^{-\ell}i) := p_{\#\mathbf{P}^\ell-1}^\ell$ for $i \geq \#\mathbf{P}^\ell$. On the space of continuous curves $f : \mathbb{R} \to \mathbb{E}$ we define the L_∞-norm as

$$\|f\|_\infty := \sup_{x\in\mathbb{R}} \|f(x)\|.$$

Then $\mathbf{G}$ is convergent at $\mathbf{P}$ if and only if the sequence f_ℓ is convergent in the L_∞-norm. A proof of this fact can be found in [2].

One easily observes that

$$\begin{aligned}\|f_{\ell+1} - f_\ell\|_\infty = \max\Bigg\{ &\max_{i=0,\ldots,\#\mathbf{P}^\ell-m-1} \max\left\{\|p_{2i}^{\ell+1} - p_i^\ell\|, \|p_{2i+1}^{\ell+1} - \frac{p_i^\ell + p_{i+1}^\ell}{2}\|\right\}, \\ &\max_{j=1,\ldots,m} \|p_{\#\mathbf{P}^{\ell+1}-1}^{\ell+1} - p_{\#\mathbf{P}^\ell-j}^\ell\|\Bigg\} =: F_\ell, \end{aligned} \tag{1}$$

where the last term comes from the end conditions. This proves, together with standard arguments, the following statement analogous to Theorem 1 in [7].

Proposition 3.2. *If the GLU-scheme* $\mathbf{G}$ *is convergent at* $\mathbf{P}$*, then* F_ℓ *is a null sequence.*

Unfortunately, F_ℓ is an artificial quantity, while the decay of $|\mathbf{G}^\ell(\mathbf{P})|_1$ towards 0 is a more natural condition. Therefore, we want to replace F_ℓ in Proposition 3.2 by $|\mathbf{G}^\ell(\mathbf{P})|_1$. The following Lemma is needed for this purpose.

Lemma 3.3. *If* $\mathbf{G}$ *is a GLU-scheme, then* F_ℓ *is a null sequence if and only if* $|\mathbf{G}^\ell(\mathbf{P})|_1$ *is a null sequence.*

Because the proof is a bit more technical, it is given in Section 4. The combination of Proposition 3.2 and Lemma 3.3 proves our first main result.

Theorem 3.4. *If the GLU-scheme* $\mathbf{G}$ *is convergent at* $\mathbf{P}$*, then* $|\mathbf{G}^\ell(\mathbf{P})|_1$ *is a null sequence.*

One could conjecture that the decay of $|\mathbf{G}^\ell(\mathbf{P})|_1$ towards 0 is already a sufficient condition for convergence. The following example disproves this thought.

Example 3.5. Let $m = 3$ and $d = 2$, which means $\mathbb{E} = \mathbb{R}^2$. Consider the scheme defined by

$$\begin{aligned} g_0(\mathbf{p}) &= \frac{7+3a}{8}p_1 + \frac{3-9a}{8}p_2 + \frac{-3+9a}{8}p_3 + \frac{1-3a}{8}p_4, \\ g_1(\mathbf{p}) &= \frac{7+2a}{16}p_1 + \frac{9-6a}{16}p_2 + \frac{1+6a}{16}p_3 + \frac{-1-2a}{16}p_4, \end{aligned}$$

for $\mathbf{p} \in \mathbb{E}^4$. Using the notation q_y for the second coordinate of $q \in \mathbb{E}$, the parameters are given by

$$a = \operatorname{sgn}\Big((p_1 - 3p_2 + 3p_3 - p_4)_y\Big)\frac{b}{b+1}, \quad b = \begin{cases} -\frac{W_{-1}\left(-\frac{\ln 4}{4}c\right)}{\ln 4} & \text{if } 0 < c < \frac{4}{e\cdot \ln 4} \\ 0 & \text{if } c = 0 \\ \frac{1}{\ln 4} & \text{if } c \geq \frac{4}{e\cdot\ln 4} \end{cases}$$

and

$$c = \begin{cases} 2\frac{\|p_1-p_2-p_3+p_4\|}{\|p_1-3p_2+3p_3-p_4\|} & \text{if } p_1 - 3p_2 + 3p_3 - p_4 \neq 0 \\ 0 & \text{else} \end{cases}.$$

Here, W_{-1} is the inverse of $x \mapsto xe^x$ for $x < -1$, see Figure 2, also known as the lower branch of the *Lambert W function*. It is defined for $y \in (-1/e, 0)$ with the image $W_{-1}(y) \in (-\infty, -1)$. Hence,

$$-\frac{1}{\ln 4}W_{-1}\left(-\frac{\ln 4}{4}c\right) \geq \frac{1}{\ln 4} > 0$$

for $0 < c < 4/(e\cdot\ln 4)$. Together, $b \geq 0$ and $a \in [-1, 1]$. Hence g_λ is an affine combination with non-linear but bounded coefficients. Now c is obviously invariant under similarities

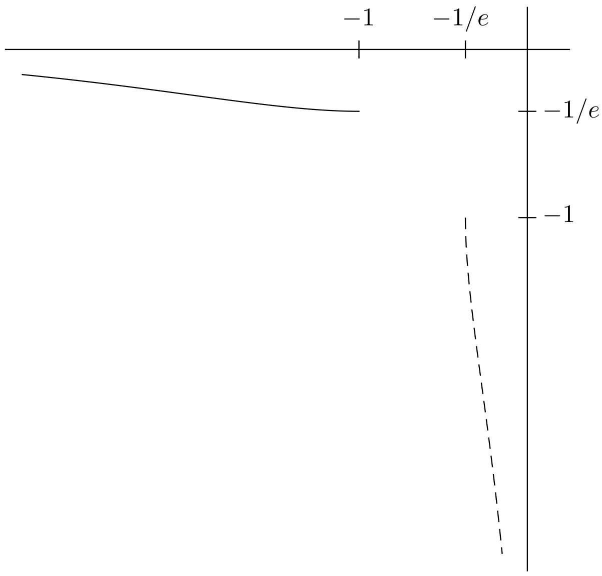

Figure 2: plot of $x \mapsto xe^x$ for $x < -1$ (solid line) and W_{-1} (dashed line).

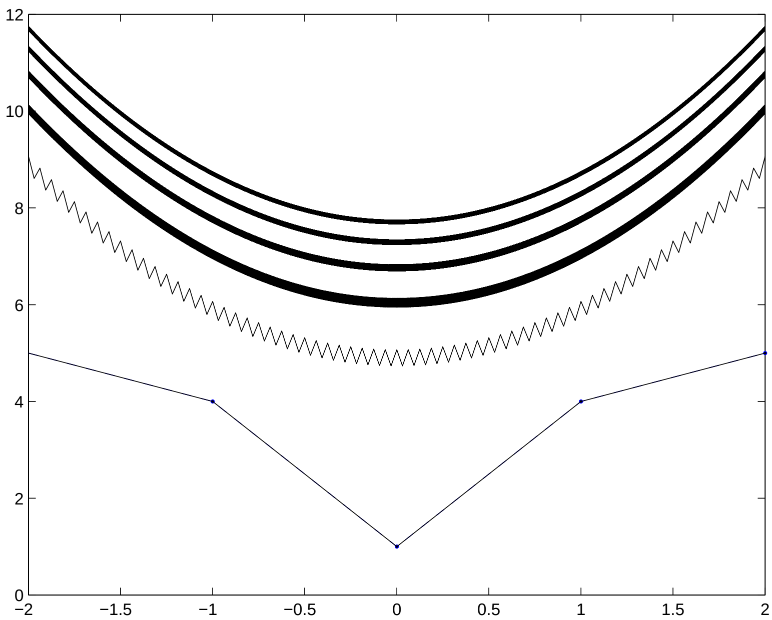

Figure 3: zoom-in of the initial chain (lowest line) and F_ℓ for $\ell = 5, 10, 15, 20, 25$ (from bottom to top) showing divergence as for large subdivision levels a non-summable increase in y-direction is still noticeable. For $\ell \geq 10$ the plot appear as thick lines due to high oscillations.

and so are b and a. Because g_λ is an affine combination with invariant coefficients, this scheme fulfills condition (G). Regarding the fact that a linear combination with bounded coefficients maps bounded data to bounded data, g_0 and g_1 are continuous in $\mathbf{n}_4$ by Lemma 2.5. Hence, $\mathbf{G}$ is a GLU-scheme.

Now we apply $\mathbf{G}$ to a special initial chain $\mathbf{P}$ with $\mathbf{G}^\ell(\mathbf{P})$ divergent but $|\mathbf{G}^\ell(\mathbf{P})|_1 \to 0$. This chain is sampled from a translated standard parabola with noise, namely $\mathbf{P} \in \mathbb{E}^7$ with $p_i = [x_0 + i, (x_0+i)^2 + 2 + (-1)^i]$, $x_0 \in \mathbb{R}$ arbitrary. One can show by induction that

$$p_i^\ell = \left[x_0 + i\frac{1}{2^\ell}, \left(x_0 + i\frac{1}{2^\ell} \right)^2 + 2\sum_{k=1}^{\ell+1} \frac{1}{k} + (-1)^i \frac{1}{\ell+1} \right], \qquad \ell \in \mathbb{N}_0. \tag{2}$$

A proof of this formula is given in Section 4. With this we have for example for the y-coordinate of p_0^ℓ

$$\lim_{\ell\to\infty} (p_0^\ell)_y = \lim_{\ell\to\infty} x_0^2 + 2\sum_{k=1}^{\ell+1} \frac{1}{k} + \frac{1}{\ell+1} = \infty,$$

showing divergence of $\mathbf{G}^\ell(\mathbf{P})$. This behaviour can also be observed in Figure 3 as high subdivision levels still have a noticable increase in the y-direction. On the other hand

$$|\mathbf{G}^\ell(\mathbf{P})|_1 \le C \left(\frac{1}{2^\ell} + \frac{1}{4^\ell} + \frac{1}{\ell+1} \right)$$

for a constant $C < \infty$, which yields $|\mathbf{G}^\ell(\mathbf{P})|_1 \to 0$.

This example shows that we have to impose a stronger condition than $|\mathbf{G}^\ell(\mathbf{P})|_1 \to 0$ to guarantee convergence of GLU-schemes. The following proposition specifies a sufficient condition in terms of summability of F_ℓ. Because this result follows word by word from Theorem 3 in [7], the proof is left out.

Proposition 3.6. *A GLU-scheme* $\mathbf{G}$ *is convergent, if* F_ℓ *is summable.*

If we want to prove the same result with $|\mathbf{G}^\ell(\mathbf{P})|_1$ being summable, we need another Lemma.

Lemma 3.7. *If* $\mathbf{G}$ *is a GLU-scheme, then* F_ℓ *is summable if and only if* $|\mathbf{G}^\ell(\mathbf{P})|_1$ *is summable.*

Proof. Anticipating the proof of Lemma 3.3, by (6) we get

$$\sum_{\ell=1}^{k} F_\ell \le (C+1)m \sum_{\ell=1}^{k} |\mathbf{G}^\ell(\mathbf{P})|_1 + \sum_{\ell=2}^{k+1} |\mathbf{G}^\ell(\mathbf{P})|_1 \le (Cm+m+1) \sum_{\ell=1}^{k+1} |\mathbf{G}^\ell(\mathbf{P})|_1.$$

Hence, summability of F_ℓ follows from summability of $|\mathbf{G}^\ell(\mathbf{P})|_1$.

On the other hand, again anticipating the inequality (7) we have

$$\sum_{\ell=1}^{k} |\mathbf{G}^\ell(\mathbf{P})|_1 \leq |\mathbf{G}(\mathbf{P})|_1 + \sum_{\ell=1}^{k} |\mathbf{G}^{\ell+1}(\mathbf{P})|_1 \leq |\mathbf{G}(\mathbf{P})|_1 + 2\sum_{\ell=1}^{k} F_\ell + \frac{1}{2}\sum_{\ell=1}^{k} |\mathbf{G}^\ell(\mathbf{P})|_1$$

and further

$$\sum_{\ell=1}^{k} |\mathbf{G}^\ell(\mathbf{P})|_1 \leq 2|\mathbf{G}(\mathbf{P})|_1 + 4\sum_{\ell=1}^{k} F_\ell.$$

Thus, if F_ℓ is summable, then so is $|\mathbf{G}^\ell(\mathbf{P})|_1$. □

Again, combining Proposition 3.6 and Lemma 3.7, we have our second main result.

Theorem 3.8. *A GLU-scheme* $\mathbf{G}$ *is convergent, if* $|\mathbf{G}^\ell(\mathbf{P})|_1$ *is summable.*

In the following, we want to use this result to show convergence of the scheme introduced in Example 2.3 for parameters $\alpha \in (2/3, 1)$.
For $\ell \in \mathbb{N}_0$ we have

$$\begin{aligned} \|p_{2i+1}^{\ell+1} - p_{2i}^{\ell+1}\| &= \|g_1(p_i^\ell, \dots, p_{i+2}^\ell) - g_0(p_i^\ell, \dots, p_{i+2}^\ell)\| \\ &= \|\alpha p_{i+1}^\ell + (1-\alpha)M - \alpha p_i^\ell - (1-\alpha)M\| \\ &\leq \alpha |\mathbf{G}^\ell(\mathbf{P})|_1 \end{aligned}$$

and

$$\|p_{2i+2}^{\ell+1} - p_{2i+1}^{\ell+1}\| \leq \|p_{2i+2}^{\ell+1} - p_{i+1}^\ell\| + \|p_{i+1}^\ell - p_{2i+1}^{\ell+1}\| =: I_1 + I_2.$$

The second term I_2 can be estimated using triangle inequality,

$$\begin{aligned} I_2 &= (1-\alpha)\frac{\Big\| \|p_{i+2}^\ell - p_{i+1}^\ell\|(p_i^\ell - p_{i+1}^\ell) + \|p_{i+1}^\ell - p_i^\ell\|(p_{i+2}^\ell - p_{i+1}^\ell)\Big\|}{\|p_{i+2}^\ell - p_{i+1}^\ell\| + \|p_{i+2}^\ell - p_i^\ell\| + \|p_{i+1}^\ell - p_i^\ell\|} \\ &\leq (1-\alpha)\max\{\|p_{i+1}^\ell - p_i^\ell\|, \|p_{i+2}^\ell - p_{i+1}^\ell\|\} \\ &\leq (1-\alpha)|\mathbf{G}^\ell(\mathbf{P})|_1 \end{aligned}$$

and analogously for the first term I_1

$$I_1 \leq (1-\alpha)\max\{\|p_{i+2}^\ell - p_{i+1}^\ell\|, \|p_{i+2}^\ell - p_i^\ell\|\} \leq 2(1-\alpha)|\mathbf{G}^\ell(\mathbf{P})|_1.$$

Summarizing all inequalities yields

$$|\mathbf{G}^{\ell+1}(\mathbf{P})|_1 \leq \max\{\alpha, 3(1-\alpha)\}|\mathbf{G}^\ell(\mathbf{P})|_1 =: C|\mathbf{G}^\ell(\mathbf{P})|_1.$$

If $\alpha \in (2/3, 1)$ then $C < 1$ and $|\mathbf{G}^\ell(\mathbf{P})|_1$ is a geometric sequence, hence summable. With Theorem 3.8 this proves convergence of the scheme for all initial data when $\alpha \in (2/3, 1)$.

We conclude the analysis of this scheme by noticing that it is non-interpolatory and does not fulfill equilinearity and as such can neither be covered by the theory developed in [7] nor by [10].

We return to the general study of convergence of GLU-schemes if summability of $|\mathbf{G}^\ell(\mathbf{P})|_1$ is not given. For a special subclass of GLU-schemes excluding Example 3.5, we can nevertheless prove that convergence follows already from $|\mathbf{G}^\ell(\mathbf{P})|_1 \to 0$. With Theorem 3.4, this gives a full classification of convergence. To formulate this result, we need further notation. According to the representation of linear schemes in terms of pairs of matrices [14], we define associated self-maps $\mathbf{g}_0, \mathbf{g}_1 : \mathbb{E}^n \to \mathbb{E}^n$ by

$$\mathbf{g}_0(p_0,\dots,p_{n-1}) := \begin{bmatrix} g_0(p_0,\dots,p_m) \\ g_1(p_0,\dots,p_m) \\ g_0(p_1,\dots,p_{m+1}) \\ g_1(p_1,\dots,p_{m+1}) \\ \vdots \\ g_0(p_m,\dots,p_{n-1}) \end{bmatrix}, \quad \mathbf{g}_1(p_0,\dots,p_{n-1}) := \begin{bmatrix} g_1(p_0,\dots,p_m) \\ g_0(p_1,\dots,p_{m+1}) \\ g_1(p_1,\dots,p_{m+1}) \\ \vdots \\ g_0(p_m,\dots,p_{n-1}) \\ g_1(p_m,\dots,p_{n-1}) \end{bmatrix},$$

where $n = 2m + 1$ is the number of points in a chain $\mathbf{p} \in \mathbb{E}^n$ such that the number of points in the subdivided chain is raised by 1, namely $\mathbf{G}(\mathbf{p}) \in \mathbb{E}^{n+1}$. For $\Lambda = [\lambda_1,\dots,\lambda_\ell] \in \{0,1\}^\ell$, we write

$$\mathbf{g}_\Lambda := \mathbf{g}_{\lambda_\ell} \circ \cdots \circ \mathbf{g}_{\lambda_1}$$

for the corresponding composition of the functions $\mathbf{g}_0, \mathbf{g}_1$.

Theorem 3.9. *Let $\mathbf{G}$ be a GLU-scheme with both refinement rules g_0 and g_1 continuous in $\mathbf{n}_{m+1}$ and $s \in \mathbb{N}$ such that*

$$\max_{\Lambda \in \{0,1\}^s} \sup_{\mathbf{p} \in B_1} |\mathbf{g}_\Lambda(\mathbf{p})|_0 \le 1. \tag{3}$$

Then $\mathbf{G}$ is convergent if and only if $|\mathbf{G}^\ell(\mathbf{P})|_1$ is a null sequence.

Remark 3.10. In Example 3.5, continuity of g_1 in $\mathbf{n}_{m+1}$ is fulfilled, but condition (3) is violated by a properly scaled version of the chain used in Example 3.5.

Proof. The necessity of $|\mathbf{G}^\ell(\mathbf{P})|_1$ being a null sequence is shown in Theorem 3.4. On the other hand, after a simple but technical computation we get accordingly to (1)

$$\begin{aligned} \|f_{\ell+k} - f_\ell\|_\infty = \max \Bigg\{ & \max_{i=0,\dots,j^*-1} \max_{r=0,\dots,2^k-1} \|p^{\ell+k}_{2^k i+r} - \left(\left(1-\frac{r}{2^k}\right) p^\ell_i + \frac{r}{2^k} p^\ell_{i+1}\right)\|, \\ & \max_{r=0,\dots,r^*} \|p^{\ell+k}_{2^k j^*+r} - \left(\left(1-\frac{r}{2^k}\right) p^\ell_{j^*} + \frac{r}{2^k} p^\ell_{j^*+1}\right)\|, \\ & \max_{i=j^*+1,\dots,\#\mathbf{P}^\ell-1} \|p^{\ell+k}_{\#\mathbf{P}^{\ell+k}-1} - p^\ell_i\| \Bigg\} =: \max\{I_1, I_2, I_3\} \end{aligned}$$

for all $k, \ell \in \mathbb{N}$, where we used the shortcut $j^* = \#\mathbf{P}^\ell - n + 1 + \lfloor 2^{-k}(n-2) \rfloor$ and $r^* = n - 2 - 2^k \lfloor 2^{-k}(n-2) \rfloor$. By the triangle inequality and with

$$\max_{r=0,\ldots,2^k-1} \| p_i^\ell - \left(1 - \frac{r}{2^k}\right) p_i^\ell - \frac{r}{2^k} p_{i+1}^\ell \| \leq |\mathbf{G}^\ell(\mathbf{P})|_1,$$

we have

$$I_1 \leq |\mathbf{G}^\ell(\mathbf{P})|_1 + \max_{i=0,\ldots,j^*-1} \max_{r=0,\ldots,2^k-1} \| p_{2^k i + r}^{\ell+k} - p_i^\ell \|. \tag{4}$$

Further, one can easily see that for each $r \in \{0, \ldots, 2^k - 1\}$ there exists $\Lambda_r \in \{0,1\}^k$ with

$$\sum_{s=1}^{k} 2^{k-s} \lambda_s = r.$$

As already stated in [10], the identity $T_j^n \mathbf{P}^k = \mathbf{g}_\Lambda(T_i^n \mathbf{P})$ holds with $j = 2^k(i + \sum_{s=1}^k 2^{-s}\lambda_s)$ for all $\Lambda = [\lambda_1, \ldots, \lambda_\ell] \in \{0,1\}^k$ and all $\mathbf{P} \in \mathbb{E}^{\geq n}$. This shows that $p_{2^k i + r}^{\ell+k}$ coincides with the first point of the subchain $T_j^n \mathbf{P}^{\ell+k} = \mathbf{g}_{\Lambda_r}(T_i^n \mathbf{P}^\ell)$, denoted by $(\mathbf{g}_{\Lambda_r}(T_i^n \mathbf{P}^\ell))_1$. Hence,

$$\| p_{2^k i + r}^{\ell+k} - p_i^\ell \| = \| (\mathbf{g}_{\Lambda_r}(T_i^n \mathbf{P}^\ell))_1 - p_i^\ell \| \leq |\mathbf{g}_{\Lambda_r}(T_i^n \mathbf{P}^\ell) - p_i^\ell|_0,$$

where the difference in the last term is meant pointwise. By assumption and Lemma 2.5, we have $\|g_\lambda(\mathbf{p})\| \leq C_\lambda |\mathbf{p}|_0$ for all $\mathbf{p} \in \mathbb{E}^{m+1}, \lambda \in \{0,1\}$. Combining these inequalities yields

$$|\mathbf{g}_\lambda(\mathbf{p})|_0 \leq C|\mathbf{p}|_0, \qquad \mathbf{p} \in \mathbb{E}^n, \ \ \lambda \in \{0,1\},$$

where $C := \max\{C_0, C_1\}$ and further $|\mathbf{g}_\mu(\mathbf{p})|_0 \leq C_s |\mathbf{p}|_0$ for all $\mathbf{p} \in \mathbb{E}^n$ with $\mu \in \{0,1\}^m$, $m \leq s$ and $C_s := \max_{m \leq s} C^m$. Choosing the representation $r = ms + \alpha$ with $m \in \mathbb{N}$, $0 \leq \alpha < s$, using condition (G) and iterating (3) m-times shows

$$|\mathbf{g}_{\Lambda_r}(T_i^n \mathbf{P}^\ell) - p_i^\ell|_0 = |\mathbf{g}_{\Lambda_r}(T_i^n \mathbf{P}^\ell - p_i^\ell)|_0 \leq |\mathbf{g}_{\Lambda_\alpha}(T_i^n \mathbf{P}^\ell - p_i^\ell)|_0 \leq C_s |T_i^n \mathbf{P}^\ell - p_i^\ell|_0,$$

where $\Lambda_\alpha = [\lambda_1, \ldots, \lambda_\alpha] \in \{0,1\}^\alpha$ are the first α entries of the index vector $\Lambda_r \in \{0,1\}^k$. A combination of this inequality with (4) and an analogous estimation of (5), anticipating the proof of Lemma 3.3, yields

$$\begin{aligned} I_1 &\leq |\mathbf{G}^\ell(\mathbf{P})|_1 + \max_{i=0,\ldots,j^*-1} \max_{r=0,\ldots,2^k-1} |\mathbf{g}_{\Lambda_r}(T_i^n \mathbf{P}^\ell) - p_i^\ell|_0 \\ &\leq |\mathbf{G}^\ell(\mathbf{P})|_1 + \max_{i=0,\ldots,j^*-1} C_s |T_i^n \mathbf{P}^\ell - p_i^\ell|_0 \\ &\leq C_s n |\mathbf{G}^\ell(\mathbf{P})|_1. \end{aligned}$$

Exactly the same argumentation holds for I_2 with $i = j^*$ and $r \in \{0, \ldots, r^*\}$. Thus

$$I_2 \leq C_s n |\mathbf{G}^\ell(\mathbf{P})|_1.$$

For the estimation of I_3 we first observe $\#\mathbf{P}^{\ell+k} = 2^k(\#\mathbf{P}^\ell - n + 1) + n - 1$ by iterating $\#\mathbf{P}^{\ell+1} = 2\#\mathbf{P}^\ell - n + 1$ from Defintion 2.1. With this, we have $\#\mathbf{P}^{\ell+k} - 1 = 2^k j^* + r^*$ and therefore analogously to the estimation of I_1 the representation $p^{\ell+k}_{\#\mathbf{P}^{\ell+k}-1} = (\mathbf{g}_{\Lambda_{r^*}}(T^n_{j^*}\mathbf{P}^\ell))_1$. Hence,

$$\begin{aligned}\|p^{\ell+k}_{\#\mathbf{P}^{\ell+k}-1} - p^\ell_i\| &\leq \|(\mathbf{g}_{\Lambda_{r^*}}(T^n_{j^*}\mathbf{P}^\ell))_1 - p^\ell_{j^*}\| + \|p^\ell_{j^*} - p^\ell_i\| \\ &\leq C_s n|\mathbf{G}^\ell(\mathbf{P})|_1 + (\#\mathbf{P}^\ell - j^* - 1)|\mathbf{G}^\ell(\mathbf{P})|_1\end{aligned}$$

for $i = j^* + 1, \ldots, \#\mathbf{P}^\ell - 1$. With $\#\mathbf{P}^\ell - j^* - 1 \leq n$ this gives the inequality $I_3 \leq (C_s + 1)n|\mathbf{G}^\ell(\mathbf{P})|_1$ and finally

$$\|f_{\ell+k} - f_\ell\|_\infty \leq (C_s + 1)n|\mathbf{G}^\ell(\mathbf{P})|_1.$$

Now, if $|\mathbf{G}^\ell(\mathbf{P})|_1$ is a null sequence, then f^ℓ is a Cauchy sequence in the L_∞-norm and hence convergent to a continuous limit. □

4 Proofs

In this section, we want to prove Lemma 3.3 and Equation (2).

Lemma 3.3. *If $\mathbf{G}$ is a GLU-scheme, then F_ℓ is a null sequence if and only if $|\mathbf{G}^\ell(\mathbf{P})|_1$ is a null sequence.*

Proof. First, we want to show that the decay of $|\mathbf{G}^\ell(\mathbf{P})|_1$ towards 0 implies the decay of F_ℓ. To do this, we estimate the norm of the subchain $T^{m+1}_i\mathbf{P}^\ell - p^\ell_i$, where the difference is meant pointwise. By the triangle inequality,

$$|T^{m+1}_i\mathbf{P}^\ell - p^\ell_i|_0 = \max_{j=i,\ldots,i+m} \|p^\ell_j - p^\ell_i\| \leq m|\mathbf{G}^\ell(\mathbf{P})|_1 \tag{5}$$

for all $i = 0, \ldots, \#\mathbf{P}^\ell - m - 1$ and analogously $|T^{m+1}_i\mathbf{P}^\ell - (p^\ell_{i+1} + p^\ell_i)/2|_0 \leq m|\mathbf{G}^\ell(\mathbf{P})|_1$. Commutation with translation and Lemma 2.5 with choosing $r = m|\mathbf{G}^\ell(\mathbf{P})|_1$ yields

$$\|p^{\ell+1}_{2i} - p^\ell_i\| = \|g_0(T^{m+1}_i\mathbf{P}^\ell) - p^\ell_i\| = \|g_0(T^{m+1}_i\mathbf{P}^\ell - p^\ell_i)\| \leq Cm|\mathbf{G}^\ell(\mathbf{P})|_1$$

and in the same way

$$\begin{aligned}\left\|p^{\ell+1}_{2i+1} - \frac{p^\ell_{i+1} + p^\ell_i}{2}\right\| &\leq \|p^{\ell+1}_{2i+1} - p^{\ell+1}_{2i}\| + \left\|p^{\ell+1}_{2i} - \frac{p^\ell_{i+1} + p^\ell_i}{2}\right\| \\ &\leq |\mathbf{G}^{\ell+1}(\mathbf{P})|_1 + Cm|\mathbf{G}^\ell(\mathbf{P})|_1.\end{aligned}$$

Dealing with the end conditions, we have $\#\mathbf{P}^{\ell+1}-2 = 2(\#\mathbf{P}^{\ell}-m-1)$ by Definition 2.1. Hence, using the abbreviation $N_\ell := \#\mathbf{P}^\ell$,

$$\begin{aligned}\|p^{\ell+1}_{N_{\ell+1}-1} - p^{\ell}_{N_\ell-j}\| \leq &\|p^{\ell+1}_{N_{\ell+1}-1} - p^{\ell+1}_{N_{\ell+1}-2}\| + \|p^{\ell+1}_{N_{\ell+1}-2} - p^{\ell}_{N_\ell-m-1}\| \\ &+ \|p^{\ell}_{N_\ell-m-1} - p^{\ell}_{N_\ell-j}\| \\ \leq &|\mathbf{G}^{\ell+1}(\mathbf{P})|_1 + \|g_0(T^{m+1}_{N_\ell-m-1}\mathbf{P}^\ell) - p^{\ell}_{N_\ell-m-1}\| + m|\mathbf{G}^{\ell}(\mathbf{P})|_1 \\ \leq &|\mathbf{G}^{\ell+1}(\mathbf{P})|_1 + (C+1)m|\mathbf{G}^{\ell}(\mathbf{P})|_1\end{aligned}$$

for all $j = 1,\ldots,m$. Collecting all results, we obtain

$$0 \leq F_\ell \leq |\mathbf{G}^{\ell+1}(\mathbf{P})|_1 + (C+1)m|\mathbf{G}^{\ell}(\mathbf{P})|_1. \tag{6}$$

Hence, if $|\mathbf{G}^{\ell}(\mathbf{P})|_1$ is a null sequence, then so is F_ℓ.

Second, we want to prove the reversed implication. To this end, we observe that each index $j = 0,\ldots,\#\mathbf{P}^{\ell+1}-1$ of $\mathbf{P}^{\ell+1}$ can be written in terms of $2i$ or $2i+1$ with $i = 0,\ldots,\#\mathbf{P}^{\ell}-m-1$. With that we have, see Figure 4,

$$\begin{aligned}\|p^{\ell+1}_{2i+1} - p^{\ell+1}_{2i}\| &\leq \left\|p^{\ell+1}_{2i+1} - \frac{p^{\ell}_{i+1}+p^{\ell}_{i}}{2}\right\| + \left\|\frac{p^{\ell}_{i+1}+p^{\ell}_{i}}{2} - p^{\ell}_{i}\right\| + \|p^{\ell}_{i} - p^{\ell+1}_{2i}\| \\ &\leq F_\ell + \frac{1}{2}|\mathbf{G}^{\ell}(\mathbf{P})|_1 + F_\ell, \qquad i = 0,\ldots,\#\mathbf{P}^{\ell}-m-1,\end{aligned}$$

and in the same way $\|p^{\ell+1}_{2i+2} - p^{\ell+1}_{2i+1}\| \leq 2F_\ell + |\mathbf{G}^{\ell}(\mathbf{P})|_1/2$. Both inequalities together yield

$$|\mathbf{G}^{\ell+1}(\mathbf{P})|_1 - \frac{1}{2}|\mathbf{G}^{\ell}(\mathbf{P})|_1 \leq 2F_\ell. \tag{7}$$

Because F_ℓ is a null sequence and thus bounded, we can also bound the sequence in (7) for all $\ell \in \mathbb{N}$ by some constant $C < \infty$. Given $N \in \mathbb{N}$, one easily verifies by induction

$$|\mathbf{G}^{\ell+1}(\mathbf{P})|_1 = \frac{1}{2^{\ell-N+1}}|\mathbf{G}^{N}(\mathbf{P})|_1 + \sum_{r=N}^{\ell}\frac{1}{2^{\ell-r}}\left(|\mathbf{G}^{r+1}(\mathbf{P})|_1 - \frac{1}{2}|\mathbf{G}^{r}(\mathbf{P})|_1\right) \tag{8}$$

for all $\ell \geq N$. Choosing $N = 1$ in (8), we have

$$\begin{aligned}|\mathbf{G}^{\ell+1}(\mathbf{P})|_1 &\leq \frac{1}{2^\ell}|\mathbf{G}(\mathbf{P})|_1 + \sum_{r=1}^{\ell}\frac{1}{2^{\ell-r}}\left(|\mathbf{G}^{r+1}(\mathbf{P})|_1 - \frac{1}{2}|\mathbf{G}^{r}(\mathbf{P})|_1\right) \\ &\leq |\mathbf{G}(\mathbf{P})|_1 + C\sum_{r=1}^{\ell}\frac{1}{2^{\ell-r}} \\ &= |\mathbf{G}(\mathbf{P})|_1 + C\sum_{s=0}^{\ell-1}\frac{1}{2^s} \\ &\leq |\mathbf{G}(\mathbf{P})|_1 + 2C =: M\end{aligned}$$

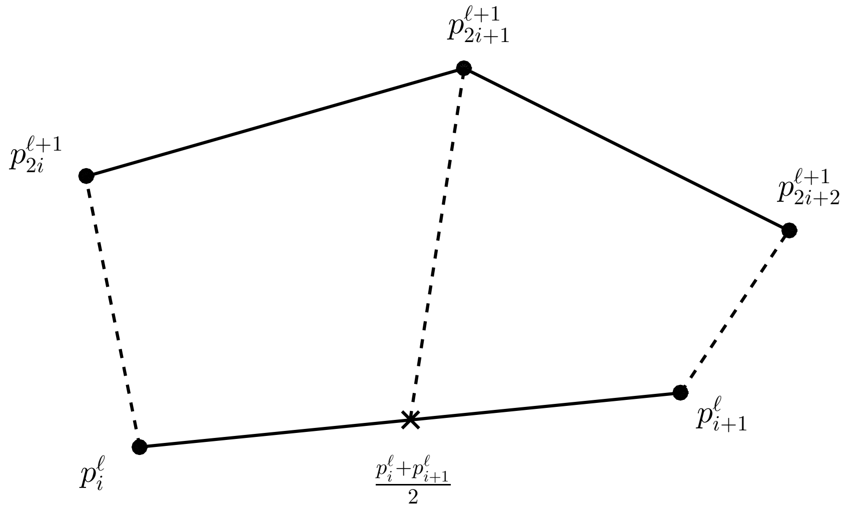

Figure 4: Illustration of $\mathbf{G}^{\ell}(\mathbf{P})$ and $\mathbf{G}^{\ell+1}(\mathbf{P})$.

for all $\ell \in \mathbb{N}$. By the fact that we are dealing with finite chains, the term $|\mathbf{G}(\mathbf{P})|_1$ is clearly finite, too, and hence $|\mathbf{G}^{\ell}(\mathbf{P})|_1$ is bounded by $M < \infty$. Using (7) again with the property of F_ℓ being a null sequence, we can find for a given $\epsilon > 0$ a natural number $N_1 \in \mathbb{N}$ with

$$|\mathbf{G}^{\ell+1}(\mathbf{P})|_1 - \frac{1}{2}|\mathbf{G}^{\ell}(\mathbf{P})|_1 \leq \frac{\epsilon}{4}$$

for all $\ell \geq N_1$. Build upon this, there exists another number $N_2 \geq N_1$ such that $M2^{-(\ell-N_1+1)} \leq \epsilon/2$ for all $\ell \geq N_2$. Finally, using (8) with $N = N_1$ we have

$$\begin{aligned} |\mathbf{G}^{\ell+1}(\mathbf{P})|_1 &\leq \frac{1}{2^{\ell-N_1+1}}|\mathbf{G}^{N_1}(\mathbf{P})|_1 + \sum_{r=N_1}^{\ell} \frac{1}{2^{\ell-r}} \left(|\mathbf{G}^{r+1}(\mathbf{P})|_1 - \frac{1}{2}|\mathbf{G}^{r}(\mathbf{P})|_1 \right) \\ &\leq M\frac{1}{2^{\ell-N_1+1}} + \frac{\epsilon}{4}\sum_{r=N_1}^{\ell} \frac{1}{2^{\ell-r}} \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{4}2 = \epsilon \end{aligned}$$

for all $\ell \geq N_2$ and hence, $|\mathbf{G}^{\ell}(\mathbf{P})|_1$ is a null sequence. □

Now we want to prove (2) by induction, which was given by

$$p_i^{\ell} = \left[x_0 + i\frac{1}{2^\ell}, \left(x_0 + i\frac{1}{2^\ell} \right)^2 + 2\sum_{k=1}^{\ell+1} \frac{1}{k} + (-1)^i \frac{1}{\ell+1} \right], \qquad \ell \in \mathbb{N}_0$$

for initial data $p_i = [x_0 + i, (x_0+i)^2 + 2 + (-1)^i]$, $x_0 \in \mathbb{R}$ arbitrary. Obviously for $\ell = 0$ the points p_i^0 coincide with the initial data p_i. Suppose (2) holds, we want to show the same representation for $p_i^{\ell+1}$. To do this, we first compute the parameter c. Using the notation $u \cdot v$ for the scalar product of vectors $u, v \in \mathbb{R}^4$ and q_x for the first coordinate of a point $q \in \mathbb{E} = \mathbb{R}^2$, we get

$$(p_i^\ell - p_{i+1}^\ell - p_{i+2}^\ell + p_{i+3}^\ell)_x = \left(x_0 + i\frac{1}{2^\ell}\right) \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{2^\ell} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} = 0.$$

The second coordinate of $p_i^\ell - p_{i+1}^\ell - p_{i+2}^\ell + p_{i+3}^\ell$ is after evaluating the scalar products equal to $(p_i^\ell - p_{i+1}^\ell - p_{i+2}^\ell + p_{i+3}^\ell)_y = 4 \cdot 4^{-\ell}$. In total,

$$p_i^\ell - p_{i+1}^\ell - p_{i+2}^\ell + p_{i+3}^\ell = \left[0, \frac{4}{4^\ell}\right].$$

An analogous computation yields

$$p_i^\ell - 3p_{i+1}^\ell + 3p_{i+2}^\ell - p_{i+3}^\ell = \left[0, 8\frac{(-1)^i}{\ell+1}\right]$$

with the observation $\operatorname{sgn}\Big((p_i^\ell - 3p_{i+1}^\ell + 3p_{i+2}^\ell - p_{i+3}^\ell)_y\Big) = (-1)^\ell$. Hence $c = (\ell+1)4^{-\ell}$. Because $(\ell+1)4^{-\ell}$ is monotonically decreasing, we estimate $0 < c \le (0+1)4^0 = 1 \le \frac{4}{e \cdot \ln 4}$. This means that b is given by

$$\begin{aligned} b = -\frac{1}{\ln 4} W_{-1}\left(-\frac{\ln 4}{4}(\ell+1)\frac{1}{4^\ell}\right) &= -\frac{1}{\ln 4} W_{-1}\left(-(\ell+1)\ln 4 e^{-(\ell+1)\ln 4}\right) \\ &= -\frac{1}{\ln 4}\Big(-(\ell+1)\ln 4\Big) \\ &= \ell + 1 \end{aligned}$$

using W_{-1} as the inverse of xe^x. Thus $a = (-1)^i(\ell+1)/(\ell+2)$. Now we are ready to investigate $p_i^{\ell+1}$, but for convenience we only show induction on $p_{2i}^{\ell+1} = g_0(T_i^4\mathbf{P}^\ell)$ while $p_{2i+1}^{\ell+1}$ is determined in the same fashion. The first coordinate is

$$\begin{aligned} \left(p_{2i}^{\ell+1}\right)_x &= \left(x_0 + i\frac{1}{2^\ell}\right)\left(\frac{1}{8}\begin{bmatrix} 7 \\ 3 \\ -3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \frac{3a}{8}\begin{bmatrix} 1 \\ -3 \\ 3 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right) \\ &\quad + \frac{1}{2^\ell}\left(\frac{1}{8}\begin{bmatrix} 7 \\ 3 \\ -3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} + \frac{3a}{8}\begin{bmatrix} 1 \\ -3 \\ 3 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}\right) \\ &= x_0 + i\frac{1}{2^\ell} \\ &= x_0 + (2i)\frac{1}{2^{\ell+1}} \end{aligned}$$

and the second coordinate after computing the scalar products when using $a = (-1)^i(\ell+1)/(\ell+2)$

$$\begin{aligned}(p_{2i}^{\ell+1})_y &= \left(x_0 + i\frac{1}{2^\ell}\right)^2 + 2\sum_{k=1}^{\ell+1}\frac{1}{k} + 3\frac{(-1)^i}{\ell+1}a \\ &= \left(x_0 + (2i)\frac{1}{2^{\ell+1}}\right)^2 + 2\sum_{k=1}^{\ell+2}\frac{1}{k} + (-1)^{2i}\frac{1}{\ell+2}.\end{aligned}$$

This completes the proof and validates (2).

5 Conclusion

We have introduced a general type of geometric subdivision schemes, called GLU-schemes, which is a generalization of the class of GLUE-schemes presented in [10] and an extension to interpolatory planar schemes analyzed in [7]. The main results concerning necessary and sufficient conditions for convergence are generalizations of those in [7]. For example, the decay of the maximal edge length towards 0 is also the necessary condition here, while summability is sufficient for convergence, too. Therefore the interpolatory property is replaced by continuity in the constant null chain $\mathbf{n}_{m+1}$, which enables a more free choice of the refinement rule g_0. Example 2.3 and its convergence analysis demonstrates this freedom and the power of the sufficient condition in comparison to [7] and [10] since their analysis can not cover non-interpolatory and non-equilinear schemes like Example 2.3. In general, a purely uniform decay of edge lengths is not enough to guarantee convergence, as Example 3.5 shows. Nevertheless this necessary condition gives a full characterization of convergence for the subclass of continuous GLU-schemes with a sort of contraction condition (3).

Future research includes a C^1-analysis for GLU-schemes or even a generalization of the tools used for G^1-analysis in [7]. Another further step could be to drop the continuity of g_0 in $\mathbf{n}_{m+1}$, because it is not clear if by further subdivision we only need to have a continuous limit for null sequences of a certain shape instead of all null sequences. Additionally, one could hope for another quantity like relative distortion in [10], such that the decay towards 0 implies convergence, while summability yields smoothness.

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