Asymptotics and numerical efficiency of the Allen-Cahn model for phase interfaces with low energy in solids

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Abstract

The accurate simulation of phase interfaces in solids requires small model error and small numerical error. If a phase field model is used and the interface carries low interface energy, then the model error is only small if the interface width in the model is chosen small. Yet, for effective numerical computation the interface width should be large. Choosing the parameters, which determine the width, is therefore an optimality problem. We study this problem for the Allen-Cahn equation coupled to the elasticity equations and we show that the numerical effort is inversely proportional to the square of the required error of the simulation. To this end we construct an asymptotic solution of second order, which yields an expansion for the kinetic relation of the model, and prove that the difference between the exact kinetic relation and the asymptotic expansion tends to zero uniformly with respect to the interface energy.

Key words phase field models, phase interfaces in solids, kinetic relation, model error, numerical efficiency
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1 Introduction

The precision, with which numerical simulations based on a phase field model describe the temporal evolution of phase transformations in a material depends on the model error and the numerical error. By the model error we mean the difference of the propagation speeds of the phase interfaces in the real material and in the model. How large this error is depends on how precise the functional dependence of the speed of the phase interface on the stress and strain fields in the model coincides with the real dependence in the material. In sharp interface models the functional giving this dependence is called kinetic relation. We extend the meaning of this term and use it also for the corresponding functional in phase field models.

The kinetic relation of a phase field model depends amongst others on the interface width, which can be chosen by adjusting the model parameters suitably. It is known that in particular for interfaces with small interface energy the model error is only small if a very small interface width is chosen.

The numerical error depends on the grid spacing. The spacing must be chosen small enough to resolve the transition of the order parameter across the diffuse interface in the

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solution of the phase field model. If the interface width is small, then a fine grid must be chosen, which means that the numerical effort in the simulation is high. For an effective simulation the interface width should be large, which in simulations of phase transitions with small interface energy conflicts with the need to choose the interface width small to obtain a small model error. To simulate the time evolution of phase transitions with low interface energy as precise and as effective as possible, the interface width must therefore be chosen in an optimal way, compromising between these two contradictory requirements.

Our main goal in this paper is to study this optimality problem for the phase field model, which consists of the system of equations of linear elasticity coupled to the Allen-Cahn phase field equation. For short we call this model here the Allen-Cahn model. To this end we must determine the kinetic relation of this model with sufficiently high accuracy and we must determine the dependence of the model error on the interface energy.

To determine the kinetic relation we construct an approximation of the exact solution of the model by an asymptotic solution. The asymptotic solution yields an asymptotic expansion of the kinetic relation. We denote the parameter of expansion by μ. The model contains a second parameter λ; the interface energy is proportional to this parameter. To study the numerical effectivity we need to know that the difference between the exact kinetic relation of the model and the asymptotic expansion tends to zero for $\mu \to 0$, uniformly with respect to λ. To verify this, we determine how the residue term, which remains when the asymptotic solution is inserted into the model equations, depends on both parameters μ and λ.

The optimality problem is of interest for this special model, which we study, because of several reasons:

The Allen-Cahn model describes the evolution of phase transitions in an elastic solid with two possible phase states of the material. Temperature effects are neglected in our model. It is the prototype of a large class of models obtained by extensions and generalizations of this model, which are used in the engineering sciences to simulate the behavior of complex and functional materials. In many cases the interface energy of phase interfaces in these materials is very small, which is shown by the fact that a microstructure of phase transitions develops. The Allen-Cahn model and its derivatives are therefore very often used in the critical case of low interface energy. From the very large literature in this field we cite here only [11, 29, 30, 31, 32], to give some examples.

To know, how effective simulations based on the prototype Allen-Cahn model are, is therefore of utmost importance.

Our second motivation to study the Allen-Cahn model is to compare the properties of this model to the properties derived in [8] of an alternative phase field model, which we call the hybrid model. The comparison is made and the consequences for the numerical efficiency of simulations based on the two models are discussed at the end of Section 2.4.

We now state the equations of the Allen-Cahn model. Let $\Omega \subseteq \mathbb{R}^3$ be a bounded open set with a sufficiently smooth boundary $\partial\Omega$. The points of Ω represent the material points of a solid elastic body. The unknown functions in the model are the displacement $u(t,x) \in \mathbb{R}$ of the material point x at time t, the Cauchy stress tensor $T(t,x) \in \mathcal{S}^3$, where $\mathcal{S}^3$ denotes the set of all symmetric 3×3-matrices, and the order parameter $S(t,x) \in \mathbb{R}$.

These unknowns must satisfy the model equations

$$-\mathrm{div}_x T = \mathsf{b}, \tag{1.1}$$

$$T = D\big(\varepsilon(\nabla_x u) - \overline{\varepsilon} S\big), \tag{1.2}$$

$$\partial_t S = -\frac{c}{(\mu\lambda)^{1/2}}\Big(\partial_S \mathsf{W}\big(\varepsilon(\nabla_x u), S\big) + \frac{1}{\mu^{1/2}}\hat{\psi}'(S) - \mu^{1/2}\lambda\Delta_x S\Big) \tag{1.3}$$

in the domain $[0,\infty)\times\Omega$. The boundary and initial conditions are

$$u(t,x) = \mathsf{U}(t,x), \qquad (t,x)\in[0,\infty)\times\partial\Omega, \tag{1.4}$$

$$\partial_{n_{\partial\Omega}} S(t,x) = 0, \qquad (t,x)\in[0,\infty)\times\partial\Omega, \tag{1.5}$$

$$S(0,x) = \mathsf{S}(x), \qquad x\in\Omega. \tag{1.6}$$

Here $\mathsf{b}(t,x)\in\mathbb{R}^3$, $\mathsf{U}(t,x)\in\mathbb{R}^3$, $\mathsf{S}(t,x)\in\mathbb{R}$ denote given data, the volume force, boundary displacement and initial data. $\partial_{n_{\partial\Omega}}$ denotes the derivative in direction of the unit normal vector $n_{\partial\Omega}$ to the boundary. The deformation gradient $\nabla_x u(t,x)$ is the 3×3–matrix of first order partial derivatives of u with respect to the components x_k of x, and the strain tensor

$$\varepsilon(\nabla_x u) = \frac{1}{2}\big(\nabla_x u + (\nabla_x u)^T\big)$$

is the symmetric part of the deformation gradient, where $(\nabla_x u)^T$ denotes the transpose matrix. The elasticity tensor $D:\mathcal{S}^3\to\mathcal{S}^3$ is a linear symmetric, positive definite mapping, $\overline{\varepsilon}\in\mathcal{S}^3$ is a given constant matrix, the transformation strain, and $\mu>0$ and $\lambda>0$ are parameters. The elastic energy is given by

$$\mathsf{W}\big(\varepsilon(\nabla_x u), S\big) = \frac{1}{2}\Big(D\big(\varepsilon(\nabla_x u) - \overline{\varepsilon}S\big)\Big) : \big(\varepsilon(\nabla_x u) - \overline{\varepsilon}S\big), \tag{1.7}$$

with the matrix scalar product $A:B = \sum_{i,j} a_{ij}b_{ij}$. Using (1.2), we obtain for the derivative

$$\partial_S \mathsf{W}(\varepsilon, S) = -\overline{\varepsilon} : D\big(\varepsilon(\nabla_x u) - \overline{\varepsilon}S\big) = -\overline{\varepsilon} : T. \tag{1.8}$$

$c>0$ is a given constant and $\hat{\psi}:\mathbb{R}\to[0,\infty)$ is a double well potential satisfying

$$\hat{\psi}(0) = \hat{\psi}(1) = 0, \qquad \hat{\psi}(\zeta) > 0 \text{ for } \zeta\neq 0,1.$$

The precise assumptions on $\hat{\psi}$, which we need in our investigations, are stated in Theorem 2.3. This completes the formulation of the model.

(1.1) and (1.2) are the equations of linear elasticity theory. This subsystem is coupled to the Allen-Cahn equation (1.3), which governs the evolution of the order parameter S. The system (1.1) – (1.3) satisfies the second law of thermodynamics. More precisely, the Clausius-Duhem inequality is satisfied with the free energy

$$\psi^*_\mu(\varepsilon, S) = \mathsf{W}(\varepsilon, S) + \frac{1}{\mu^{1/2}}\hat{\psi}(S) + \frac{\mu^{1/2}\lambda}{2}|\nabla_x S|^2. \tag{1.9}$$

From this expression we see that the parameter λ determines the energy density of the phase interface. The scaling $\frac{c}{(\mu\lambda)^{1/2}}$ on the right hand side of (1.3) is necessary for otherwise the propagation speed of the diffuse interface would tend to zero for $\mu\to 0$ or $\lambda\to 0$.

We assume that the parameter μ and λ vary in intervals $(0, \mu_0]$ and $(0, \lambda_0]$, respectively, with $\mu_0 > 0$ and $\lambda_0 > 0$ chosen sufficiently small. For all values of μ and λ in these intervals we construct an approximate solution $(u^{(\mu)}, T^{(\mu)}, S^{(\mu)})$ of the equations (1.1) – (1.5) in the bounded domain

$$Q = [t_1, t_2] \times \Omega \subseteq \mathbb{R}^4, \tag{1.10}$$

where $0 \leq t_1 < t_2 < \infty$ are given, fixed times. The approximate solution satisfies these equations up to a residue, which tends to zero for $\mu \to 0$, hence $(u^{(\mu)}, T^{(\mu)}, S^{(\mu)})$ is an asymptotic solution with respect to μ. Of course, the asymptotic solution also depends on λ, hence $(u^{(\mu)}, T^{(\mu)}, S^{(\mu)}) = (u^{(\mu\lambda)}, T^{(\mu\lambda)}, S^{(\mu\lambda)})$, but for simplicity we mostly drop the parameter λ in the notation. The initial condition (1.6) is stated only for completeness, since the initial condition at time $t = t_1$ is implicitly given by our construction. We choose this initial condition and the time interval $[t_1, t_2]$ such that the diffusive phase interface does not reach the boundary in this interval of time.

The asymptotic solution, which we construct, is of second order. To define what we mean by the order of our asymptotic solution, we first sketch the construction of this asymptotic solution, which is carried out precisely in Sections 2 and 3, and how we determine the kinetic relation from this asymptotic solution. The construction starts from a family $t \to \Gamma^{(\mu)}(t) \subseteq \Omega$ of two dimensional regular surfaces, which moves with an as yet unknown normal speed $s^{(\mu)} = s^{(\mu)}(t, x) \in \mathbb{R}$, where $(t, x) \in \Gamma^{(\mu)}(t)$. Therefore

$$\Gamma^{(\mu)} = \{(t, x) \in Q \mid x \in \Gamma^{(\mu)}(t)\}$$

is a regular three dimensional surface in Q. As usual, to construct the functions $u^{(\mu)}$, $T^{(\mu)}$ and $S^{(\mu)}$ we use an ansatz in the form of an asymptotic expansion in a neighborhood of the given surface $\Gamma^{(\mu)}$, the inner expansion, and another asymptotic expansion away from this surface, the outer expansion. The inner expansion for $S^{(\mu)}$ is chosen such that $S^{(\mu)}$ is a transition function, which transits from 0 to 1 in a neighborhood of the surface $\Gamma^{(\mu)}$ and such that this surface is a level set of $S^{(\mu)}$:

$$\Gamma^{(\mu)} = \big\{(t, x) \in Q \ \big|\ S^{(\mu)}(t, x) = \frac{1}{2}\big\}. \tag{1.11}$$

We insert the expansions into the model equations (1.1) – (1.3). It turns out that this ansatz defines an asymptotic solution $(u^{(\mu)}, T^{(\mu)}, S^{(\mu)})$ of these equations only if the normal speed $s^{(\mu)}$ of $\Gamma^{(\mu)}(t)$ has a certain definite value. Since by (1.11) the surface $\Gamma^{(\mu)}(t)$ is a level set of the order parameter $x \mapsto S^{(\mu)}(t, x)$ at time t, it is clear that $s^{(\mu)}$ is the propagation speed of the diffuse interface defined by the transition region of the approximate order parameter $S^{(\mu)}$ and is an approximation to the propagation speed of the diffuse interface defined by the exact solution of the phase field model. We prove in Section 6 that the error of approximation tends to zero for $\mu \to 0$.

Therefore the law determining the normal speed $s^{(\mu)}$ is an approximation of the exact kinetic relation of the Allen-Cahn phase field model. This law has the form of an asymptotic expansion in powers of $\mu^{1/2}$. We therefore call it the asymptotic expansion of the kinetic relation of the Allen-Cahn phase field model. The law defines an evolution problem for the family $t \mapsto \Gamma^{(\mu)}(t)$ of surfaces. In investigations of the asymptotics of phase field models this evolution problem is usually considered to be a sharp interface problem, but because of the complicated form, which this evolution problem takes for an asymptotic solution of second order, we refrain from this view.

Since the components $u^{(\mu)}$, $T^{(\mu)}$ and $S^{(\mu)}$ need different numbers of terms in their ansatz and since the rate, with which the residue tends to zero for $\mu \to 0$, depends on the norm used in the estimates, we use the expansion of the kinetic relation to define the order of the asymptotic solution:

Definition 1.1 *We call a function* $(u^{(\mu)}, T^{(\mu)}, S^{(\mu)})$ *asymptotic solution of* (1.1) – (1.3) *of order* $m+1$*, if the propagation speed* $s^{(\mu)}$ *has an expansion of the form*

$$s^{(\mu)}(t,x) = \sum_{k=0}^{\ell_0} s_{0k}(t,x)\lambda^{k/2} + \mu^{1/2}\sum_{k=0}^{\ell_1} s_{1k}(t,x)\lambda^{k/2} + \dots \mu^{m/2}\sum_{k=0}^{\ell_m} s_{mk}(t,x)\lambda^{k/2}, \quad (1.12)$$

for $(t,x) \in \Gamma^{(\mu)}$*, where the terms* s_{jk} *are independent of* μ *and* λ*.*

This definition differs from the standard convention, which would be to call this an asymptotic solution of order m. We use this different defintion, since to us it seems to be more intuitive.

Though we construct an asymptotic solution of second order, the error of approximation of the boundary conditions (1.5) tends to zero not faster than in an asymptotic solution of first order; to obtain a higher order of approximation of these boundary conditions we had to include boundary expansions in the asymptotic solution. However, the boundary expansions do not influence the asymptotic expansion of the kinetic relation. Since it is our goal to determine this kinetic relation, we avoid the highly technical construction of boundary expansions.

The paper is organized as follows. The main results are contained in Sections 2, where we first give the evolution law for the family of surfaces $t \mapsto \Gamma^{(\mu)}(t)$. Subsequently we specify in Theorem 2.3 properties of the asymptotic solution of second order, which is constructed in later sections. In particular, we obtain an asymptotic expansion of the kinetic relation of the Allen-Cahn model and we find the scaling law for the width of the diffuse interface.

These properties are needed in Section 2.4, which is the central part of our paper. Here we first give a precise definition for the model error $\mathcal{E}$ and go on to deduce a lower bound for the numerical effort needed to simulate the propagation of an interface without interface energy as a function of the prescribed total error of the simulation. This estimate is stated in Corollary 2.9. As stated above, to prove this corollary we need an estimate for the difference between the exact kinetic relation and the asymptotic expansion, which is uniform with repsect to λ. This estimate is stated in Theorem 2.8. The proof of this theorem uses properties of the asymptotic solution and is therefore postponed to Section 6.

Section 2.4 closes with a short comparison of the numerical effort needed in simulations based on the Allen-Cahn model to simulations based on the hybrid model.

Sections 3 – 5 contain the proof of Theorem 2.3. In Section 3 we construct the function $(u^{(\mu)}, T^{(\mu)}, S^{(\mu)})$. That is, we state the inner and outer expansions which define the function $(u^{(\mu)}, T^{(\mu)}, S^{(\mu)})$. In these asymptotic expansions functions appear, which are obtained as solutions of systems of algebraic and differential equations. These systems are also stated in Section 3. The system for the outer expansion can be readily solved, the solution of the system of ordinary differential equations for the inner expansion is more involved and is discussed in Section 4. In two equations of this system a linear differential operator appears with kernel different from $\{0\}$. In order that these differential equations be solvable the right hand sides must therefore satisfy orthogonality conditions. The right hand sides contain the coefficients of the asymptotic expansion of the kinetic relation. The orthogonality conditions dictate the values of these coefficients; therefrom the asymptotic expansion of the kinetic relation originates. Finally, in Section 5 we verify that $(u^{(\mu)}, T^{(\mu)}, S^{(\mu)})$ is really an asymptotic solution of the model equations (1.1) – (1.3) and prove the necessary estimates.

In the bibliography of [8] we gave many references to the literature on existence, uniqueness and asymptotics for models containing the Allen-Cahn and Cahn-Hilliard equations. We refer the reader to that bibliography and discuss here only some publications, which are of interest in the construction of asymptotic solutions.

We believe that for the model (1.1) – (1.3) an asymptotic solution was constructed and used to identify the associated sharp interface problem for the first time in [23], following earlier such investigations for other phase field models. For example, in [13] these investigations were carried out for a model from solidification theory, which consists ot the Allen-Cahn equation coupled to the heat equation.

The considerations in [13, 23] are formal, since it is not shown that the asymptotic solution converges to an exact solution of the model equations for $\mu \to 0$. Under the assumption that the associated sharp interface problems have smooth solutions, this was proved in [28] for the Allen-Cahn equation, in [10] for the Cahn-Hilliard equation, in [14] for the model from solidification theory and in [1] for a model consisting of the Cahn-Hilliard equation coupled with the elasticity equations. The proofs use variants of a spectral estimate derived in [16]. For the model from solidification theory the associated sharp interface model is the Mullins-Sekerka model with surface tension.

In [15] an asymptotic solution for the Cahn-Hilliard equation has recently been constructed with a method different from the one used in [10], and which is similar to our method.

There are several papers, whose motivation is to improve the efficiency of numerical simulations: If in the asymptotic expansion (1.12) the coefficient of $\mu^{1/2}$ vanishes, then the difference between the approximate propagation speed $s^{(\mu)}$ and the propagation speed of the diffuse interface defined by the exact solution of the phase field model tends to zero like $O(\mu)$. As will become clear from the discussion in Section 2.4, this improves the effectivity of numerical simulations based on the phase feld model. This idea lies behind the investigations in [25], where for the phase field model consisting of the Allen-Cahn equation coupled to the heat equation it is shown that if one introduces a μ dependent kinetic coefficient and chooses the double well potential and coupling term suitably, then one can achieve that the coefficient of $\mu^{1/2}$ vanishes. This result has been improved and generalized in [9, 17, 24]. A similar idea is also present in [22].

Since the construction of asymptotic solutions is based on sharp interface problems, a rigorous analysis of these problems is of special interest. Of particular interest is the Hele-Shaw problem with surface tension, since this is the sharp interface problem associated with the Cahn-Hilliard equation. Existence, uniqueness, and regularity of classical solutions of this problem have been investigated in [18, 19, 20]. In [21] it is shown that if the initial data are close to a sphere then a classical solution exists and converges to spheres. Existence of solutions to the Mullins-Sekerka problem mentioned above has been shown in [27].

2 The kinetic relation

2.1 Notations

To state the main results we need some notations. Let $\mu > 0$ and assume that $\Gamma^{(\mu)}$ is an orientable, three dimensional C^k–manifold with $k \geq 1$ sufficiently large embedded in Q such that $\Gamma^{(\mu)}(t)$ is a regular two dimensional surface in Ω for every $t \in [t_1, t_2]$. In the following we drop the superscript μ and write $\Gamma = \Gamma^{(\mu)}$, $\Gamma(t) = \Gamma^{(\mu)}(t)$ to simplify the notation, but the manifolds do actually depend on μ. Let

$$n : \Gamma \to \mathbb{R}^3 \tag{2.1}$$

be a continuous vector field such that $n(t,x) \in \mathbb{R}^3$ is a unit normal vector to $\Gamma(t)$ at $x \in \Gamma(t)$, for every $t \in [t_1, t_2]$. For $\delta > 0$ and $t \in [t_1, t_2]$ define the sets

$$\mathcal{U}_\delta(t) = \{x \in \Omega \mid \operatorname{dist}(x, \Gamma(t)) < \delta\} \quad \text{and} \quad \mathcal{U}_\delta = \{(t,x) \in Q \mid x \in \mathcal{U}_\delta(t)\}. \tag{2.2}$$

We assume that there is $\delta > 0$ such that $\mathcal{U}_\delta \subseteq Q$. Since Γ is a regular C^1–manifold in Q, then δ can be chosen sufficiently small such that for all $t \in [t_1, t_2]$ the mapping

$$(\eta, \xi) \mapsto x(t, \eta, \xi) = \eta + \xi n(t, \eta) : \Gamma(t) \times (-\delta, \delta) \to \mathcal{U}_\delta(t) \tag{2.3}$$

is bijective. We say that this mapping defines new coordinates (η, ξ) in $\mathcal{U}_\delta(t)$ and (t, η, ξ) in $\mathcal{U}_\delta$. If no confusion is possible we switch freely between the coordinates (t,x) and (t, η, ξ). In particular, if $(t,x) \mapsto w(t,x)$ is a function defined on $\mathcal{U}_\delta$ we write $w(t, \eta, \xi)$ for $w\big(t, x(t, \eta, \xi)\big)$, as usual.

We use the standard convention and denote for a function w defined on a subset U of Q by $w(t)$ the function $x \mapsto w(t,x)$, which is defined on the set $\{x \mid (t,x) \in U\} \subseteq \mathbb{R}^3$.

If w is a function defined on $\mathcal{U}_\delta(t) \setminus \Gamma(t)$, we set for $\eta \in \Gamma(t)$

$$
\begin{aligned}
w^{(\pm)}(\eta) &= \lim_{\substack{\xi \to 0 \\ \xi > 0}} w\big(\eta \pm \xi n(t, \eta)\big), \\
(\partial_n^i w)^{(+)}(\eta) &= \lim_{\substack{\xi \to 0 \\ \xi > 0}} \frac{\partial^i}{\partial \xi^i} w\big(\eta + \xi n(t, \eta)\big), \quad i \in \mathbb{N}, \\
(\partial_n^i w)^{(-)}(\eta) &= \lim_{\substack{\xi \to 0 \\ \xi < 0}} \frac{\partial^i}{\partial \xi^i} w\big(\eta + \xi n(t, \eta)\big), \quad i \in \mathbb{N}, \\
[w](\eta) &= w^{(t)}(\eta) - w^{(-)}(\eta), \\
[\partial_n^i w](\eta) &= (\partial_n^i w)^{(+)}(\eta) - (\partial_n^i w)^{(-)}(\eta), \\
\langle w \rangle(\eta) &= \frac{1}{2}\big(w^{(+)}(\eta) + w^{(-)}(\eta)\big),
\end{aligned}
$$

provided that the one-sided limits in these equations exist. If w is defined on $\mathcal{U}_\delta \setminus \Gamma$, we set

$$w^{(\pm)}(t, \eta) = \big(w^{(\pm)}(t)\big)(\eta), \qquad (\partial_n^i w)^{(\pm)}(t, \eta) = \big((\partial_n^i w)^{(\pm)}(t)\big)(\eta),$$

and define $[w](t, \eta)$, $\langle w \rangle(t, \eta)$, $[\partial_n^i w](t, \eta)$ as above. Let $\tau_1(\eta)$, $\tau_2(\eta) \in \mathbb{R}^3$ be two orthogonal unit vectors to $\Gamma(t)$ at $\eta \in \Gamma(t)$. For functions $w : \Gamma(t) \to \mathbb{R}$, $W : \Gamma(t) \to \mathbb{R}^3$ we define the surface gradients by

$$
\begin{aligned}
\nabla_\Gamma w &= (\partial_{\tau_1} w)\tau_1 + (\partial_{\tau_2} w)\tau_2, && (2.4) \\
\nabla_\Gamma W &= (\partial_{\tau_1} W) \otimes \tau_1 + (\partial_{\tau_2} W) \otimes \tau_2, && (2.5)
\end{aligned}
$$

where for vectors $c, d \in \mathbb{R}^3$ a 3×3-matrix is defined by

$$c \otimes d = (c_i d_j)_{i,j=1,2,3}.$$

With (2.4), (2.5) we have for functions $w : \mathcal{U}_\delta(t) \to \mathbb{R}$ and $W : \mathcal{U}_\delta(t) \to \mathbb{R}^3$ at $\eta \in \Gamma(t)$ the decompositions

$$\nabla_x w = (\partial_n w)n + \nabla_\Gamma w, \tag{2.6}$$

$$\nabla_x W = (\partial_n W) \otimes n + \nabla_\Gamma W, \tag{2.7}$$

where $n = n(t, \eta)$ is the unit normal vector to $\Gamma(t)$.

The normal speed of the family of surfaces $t \mapsto \Gamma(t)$ is of fundamental importance in this paper. Therefore we give a precise definition.

Definition 2.1 *Let $m(t, \eta) = \big(m'(t, \eta), m''(t, \eta)\big) \in \mathbb{R} \times \mathbb{R}^3$ be a normal vector to Γ at $(t, \eta) \in \Gamma$. The normal speed of the family of surfaces $t \mapsto \Gamma(t)$ at $\eta \in \Gamma(t)$ is defined by*

$$s(t, \eta) = \frac{-m'(t, \eta)}{m''(t, \eta) \cdot n(t, \eta)}, \tag{2.8}$$

with the unit normal vector $n(t, \eta) \in \mathbb{R}^3$ to $\Gamma(t)$.

Note that with this definion the speed is measured positive in the direction of the normal vector field n. Since $m''(t, \eta) \in \mathbb{R}^3$ is a normal vector to $\Gamma(t)$, the denominator in (2.8) is different from zero.

If $\omega = (\omega', \omega'') \in \mathbb{R} \times \mathbb{R}^3$ is a tangential vector to Γ at (t, η) with $\omega' \neq 0$, then with the unit normal $n(t, \eta) \in \mathbb{R}^3$ to $\Gamma(t)$ the vector $(-\omega'' \cdot n, \omega' n)$ is a normal vector to Γ at (t, η), hence (2.8) implies that the normal speed at $\eta \in \Gamma(t)$ is given by

$$s(t, \eta) = \frac{n \cdot \omega''}{\omega' n \cdot n} = \frac{n \cdot \omega''}{\omega'}. \tag{2.9}$$

For later use we prove the following

Lemma 2.2 *Let $x \in \mathcal{U}_\delta(t_0)$ be a point having the representation $x = \eta + n(t, \eta)\xi$ in the (η, ξ)–coordinates, where $\eta = \eta(t, x) \in \Gamma(t)$ and $\xi = \xi(t, x)$. Then the normal speed satisfies*

$$s(t_0, \eta) = n(t_0, \eta) \cdot \partial_t \eta(t_0, x) = -\partial_t \, \xi(t_0, x). \tag{2.10}$$

The tangential component of the vector $\partial_t \eta(t_0, x) \in \mathbb{R}^3$ to the surface $\Gamma(t_0)$ is equal to $-\xi \partial_t n\big(t_0, \eta(t_0, x)\big)$.

Proof: By definition of $\mathcal{U}_\delta$, there is a neighborhood U of t_0 in $[t_1, t_2]$ such that $\{x\} \times U \subseteq \mathcal{U}_\delta$, which implies that x has the representation

$$x = \eta(t, x) + \xi(t, x) n\big(t, \eta(t, x)\big).$$

for all $t \in U$. We differentiate this equation and obtain

$$0 = \partial_t x = n \, \partial_t \xi + \xi \, \partial_t n + \partial_t \eta. \tag{2.11}$$

From $0 = \partial_t 1 = \partial_t |n|^2 = 2n \cdot \partial_t n$ we see that $\partial_t n$ is tangential to $\Gamma(t)$, hence (2.11) implies that the tangential component of $\partial_t \eta$ is equal to $-\xi \partial_t n$. Multiplication of (2.11) with n yields

$$\partial_t \xi = -n \cdot \partial_t \eta. \tag{2.12}$$

Since $\partial_t\big(t, \eta(t, x)\big) = \big(1, \partial_t \eta(t, x)\big)$ is a tangential vector to Γ, it follows from (2.9) that $s = \frac{n \cdot \partial_t \eta}{1} = n \cdot \partial_t \eta$, which together with (2.12) implies (2.10). ■

2.2 The evolution problem for the level set $\Gamma^{(\mu)}$

The level set $\Gamma = \Gamma^{(\mu)}$ of $S^{(\mu)}$ defined in (1.11) is determined by an evolution problem for the family of surfaces $t \mapsto \Gamma(t)$. To state this evolution problem let $\mathcal{N}$ be the operator, which assigns the normal speed to the family $t \mapsto \Gamma(t)$, i.e.

$$s(t,x) = \mathcal{N}(\Gamma)(t,x),$$

with $s(t,x) = s^{(\mu)}(t,x)$ defined by (2.8). The evolution problem is given by

$$\mathcal{N}(\Gamma)(t) = \mathcal{K}^{(\mu)}\big(\Gamma(t)\big), \qquad t_1 \le t \le t_2\,, \tag{2.13}$$

where $\mathcal{K}^{(\mu)}$ is the non-local evolution operator, which has the form

$$\mathcal{K}^{(\mu)}\big(\Gamma(t)\big)(x) = s_0\big(\hat{T}, \kappa_\Gamma, \lambda^{1/2}\big)(t,x) + \mu^{1/2} s_1\big(\hat{u}, \hat{T}, \check{T}, S_0, S_1, \lambda^{1/2}\big)(t,x), \tag{2.14}$$

for $x \in \Gamma(t)$. Here $(\hat{u}, \hat{T}, \check{u}, \check{T}, S_0, S_1)$ is the solution of a transmission-boundary value problem for a coupled system of elliptic partial differential equations and ordinary differential equations, which can be solved recursively. $\kappa_\Gamma(t,x)$ denotes twice the mean curvature of the surface $\Gamma(t)$ at $x \in \Gamma(t)$. With the principle curvatures κ_1, κ_2 of $\Gamma(t)$ at $x \in \Gamma(t)$ we thus have

$$\kappa_\Gamma(t,x) = \kappa_1(t,x) + \kappa_2(t,x).$$

The transmission condition is posed on $\Gamma(t)$. Therefore the functions $\hat{u}$, $\hat{T}$, $\check{u}$, $\check{T}$ and S_1 depend on $\Gamma(t)$. We first state and discuss the transmission-boundary value problem. The precise form of the functions s_0 and s_1 is given in Theorem 2.3 following below.

Let $\hat{S}:\ Q \setminus \Gamma \to \{0,1\}$ be a piecewise constant function, which only takes the values 0 and 1 with a jump across Γ. The sets

$$\gamma = \{(t,x) \in Q \setminus \Gamma \mid \hat{S}(t,x) = 0\}, \qquad \gamma(t) = \{x \in \Omega \setminus \Gamma(t) \mid (t,x) \in \gamma\},$$
$$\gamma' = \{(t,x) \in Q \setminus \Gamma \mid \hat{S}(t,x) = 1\}, \qquad \gamma'(t) = \{x \in \Omega \setminus \Gamma(t) \mid (t,x) \in \gamma'\}$$

yield partitions $Q = \gamma \cup \Gamma \cup \gamma'$ and $\Omega = \gamma(t) \cup \Gamma(t) \cup \gamma'(t)$ of Q and Ω, respectively. If x belongs to $\gamma(t)$ or $\gamma'(t)$, then the crystal structure at the material point x at time t belongs to phase 1 or phase 2, respectively. We assume that the normal vector field n given in (2.1) is such that the vector $n(t,x)$ points into the set $\gamma'(t)$ for every $x \in \Gamma(t)$.

The transmission-boundary value problem can be separated into two transmission-boundary value problems for the elasticity equations and a boundary value problem for a coupled system of two ordinary differential equations. To state the complete problem we fix $t \in [t_1, t_2]$ and assume that $\Gamma(t)$ is known. In the first transmission-boundary problem the unknowns are the displacement $x \mapsto \hat{u}(t,x) \in \mathbb{R}^3$ and the stress tensor $x \mapsto \hat{T}(t,x) \in \mathcal{S}^3$, which must satisfy the equations

$$-\mathrm{div}_x \hat{T} = \mathsf{b}, \tag{2.15}$$
$$\hat{T} = D\big(\varepsilon(\nabla_x \hat{u}) - \bar{\varepsilon}\hat{S}\big), \tag{2.16}$$
$$[\hat{u}] = 0, \tag{2.17}$$
$$[\hat{T}]n = 0, \tag{2.18}$$
$$\hat{u}(t)\big|_{\partial\Omega} = \mathsf{U}(t), \tag{2.19}$$

with b and U given in (1.1) and (1.4). In the second transmission-boundary problem the unknowns are the displacement $x \mapsto \check{u}(t,x) \in \mathbb{R}^3$ and the stress tensor $x \mapsto \check{T}(t,x) \in \mathcal{S}^3$, and the problem is

$$-\mathrm{div}_x \check{T} = 0, \tag{2.20}$$

$$\check{T} = D\Big(\varepsilon(\nabla_x \check{u}) - \overline{\varepsilon}\,\frac{\hat{T} : \overline{\varepsilon}}{\hat{\psi}''(\hat{S})}\Big), \tag{2.21}$$

$$[\check{u}] = 0, \tag{2.22}$$

$$[\check{T}]n = 0, \tag{2.23}$$

$$\check{u}(t)\big|_{\partial\Omega} = 0. \tag{2.24}$$

The equations (2.15), (2.16) and (2.20), (2.21) must hold on the set $\Omega \setminus \Gamma(t)$, whereas the equations (2.17), (2.18) and (2.22), (2.23) are posed on $\Gamma(t)$.

In the boundary value problem for the ordinary differential equations the unknowns are $S_0 : \mathbb{R} \to \mathbb{R}$, $S_1 : \Gamma \times \mathbb{R} \to \mathbb{R}$ and $s_0 : \Gamma \to \mathbb{R}$. We use the notations $S_1'(t,\eta,\zeta) = \partial_\zeta S_1(t,\eta,\zeta)$, $S_1''(t,\eta,\zeta) = \partial_\zeta^2 S_1(t,\eta,\zeta)$. In this problem not only t, but also $\eta \in \Gamma(t)$ is a parameter. For all $\zeta \in \mathbb{R}$ and all values of the parameter $\eta \in \Gamma(t)$ the unknowns must satisfy the coupled ordinary differential equations

$$\hat{\psi}'\big(S_0(\zeta)\big) - S_0''(\zeta) = 0, \tag{2.25}$$

$$\hat{\psi}''\big(S_0(\zeta)\big)S_1(t,\eta,\zeta) - S_1''(t,\eta,\zeta) = F_1(t,\eta,\zeta), \tag{2.26}$$

and the boundary conditions

$$S_0(0) = \frac{1}{2}, \quad \lim_{\zeta \to -\infty} S_0(\zeta) = 0, \quad \lim_{\zeta \to \infty} S_0(\zeta) = 1, \tag{2.27}$$

$$\lim_{\zeta \to -\infty} S_1(t,\eta,\zeta) = \frac{\overline{\varepsilon} : \hat{T}^{(-)}(t,\eta)}{\hat{\psi}''(0)}, \tag{2.28}$$

$$\lim_{\zeta \to +\infty} S_1(t,\eta,\zeta) = \frac{\overline{\varepsilon} : \hat{T}^{(+)}(t,\eta)}{\hat{\psi}''(1)}, \tag{2.29}$$

$$S_1(t,\eta,0) = 0, \tag{2.30}$$

with the right hand side of (2.26) given by

$$\begin{aligned} F_1(t,\eta,\zeta) = \overline{\varepsilon} : \Big([\hat{T}](t,\eta)S_0(\zeta) + \hat{T}^{(-)}(t,\eta)\Big) \\ + \Big(\frac{s_0(t,\eta)}{c} - \lambda^{1/2}\kappa_\Gamma(t,\eta)\Big)S_0'(\zeta), \end{aligned} \tag{2.31}$$

where the constant $c > 0$ is given in (1.3).

The linear elliptic system (2.15), (2.16) differs from the standard elasticity system only by the term $-D\overline{\varepsilon}\hat{S}$. This term is known since $\Gamma(t)$ is given. Under suitable regularity assumptions for the given functions b and U and very mild assumptions on the regularity of the interface $\Gamma(t)$ the problem has a unique weak solution $(\hat{u}, \hat{T})$. This can be proved by standard methods from functional analysis. Of course, the regularity of the solution depends on the regularity of b, U and $\Gamma(t)$.

After insertion of the stress tensor $\hat{T}$ from this solution into (2.21), the equations (2.20) – (2.24) form a transmission-boundary value problem of the same type as (2.15) – (2.19), with unique solution $(\check{u}, \check{T})$ determined by the same methods.

We also insert $\hat{T}$ into (2.28), (2.29) and (2.31), which determines the right hand side of the differential equation (2.26) and the boundary conditions (2.28), (2.29) posed at $\pm\infty$. The nonlinear differential equation (2.25) has a unique solution S_0 satisfying the boundary conditions (2.27). By insertion of S_0 into (2.26) and (2.31), equation (2.26) becomes a linear differential equation for S_1, however with an additional unknown function s_0 in the right hand side. This function is constant with respect to ζ. We sketch here the procedure used to determine s_0. This procedure is standard in investigations of the asymptotics of phase field models:

The second order differential operator $\big(\hat{\psi}''(S_0) - \partial_\zeta^2\big)$ is selfadjoint in the Hilbert space $L^2(\mathbb{R})$ with a one dimensional kernel spanned by the function S_0'. This is seen by differentiating the equation (2.25). From functional analysis we thus know that for $F_1 \in L^2(\mathbb{R})$ the differential equation $\big(\hat{\psi}''(S_0) - \partial_\zeta^2\big)w = F_1$ has a solution in $w \in L^2(\mathbb{R})$ if and only if the orthogonality condition

$$\int_{-\infty}^{\infty} F_1(t, \eta, \zeta) S_0'(\zeta)\, d\zeta = 0 \tag{2.32}$$

holds. It turns out that though the function F_1 defined in (2.31) does not in general belong to $L^2(\mathbb{R})$ and the solution $S_1(t, \eta, \cdot)$ is not sought in $L^2(\mathbb{R})$, which is seen from the boundary conditions (2.28), (2.29), the orthogonality condition (2.32) is sufficient for the solution S_1 to exist. Comparison with (2.31) shows that (2.32) can be satisfied by choosing the constant $s_0(t, \eta)$ suitably. This defines the function $s_0 : \Gamma \to \mathbb{R}$ uniquely. Since F_1 depends on $\hat{T}$, κ_Γ and $\lambda^{1/2}$, it follows that also s_0 is a function of these variables:

$$s_0(t, \eta) = s_0\big(\hat{T}, \kappa_\Gamma, \lambda^{1/2}\big)(t, x).$$

The explicit expression for s_0 obtained in this way is stated below in (2.39). In fact, $s_0(\hat{T}, \kappa_\Gamma, \lambda^{1/2})$ is the first term on the right hand side in the expression (2.14) for $\mathcal{K}^{(\mu)}\big(\Gamma(t)\big)$.

The procedure sketched here is discussed precisely in Section 4.3 when we determine the second term s_1 in (2.14), which is obtained from a similar, but more complicated boundary value problem.

2.3 The asymptotic solution and the kinetic relation

To state the properties of the asymptotic solution and the kinetic relation in Theorem 2.3, we introduce some definitions.

We need in our investigations that the second derivatives $\hat{\psi}''(0)$ and $\hat{\psi}''(1)$ of the double well potential at the minima 0 and 1 are positive, and we set

$$a = \min\Big\{\sqrt{\hat{\psi}''(0)}, \sqrt{\hat{\psi}''(1)}\Big\}.$$

Depending on the parameters λ and μ, we partition Q into the inner neighborhood $Q_{\text{inn}}^{(\mu\lambda)}$ of Γ, into the matching region $Q_{\text{match}}^{(\mu\lambda)}$ and into the outer region $Q_{\text{out}}^{(\mu\lambda)}$. These sets are defined by

$$
\begin{aligned}
Q_{\text{inn}}^{(\mu\lambda)} &= \Big\{(t,\eta,\xi)\in\mathcal{U}_\delta \;\Big|\; |\xi| < \frac{3}{2}\frac{(\mu\lambda)^{1/2}|\ln\mu|}{a}\Big\},\\
Q_{\text{match}}^{(\mu\lambda)} &= \Big\{(t,\eta,\xi)\in\mathcal{U}_\delta \;\Big|\; \frac{3}{2}\frac{(\mu\lambda)^{1/2}|\ln\mu|}{a} \le |\xi| \le \frac{3(\mu\lambda)^{1/2}|\ln\mu|}{a}\Big\},\\
Q_{\text{out}}^{(\mu\lambda)} &= Q\setminus\big(Q_{\text{inn}}^{(\mu\lambda)}\cup Q_{\text{match}}^{(\mu\lambda)}\big).
\end{aligned}
\tag{2.33}
$$

We always assume that the parameters λ and μ satisfy $0<\lambda\le\lambda_0$ and $0<\mu\le\mu_0$, where λ_0, μ_0 are fixed constants satisfying

$$
\mu_0 \le e^{-2}, \qquad \frac{3(\mu_0\lambda_0)^{1/2}|\ln\mu_0|}{a} < \delta.
$$

The first condition is imposed for purely technical reasons and guarantees that the function $\mu\mapsto\mu^{1/2}|\ln\mu|$ is increasing, the second condition guarantees that $Q_{\text{inn}}^{(\mu\lambda)}, Q_{\text{match}}^{(\mu\lambda)}\subset\mathcal{U}_\delta$ and that $Q_{\text{out}}^{(\mu\lambda)}\cap\mathcal{U}_\delta$ is a nonempty, relatively open subset of $\mathcal{U}_\delta$.

By (2.17), the function $\hat{u}: Q\to\mathbb{R}^3$ is continuous at every point $(t,\eta)\in\Gamma$, but the first and higher derivatives of $\hat{u}$ in the direction of the normal vector $n(t,x)$ can jump across Γ. For these jumps we write

$$
u^*(t,\eta) = [\partial_n\hat{u}](t,\eta), \tag{2.34}
$$

$$
a^*(t,\eta) = [\partial_n^2\hat{u}](t,\eta). \tag{2.35}
$$

We also set

$$
c_1 = \int_0^1 \sqrt{2\hat{\psi}(\vartheta)}\,d\vartheta. \tag{2.36}
$$

Theorem 2.3 *Suppose that the double well potential $\hat{\psi}\in C^5(\mathbb{R})$ satisfies*

$$
\begin{aligned}
\hat{\psi}(r) > 0, &\quad \textit{for } 0<r<1,\\
\hat{\psi}(r) = \hat{\psi}'(r) = 0, &\quad \textit{for } r=0,1,\\
a = \min\Big\{\sqrt{\hat{\psi}''(0)}, \sqrt{\hat{\psi}''(1)}\Big\} > 0.&
\end{aligned}
\tag{2.37}
$$

Moreover, suppose that $\hat{\psi}$ satisfies the symmetry condition

$$
\hat{\psi}(\tfrac{1}{2}-\zeta) = \hat{\psi}(\tfrac{1}{2}+\zeta), \quad \zeta\in\mathbb{R}. \tag{2.38}
$$

Assume that there is a solution Γ of the evolution problem (2.13), (2.14) with $s_0 = s_0(\hat{T},\kappa_\Gamma,\lambda^{1/2}):\Gamma\to\mathbb{R}$ given by

$$
s_0 = \frac{c}{c_1}\Big(-\overline{\varepsilon} : \langle\hat{T}\rangle + \lambda^{1/2}c_1\kappa_\Gamma\Big), \tag{2.39}
$$

and with $s_1 = s_1(\hat{u},\hat{T},\check{T},S_0,S_1,\lambda^{1/2}):\Gamma\to\mathbb{R}$ defined by

$$
s_1 = s_{10} + \lambda^{1/2}s_{11} = s_{10}(\hat{T},\check{T},S_0,S_1) + \lambda^{1/2}s_{11}(\hat{u},S_0), \tag{2.40}
$$

where

$$s_{10} = \frac{c}{c_1}\Bigg(-\overline{\varepsilon}:\langle\check{T}\rangle + \overline{\varepsilon}:[\hat{T}]\Big(\Big\langle\frac{\overline{\varepsilon}:\hat{T}}{\hat{\psi}''(\hat{S})}\Big\rangle - \int_{-\infty}^{\infty} S_1 S_0'\,d\zeta\Big)$$

$$+\frac{1}{c_1}\overline{\varepsilon}:\langle\hat{T}\rangle\int_{-\infty}^{\infty} S_1' S_0'\,d\zeta + \frac{1}{2}\int_{-\infty}^{\infty}\hat{\psi}'''(S_0) S_1^2 S_0'\,d\zeta\Bigg), \tag{2.41}$$

$$s_{11} = -\frac{c}{c_1}\,\overline{\varepsilon}:D\varepsilon(a^*\otimes n + \nabla_\Gamma u^*)\int_{-\infty}^{\infty} S_0(\zeta)S_0(-\zeta)\,d\zeta. \tag{2.42}$$

In (2.39) *and* (2.41), (2.42) *we have* $S_0 = S_0(\zeta)$ *and* $S_1 = S_1(t,\eta,\zeta)$*, for all other functions the argument is* (t,η)*. The positive constant* c *is defined in* (1.3)*. The notations* $[\cdot]$ *and* $\langle\cdot\rangle$ *are introduced in Section 2.1. In particular, we have*

$$\Big\langle\frac{\overline{\varepsilon}:\hat{T}}{\hat{\psi}''(\hat{S})}\Big\rangle = \frac{1}{2}\Big(\frac{\overline{\varepsilon}:\hat{T}^{(+)}}{\hat{\psi}''(1)} + \frac{\overline{\varepsilon}:\hat{T}^{(-)}}{\hat{\psi}''(0)}\Big).$$

With these functions the normal speed $s(t,\eta)$ *of* $\Gamma(t)$ *at* $\eta\in\Gamma(t)$ *is thus given by*

$$s(t,\eta) = s_0(t,\eta) + \mu^{1/2}s_1(t,\eta,\lambda^{1/2}) = s_0(t,\eta) + \mu^{1/2}\big(s_{10}(t,\eta) + \lambda^{1/2}s_{11}(t,\eta)\big). \tag{2.43}$$

We assume moreover that the solution Γ *is a* C^5*–manifold and that the functions* $\hat{u}$ *and* $\check{u}$ *defined by the evolution problem satisfy* $\hat{u}\in C^4(\gamma\cup\gamma',\mathbb{R}^3)$*,* $\check{u}\in C^3(\gamma\cup\gamma',\mathbb{R}^3)$ *and that* $\hat{u}$ *has* C^4*–extensions,* $\check{u}$ *has* C^3*–extensions from* γ *to* $\gamma\cup\Gamma$ *and from* γ' *to* $\gamma'\cup\Gamma$*. For the given right hand side of* (1.1) *we assume that* $\mathsf{b}\in C^1(\overline{Q})$*.*

Under these assumptions there is an approximate solution $(u^{(\mu)},T^{(\mu)},S^{(\mu)})$ *of the Allen-Cahn model* (1.1) – (1.5)*, for which* Γ *is the level set*

$$\Gamma = \Big\{(t,x)\in Q \;\Big|\; S^{(\mu)}(t,x) = \frac{1}{2}\Big\}, \tag{2.44}$$

and which satisfies the equations

$$-\mathrm{div}_x T^{(\mu)} = \mathsf{b} + f_1^{(\mu\lambda)}, \tag{2.45}$$

$$T^{(\mu)} = D\big(\varepsilon(\nabla_x u^{(\mu)}) - \overline{\varepsilon}S^{(\mu)}\big), \tag{2.46}$$

$$\partial_t S^{(\mu)} + \frac{c}{(\mu\lambda)^{\frac{1}{2}}}\Big(\partial_S\mathsf{W}\big(\varepsilon(\nabla_x u^{(\mu)}),S^{(\mu)}\big) + \frac{1}{\mu^{\frac{1}{2}}}\hat{\psi}'(S^{(\mu)}) - \mu^{\frac{1}{2}}\lambda\Delta_x S^{(\mu)}\Big) = f_2^{(\mu\lambda)}, \tag{2.47}$$

$$u^{(\mu)}(t,x) = \mathsf{U}(t,x), \qquad (t,x)\in[t_1,t_2]\times\partial\Omega, \tag{2.48}$$

$$\partial_{n_{\partial\Omega}} S^{(\mu)}(t,x) = f_3^{(\mu\lambda)}, \qquad (t,x)\in[t_1,t_2]\times\partial\Omega, \tag{2.49}$$

where to the right hand sides $f_1^{(\mu\lambda)},\ldots,f_3^{(\mu\lambda)}$ *there exist nonnegative constants* $K_1,\ldots,K_5$ *such that for all* $\mu\in(0,\mu_0]$ *and all* $\lambda\in(0,\lambda_0]$

$$\|f_1^{(\mu\lambda)}\|_{L^\infty(Q_{\mathrm{inn}}^{(\mu\lambda)}\cup Q_{\mathrm{match}}^{(\mu\lambda)})} \le |\ln\mu|^2\Big(\frac{\mu}{\lambda}\Big)^{\frac{1}{2}}K_1\,, \tag{2.50}$$

$$\|f_1^{(\mu\lambda)}\|_{L^\infty(Q_{\mathrm{out}}^{(\mu\lambda)})} \le \mu^{\frac{3}{2}}K_2\,, \tag{2.51}$$

$$\|f_2^{(\mu\lambda)}\|_{L^\infty(Q_{\mathrm{inn}}^{(\mu\lambda)}\cup Q_{\mathrm{match}}^{(\mu\lambda)})} \le |\ln\mu|^2\Big(\frac{\mu}{\lambda}\Big)^{\frac{1}{2}}K_3\,, \tag{2.52}$$

$$\|f_2^{(\mu\lambda)}\|_{L^\infty(Q_{\mathrm{out}}^{(\mu\lambda)})} \le \frac{\mu}{\lambda^{1/2}}K_4\,, \tag{2.53}$$

$$\|f_3^{(\mu\lambda)}\|_{L^\infty(\partial\Omega)} \le \mu^{\frac{1}{2}}K_5\,, \tag{2.54}$$

In the neighborhood $Q_{\rm inn}^{(\mu\lambda)}$ *of* Γ *the order parameter in the approximate solution is of the form*

$$S^{(\mu)}(t,x) = S_0\Big(\frac{\xi}{(\mu\lambda)^{1/2}}\Big) + \mu^{1/2} S_1\Big(t,\eta,\frac{\xi}{(\mu\lambda)^{1/2}}\Big) + \mu S_2\Big(t,\eta,\frac{\xi}{(\mu\lambda)^{1/2}}\Big), \tag{2.55}$$

where the monotonically increasing transition profile $S_0 : \mathbb{R} \to \mathbb{R}$ *and the function* $S_1 : \Gamma \times \mathbb{R} \to \mathbb{R}$ *are given as solution of the coupled problem* (2.25) – (2.31), *and where* $S_2 : \Gamma \times \mathbb{R} \to \mathbb{R}$ *satisfyies* $S_2(t,\eta,0) = 0$ *and*

$$|S_2(t,\eta,\zeta)| \le C(1+|\zeta|), \qquad \text{for } (t,\eta,\zeta) \in \Gamma \times \mathbb{R}, \tag{2.56}$$

with a constant C *independent of* (t,η,ζ).

We mention that the positive constant c in (1.3) does not play a major role in the analysis and could be replaced by 1. We refrain from replacing it to show how c appears in the kinetic relation.

The **proof** of this theorem forms the content of Sections 3 – 5. We remark that the symmetry assumption (2.38) for the double well potential $\hat{\psi}$ serves to simplify the computations in the derivation of the asymptotic solution. Without this assumption the term s_1 in the kinetic relation (2.43) would contain other terms in addition to the terms s_{10} and s_{11} given in (2.41) an (2.42).

The regularity properties of Γ and of $\hat{u}$, $\check{u}$ are of course not independent, since $\hat{u}$ and $\check{u}$ are solutions of the elliptic transmission problems (2.15) – (2.19) and (2.20) – (2.24), respectively. Therefore the regularity theory of elliptic equations shows that $\hat{u}$ and $\check{u}$ automatiucally have the differentiability properties assumed in the theorem if the manifold Γ and the right hand side b are sufficiently smooth.

Since by definition of $Q_{\rm inn}^{(\mu\lambda)}$ and $Q_{\rm match}^{(\mu\lambda)}$ in (2.33) we have

$$\text{meas}(Q_{\rm inn}^{(\mu\lambda)} \cup Q_{\rm match}^{(\mu\lambda)}) \le C_3 (\mu\lambda)^{1/2} |\ln\mu|,$$

we immediately obtain from (2.50) – (2.53) the following

Corollary 2.4 *There are constants* K_6, K_7 *such that for all* $0 < \mu \le \mu_0$ *and all* $0 < \lambda \le \lambda_0$

$$\|f_1^{(\mu\lambda)}\|_{L^1(Q)} \le |\ln\mu|^3 \mu K_6\,, \tag{2.57}$$

$$\Big\|f_2^{(\mu\lambda)}\|_{L^1(Q)} \le \frac{|\ln\mu|^3\,\mu}{\lambda^{1/2}} K_7\,. \tag{2.58}$$

The leading term s_0 given in (2.39) can be written in a more common and more general form. To give this form, we need a result on the jump of the Eshelby tensor. The Eshelby tensor to the solution $(\hat{u},\hat{T})$ of the transmission problem (2.15) – (2.19) is defined by

$$\hat{C}(\nabla_x\hat{u},\hat{S}) = \psi_\mu\big(\varepsilon(\nabla_x\hat{u}),\hat{S}\big)I - (I+\nabla_x\hat{u})^T\hat{T}, \tag{2.59}$$

where $I \in \mathcal{S}^3$ is the unit matrix and where

$$\psi_\mu(\varepsilon,S) = \mathsf{W}(\varepsilon,S) + \frac{1}{\mu^{1/2}}\hat{\psi}(S) \tag{2.60}$$

is that part of the free energy ψ_μ^* defined in (1.9) without gradient term. The last term on the right hand side of (2.59) is a matrix product. We use the standard convention to denote the matrix product of two matrices $A \in \mathbb{R}^{k\times m}$ and $B \in \mathbb{R}^{m\times\ell}$ by $AB \in \mathbb{R}^{k\times\ell}$.

Lemma 2.5 *Let* $(\hat{u},\hat{T})$ *be the solution of the transmission problem* (2.15) – (2.19) *and let* n *be a unit normal vector field to* $\Gamma(t)$*. Then the jump* $[\hat{C}]$ *of the Eshelby tensor to* $(\hat{u},\hat{T})$ *across* Γ *satisfies*

$$n\cdot[\hat{C}]n = \frac{1}{\mu^{1/2}}\,[\hat{\psi}(\hat{S})] - \overline{\varepsilon} : \langle\hat{T}\rangle. \tag{2.61}$$

This result is known [2]. We gave a proof in [7], but since the proof is short and since the definition (2.59) of the Eshelby tensor differs slightly from the definitions used in [7], we give a proof in Section 2.5 for completeness.

Corollary 2.6 *The leading term* s_0 *of the kinetic relation defined in* (2.39) *satisfies*

$$s_0 = \frac{c}{c_1}\Big(n\cdot[\hat{C}]n + \lambda^{1/2}c_1\kappa_\Gamma\Big). \tag{2.62}$$

This corollary follows immediately from (2.61), since by assumption (2.37) we have $[\hat{\psi}(\hat{S})] = \hat{\psi}(1) - \hat{\psi}(0) = 0$, which implies that $n\cdot[\hat{C}]n = -\overline{\varepsilon} : \langle\hat{T}\rangle$. ■

2.4 Consequences for numerical simulations

In this section we discuss the consequences of Theorem 2.3 for numerical simulations of interfaces with small interface energy. We show that the numerical effort can be made as small as possible by choosing the parameters μ and λ in an optimal way, and we derive a lower bound for the numerical effort with the optimized parameters. It will turn out that this effort grows inversely with the square of the total error of the simulation. At the end we also compare the numerical efforts for simulations based on the Allen-Cahn and hybrid model.

In many functional materials the phase interfaces consist only of a few atomic layers. For interfaces with such small width mathematical models with sharp interface are appropriate. We therefore base the following considerations on the hypothesis that the propagation speed of the interface in the sharp interface model is a good approximation to the propagation speed of the interface in the real material. The model error of the Allen-Cahn model is then the difference of the propagation speed of the sharp interface and the propagation speed of the diffuse interface in the phase field model. The parameters μ and λ in the Allen-Cahn model should be chosen such that this model error is small and such that numerical simulations based on the Allen-Cahn model are effective.

To make this precise we must first determine the sharp interface model to be used. The model consists of the transmission problem (2.15) – (2.19) combined with a kinetic relation. To find this relation, one proceeds in the usual way and uses that by the second law of thermodynamics the Clausius-Duhem inequality

$$\partial_t\psi_{\text{sharp}} + \operatorname{div}_x q_{\text{sharp}} \le \hat{u}_t\cdot\mathsf{b}$$

must be satisfied to impose restrictions on the form of the kinetic relation. Here ψ_{sharp} denotes the free energy in the sharp interface problem and q_{sharp} is the flux of the free energy. We use the standard free energy and flux

$$\begin{aligned}\psi_{\text{sharp}}\big(\varepsilon(\nabla_x\hat{u}),\hat{S}\big) &= \mathsf{W}\big(\varepsilon(\nabla_x\hat{u}),\hat{S}\big) + \lambda^{1/2}c_1\int_{\Gamma(t)}\mathrm{d}\sigma, \qquad (2.63)\\ q_{\text{sharp}}(\hat{T},\hat{S}) &= -\hat{T}\cdot\hat{u}_t\,.\end{aligned}$$

The last term on the right hand side of (2.63) is the interface energy, hence $\lambda^{1/2}c_1$ is the interface energy density. It is well known that if $\big(\hat{u}(t), \hat{T}(t)\big)$ is a solution of the transmission problem (2.15) – (2.19) at time t and if the interface $\Gamma(t)$ in this problem moves with the given normal speed $s_{\mathrm{sharp}}(t,x)$ at $x \in \Gamma(t)$, then the Clausius-Duhem inequality holds if and only if the inequality

$$s_{\mathrm{sharp}}(t,x)\Big(n(t,x)\cdot[\hat{C}](t,x)n(t,x) + \lambda^{1/2}c_1\kappa_\Gamma(t,x)\Big) \geq 0 \qquad (2.64)$$

is satisfied at every point $x \in \Gamma(t)$. A proof of this well known result is given in [3], however only for the case where $\lambda = 0$ in (2.63). The proof can be readily generalized to the case $\lambda > 0$.

A simple linear kinetic relation, for which (2.64) obviously holds, is

$$s_{\mathrm{sharp}} = \frac{c}{c_1}\big(n\cdot[\hat{C}]n + \lambda^{1/2}c_1\kappa_\Gamma\big). \qquad (2.65)$$

The sharp interface problem thus consists of the transmission problem (2.15) – (2.19) combined with the kinetic relation (2.65). For this problem the Clausius-Duhem inequality is satisfied.

We can now define the model error. To this end let $\big(u^{(\mu)}, T^{(\mu)}, S^{(\mu)}\big)$ be the asymptotic solution in the domain $Q = [t_1, t_2] \times \Omega$ constructet in Theorem 2.3, where by (2.44), the manifold Γ is the level set $\{S^{(\mu)} = \frac{1}{2}\}$. Let $\hat{t} \in [t_1, t_2]$ be a fixed time and let$\big(u^{(\mu)}_{\mathrm{AC}}, T^{(\mu)}_{\mathrm{AC}}, S^{(\mu)}_{\mathrm{AC}}\big)$ be the exact solution of the Allen-Cahn model (1.1) – (1.3) in the domain $[\hat{t}, t_2] \times \Omega$, which satisfies the boundary and initial conditions

$$\begin{aligned} u^{(\mu)}_{\mathrm{AC}}(t,x) &= \mathsf{U}(t,x), \qquad (t,x) \in [\hat{t}, t_2] \times \partial\Omega, &(2.66)\\ \partial_{n_{\partial\Omega}} S^{(\mu)}_{\mathrm{AC}}(t,x) &= f_3^{(\mu\lambda)}(t,x), \qquad (t,x) \in [\hat{t}, t_2] \times \partial\Omega, &(2.67)\\ S^{(\mu)}_{\mathrm{AC}}(\hat{t},x) &= S^{(\mu)}(\hat{t},x), \quad x \in \Omega, &(2.68)\end{aligned}$$

where $f_3^{(\mu\lambda)}$ is the right hand side of (2.49). The level set of the order parameter $S^{(\mu)}_{\mathrm{AC}}$ is denoted by

$$\Gamma_{\mathrm{AC}} = \Big\{(t,x) \in Q \;\Big|\; S^{(\mu)}_{\mathrm{AC}}(t,x) = \frac{1}{2}\Big\}. \qquad (2.69)$$

Note that these functions and manifolds also depend on the parameter λ, hence we really have $\big(u^{(\mu)}, T^{(\mu)}, S^{(\mu)}\big) = \big(u^{(\mu\lambda)}, T^{(\mu\lambda)}, S^{(\mu\lambda)}\big)$, $\big(u^{(\mu)}_{\mathrm{AC}}, T^{(\mu)}_{\mathrm{AC}}, S^{(\mu\lambda)}_{\mathrm{AC}}\big) = \big(u^{(\mu\lambda)}_{\mathrm{AC}}, T^{(\mu\lambda)}_{\mathrm{AC}}, S^{(\mu\lambda)}_{\mathrm{AC}}\big)$, $\Gamma = \Gamma^{(\mu\lambda)}$, $\Gamma_{\mathrm{AC}} = \Gamma^{(\mu\lambda)}_{\mathrm{AC}}$, but in this section we mostly drop the parameters μ and λ for simplicity in notation, since no confusion is possible.

Let $\Gamma_{\mathrm{sharp}} \subseteq Q$ be the sharp interface in the solution of the sharp interface problem (2.15) – (2.19), (2.65), which satisfies the initial condition

$$\Gamma_{\mathrm{sharp}}(\hat{t}) = \Gamma(\hat{t}). \qquad (2.70)$$

The normal speeds of the different surfaces are

$$s = s^{(\mu\lambda)} = \mathcal{N}\big(\Gamma^{(\mu\lambda)}\big), \qquad s_{\mathrm{AC}} = s^{(\mu\lambda)}_{\mathrm{AC}} = \mathcal{N}\big(\Gamma^{(\mu\lambda)}_{\mathrm{AC}}\big), \qquad s_{\mathrm{sharp}} = \mathcal{N}\big(\Gamma_{\mathrm{sharp}}\big),$$

where $\mathcal{N}$ is the normal speed operator introduced at the beginning of Section 2.2. Of course, s_{sharp} is given by (2.65). Note that the functions $s^{(\mu\lambda)}(\hat{t})$, $s^{(\mu\lambda)}_{\mathrm{AC}}(\hat{t})$, $s_{\mathrm{sharp}}(\hat{t})$ are defined on the same set, since the initial condition (2.68) and (2.70) together imply $\Gamma_{\mathrm{AC}}(\hat{t}) = \Gamma(\hat{t}) = \Gamma_{\mathrm{sharp}}(\hat{t})$.

Definition 2.7 *We call the function* $\mathcal{E} = \mathcal{E}^{(\mu\lambda)}(\hat{t}) : \Gamma(\hat{t}) \to \mathbb{R}$ *defined by*

$$\mathcal{E} = s_{\mathrm{AC}}(\hat{t}) - s_{\mathrm{sharp}}(\hat{t}) \tag{2.71}$$

the model error of the Allen-Cahn model at time $\hat{t}$ *to the parameters* μ *and* λ.

We can now discuss the choice of the parameters μ and λ. Since (2.65) coincides with the leading term s_0 in the asymptotic expansion (2.43) of the kinetic relation of the Allen-Cahn model, which is seen from (2.62), we have

$$s_{\mathrm{sharp}} = s_0. \tag{2.72}$$

Therefore (2.43) yields

$$\mathcal{E} = s_{\mathrm{AC}} - s_{\mathrm{sharp}} = s_{\mathrm{AC}} - s_0 = (s_{\mathrm{AC}} - s) + (s - s_0) = (s_{\mathrm{AC}} - s) + \mu^{1/2}(s_{10} + \lambda^{1/2} s_{11}). \tag{2.73}$$

The difference $s_{\mathrm{AC}} - s$ between the propagation speeds of the exact solution and the asymptotic solution tends to zero for $\mu \to 0$ faster than the term $\mu^{1/2} s_{10}$, and the convergence is uniform with respect to λ. This is the basic result, which allows to discuss the optimal choice of μ and λ. The precise result is

Theorem 2.8 *There is a constant* $C_{\mathcal{E}} > 0$ *such that for all* $0 < \mu \le \mu_0$ *and all* $0 < \lambda \le \lambda_0$ *we have the estimate*

$$\|s_{\mathrm{AC}}(\hat{t}) - s(\hat{t})\|_{L^2(\Gamma(\hat{t}))} \le C_{\mathcal{E}} |\ln \mu|^3 \mu. \tag{2.74}$$

The **proof** of this theorem is given in Section 6.

(2.73) and (2.74) together yield

$$\|\mathcal{E}^{(\mu\lambda)}\|_{L^2(\Gamma(\hat{t}))} \le C\mu^{1/2}, \tag{2.75}$$

with a constant C, which can be chosen independently of λ. By this inequality, $\mu^{1/2}$ controls the model error. Therefore we write $F = \mu^{1/2}$ and call F the error parameter. Moreover, since $\lambda^{1/2} c_1$ is the interface energy density, we call $E = \lambda^{1/2}$ the interface energy parameter. Also, since by (2.55) the interface width is proportional to $(\mu\lambda)^{1/2}$, we call $B = (\mu\lambda)^{1/2}$ the interface width parameter. These three parameters and the propagation speed s_{AC} are connected by the fundamental relations

$$B = EF, \tag{2.76}$$

$$s_{\mathrm{AC}} = \frac{c}{c_1} n \cdot [\hat{C}] n + c\kappa_\Gamma E + \mathcal{E}[E, F], \tag{2.77}$$

$$\|\mathcal{E}[E, F]\|_{L^2(\Gamma(\hat{t}))} \le CF, \tag{2.78}$$

where we use the notation $\mathcal{E}[E, F] = \mathcal{E}^{(\mu\lambda)}$. The first equation is an immediate consequence of the definition of the parameters, the second is obtained by insertion of (2.65) into (2.71), and the last inequality is just a restatement of (2.75).

Now assume that we want to use a phase field model to numerically simulate the propagation of a phase interface. In such a simulation the numerical effort is proportional to h^{-p}, where h denotes the grid spacing and where the power $p > 1$ depends on whether we want to simulate a problem in 2–d or in 3–d and it depends on the numerical scheme we use. In order for the simulation to be precise, we must guarantee that the model error and the numerical error are small. To make the numerical error small, we must choose the grid spacing h small enough to resolve the transition of the order parameter across the interface, which means that we must choose $h < B$, hence we have $h^{-p} > B^{-p}$. Therefore we see that the numerical effort of a simulation based on a phase field model is measured by the number B^{-p}. We call the number

$$e_{\text{num}} = B^{-p}$$

the parameter of numerical effort. For a simulation based on the Allen-Cahn model we see from (2.76) that the numerical effort is

$$e_{\text{num}} = (EF)^{-p}.$$

Assume that the interface, which we want to simulate with the Allen-Cahn model, has very small interface energy density. As mentioned in the introduction, such interfaces are common in metallic or functional materials. For such materials the interface energy parameter E is small. To make the model error small, we must also choose the error parameter F small, which means that the numerical effort parameter $e_{\text{num}} = (EF)^{-p}$ is very large as a product of two large numbers E^{-p} and F^{-p}.

To be more specific, we consider an interface without interface energy, which means that the free energy ψ_{sharp} does not contain the last term on the right hand side of (2.63). From (2.65) we see that the propagation speed of the sharp interface with zero interface energy density is

$$s_{\text{sharp}} = \frac{c}{c_1} n \cdot [\hat{C}]n.$$

From this equation and from (2.77) we see that in this case the total model error, which we denote by $\mathcal{E}_{\text{total}}$, is

$$\mathcal{E}_{\text{total}} = s_{\text{AC}} - s_{\text{sharp}} = c\kappa_\Gamma E + \mathcal{E}[E, F].$$

This means that the term $c\kappa_\Gamma E$ is now part of the total model error. This term does not vanish identically, since we cannot set $\lambda = 0$ in the Allen-Cahn equation (1.3). Instead the values of λ and of $E = \lambda^{1/2}$ must be positive.

If we prescribe the L^2-norm $\mathcal{E}_{L^2} = \|\mathcal{E}_{\text{total}}\|_{L^2(\Gamma(\hat{t}))}$ of the total model error, we must therefore choose the parameters E and F such that

$$c\|\kappa_\Gamma\|_{L^2(\Gamma(\hat{t}))} E + \|\mathcal{E}[E, F]\|_{L^2(\Gamma(\hat{t}))} \leq \mathcal{E}_{L^2}, \tag{2.79}$$

$$EF \stackrel{!}{=} \max, \tag{2.80}$$

where the second condition is imposed by the requirement to make the numerical effort $e_{\text{num}} = (EF)^{-p}$ as small as possible. To discuss this optimization problem, we assume first that the term s_{10} in the asymptotic expansion (2.43) of the kinetic relation of the Allen-Cahn model is not identically equal to zero. In this case we conclude from (2.73) and (2.74) by the inverse triangle inequality that for sufficiently small $\lambda^{1/2} = E$ and for sufficiently small $\mu^{1/2} = F$

$$\begin{aligned}\|\mathcal{E}[E,F]\|_{L^2(\Gamma(\hat t))} &= \|\mu^{1/2}s_{10} + (\mu\lambda)^{1/2}s_{11} + (s_{\rm AC} - s)\|_{L^2(\Gamma(\hat t))}\\ &\geq \mu^{1/2}\|s_{10}\|_{L^2(\Gamma(\hat t))} - (\mu\lambda)^{1/2}\|s_{11}\|_{L^2(\Gamma(\hat t))} - \|s_{\rm AC} - s\|_{L^2(\Gamma(\hat t))}\\ &\geq \mu^{1/2}\big(\|s_{10}\|_{L^2(\Gamma(\hat t))} - \lambda^{1/2}\|s_{11}\|_{L^2(\Gamma(\hat t))} - C_{\mathcal{E}}|\ln\mu|^3\mu^{1/2}\big)\\ &\geq \mu^{1/2}\Big(\|s_{10}\|_{L^2(\Gamma(\hat t))} - \frac{1}{2}\|s_{10}\|_{L^2(\Gamma(\hat t))}\Big) = \frac{1}{2}\|s_{10}\|_{L^2(\Gamma(\hat t))}F.\end{aligned}$$

This inequality and (2.79) imply that the solution (E,F) of the optimization problem (2.79), (2.80) satisfies

$$F \leq \frac{2}{\|s_{10}\|_{L^2(\Gamma(\hat t))}}\|\mathcal{E}[E,F]\|_{L^2(\Gamma(\hat t))} \leq \frac{2}{\|s_{10}\|_{L^2(\Gamma(\hat t))}}\,\mathcal{E}_{L^2} \qquad \text{and} \qquad E \leq \frac{1}{c\|\kappa_\Gamma\|_{L^2(\Gamma(\hat t))}}\,\mathcal{E}_{L^2}.$$

From this result we obtain

Corollary 2.9 *Let $\mathcal{E}_{\max}$ denote the total model error of the Allen-Cahn model in the simulation of an interface without interface energy. If the term s_{10} in the asymptotic expansion* (2.43) *of the kinetic relation of the Allen-Cahn model is not identically equal to zero, then the interface width B satisfies*

$$B = EF \leq \frac{2}{c\|s_{10}\|_{L^2(\Gamma(\hat t))}\|\kappa_\Gamma\|_{L^2(\Gamma(\hat t))}}\,\mathcal{E}^2_{L^2}\,. \tag{2.81}$$

In a numerical simulation of an interface without interface energy the parameter of numerical effort satisfies

$$e_{\rm num} \geq \left(\frac{c\|s_{10}\|_{L^2(\Gamma(\hat t))}\|\kappa_\Gamma\|_{L^2(\Gamma(\hat t))}}{2\,\mathcal{E}^2_{L^2}}\right)^p \tag{2.82}$$

with a power $p > 1$ depending on the space dimension and the numerical method used.

The interface width thus decreases with the square of the model error. Since the time step in a simulation must be decreased when the grid spacing h in x–direction is decreased, the number p can be larger than 4 in a three dimensional simulation. From (2.82) we thus see that the numerical effort grows very rapidely when the required accuracy is increased. The Allen-Cahn model is therefore ineffective when used to accurately simulate interfaces with low interface energy.

If the term s_{10} vanishes identically, then the same considerations show that instead of (2.81) and (2.82) we would have $B = O(\mathcal{E}^{3/2}_{L^2})$ and $e_{\rm num} \geq C\mathcal{E}_{L^2}^{-\frac{3}{2}p}$. The numerical effort would still grow fast when the required accuracy is increased, though less fast than for $s_{10} \neq 0$. However, a close investigation of the terms in the definition (2.41) of s_{10}, which we do not present here, shows that only in very exceptional situations one can expect that s_{10} vanishes identically.

In Corollary 2.9 we assumed that the mesh is globally refined. Of course, one can improve the effectivity of simulations by using local mesh refinement in the neighborhood of the interface. We do not discuss this question of numerical analysis here, but Corollary 2.9 in fact shows that adaptive mesh refinement and other advanced numerical techniques are needed to make precise simulations of interfaces with small energy based on the Allen-Cahn model effective.

Comparison to the hybrid phase field model With Corollary 2.9 we can refine the comparison given in [8] of the Allen-Cahn model and another phase field model, which we call the hybrid model. The hybrid model was introduced and discussed in [3, 4, 5, 6, 8]. By formal construction of asymptotic solutions we showed in [8] the following result:

Let $B_{\text{AC}}(\mathcal{E}_{L^2})$ and $B_{\text{hyb}}(\mathcal{E}_{L^2})$ be the interface widths in the Allen-Cahn model and the hybrid model, respectively, which result when the model parameters are adjusted to model an interface without interface energy with the total model error $\mathcal{E}_{L^2}$. Then we have for $\mathcal{E}_{L^2} \to 0$ that

$$B_{\text{hyb}}(\mathcal{E}_{L^2}) = O(\mathcal{E}_{L^2}), \qquad B_{AC}(\mathcal{E}_{L^2}) = o(1)O(\mathcal{E}_{L^2}) = o(\mathcal{E}_{L^2}).$$

The Landau symbol $o(1)$ denotes terms, which tend to zero for $\mathcal{E}_{\max} \to 0$. The result for the Allen-Cahn model was obtained under an assumption, which corresponds to our Assumption A, however without giving a formal justification of this assumption.

To achieve a prescribed small value $\mathcal{E}_{L^2}$ of the total model error we must therefore choose the interface width in the Allen-Cahn model smaller than in the hybrid model. Consequently, the hybrid model is numerically more effective, but how much more depends on the rate of decay of the $o(\mathcal{E}_{L^2})$ term in the result for the Allen-Cahn model. In [8] we could not determine this decay rate, since the asymptotic solution constructed in [8] for the Allen-Cahn model was only of first order.

Corollary 2.9 yields this decay rate. From the result for the hybrid model and from Corollary 2.9 we thus obtain for the parameters $e_{\text{num}}^{\text{hyb}}$ and $e_{\text{num}}^{\text{AC}}$ of the hybrid model and the Allen-Cahn model, respectively, that

$$e_{\text{num}}^{\text{hyb}} \le C\mathcal{E}_{L^2}^{-p}, \qquad e_{\text{num}}^{\text{AC}} \ge C\mathcal{E}_{L^2}^{-2p},$$

which shows that when the prescribed error $\mathcal{E}_{L^2}$ is small, the hybrid model can be quite considerably more effectice in numerical simulations of interfaces with low interface energy or no interface energy than the Allen-Cahn model.

2.5 The jump of solutions of the transmission problems

In this section we prove Lemma 2.5. In the proof we need results on the jump of the solution $(\hat{u}, \hat{T})$ of the transmission problem (2.15) – (2.19), which we also need in later sections to construct the asymptotic solution. We proof these results first. We conclude the section with results on the jump of the solution $(\check{u}, \check{T})$ of the transmission problem (2.20) – (2.24), which are also used in the following sections.

We define a scalar product $\alpha :_D \beta$ on $\mathcal{S}^3$ by $\alpha :_D \beta = \alpha : (D\beta)$, for $\alpha, \beta \in \mathcal{S}^3$. For a unit vector $n \in \mathbb{R}^3$ let a linear subspace of $\mathcal{S}^3$ be given by

$$\mathcal{S}_n^3 = \Big\{ \frac{1}{2}(\omega \otimes n + n \otimes \omega) \,\big|\, \omega \in \mathbb{R}^3 \Big\}, \tag{2.83}$$

let $P_n : \mathcal{S}^3 \to \mathcal{S}^3$ be the projector onto $\mathcal{S}_n^3$, which is orthogonal with respect to the scalar product $\alpha :_D \beta$ and let $Q_n = I - P_n$.

Lemma 2.10 *Let $\omega^* \in \mathbb{R}^3$ be a vector. This vector satisfies $\Big(D\big(\varepsilon(\omega^* \otimes n) - \overline{\varepsilon}\big)\Big)n = 0$ if and only if $\varepsilon(\omega^* \otimes n) = P_n \overline{\varepsilon}$ holds.*

This lemma is proved in [7, Lemma 2.2].

Lemma 2.11 *Let $(\hat{u}, \hat{T})$ be a solution of the transmission problem* (2.15) – (2.19). *Assume that $\hat{u}$ is continuous in Q and that the limits $(\nabla_x \hat{u})^{(\pm)}$ exist and define continuous extensions of $\nabla_x \hat{u}$ from the set γ' to $\gamma' \cup \Gamma$ and from the set γ to $\gamma \cup \Gamma$, respectively. Then we have*

$$[\varepsilon(\nabla_x \hat{u})] = \varepsilon(u^* \otimes n), \qquad [\hat{T}] = D\big(\varepsilon(u^* \otimes n) - \bar{\varepsilon}\big), \tag{2.84}$$

$$[\varepsilon(\nabla_x \hat{u})] = P_n \bar{\varepsilon}, \qquad [\hat{T}] = -D Q_n \bar{\varepsilon}, \tag{2.85}$$

Proof: Equation (2.84) is proved in [8, Lemma 2.2], (2.85) is proved in [7]. For completeness we give the short proofs here.

Since by assumption $\hat{u}$ is continuous across Γ and since $\nabla_x \hat{u}$ has continuous extensions from both sides of Γ onto Γ, the surface gradients $(\nabla_\Gamma \hat{u})^{(+)}$ and $(\nabla_\Gamma \hat{u})^{(-)}$ on both sides of Γ coincide, hence $[\nabla_\Gamma \hat{u}] = 0$. Using the decomposition (2.7) and the definition (2.34) of u^* we therefore obtain

$$[\nabla_x \hat{u}] = [(\partial_n \hat{u}) \otimes n + \nabla_\Gamma \hat{u}] = [(\partial_n \hat{u}) \otimes n] + [\nabla_\Gamma \hat{u}] = [\partial_n \hat{u}] \otimes n = u^* \otimes n. \tag{2.86}$$

Thus, by (2.16),

$$D\big(\varepsilon(u^* \otimes n) - \bar{\varepsilon}\big) = D\big([\varepsilon(\nabla_x \hat{u})] - \bar{\varepsilon}[\hat{S}]\big) = \big[D\big(\varepsilon(\nabla_x \hat{u}) - \bar{\varepsilon}\hat{S}\big)\big] = [\hat{T}].$$

This proves (2.84). From (2.18) and (2.84) we infer that

$$0 = [\hat{T}]n = \Big(D\big(\varepsilon(u^* \otimes n) - \bar{\varepsilon}\big)\Big)n,$$

so that $[\varepsilon(\nabla_x \hat{u})] = \varepsilon(u^* \otimes n) = P_n \bar{\varepsilon}$, by Lemma 2.10. Therefore we find

$$[\hat{T}] = D\big([\varepsilon(\nabla_x \hat{u})] - \bar{\varepsilon}\big) = D(P_n \bar{\varepsilon} - \bar{\varepsilon}) = -D Q_n \bar{\varepsilon},$$

which proves (2.85). ∎

Proof of Lemma 2.5 Note first that for matrices $A^{(+)}$, $A^{(-)}$, $B^{(+)}$, $B^{(-)}$ and for any product $\bullet$ satisfying the distributive law we obtain with the notations $[A] = A^{(+)} - A^{(-)}$, $\langle A \rangle = \frac{1}{2}\big(A^{(+)} + A^{(-)}\big)$ that

$$[A \bullet B] = [A] \bullet \langle B \rangle + \langle A \rangle \bullet [B].$$

In the following computations we apply this property two times.

W and ψ_μ are defined in (1.7) and (2.60). From these definitions, from the symmetry of the elasticity tensor $D : \mathcal{S}^3 \to \mathcal{S}^3$ and from $[\hat{S}] = 1$ we obtain with $\hat{\varepsilon} = \varepsilon(\nabla_x \hat{u})$ that

$$\begin{aligned}
[\psi_\mu] &= \frac{1}{2}\big\langle D(\hat{\varepsilon} - \bar{\varepsilon}\hat{S})\big\rangle : [\hat{\varepsilon} - \bar{\varepsilon}\hat{S}] + \frac{1}{2}\big[D(\hat{\varepsilon} - \bar{\varepsilon}\hat{S})\big] : \langle \hat{\varepsilon} - \bar{\varepsilon}\hat{S} \rangle + \frac{1}{\mu^{1/2}}[\hat{\psi}(\hat{S})] \\
&= \langle D(\hat{\varepsilon} - \bar{\varepsilon}\hat{S})\rangle : [\hat{\varepsilon} - \bar{\varepsilon}\hat{S}] + \frac{1}{\mu^{1/2}}[\hat{\psi}(\hat{S})] \\
&= \langle \hat{T} \rangle : [\hat{\varepsilon}] - \langle \hat{T} \rangle : \bar{\varepsilon} + \frac{1}{\mu^{1/2}}[\hat{\psi}(\hat{S})].
\end{aligned} \tag{2.87}$$

The last equality sign is obtained from (2.16). We employ (2.18) and (2.86) to compute

$$n \cdot \big[(I + \nabla_x \hat u)^T \hat T\big] n = n \cdot [\hat T] n + \big([\nabla_x \hat u] n\big) \cdot \langle \hat T\rangle n + \big(\langle \nabla_x \hat u\rangle n\big) \cdot [\hat T] n$$
$$= \big((u^* \otimes n) n\big) \cdot \langle \hat T\rangle n = u^* \cdot \langle \hat T\rangle n = (u^* \otimes n) : \langle \hat T\rangle = [\hat\varepsilon] : \langle \hat T\rangle. \quad (2.88)$$

In the last step we used the symmetry of $\langle \hat T\rangle$ and (2.84). Combination of (2.59), (2.87), (2.88) yields

$$n \cdot [\hat C] n = [\psi_\mu] - n \cdot \big[(I + \nabla_x \hat u)^T \hat T\big] n = \frac{1}{\mu^{1/2}} [\hat\psi(\hat S)] - \langle \hat T\rangle : \overline{\varepsilon},$$

which is (2.61). ■

Lemma 2.12 *Let* $(\hat u, \hat T)$ *and* $(\check u, \check T)$ *be solutions of the transmission problems* (2.15) – (2.19) *and* (2.20) – (2.23), *respectively. Assume that* $\hat u$ *and* $\check u$ *are continuous in* Q *and that the limits* $(\nabla_x \hat u)^{(\pm)}$ *and* $(\nabla_x \check u)^{(\pm)}$ *exist and define continuous extensions of* $\nabla_x \hat u$ *and of* $\nabla_x \check u$ *from the set* γ' *to* $\gamma' \cup \Gamma$ *and from the set* γ *to* $\gamma \cup \Gamma$, *respectively. Then we have*

$$[\partial_n \check u] = \Big[\frac{\overline{\varepsilon} : \hat T}{\hat\psi''(\hat S)}\Big] u^*, \qquad [\nabla_x \check u] = \Big[\frac{\overline{\varepsilon} : \hat T}{\hat\psi''(\hat S)}\Big] u^* \otimes n. \quad (2.89)$$

Proof: The decomposition (2.7) yields

$$[\nabla_x \check u] = [(\partial_n \check u) \otimes n + \nabla_\Gamma \check u] = [(\partial_n \check u) \otimes n] + [\nabla_\Gamma \check u] = [\partial_n \check u] \otimes n. \quad (2.90)$$

From this equation and from (2.21), (2.23) we infer

$$0 = [\check T] n = \Big(D\big([\varepsilon(\nabla_x \check u)] - \Big[\frac{\overline{\varepsilon} : \hat T}{\hat\psi''(\hat S)}\Big]\overline{\varepsilon}\big)\Big) n = \Big(D\big(\varepsilon\big([\partial_n \check u] \otimes n\big) - \Big[\frac{\overline{\varepsilon} : \hat T}{\hat\psi''(\hat S)}\Big]\overline{\varepsilon}\big)\Big) n.$$

Thus, Lemma 2.10, the linearity of the projector P_n and (2.84), (2.85) imply

$$\varepsilon\big([\partial_n \check u] \otimes n\big) = \Big[\frac{\overline{\varepsilon} : \hat T}{\hat\psi''(\hat S)}\Big] P_n \overline{\varepsilon} = \Big[\frac{\overline{\varepsilon} : \hat T}{\hat\psi''(\hat S)}\Big] \varepsilon(u^* \otimes n) = \varepsilon\Big(\Big[\frac{\overline{\varepsilon} : \hat T}{\hat\psi''(\hat S)}\Big] u^* \otimes n\Big),$$

whence

$$\Big([\partial_n \check u] - \Big[\frac{\overline{\varepsilon} : \hat T}{\hat\psi''(\hat S)}\Big] u^*\Big) \otimes n + n \otimes \Big([\partial_n \check u] - \Big[\frac{\overline{\varepsilon} : \hat T}{\hat\psi''(\hat S)}\Big] u^*\Big) = 0.$$

We multiply this equation from the right with n and obtain

$$\Big([\partial_n \check u] - \Big[\frac{\overline{\varepsilon} : \hat T}{\hat\psi''(\hat S)}\Big] u^*\Big) + n\Big([\partial_n \check u] - \Big[\frac{\overline{\varepsilon} : \hat T}{\hat\psi''(\hat S)}\Big] u^*\Big) \cdot n = 0,$$

which means that $[\partial_n \check u] - \big[\frac{\overline{\varepsilon} : \hat T}{\hat\psi''(\hat S)}\big] u^*$ is a multiple of n. Scalar multiplication of the last equation with n yields $\big([\partial_n \check u] - \big[\frac{\overline{\varepsilon} : \hat T}{\hat\psi''(\hat S)}\big] u^*\big) \cdot n = 0$, from which we now conclude that the first equation in (2.89) holds. The second equation is obtained from (2.90). ■

3 The asymptotic solution

This section forms the first part of the proof Theorem 2.3. We state in this section the form of the asymptotic solution $(u^{(\mu)}, T^{(\mu)}, S^{(\mu)})$. The proof continues in Section 4, where we study properties of the functions $S_0, \ldots, S_2$ appearing in the asymptotic solution. In Section 5 we use these properties to show that $(u^{(\mu)}, T^{(\mu)}, S^{(\mu)})$ is an asymptotic solution by verifying that the estimates (2.50) –(2.53) hold. This concludes the proof of Theorem 2.3.

3.1 Notations

Before we can start with the construction of the asymptotic solution we must introduce more definitions and notations. In particular, we must introduce parallel manifolds to the manifold Γ and we must extend the definition of the surface gradients for functions defined on Γ, which are given in Section 2.1, to functions defined on the parallel manifolds. Theses definitions and notations are needed throughout the remaining sections.

Let $\delta > 0$ be the number from (2.2). For ξ satisfying $-\delta < \xi < \delta$

$$\Gamma_\xi = \{(t, \eta + n(t,\eta)\xi) \mid (t,\eta) \in \Gamma\}$$

is a three dimensional parallel manifold of Γ embedded in $\mathcal{U}_\delta$, and

$$\Gamma_\xi(t) = \{x \in \Omega \mid (t,x) \in \Gamma_\xi\}$$

is a two-dimensional parallel surface of $\Gamma(t)$ embedded in $\mathcal{U}_\delta(t)$. Let $\tau_1,\ \tau_2 \in \mathbb{R}^3$ be two orthogonal unit vectors tangent to $\Gamma_\xi(t)$ at $x \in \Gamma_\xi(t)$. For functions $w : \Gamma_\xi(t) \to \mathbb{R}$, $W : \Gamma_\xi(t) \to \mathbb{R}^3$ and $\hat{W} : \Gamma_\xi(t) \to \mathbb{R}^{3\times 3}$ we define the surface gradient and the surface divergence on $\Gamma_\xi(t)$ by

$$\nabla_{\Gamma_\xi} w = (\partial_{\tau_1} w)\tau_1 + (\partial_{\tau_2} w)\tau_2, \tag{3.1}$$

$$\nabla_{\Gamma_\xi} W = (\partial_{\tau_1} W) \otimes \tau_1 + (\partial_{\tau_2} W) \otimes \tau_2, \tag{3.2}$$

$$\operatorname{div}_{\Gamma_\xi} W = \tau_1 \cdot \partial_{\tau_1} W + \tau_2 \cdot \partial_{\tau_2} W = \sum_{i=1}^{2} \tau_i \cdot (\nabla_{\Gamma_\xi} W)\tau_i, \tag{3.3}$$

$$\operatorname{div}_{\Gamma_\xi} \hat{W} = (\partial_{\tau_1} \hat{W})\tau_1 + (\partial_{\tau_2} \hat{W})\tau_2. \tag{3.4}$$

Clearly, with ∇_Γ defined in (2.4) and (2.5) we have $\nabla_{\Gamma_0} = \nabla_\Gamma$. For brevity we write $\operatorname{div}_\Gamma = \operatorname{div}_{\Gamma_0}$. If w, W, $\hat{W}$ are defined on Γ_ξ, we define $\nabla_{\Gamma_\xi} w : \Gamma_\xi \mapsto \mathbb{R}^3$, $\nabla_{\Gamma_\xi} W : \Gamma_\xi \mapsto \mathbb{R}^{3\times 3}$, $\operatorname{div}_{\Gamma_\xi} W : \Gamma_\xi \mapsto \mathbb{R}$, $\operatorname{div}_{\Gamma_\xi} \hat{W} : \Gamma_\xi \mapsto \mathbb{R}^3$ by applying the operators ∇_{Γ_ξ} and $\operatorname{div}_{\Gamma_\xi}$ to the restrictions $w|_{\Gamma_\xi(t)}$, $W|_{\Gamma_\xi(t)}$, $\hat{W}|_{\Gamma_\xi(t)}$ for every t. With these definitions we have the splittings

$$\nabla_x w(t,x) = \partial_\xi w(t,\eta,\xi)\, n(t,\eta) + \nabla_{\Gamma_\xi} w(t,\eta,\xi), \tag{3.5}$$

$$\nabla_x W(t,x) = \partial_\xi W(t,\eta,\xi) \otimes n(t,\eta) + \nabla_{\Gamma_\xi} W(t,\eta,\xi), \tag{3.6}$$

$$\operatorname{div}_x \hat{W}(t,x) = \big(\partial_\xi \hat{W}(t,\eta,\xi)\big) n(t,\eta) + \operatorname{div}_{\Gamma_\xi} \hat{W}(t,\eta,\xi), \tag{3.7}$$

where, as usual, $W(t,\eta,\xi) = W\big(t, \eta + n(t,\eta)\xi\big)$.

The operators ∇_Γ and div_Γ can be applied to functions defined on subsets of Γ. In contrast, the operator ∇_η introduced next can be applied to functions defined on $\Gamma \times J$, where $J \subseteq \mathbb{R}$ is an interval. For $w : \Gamma \times J \to \mathbb{R}$, $W : \Gamma \times J \to \mathbb{R}^3$, $\hat{W} : \Gamma \times J \to \mathbb{R}^{3\times 3}$ consider the functions $\eta \mapsto w_{t,\xi}(\eta) = w(t,\eta,\xi)$, $\eta \mapsto W_{t,\xi}(\eta) = W(t,\eta,\xi)$, $\eta \mapsto \hat{W}_{t,\xi}(\eta) = \hat{W}(t,\eta,\xi)$, which are defined on $\Gamma(t)$. To these functions the operators ∇_Γ and div_Γ can be applied. We set

$$\nabla_\eta w(t,\eta,\xi) = \nabla_\Gamma w_{t,\xi}(\eta) \in \mathbb{R}^3, \tag{3.8}$$

$$\nabla_\eta W(t,\eta,\xi) = \nabla_\Gamma W_{t,\xi}(\eta) \in \mathbb{R}^{3\times 3}, \tag{3.9}$$

$$\mathrm{div}_\eta W(t,\eta,\xi) = \mathrm{div}_\Gamma W_{t,\xi}(\eta) \in \mathbb{R}, \tag{3.10}$$

$$\mathrm{div}_\eta \hat{W}(t,\eta,\xi) = \mathrm{div}_\Gamma \hat{W}_{t,\xi}(\eta) \in \mathbb{R}^3. \tag{3.11}$$

If W is defined on $\mathcal{U}_\delta$, then $(t,\eta,\xi) \to W(t,\eta,\xi) = W(t,\eta + n(t,\eta)\xi)$ is defined on $\Gamma \times (-\delta,\delta)$. Consequently, the gradient $\nabla_\eta W$ is defined. The connection between $\nabla_\eta W$ and $\nabla_{\Gamma_\xi} W = \nabla_{\Gamma_\xi} W\big|_{\Gamma_\xi}$ is given by the chain rule, which yields

$$\nabla_\eta W(t,\eta,\xi) = \big(\nabla_{\Gamma_\xi} W(t,\eta + n(t,\eta)\xi)\big)\big(I + \xi\, \nabla_\eta n(t,\eta)\big). \tag{3.12}$$

In particular, we have $\nabla_\eta W(t,\eta,0) = \nabla_\Gamma W(t,\eta)$. Similar formulas and relations hold for $\nabla_\eta w$, $\mathrm{div}_\eta W$, $\mathrm{div}_\eta \hat{W}$. If $W : \mathcal{U}_\delta \to \mathbb{R}^3$ is constant on all the lines normal to $\Gamma(t)$, for all t, we have $W(t,\eta,\xi) = W(t,\eta)$. For such functions we sometimes interchangeably use the notations $\nabla_\eta W$ and $\nabla_\Gamma W$. Similarly, we interchangeably use the notations $\nabla_\eta w$ and $\nabla_\Gamma w$, $\mathrm{div}_\eta W$ and $\mathrm{div}_\Gamma W$, $\mathrm{div}_\eta \hat{W}$ and $\mathrm{div}_\Gamma \hat{W}$ if w and $\hat{W}$ are independent of ξ.

Note that by (3.12) we have for $x \in \Gamma_\xi(t)$ that

$$\nabla_{\Gamma_\xi} W(t,x) = \big(\nabla_\eta W(t,\eta,\xi)\big) A(t,\eta,\xi), \tag{3.13}$$

where $A(t,\eta,\xi) \in \mathbb{R}^{3\times 3}$ is the inverse of the linear mapping $\big(I + \xi \nabla_\eta n(t,\eta)\big) : \mathbb{R}^3 \to \mathbb{R}^3$. From the mean value theorem we obtain the expansion

$$A(t,\eta,\xi) = I + \xi\, R_A(t,\eta,\xi) \tag{3.14}$$

where the remainder term $R_A(t,\eta,\xi) \in \mathbb{R}^{3\times 3}$ is bounded when (t,η,ξ) varies in $\Gamma\times(-\delta,\delta)$. Insertion into (3.13) yields

$$\nabla_{\Gamma_\xi} W(t,x) = \nabla_\eta W(t,\eta,\xi)\,\big(I + \xi R_A(t,\eta,\xi)\big). \tag{3.15}$$

For $w : \mathcal{U}_\delta \to \mathbb{R}$ we consider $\nabla_{\Gamma_\xi} w$ and $\nabla_\eta w$ to be column vectors. For such w the equation corresponding to (3.15) is

$$\nabla_{\Gamma_\xi} w(t,x) = A^T(t,\eta,\xi) \nabla_\eta w(t,\eta,\xi) = \big(I + \xi\, R_A^T(t,\eta,\xi)\big)\, \nabla_\eta w(t,\eta,\xi). \tag{3.16}$$

Furthermore, (3.3), (3.15) and (3.10) together yield for $W : \mathcal{U}_\delta \to \mathbb{R}^3$ that

$$\mathrm{div}_{\Gamma_\xi} W = \sum_{i=1}^{2} \tau_i \cdot \big((\nabla_\eta W)(I + \xi R_A)\tau_i\big) = \mathrm{div}_\eta W + \xi\, \mathrm{div}_{\Gamma,\xi} W, \tag{3.17}$$

with the remainder term

$$\mathrm{div}_{\Gamma,\xi} W(t,\eta,\xi) = \sum_{i=1}^{2} \tau_i \cdot \big((\nabla_\eta W) R_A \tau_i\big) = \sum_{i=1}^{2} \tau_i \cdot \big((\nabla_\Gamma W_{t,\xi}) R_A \tau_i\big), \tag{3.18}$$

and this equation implies for $\hat{W} : \mathcal{U}_\delta \to \mathbb{R}^{3\times 3}$ that

$$\operatorname{div}_{\Gamma_\xi} \hat{W}(t,\eta,\xi) = \operatorname{div}_\eta \hat{W} + \xi \operatorname{div}_{\Gamma,\xi} \hat{W}, \tag{3.19}$$

where $\operatorname{div}_{\Gamma,\xi} \hat{W} = \sum_{i,j=1}^{2} (\partial_{\tau_j} \hat{W}_{t,\xi}) \tau_i (\tau_j \cdot R_A \tau_i)$. The terms $\operatorname{div}_{\Gamma,\xi} W$ and $\operatorname{div}_{\Gamma,\xi} \hat{W}$ are bounded when (t,η,ξ) varies in $\Gamma \times (-\delta,\delta)$.

For functions w with values in $\mathbb{R}$ we define the second gradients $\nabla^2_{\Gamma_\xi} w$, $\nabla^2_\eta w$ by applying the operators ∇_{Γ_ξ}, ∇_η to the vector functions $\nabla_{\Gamma_\xi} w$, $\nabla_\eta w$. For W with values in $\mathbb{R}^3$ we define second gradients $\nabla^2_{\Gamma_\xi} W$, $\nabla^2_\eta W$ by applying these operators to the rows of $\nabla_{\Gamma_\xi} W$, $\nabla_\eta W$. We remark that

$$\Delta_{\Gamma_\xi} w = \operatorname{div}_{\Gamma_\xi} \nabla_{\Gamma \xi}\, w$$

is the surface Laplacian.

Definition 3.1 *Let $I \subseteq \mathbb{R}$ be an interval. For $k, m \in \mathbb{N}_0$ and $p = 1, 3$ we define the space*

$$\begin{aligned} &C^k\big(I, C^m(\Gamma, \mathbb{R}^p)\big) \\ &\quad = \{(t,\eta,\xi) \to w(t,\eta,\xi) : \Gamma \times I \to \mathbb{R}^p \mid \partial_\xi^\ell \partial_t^i \nabla_\eta^j w \in C(\Gamma \times I),\ \ell \le k,\ i + j \le m\}. \end{aligned}$$

3.2 Construction of the asymptotic solution

We start with the construction of the asymptotic solution $(u^{(\mu)}, T^{(\mu)}, S^{(\mu)})$. We assume that the hypotheses of Theorem 2.3 are satisfied. In particular, we assume that there is a sufficiently smooth solution $\Gamma = \Gamma^{(\mu)}$ of the evolution problem (2.13), (2.14), with $s_0(\hat{T}, \kappa_\Gamma, \lambda^{1/2})$, $s_1(\hat{u}, \hat{T}, \check{T}, S_0, S_1, \lambda^{1/2})$ defined in (2.39) – (2.42). By this assumption, the function $(\hat{u}, \hat{T}, \check{u}, \check{T}, S_0, S_1)$ is known as a solution of the transmission-boundary value problem (2.15) – (2.31). We use the notation

$$1^+(r) = \begin{cases} 1, & r > 0 \\ 0, & r \le 0 \end{cases}, \qquad 1^-(r) = 1 - 1^+(r), \qquad r^\pm = r\, 1^\pm(r). \tag{3.20}$$

Let $\phi \in C_0^\infty((-2,2))$ be a function satisfying $0 \le \phi(r) \le 1$ for all $r \in \mathbb{R}$ and $\phi(r) = 1$ for $|r| \le 1$. With the constant a from (2.37) we define a function $\phi_{\mu\lambda} : Q \to [0,1]$ by

$$\begin{aligned} &\phi_{\mu\lambda}(t,x) = \phi_{\mu\lambda}(t,\eta,\xi) = \phi\Big(\frac{2a\xi}{3(\mu\lambda)^{1/2}|\ln \mu|}\Big), \quad \text{for } (t,x) \in \mathcal{U}_\delta, \\ &\phi_{\mu\lambda}(t,x) = 0, \qquad \text{otherwise}. \end{aligned} \tag{3.21}$$

By (2.33), $\phi_{\mu\lambda}$ is equal to 1 in $Q^{(\mu\lambda)}_{\text{inn}}$, transists smoothly from 1 to 0 in $Q^{(\mu\lambda)}_{\text{match}}$, and vanishes in $Q^{(\mu\lambda)}_{\text{out}}$. With this function the asymptotic solution is defined by

$$u^{(\mu)}(t,x) = u_1^{(\mu)}(t,x)\, \phi_{\mu\lambda}(t,x) + u_2^{(\mu)}(t,x)\, \big(1 - \phi_{\mu\lambda}(t,x)\big), \tag{3.22}$$

$$S^{(\mu)}(t,x) = S_1^{(\mu)}(t,x)\, \phi_{\mu\lambda}(t,x) + S_2^{(\mu)}(t,x)\, \big(1 - \phi_{\mu\lambda}(t,x)\big), \tag{3.23}$$

$$T^{(\mu)}(t,x) = D\Big(\varepsilon\big(\nabla_x u^{(\mu)}(t,x)\big) - \overline{\varepsilon} S^{(\mu)}(t,x)\Big), \tag{3.24}$$

where $u_1^{(\mu)}$, $S_1^{(\mu)}$ are components of the inner expansion $\big(u_1^{(\mu)}, T_1^{(\mu)}, S_1^{(\mu)}\big)$ defined in $\mathcal{U}_\delta$, and $u_2^{(\mu)}$, $S_2^{(\mu)}$ are components of the outer expansion $\big(u_2^{(\mu)}, T_2^{(\mu)}, S_2^{(\mu)}\big)$ defined in $Q \setminus \Gamma$. The function $\big(u^{(\mu)}, T^{(\mu)}, S^{(\mu)}\big)$ is equal to the inner expansion $\big(u_1^{(\mu)}, T_1^{(\mu)}, S_1^{(\mu)}\big)$ in the region $Q_{\rm inn}^{(\mu\lambda)}$ and equal to the outer expansion $\big(u_2^{(\mu)}, T_2^{(\mu)}, S_2^{(\mu)}\big)$ in the region $Q_{\rm out}^{(\mu\lambda)}$. In the region $Q_{\rm match}^{(\mu\lambda)}$ both expansions are matched.

The outer expansion The outer expansion is defined as follows. With the solutions $(\hat{u}, \hat{T})$ of the transmission problem (2.15) – (2.19) and $(\check{u}, \check{T})$ of the transmission problem (2.20) – (2.24) we set for $(t, x) \in Q \setminus \Gamma$

$$u_2^{(\mu)}(t,x) = \hat{u}(t,x) + \mu^{1/2}\check{u}(t,x) + \mu\,\tilde{u}(t,x), \tag{3.25}$$

$$S_2^{(\mu)}(t,x) = \hat{S}(t,x) + \mu^{1/2}\tilde{S}_1(t,x) + \mu\tilde{S}_2(t,x) + \mu^{3/2}\tilde{S}_3(t,x), \tag{3.26}$$

$$T_2^{(\mu)}(t,x) = D\Big(\varepsilon\big(\nabla_x u_2^{(\mu)}(t,x)\big) - \overline{\varepsilon} S_2^{(\mu)}(t,x)\Big). \tag{3.27}$$

The functions $\tilde{u}$, $\tilde{S}_1, \ldots, \tilde{S}_3$ and another unknown function $\tilde{T}$ solve the system of algebraic and partial differential equations

$$-\mathrm{div}_x \tilde{T} = 0, \tag{3.28}$$

$$\tilde{T} = D\big(\varepsilon(\nabla_x \tilde{u}) - \overline{\varepsilon}\tilde{S}_2\big), \tag{3.29}$$

$$-\hat{T} : \overline{\varepsilon} + \hat{\psi}''(\hat{S})\tilde{S}_1 = 0, \tag{3.30}$$

$$-\check{T} : \overline{\varepsilon} + \hat{\psi}''(\hat{S})\tilde{S}_2 + \frac{1}{2}\hat{\psi}'''(\hat{S})\tilde{S}_1^2 = 0, \tag{3.31}$$

$$\begin{aligned} -\tilde{T} : \overline{\varepsilon} + \hat{\psi}''(\hat{S})\tilde{S}_3 + \hat{\psi}'''(\hat{S})\tilde{S}_1\tilde{S}_2 + \frac{1}{6}\hat{\psi}^{(IV)}(\hat{S})\tilde{S}_1^3 & \\ + \frac{\lambda^{1/2}}{c}\partial_t \tilde{S}_1 - \lambda\Delta_x \tilde{S}_1 &= 0, \end{aligned} \tag{3.32}$$

in the set $Q \setminus \Gamma$. Moreover, $\tilde{u}$ satisfies the boundary conditions

$$\tilde{u}(t,x) = 0, \qquad (t,x) \in [t_1, t_2] \times \partial\Omega, \tag{3.33}$$

$$\tilde{u}^{(-)}(t,\eta) = \lambda^{1/2} u^*(t,\eta) \int_{-\infty}^{0} S_1(t,\eta,\zeta) - \frac{\overline{\varepsilon} : \hat{T}^{(-)}(t,\eta)}{\hat{\psi}''(1)}\, d\zeta, \quad (t,\eta) \in \Gamma, \tag{3.34}$$

$$\begin{aligned} \tilde{u}^{(+)}(t,\eta) &= \lambda^{1/2} u^*(t,\eta) \int_{0}^{\infty} S_1(t,\eta,\zeta) - \frac{\overline{\varepsilon} : \hat{T}^{(+)}(t,\eta)}{\hat{\psi}''(1)}\, d\zeta \\ &\quad + \lambda\, a^*(t,\eta) \int_{-\infty}^{\infty} \Big(\int_{-\infty}^{\zeta} S_0(\vartheta)\, d\vartheta - \zeta^+ \Big) d\zeta, \quad (t,\eta) \in \Gamma. \end{aligned} \tag{3.35}$$

Since by assumption Γ, $\hat{T}$, $\check{T}$, S_1 are known from the evolution problem, this system can be solved recursively. To see this, note that (3.30) yields

$$\tilde{S}_1 = \frac{\hat{T} : \overline{\varepsilon}}{\hat{\psi}''(\hat{S})}. \tag{3.36}$$

We insert this equation into (3.31) and solve this equation for $\tilde{S}_2$ to obtain

$$\tilde{S}_2 = \frac{\check{T} : \overline{\varepsilon}}{\hat{\psi}''(\hat{S})} - \frac{\hat{\psi}'''(\hat{S})}{2\hat{\psi}''(\hat{S})} \Big(\frac{\hat{T} : \overline{\varepsilon}}{\hat{\psi}''(\hat{S})} \Big)^2. \tag{3.37}$$

Using this function in (3.29), we can determine $\tilde{u}$ and $\tilde{T}$ from the boundary value problem (3.28), (3.29), (3.33) – (3.35). Finally, we can solve (3.32) for $\tilde{S}_3$.

The inner expansion The inner expansion $(u_1^{(\mu)}, T_1^{(\mu)}, S_1^{(\mu)})$ is essentially obtained by smoothing the jumps of the functions $\hat{u}$, $\check{u}$ and $\hat{S}$ from the evolution problem for the surface Γ. Before we can define the inner expansion we must therefore study in the next two lemmas the jumps of $\hat{u}$ and $\check{u}$ across Γ.

Let $u^* = [\partial_n \hat{u}]$ and $a^* = [\partial_n^2 \hat{u}]$ be the jumps of derivatives of $\hat{u}$ across Γ. These functions are introduced in (2.34), (2.35). For $(t,x) = (t, x(t,\eta,\xi)) \in \mathcal{U}_\delta$ we decompose $\hat{u}$ and $\check{u}$ in the form

$$\hat{u}(t,x) = u^*(t,\eta)\, \xi^+ + a^*(t,\eta) \frac{1}{2} (\xi^+)^2 + \hat{v}(t,x), \tag{3.38}$$

$$\check{u}(t,x) = u^*(t,\eta) \Big(\frac{\overline{\varepsilon} : \hat{T}^{(+)}(t,\eta)}{\hat{\psi}''(1)}\, \xi^+ + \frac{\overline{\varepsilon} : \hat{T}^{(-)}(t,\eta)}{\hat{\psi}''(0)}\, \xi^- \Big) + \check{v}(t,x), \tag{3.39}$$

where ξ^+, ξ^- are defined in (3.20) and where the remainder terms $\hat{v}$ and $\check{v}$ are defined by (3.38), (3.39). The decomposition (3.38) is motivated by the fact that

$$[\partial_n^i \hat{v}] = 0, \qquad i = 0, 1, 2, \tag{3.40}$$

which follows immediately from (3.38), (2.17) and (2.34), (2.35). The first two terms on the right hand side of (3.38) thus serve to separate off the jumps of the first and second derivatives of $\hat{u}$ at Γ. Similarly, the normal derivatives of first order of $\check{v}$ do not jump across Γ. More precisely, we have the following result.

Lemma 3.2 *Let $(\hat{u}, \hat{T})$ and $(\check{u}, \check{T})$ be solutions of the transmission problems* (2.15) – (2.19) *and* (2.20) – (2.24)*, respectively.*
(i) $\check{v}$ defined in (3.39) *satisfies*

$$[\partial_n^i \check{v}] = 0, \qquad i = 0, 1. \tag{3.41}$$

(ii) Assume that Γ is a C^5–manifold. Suppose that $\hat{u} \in C^4(\gamma \cup \gamma', \mathbb{R}^3)$, $\check{u} \in C^3(\gamma \cup \gamma', \mathbb{R}^3)$ and that $\hat{u}$ has C^4–extensions, $\check{u}$ has C^3–extensions from γ to $\gamma \cup \Gamma$ and from γ' to $\gamma' \cup \Gamma$. With the function spaces introduced in Definition 3.1 we then have

$$\hat{v} \in C^2\big((-\delta,\delta), C^2(\Gamma)\big) \cap C^3\big((-\delta,0], C^1(\Gamma)\big) \cap C^3\big([0,\delta), C^1(\Gamma)\big), \tag{3.42}$$

$$\check{v} \in C^1\big((-\delta,\delta), C^2(\Gamma)\big) \cap C^3\big(\Gamma \times (-\delta,0]\big) \cap C^3\big(\Gamma \times [0,\delta)\big). \tag{3.43}$$

Proof: To prove (3.41) note that by definition of $[\partial_n w]$ in Section 2.1 and by definiton of $\xi^\pm$ in (3.20) we have

$$\Big[\partial_n \Big(\frac{\overline{\varepsilon} : \hat{T}^{(+)}}{\hat{\psi}''(1)}\, \xi^+ + \frac{\overline{\varepsilon} : \hat{T}^{(-)}}{\hat{\psi}''(0)}\, \xi^- \Big) u^* \Big] = \Big(\frac{\overline{\varepsilon} : \hat{T}^{(+)}}{\hat{\psi}''(1)} - \frac{\overline{\varepsilon} : \hat{T}^{(-)}}{\hat{\psi}''(0)} \Big) u^* = \Big[\frac{\overline{\varepsilon} : \hat{T}}{\hat{\psi}''(\hat{S}} \Big] u^*.$$

From this equation, from the first equation in (2.89) and from (3.39) we obtain (3.41) for $i = 1$. For $i = 0$ equation (3.41) is an immediate consequence of (3.39) and (2.22). This proves (i).

(ii) Since Γ is a C^5–manifold, the coordinate mapping $(t,\eta,\xi) \mapsto (t,x(t,\eta,\xi)) = \big(t,\eta+\xi\, n(t,\eta)\big)$ and the inverse mapping $(t,x) \mapsto \big(t,\eta(t,x),\xi(t,x)\big)$ are C^4. It follows from this differentiability property of the coordinate mapping and from our differentiability assumptions for $\hat{u}$ that $(t,\eta,\xi) \mapsto \hat{u}(t,\eta,\xi)$ is C^4 in $\Gamma \times (-\delta, 0]$ and in $\Gamma \times [0,\delta)$, and that $(t,\eta) \mapsto u^*(t,\eta) = n(t,\eta) \cdot [\nabla_x \hat{u}](t,\eta)$ belongs to $C^3(\Gamma)$ and $(t,\eta) \mapsto a^*(t,\eta) = n(t,\eta) \cdot [\partial_n \nabla_x \hat{u}](t,\eta)$ belongs to $C^2(\Gamma)$. Since by (3.38) we have

$$\hat{v}(t,\eta,\xi) = \hat{u}(t,\eta,\xi) - u^*(t,\eta)\xi^+ - a^*(t,\eta)\frac{1}{2}(\xi^+)^2,$$

these properties imply that

$$\hat{v} \in \bigcap_{m=0}^{1} \Big(C^{2+m}\big((-\delta,0], C^{2-m}(\Gamma)\big) \cap C^{2+m}\big([0,\delta), C^{2-m}(\Gamma)\big)\Big).$$

From this relation and from (3.40) we conclude that (3.42) holds. Relation (3.43) is obtained in the same way, using (3.41) instead of (3.40). ∎

For brevity in notation we define

$$\hat{\sigma}(\xi) = \hat{\sigma}(t,\eta,\xi) = \overline{\varepsilon} : D\varepsilon\big(\nabla_x \hat{v}(t,x)\big), \qquad \hat{\sigma}'(\xi) = \partial_\xi \hat{\sigma}(t,\eta,\xi), \tag{3.44}$$

$$\check{\sigma}(\xi) = \check{\sigma}(t,\eta,\xi) = \overline{\varepsilon} : D\varepsilon\big(\nabla_x \check{v}(t,x)\big). \tag{3.45}$$

Later we need the following result, which shows how $\check{\sigma}(t,\eta,0)$ can be computed from the limit values of $\hat{T}$ and $\check{T}$ at Γ.

Lemma 3.3 *The function $\check{\sigma}$ defined in* (3.45) *satisfies*

$$\overline{\varepsilon} : \check{T}^{(+)} = \check{\sigma}(0) + \overline{\varepsilon} : [\hat{T}]\, \frac{\overline{\varepsilon} : \hat{T}^{(+)}}{\hat{\psi}''(1)}, \tag{3.46}$$

$$\overline{\varepsilon} : \check{T}^{(-)} = \check{\sigma}(0) + \overline{\varepsilon} : [\hat{T}]\, \frac{\overline{\varepsilon} : \hat{T}^{(-)}}{\hat{\psi}''(0)}, \tag{3.47}$$

$$\check{\sigma}(0) = \overline{\varepsilon} : \langle \check{T} \rangle - \overline{\varepsilon} : [\hat{T}] \Big\langle \frac{\overline{\varepsilon} : \hat{T}}{\hat{\psi}''(\hat{S})} \Big\rangle. \tag{3.48}$$

Proof: We apply the decomposition (2.7) of the gradient to the function $W(t,\eta,\xi) = u^*(t,\eta)\Big(\frac{\overline{\varepsilon}:\hat{T}^{(+)}(t,\eta)}{\hat{\psi}''(1)}\xi^+ + \frac{\overline{\varepsilon}:\hat{T}^{(-)}(t,\eta)}{\hat{\psi}''(0)}\xi^-\Big)$. This yields

$$(\nabla_x W)^{(+)}(t,\eta) = \big((\partial_n W) \otimes n + \nabla_\Gamma W\big)^{(+)} = u^* \otimes n\, \frac{\overline{\varepsilon} : \hat{T}^{(+)}}{\hat{\psi}''(1)}.$$

(3.39) thus implies

$$(\nabla_x \check{u})^{(+)} = u^* \otimes n\, \frac{\overline{\varepsilon} : \hat{T}^{(+)}}{\hat{\psi}''(1)} + \nabla_x \check{v}.$$

Insertion of this equations into (2.21) yields

$$\check{T}^{(+)} = D\big(\varepsilon(u^* \otimes n) - \overline{\varepsilon}\big) \frac{\overline{\varepsilon} : \hat{T}^{(+)}}{\hat{\psi}''(1)} + D\varepsilon(\nabla_x \check{v}).$$

We take the scalar product with $\overline{\varepsilon}$ on both sides of this equation and note (2.84) and the definition of $\check{\sigma}$ in (3.45) to obtain (3.46). Equation (3.47) is obtained in the same way. To prove (3.48), we add (3.46) and (3.47) and solve the resulting equation for $\check{\sigma}(0)$. This proves the lemma. ∎

Definition of the inner expansion We can now construct the inner expansion $(u_1^{(\mu)}, T_1^{(\mu)}, S_1^{(\mu)})$. With the remainder terms $\hat{v}$, $\check{v}$ introduced in (3.38), (3.39) we set in the neighborhood $\mathcal{U}_\delta$ of Γ

$$\begin{aligned} u_1^{(\mu)}(t,x) &= (\mu\lambda)^{1/2} u_0\big(t,\eta,\frac{\xi}{(\mu\lambda)^{1/2}}\big) + \mu\lambda^{1/2} u_1\big(t,\eta,\frac{\xi}{(\mu\lambda)^{1/2}}\big) + \mu\lambda u_2\big(t,\eta,\frac{\xi}{(\mu\lambda)^{1/2}}\big) \\ &\quad + \hat{v}(t,x) + \mu^{1/2}\check{v}(t,x), \qquad (3.49) \\ S_1^{(\mu)}(t,x) &= S_0\big(\frac{\xi}{(\mu\lambda)^{1/2}}\big) + \mu^{1/2} S_1\big(t,\eta,\frac{\xi}{(\mu\lambda)^{1/2}}\big) + \mu S_2\big(t,\eta,\frac{\xi}{(\mu\lambda)^{1/2}}\big), \qquad (3.50) \\ T_1^{(\mu)}(t,x) &= D\Big(\varepsilon\big(\nabla_x u_1^{(\mu)}(t,x)\big) - \bar{\varepsilon} S_1^{(\mu)}(t,x)\Big), \qquad (3.51) \end{aligned}$$

where by assumption S_0, S_1 are known from the evolution problem for Γ, and where the functions $u_0, \ldots, u_2$ are defined by

$$u_0(t,\eta,\zeta) = u^*(t,\eta) \int_{-\infty}^{\zeta} S_0(\vartheta)\, d\vartheta, \qquad (3.52)$$

$$u_1(t,\eta,\zeta) = u^*(t,\eta) \int_0^{\zeta} S_1(t,\eta,\vartheta)\, d\vartheta, \qquad (3.53)$$

$$u_2(t,\eta,\zeta) = a^*(t,\eta) \int_{-\infty}^{\zeta} \int_{-\infty}^{\vartheta} S_0(\vartheta_1)\, d\vartheta_1 d\vartheta. \qquad (3.54)$$

The function $S_2 = S_2(t,\eta,\zeta)$ together with another unknown function $s_1 = s_1(t,\eta)$ solve a boundary value problem. To state this boundary value problem let $\kappa(t,\eta,\xi)$ denote twice the mean curvature of the surface $\Gamma_\xi(t)$ at $\eta \in \Gamma_\xi(t)$. With the notation introduced in Section 2.2 we thus have $\kappa(t,\eta,0) = \kappa_\Gamma(t,\eta)$. We write $\kappa'(0) = \partial_\xi \kappa(t,\eta,0)$.

The boundary value problem for S_2 and s_1 consists of the ordinary differential equation

$$\hat{\psi}''\big(S_0(\zeta)\big) S_2(t,\eta,\zeta) - S_2''(t,\eta,\zeta) = F_2(t,\eta,\zeta), \qquad (3.55)$$

with the right hand side given by

$$\begin{aligned} F_2(t,\eta,\zeta) &= \check{\sigma}(0) + \bar{\varepsilon} : [\hat{T}]\, S_1 - \frac{1}{c_1}\bar{\varepsilon} : \langle \hat{T} \rangle S_1' - \frac{1}{2}\hat{\psi}'''(S_0) S_1^2 \\ &\quad + \lambda^{1/2}\Big(\hat{\sigma}'(0)\zeta + \bar{\varepsilon} : D\varepsilon(a^* \otimes n + \nabla_\Gamma u^*) \int_{-\infty}^{\zeta} S_0(\vartheta)\, d\vartheta\Big) \\ &\quad + \Big(\frac{s_1}{c} - \lambda\kappa'(0)\zeta\Big) S_0', \qquad (3.56) \end{aligned}$$

and of boundary conditions. To formulate these boundary conditions, we choose $\varphi \in C^\infty(\mathbb{R}, [0,1])$ such that

$$\varphi(\zeta) = \begin{cases} 0, & \zeta \le 1, \\ 1, & \zeta \ge 2, \end{cases} \qquad (3.57)$$

set

$$\varphi_+(\zeta) = \frac{\varphi(\zeta)}{\hat{\psi}''(1)}, \qquad \varphi_-(\zeta) = \frac{\varphi(-\zeta)}{\hat{\psi}''(0)}, \qquad (3.58)$$

and define

$$\rho_2(t,\eta,\zeta) = \varphi_-(\zeta)\bigg(\overline{\varepsilon} : \check{T}^{(-)} - \frac{\hat{\psi}'''(0)}{2}\Big(\frac{\overline{\varepsilon} : \hat{T}^{(-)}}{\hat{\psi}''(0)}\Big)^2 + \lambda^{1/2}\,\hat{\sigma}'(0)\zeta\bigg)$$
$$+ \varphi_+(\zeta)\bigg(\overline{\varepsilon} : \check{T}^{(+)} - \frac{\hat{\psi}'''(1)}{2}\Big(\frac{\overline{\varepsilon} : \hat{T}^{(+)}}{\hat{\psi}''(1)}\Big)^2 + \lambda^{1/2}\hat{\sigma}'(0)\zeta$$
$$+ \lambda^{1/2}\overline{\varepsilon} : D\varepsilon(a^* \otimes n + \nabla_\Gamma u^*)\zeta^+\bigg). \tag{3.59}$$

With this function the boundary conditions are

$$S_2(t,\eta,0) = 0, \tag{3.60}$$
$$\lim_{\zeta\to\pm\infty}\big(S_2(t,\eta,\zeta) - \rho_2(t,\eta,\zeta)\big) = 0. \tag{3.61}$$

The function $s_1 = s_1(t,\eta)$ in (3.56) is independent of ζ. It is determined by the procedure sketched at the end of Section 2.2, which we apply of course to the boundary value problem (3.55), (3.56), (3.60), (3.61) instead of the problem (2.26), (2.28) – (2.31). This is discussed precisely in Section 4.3. The function s_1, whose explicit expression is given in (2.40) – (2.42), forms the second term in the definition (2.14) of the evolution operator $\mathcal{K}^{(\mu)}$.

4 The functions $S_0, \dots, S_2$ from the inner expansion

The functions $\tilde{S}_1, \dots, \tilde{S}_3$ in the outer expansion can be determined explicitly from (3.30) – (3.32), whereas the functions $S_0, \dots, S_2$ in the inner expansion are determined as solutions of three coupled boundary value problems to linear and nonlinear ordinary differential equations. It is not obvious that these solutions exist and what properties they have. We study these solutions in this section.

4.1 The function S_0

The first boundary value problem determining S_0 is given by (2.25), (2.27),

Lemma 4.1 *Assume that the double well potential $\hat{\psi}$ satisfies* (2.37)*. Then S_0 is a solution of the boundary value problem* (2.25), (2.27)*, if and only if S_0 satisfies the initial value problem*

$$S_0'(\zeta) = \sqrt{2\hat{\psi}\big(S_0(\zeta)\big)}, \quad \zeta \in \mathbb{R}, \qquad S_0(0) = \frac{1}{2}. \tag{4.1}$$

Proof: Let S_0 be a solution of (2.25), (2.27). We multiply (2.25) by S_0' and obtain

$$\frac{d}{d\zeta}\Big(\hat{\psi}(S_0) - \frac{1}{2}(S_0')^2\Big) = 0,$$

or

$$\hat{\psi}(S_0) - \frac{1}{2}(S_0')^2 = C_1. \tag{4.2}$$

By (2.27) we have $\lim_{\zeta\to\infty} S_0(\zeta) = 1$. From (4.2) and from (2.37) we thus obtain that $\lim_{\zeta\to\infty}\big(S_0'(\zeta)\big)^2 = -2C_1$. Using again (2.27), we infer from this limit relation that

$\lim_{\zeta\to\infty} S_0'(\zeta) = 0$, hence $C_1 = 0$. We solve (4.2) for S_0' and use that because of the boundary conditions (2.27) the function S_0 must be increasing, hence S_0' must be non-negative. This shows that a solution of (2.25), (2.27) must satisfy the initial value problem (4.1).

To prove the converse we differentiate the differential equation in (4.1) and obtain (2.25). We leave it to the reader to verify that the solution of (4.1) satisfies the boundary conditions (2.27). ■

Theorem 4.2 *Assume that* $\hat{\psi} \in C^3([0,1],\mathbb{R})$ *has the properties* (2.37). *Then there is a unique solution* $S_0 \in C^4(\mathbb{R},(0,1))$ *of the initial value problem* (4.1). *This solution is strictly increasing and satisfies* (2.25) *and* (2.27). *Moreover, there are constants* $K_1,\ldots,K_3 > 0$ *such that for* $a > 0$ *defined in* (2.37)

$$0 < S_0(\zeta) \le K_1 e^{-a|\zeta|}, \quad -\infty < \zeta \le 0, \tag{4.3}$$

$$1 - K_2 e^{-a\zeta} \le S_0(\zeta) < 1, \quad 0 \le \zeta < \infty, \tag{4.4}$$

$$|\partial^i S_0(\zeta)| \le K_3 e^{-a|\zeta|}, \quad -\infty < \zeta < \infty,\ i = 1,\ldots,4\,. \tag{4.5}$$

This theorem follows immediately from the standard theory of ordinary differential equations, and we omit the proof.

Lemma 4.3 *If* $\hat{\psi}$ *satisfies the symmetry condition* (2.38), *then the solution* S_0 *of* (4.1) *satisfies for all* $\zeta \in \mathbb{R}$

$$S_0(-\zeta) = 1 - S_0(\zeta), \qquad S_0'(\zeta) = S_0'(-\zeta), \tag{4.6}$$

$$\int_{-\infty}^{\zeta} S_0(\vartheta)\, d\vartheta = \int_{-\infty}^{-|\zeta|} S_0(\vartheta)\, d\vartheta + \zeta^+, \tag{4.7}$$

$$\Big| \int_{-\infty}^{\zeta} S_0(\vartheta)\, d\vartheta - \zeta^+ \Big| \le \frac{K_1}{a} e^{-a|\zeta|}, \tag{4.8}$$

$$|\hat{\psi}''\big(S_0(\zeta)\big) - \hat{\psi}''\big(\hat{S}(\zeta)\big)| \le K_4 e^{-a|\zeta|}. \tag{4.9}$$

Proof: If the symmetry condition (2.38) holds and if S_0 is a solution of the initial value problem (4.1), then also $\zeta \mapsto \big(1 - S_0(-\zeta)\big)$ is a solution. To see this, note that (2.38) and (4.1) imply

$$\begin{aligned}\sqrt{2\hat{\psi}\big(1 - S_0(-\zeta)\big)} &= \sqrt{2\hat{\psi}\Big(\frac{1}{2} + \Big(\frac{1}{2} - S_0(-\zeta)\Big)\Big)} = \sqrt{2\hat{\psi}\Big(\frac{1}{2} - \Big(\frac{1}{2} - S_0(-\zeta)\Big)\Big)} \\ &= \sqrt{2\hat{\psi}\big(S_0(-\zeta)\big)} = (\partial_\zeta S_0)(-\zeta) = \partial_\zeta\big(1 - S_0(-\zeta)\big),\end{aligned}$$

whence $1 - S_0(-\zeta)$ satisfies the differential equation in (4.1). Since we obviously have $1 - S_0(0) = \frac{1}{2}$, we see that $1 - S_0(-\zeta)$ is a solution of (4.1). Since the solution of this initial value problem is unique, we infer that $S_0(\zeta) = 1 - S_0(-\zeta)$ holds, which implies $S_0'(\zeta) = S_0'(-\zeta)$.

To prove (4.7), note that (4.6) implies for $\zeta > 0$

$$\begin{aligned}\int_{-\infty}^{\zeta} S_0(\vartheta)\, d\vartheta &= \int_{-\infty}^{-\zeta} S_0(\vartheta)\, d\vartheta + \int_{-\zeta}^{\zeta} S_0(\vartheta)\, d\vartheta \\ &= \int_{-\infty}^{-\zeta} S_0(\vartheta)\, d\vartheta + \int_{0}^{\zeta} S_0(\vartheta) + S_0(-\vartheta)\, d\vartheta \\ &= \int_{-\infty}^{-\zeta} S_0(\vartheta)\, d\vartheta + \int_{0}^{\zeta} S_0(\vartheta) + \big(1 - S_0(\vartheta)\big)\, d\vartheta = \int_{-\infty}^{-\zeta} S_0(\vartheta)\, d\vartheta + \zeta.\end{aligned}$$

From this equation we immediately obtain (4.7). The inequality (4.8) follows from (4.7) and from (4.3), which yield

$$\Big| \int_{-\infty}^{\zeta} S_0(\vartheta)\, d\vartheta - \zeta^+ \Big| = \Big| \int_{-\infty}^{-|\zeta|} S_0(\vartheta)\, d\vartheta \Big| \le \int_{-\infty}^{-|\zeta|} K_1 e^{-a|\vartheta|} d\vartheta = \frac{K_1}{a} e^{-a|\zeta|}.$$

For the proof of (4.9) note that $\hat{S}(\zeta) = 1$ for $\zeta > 0$. Consequently, the mean value theorem and (4.4) together imply for $\zeta > 0$ that

$$|\hat{\psi}''\big(S_0(\zeta)\big) - \hat{\psi}''\big(\hat{S}(\zeta)\big)| = |\hat{\psi}''\big(S_0(\zeta)\big) - \hat{\psi}''(1)| \le |\hat{\psi}'''(r^*)(S_0(\zeta) - 1)| \le C K_2 e^{-a|\zeta|},$$

with a suitable number r^* between $S_0(\zeta)$ and 1. For $\zeta < 0$ an analogous estimate is obtained using (4.3) and noting that $\hat{S}(\zeta) = 0$ if $\zeta < 0$. ■

4.2 The functions S_1 and S_2

The solutions S_1 and S_2 of the second and third boundary value problems are studied in this section. The second problem determing S_1 is given by the equations (2.26), (2.28) – (2.31), the third problem, which determines S_2, consists of the equations (3.55), (3.56), (3.60), (3.61). The properties of S_1 and S_2, which we need in Section 5, are summarized in the next two theorems.

To state the first theorem we need the function $\rho_1 : \Gamma \times \mathbb{R} \to \mathbb{R}$, which is defined by

$$\rho_1(t, \eta, \zeta) = \overline{\varepsilon} : \hat{T}^{(-)}(t, \eta)\, \varphi_-(\zeta) + \overline{\varepsilon} : \hat{T}^{(+)}(t, \eta)\, \varphi_+(\zeta), \tag{4.10}$$

with $\varphi_\pm$ introduced in (3.58).

Theorem 4.4 *Assume that $\hat{\psi}$ belongs to $C^5([0,1], \mathbb{R})$ and satisfies the assumptions* (2.37) *and the symmetry condition* (2.38). *Suppose that the function $s_0 = s_0(t, \eta)$ in* (2.31) *is given by* (2.39). *Let S_0 be the solution of the boundary value problem* (2.25), (2.27), *which exists by Theorem 4.2.*

Then for every $(t, \eta) \in \Gamma$ there is a unique solution $\zeta \mapsto S_1(t, \eta, \zeta) : \mathbb{R} \to \mathbb{R}$ of the boundary value problem (2.26), (2.28) – (2.31). *The function S_1 belongs to the space $C^2(\mathbb{R}, C^2(\Gamma, \mathbb{R}))$. Moreover, there are constants $K_1, \ldots, K_3$ such that for the constant a defined in* (2.37) *and for all $(t, \eta, \zeta) \in \Gamma \times \mathbb{R}$ the estimates*

$$\|D^\alpha_{(t,\eta)} S_1\|_{L^\infty(\Gamma\times\mathbb{R})} \le K_1, \qquad |\alpha| \le 2, \tag{4.11}$$

$$\big|\partial_\zeta^j D^\alpha_{(t,\eta)}\big(S_1(t,\eta,\zeta) - \rho_1(t,\eta,\zeta)\big)\big| \le K_2\, e^{-a|\zeta|}, \quad 0 \le j, |\alpha| \le 2, \tag{4.12}$$

$$\big|\partial_\zeta^j D^\alpha_{(t,\eta)} S_1(t,\eta,\zeta)\big| \le K_3\, e^{-a|\zeta|}, \quad |\alpha| \le 2,\ j = 1, 2, \tag{4.13}$$

hold.

We do not give the proof of this theorem, since it is almost the same as the proof of Theorem 1.2 in [7]. Morover, it is obtained from the proof of the following theorem by simplification. The main difference between the two proofs is that the right hand side F_1 of the differential equation (2.26) for S_1 is bounded, whereas the right hand side F_2 of the differential equation (3.55) for S_2 grows linearly for $\zeta \to \pm\infty$.

Theorem 4.5 *Assume that $\hat{\psi}$ satisfies the assumptions given in the Theorem 4.4. Let S_0 be the solution of the boundary value problem (2.25), (2.27), and let S_1 be the solution of the boundary value problem (2.26), (2.28) – (2.31). Suppose that the function $s_1 = s_1(t,\eta)$ in (3.56) satisfies (2.40) with s_{10}, s_{11} given in (2.41), (2.42).*

(i) Then for every $(t,\eta) \in \Gamma$ there is a unique solution $\zeta \mapsto S_2(t,\eta,\zeta) : \mathbb{R} \to \mathbb{R}$ of the boundary value problem (3.55), (3.56), (3.60), (3.61). The function S_2 belongs to $C^2(\mathbb{R}, C^2(\Gamma,\mathbb{R}))$, and there are constants $K_4, \ldots, K_6$ such that for the constant a defined in (2.37) and for all $(t,\eta,\zeta) \in \Gamma \times \mathbb{R}$ the estimates

$$
\begin{aligned}
|\partial_\zeta^j D^\alpha_{(t,\eta,\zeta)} S_2(t,\eta,\zeta)| &\leq K_4(1+|\zeta|)^{1-j}, \quad |\alpha| \leq 2,\ j = 0,1, && (4.14)\\
\big|\partial_\zeta^j D^\alpha_{(t,\eta)} \big(S_2(t,\eta,\zeta) - \rho_2(t,\eta,\zeta)\big)\big| &\leq K_5(1+|\zeta|)\, e^{-a|\zeta|}, \quad 0 \leq j, |\alpha| \leq 2, && (4.15)\\
\big|\partial_\zeta^2 D^\alpha_{(t,\eta)} S_2(t,\eta,\zeta)\big| &\leq K_6(1+|\zeta|)\, e^{-a|\zeta|}, \qquad |\alpha| \leq 2, && (4.16)
\end{aligned}
$$

hold, where ρ_2 is defined in (3.59).

(ii) S_2 is the only solution of the differential equation (3.55) with F_2 given by (3.56), which satisfies (3.60) and for which constants $C, \theta > 0$ exist such that

$$
|S_2(t,\eta,\zeta)| \leq C e^{(a-\theta)|\zeta|}, \qquad \zeta \in \mathbb{R}, \tag{4.17}
$$

holds.

4.3 Proof of Theorem 4.5

In this section we give the proof of Theorem 4.5, which is divided into five parts:

(I) Reduction of the boundary value problem for S_2 to a problem in L^2. With ρ_2 defined in (3.59) we make the ansatz

$$
S_2(t,\eta,\zeta) = w(t,\eta,\zeta) + \rho_2(t,\eta,\zeta). \tag{4.18}
$$

Insertion of this ansatz into the equations (3.55) and (3.60), (3.61) shows that S_2 is a solution of the problem given by these equations if and only if w solves the equations

$$
\begin{aligned}
\hat{\psi}''\big(S_0(\zeta)\big) w(t,\eta,\zeta) - \partial_\zeta^2 w(t,\eta,\zeta) = F_2(t,\eta,\zeta) + F_3(t,\eta,\zeta), && (4.19)\\
w(t,\eta,0) = 0, && (4.20)\\
\lim_{\zeta\to\pm\infty} w(t,\eta,\zeta) = 0, && (4.21)
\end{aligned}
$$

where F_2 is given by (3.56) and where

$$
F_3 = -(\hat{\psi}''(S_0) - \partial_\zeta^2)\rho_2. \tag{4.22}
$$

To get (4.20) we used that $\varphi_+(0) = \varphi_-(0) = 0$, which by (3.59) implies $\rho_2(t,\eta,0) = 0$. To show that the solution S_2 of the problem (3.55) and (3.60), (3.61) exists, it therefore suffices to prove that the reduced problem (4.19) – (4.22) has a solution.

(II) Spectral theory For this proof note that $\hat\psi''(S_0) - \partial_\zeta^2$ is a linear self-adjoint differential operator in $L^2(\mathbb{R})$. From the spectral theory of such operators we know that the continuous spectrum of $\hat\psi''(S_0) - \partial_\zeta^2$ is contained in the interval $[a_0, \infty)$, where

$$a_0 = \min\Big\{ \lim_{\zeta\to-\infty} \hat\psi''\big(S_0(\zeta)\big), \lim_{\zeta\to\infty} \hat\psi''\big(S_0(\zeta)\big) \Big\},$$

and that the part of the spectrum in $(-\infty, a_0)$ is a pure point spectrum. (4.9) yields $\lim_{\zeta\to-\infty} \hat\psi''\big(S_0(\zeta)\big) = \hat\psi''(0)$, $\lim_{\zeta\to\infty} \hat\psi''\big(S_0(\zeta)\big) = \hat\psi''(1)$, hence the assumption (2.37) implies $a_0 = a^2 > 0$. Therefore 0 does not belong to the continuous spectrum. From the spectral theory we also know that for every $\omega \in \mathbb{C}$, which is not in the continuous spectrum, the differential equation $\big(\hat\psi''(S_0) - \partial_\zeta^2\big)w - \omega w = f$ has a solution $w \in L^2(\mathbb{R})$, if and only if $f \in L^2(\mathbb{R})$ is orthogonal to the kernel of the operator $\psi''(S_0) - \partial_\zeta^2 - \omega$. This implies in particular, that for every $(t, \eta) \in \Gamma$ the differential equation (4.19) has a solution $w(t, \eta, \cdot) \in L^2(\mathbb{R})$, if the right hand side $\zeta \mapsto f(\zeta) = F_2(t, \eta, \zeta) + F_3(t, \eta, \zeta)$ belongs to $L^2(\mathbb{R})$ and is orthogonal to the kernel of the operator $\hat\psi''(S_0) - \partial_\zeta^2$. To show that the problem (4.19) – (4.21) has a solution, we therefore verify in the next two parts of the proof that $F_2 + F_3$ satisfies these two conditions.

(III) The asymptotic behavior of $F_2 + F_3$ at infinity. We first show that the right hand side $F_2 + F_3$ of (4.19) decays exponentially at $\pm\infty$, which implies that $F_2 + F_3 \in L^2(\mathbb{R})$. To simplify the notation we define

$$\overline{\varphi}_+(\zeta) = \varphi_+(\zeta)\hat\psi''\big(S_0(\zeta)\big), \qquad \overline{\varphi}_-(\zeta) = \varphi_-(\zeta)\hat\psi''\big(S_0(\zeta)\big), \tag{4.23}$$

with φ_+, φ_- given in (3.58). For these functions we obtain from (4.9) that

$$|\overline{\varphi}_- - \hat\psi''(0)\varphi_-| = \varphi_-(\zeta)\,|\hat\psi''(S_0) - \hat\psi''(0)| \le CK_4 e^{-a|\zeta|}, \tag{4.24}$$

$$|\overline{\varphi}_+ - \hat\psi''(1)\varphi_+| = \varphi_+(\zeta)\,|\hat\psi''(S_0) - \hat\psi''(1)| \le CK_4 e^{-a|\zeta|}, \tag{4.25}$$

for all $\zeta \in \mathbb{R}$. Since $\hat\psi''(0)\varphi_-(\zeta) = 1$ for $\zeta \le -2$ and $\hat\psi''(1)\varphi_+(\zeta) = 1$ for $\zeta \ge 2$, these estimates imply

$$|1 - \overline{\varphi}_-(\zeta)| \le CK_4 e^{-a|\zeta|}, \qquad -\infty < \zeta \le 0. \tag{4.26}$$

$$|1 - \overline{\varphi}_+(\zeta)| \le CK_4 e^{-a|\zeta|}, \qquad 0 \le \zeta < \infty, \tag{4.27}$$

$$|1 - \overline{\varphi}_-(\zeta) - \overline{\varphi}_+(\zeta)| \le CK_4 e^{-a|\zeta|}, \qquad \zeta \in \mathbb{R}. \tag{4.28}$$

To get the last estimate we combined the first two estimates and noted that $\overline{\varphi}_-(\zeta) = 0$ for $\zeta \ge -1$ and $\overline{\varphi}_+(\zeta) = 0$ for $\zeta \le 1$.

Note that by (3.56), (3.59) and (4.22) the function $F_2 + F_3$ can be decomposed in the form

$$F_2 + F_3 = F_2 - \hat\psi''(S_0)\rho_2 + \partial_\zeta^2 \rho_2 = \sum_{j=1}^{5} I_j, \tag{4.29}$$

where

$$I_1 \;=\; \breve{\sigma}(0) + \overline{\varepsilon} : [\hat{T}]\, S_1 - \overline{\varepsilon} : \check{T}^{(-)}\, \overline{\varphi}_- - \overline{\varepsilon} : \check{T}^{(+)}\, \overline{\varphi}_+\,, \tag{4.30}$$

$$I_2 \;=\; -\frac{\hat{\psi}'''(S_0)}{2} S_1^2 + \frac{\hat{\psi}'''(0)}{2}\Big(\frac{\overline{\varepsilon} : \hat{T}^{(-)}}{\hat{\psi}''(0)}\Big)^2 \overline{\varphi}_- + \frac{\hat{\psi}'''(1)}{2}\Big(\frac{\overline{\varepsilon} : \hat{T}^{(+)}}{\hat{\psi}''(1)}\Big)^2 \overline{\varphi}_+\,, \tag{4.31}$$

$$I_3 \;=\; \lambda^{1/2}\, \hat{\sigma}'(0)\zeta\,\big(1 - \overline{\varphi}_- - \overline{\varphi}_+\big), \tag{4.32}$$

$$I_4 \;=\; \lambda^{1/2}\, \overline{\varepsilon} : D\varepsilon(a^* \otimes n + \nabla_x u^*)\Big(\int_{-\infty}^{\zeta} S_0(\vartheta)\, d\vartheta - \zeta^+ \overline{\varphi}_+\Big), \tag{4.33}$$

$$I_5 \;=\; -\frac{1}{c_1}\, \overline{\varepsilon} : \langle \hat{T} \rangle S_1' + \Big(\frac{s_1}{c} - \lambda\kappa'(0)\zeta\Big) S_0' + \partial_\zeta^2 \rho_2\,. \tag{4.34}$$

We show that everyone of these terms decays to zero for $\zeta \to \pm\infty$. To verify this for the first term we insert (3.46) and (3.47) into (4.30), which results in

$$I_1 = \breve{\sigma}(0) + \overline{\varepsilon} : [\hat{T}]\, S_1 - \breve{\sigma}(0)(\overline{\varphi}_+ + \overline{\varphi}_-) - \overline{\varepsilon} : [\hat{T}]\,\Big(\frac{\overline{\varepsilon} : \hat{T}^{(-)}}{\hat{\psi}''(0)} \overline{\varphi}_- + \frac{\overline{\varepsilon} : \hat{T}^{(+)}}{\hat{\psi}''(1)} \overline{\varphi}_+\Big).$$

We introduce the terms $\hat{\psi}''(0)\varphi_-$ and $\hat{\psi}''(1)\varphi_+$ into this equation. Noting the definition of ρ_1 in (4.10), this leads to

$$\begin{aligned} |I_1| &\le \big|\breve{\sigma}(0)\,(1 - \overline{\varphi}_+ + \overline{\varphi}_-) + \overline{\varepsilon} : [\hat{T}]\,\big(S_1 - \rho_1\big)\big| \\ &\quad + \Big|\overline{\varepsilon} : [\hat{T}]\,\Big(\frac{\overline{\varepsilon} : \hat{T}^{(-)}}{\hat{\psi}''(0)}(\overline{\varphi}_- - \hat{\psi}''(0)\varphi_-) + \frac{\overline{\varepsilon} : \hat{T}^{(+)}}{\hat{\psi}''(1)}(\overline{\varphi}_+ - \hat{\psi}''(1)\varphi_+)\Big)\Big| \le Ce^{-a|\zeta|}, \end{aligned} \tag{4.35}$$

for all $\zeta \in \mathbb{R}$, where we applied the estimates (4.12), (4.24), (4.25) and (4.28).

Next we estimate I_2. By definition we have $\overline{\varphi}_+(\zeta) = 0$ for $\zeta \le 1$. From (4.31) we thus have on the half axis $-\infty < \zeta \le 0$ that

$$\begin{aligned} I_2 &= -\frac{\hat{\psi}'''(S_0)}{2} S_1^2 + \frac{\hat{\psi}'''(0)}{2}\Big(\frac{\overline{\varepsilon} : \hat{T}^{(-)}}{\hat{\psi}''(0)}\Big)^2 \overline{\varphi}_- \\ &= \frac{\hat{\psi}'''(0) - \hat{\psi}'''(S_0)}{2} S_1^2 - \frac{\hat{\psi}'''(0)}{2} S_1^2 (1 - \overline{\varphi}_-) - \frac{\hat{\psi}'''(0)}{2}\Big(S_1^2 - \Big(\frac{\overline{\varepsilon} : \hat{T}^{(-)}}{\hat{\psi}''(0)}\Big)^2\Big)\overline{\varphi}_- \\ &= I_{21} + I_{22} + I_{23}. \end{aligned} \tag{4.36}$$

To estimate I_{21} we apply the mean value theorem to $\hat{\psi}'''$ and use (4.3) and (4.11), to estimate I_{22} we use (4.26) and (4.11). The result is

$$|I_{21} + I_{22}| \le CK_1 e^{-a|\zeta|}, \quad -\infty < \zeta \le 0. \tag{4.37}$$

To estimate I_{23} note that by (3.57), (3.58) and (4.10) we have for $-\infty < \zeta \le -2$ that

$$\rho_1(t, \eta, \zeta) = \varphi_-(\zeta)\big(\overline{\varepsilon} : \hat{T}^{(-)}(t, \eta)\big) = \frac{\overline{\varepsilon} : \hat{T}^{(-)}(t, \eta)}{\hat{\psi}''(0)}.$$

With this equation we infer from (4.11) and (4.12) with $\alpha, j = 0$ that

$$|I_{23}| = \Big|\frac{\hat{\psi}'''(0)}{2}\Big(S_1 - \frac{\overline{\varepsilon} : \hat{T}^{(-)}}{\hat{\psi}''(0)}\Big)\Big(S_1 + \frac{\overline{\varepsilon} : \hat{T}^{(-)}}{\hat{\psi}''(0)}\Big)\overline{\varphi}_-\Big| \le Ce^{-a|\zeta|}, \quad -\infty < \zeta \le 0. \tag{4.38}$$

(4.36) – (4.38) together imply that $|I_2(\zeta)| \le Ce^{-a|\zeta|}$ for $-\infty < \zeta \le 0$. On the half axis $0 \le \zeta < \infty$ we estimate I_2 analogously. This proves that

$$|I_2| \le Ce^{-a|\zeta|}, \qquad -\infty < \zeta < \infty. \tag{4.39}$$

The estimate for I_3 is obtained by application of (4.28) to (4.32), which immediately yields

$$|I_3| \le C(1+|\zeta|)e^{-a|\zeta|}, \qquad -\infty < \zeta < \infty. \tag{4.40}$$

To study the asymptotic behavior of I_4 note that (4.8) and (4.27) together imply

$$\begin{aligned}\Big|\int_{-\infty}^{\zeta} S_0(\vartheta)\,d\vartheta - \zeta^+\overline{\varphi}_+(\zeta)\Big| &\le \Big|\int_{-\infty}^{\zeta} S_0(\vartheta)\,d\vartheta - \zeta^+\Big| + \zeta^+|1-\overline{\varphi}_+(\zeta)| \\ &\le \Big(\frac{1}{a}K_1 + \zeta^+ CK_4\Big)e^{-a|\zeta|}.\end{aligned}$$

Insertion of this inequality into (4.33) results in

$$|I_4| \le C(1+|\zeta|)\,e^{-a|\zeta|}. \tag{4.41}$$

It remains to investigate I_5. Note first that the third term on the right hand side of (4.34) satisfies

$$\partial_\zeta^2 \rho(t,\eta,\zeta) = 0, \qquad \text{for } |\zeta| \ge 2. \tag{4.42}$$

To show this it suffices to remark that the functions $\varphi_\pm$ are constant on the intervals $(-\infty,-2)$ and $(2,\infty)$, from which we see by inspection of (3.59) that on these intervals the function $\zeta \mapsto \rho_2(t,\eta,\zeta)$ is a sum of constant and linear terms, whence (4.42) follows. If we estimate the first term on the right hand side of (4.34) by employing (4.13) with $\alpha = 0$, $j = 1$ and the second term by using (4.5), we obtain together with (4.42) that

$$|I_5| \le C(1+|\zeta|)e^{-a|\zeta|}. \tag{4.43}$$

We combine (4.29), (4.35), (4.39) – (4.41) and (4.43) to derive the estimate

$$|F_2(t,\eta,\zeta) + F_3(t,\eta,\zeta)| \le C(1+|\zeta|)e^{-a|\zeta|}, \qquad \text{for all } \zeta \in \mathbb{R}, \tag{4.44}$$

which shows in particular that the right hand side of (4.19) belongs to $L^2(\mathbb{R})$.

(IV) The orthogonality condition determining s_1. Next we must show that the right hand side of (4.19) is orthogonal to the kernel of $\hat{\psi}''(S_0) - \partial_\zeta^2$. This kernel is different from $\{0\}$, since S_0' belongs to the kernel. This is immediately seen by differentiation of (2.25), which yields

$$\hat{\psi}''\big(S_0(\zeta)\big)S_0'(\vartheta) - \partial_\zeta^2 S_0'(\zeta) = 0. \tag{4.45}$$

Since by (4.5) the function S_0' is in the domain of definition of $\hat{\psi}''(S_0) - \partial_\zeta^2$, it belongs to the kernel of this operator.

The theory of linear ordinary differential equations of second order implies now that the kernel is one-dimensional, hence every function from the kernel is a multiple of S_0'. Therefore the right hand side F_2+F_3 of (4.19) is orthogonal to the kernel if it is orthogonal to S_0'. Note that the integrals $\int_{-\infty}^{\infty} F_2 S_0'\,d\zeta$ and $\int_{-\infty}^{\infty} F_3 S_0'\,d\zeta$ both exist, since F_2 and F_3

grow at most linearly for $\zeta \to \pm\infty$, whereas by (4.5) the function S_0' decays exponentially at $\pm\infty$. Therefore we obtain from (4.22) and (4.45) by partial integration that

$$\int_{-\infty}^{\infty} (F_2 + F_3) S_0'\, d\zeta = \int_{-\infty}^{\infty} F_2 S_0'\, d\zeta - \int_{-\infty}^{\infty} \Big(\big(\hat\psi''(S_0) - \partial_\zeta^2\big)\rho_2 \Big) S_0'\, d\zeta$$
$$= \int_{-\infty}^{\infty} F_2 S_0'\, d\zeta - \int_{-\infty}^{\infty} \rho_2 \big(\hat\psi''(S_0) - \partial_\zeta^2\big) S_0'\, d\zeta = \int_{-\infty}^{\infty} F_2 S_0'\, d\zeta. \quad (4.46)$$

To study the last integral on the right hand side note that by (4.6) the function S_0' is even, which implies that

$$\int_{-\infty}^{\infty} \hat\sigma'(t,\eta,0)\zeta\, S_0'(\zeta)\, d\zeta = 0, \quad \text{and} \quad \int_{-\infty}^{\infty} \kappa'(t,\eta,0)\zeta\, \big(S_0'(\zeta)\big)^2 d\zeta = 0. \quad (4.47)$$

Moreover, since by Lemma 4.1 the function S_0' satisfies (4.1), we obtain by substitution of $\vartheta = S_0(\zeta)$

$$\int_{-\infty}^{\infty} S_0'(\zeta) S_0'(\zeta)\, d\zeta = \int_{-\infty}^{\infty} \sqrt{2\hat\psi\big(S_0(\zeta)\big)}\, S_0'(\zeta)\, d\zeta = \int_0^1 \sqrt{2\hat\psi(\vartheta)}\, d\vartheta = c_1, \quad (4.48)$$

where the last equality sign holds by definition of c_1 in (2.36). Finally, by partial integration,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\zeta} S_0(\vartheta)\, d\vartheta\, S_0'(\zeta)\, d\zeta = \lim_{\zeta_1 \to \infty} \Big(\int_{-\infty}^{\zeta_1} S_0(\vartheta)\, d\vartheta\, S_0(\zeta_1) - \int_{-\infty}^{\zeta_1} S_0(\zeta)^2\, \zeta \Big)$$
$$= \lim_{\zeta_1 \to \infty} \int_{-\infty}^{\zeta_1} S_0(\zeta)\big(S_0(\zeta_1) - S_0(\zeta)\big)\, d\zeta$$
$$= \int_{-\infty}^{\infty} S_0(\zeta)\big(1 - S_0(\zeta)\big)\, d\zeta = \int_{-\infty}^{\infty} S_0(\zeta) S_0(-\zeta)\, d\zeta. \quad (4.49)$$

In the second last step we used that S_0 is increasing. The equality sign thus follows from the theorem of Beppo Levi. The last equality sign is obtained from (4.6).

The equations (3.56) and (4.47) – (4.49) yield

$$\int_{-\infty}^{\infty} F_2(t,\eta,\zeta) S_0'(\zeta)\, d\zeta = \int_{-\infty}^{\infty} \big(\check\sigma(0) + \overline{\varepsilon} : [\hat T] S_1\big) S_0'\, d\zeta - \frac{1}{c_1} \overline{\varepsilon} : \langle \hat T \rangle \int_{-\infty}^{\infty} S_1' S_0'\, d\zeta$$
$$- \frac{1}{2} \int_{-\infty}^{\infty} \hat\psi'''(S_0) S_1^2 S_0'\, d\zeta + \lambda^{1/2}\, \overline{\varepsilon} : D\varepsilon(a^* \otimes n + \nabla_\Gamma u^*) \int_{-\infty}^{\infty} \int_{-\infty}^{\zeta} S_0(\vartheta)\, d\vartheta\ S_0'(\zeta)\, d\zeta$$
$$+ \int_{-\infty}^{\infty} \Big(\lambda^{1/2} \hat\sigma'(0)\zeta + \big(\frac{s_1}{c} - \lambda\kappa'(0)\zeta\big) S_0' \Big) S_0'(\zeta) d\zeta$$
$$= \check\sigma(0) + \overline{\varepsilon} : [\hat T] \int_{-\infty}^{\infty} S_1 S_0'\, d\zeta - \frac{1}{c_1} \overline{\varepsilon} : \langle \hat T \rangle \int_{-\infty}^{\infty} S_1' S_0'\, d\zeta - \frac{1}{2} \int_{-\infty}^{\infty} \hat\psi'''(S_0) S_1^2 S_0'\, d\zeta$$
$$+ \lambda^{1/2}\, \overline{\varepsilon} : D\varepsilon(a^* \otimes n + \nabla_\Gamma u^*) \int_{-\infty}^{\infty} S_0(\zeta) S_0(-\zeta)\, d\zeta + c_1 \frac{s_1}{c}\, .$$
$$= -\frac{c_1}{c} s_{10} - \lambda^{1/2} \frac{c_1}{c} s_{11} + \frac{c_1}{c} s_1 = \frac{c_1}{c}(-s_{10} - \lambda^{1/2} s_{11} + s_1).$$

To get the second last equality sign we inserted (3.48) for $\check\sigma(0)$ and used (2.41), (2.42). The right hand side of this equation vanishes if and only if s_1 satisfies (2.40). From (4.46) we thus infer that $F_1 + F_3$ is orthogonal to the kernel of $\hat{\psi}''(S_0) - \partial_\zeta^2$ if and only if (2.40) holds.

Consequently, from part (II) of the proof we conclude that the differential equation (4.19) has a solution w in $L^2(\mathbb{R})$ if and only if s_1 satisfies (2.40). In fact, there is exactly one such w, which also satisfies (4.20). To prove this assume that $\tilde{w} \in L^2(\mathbb{R})$ is a special solution of (4.19). Then we obtain every solution contained in $L^2(\mathbb{R})$ in the form $w = \tilde{w} + \beta S_0'$ with an arbitrary constant $\beta \in \mathbb{R}$. Since (4.1) and (2.37) yield

$$S_0'(0) = \sqrt{2\hat{\psi}\big(S_0(0)\big)} = \sqrt{2\hat{\psi}\Big(\frac{1}{2}\Big)} > 0,$$

we can choose β such that

$$w(t, \eta, 0) = \tilde{w}(t, \eta, 0) + \beta S_0'(0) = 0,$$

which is (4.20). This equation determines β uniquely, hence w is the unique solution of (4.19) and (4.20) in $L^2(\mathbb{R})$.

(V) Existence of the solution, estimates (4.18) – (4.20). We show next that this function w satisfies (4.21). To this end we need the following

Lemma 4.6 *Let $\hat{a}_- > 0$, $\hat{a}_+ > 0$ and set $\hat{a} = \sqrt{\min\{\hat{a}_-, \hat{a}_+\}}$. Let $g : \mathbb{R} \to \mathbb{R}$ be a smooth function and let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that*

$$|g(\zeta) - \hat{a}_-| \le Ce^{-\hat{a}|\zeta|}, \qquad \text{for } \zeta < 0, \tag{4.50}$$

$$|g(\zeta) - \hat{a}_+| \le Ce^{-\hat{a}|\zeta|}, \qquad \text{for } \zeta > 0, \tag{4.51}$$

$$|f(\zeta)| \le C(1 + |\zeta|)e^{-\hat{a}|\zeta|}, \quad \text{for } \zeta \in \mathbb{R}. \tag{4.52}$$

Let $\hat{w}$ be a solution of

$$g(\zeta)\hat{w}(\zeta) - \partial^2 \hat{w}(\zeta) = f(\zeta), \qquad \zeta \in \mathbb{R}. \tag{4.53}$$

(i) Then $\hat{w}$ belongs to the space $C^2(\mathbb{R})$. If $\hat{w} \in L^2(\mathbb{R})$, then there is $C > 0$ such that

$$|\partial_\zeta^j \hat{w}(\zeta)| \le C(1 + |\zeta|)e^{-\hat{a}|\zeta|}, \qquad \text{for all } \zeta \in \mathbb{R}, \text{ for } j = 0, 1, 2. \tag{4.54}$$

(ii) If there are $C, \theta > 0$ such that

$$|\hat{w}(\zeta)| \le Ce^{(\hat{a}-\theta)|\zeta|} \tag{4.55}$$

holds for all $\zeta \in \mathbb{R}$, then $\hat{w} \in L^2(\mathbb{R})$.

This is a standard result from the theory of ordinary differential equations, and we omit the proof.

To show that w satisfies (4.21), we apply this lemma with $\hat{a}_- = \hat{\psi}''(0)$, $\hat{a}_+ = \hat{\psi}''(1)$, $g(\zeta) = \hat{\psi}''\big(S_0(\zeta)\big)$ and $f(\zeta) = F_2(t, \eta, \zeta) + F_3(t, \eta, \zeta)$. Then we have $\hat{a} = \sqrt{\min\{\hat{\psi}''(0), \hat{\psi}''(1)\}} = a$, by (2.37), and from (4.9) and (4.44) we see that (4.50) – (4.52) hold for this choice of functions and constants. Moreover, with this choice of functions the differential equation (4.53) is equal to (4.19). Since $w \in L^2(\mathbb{R})$ is a solution of

(4.19), we see that all assumptions for part (i) of Lemma 4.6 are satisfied, hence (4.54) holds for w, which means that

$$|\partial_\zeta^j w(\zeta)| \leq C(1+|\zeta|)e^{-a|\zeta|}, \qquad \zeta \in \mathbb{R},\ j=0,1,2, \tag{4.56}$$

and this in particular implies that w satisfies (4.21).

We have now found a unique solution $w \in L^2(\mathbb{R})$ of (4.19) – (4.21). By part (I) of this proof this means that S_2 given by (4.18) is a solution of the boundary value problem (3.55), (3.56), (3.60) and (3.61). Since by (4.18) we have $w = S_2 - \rho_2$, the inequality (4.56) shows that S_2 satisfies (4.15) for $\alpha = 0$.

To verify that S_2 satisfies (4.15) for $\alpha \neq 0$, it must first be shown that S_2 is two times continuously differentiable with respect to (t,η). This follows if we can show that $w = S_2 - \rho_2$ is two times continuoulsly differentiable with respect to (t,η), since by our regularity assumptions the function ρ_2 has this differentiability property. To prove this differentiability of w, we write (4.19), (4.20) as a perturbation problem for the linear equation

$$Aw = f(t,\eta)$$

in $L^2(\mathbb{R})$, where $A = (\hat{\psi}''(S_0) - \partial_\zeta^2)$ is the linear differential operator on the left hand side of (4.19) and $f(t,\eta) = F_2(t,\eta,\cdot) + F_3(t,\eta,\cdot) \in L^2(\mathbb{R})$ is the function on the right hand side of (4.19), which depends two times continuously differentiable on (t,η) and satisfies the estimate (4.44) for every $(t,\eta) \in \Gamma$. Since 0 is an eigenvalue of A and since $f(t,\eta)$ is orthogonal to the kernel of A for every (t,η), this linear equation has infinitely many solutions and the solution set is affine. The condition (4.20) defines a linear subspace, which is closed in the Sobolev space $H^1(\mathbb{R})$, and which intersects the solution set in exactly one point w, which is the solution of (4.19), (4.20).

To the problem set in this way we can apply the pertubation theory of linear operators. The theory yields that w is two times continuously differentiable with respect to (t,η). We avoid the details but refer to standard texts on the pertubation theory of linear operators, for example [26].

With this knowledge we can derive the estimate (4.15) for $\alpha \neq 0$ by applying the differential operator $D^\alpha_{(t,\eta)}$ with $1 \leq |\alpha| \leq 2$ to the differential equation (4.19) and obtain

$$\hat{\psi}''(S_0)(D^\alpha_{(t,\eta)}w) - \partial_\zeta^2(D^\alpha_{(t,\eta)}w) = D^\alpha_{(t,\eta)}(F_2+F_3). \tag{4.57}$$

This is a differential equation for the function $D^\alpha_{(t,\eta)}w$ with right hand side satisfying the estimate

$$|D^\alpha_{(t,\eta)}(F_2+F_3)| \leq C(1+|\zeta|)e^{-a|\zeta|}. \tag{4.58}$$

The proof of this estimate proceeds in the same way as the proof of the corresponding estimate for $\alpha = 0$, which we gave in part (III). Essentially one has to replace the terms appearing in F_2 and F_3, which depend on (t,η), by their derivatives. To avoid repetition of many technical details, we omit this proof.

The differential equation (4.57) has the same form as the differential equation (4.19). From (4.58) we see that the assumption (4.52) holds, hence we can apply Lemma 4.6 (i) to this differential equation, from which we see that w belongs to the space $C^2(\mathbb{R}, C^2(\Gamma,\mathbb{R}))$ and that the inequality (4.54) holds with $\hat{w}$ replaced by $D^\alpha_{(t,\eta)}w$, whence we have

$$|\partial_\zeta^j D^\alpha_{(t,\eta)} w(\zeta)| \leq C(1+|\zeta|)e^{-a|\zeta|}, \qquad \zeta \in \mathbb{R},\ 0 \leq j \leq 2,\ 1 \leq |\alpha| \leq 2.$$

Since $w = S_2 - \rho_2$, this is inequality (4.15) with $\alpha \neq 0$. Therefore we proved that (4.15) holds for all $0 \leq |\alpha| \leq 2$.

The inequality (4.16) is a consequence of (4.15) and of (4.42), the inequality (4.14) follows by combination of (4.15) with the estimate

$$|\partial_\zeta^j D^\alpha_{(t,\eta)} \rho_2(t,\eta,\zeta)| \leq C(1+|\zeta|)^{1-j}, \qquad j = 0, 1,$$

which is seen to hold by inspection of (3.59).

We have now proved statement (i) of Theorem 4.5, and it remains to verify (ii). That is, we have to show that S_2 is the only solution of (3.55), (3.56), (3.60) satisfying (4.17). Indeed, from (4.14) it follows that S_2 satisfies (4.17). Assume that S_2^* is a second solution satisfying (4.17). Then $\hat{w} = S_2 - S_2^*$ fulfills (4.55) and the equation

$$\hat{\psi}''(S_0)\hat{w} - \partial^2 \hat{w} = 0.$$

Lemma 4.6 (ii) thus yields $\hat{w} \in L^2(\mathbb{R})$. Consequently, by (4.18) we have $S_2^* = w + \hat{w} + \rho_2$, where $w + \hat{w} \in L^2(\mathbb{R})$ is a solution of (4.19), (4.20). At the end of part (IV) of this proof we showed that w is the only solution of (4.19), (4.20) in $L^2(\mathbb{R})$, whence $\hat{w} = 0$, hence $S_2^* = S_2$.

The proof of Theorem 4.5 is complete. ∎

5 Proof of the estimates (2.50) – (2.53) in Theorem 2.3

The proof of (2.50) – (2.53) is straightforward: We insert the function $(u^{(\mu)}, T^{(\mu)}, S^{(\mu)})$ defined in Section 3 into the model equations (1.1) and (1.3) and compute the residues. However, the necessary computations are long. Therefore we divide them into four parts:

In Section 5.1 we compute for the functions $T_1^{(\mu)}$ and $T_2^{(\mu)}$ from the inner and outer expansions of $T^{(\mu)}$ the residues $\text{div}_x T_1^{(\mu)} + \mathsf{b}$ and $\text{div}_x T_2^{(\mu)} + \mathsf{b}$ separately. Likewise, in Section 5.2 we insert the inner expansion $(u_1^{(\mu)}, T_1^{(\mu)}, S_1^{(\mu)})$ and the outer expansion $(u_2^{(\mu)}, T_2^{(\mu)}, S_2^{(\mu)})$ into (1.3) and compute and estimate the residues separately. With these residues we can prove (2.50) – (2.53) in the regions $Q_{\text{inn}}^{(\mu\lambda)}$ and $Q_{\text{out}}^{(\mu\lambda)}$, but in the matching region $Q_{\text{match}}^{(\mu\lambda)}$ we need auxiliary estimates, which are derived in Section 5.3. All the estimates are put together in Section 5.4 to complete the proof.

5.1 Asymptotic expansion of $\text{div}_x \boldsymbol{T}^{(\mu)} + \mathsf{b}$

Lemma 5.1 *Let $(\hat{u}, \hat{T})$ be the solution of the transmission problem* (2.15) – (2.19). *With the splitting* (3.38) *of $\hat{u}$ the stress tensor field $\hat{T}$ satisfies in the neighborhood $\mathcal{U}_\delta$ of* Γ

$$\hat{T} = [\hat{T}]\hat{S} + D\varepsilon(\nabla_x \hat{v}) + D\varepsilon\big(a^* \otimes n + \nabla_{\Gamma_\xi} u^*\big)\xi^+ + D\varepsilon(\nabla_{\Gamma_\xi} a^*)\frac{1}{2}(\xi^+)^2. \tag{5.1}$$

With $\hat{\sigma}'$ defined in (3.44) *and with a remainder term $R_{\overline{\varepsilon}:\hat{T}} \in L^\infty(\mathcal{U}_\delta)$ we have for $(t,\eta,\xi) \in \mathcal{U}_\delta$*

$$\begin{aligned} \overline{\varepsilon} : \hat{T}(t,\eta,\xi) &= \overline{\varepsilon} : \hat{T}^{(+)}(t,\eta) + \hat{\sigma}'(t,\eta,0)\,\xi \\ &\quad + \overline{\varepsilon} : D\varepsilon\big(a^*(t,\eta) \otimes n(t,\eta) + \nabla_\Gamma u^*(t,\eta)\big)\,\xi + R_{\overline{\varepsilon}:\hat{T}}(t,\eta,\xi)\,\xi^2, \quad \xi > 0, \end{aligned} \tag{5.2}$$

$$\overline{\varepsilon} : \hat{T}(t,\eta,\xi) = \overline{\varepsilon} : \hat{T}^{(-)}(t,\eta) + \hat{\sigma}'(t,\eta,0)\,\xi + R_{\overline{\varepsilon}:\hat{T}}(t,\eta,\xi)\,\xi^2, \qquad \xi < 0. \tag{5.3}$$

The functions u^* and a^* introduced in (2.34), (2.35) *and the normal vector n satisfy on the interface Γ*

$$\Big(D\big(\varepsilon(u^*\otimes n)-\overline{\varepsilon}\big)\Big)n = 0, \tag{5.4}$$

$$\big(D\varepsilon(a^*\otimes n+\nabla_\Gamma u^*)\big)n+\mathrm{div}_\Gamma D\varepsilon(u^*\otimes n) = 0. \tag{5.5}$$

Proof: With the splitting (3.6) of the gradient operator we compute from (3.38) that

$$\begin{aligned}\nabla_x\hat{u}(t,x) &= \partial_\xi\hat{u}\otimes n+\nabla_{\Gamma_\xi}\hat{u}\\ &= \big(u^*(t,\eta)1^+(\xi)+a^*(t,\eta)\xi^+\big)\otimes n(t,\eta)+\nabla_{\Gamma_\xi}u^*(t,\eta)\xi^+\\ &\quad+\nabla_{\Gamma_\xi}a^*(t,\eta)\frac{1}{2}(\xi^+)^2+\nabla_x\hat{v}(t,x).\end{aligned}$$

We insert this equation into (2.16), note that $\hat{S}(t,x)=1^+(\xi)$, and employ that by (2.84)

$$D\big(\varepsilon(u^*\otimes n)-\overline{\varepsilon}\big)\hat{S}=[\hat{T}]\hat{S}$$

to obtain (5.1).

By (3.15) we have

$$\nabla_{\Gamma_\xi}u^*(t,\eta)\xi^+=\nabla_\Gamma u^*(t,\eta)\xi^++(\nabla_\Gamma u^*(t,\eta))R_A(t,\eta,\xi)(\xi^+)^2, \tag{5.6}$$

where we used our convention to identify $\nabla_\Gamma u^*$ and $\nabla_\eta u^*$, since u^* does not depend on ξ. Noting the definition of $\hat{\sigma}$ in (3.44), we obtain from (5.1) and from (5.6) that

$$\begin{aligned}\overline{\varepsilon}:\hat{T}(t,\eta,\xi) &= \overline{\varepsilon}:[\hat{T}](t,\eta)1^+(\xi)+\hat{\sigma}(t,\eta,\xi)\\ &\quad+\overline{\varepsilon}:D\varepsilon\big(a^*(t,\eta)\otimes n(t,\eta)+\nabla_\Gamma u^*(t,\eta)\big)\xi^+\\ &\quad+\overline{\varepsilon}:D\varepsilon\Big(\nabla_\Gamma u^*(t,\eta)R_A(t,\eta,\xi)+\frac{1}{2}\nabla_{\Gamma_\xi}a^*(t,\eta)\Big)(\xi^+)^2.\end{aligned} \tag{5.7}$$

(3.38) and (2.16) together imply for $\xi<0$ that

$$\hat{T}(t,\eta,\xi)=D\varepsilon\big(\nabla_x\hat{v}(t,\eta,\xi)\big), \tag{5.8}$$

whence

$$\overline{\varepsilon}:\hat{T}(t,\eta,\xi)=\overline{\varepsilon}:D\varepsilon\big(\nabla_x\hat{v}(t,\eta,\xi)\big)=\hat{\sigma}(t,\eta,\xi),\quad \xi<0,$$

and therefore

$$\overline{\varepsilon}:\hat{T}^{(-)}(t,\eta)=\hat{\sigma}(t,\eta,0),\quad \overline{\varepsilon}:\hat{T}^{(+)}(t,\eta)=\overline{\varepsilon}:[\hat{T}](t,\eta)+\hat{\sigma}(t,\eta,0). \tag{5.9}$$

By Taylor's formula we can express $\hat{\sigma}$ in the form

$$\hat{\sigma}(t,\eta,\xi)=\hat{\sigma}(t,\eta,0)+\partial_\xi\hat{\sigma}(t,\eta,0)\xi+\partial_\xi^2\hat{\sigma}(t,\eta,\xi^*)\xi^2.$$

We expand $\hat{\sigma}$ in (5.7) with this formula and note the equations (5.9) to obtain (5.2) and (5.3).

(5.4) is an immediate consequence of (2.84) and (2.18). To prove (5.5) we apply (3.7) to calculate from (5.1) that

$$\begin{aligned} 0 &= \operatorname{div}_x \hat{T} + \mathsf{b} \\ &= \partial_\xi([\hat{T}]n\hat{S}) + \operatorname{div}_{\Gamma_\xi}[\hat{T}]\hat{S} \\ &\quad + \partial_\xi\big(D\varepsilon(a^*\otimes n + \nabla_{\Gamma_\xi}u^*)\xi^+\big)n + \partial_\xi\big(\frac{1}{2}D\varepsilon(\nabla_{\Gamma_\xi}a^*)(\xi^+)^2\big)n \\ &\quad + \operatorname{div}_{\Gamma_\xi} D\varepsilon\Big(\big(a^*\otimes n + \nabla_{\Gamma_\xi}u^*\big)\xi^+ + \frac{1}{2}(\nabla_{\Gamma_\xi}a^*)(\xi^+)^2\Big) \\ &\quad + \operatorname{div}_x D\varepsilon(\nabla_x \hat{v}) + \mathsf{b}. \end{aligned} \tag{5.10}$$

From this equation we obtain for $\xi < 0$ that

$$\operatorname{div}_x D\varepsilon(\nabla_x\hat{v}) + \mathsf{b} = \operatorname{div}_x\hat{T} + \mathsf{b} = 0. \tag{5.11}$$

By assumption in Theorem 2.3, the function b is continuous at Γ. Moreover, by Lemma 3.2 and the differentiability properties of $\hat{v}$ required in Theorem 2.3 the function $\hat{v}$ is two times continuously differentiable at Γ. Therefore we infer from (5.11) that

$$\big(\operatorname{div}_x D\varepsilon(\nabla_x\hat{v}) + \mathsf{b}\big)^{(+)} = \big(\operatorname{div}_x D\varepsilon(\nabla_x\hat{v}) + \mathsf{b}\big)^{(-)} = 0, \quad \text{on } \Gamma. \tag{5.12}$$

With this equation and with $[\hat{T}]n = 0$, by (2.18), we conclude from (5.10) that

$$0 = \lim_{\xi\to 0+}(\operatorname{div}_x\hat{T} + \mathsf{b}) = \big(D\varepsilon(a^*\otimes n + \nabla_\Gamma u^*)\big)n + \operatorname{div}_\Gamma[\hat{T}].$$

From this relation and from $\operatorname{div}_\Gamma[\hat{T}] = \operatorname{div}_\Gamma D\varepsilon(u^*\otimes n)$, which is a consequence of (2.84), we obtain (5.5). ■

Next we study the stress field $T_1^{(\mu)}$ in the inner expansion.

Lemma 5.2 *Let $u_1^{(\mu)}$, $S_1^{(\mu)}$, $T_1^{(\mu)}$ be given in (3.49) – (3.51), let u_0, u_1, u_2 be defined in (3.52) – (3.54), and let R_A be the remainder term from (3.14). We set $\zeta = \frac{\xi}{(\mu\lambda)^{1/2}}$. Then we have for (t,η,ξ) from the neighborhood $\mathcal{U}_\delta$ of Γ*

$$\begin{aligned} T_1^{(\mu)}(t,\eta,\xi) &= [\hat{T}](S_0 + \mu^{1/2}S_1) + D\varepsilon\big(\nabla_x(\hat{v} + \mu^{1/2}\check{v})\big) \\ &\quad + (\mu\lambda)^{1/2} D\varepsilon(a^*\otimes n + \nabla_\Gamma u^*)\int_{-\infty}^{\zeta} S_0(\vartheta)\,d\vartheta + \mu R_{T_1}(\lambda,t,\eta,\xi,\zeta), \end{aligned} \tag{5.13}$$

where

$$R_{T_1}(\lambda,t,\eta,\xi,\zeta) = D\Big(\varepsilon\big(\nabla_{\Gamma_\xi}(\lambda^{1/2}u_1 + \lambda u_2) + \lambda\zeta(\nabla_\eta u_0)R_A\big) - \overline{\varepsilon}S_2\Big). \tag{5.14}$$

The argument of $[\hat{T}]$, u^*, a^*, n is (t,η), the argument of S_1, S_2, u_0, u_1, u_2 is $(t,\eta,\frac{\xi}{(\mu\lambda)^{1/2}})$, the argument of $\nabla_x\hat{v}$, $\nabla_x\check{v}$, R_A is (t,η,ξ), and the argument of S_0 outside of the integral is $\frac{\xi}{(\mu\lambda)^{1/2}}$. Moreover, we have

$$\begin{aligned} \operatorname{div}_x T_1^{(\mu)} + \mathsf{b} &= \operatorname{div}_x D\varepsilon(\nabla_x\hat{v}) + \mathsf{b} + \xi\operatorname{div}_{\Gamma,\xi}[\hat{T}]S_0 + \mu^{1/2}\operatorname{div}_{\Gamma_\xi}\big([\hat{T}]S_1\big) \\ &\quad + \mu^{1/2}\operatorname{div}_x D\varepsilon(\nabla_x\check{v}) + (\mu\lambda)^{1/2}\operatorname{div}_{\Gamma_\xi} D\varepsilon(a^*\otimes n + \nabla_\Gamma u^*)\int_{-\infty}^{\zeta} S_0(\vartheta)\,d\vartheta \\ &\quad + \mu\operatorname{div}_x R_{T_1}\,. \end{aligned} \tag{5.15}$$

With $\hat\sigma(0)$, $\hat\sigma'(0)$, $\check\sigma(0)$ defined in (3.44), (3.45) we have

$$
\begin{aligned}
&\partial_S \mathsf{W}\big(\varepsilon(\nabla_x u_1^{(\mu)}), S_1^{(\mu)}\big)\\
&\quad = -\overline{\varepsilon} : T_1^{(\mu)}(t,\eta,\xi) = -\overline{\varepsilon} : [\hat T](S_0 + \mu^{1/2} S_1) - \hat\sigma(0) - (\mu\lambda)^{1/2}\hat\sigma'(0)\zeta - \mu^{1/2}\check\sigma(0)\\
&\qquad\qquad - (\mu\lambda)^{1/2}\overline{\varepsilon} : D\varepsilon(a^*\otimes n + \nabla_\Gamma u^*)\int_{-\infty}^{\zeta} S_0(\vartheta)d\vartheta - \mu R_W(\lambda,t,\eta,\xi,\zeta), \quad (5.16)
\end{aligned}
$$

where

$$
R_W(\lambda,t,\eta,\xi,\zeta) = \overline{\varepsilon} : R_{T_1}(\lambda,t,\eta,\xi,\zeta) + \lambda\partial_\xi^2\hat\sigma(t,\eta,\hat\xi)\zeta^2 + \lambda^{1/2}\partial_\xi\check\sigma(t,\eta,\check\xi)\zeta, \qquad (5.17)
$$

with suitable $\hat\xi$, $\check\xi$ between 0 and ξ and with R_{T_1} defined in (5.14).

Proof: With the splitting (3.6) of the gradient operator we obtain by definition of $u_1^{(\mu)}$ in (3.49), (3.52) – (3.54) that

$$
\begin{aligned}
\nabla_x u_1^{(\mu)} &= (\mu\lambda)^{1/2}\nabla_x u_0 + \mu\lambda^{1/2}\nabla_x u_1 + \mu\lambda\nabla_x u_2 + \nabla_x(\hat v + \mu^{1/2}\check v)\\
&= (u^*\otimes n)S_0 + \mu^{1/2}u^*\otimes n S_1 + (\mu\lambda)^{1/2} a^*\otimes n\int_{-\infty}^{\zeta} S_0(\vartheta)d\vartheta\\
&\quad + (\mu\lambda)^{1/2}\nabla_{\Gamma_\xi} u_0 + \mu\lambda^{1/2}\nabla_{\Gamma_\xi} u_1 + \mu\lambda\nabla_{\Gamma_\xi} u_2 + \nabla_x(\hat v + \mu^{1/2}\check v). \quad (5.18)
\end{aligned}
$$

(3.52) and (3.15) together yield

$$
\nabla_{\Gamma_\xi} u_0 = \nabla_\eta u_0(I + \xi R_A) = \nabla_\Gamma u^*\int_{-\infty}^{\zeta} S_0(\vartheta)d\vartheta + (\mu\lambda)^{1/2}\zeta\,(\nabla_\eta u_0)R_A. \qquad (5.19)
$$

We insert (5.18), (5.19) and (3.50) into (3.51). From the resulting equation we obtain (5.13) and (5.14) if we also note that by (2.84)

$$
\begin{aligned}
&D\big(\varepsilon(u^*\otimes n)(S_0 + \mu^{1/2}S_1) - \overline{\varepsilon}(S_0 + \mu^{1/2}S_1)\big)\\
&\qquad\qquad = D\big(\varepsilon(u^*\otimes n) - \overline{\varepsilon}\big)(S_0 + \mu^{1/2}S_1) = [\hat T](S_0 + \mu^{1/2}S_1).
\end{aligned}
$$

To prove (5.15) we employ the splitting (3.7) of the divergence operator and (3.19) to compute from (5.13)

$$
\begin{aligned}
\operatorname{div}_x T_1^{(\mu)} + \mathsf{b} &= \partial_\xi([\hat T]nS_0) + (\operatorname{div}_\Gamma[\hat T])S_0 + \xi\big(\operatorname{div}_{\Gamma,\xi}[\hat T]\big)S_0\\
&\quad + \mu^{1/2}\big(\partial_\xi([\hat T]nS_1) + \operatorname{div}_{\Gamma_\xi}([\hat T]S_1)\big)\\
&\quad + \operatorname{div}_x D\varepsilon(\nabla_x\hat v) + \mathsf{b} + \mu^{1/2}\operatorname{div}_x D\varepsilon(\nabla_x\check v) + \big(D\varepsilon(a^*\otimes n + \nabla_\Gamma u^*)\big)nS_0\\
&\quad + (\mu\lambda)^{1/2}\operatorname{div}_{\Gamma_\xi} D\varepsilon(a^*\otimes n + \nabla_\Gamma u^*)\int_{-\infty}^{\zeta} S_0(\vartheta)d\vartheta + \mu\operatorname{div}_x R_{T_1}. \quad (5.20)
\end{aligned}
$$

By (2.84) and (5.5) we have

$$
\big(D\varepsilon(a^*\otimes n + \nabla_\Gamma u^*)\big)nS_0 + \operatorname{div}_\Gamma[\hat T]S_0 = \Big(\big(D\varepsilon(a^*\otimes n + \nabla_\Gamma u^*)\big)n + \operatorname{div}_\Gamma D\varepsilon(u^*\otimes n)\Big)S_0 = 0.
$$

With this equation and with (2.18) we obtain (5.15) from (5.20).

(5.16), (5.17) follow immediately from (1.8), which implies $\partial_S \mathsf{W}\big(\varepsilon(\nabla_x u_1^{(\mu)}), S_1^{(\mu)}\big) = -\bar{\varepsilon} : T_1^{(\mu)}$, and from (5.13), (5.14), using the Taylor expansions

$$\check{\sigma}(\xi) = \check{\sigma}(0) + \partial_\xi \check{\sigma}(\check{\xi})\xi = \check{\sigma}(0) + (\mu\lambda)^{1/2}\partial_\xi \check{\sigma}(\check{\xi})\zeta,$$
$$\hat{\sigma}(\xi) = \hat{\sigma}(0) + \partial_\xi \hat{\sigma}(0)\xi + \partial_\xi^2 \hat{\sigma}(\hat{\xi})\xi^2 = \hat{\sigma}(0) + (\mu\lambda)^{1/2}\hat{\sigma}'(0)\zeta + \mu\lambda\hat{\sigma}''(\hat{\xi})\zeta^2.$$

This completes the proof of Lemma 5.2. ■

Corollary 5.3 *Let $Q_{\text{inn}}^{(\mu\lambda)}$ and $Q_{\text{match}}^{(\mu\lambda)}$ be defined in* (2.33) *and let $T_1^{(\mu)}$ be given by* (3.51). *Then there is a constant C such that for all $0 < \mu \le \mu_0$ and all $0 < \lambda \le \lambda_0$*

$$\|\operatorname{div}_x T_1^{(\mu)} + \mathsf{b}\|_{L^\infty(Q_{\text{inn}}^{(\mu\lambda)} \cup Q_{\text{match}}^{(\mu\lambda)})} \le C\Big(\frac{\mu}{\lambda}\Big)^{1/2} |\ln \mu|^2. \tag{5.21}$$

Proof: We estimate the terms on the right hand side of (5.15). Note first that if $(t, \eta, \xi) \in Q_{\text{inn}}^{(\mu\lambda)} \cup Q_{\text{match}}^{(\mu\lambda)}$ and $\zeta = \frac{\xi}{(\mu\lambda)^{1/2}}$, then

$$|\xi| \le \frac{3}{a}(\mu\lambda)^{1/2}|\ln \mu|, \quad |\zeta| \le \frac{3}{a}|\ln \mu|. \tag{5.22}$$

This follows from (2.33). With these inequalities the first two terms on the right hand side of (5.15) can be estimated as follows: From the differentiability properties of $\hat{v}$ and b, which in Theorem 2.3 are assumed to hold, it follows by Lemma 3.2 that

$$\operatorname{div}_x D\varepsilon(\nabla_x \hat{v}) + \mathsf{b} \in C(\mathcal{U}_\delta) \cap C^1\big((-\delta, 0], C(\Gamma)\big) \cap C^1\big([0, \delta), C(\Gamma)\big). \tag{5.23}$$

Because of this differentiability property we can apply the mean value theorem to $\operatorname{div}_x D\varepsilon(\nabla_x \hat{v}) + \mathsf{b}$, which together with (5.12) and (5.22) yields for all $(t, \eta, \xi) \in Q_{\text{inn}}^{(\mu\lambda)} \cup Q_{\text{match}}^{(\mu\lambda)}$ with $\xi \ge 0$ that

$$\begin{aligned}\Big|\big(\operatorname{div}_x D\varepsilon(\nabla_x \hat{v}) + \mathsf{b}\big)(t, \eta, \xi)\Big| &= \Big|\big(\operatorname{div}_x D\varepsilon(\nabla_x \hat{v}) + \mathsf{b}\big)^{(+)} + \partial_\xi\big(\operatorname{div}_x D\varepsilon(\nabla_x \hat{v}) + \mathsf{b}\big)(t, \eta, \xi^*)\xi\Big| \\ &= \big|\partial_\xi\big(\operatorname{div}_x D\varepsilon(\nabla_x \hat{v}) + \mathsf{b}\big)(t, \eta, \xi^*)\big|\xi \le C_1\xi \le C_1\frac{3}{a}(\mu\lambda)^{1/2}|\ln \mu|, \end{aligned} \tag{5.24}$$

with a suitable number ξ^* between 0 and ξ. Since by (5.11) the term $\operatorname{div}_x D\varepsilon(\nabla_x \hat{v}) + \mathsf{b}$ vanishes for $\xi < 0$, the inequality (5.24) holds for all $(t, \eta, \xi) \in Q_{\text{inn}}^{(\mu\lambda)} \cup Q_{\text{match}}^{(\mu\lambda)}$.

To estimate the last term in (5.15) note that (4.8), (4.11) and (5.22) yield

$$0 \le \int_{-\infty}^{\zeta} S_0(\vartheta)d\vartheta \le \zeta^+ + C_2 \le C_3\frac{3}{a}|\ln \mu|, \tag{5.25}$$

$$0 \le \int_{-\infty}^{\zeta}\int_{-\infty}^{\vartheta} S_0(\vartheta_1)d\vartheta_1 d\vartheta \le \frac{1}{2}(\zeta^+)^2 + C_4 \le C_5\Big(\frac{3}{a}|\ln \mu|\Big)^2,$$

$$\Big|\int_0^\zeta S_1(t, \eta, \vartheta)d\vartheta\Big| \le C_6|\zeta| \le C_6\frac{3}{a}|\ln \mu|.$$

Using these inequalities, the definitions of u_0, u_1, u_2 in (3.52) – (3.54) and the inequality (4.14) we obtain from (5.14) that

$$|R_{T_1}| \le \big(C_7 + \frac{C_8}{a^2}\big)|\ln \mu|^2, \tag{5.26}$$

$$|\mu \operatorname{div}_x R_{T_1}| \le \mu^{1/2}\lambda^{-1/2}\big(C_9 + \frac{C_{10}}{a^2}\big)|\ln \mu|^2. \tag{5.27}$$

(5.26) is used later, (5.27) is the desired estimate for the last term in (5.15).

To estimate the other terms in (5.15) we apply (4.3), (4.4), (4.11), (5.22) and(5.25). Together with (5.24) and (5.27) we find for $(t,\eta,\xi)\in Q^{(\mu\lambda)}_{\rm inn}\cup Q^{(\mu\lambda)}_{\rm match}$ that

$$\Big|(\mathrm{div}_x T_1^{(\mu)}+\mathsf{b})(t,\eta,\xi)\Big| \le \Big(\frac{C_{11}}{a^2}+C_{12}\Big)\Big(\mu^{1/2}+(\mu\lambda)^{1/2}|\ln\mu|+\mu^{1/2}\lambda^{-1/2})|\ln\mu|^2\Big),$$

which implies (5.21). ■

In the next lemma we study the outer expansion $T_2^{(\mu)}$.

Lemma 5.4 *Let* $(\tilde u,\tilde T)$ *be the solution of the boundary value problem, which consists of the elliptic system* (3.28), (3.29) *with* $\tilde S_2$ *given by* (3.37), *and of the boundary conditions* (3.33) – (3.35). *Let* $\tilde S_3$ *be the solution of* (3.32). *Then* $T_2^{(\mu)}$ *defined in* (3.27) *satisfies on* $Q\setminus\Gamma$

$$T_2^{(\mu)} = \hat T+\mu^{1/2}\check T+\mu\tilde T-\mu^{3/2}D\bar\varepsilon\tilde S_3, \tag{5.28}$$

$$\mathrm{div}_x T_2^{(\mu)}+\mathsf{b} = -\mu^{3/2}\mathrm{div}_x(D\bar\varepsilon\tilde S_3). \tag{5.29}$$

Proof: Insertion of (3.25) and (3.26) into (3.27) yields

$$\begin{aligned}T_2^{(\mu)} = D\big(\varepsilon(\nabla_x\hat u)-\bar\varepsilon\hat S\big)+\mu^{1/2}D\big(\varepsilon(\nabla_x\check u)-\bar\varepsilon\tilde S_1\big)\\ +\mu D\big(\varepsilon(\nabla_x\tilde u)-\bar\varepsilon\tilde S_2\big)-\mu^{3/2}D\bar\varepsilon\tilde S_3.\end{aligned}$$

Using (3.36), we see from this equation and from (2.16), (2.21), (3.29) that (5.28) holds. (5.29) is an immediate consequence of (5.28) and (2.15), (2.20), (3.28). ■

5.2 Asymptotic expansion of $S_t+c(\mathsf{W}_S+\hat\psi-\Delta_x S)$

In this section we compute the form of the residue

$$(\mu\lambda)^{1/2}\partial_t S+c\Big(\partial_S\mathsf{W}\big(\varepsilon(\nabla_x u),S\big)+\frac{1}{\mu^{1/2}}\hat\psi'(S)-\mu^{1/2}\lambda\Delta_x S\Big), \tag{5.30}$$

which is obtained when we either insert for (u,S) the inner expansion $\big(u_1^{(\mu)},S_1^{(\mu)}\big)$ or the outer expansion $\big(u_2^{(\mu)},S_2^{(\mu)}\big)$ of the asymptotic solution $\big(u^{\mu)},S^{(\mu)}\big)$.

For functions $(t,x)\mapsto w(t,x)$ defined in a neighborhood of Γ we write $w(t,\eta,\xi)=w(t,x)$ with $x=\eta+n(t,\eta)\xi$, as always. However, in the following computations this slight abuse of notation could lead to confusion when we consider derivatives with respect to t. To avoid this, we introduce the notations

$$w_{|t}(t,x)=w_{|t}(t,\eta,\xi)=\partial_r w(r,\eta,\xi)_{|r=t}\,,\quad (\partial_t w)(t,\eta,\xi)=\partial_t w(t,x).$$

As introduced previously, for $i=0,1,2$ we write $S_i'(t,\eta,\zeta)=\partial_\zeta S_i(t,\eta,\zeta)$ and $S_i''(t,\eta,\zeta)=\partial_\zeta^2 S_i(t,\eta,\zeta)$.

Inner expansion We first compute (5.30) for $(u,S) = \big(u_1^{(\mu)}, S_1^{(\mu)}\big)$. To this end we need

Lemma 5.5 *Let $s^{(\mu)}(t,\eta)$ be the normal speed of the phase interface $\Gamma(t)$ at $\eta \in \Gamma(t)$, let ∇_η be the operator defined in (3.8), and let w be a function defined in a neighborhood of Γ. Then we have*

$$\partial_t w(t,x) = w_{|t}(t,\eta,\xi) - \xi(\partial_t n)(t,\eta)\cdot\nabla_\eta w(t,\eta,\xi) - s^{(\mu)}(t,\eta)\partial_\xi w(t,\eta,\xi).$$

Proof: By definition, $\nabla_\eta w(t,\eta,\xi)$ is a tangential vector to $\Gamma(t)$. Lemma 2.2 thus yields

$$\begin{aligned}\partial_t w(t,x) = \partial_t w(t,\eta,\xi) &= w_{|t}(t,\eta,\xi) + \partial_t\eta\cdot\nabla_\eta w(t,\eta,\xi) + \partial_\xi w(t,\eta,\xi)\partial_t\xi\\ &= w_{|t}(t,\eta,\xi) - \xi(\partial_t n)(t,\eta)\cdot\nabla_\eta w(t,\eta,\xi) - s^{(\mu)}(t,\eta)\partial_\xi w(t,\eta,\xi).\end{aligned}$$

This proves the lemma. ■

We apply this lemma to the function $S_1^{(\mu)}$ defined in (3.50) to obtain

$$\partial_t S_1^{(\mu)}(t,x) = S_{1|t}^{(\mu)}(t,\eta,\xi) - \xi(\partial_t n)(t,\eta)\cdot\nabla_\eta S_1^{(\mu)}(t,\eta,\xi) - s^{(\mu)}(t,\eta)\partial_\xi S_1^{(\mu)}(t,\eta,\xi).$$

From this equation and from the asymptotic expansion (2.43) of $s^{(\mu)}$ we conclude for the first term in (5.30) that

$$\begin{aligned}&(\mu\lambda)^{1/2}\partial_t S_1^{(\mu)}(t,x)\\ &\quad= (\mu\lambda)^{1/2}\partial_t\Big(S_0\big(\frac{\xi}{(\mu\lambda)^{1/2}}\big) + \mu^{1/2}S_1\big(t,\eta,\frac{\xi}{(\mu\lambda)^{1/2}}\big) + \mu S_2\big(t,\eta,\frac{\xi}{(\mu\lambda)^{1/2}}\big)\Big)\\ &\quad= -(s_0+\mu^{1/2}s_1)(S_0' + \mu^{1/2}S_1' + \mu S_2') + \mu\lambda^{1/2}\tilde R_{S_t}\\ &\quad= -s_0S_0' - \mu^{1/2}(s_1S_0' + s_0S_1') + \mu R_{S_t},\end{aligned}\tag{5.31}$$

with

$$\begin{aligned}\tilde R_{S_t}(\mu,\lambda,t,\eta,\xi) &= (S_{1|t} + \mu^{1/2}S_{2|t}) - \xi(\partial_t n)\cdot\nabla_\eta(S_1+\mu^{1/2}S_2),\\ R_{S_t}(\mu,\lambda,t,\eta,\xi) &= \lambda^{1/2}\tilde R_{S_t} - s_1S_1' - (s_0+\mu^{1/2}s_1)S_2'.\end{aligned}\tag{5.32}$$

For the third term in (5.30) we get from Taylor's formula and from (3.50)

$$\frac{1}{\mu^{1/2}}\hat\psi'(S_1^{(\mu)}) = \frac{1}{\mu^{1/2}}\hat\psi'(S_0) + \hat\psi''(S_0)S_1 + \mu^{1/2}\big(\hat\psi''(S_0)S_2 + \frac12\hat\psi'''(S_0)S_1^2\big) + \mu R_{\hat\psi'}\tag{5.33}$$

where

$$R_{\hat\psi'} = \frac12\hat\psi'''(S_0)\big(2S_1S_2 + \mu^{1/2}S_2^2\big) + \frac16\hat\psi^{(IV)}\big(S_0 + \vartheta(\mu^{1/2}S_1 + \mu S_2)\big)(S_1+\mu^{1/2}S_2)^3.\tag{5.34}$$

with suitable $0<\vartheta<1$. Observe next that

$$\Delta_x S_1^{(\mu)}(t,x) = \partial_\xi^2 S_1^{(\mu)}(t,\eta,\xi) - \kappa(t,\eta,\xi)\partial_\xi S_1^{(\mu)}(t,\eta,\xi) + \Delta_{\Gamma_\xi}S_1^{(\mu)}(t,\eta,\xi),\tag{5.35}$$

where $\Delta_{\Gamma_\xi} = \operatorname{div}_{\Gamma_\xi} \nabla_{\Gamma_\xi}$ denotes the surface Laplacian and where $\kappa(t,\eta,\xi)$ is twice the mean curvature of the surface $\Gamma_\xi(t)$ at $\eta \in \Gamma_\xi(t)$. With the notation $\kappa'(t,\eta,0) = \partial_\xi \kappa(t,\eta,0)$ we obtain from Taylor's formula

$$\begin{aligned} \kappa(t,\eta,\xi) &= \kappa(t,\eta,0) + \partial_\xi \kappa(t,\eta,0)\xi + \frac{1}{2}\partial_\xi^2 \kappa(t,\eta,\xi^*)\xi^2 \\ &= \kappa_\Gamma(t,\eta) + (\mu\lambda)^{1/2}\kappa'(t,\eta,0)\zeta + \mu\lambda R_\kappa(t,\eta,\xi)\zeta^2, \end{aligned} \tag{5.36}$$

where $R_\kappa(t,\eta,\xi) = \frac{1}{2}\partial_\xi^2\kappa(t,\eta,\xi^*)$ is the remainder term, with suitable ξ^* between 0 and ξ. We insert (3.50) and (5.36) into (5.35) and obtain for the fourth term in (5.30) that

$$\begin{aligned} &\mu^{1/2}\lambda\Delta_x S_1^{(\mu)} = \mu^{-1/2}S_0'' + S_1'' + \mu^{1/2}S_2'' \\ &\quad - \lambda^{1/2}\Big(\kappa_\Gamma + (\mu\lambda)^{1/2}\kappa'\zeta + \mu\lambda R_\kappa\zeta^2\Big)(S_0' + \mu^{1/2}S_1') - \mu\lambda^{1/2}\kappa(\xi)S_2' + \mu^{1/2}\lambda\Delta_{\Gamma_\xi}S_1^{(\mu)} \\ &\qquad = \mu^{-1/2}S_0'' + \big(S_1'' - \lambda^{1/2}\kappa_\Gamma S_0'\big) + \mu^{1/2}\Big(S_2'' - \lambda^{1/2}\kappa_\Gamma S_1' - \lambda\kappa'\zeta S_0'\Big) + \mu R_\Delta, \end{aligned} \tag{5.37}$$

where

$$R_\Delta = -\lambda\kappa'\zeta S_1' - \lambda^{1/2}\kappa(\xi)S_2' + \lambda\Delta_{\Gamma_\xi}(S_1 + \mu^{1/2}S_2) + \lambda^{3/2}R_\kappa\zeta^2(S_0' + \mu^{1/2}S_1'). \tag{5.38}$$

From (5.31), (5.16), (5.33) and (5.37) we obtain

$$\begin{aligned} &(\mu\lambda)^{1/2}\partial_t S_1^{(\mu)} + c\Big(\partial_S \mathsf{W}\big(\varepsilon(\nabla_x u_1^{(\mu)}), S_1^{(\mu)}\big) + \frac{1}{\mu^{1/2}}\hat\psi'(S_1^{(\mu)}) - \mu^{1/2}\lambda\Delta_x S_1^{(\mu)}\Big) \\ &\quad = \frac{c}{\mu^{1/2}}\big(\hat\psi'(S_0) - S_0''\big) + c\Big(\hat\psi''(S_0)S_1 - S_1'' - \overline{\varepsilon} : [\hat T]S_0 - \hat\sigma(0) + (\lambda^{1/2}\kappa_\Gamma - \frac{s_0}{c})S_0'\Big) \\ &\quad + c\mu^{1/2}\Bigg(\hat\psi''(S_0)S_2 - S_2'' - \overline{\varepsilon} : [\hat T]S_1 - \check\sigma(0) + \big(\lambda^{1/2}\kappa_\Gamma - \frac{s_0}{c}\big)S_1' \\ &\qquad\qquad - \lambda^{1/2}\Big(\hat\sigma'(0)\zeta + \overline{\varepsilon} : D\varepsilon(a^* \otimes n + \nabla_\Gamma u^*)\int_{-\infty}^{\zeta} S_0(\vartheta)d\vartheta\Big) \\ &\qquad\qquad + \frac{1}{2}\hat\psi'''(S_0)S_1^2 + \big(\lambda\kappa'\zeta - \frac{s_1}{c}\big)S_0'\Bigg) + \mu R_{S_t + c(\ldots)}\,, \end{aligned} \tag{5.39}$$

where

$$R_{S_t+c(\ldots)} = R_{S_t} + c(-R_W + R_{\hat\psi'} - R_\Delta), \tag{5.40}$$

with R_{S_t}, R_W, $R_{\hat\psi'}$ and R_Δ given in (5.32), (5.17), (5.34) and (5.38), respectively.

Corollary 5.6 *Let s_0 be given by* (2.39) *and assume that the functions S_0, S_1 and S_2 satisfy the ordinary differential equations* (2.25), (2.26), (3.55), *with F_1, F_2 given by* (2.31), (3.56). *Assume moreover that the conditions* (2.27) – (2.30) *and* (3.60), (3.61) *hold. Then there is a constant K such that the interior expansion $\big(u_1^{(\mu)}, S_1^{(\mu)}, T_1^{(\mu)}\big)$ defined in* (3.49) – (3.51) *satisfies for all $(t,\eta,\xi) \in Q_{\text{inn}}^{(\mu\lambda)} \cup Q_{\text{match}}^{(\mu\lambda)}$ and all $0 < \mu \le \mu_0$, $0 < \lambda \le \lambda_0$ the inequality*

$$\begin{aligned} &\Big|\partial_t S_1^{(\mu)} + \frac{c}{(\mu\lambda)^{1/2}}\Big(\partial_S \mathsf{W}\big(\varepsilon(\nabla_x u_1^{(\mu)}), S_1^{(\mu)}\big) + \frac{1}{\mu^{1/2}}\hat\psi'(S_1^{(\mu)}) - \mu^{1/2}\lambda\Delta_x S_1^{(\mu)}\Big)\Big| \\ &\qquad = \Big(\frac{\mu}{\lambda}\Big)^{1/2}|R_{S_t} + c(-R_W + R_{\hat\psi'} - R_\Delta)| \le K\Big(\frac{\mu}{\lambda}\Big)^{1/2}|\ln\mu|^2. \end{aligned} \tag{5.41}$$

Proof: From (2.39) we obtain

$$\lambda^{1/2}\kappa_\Gamma - \frac{s_0}{c} = \frac{1}{c_1}\overline{\varepsilon} : \langle \hat{T}\rangle,$$

and by (5.9) we have $\overline{\varepsilon} : \hat{T}^{(-)} = \hat{\sigma}(0)$. After insertion of these two equations into (5.39), the latter equation takes the form

$$\begin{aligned}(\mu\lambda)^{1/2}\partial_t S_1^{(\mu)} &+ c\Big(\partial_S \mathsf{W}\big(\varepsilon(\nabla_x u_1^{(\mu)}), S_1^{(\mu)}\big) + \frac{1}{\mu^{1/2}}\hat{\psi}'(S_1^{(\mu)}) - \mu^{1/2}\lambda\Delta_x S_1^{(\mu)}\Big)\\ &= \frac{c}{\mu^{1/2}}\Big(\hat{\psi}'(S_0) - S_0''\Big) + c\Big(\hat{\psi}''(S_0)S_1 - S_1'' - F_1\Big) + c\mu^{1/2}\Big(\hat{\psi}''(S_0)S_2 - S_2'' - F_2\Big)\\ &\qquad + \mu R_{S_t+c(\dots)} = \mu\big(R_{S_t} + c(-R_W + R_{\hat{\psi}'} - R_\Delta)\big). \qquad (5.42)\end{aligned}$$

Here we also used (2.25), (2.26) and (3.55). Noting the inequalities (5.22) for ξ and ζ, the inequalities (4.3) – (4.5), (4.11), (4.13), (4.14) for S_0, S_1, S_2, and the inequality (5.26) for the term R_{T_1}, which appears in R_W, we see by inspection of every term in (5.32), (5.17), (5.34) and (5.38) that the inequality

$$\big|R_{S_t} + c(-R_W + R_{\hat{\psi}'} - R_\Delta)\big| \le K|\ln\mu|^2 \qquad (5.43)$$

holds. To obtain inequality (5.41) we divide (5.42) by $(\mu\lambda)^{1/2}$ and estimate the right hand side of the resulting equation using (5.43). ■

Outer expansion Next we compute (5.30) for $(u,S) = \big(u_2^{(\mu)}, S_2^{(\mu)}\big)$. Note first that (1.8) and (5.28) yield

$$\partial_S \mathsf{W}\big(\varepsilon(\nabla_x u_2^{(\mu)}), S_2^{(\mu)}\big) = -\overline{\varepsilon} : (\hat{T} + \mu^{1/2}\check{T} + \mu\tilde{T}) + \mu^{3/2}\overline{\varepsilon} : D\overline{\varepsilon}\tilde{S}_3\,. \qquad (5.44)$$

Also, Taylor's formula and (3.26) yield for a suitable $0 < \vartheta(t,x) < 1$

$$\begin{aligned}\hat{\psi}'(S_2^{(\mu)}) &= \hat{\psi}'(\hat{S}) + \hat{\psi}''(\hat{S})\big(\mu^{1/2}\tilde{S}_1 + \mu\tilde{S}_2 + \mu^{3/2}\tilde{S}_3\big)\\ &\quad + \frac{1}{2}\hat{\psi}'''(\hat{S})\big(\mu^{1/2}\tilde{S}_1 + \mu\tilde{S}_2 + \mu^{3/2}\tilde{S}_3\big)^2 + \frac{1}{6}\hat{\psi}^{(IV)}(\hat{S})\big(\mu^{1/2}\tilde{S}_1 + \mu\tilde{S}_2 + \mu^{3/2}\tilde{S}_3\big)^3\\ &\quad + \frac{1}{24}\hat{\psi}^{(V)}\big(\hat{S} + \vartheta(\mu^{1/2}\tilde{S}_1 + \mu\tilde{S}_2 + \mu^{3/2}\tilde{S}_3)\big)\big(\mu^{1/2}\tilde{S}_1 + \mu\tilde{S}_2 + \mu^{3/2}\tilde{S}_3\big)^4\\ &= \mu^{1/2}\hat{\psi}''(\hat{S})\tilde{S}_1 + \mu\Big(\hat{\psi}''(\hat{S})\tilde{S}_2 + \frac{1}{2}\hat{\psi}'''(\hat{S})\tilde{S}_1^2\Big)\\ &\quad + \mu^{3/2}\Big(\hat{\psi}''(\hat{S})\tilde{S}_3 + \hat{\psi}'''(\hat{S})\tilde{S}_1\tilde{S}_2 + \frac{1}{6}\hat{\psi}^{(IV)}(\hat{S})\tilde{S}_1^3\Big) + \mu^2\,\overline{R}_{\hat{\psi}'}\,. \qquad (5.45)\end{aligned}$$

Here we used that $\hat{\psi}'(\hat{S}) = 0$, by (2.37). Equations (5.44) and (5.45) imply that in the domain $Q\setminus\Gamma$

$$\begin{aligned}(\mu\lambda)^{1/2}\partial_t S_2^{(\mu)} &+ c\Big(\partial_S \mathsf{W}\big(\varepsilon(\nabla_x u_2^{(\mu)}), S_2^{(\mu)}\big) + \frac{1}{\mu^{1/2}}\hat{\psi}'(S_2^{(\mu)}) - \mu^{1/2}\lambda\Delta_x S_2^{(\mu)}\Big)\\ &= c\Big(-\overline{\varepsilon} : \hat{T} + \hat{\psi}''(\hat{S})\tilde{S}_1\Big)\\ &\quad + c\mu^{1/2}\Big(-\overline{\varepsilon} : \check{T} + \hat{\psi}''(\hat{S})\tilde{S}_2 + \frac{1}{2}\hat{\psi}'''(\hat{S})\tilde{S}_1^2\Big)\\ &\quad + c\mu\Big(-\overline{\varepsilon} : \tilde{T} + \hat{\psi}''(\hat{S})\tilde{S}_3 + \hat{\psi}'''(\hat{S})\tilde{S}_1\tilde{S}_2 + \frac{1}{6}\hat{\psi}^{(IV)}(\hat{S})\tilde{S}_1^3 - \lambda\Delta_x\tilde{S}_1 + \frac{\lambda^{1/2}}{c}\partial_t\tilde{S}_1\Big)\\ &\quad + \mu^{3/2}\,\overline{R}_{S_t+c(\dots)}\,, \qquad (5.46)\end{aligned}$$

where

$$\overline{R}_{S_t+c(\dots)} = c\left(\overline{\varepsilon} : D\overline{\varepsilon}\tilde{S}_3 + \overline{R}_{\hat{\psi}'} - \lambda\Delta_x(\tilde{S}_2 + \mu^{1/2}\tilde{S}_3)\right) + \lambda^{1/2}\partial_t(\tilde{S}_2 + \mu^{1/2}\tilde{S}_3). \tag{5.47}$$

Here we used that the function $\hat{S}$ has the constant values 0 in γ and 1 in γ'.

Corollary 5.7 *Assume that the functions $\tilde{S}_1$, $\tilde{S}_2$ and $\tilde{S}_3$ satisfy* (3.30) – (3.32)*. Then there is a constant K such that for all $(t,x) \in Q \setminus \Gamma$ and all $0 < \mu \le \mu_0$, $0 < \lambda \le \lambda_0$*

$$\left|\partial_t S_2^{(\mu)} + \frac{c}{(\mu\lambda)^{1/2}}\Big(\partial_S \mathsf{W}\big(\varepsilon(\nabla_x u_2^{(\mu)}), S_2^{(\mu)}\big) + \frac{1}{\mu^{1/2}}\hat{\psi}'(S_2^{(\mu)}) - \mu^{1/2}\lambda\Delta_x S_2^{(\mu)}\Big)\right| \le K\frac{\mu}{\lambda^{1/2}}\,. \tag{5.48}$$

Proof: By (3.30) – (3.32), the brackets on the right hand side of equation (5.46) vanish. Therefore, if we divide the latter equation by $(\mu\lambda)^{1/2}$, we obtain

$$\partial_t S_2^{(\mu)} + \frac{c}{(\mu\lambda)^{1/2}}\Big(\partial_S \mathsf{W}\big(\varepsilon(\nabla_x u_2^{(\mu)}), S_2^{(\mu)}\big) + \frac{1}{\mu^{1/2}}\hat{\psi}'(S_2^{(\mu)}) - \mu^{1/2}\lambda\Delta_x S_2^{(\mu)}\Big) = \frac{\mu}{\lambda^{1/2}}\overline{R}_{S_t+c(\dots)}.$$

$\tilde{S}_1$ and $\tilde{S}_2$ are given in (3.36), (3.37), and the function $\tilde{S}_3$ is obtained by solving (3.32) for this function. From these equations we see by our general regularity assumptions that $\|(\tilde{S}_1, \tilde{S}_2, \tilde{S}_3)\|_{L^\infty(Q\setminus\Gamma)} \le K_1$, with the constant K_1 independent of μ. Using this, we see by inspection of every term in (5.47) that $\|\overline{R}_{S_t+c(\dots)}\|_{L^\infty(Q\setminus\Gamma)} \le K$, with K independent of μ and λ. This inequality and the equation above imply (5.48). ∎

5.3 Auxiliary estimates needed in the matching region

The following auxiliary estimates are needed to prove (2.50) and (2.52) in the matching region $Q_{\text{match}}^{(\mu\lambda)}$.

Lemma 5.8 *The functions $u_2^{(\mu)}$, $T_2^{(\mu)}$, $S_2^{(\mu)}$ defined in* (3.25) – (3.27) *and $u_1^{(\mu)}$, $T_1^{(\mu)}$, $S_1^{(\mu)}$ defined in* (3.49) – (3.51) *satisfy*

$$\|S_1^{(\mu)} - S_2^{(\mu)}\|_{L^\infty(Q_{\text{match}}^{(\mu\lambda)})} \le K\mu^{3/2}|\ln\mu|^2, \tag{5.49}$$

$$\|D_x^\alpha(S_1^{(\mu)} - S_2^{(\mu)})\|_{L^\infty(Q_{\text{match}}^{(\mu\lambda)})} \le K\lambda^{-\frac{|\alpha|}{2}}\mu^{\frac{3-|\alpha|}{2}}, \quad 1 \le |\alpha| \le 2, \tag{5.50}$$

$$\|\partial_t(S_1^{(\mu)} - S_2^{(\mu)})\|_{L^\infty(Q_{\text{match}}^{(\mu\lambda)})} \le K\lambda^{-1/2}\mu, \tag{5.51}$$

$$\|u_1^{(\mu)} - u_2^{(\mu)}\|_{L^\infty(Q_{\text{match}}^{(\mu\lambda)})} \le K\lambda^{1/2}\mu^{3/2}|\ln\mu|, \tag{5.52}$$

$$\|\nabla_x(u_1^{(\mu)} - u_2^{(\mu)})\|_{L^\infty(Q_{\text{match}}^{(\mu\lambda)})} \le K\mu, \tag{5.53}$$

$$\|T_1^{(\mu)} - T_2^{(\mu)}\|_{L^\infty(Q_{\text{match}}^{(\mu\lambda)})} \le K\mu, \tag{5.54}$$

for all $\mu \in (0,\mu_0]$, $\lambda \in (0,\lambda_0]$. Here α denotes a multi-index and K denotes a positive constant, which does not necessarily have the same value in the six estimates.

Proof: Since the proofs of these estimates are long and technical, we present here only the proofs of (5.49) and (5.52). The proofs of the estimates (5.50), (5.51) and (5.53) run along the same lines. (5.54) is an immediate consequence of the definitions (3.27), (3.51) of $T_2^{(\mu)}$ and $T_1^{(\mu)}$, and of the estimates (5.53), (5.49).

In this proof we mostly drop the arguments t and η to simplify the notation. As usual we write $\zeta = \frac{\xi}{(\mu\lambda)^{1/2}}$. We need that for $(t,\eta,\xi) \in Q_{\text{match}}^{(\mu\lambda)}$ the inequalities

$$\frac{3}{2}\frac{|\ln\mu|}{a} \leq \Big|\frac{\xi}{(\mu\lambda)^{1/2}}\Big| = |\zeta| \leq 3\frac{|\ln\mu|}{a}, \tag{5.55}$$

hold, by definition of $Q_{\text{match}}^{(\mu\lambda)}$ in (2.33).

We begin with the proof of (5.49). By definition of $S_1^{(\mu)}$ and $S_2^{(\mu)}$ in (3.50) and (3.26) we have

$$\begin{aligned} |S_1^{(\mu)} - S_2^{(\mu)}| &= |S_0 + \mu^{1/2}S_1 + \mu S_2 - \hat{S} - \mu^{1/2}\tilde{S}_1 - \mu\tilde{S}_2 - \mu^{3/2}\tilde{S}_3| \\ &\leq |S_0 - \hat{S}| + \mu^{1/2}|S_1 + \mu^{1/2}S_2 - \tilde{S}_1 - \mu^{1/2}\tilde{S}_2| + \mu^{3/2}|\tilde{S}_3|. \end{aligned} \tag{5.56}$$

To estimate the first term on the right hand side note that since $\hat{S}(t,x) = \hat{S}(\xi) = 1^+(\xi)$, relations (4.3), (4.4), and (5.55) imply for $(t,\eta,\xi) \in Q_{\text{match}}^{(\mu\lambda)}$ that

$$\Big|S_0\Big(\frac{\xi}{(\mu\lambda)^{1/2}}\Big) - \hat{S}(\xi)\Big| \leq K_1 e^{-a|\zeta|} \leq K_1 e^{-\frac{3}{2}|\ln\mu|} = K_1\mu^{3/2}. \tag{5.57}$$

To estimate the second term on the right hand side of (5.56) we introduce the notations

$$\tilde{\rho}_1(\zeta) = \begin{cases} \frac{\overline{\varepsilon}:\hat{T}^{(-)}}{\hat{\psi}''(0)}, & \zeta < 0, \\ \frac{\overline{\varepsilon}:\hat{T}^{(+)}}{\hat{\psi}''(1)}, & \zeta > 0, \end{cases} \tag{5.58}$$

$$\tilde{\rho}_2(\zeta) = \begin{cases} \frac{1}{\hat{\psi}''(0)}\Big(\overline{\varepsilon} : \check{T}^{(-)} - \frac{\hat{\psi}'''(0)}{2}\Big(\frac{\overline{\varepsilon}:\hat{T}^{(-)}}{\hat{\psi}''(0)}\Big)^2 + \lambda^{\frac{1}{2}}\hat{\sigma}'(0)\zeta\Big), & \zeta < 0, \\ \frac{1}{\hat{\psi}''(1)}\Big(\overline{\varepsilon} : \check{T}^{(+)} - \frac{\hat{\psi}'''(1)}{2}\Big(\frac{\overline{\varepsilon}:\hat{T}^{(+)}}{\hat{\psi}''(1)}\Big)^2 + \lambda^{\frac{1}{2}}\hat{\sigma}'(0)\zeta \\ \qquad + \lambda^{\frac{1}{2}}\overline{\varepsilon} : D\varepsilon(a^* \otimes n + \nabla_\Gamma u^*)\zeta\Big), & \zeta > 0. \end{cases} \tag{5.59}$$

By definition of the functions ρ_1, ρ_2 in (4.10) and (3.59), and by definition of φ, $\varphi_\pm$ in (3.57) and (3.58), we have

$$\tilde{\rho}_i(\zeta) - \rho_i(\zeta) = (1 - \varphi(-\zeta) - \varphi(\zeta))\tilde{\rho}_i(\zeta) = 0, \qquad \text{for } i = 1,2 \text{ and } |\zeta| \geq 2.$$

We can therefore choose a suitable constant $\tilde{K}$ such that $\big|\tilde{\rho}_i(\zeta) - \rho_i(\zeta)\big| \leq \tilde{K}e^{-a|\zeta|}$ holds for $i = 1,2$ and all $\zeta \in \mathbb{R}$. From this inequality and from the estimates (4.12), (4.15) we conclude that

$$|S_1(\zeta) - \tilde{\rho}_1| \leq |S_1(\zeta) - \rho_1(\zeta)| + |\rho_1(\zeta) - \tilde{\rho}_1| \leq (K_2 + \tilde{K})e^{-a|\zeta|}, \tag{5.60}$$

$$|S_2(\zeta) - \tilde{\rho}_2| \leq |S_2(\zeta) - \rho_2(\zeta)| + |\rho_2(\zeta) - \tilde{\rho}_2| \leq \big(K_5(1 + |\zeta|) + \tilde{K}\big)e^{-a|\zeta|}, \tag{5.61}$$

for $\zeta \in \mathbb{R}$.

Now we proceed to estimate the second term on the right hand side of (5.56). We insert the functions $\tilde{\rho}_1$ and $\tilde{\rho}_2$ into this term, use the expressions for $\tilde{S}_1$ and $\tilde{S}_2$ given in

(3.36), (3.37), and employ the triangle inequality to obtain

$$
\begin{aligned}
&|S_1+\mu^{1/2}S_2-\tilde S_1-\mu^{1/2}\tilde S_2| \\
&\qquad \le \left|\tilde\rho_1+\mu^{1/2}\tilde\rho_2-\frac{\overline{\varepsilon}:\hat T}{\hat\psi''(\hat S)}-\mu^{1/2}\left(\frac{\overline{\varepsilon}:\check T}{\hat\psi''(\hat S)}-\frac{\hat\psi'''(\hat S)}{2\hat\psi''(\hat S)}\Big(\frac{\overline{\varepsilon}:\hat T}{\hat\psi''(\hat S)}\Big)^2\right)\right| \\
&\qquad\qquad +|S_1-\tilde\rho_1|+\mu^{1/2}|S_2-\tilde\rho_2| = |I_1|+|I_2|+|I_3|. \qquad (5.62)
\end{aligned}
$$

By (5.60), (5.61) and (5.55) we have for $(t,\eta,\xi)\in Q^{(\mu\lambda)}_{\text{match}}$

$$
\begin{aligned}
|I_2|+|I_3| &= \big|S_1(t,\eta,\zeta)-\tilde\rho_1(t,\eta,\zeta)\big|+\mu^{1/2}\big|S_2(t,\eta,\zeta)-\tilde\rho_2(t,\eta,\zeta)\big| \\
&\le (K_2+\tilde K)e^{-a|\zeta|}+\mu^{1/2}\big(K_5(1+|\zeta|)+\tilde K\big)e^{-a|\zeta|} \\
&\le \left(C_1+\mu^{1/2}C_2\Big(1+3\frac{|\ln\mu|}{a}\Big)\right)e^{-\frac{3}{2}|\ln\mu|}\le C_3\,\mu^{3/2}. \qquad (5.63)
\end{aligned}
$$

To find an estimate for $|I_1|$ note that the definitions of $\tilde\rho_1$, $\tilde\rho_2$ in (5.58), (5.59) yield for $(t,\eta,\xi)\in Q^{(\mu\lambda)}_{\text{match}}$ with $\xi>0$

$$
\begin{aligned}
I_1 &= \tilde\rho_1\Big(\frac{\xi}{(\mu\lambda)^{1/2}}\Big)+\mu^{\frac12}\tilde\rho_2\Big(\frac{\xi}{(\mu\lambda)^{1/2}}\Big)-\frac{\overline{\varepsilon}:\hat T(\xi)}{\hat\psi''(1)}-\mu^{\frac12}\left(\frac{\overline{\varepsilon}:\check T(\xi)}{\hat\psi''(1)}-\frac{\hat\psi'''(1)}{2\hat\psi''(1)}\Big(\frac{\overline{\varepsilon}:\hat T(\xi)}{\hat\psi''(1)}\Big)^2\right) \\
&= \frac{1}{\hat\psi''(1)}\Bigg(\overline{\varepsilon}:\hat T^{(+)}-\overline{\varepsilon}:\hat T(\xi)+\sigma'(0)\xi+\overline{\varepsilon}:D\varepsilon(a^*\otimes n+\nabla_\Gamma u^*)\xi \qquad (5.64) \\
&\qquad +\mu^{\frac12}\big(\overline{\varepsilon}:\check T^{(+)}-\overline{\varepsilon}:\check T(\xi)\big)-\mu^{\frac12}\frac{\hat\psi'''(1)}{2}\left(\Big(\frac{\overline{\varepsilon}:\hat T^{(+)}}{\hat\psi''(1)}\Big)^2-\Big(\frac{\overline{\varepsilon}:\hat T(\xi)}{\hat\psi''(1)}\Big)^2\right)\Bigg).
\end{aligned}
$$

(5.2) and (5.55) together yield

$$
\begin{aligned}
&|\overline{\varepsilon}:\hat T^{(+)}+\sigma'(0)\xi+\overline{\varepsilon}:D\varepsilon(a^*\otimes n+\nabla_\Gamma u^*)\xi-\overline{\varepsilon}:\hat T(\xi)| \\
&\qquad\qquad = |R_{\overline{\varepsilon}:\hat T}(\xi)\xi^2|\le C_4\,\mu\lambda|\ln\mu|^2. \qquad (5.65)
\end{aligned}
$$

Since $\check T^{(+)}=\check T(0+)$, $\hat T^{(+)}=\hat T(0+)$, the mean value theorem and (5.55) imply

$$
\begin{aligned}
&\mu^{1/2}\left|\overline{\varepsilon}:\big(\check T^{(+)}-\check T(\xi)\big)-\frac{\hat\psi'''(1)}{2\big(\hat\psi''(1)\big)^2}\Big((\overline{\varepsilon}:\hat T^{(+)})^2-\big(\overline{\varepsilon}:\hat T(\xi)\big)^2\Big)\right| \\
&\qquad\qquad = \mu^{1/2}|R(\xi)\xi|\le C_5\,\mu\lambda^{1/2}|\ln\mu|, \qquad (5.66)
\end{aligned}
$$

where the remainder term R belongs to $L^\infty(\mathcal U_\delta)$. Combination of (5.64) – (5.66) yields for $(t,\eta,\xi)\in Q^{(\mu\lambda)}_{\text{match}}$ with $\xi>0$ that

$$
|I_1|\le \mu\big(C_4\lambda|\ln\mu|^2+C_5\lambda^{1/2}|\ln\mu|\big)\le C_6\,\lambda^{1/2}|\ln\mu|^2\mu. \qquad (5.67)
$$

From the definitions of $\tilde\rho_1$, $\tilde\rho_2$ in (5.58) and (5.59) we see that for $(t,\eta,\xi)\in Q^{(\mu\lambda)}_{\text{match}}$ with $\xi<0$ the term I_1 takes the form

$$
\begin{aligned}
I_1 &= \frac{1}{\hat\psi''(0)}\Bigg(\overline{\varepsilon}:\hat T^{(-)}-\overline{\varepsilon}:\hat T(\xi)+\sigma'(0)\xi \\
&\qquad +\mu^{1/2}\big(\overline{\varepsilon}:\check T^{(-)}-\overline{\varepsilon}:\check T(\xi)\big)-\mu^{1/2}\frac{\hat\psi'''(0)}{2}\left(\Big(\frac{\overline{\varepsilon}:\hat T^{(-)}}{\hat\psi''(0)}\Big)^2-\Big(\frac{\overline{\varepsilon}:\check T(\xi)}{\hat\psi''(0)}\Big)^2\right)\Bigg).
\end{aligned}
$$

Using (5.3) instead of (5.2), we see as above that the estimate (5.67) also holds in this case, whence the estimate (5.67) is valid for all $(t, \eta, \xi) \in Q^{(\mu\lambda)}_{\text{match}}$.

To finish the proof of (5.49), we combine (5.56) with (5.57), (5.62), (5.63) and (5.67) to obtain the estimate

$$|S^{(\mu)}_1 - S^{(\mu)}_2| \le K_1\mu^{3/2} + \mu^{1/2}\big(C_6\,\lambda^{1/2}|\ln\mu|^2\mu + C_3\,\mu^{3/2}\big) + \mu^{3/2}|\tilde S_3|,$$

which implies (5.49).

Next we prove (5.52). From (3.25) and (3.38), (3.39) we conclude for $(t, x) \in \mathcal{U}_\delta$ that

$$\begin{aligned} u^{(\mu)}_2(t, x) &= \hat u(t, x) + \mu^{1/2}\check u(t, x) + \mu\tilde u(t, x) \\ &= u^*\xi^+ + a^*\frac{1}{2}(\xi^+)^2 + \mu^{1/2}u^*\Big(\frac{\overline{\varepsilon} : \hat T^{(+)}}{\hat\psi''(1)}\xi^+ + \frac{\overline{\varepsilon} : \hat T^{(-)}}{\hat\psi''(0)}\xi^-\Big) \\ &\qquad + \hat v(t, x) + \mu^{1/2}\check v(t, x) + \mu\tilde u(t, x). \end{aligned}$$

Combination of this equation with (3.49) and insertion of (3.52) – (3.54) yields with $\zeta = \frac{\xi}{(\mu\lambda)^{1/2}}$ that

$$\begin{aligned} &u^{(\mu)}_1(t, x) - u^{(\mu)}_2(t, x) \\ &\quad= (\mu\lambda)^{1/2}u_0\Big(\frac{\xi}{(\mu\lambda)^{1/2}}\Big) + \mu\lambda^{1/2}u_1\Big(\frac{\xi}{(\mu\lambda)^{1/2}}\Big) + \mu\lambda u_2\Big(\frac{\xi}{(\mu\lambda)^{1/2}}\Big) \\ &\qquad - \bigg(u^*\xi^+ + a^*\frac{1}{2}(\xi^+)^2 + \mu^{1/2}u^*\Big(\frac{\overline{\varepsilon} : \hat T^{(+)}}{\hat\psi''(1)}\xi^+ + \frac{\overline{\varepsilon} : \hat T^{(-)}}{\hat\psi''(0)}\xi^-\Big) + \mu\tilde u(\xi)\bigg) \\ &\quad= (\mu\lambda)^{1/2}u^*\Big(\int_{-\infty}^{\zeta} S_0(\vartheta)\,\mathrm{d}\vartheta - \zeta^+\Big) + \mu\lambda\, a^*\int_{-\infty}^{\zeta}\Big(\int_{-\infty}^{\vartheta} S_0(\vartheta_1)\,d\vartheta_1 - \vartheta^+\Big)\,d\vartheta \\ &\qquad + \mu\lambda^{1/2}u^*\int_0^{\zeta}\Big(S_1(\vartheta) - \frac{\overline{\varepsilon} : \hat T^{(+)}}{\hat\psi''(1)}1^+(\vartheta) - \frac{\overline{\varepsilon} : \hat T^{(-)}}{\hat\psi''(0)}1^-(\vartheta)\Big)d\vartheta - \mu\tilde u(\xi) \\ &\quad= (\mu\lambda)^{1/2}J_1(\zeta) + \mu\big(J_2(\zeta) + J_3(\zeta) - \tilde u(\xi)\big). \end{aligned} \tag{5.68}$$

To estimate the right hand side we use the boundary condition (3.35) for $\tilde u$, note that by (5.58) the equation $\tilde\rho_1(\zeta) = \frac{\overline{\varepsilon}:\hat T^{(+)}}{\hat\psi''(1)}$ holds for $\zeta > 0$, and employ the inequalities (5.60), (4.8) to compute for $(t, \eta, \xi) \in Q^{(\mu\lambda)}_{\text{match}}$ with $\xi > 0$

$$\begin{aligned} |J_2(\zeta) + J_3(\zeta) - \tilde u^{(+)}| &= \Big|J_2(\zeta) - \lambda a^*\int_{-\infty}^{\infty}\Big(\int_{-\infty}^{\vartheta} S_0(\vartheta_1)\,d\vartheta_1 - \vartheta^+\Big)\,d\vartheta \\ &\qquad + J_3(\zeta) - \lambda^{1/2}u^*\int_0^{\infty}\Big(S_1(\vartheta) - \frac{\overline{\varepsilon} : \hat T^{(+)}}{\hat\psi''(1)}\Big)\,d\vartheta\Big| \\ &= \Big|\lambda a^*\int_{\zeta}^{\infty}\Big(\int_{-\infty}^{\vartheta} S_0(\vartheta_1)\vartheta_1 - \vartheta^+\Big)\,d\vartheta + \lambda^{1/2}u^*\int_{\zeta}^{\infty}\big(S_1(\vartheta) - \tilde\rho_1(\zeta)\big)\,d\vartheta\Big| \\ &\le \lambda|a^*|\int_{\zeta}^{\infty}\frac{K_1}{a}e^{-a\vartheta}\,d\vartheta + \lambda^{1/2}|u^*|\int_{\zeta}^{\infty}(K_2 + \tilde K)e^{-a\vartheta}\,d\vartheta \\ &= \Big(\lambda|a^*|\frac{K_1}{a^2} + \lambda^{1/2}|u^*|\frac{K_2 + \tilde K}{a}\Big)e^{-a\zeta} \le C_1\lambda^{1/2}\big(\|a^*\|_{L^\infty(\Gamma)} + \|u^*\|_{L^\infty(\Gamma)}\big)e^{-a\zeta}. \end{aligned} \tag{5.69}$$

The mean value theorem implies for $\xi > 0$

$$\tilde{u}(t,x) = \tilde{u}(t,\eta,\xi) = \tilde{u}(t,\eta,0+) + R_{\tilde{u}}(t,\eta,\xi)\xi.$$

Since $\tilde{u}(t,\eta,0+) = \tilde{u}^{(+)}(t,\eta)$, we infer from this equation and from (5.69), (5.55) for all $(t,\eta,\xi) \in Q^{(\mu\lambda)}_{\text{match}}$ with $\xi > 0$ that

$$\begin{aligned}
|J_2(\zeta) + J_3(\zeta) - \tilde{u}(\xi)| &\le |J_2(\zeta) + J_3(\zeta) - \tilde{u}^{(+)}| + |\tilde{u}(\xi) - \tilde{u}^{(+)}| \\
&\le C_1\lambda^{1/2}\big(\|a^*\|_{L^\infty(\Gamma)} + \|u^*\|_{L^\infty(\Gamma)}\big)e^{-a\zeta} + \|R_{\tilde{u}}\|_{L^\infty(\mathcal{U}_\delta)}\xi \\
&\le C_1\lambda^{1/2}\big(\|a^*\|_{L^\infty(\Gamma)} + \|u^*\|_{L^\infty(\Gamma)}\big)e^{-\frac{3}{2}|\ln\mu|} + \|R_{\tilde{u}}\|_{L^\infty(\mathcal{U}_\delta)}\frac{3}{a}(\mu\lambda)^{1/2}|\ln\mu| \\
&\le C_2\lambda^{1/2}\mu^{1/2}|\ln\mu|. \qquad (5.70)
\end{aligned}$$

Using the boundary condition (3.34) for $\tilde{u}$ instead of (3.35), we see by the analogous computation that (5.70) also holds for $(t,\eta,\xi) \in Q^{(\mu\lambda)}_{\text{match}}$ with $\xi < 0$.

Now use (5.70) to estimate the second term on the right hand side of (5.68). The first term is estimated by (4.8). Because (5.55) yields $e^{-a|\zeta|} \le e^{-\frac{3}{2}|\ln\mu|} = \mu^{3/2}$, we obtain for $(t,\eta,\xi) \in Q^{(\mu\lambda)}_{\text{match}}$

$$|u_1^{(\mu)}(t,x) - u_2^{(\mu)}(t,x)| \le (\mu\lambda)^{1/2}|u^*|\,\Big|\int_{-\infty}^{\zeta} S_0(\vartheta)\,d\vartheta - \zeta^+\Big| + \mu\big|J_2(\zeta) + J_3(\zeta) - \tilde{u}(\xi)\big|$$

$$\le (\mu\lambda)^{1/2}\big(\max_\Gamma |u^*|\big)\,\frac{K_1}{a}e^{-a|\zeta|} + C_2\lambda^{1/2}\mu^{3/2}|\ln\mu| \le C_1\lambda^{1/2}\mu^2 + C_2\lambda^{1/2}\mu^{3/2}|\ln\mu|,$$

which implies (5.52). ■

5.4 End of the proof of Theorem 2.3

To complete the proof of Theorem 2.3 note first that (3.22), (3.23) imply

$$u^{(\mu)}|_{\partial\Omega} = u_2^{(\mu)}|_{\partial\Omega}, \qquad \partial_{n_{\partial\Omega}}S^{(\mu)}|_{\partial\Omega)} = \partial_{n_{\partial\Omega}}S_2^{(\mu)}|_{\partial\Omega}, \qquad S^{(\mu)}|_{Q^{(\mu\lambda)}_{\text{inn}}} = S_1^{(\mu)}|_{Q^{(\mu\lambda)}_{\text{inn}}}.$$

Therefore (2.55) follows from the definition of $S_1^{(\mu)}$ in (3.50), equation (2.44) follows from (2.27), (2.30), (3.60), and (2.48) is a consequence of the definition of $u_2^{(\mu)}|_{\partial\Omega}$ in (3.25) and of (2.19), (2.24), (3.33). Moreover, the estimate (2.54) for the right hand side $f_3^{(\mu\lambda)}$ of (2.49) follows from the definition of $S_2^{(\mu)}|_{\partial\Omega}$ in (3.26) and from $\partial_{n_{\partial\Omega}}\hat{S}|_{\partial\Omega} = 0$. This last equation holds, since by assumption $\Gamma(t) \subseteq \Omega$, which implies that $\hat{S}(t)$ is identically equal to 0 or 1 in a neighborhood of $\partial\Omega$.

(2.46) follows from the definition of $T^{(\mu)}$ in (3.24); equation (2.56) is an immediate consequence of (4.14).

It remains to verify the estimates (2.50) – (2.53) for the right hand sides $f_1^{(\mu\lambda)}$, $f_2^{(\mu\lambda)}$ of the equations (2.45) and (2.47). To this end we put together all the estimates derived in Sections 5.1 – 5.3. We start with the proof of (2.50) and (2.51).

Equation (3.22) yields

$$\nabla_x u^{(\mu)} = \nabla_x u_1^{(\mu)}\phi_{\mu\lambda} + \nabla_x u_2^{(\mu)}(1-\phi_{\mu\lambda}) + \big(u_1^{(\mu)} - u_2^{(\mu)}\big)\otimes\nabla_x\phi_{\mu\lambda}.$$

We insert this equation into (3.24) and use (3.23) and (3.27), (3.51) to obtain

$$T^{(\mu)} = T_1^{(\mu)}\phi_{\mu\lambda} + T_2^{(\mu)}(1-\phi_{\mu\lambda}) + D\varepsilon\big((u_1^{(\mu)} - u_2^{(\mu)}) \otimes \nabla_x\phi_{\mu\lambda}\big). \tag{5.71}$$

The function $\phi_{\mu\lambda}$ defined in (3.21) is independent of (t,η). The decomposition (3.5) of the gradient thus yields

$$\nabla_x\phi_{\mu\lambda} = \frac{2a}{3(\lambda\mu)^{1/2}|\ln\mu|}\phi'_{\mu\lambda}n, \tag{5.72}$$

with the unit normal vector $n = n(t,\eta)$ to $\Gamma_\xi(t)$ at $\eta \in \Gamma_\xi(t)$. We write

$$\phi'_{\mu\lambda} = \phi'\Big(\frac{2a\xi}{3(\mu\lambda)^{1/2}|\ln\mu|}\Big), \qquad \phi''_{\mu\lambda} = \phi''\Big(\frac{2a\xi}{3(\mu\lambda)^{1/2}|\ln\mu|}\Big),$$

by a slight abuse of notation. With (5.71) and (5.72) we compute

$$\begin{aligned}
\operatorname{div}_x T^{(\mu)} + \mathsf{b} &= (\operatorname{div}_x T_1^{(\mu)} + \mathsf{b})\phi_{\mu\lambda} + (\operatorname{div}_x T_2^{(\mu)} + \mathsf{b})(1-\phi_{\mu\lambda})\\
&\quad + \Big((T_1^{(\mu)} - T_2^{(\mu)})n + \operatorname{div}_x D\varepsilon\big((u_1^{(\mu)} - u_2^{(\mu)}) \otimes n\big)\Big)\frac{2a}{3(\lambda\mu)^{1/2}|\ln\mu|}\phi'_{\mu\lambda}\\
&\quad + \Big(D\varepsilon\big((u_1^{(\mu)} - u_2^{(\mu)}) \otimes n\big)\Big)n\Big(\frac{2a}{3(\lambda\mu)^{1/2}|\ln\mu|}\Big)^2\phi''_{\mu\lambda}.
\end{aligned} \tag{5.73}$$

Inequality (2.51) is an immediate consequence of this equation and of (5.29), since $\phi_{\mu\lambda} = 0$ in $Q_{\text{out}}^{(\mu\lambda)}$, and (2.50) is obtained by estimating the right hand side of (5.73) using the obvious inequality

$$|\operatorname{div}_x D\varepsilon\big((u_1^{(\mu)} - u_2^{(\mu)}) \otimes n\big)| \le C\big(|\nabla_x(u_1^{(\mu)} - u_2^{(\mu)})| + |u_1^{(\mu)} - u_2^{(\mu)}|\big). \tag{5.74}$$

and the equation and inequalities (5.21), (5.29), (5.52) – (5.54).

We next proof (2.52) and (2.53). The inequality (2.53) follows immediately from (5.48), since $\phi_{\mu\lambda} = 0$ on $Q_{\text{out}}^{(\mu\lambda)}$, which by (3.22) and (3.23) implies $(u^{(\mu)}, S^{(\mu)}) = (u_2^{(\mu)}, S_2^{(\mu)})$ on $Q_{\text{out}}^{(\mu\lambda)} \subseteq Q \setminus \Gamma$.

It remains to verify (2.52). Since $\mathsf{W}_S(\varepsilon, S) = -\overline{\varepsilon} : D(\varepsilon - \overline{\varepsilon}S)$, by (1.8), it follows from (3.24), (3.51), and (5.71) that

$$\begin{aligned}
\mathsf{W}_S\big(\varepsilon(\nabla_x u^{(\mu)}), S^{(\mu)}\big) - \mathsf{W}_S\big(\varepsilon(\nabla_x u_1^{(\mu)}), S_1^{(\mu)}\big) &= -\overline{\varepsilon} : (T^{(\mu)} - T_1^{(\mu)})\\
&= -\overline{\varepsilon} : (T_2^{(\mu)} - T_1^{(\mu)})(1-\phi_{\mu\lambda}) - \overline{\varepsilon} : D\varepsilon\big((u_1^{(\mu)} - u_2^{(\mu)}) \otimes \nabla_x\phi_{\mu\lambda}\big).
\end{aligned}$$

The mean value theorem and (3.23) imply

$$\hat\psi'(S^{(\mu)}) - \hat\psi'(S_1^{(\mu)}) = \hat\psi''\Big(S_1^{(\mu)} + \vartheta(S_2^{(\mu)} - S_1^{(\mu)})(1-\phi_{\mu\lambda})\Big)(S_2^{(\mu)} - S_1^{(\mu)})(1-\phi_{\mu\lambda}),$$

for a suitable $0 < \vartheta(t,x) < 1$, and

$$\begin{aligned}
\Delta_x S^{(\mu)} - \Delta_x S_1^{(\mu)} &= \Delta_x(S_2^{(\mu)} - S_1^{(\mu)})(1-\phi_{\mu\lambda}) + 2\nabla_x(S_1^{(\mu)} - S_2^{(\mu)}) \cdot \nabla_x\phi_{\mu\lambda}\\
&\quad + (S_1^{(\mu)} - S_2^{(\mu)})\Delta_x\phi_{\mu\lambda}.
\end{aligned} \tag{5.75}$$

The right hand sides of the last three equations vanish on the set $Q_{\mathrm{inn}}^{(\mu\lambda)}$, since $\phi_{\mu\lambda} = 1$ on this set. On the set $Q_{\mathrm{match}}^{(\mu\lambda)}$ the right hand sides of these equations can be estimated using (5.49), (5.50), (5.52), (5.54), (5.72). In the estimation of (5.75) we also note that since $\phi_{\mu\lambda}$ is independent of (t, η), analogous to (5.35) the equation

$$\Delta_x \phi_{\mu\lambda} = -\kappa \partial_\xi \phi_{\mu\lambda} + \partial_\xi^2 \phi_{\mu\lambda} = -\kappa \frac{2a}{3(\lambda\mu)^{1/2} |\ln \mu|} \phi'_{\mu\lambda} + \Big(\frac{2a}{3(\lambda\mu)^{1/2} |\ln \mu|} \Big)^2 \phi''_{\mu\lambda}$$

holds, with twice the mean curvature $\kappa(t, \eta, \xi)$ of the surface $\Gamma_\xi(t)$ at $\eta \in \Gamma_\xi(t)$. Together we obtain that on $Q_{\mathrm{inn}}^{(\mu\lambda)} \cup Q_{\mathrm{match}}^{(\mu\lambda)}$ the inequality

$$\begin{aligned} \Big| \Big(\mathsf{W}_S\big(\varepsilon(\nabla_x u^{(\mu)}), S^{(\mu)}\big) + \frac{1}{\mu^{1/2}} \hat{\psi}'(S^{(\mu)}) - \mu^{1/2} \lambda \Delta_x S^{(\mu)} \Big) \\ - \Big(\mathsf{W}_S\big(\varepsilon(\nabla_x u_1^{(\mu)}), S_1^{(\mu)}\big) + \frac{1}{\mu^{1/2}} \hat{\psi}'(S_1^{(\mu)}) - \mu^{1/2} \lambda \Delta_x S_1^{(\mu)} \Big) \Big| \\ \le K\big(\mu + \mu |\ln \mu|^2 + \mu^{1/2} \lambda \mu^{1/2} \lambda^{-1} \big) \le K \mu |\ln \mu|^2 \end{aligned} \tag{5.76}$$

holds. Similarly, (3.23) implies

$$\partial_t S^{(\mu)} - \partial_t S_1^{(\mu)} = \partial_t (S_2^{(\mu)} - S_1^{(\mu)})(1 - \phi_{\mu\lambda}) + (S_1^{(\mu)} - S_2^{(\mu)}) \partial_t \phi_{\mu\lambda}.$$

The right hand side of this equation vanishes on $Q_{\mathrm{inn}}^{(\mu\lambda)}$. To estimate the right hand side on the set $Q_{\mathrm{match}}^{(\mu\lambda)}$ we use the inequalities (5.49), (5.51) and the equation

$$\partial_t \phi_{\mu\lambda} = -\frac{2as^{(\mu)}}{3(\mu\lambda)^{1/2} |\ln \mu|} \phi'_{\mu\lambda},$$

which follows from (3.21) and Lemma 5.5. The result is

$$|\partial_t S^{(\mu)} - \partial_t S_1^{(\mu)}| \le K \lambda^{-1/2} \mu |\ln \mu|, \qquad \text{on } Q_{\mathrm{inn}}^{(\mu\lambda)} \cup Q_{\mathrm{match}}^{(\mu\lambda)}. \tag{5.77}$$

By combination of (5.41), (5.76) and (5.77) we see that the inequality

$$\begin{aligned} \Big| \partial_t S^{(\mu)} + \frac{c}{(\mu\lambda)^{1/2}} \Big(\mathsf{W}_S\big(\varepsilon(\nabla_x u^{(\mu)}), S^{(\mu)}\big) + \frac{1}{\mu^{1/2}} \hat{\psi}'(S^{(\mu)}) - \mu^{1/2} \lambda \Delta_x S^{(\mu)} \Big) \Big| \\ \le \Big| \partial_t S_1^{(\mu)} + \frac{c}{(\mu\lambda)^{1/2}} \Big(\mathsf{W}_S\big(\varepsilon(\nabla_x u_1^{(\mu)}), S_1^{(\mu)}\big) + \frac{1}{\mu^{1/2}} \hat{\psi}'(S_1^{(\mu)}) - \mu^{1/2} \lambda \Delta_x S_1^{(\mu)} \Big) \Big| \\ + K \lambda^{-1/2} \mu |\ln \mu| + \frac{c}{(\mu\lambda)^{1/2}} K \mu |\ln \mu|^2 \\ \le K \Big(\frac{\mu}{\lambda} \Big)^{1/2} |\ln \mu|^2 + K \frac{\mu}{\lambda^{1/2}} |\ln \mu| + cK \Big(\frac{\mu}{\lambda} \Big)^{1/2} |\ln \mu|^2 \le K_1 \Big(\frac{\mu}{\lambda} \Big)^{1/2} |\ln \mu|^2 \end{aligned}$$

holds on the set $Q_{\mathrm{inn}}^{(\mu\lambda)} \cup Q_{\mathrm{match}}^{(\mu\lambda)}$. This proves (2.52) and completes the proof of Theorem 2.3. ■

6 Proof of Theorem 2.8

This section contains the proof of Theorem 2.8. We follow the convention introduced in Section 2.4 and often drop the indices μ and λ in the notations. (u, T, S) denotes the asymptotic solution constructed in Theorem 2.3 and $(u_{\text{AC}}, T_{\text{AC}}, S_{\text{AC}})$ denotes the exact solution of (1.1) – (1.3), (2.66) – (2.68). For the proof we need a lemma and a theorem, which we state first.

Lemma 6.1 *For all $x \in \Gamma(\hat{t})$ the propagation speeds s_{AC} and s satisfy*

$$s_{\text{AC}}(\hat{t}, x) - s(\hat{t}, x) = \frac{1}{|\nabla_x S(\hat{t}, x)|}\Big(f_2^{(\mu\lambda)}(\hat{t}, x) - \frac{c}{(\mu\lambda)^{1/2}}\,\overline{\varepsilon} : \big(T_{\text{AC}}(\hat{t}, x) - T(\hat{t}, x)\big)\Big), \quad (6.1)$$

where $f_2^{(\mu\lambda)}$ is the right hand side of equation (2.47).

Proof: Since the manifold Γ is a level set of S and since by (2.69) the manifold Γ_{AC} is a level set of S_{AC}, it follows that $(\partial_t S(\hat{t}, x), \nabla_x S(\hat{t}, x))$ and $(\partial_t S_{\text{AC}}(\hat{t}, x), \nabla_x S_{\text{AC}}(\hat{t}, x))$ are normal vectors to the respective manifolds at $(\hat{t}, x)$. Moreover, (2.68) implies that $\nabla_x S_{\text{AC}}(\hat{t}, x) = \nabla_x S(\hat{t}, x)$. From (2.8) we thus infer that

$$s(\hat{t}, x) = \frac{-\partial_t S(\hat{t}, x)}{\nabla_x S(\hat{t}, x) \cdot n(\hat{t}, x)} = \frac{-\partial_t S(\hat{t}, x)}{|\nabla_x S(\hat{t}, x)|}, \quad (6.2)$$

$$s_{\text{AC}}(\hat{t}, x) = \frac{-\partial_t S_{\text{AC}}(\hat{t}, x)}{|\nabla_x S_{\text{AC}}(\hat{t}, x)|} = \frac{-\partial_t S_{\text{AC}}(\hat{t}, x)}{|\nabla_x S(\hat{t}, x)|}. \quad (6.3)$$

For brevity we do not write the argument $(\hat{t}, x)$ in the following computation. In (6.2) we eliminate $\partial_t S$ with the help of (2.47), and in (6.3) we replace $\partial_t S_{\text{AC}}$ by the right hand side of (1.3). Together with another application of (2.68) this results in

$$\begin{aligned} s_{\text{AC}} - s = \frac{c}{(\mu\lambda)^{1/2}|\nabla_x S|}\Big(\big(\partial_S \mathsf{W}\big(\varepsilon(\nabla_x u_{\text{AC}}), S_{\text{AC}}\big) + \frac{1}{\mu^{1/2}}\hat{\psi}'(S) - \mu^{1/2}\lambda\Delta_x S\big) \\ - \big(\partial_S \mathsf{W}\big(\varepsilon(\nabla_x u), S\big) + \frac{1}{\mu^{1/2}}\hat{\psi}'(S) - \mu^{1/2}\lambda\Delta_x S\big)\Big) + \frac{1}{|\nabla_x S|} f_2^{(\mu\lambda)} \\ = \frac{1}{|\nabla_x S|} f_2^{(\mu\lambda)} - \frac{c}{(\mu\lambda)^{1/2}|\nabla_x S|}\Big(\overline{\varepsilon} : T_{\text{AC}} - \overline{\varepsilon} : T\Big). \end{aligned}$$

which is (6.1). In the last step we used that by (1.8) and (2.46) we have

$$\partial_S \mathsf{W}\big(\varepsilon(\nabla_x u_{\text{AC}}), S_{\text{AC}}\big) = -\overline{\varepsilon} : T_{\text{AC}}, \quad \text{and} \quad \partial_S \mathsf{W}\big(\varepsilon(\nabla_x u), S\big) = -\overline{\varepsilon} : T.$$

The proof of Lemma 6.1 is complete. ■

Theorem 6.2 *Suppose that the order of differentiability of $\hat{\psi}$, Γ, $\hat{u}$, $\check{u}$, b, is higher by two than required in Theorem 2.3. Assume that the principal curvatures $\kappa_1^{(\lambda\mu)}$, $\kappa_2^{(\lambda\mu)}$ of the regular C^1–manifold $\Gamma(\hat{t}) = \Gamma^{(\mu\lambda)}(\hat{t})$ are bounded, uniformly with respect to $\mu \in (0, \mu_0]$ and $\lambda \in (0, \lambda_0]$, and that there is an open subset $\Omega' \subset\subset \Omega$ and $\delta > 0$ such that the neighborhood $\mathcal{U}_\delta^{(\mu\lambda)}(\hat{t})$ of $\Gamma^{(\mu\lambda)}(\hat{t})$ defined in (2.2) satisfies $\mathcal{U}_\delta^{(\mu\lambda)}(\hat{t}) \subseteq \Omega'$. Then there is a constant K_5 such that for all $\mu \in (0, \mu_0]$ and all $\lambda \in (0, \lambda_0]$*

$$\|T_{\text{AC}} - T\|_{L^2(\Gamma(\hat{t}))} \le K_5 |\ln \mu|^3 \mu. \quad (6.4)$$

We postpone the proof of this theorem and first finish the proof of Theorem 2.8.

End of the proof of Theorem 2.8 By (2.6) and (2.55) we have for $x = (\eta, 0) \in \Gamma(\hat{t})$ that

$$\begin{aligned}\nabla_x S(\hat{t}, x) &= n(\partial_n S)(\hat{t}, x) + \nabla_\Gamma S(\hat{t}, x) \\ &= \frac{1}{(\mu\lambda)^{1/2}} \Big(S_0'(\zeta) + \mu^{1/2} \partial_\zeta S_1(t, \eta, \zeta) + \mu \partial_\zeta S_2(t, \eta, \zeta) \Big)_{|\zeta=0} n(\hat{t}, \eta) \\ &\qquad + \mu^{1/2} \nabla_\Gamma S_1(\hat{t}, \eta, 0) + \mu \nabla_\Gamma S_2(\hat{t}, \eta, 0). \end{aligned} \tag{6.5}$$

(4.1) implies

$$S_0'(0) = \sqrt{2\hat{\psi}(S_0(0))} = \sqrt{2\hat{\psi}(1/2)} > 0,$$

whence, from (6.5) for $\mu \in (0, \mu_0]$ and $\lambda \in (0, \lambda_0]$ with μ_0 sufficiently small,

$$\begin{aligned}|\nabla_x S(\hat{t}, x)| &\geq \frac{1}{(\mu\lambda)^{1/2}} \Big(\sqrt{2\hat{\psi}(1/2)} - \mu^{1/2} |\partial_\zeta S_1(t, \eta, 0)| - \mu |\partial_\zeta S_2(t, \eta, 0)| \Big) \\ &\quad - \mu^{1/2} |\nabla_\Gamma S_1(\hat{t}, \eta, 0)| - \mu |\nabla_\Gamma S_2(\hat{t}, \eta, 0)| \geq \frac{1}{2(\mu\lambda)^{1/2}} \sqrt{2\hat{\psi}(1/2)}. \end{aligned} \tag{6.6}$$

Combination of (6.1) with the inequalities (2.52), (6.4) and (6.6) yields

$$\begin{aligned}&\|s_{\mathrm{AC}}(\hat{t}) - s(\hat{t})\|_{L^2(\Gamma(\hat{t}))} \\ &\quad \leq \frac{1}{\min_{\Gamma(\hat{t})} |\nabla_x S(\hat{t})|} \Big(\|f_2^{(\mu\lambda)}(\hat{t})\|_{L^2(\Gamma(\hat{t}))} + \frac{c|\bar{\varepsilon}|}{(\mu\lambda)^{1/2}} \|T_{\mathrm{AC}}(\hat{t}) - T(\hat{t})\|_{L^2(\Gamma(\hat{t}))} \Big) \\ &\quad \leq \frac{2(\mu\lambda)^{1/2}}{\sqrt{2\hat{\psi}(1/2)}} \Big(|\ln \mu|^2 \Big(\frac{\mu}{\lambda}\Big)^{1/2} K_3 \,\mathrm{meas}(\Gamma(\hat{t}))^{1/2} + \frac{c|\bar{\varepsilon}|}{(\mu\lambda)^{1/2}} K_5 |\ln \mu|^3 \mu \Big) \\ &\qquad\qquad \leq K_6 |\ln \mu|^2 \mu + K_7 |\ln \mu|^3 \mu. \end{aligned}$$

(2.74) follows from this estimate. The proof of Theorem 2.8 is complete. ■

Proof of Theorem 6.2: Note that the function $\big(u_{\mathrm{AC}}(\hat{t}), T_{\mathrm{AC}}(\hat{t})\big)$ solves the equations (1.1), (1.2) in Ω with $S_{\mathrm{AC}}(\hat{t}) = S(\hat{t})$, by the initial condition (2.68). Moreover, (2.66) holds for $u_{\mathrm{AC}}(\hat{t})$. From the equations (2.45), (2.46), (2.48) we thus conclude that the difference $(u_{\mathrm{AC}} - u, T_{\mathrm{AC}} - T)$ satisfies

$$-\mathrm{div}_x (T_{\mathrm{AC}} - T)(\hat{t}) = -f_1^{(\mu\lambda)}(\hat{t}), \tag{6.7}$$

$$(T_{\mathrm{AC}} - T)(\hat{t}) = D\varepsilon\big(\nabla_x (u_{\mathrm{AC}} - u)(\hat{t})\big), \tag{6.8}$$

$$(u_{\mathrm{AC}} - u)(\hat{t}, x) = 0, \qquad x \in \partial\Omega. \tag{6.9}$$

This is the Dirichlet boundary value problem for the elliptic system of elasticity theory in the domain Ω. It suggests itself to derive the inequality (6.4) by using the L^2–regularity theory of elliptic systems, which allows to estimate the norm $\|T_{\mathrm{AC}} - T\|_{L^2(\Gamma(\hat{t}))}$ by the L^2–norm of the right hand side $-f_1^{(\mu\lambda)}(\hat{t})$ of (6.7). To apply this theory directly we would need that the L^2–norm of $f_1^{(\mu\lambda)}(\hat{t})$ decays to zero for $\mu \to 0$ uniformly with respect to λ.

However, the relation $\mathrm{meas}\big(Q^{(\mu\lambda)}_{\mathrm{inn}}(\hat t) \cup Q^{(\mu\lambda)}_{\mathrm{match}}(\hat t)\big) \le C_1(\mu\lambda)^{1/2}|\ln\mu|$, which follows from (2.33), and the estimates (2.50), (2.51) yield

$$\|f_1^{(\mu\lambda)}(\hat t)\|_{L^2(\Omega)} \le |\ln\mu|^{5/2}\frac{\mu^{3/4}}{\lambda^{1/4}}K_1 C_1^{1/2}.$$

The right hand side does not decay to zero for $\mu \to \infty$ uniformly with respect to λ, but blows up for $\lambda \to 0$. Therefore direct application of the L^2–regularity theory is not possible. Before giving the detailed proof of (6.4) we sketch how to circumvent this difficulty.

Set

$$\mathsf{A}(\mu) = \frac{3}{a}\mu^{1/2}|\ln\mu|, \tag{6.10}$$

where $a > 0$ is the constant defined in (2.37). By (2.33) we have

$$Q^{(\mu\lambda)}_{\mathrm{inn}}(\hat t) \cup Q^{(\mu\lambda)}_{\mathrm{match}}(\hat t) = \Big\{(\eta,\xi) \in \mathcal{U}_\delta(\hat t) \,\Big|\, |\xi| \le \mathsf{A}(\mu)\lambda^{1/2}\Big\}. \tag{6.11}$$

Define

$$\begin{aligned} f_{11}^{(\mu\lambda)}(x) &= \begin{cases} f_1^{(\mu\lambda)}(\hat t, x), & x \in Q^{(\mu\lambda)}_{\mathrm{inn}}(\hat t) \cup Q^{(\mu\lambda)}_{\mathrm{match}}(\hat t) \\ 0, & x \in Q^{(\mu\lambda)}_{\mathrm{out}}(\hat t), \end{cases} \\ f_{12}^{(\mu\lambda)}(x) &= \begin{cases} 0, & x \in Q^{(\mu\lambda)}_{\mathrm{inn}}(\hat t) \cup Q^{(\mu\lambda)}_{\mathrm{match}}(\hat t) \\ f_1^{(\mu\lambda)}(\hat t, x), & x \in Q^{(\mu\lambda)}_{\mathrm{out}}(\hat t), \end{cases} \end{aligned} \tag{6.12}$$

hence $f_1^{(\mu\lambda)}(\hat t) = f_{11}^{(\mu\lambda)} + f_{12}^{(\mu\lambda)}$. For $x = x(\hat t, \eta, \xi) \in \mathcal{U}_\delta(\hat t)$ we write as usual $f_{11}^{(\mu\lambda)}(x) = f_{11}^{(\mu\lambda)}(\eta,\xi)$. From (2.50) and (6.11) we obtain for $\eta \in \Gamma(\hat t)$ and $-\mathsf{A}(\mu)\lambda^{1/2} \le \xi \le \mathsf{A}(\mu)\lambda^{1/2}$ that

$$|f_{11}^{(\mu\lambda)}(\eta,\xi)| \le |\ln\mu|^2\left(\frac{\mu}{\lambda}\right)^{1/2} K_1. \tag{6.13}$$

Define $\delta_*^{(\mu\lambda)} : \Gamma(\hat t) \to \mathbb{R}$ by

$$\delta_*^{(\mu\lambda)}(\eta) = \int_{-\mathsf{A}(\mu)}^{\mathsf{A}(\mu)} \lambda^{1/2} f_{11}^{(\mu\lambda)}\big(\eta, \lambda^{1/2}\zeta\big)\, d\zeta. \tag{6.14}$$

(6.13) and (6.10) together imply that

$$|\delta_*^{(\mu\lambda)}(\eta)| \le 2\mathsf{A}(\mu)\lambda^{1/2}|\ln\mu|^2\left(\frac{\mu}{\lambda}\right)^{1/2} K_1 = \frac{6}{a}K_1\mu|\ln\mu|^3, \tag{6.15}$$

for all $\eta \in \Gamma(\hat t)$. Examination of the boundary value problem (6.7) – (6.9) suggests that for μ fixed and $\lambda \to 0$ the solution $(u_{\mathrm{AC}} - u, T_{\mathrm{AC}} - T)$ converges to the solution $(u_*, T_*) : \Omega \to \mathbb{R}^3 \times \mathcal{S}^3$ of the transmission problem

$$\begin{aligned} -\mathrm{div}_x T_* &= 0, && (6.16)\\ T_* &= D\varepsilon(\nabla_x u_*), && (6.17)\\ [T_*]n &= \delta_*^{(\mu)}, \qquad \text{on } \Gamma(\hat t), && (6.18)\\ [u_*] &= 0, \qquad \text{on } \Gamma(\hat t), && (6.19)\\ u_*(x) &= 0, \qquad x \in \partial\Omega, && (6.20) \end{aligned}$$

where $\delta_*^{(\mu)}(\eta) = \lim_{\lambda\to 0} \delta_*^{(\mu\lambda)}(\eta)$ for $\eta \in \Gamma(\hat t)$. If this limit exists, it follows from (6.15) that

$$|\delta_*^{(\mu)}(\eta)| \le \frac{6}{a} K_1 \mu |\ln \mu|^3.$$

This implies that the solution (u_*, T_*) will be bounded by $C\mu|\ln\mu|^3$ with a suitable constant C, and this limit behavior suggests that though the L^2–norm of $f_1^{(\mu\lambda)}(\hat t)$ blows up for $\lambda \to 0$, the solution $(u_{\rm AC} - u, T_{\rm AC} - T)(\hat t)$ of (6.7) – (6.9) is bounded by $C\mu|\ln\mu|^3$ with C independent of λ. The reason for the blow up of $\|f_1^{(\mu\lambda)}(\hat t)\|_{L^2(\Omega)}$ for $\lambda \to 0$ is therefore not that the norm of the solution $(u_{\rm AC} - u, T_{\rm AC} - T)(\hat t)$ would blow up, but that the solution looses regularity in a neighborhood of the surface $\Gamma(\hat t)$, which is shown by the equation (6.18) for the limit solution. This equation implies that T_* does not belong to the Sobolev space $W^{1,2}(\Omega)$.

In the following proof we do not study the limit (u_*, T_*). Instead, based on the idea of the behavior of the regularity of $(u_{\rm AC} - u, T_{\rm AC} - T)(\hat t)$, we decompose this function in the form

$$(u_{\rm AC} - u, T_{\rm AC} - T)(\hat t) = \big(u^{(\lambda)}, T^{(\lambda)}\big) + \big(u_*^{(\lambda)}, T_*^{(\lambda)}\big),$$

where $(u^{(\lambda)}, T^{(\lambda)}) = (u_\mu^{(\lambda)}, T_\mu^{(\lambda)})$ is bounded by $C\mu|\ln\mu|^3$, uniformly with respect to λ, and for $\lambda \to 0$ has the same regularity behavior as $(u_{\rm AC} - u, T_{\rm AC} - T)$, but otherwise does not approximate $(u_{\rm AC} - u, T_{\rm AC} - T)$. The construction is such that the difference $(u_*^{(\lambda)}, T_*^{(\lambda)}) = (u_{*\mu}^{(\lambda)}, T_{*\mu}^{(\lambda)}) = (u_{\rm AC} - u, T_{\rm AC} - T) - (u^{(\lambda)}, T^{(\lambda)})$ does not loose its regularity for $\lambda \to 0$. Hence, we can use the standard L^2–theory for elliptic equations to show that also $(u_*^{(\lambda)}, T_*^{(\lambda)})$ is bounded by $C\mu|\ln\mu|^3$ independently of λ.

To construct $(u^{(\lambda)}, T^{(\lambda)})$ let $\mathcal{U}_\delta(\hat t)$ be the neighborhood of $\Gamma(\hat t)$ defined in (2.2) and let $\phi_* \in C_0^\infty\big((-\delta, \delta)\big)$ be a function satisfying

$$\phi_*(\xi) = 1, \quad -\delta/2 \le \xi \le \delta/2. \tag{6.21}$$

We set

$$u^{(\lambda)}(x) = \begin{cases} \lambda^{1/2} V\Big(\lambda, \eta, \dfrac{\xi}{\lambda^{1/2}}\Big)\phi_*(\xi), & x = x(\hat t, \eta, \xi) \in \mathcal{U}_\delta(\hat t), \\ 0, & x \in \Omega \setminus \mathcal{U}_\delta(\hat t), \end{cases} \tag{6.22}$$

$$T^{(\lambda)}(x) = D\varepsilon\big(\nabla_x u^{(\lambda)}(x)\big), \qquad x \in \Omega, \tag{6.23}$$

where the function $\zeta \mapsto V(\lambda, \eta, \zeta) : [-\frac{\delta}{\lambda^{1/2}}, \frac{\delta}{\lambda^{1/2}}] \to \mathbb{R}^3$ is constructed as follows: We use the notations $V' = \partial_\zeta V$, $V'' = \partial_\zeta^2 V$. In the interval $[-\mathsf{A}(\mu), \mathsf{A}(\mu)]$ the function V is the solution of the boundary value problem

$$\Big(D\varepsilon\big(V''(\lambda, \eta, \zeta) \otimes n\big)\Big) n = \lambda^{1/2} f_{11}^{(\mu\lambda)}(\eta, \lambda^{1/2}\zeta), \qquad -\mathsf{A}(\mu) \le \zeta \le \mathsf{A}(\mu), \tag{6.24}$$

$$V\big(\lambda, \eta, \pm\mathsf{A}(\mu)\big) = 0, \tag{6.25}$$

where $n = n(\eta)$ is the unit normal vector to $\Gamma(\hat t)$ at $\eta \in \Gamma(\hat t)$. The equation (6.24) is a second order linear system of ordinary differential equations for the three components of V, which can be written in the form

$$BV'' = \lambda^{1/2} f_{11}^{(\mu\lambda)}, \tag{6.26}$$

with a 3×3–matrix $B = B(\eta)$ defined by the equation

$$B\omega = \big(D\varepsilon(\omega\otimes n)\big)n, \tag{6.27}$$

which must hold for all $\omega\in\mathbb{R}^3$. The matrix B is symmetric and positive definite uniformly with respect to η. To see this, note that since the elasticity tensor $D:\mathcal{S}^3\to\mathcal{S}^3$ is a linear, symmetric, positive definite mapping, we compute for $\omega_1,\omega_2\in\mathbb{R}^3$

$$\begin{aligned}(B\omega_1)\cdot\omega_2 &= \Big(\big(D\varepsilon(\omega_1\otimes n)\big)n\Big)\cdot\omega_2\\ &= (\omega_2\otimes n):D\varepsilon(\omega_1\otimes n) = \varepsilon(\omega_2\otimes n):D\varepsilon(\omega_1\otimes n)\\ &= \big(D\varepsilon(\omega_2\otimes n)\big):\varepsilon(\omega_1\otimes n) = \Big(\big(D\varepsilon(\omega_2\otimes n)\big)n\Big)\cdot\omega_1 = (B\omega_2)\cdot\omega_1.\end{aligned}$$

This shows that B is symmetric. For $\omega\in\mathbb{R}^3$ we have with a suitable constant $C_0>0$, which only depends on D but is independent of η, that

$$(B\omega)\cdot\omega = \varepsilon(\omega\otimes n):D\varepsilon(\omega\otimes n)\ge C_0|\varepsilon(\omega\otimes n)|^2\ge\frac{C_0}{2}|\omega|^2,$$

hence B is positive definite uniformly with respect to $\eta\in\Gamma(\hat t)$.

Therefore the boundary value problem (6.24), (6.25) has a unique solution V on $[-\mathsf{A}(\mu),\mathsf{A}(\mu)]$. To extend $\zeta\mapsto V(\lambda,\eta,\zeta)$ to all of $[-\frac{\delta}{\lambda^{1/2}},\frac{\delta}{\lambda^{1/2}}]$, we continue V to the intervals $\big(-\frac{\delta}{\lambda^{1/2}},-\mathsf{A}(\mu)\big)$ and $\big(\mathsf{A}(\mu),\frac{\delta}{\lambda^{1/2}}\big)$ by affine functions:

$$V(\lambda,\eta,\zeta) = \begin{cases}\big(\zeta+\mathsf{A}(\mu)\big)V'\big(\lambda,\eta,-\mathsf{A}(\mu)\big), & -\frac{\delta}{\lambda^{1/2}}\le\zeta\le-\mathsf{A}(\mu),\\ \big(\zeta-\mathsf{A}(\mu)\big)V'\big(\lambda,\eta,\mathsf{A}(\mu)\big), & \mathsf{A}(\mu)\le\zeta\le\frac{\delta}{\lambda^{1/2}}\,.\end{cases} \tag{6.28}$$

By this extension, $\zeta\mapsto V(\lambda,\eta,\zeta)$ is continuously differentiable at $\zeta=\pm\mathsf{A}(\mu)$. For $x=x(\hat t,\eta,\xi)\in\mathcal{U}_\delta$ we use the notation

$$V(\lambda,x) = V\big(\lambda,\eta,\frac{\xi}{\lambda^{1/2}}\big).$$

In the remaining part of the proof of Theorem 6.2 we need the following lemma, which we prove first.

Lemma 6.3 *There are constants $C_1,\dots,C_4$ such that for all $\mu\in(0,\mu_0]$, $\lambda\in(0,\lambda_0]$, $(\eta,\zeta)\in\Gamma(\hat t)\times\big(-\frac{\delta}{\lambda^{1/2}},\frac{\delta}{\lambda^{1/2}}\big)$ and $x\in\mathcal{U}_\delta(\hat t)$ the estimates*

$$\begin{aligned}|\nabla_\eta^j V'(\lambda,\eta,\zeta)| &\le C_1|\ln\mu|^3\mu, && j=0,1,2, && (6.29)\\ |\lambda^{1/2}\nabla_\eta^j V(\lambda,\eta,\zeta)| &\le C_2|\ln\mu|^3\mu, && j=0,1,2, && (6.30)\\ |\lambda^{1/2}\nabla_x V(\lambda,x)| &\le C_3|\ln\mu|^3\mu, && && (6.31)\\ |\lambda^{1/2}\partial_{x_k}\nabla_{\Gamma_\xi}V(\lambda,x)| &\le C_4|\ln\mu|^3\mu, && k=1,\dots,3, && (6.32)\end{aligned}$$

hold. Moreover, there is a function $g^{(\mu\lambda)}:\Omega\to\mathbb{R}^3$ and a constant C_5 such that $T^{(\lambda)}$ defined in (6.23) *satisfies*

$$\mathrm{div}_x T^{(\lambda)} = f_{11}^{(\mu\lambda)} + g^{(\mu\lambda)}, \tag{6.33}$$

with

$$|g^{(\mu\lambda)}(x)|\le C_5|\ln\mu|^3\mu, \tag{6.34}$$

for all $x\in\Omega$ and all $\mu\in(0,\mu_0]$, $\lambda\in(0,\lambda_0]$.

Proof: In the following computations we drop the arguments λ and η. Integration of (6.26) yields

$$BV'(\zeta) = \int_{-\mathsf{A}(\mu)}^{\zeta} \lambda^{1/2} f_{11}^{(\mu\lambda)}(\lambda^{1/2}\vartheta)d\vartheta + BV'(-\mathsf{A}(\mu)), \tag{6.35}$$

$$BV(\zeta) = \int_{-\mathsf{A}(\mu)}^{\zeta} \int_{-\mathsf{A}(\mu)}^{\vartheta_1} \lambda^{1/2} f_{11}^{(\mu\lambda)}(\lambda^{1/2}\vartheta)d\vartheta d\vartheta_1 + \big(\zeta + \mathsf{A}(\mu)\big)BV'(-\mathsf{A}(\mu)), \tag{6.36}$$

where we used the boundary condition (6.25) to get the second equation. Since $V(\mathsf{A}(\mu)) = 0$, the relations (6.36) and (6.13) together yield

$$\begin{aligned} 2\mathsf{A}(\mu)|BV'(-\mathsf{A}(\mu))| &= \Big| - \int_{-\mathsf{A}(\mu)}^{\mathsf{A}(\mu)} \int_{-\mathsf{A}(\mu)}^{\vartheta_1} \lambda^{1/2} f_{11}^{(\mu\lambda)} d\vartheta d\vartheta_1 \Big| \\ &\leq \int_{-\mathsf{A}(\mu)}^{\mathsf{A}(\mu)} \int_{-\mathsf{A}(\mu)}^{\vartheta_1} |\ln\mu|^2 \mu^{1/2} K_1 d\vartheta d\vartheta_1 = 2\mathsf{A}(\mu)^2 |\ln\mu|^2 \mu^{1/2} K_1\,, \end{aligned}$$

hence, by (6.10),

$$|BV'(-\mathsf{A}(\mu))| \leq \mathsf{A}(\mu)|\ln\mu|^2\mu^{1/2}K_1 = \frac{3}{a}K_1|\ln\mu|^3\mu.$$

Since $B = B(\eta)$ is positive definite uniformly with respect to η, this inequality implies the estimate (6.29) for $j = 0$ and $-\mathsf{A}(\mu) \leq \zeta \leq \mathsf{A}(\mu)$. Since by definition (6.28) we have $V'(\zeta) = V'(-\mathsf{A}(\mu))$ for $\zeta \leq -\mathsf{A}(\mu)$ and $V'(\zeta) = V'(\mathsf{A}(\mu))$ for $\mathsf{A}(\mu) \leq \zeta$, the estimate (6.29) with $j = 0$ holds also for the values of ζ outside of the interval $[-\mathsf{A}(\mu), \mathsf{A}(\mu)]$.

To prove (6.30) for $j = 0$ we use that $V(-\mathsf{A}(\mu)) = 0$. By integration we thus obtain from (6.29) for $\zeta \in [\frac{-\delta}{\lambda^{1/2}}, \frac{\delta}{\lambda^{1/2}}]$ that

$$|V(\zeta)| = \Big| \int_{-\mathsf{A}(\mu)}^{\zeta} V'(\vartheta)d\vartheta \Big| \leq |\zeta + \mathsf{A}(\mu)|C_1|\ln\mu|^3\mu \leq \big(\lambda^{-1/2}\delta + \mathsf{A}(\mu)\big)C_1|\ln\mu|^3\mu,$$

which implies (6.30) for $j = 0$.

To verify (6.29) and (6.30) for $j = 1, 2$ we differentiate the differential equation (6.26) and the boundary condition (6.25) with respect to η. For $j = 1$ we obtain the differential equation

$$B(\eta)(\partial_{\eta_k}V)'' = \lambda^{1/2}\big(\partial_{\eta_k} f_{11}^{(\mu\lambda)} - \partial_{\eta_k}B(\eta)B(\eta)^{-1}f_{11}^{(\mu\lambda)}\big),$$

and a similar equation for $j = 2$. We then use the estimate

$$|\nabla_\eta^j f_{11}^{(\mu\lambda)}(\eta,\xi)| = |\nabla_\eta^j f_1^{(\mu\lambda)}(\eta,\xi)| \leq |\ln\mu|^2\Big(\frac{\mu}{\lambda}\Big)^{1/2}K, \qquad j = 1, 2.$$

This estimate is obtained by differentiation with respect to η of the asymptotic expansions in Section 5.1 leading to Corollary 5.3. Under the regularity assumptions in Theorem 6.2 these derivatives exist. With this estimate we can employ the same arguments as above for the case $j = 0$ to derive (6.29) and (6.30) for $j = 1, 2$.

To prove (6.31) we use the decomposition (3.6) of the gradient and (3.13) to compute

$$\nabla_x V(\lambda, x) = \partial_\xi V\big(\lambda, \eta, \frac{\xi}{\lambda^{1/2}}\big) \otimes n + \nabla_{\Gamma_\xi} V\big(\lambda, \eta, \frac{\xi}{\lambda^{1/2}}\big) = \lambda^{-\frac{1}{2}}V' \otimes n + (\nabla_\eta V)A(\hat{t}, \eta, \xi).$$

The right hand side is estimated by (6.29) and (6.30) to obtain (6.31). The estimate (6.32) is obtained from (6.29) and (6.30) by similar decompositions.

To prove (6.33), (6.34) note that by (6.22), (6.23) we have $T^{(\lambda)} = D\varepsilon(\nabla_x u^{(\lambda)}) = D\varepsilon\big(\nabla_x(\lambda^{1/2}V\phi_*)\big)$. Using (3.6) and (3.7) we therefore obtain by a similar computation as in (5.73) that

$$\begin{aligned}
\mathrm{div}_x T^{(\lambda)} &= \mathrm{div}_x D\varepsilon\Big(\nabla_x\big(\lambda^{1/2}V(\lambda,\eta,\frac{\xi}{\lambda^{1/2}})\phi_*(\xi)\big)\Big)\\
&= \Big(\lambda^{-1/2}\big(D\varepsilon(V''\otimes n)\big)n + \mathrm{div}_{\Gamma_\xi} D\varepsilon(V'\otimes n)\Big)\phi_*\\
&\quad + \lambda^{1/2}\Big(D\varepsilon(\partial_\xi\nabla_{\Gamma_\xi}V)\big)n + \mathrm{div}_{\Gamma_\xi} D\varepsilon(\nabla_{\Gamma_\xi}V)\Big)\phi_*\\
&\quad + \Big(\big(D\varepsilon(\lambda^{1/2}\nabla_x V)\big)n + \mathrm{div}_x D\varepsilon(\lambda^{1/2}V\otimes n)\Big)\phi_*'\\
&\quad + \Big(D\varepsilon(\lambda^{1/2}V\otimes n)\Big)n\,\phi_*''\\
&= f_{11}^{(\mu\lambda)} + g^{(\mu\lambda)}.
\end{aligned} \tag{6.37}$$

In the last step we used the differential equation (6.24) and noted that for $\xi \in ([-\delta, -\mathsf{A}(\mu)\lambda^{1/2}] \cup [\mathsf{A}(\mu)\lambda^{1/2}, \delta])$ we have $V''\big(\lambda,\eta,\frac{\xi}{\lambda^{1/2}}\big) = 0$, by definition of V for such values of ξ in (6.28). We also used that $\phi_*(\xi) = 1$ for $\xi \in [-\mathsf{A}(\mu)\lambda^{1/2}, \mathsf{A}(\mu)\lambda^{1/2}]$, which follows from (6.21) and (6.10), since we have chosen μ_0 and λ_0 small enough such that $\mathsf{A}(\mu)\lambda^{1/2} < \delta/2$ for all $0 < \mu \le \mu_0$ and $0 < \lambda \le \lambda_0$.

The function $g^{(\mu\lambda)}$ is the sum of terms number 2 to 7 in the middle expression of equation (6.37). If we examine everyone of these six terms and apply (6.29) – (6.32) and also note that the functions ϕ_*, ϕ_*' and ϕ_*'' are bounded independently of μ and λ and vanish outside of $\mathcal{U}_\delta(\hat t)$, which follows from $\phi_* \in C_0^\infty((-\delta,\delta))$, we see that (6.34) holds for $g^{(\mu\lambda)}$. This completes the proof of Lemma 6.3. ■

To conclude the proof of Theorem 6.2 let $(u_*^{(\lambda)}, T_*^{(\lambda)})$ be the solution of the boundary value problem

$$-\mathrm{div}_x T_*^{(\lambda)} = g^{(\mu\lambda)} - f_{12}^{(\mu\lambda)}, \tag{6.38}$$

$$T_*^{(\lambda)} = D\varepsilon(\nabla_x u_*^{(\lambda)}), \tag{6.39}$$

$$u_*^{(\lambda)}(x) = 0, \qquad x \in \partial\Omega. \tag{6.40}$$

From these equations and from (6.22), (6.23), (6.33) we see that the function $(u^{(\lambda)} + u_*^{(\lambda)}, T^{(\lambda)} + T_*^{(\lambda)})$ satisfies

$$\begin{aligned}
-\mathrm{div}_x(T^{(\lambda)} + T_*^{(\lambda)}) &= -f_{11}^{(\mu\lambda)} - g^{(\mu\lambda)} + g^{(\mu\lambda)} - f_{12}^{(\mu\lambda)} = -f_1^{(\mu\lambda)}(\hat t),\\
(T^{(\lambda)} + T_*^{(\lambda)}) &= D\varepsilon\big(\nabla_x(u^{(\lambda)} + u_*^{(\lambda)})\big),\\
(u^{(\lambda)} + u_*^{(\lambda)})(x) &= 0, \qquad x \in \partial\Omega,
\end{aligned}$$

hence $(u^{(\lambda)} + u_*^{(\lambda)}, T^{(\lambda)} + T_*^{(\lambda)})$ is equal to the unique solution of the boundary value problem (6.7) – (6.9), which means that $(u^{(\lambda)} + u_*^{(\lambda)}, T^{(\lambda)} + T_*^{(\lambda)}) = (u_{\mathrm{AC}} - u, T_{\mathrm{AC}} - u)(\hat t)$. Consequently, we have

$$\|T_{\mathrm{AC}} - T\|_{L^2(\Gamma(\hat t))} \le \|T^{(\lambda)}\|_{L^2(\Gamma(\hat t))} + \|T_*^{(\lambda)}\|_{L^2(\Gamma(\hat t))}. \tag{6.41}$$

To estimate $\|T_*^{(\lambda)}\|_{L^2(\Gamma(\hat{t}))}$ we can use the theory of interior regularity for the elliptic boundary value problem (6.38) – (6.40). By this theory there is a constant C such that $\|u_*^{(\lambda)}\|_{W^{2,2}(\Omega')} \leq C\|g^{(\mu\lambda)} - f_{12}^{(\mu\lambda)}\|_{L^2(\Omega)}$, where Ω' is the subdomain of Ω introduced in Theorem 6.2, hence by the Sobolev embedding theorem and by (6.39),

$$\|T_*^{(\lambda)}\|_{L^2(\Gamma(\hat{t}))} \leq C_1\|T_*^{(\lambda)}\|_{W^{1,2}(\Omega')} \leq C_2\|g^{(\mu\lambda)} - f_{12}^{(\mu\lambda)}\|_{L^2(\Omega)}, \tag{6.42}$$

where by our assumptions on $\Gamma^{(\mu\lambda)}(\hat{t})$ in Theorem 6.2 the constants C_1, C_2 can be chosen independently of μ and λ. By definition of $f_{12}^{(\mu\lambda)}$ in (6.12) and by (2.51) we have $|f_{12}^{(\mu\lambda)}(x)| \leq \mu^{3/2}K_2$ for all $x \in \Omega$. From this inequality, from (6.34) and from (6.42) we conclude that

$$\begin{aligned}\|T_*^{(\lambda)}\|_{L^2(\Gamma(\hat{t}))} &\leq C_2\Big(\int_\Omega \big(|g^{(\mu\lambda)}(x)| + |f_{12}^{(\mu\lambda)}(x)|\big)^2 dx\Big)^{1/2} \\ &\leq C_2(C_5|\ln\mu|^3\mu + \mu^{3/2}K_2)\Big(\int_\Omega dx\Big)^{1/2} \leq K|\ln\mu|^3\mu. \end{aligned}\tag{6.43}$$

From (6.23), (6.22) and from the inequalities (6.30), (6.31) we infer that

$$\begin{aligned}|T^{(\lambda)}(x)| &\leq C|\nabla_x u^{(\lambda)}(x)| = C\big|\nabla_x\big(\lambda^{1/2}V(\lambda,x)\phi_*(\xi)\big)\big| \\ &= C\big|\big(\lambda^{1/2}\nabla_x V(\lambda,x)\big)\phi_*(\xi) + \lambda^{1/2}V(\lambda,x)\otimes\big(n\phi_*'(\xi)\big)\big| \leq K'|\ln\mu|^3\mu,\end{aligned}$$

whence

$$\|T^{(\lambda)}\|_{L^2(\Gamma(\hat{t}))} \leq K''|\ln\mu|^3\mu.$$

Combination of this inequality with (6.41) and (6.43) yields (6.4). The proof of Theorem 6.2 is complete. ∎

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