Resolvent estimates of the Stokes system with Navier boundary conditions in general unbounded domains

Reinhard Farwig* Veronika Rosteck†

Abstract

Consider the Stokes resolvent system in general unbounded domains $\Omega \subset \mathbb{R}^n$, $n \geq 2$, with boundary of uniform class C^3, and Navier slip boundary condition. The main result is the resolvent estimate in function spaces of the type $\tilde{L}^q$ defined as $L^q \cap L^2$ when $q \geq 2$, but as $L^q + L^2$ when $1 < q < 2$, adapted to the unboundedness of the domain. As a consequence we get that the Stokes operator generates an analytic semigroup on a solenoidal subspace $\tilde{L}^q_\sigma(\Omega)$ of $\tilde{L}^q(\Omega)$.

Key Words: Stokes resolvent system; Navier boundary condition; general unbounded domains; spaces $\tilde{L}^q(\Omega)$

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1 Introduction and main result

Given an unbounded domain $\Omega \subset \mathbb{R}^n$ we consider for a prescribed external force $\boldsymbol{f} : \Omega \times (0,T) \to \mathbb{R}^n$ the Stokes resolvent system with Navier boundary condition

$$\begin{aligned} \lambda \boldsymbol{u} - \Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} \quad & \text{in } \Omega \\ \operatorname{div} \boldsymbol{u} = 0 \quad & \text{in } \Omega \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad & \text{on } \partial\Omega \\ \alpha \boldsymbol{u} + \beta (\mathbf{T}(\boldsymbol{u}, p)\boldsymbol{n})_\tau = \mathbf{0} \quad & \text{on } \partial\Omega. \end{aligned} \tag{1.1}$$

*Fachbereich Mathematik, Technische Universität Darmstadt *and* International Research Training Group (IRTG 1529) Darmstadt-Tokyo, 64289 Darmstadt, Germany, farwig@mathematik.tu-darmstadt.de

†Fachbereich Mathematik, Technische Universität Darmstadt, 64289 Darmstadt, Germany, rosteckva@gmx.de

Here $\boldsymbol{u} : \Omega \to \mathbb{R}^n$, $p : \Omega \to \mathbb{R}$ are the unknown velocity field and pressure, respectively, and the complex number λ is contained in the sector

$$\mathcal{S}_\varepsilon = \{\lambda \in \mathbb{C}\backslash\{0\} : |\arg \lambda| < \pi/2 + \varepsilon\},\ 0 < \varepsilon < \pi/2.$$

The tensor $\mathsf{T} = \mathsf{T}(\boldsymbol{u}, p) = -p\mathsf{I} + \mathsf{S}(\boldsymbol{u}) = -p\mathsf{I} + 2\nu\mathsf{D}(\boldsymbol{u})$ is the Cauchy stress tensor where $\mathsf{D}(\boldsymbol{u}) = \frac{1}{2}(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^\top)$ denotes the symmetric part of the velocity gradient and $\nu > 0$ is the viscosity. As usual in the analysis of the (linear) Stokes system we set $\nu = 1$ for simplicity and obtain for the viscous stress tensor

$$\mathsf{S}(\boldsymbol{u}) = \nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^\top.$$

Let $\boldsymbol{n}$ denote the unit outer normal to $\partial\Omega$, and let the subscript τ indicate the tangential component of a vector field on $\partial\Omega$. The constants $\alpha \in [0, 1)$ and $\beta \in (0, 1]$ satisfy $\alpha+\beta = 1$. Hence the boundary condition $\alpha\boldsymbol{u}+\beta(\mathsf{T}(\boldsymbol{u}, p)\boldsymbol{n})_\tau = \boldsymbol{0}$ (called Navier or Robin condition or of third type) simplifies to

$$B_{\alpha,\beta}(\boldsymbol{u}) := \alpha\boldsymbol{u} + \beta(\mathsf{S}(\boldsymbol{u})\boldsymbol{n})_\tau = \boldsymbol{0} \tag{1.2}$$

and describes three different physical cases. For $\alpha = 0$ and $\beta = 1$ we obtain the so-called no-stick or perfect slip condition, meaning that the fluid is subject to no tangential stresses at the boundary; this case is similar to the Neumann or perfect insulation condition for the heat equation. When $0 < \alpha, \beta < 1$, tangential stresses at the boundary are proportional to the tangential velocity $\boldsymbol{u}_\tau = \boldsymbol{u}$ on $\partial\Omega$ (recall the impermeability condition $\boldsymbol{u} \cdot \boldsymbol{n} = \boldsymbol{0}$ on $\partial\Omega$); for the Laplacian this condition is called the third or oblique boundary condition. Finally, if $\beta \to 0_+$, the Navier condition becomes the no-slip or Dirichlet condition and describes the adhesion of particles on the boundary.

The boundary condition (1.2) was introduced by Navier in [23] and is in particular reasonable for problems dealing with coating flows, fibre spinning and microfluids or flows in semiconductor melts; for references see [29].

Let us mention some known results for the Stokes resolvent system with Dirichlet boundary condition $\boldsymbol{u} = \boldsymbol{0}$. A typical strategy is to start with the whole space case solved with the help of Fourier transforms and multiplier theory and continue with the half space and bent half spaces. Finally, a cut-off procedure allows to solve the problem in bounded domains ([13]). For a potential theoretic approach see [30], and, e.g., [18] for a method using pseudodifferential operators. Moreover, we refer to [13] for results in exterior domains, to [9, 10] for infinite cylinders, and to [1, 2] for layers. Resolvent estimates in weighted function spaces for (bent) half spaces and aperture domains are considered in [14, 15, 16]. The main argument is that a cut-off procedure reduces the problem to *finitely many* bent half spaces and the whole space case. However, this techniques excludes many other interesting unbounded domains, e.g., domains with several exits to infinity, with infinitely many holes, with spiraling exits etc.

The Navier boundary condition (1.2) was first considered by Giga in [19] for a bounded domain as a special case of a more general condition. For the case of a half space Saal [25] showed that the Stokes operator generates an analytic semigroup and admits a bounded H^∞-calculus. In [26] Shibata and Shimada proved the unique solvability of the Stokes resolvent system with the Navier boundary condition for bounded and exterior domains. This is done by a cut-off technique, where - as for the Dirichlet case - existence and uniqueness are proven successively for the whole space, the half space, bent half spaces and a bounded (or exterior) domain. We note that an inhomogeneous divergence as well as non-zero boundary conditions are included; this will also be used in our analysis. Shimada [29] even proved the strong unique solvability of the instationary system with Navier boundary condition for a bounded domain.

For the case of the Neumann boundary condition where $\boldsymbol{n} \cdot \mathsf{T}(\boldsymbol{u}, p) = \boldsymbol{\varphi}$ is prescribed on $\partial\Omega$ similar results were obtained by Shibata and Shimizu, see [27] for the resolvent equation in bounded and exterior domains and [28] for the instationary system in a bounded domain. The Neumann condition was also treated in several papers of Solonnikov and Grubb (e.g. in [20]) using pseudo-differential operators.

Due to counter-examples by Bogovskij and Maslennikova [3, 22] the Helmholtz decomposition of vector fields in $L^q(\Omega)$, $1 < q < \infty$, on an unbounded smooth domain may fail unless $q = 2$. By analogy, a bounded Helmholtz projection P_q with the properties required to define the Stokes operator $A_q = -P_q\Delta$ when $q \neq 2$ may not exist. Therefore, in [4, 5, 6, 7, 8] H. Kozono, H. Sohr and the first author of this article introduced the spaces

$$\tilde{L}^q(\Omega) := \begin{cases} L^q(\Omega) + L^2(\Omega), & \text{if } 1 \le q < 2, \\ L^q(\Omega) \cap L^2(\Omega), & \text{if } 2 \le q \le \infty. \end{cases} \tag{1.3}$$

The corresponding norm is defined as $\|u\|_{\tilde{L}^q} = \max\{\|u\|_q, \|u\|_2\}$ when $q \ge 2$, and as $\inf\{\|u_1\|_q + \|u_2\|_2 : u = u_1 + u_2, u_1 \in L^q(\Omega), u_2 \in L^2(\Omega)\}$ when $1 \le q < 2$. For bounded domains we have that $\tilde{L}^q(\Omega) = L^q(\Omega)$ with equivalent norms. We note that functions in $\tilde{L}^q(\Omega)$ locally behave like L^q-functions, but globally exploit L^2-properties. By well-known results of interpolation theory, $\tilde{L}^q(\Omega)' \cong \tilde{L}^{q'}(\Omega)$ when $1 \le q < \infty$.

By analogy, function spaces like $\tilde{L}^q_\sigma(\Omega)$ of solenoidal vector fields and $\tilde{W}^{k,q}(\Omega)$ of weakly differentiable functions will be defined. In [6] the authors showed for general uniformly smooth domain $\Omega \subset \mathbb{R}^n$ that in $\tilde{L}^q(\Omega)$, $1 < q < \infty$, the corresponding Helmholtz projection $\tilde{P}_q$ is a well-defined bounded projection. Then the Stokes operator $\tilde{A}_q = -\tilde{P}_q \Delta$ with Dirichlet boundary condition generates an analytic semigroup ([5, 8]) on $\tilde{L}^q_\sigma(\Omega)$ and has the property of maximal regularity ([7]). Moreover, Kunstmann [21] showed that $\tilde{A}_q$ admits a bounded H^∞-calculus. These results are applied by Riechwald and the first author in [11] in order to develop the theory of very weak solutions to the Navier-Stokes equations with Dirichlet boundary condition in uniformly smooth domains.

To work in general unbounded domains we use the exhaustion method, i.e., we approximate Ω from the interior by a sequence of increasing bounded domains. In the case of the Dirichlet boundary condition $\boldsymbol{u} = \mathbf{0}$ on $\partial\Omega$ (see [5]) the boundary condition is included in the definition of the space $W^{1,q}_0(\Omega)$ as a closure of C_0^∞-functions. A similar approach cannot directly be applied to the Navier boundary condition. Moreover, we do not have a global trace theorem at hand for general unbounded domains. Therefore, we pose some restrictions on the domain Ω, see Assumption 1.1 below. Actually, it is not clear whether there are uniform C^3-domains *not* fulfilling Assumption 1.1; for the definition of uniform C^3-domain Ω and further notation we refer to Definition 2.1 in Sect. 2 below.

Assumption 1.1. A uniform C^3-domain $\Omega \subset \mathbb{R}^n$ of type $(\tilde{\alpha}, \tilde{\beta}, K)$ in this article is assumed to have the following representation: There exists a sequence $\{\Omega_j\}_{j\in\mathbb{N}}$ of bounded uniform C^3-domains of type $\tau_\Omega = (\tilde{\alpha}, \tilde{\beta}, K)$ such that $\Omega_j \subset \Omega$ and

- $\Omega_j \subset \Omega_{j+1}$ for all $j \in \mathbb{N}$ and $\Omega = \bigcup_{j=1}^\infty \Omega_j$,
- $\Gamma_j := \partial\Omega_j \cap \partial\Omega \neq \emptyset$ for all $j \in \mathbb{N}$,
- $\Gamma_j \subset \Gamma_{j+1}$ for all $j \in \mathbb{N}$ and $\partial\Omega = \bigcup_{j=1}^\infty \Gamma_j$.

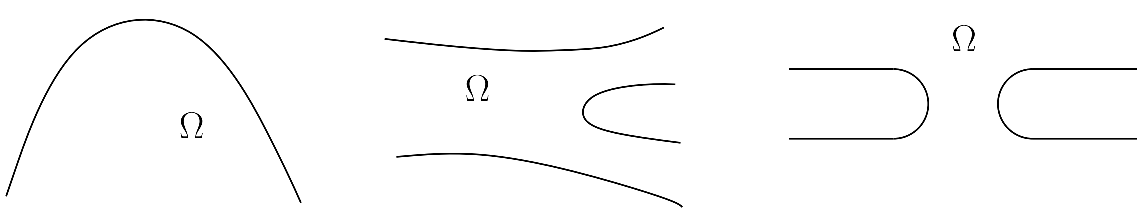

Figure 1: Examples of uniformly smooth domains satisfying Assumption 1.1.

For $1 < q < \infty$ let $P_q : L^q(\Omega) \to L^q_\sigma(\Omega)$ denote the Helmholtz projection. To define the Stokes operator with Navier boundary condition $B_{\alpha,\beta}$ we introduce for $1 < q < \infty$ the Sobolev space

$$W^{2,q}_B(\Omega) = W^{2,q}_{B_{\alpha,\beta}}(\Omega) = \big\{\boldsymbol{u} \in W^{2,q}(\Omega) : B_{\alpha,\beta}(\boldsymbol{u}) = \mathbf{0} \text{ on } \partial\Omega\big\}.$$

The boundary condition for the space $W^{2,q}_B(\Omega)$ is understood locally in the sense of usual traces. Then for a bounded domain Ω the domain of the Stokes operator $A_q = -P_q\Delta$ is given by

$$D^q(\Omega) = D(A_q) = L^q_\sigma(\Omega) \cap W^{2,q}_{B_{\alpha,\beta}}(\Omega).$$

However, this definition is not suitable for general unbounded domains. For this reason, let

$$\tilde{D}^q(\Omega) = D(\tilde{A}_q) = \begin{cases} D(A_q) \cap D(A_2), & 2 \le q < \infty, \\ D(A_q) + D(A_2), & 1 < q < 2. \end{cases} \tag{1.4}$$

Then we define with the help of the Helmholtz projection $\tilde{P}_q$ the Stokes operator with Navier boundary condition for a general uniformly smooth domain as

$$\tilde{A}_q = -\tilde{P}_q\Delta : D(\tilde{A}_q) \subset \tilde{L}^q_\sigma(\Omega) \to \tilde{L}^q_\sigma(\Omega). \tag{1.5}$$

We will write either $D(\tilde{A}_q)$ or $\tilde{D}^q(\Omega)$; if there can be no confusion concerning the domain Ω, we write $D(\tilde{A}_q)$, otherwise, we use the notation $\tilde{D}^q(\Omega)$.

Now we can rewrite the system (1.1) as the abstract resolvent problem

$$\lambda\boldsymbol{u} + \tilde{A}_q\boldsymbol{u} = \tilde{P}_q\boldsymbol{f}.$$

We are interested in the unique solvability of this equation for a right-hand side in $\tilde{L}^q_\sigma(\Omega)$ and in properties of the Stokes operator $\tilde{A}_q$. In particular, we show that $-\tilde{A}_q$ generates an analytic semigroup in $\tilde{L}^q_\sigma(\Omega)$. Our main result reads as follows:

Theorem 1.2 (Resolvent problem for $\tilde{A}_q$)**.** *Let $1 < q < \infty$, $0 < \varepsilon < \frac{\pi}{2}$, $\delta > 0$. Let $\Omega \subset \mathbb{R}^n$, $n \ge 2$, be a uniform C^3-domain of type $\tau_\Omega = (\tilde{\alpha}, \tilde{\beta}, K)$ and let Assumption 1.1 be satisfied. Then the following assertions hold:*

(i) *The sector $\mathcal{S}_\varepsilon$ is contained in the resolvent set of $-\tilde{A}_q$, i.e., the resolvent*

$$(\lambda + \tilde{A}_q)^{-1} : \tilde{L}^q_\sigma(\Omega) \to \tilde{L}^q_\sigma(\Omega)$$

exists as a bounded operator for $\lambda \in \mathcal{S}_\varepsilon$. Moreover, for $\boldsymbol{f} \in \tilde{L}^q_\sigma(\Omega)$ there exists a unique $\boldsymbol{u} = (\lambda + \tilde{A}_q)^{-1}\boldsymbol{f}$ satisfying the estimate

$$\|\lambda\boldsymbol{u}\|_{\tilde{L}^q(\Omega)} + \|\boldsymbol{u}\|_{\tilde{W}^{2,q}(\Omega)} \le C\|\boldsymbol{f}\|_{\tilde{L}^q(\Omega)} \tag{1.6}$$

for all $\lambda \in \mathcal{S}_\varepsilon$ with $|\lambda| \ge \delta$, where $C = C(q, \varepsilon, \delta, \tau_\Omega) > 0$.

(ii) *For given* $\boldsymbol{f} \in \tilde{L}^q(\Omega)$ *and* $\lambda \in \mathcal{S}_\varepsilon$ *the Stokes resolvent system* (1.1) *has a unique solution* $(\boldsymbol{u}, \nabla p) \in D(\tilde{A}_q) \times \tilde{L}^q(\Omega)$ *defined by* $\boldsymbol{u} = (\lambda + \tilde{A}_q)^{-1}\tilde{P}_q\boldsymbol{f}$ *and* $\nabla p = (I - \tilde{P}_q)(\boldsymbol{f} + \Delta \boldsymbol{u})$, *and satisfying*

$$\|\lambda \boldsymbol{u}\|_{\tilde{L}^q(\Omega)} + \|\boldsymbol{u}\|_{\tilde{W}^{2,q}(\Omega)} + \|\nabla p\|_{\tilde{L}^q(\Omega)} \leq C\|\boldsymbol{f}\|_{\tilde{L}^q(\Omega)} \tag{1.7}$$

for $|\lambda| \geq \delta$ *with* $C = C(q, \varepsilon, \delta, \tau_\Omega) > 0$.

(iii) *The Stokes operator* $\tilde{A}_q : D(\tilde{A}_q) \to \tilde{L}^q_\sigma(\Omega)$ *is a densely defined closed operator, and* $-\tilde{A}_q$ *generates an analytic semigroup* $\{e^{-t\tilde{A}_q}\}_{t\geq 0}$ *in* $\tilde{L}^q_\sigma(\Omega)$ *satisfying the estimate*

$$\|e^{-t\tilde{A}_q}\boldsymbol{f}\|_{\tilde{L}^q(\Omega)} \leq Ce^{\delta t}\|\boldsymbol{f}\|_{\tilde{L}^q(\Omega)} \tag{1.8}$$

for $\boldsymbol{f} \in \tilde{L}^q_\sigma(\Omega)$, $t \geq 0$, *where* $C = C(q, \delta, \tau_\Omega) > 0$.

(iv) *For the adjoint operator the duality relations* $\tilde{A}'_q = \tilde{A}_{q'}$, $\langle \tilde{A}_q\boldsymbol{u}, \boldsymbol{v}\rangle = \langle \boldsymbol{u}, \tilde{A}_{q'}\boldsymbol{v}\rangle$ *for all* $\boldsymbol{u} \in D(\tilde{A}_q)$, $\boldsymbol{v} \in D(\tilde{A}_{q'})$, *hold.*

Corollary 1.3 (Equivalent norms on $D(\tilde{A}_q)$)**.** *Let* $1 < q < \infty$, *let* $\Omega \subset \mathbb{R}^n$, $n \geq 2$, *be a uniform* C^3*-domain, and let Assumption 1.1 hold. Then the norms*

$$\|\cdot\|_{\tilde{W}^{2,q}(\Omega)},\quad \|\cdot\|_{\tilde{L}^q(\Omega)} + \|\tilde{A}_q\cdot\|_{\tilde{L}^q(\Omega)},\quad \|\cdot\|_{\tilde{L}^q(\Omega)} + \|(1+\tilde{A}_q)\cdot\|_{\tilde{L}^q(\Omega)},\quad \|(1+\tilde{A}_q)\cdot\|_{\tilde{L}^q(\Omega)}$$

are equivalent on $D(1 + \tilde{A}_q) := D(\tilde{A}_q)$ *with a constant depending on* Ω *only through* τ_Ω.

Corollary 1.4 ($\tilde{L}^r$-$\tilde{L}^q$–estimate)**.** *Let* $1 < q \leq r < \infty$, $\delta > 0$, *let* $\Omega \subset \mathbb{R}^n$, $n \geq 2$, *be a uniform* C^3*-domain, and let Assumption 1.1 be satisfied. Then the estimate*

$$\|e^{-t\tilde{A}_q}\boldsymbol{u}\|_{\tilde{L}^r(\Omega)} \leq C\Big(\frac{1+t}{t}\Big)^\gamma e^{\delta t}\|\boldsymbol{u}\|_{\tilde{L}^q(\Omega)}, \quad \boldsymbol{u} \in \tilde{L}^q_\sigma(\Omega),\, t > 0, \tag{1.9}$$

with a constant $C = C(\delta, q, \tau_\Omega) > 0$, *holds true in the following cases*:

(i) *If* $q < \frac{n}{2}$ *and* $q \leq r \leq \frac{nq}{n-2q}$ *where* $0 \leq \gamma = \frac{n}{2}\big(\frac{1}{q} - \frac{1}{r}\big) \leq 1$.

(ii) *If* $q \geq \frac{n}{2}$ *and* $q \leq r$ *where* $1 \geq \gamma \geq 1 - \frac{q}{r} \geq 0$.

This article is organized as follows. In Sect. 2 we describe several preliminaries and recall necessary results for the bounded domain case. Then Sect. 3 contains the proof of Theorem 1.2 and of the Corollaries 1.3 and 1.4.

2 Preliminaries

Let us recall the definition of a uniform C^k-domain and its essential properties.

Definition 2.1. *A domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is called a uniform C^k-domain of type $\tau_\Omega = (\tilde{\alpha}, \tilde{\beta}, K)$, where $k \in \mathbb{N}$, $k \geq 2$, $\tilde{\alpha} > 0$, $\tilde{\beta} > 0$ and $K > 0$, if for each $x_0 \in \partial\Omega$ there exist - after a translation and rotation - a Cartesian coordinate system with origin at x_0 and coordinates $y = (y', y_n)$, $y' = (y_1, \dots, y_{n-1})$, and a C^k-function $h(y')$, $|y'| \leq \tilde{\alpha}$, with $\|h\|_{C^k} \leq K$ such that the neighborhood*

$$U_{\tilde{\alpha},\tilde{\beta},h}(x_0) := \{y \in \mathbb{R}^n : h(y') - \tilde{\beta} < y_n < h(y') + \tilde{\beta},\ |y'| < \tilde{\alpha}\}$$

of x_0 satisfies $U_{\tilde{\alpha},\tilde{\beta},h}(x_0) \cap \partial\Omega = \{y = (y', y_n) \in \mathbb{R}^n : h(y') = y_n,\ |y'| < \tilde{\alpha}\}$ and

$$U^-_{\tilde{\alpha},\tilde{\beta},h}(x_0) := \{y \in \mathbb{R}^n : h(y') - \tilde{\beta} < y_n < h(y'),\ |y'| < \tilde{\alpha}\} = U_{\tilde{\alpha},\tilde{\beta},h}(x_0) \cap \Omega.$$

Notice that the constants $\tilde{\alpha}, \tilde{\beta}, K$ do not depend on $x_0 \in \partial\Omega$. Without loss of generality we choose the new coordinate system $y = (y', y_n)$ in a way that the axes of y' are tangential to $\partial\Omega$ in x_0. Thus we have $h(0) = 0$, $\nabla' h(0) = 0$, and due to a continuity argument for each given constant $M > 0$ we can choose $\tilde{\alpha} > 0$ sufficiently small such that

$$\|h\|_{C^1} \leq M. \tag{2.1}$$

Considering a uniform C^3-domain of type $\tau_\Omega = (\tilde{\alpha}, \tilde{\beta}, K)$ there exists a covering of $\overline{\Omega}$ by open balls $B_j = B_r(x_j)$ where $x_j \in \overline{\Omega}$ and $r = r(\tau_\Omega) > 0$, i.e. $\overline{\Omega} \subset \bigcup_j B_j$, such that with appropriate functions $h_j \in C^3$

$$\overline{B}_j \subset U_{\tilde{\alpha},\tilde{\beta},h_j}(x_j) \quad \text{if} \quad x_j \in \partial\Omega, \qquad \overline{B}_j \subset \Omega \quad \text{if} \quad x_j \in \Omega.$$

The index j runs from 1 to some finite number $N \in \mathbb{N}$ if Ω is bounded and $j \in \mathbb{N}$ for Ω unbounded. The covering $\{B_j\}$ can be established in such a way that no more than some fixed number $N_0 = N_0(\tau_\Omega)$ of the balls have a nonempty intersection. Moreover, there exists a partition of unity $\{\varphi_j\}$, $\varphi_j \in C_0^\infty(\mathbb{R}^n)$, related to this covering, such that

$$0 \leq \varphi_j \leq 1, \quad \operatorname{supp} \varphi_j \subset B_j, \quad \sum_j \varphi_j = 1 \text{ on } \overline{\Omega} \tag{2.2}$$

$$\|\nabla\varphi_j\|_\infty,\ \|\nabla^2\varphi_j\|_\infty,\ \|\nabla^3\varphi_j\|_\infty \leq C = C(\tau_\Omega) \tag{2.3}$$

uniformly in j. Without loss of generality let us assume for $x_j \in \Omega$ that $\operatorname{supp} \varphi_j \subset B_j^-$, where B_j^- denotes the lower half-ball of B_j.

If Ω is unbounded, then it can be expressed as a union of countably many bounded uniform C^3-domains $\Omega_k \subset \Omega$, $k \in \mathbb{N}$, such that $\Omega_k \subset \Omega_{k+1}$ for all $k \in \mathbb{N}$

and $\Omega = \bigcup_{k=1}^{\infty} \Omega_k$. Each of these subdomains is of the same type $(\tilde{\alpha}', \tilde{\beta}', K')$ and we may assume that $\tilde{\alpha} = \tilde{\alpha}', \tilde{\beta} = \tilde{\beta}', K = K'$, i.e. $\tau_{\Omega_k} = \tau_\Omega$. Under Assumption 1.1 to hold in Theorem 1.2 we even suppose that $\Gamma_j := \partial\Omega_j \cap \partial\Omega \neq \emptyset$, $\Gamma_j \subset \Gamma_{j+1}$ for all $j \in \mathbb{N}$, and $\partial\Omega = \bigcup_{j=1}^{\infty} \Gamma_j$.

Using the above-mentioned procedure we are able to localize our problem to domains of the form

$$H' := H'_{\tilde{\alpha},\tilde{\beta},r,h} = \{y \in \mathbb{R}^n : h(y') - \tilde{\beta} < y_n < h(y'),\ |y'| < \tilde{\alpha}\} \cap B_r(0),$$

where we assume $\overline{B_r(0)} \subset \{y \in \mathbb{R}^n : h(y') - \tilde{\beta} < y_n < h(y') + \tilde{\beta},\ |y'| < \tilde{\alpha}\}$, and the function $h \in C_0^3(B_r'(0))$ satisfies $h(0) = 0$, $\nabla' h(0) = 0$, and the smallness asumption $\|h\|_{C^1} \leq M$ is satisfied for some given $M > 0$. Here $B_r'(0)$, $0 < r = r(\tau_\Omega) < \tilde{\alpha}$, denotes an $(n-1)$-dimensional ball.

Furthermore, again thanks to the properties of the support of the φ_j's we can even work in the domains H, with

$$H \subset H' \quad \text{and} \quad \partial H = \partial_1 H \dot{\cup} \partial_2 H, \text{ where } \partial_1 H \subset \{y \in \mathbb{R}^n : y_n = h(y')\}. \tag{2.4}$$

We evidently choose H so that $\operatorname{supp}\varphi_j \cap \overline{H'} \subset \overline{H}$ and $\operatorname{dist}(\operatorname{supp}\varphi_j, \partial_2 H) > 0$, see Figure 2, and that H is a uniform C^3-domain of type τ_Ω. Such a domain H is obviously bounded and (uniformly) star shaped with respect to some ball

$$B_{r'}(x_0) \subset H \quad \text{where } 0 < \tilde{r} = \tilde{r}(\tau_\Omega) \leq r' \leq r = r(\tau_\Omega). \tag{2.5}$$

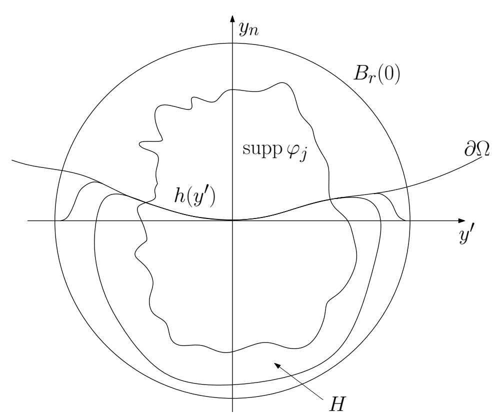

Figure 2: Illustration of a local domain H.

Let us introduce the following spaces of Sobolev type. Given $1 < q, q' < \infty$ such that $1 = \frac{1}{q} + \frac{1}{q'}$, let $W^{1,q}(\Omega)$ and $W_0^{1,q}(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{W^{1,q}}}$ with norm $\|\cdot\|_{W^{1,q}}$

denote the usual L^q-Sobolev spaces. Then

$$\begin{aligned} W^{-1,q}(\Omega) &= \big(W_0^{1,q'}(\Omega)\big)' \\ \hat W^{1,q}(\Omega) &= \big\{u \in L^q_{\mathrm{loc}}(\overline{\Omega}) : \nabla u \in L^q(\Omega)\big\} \\ \hat W^{-1,q}(\Omega) &= \big(\hat W^{1,q'}(\Omega)\big)'. \end{aligned}$$

In the space $\hat W^{1,q}(\Omega)$ we identify two elements differing by a constant and equip it with the norm $\|\nabla \cdot\|_{L^q(\Omega)}$. If Ω is bounded, we may identify

$$\hat W^{1,q}(\Omega) = W^{1,q}(\Omega) \cap L^q_0(\Omega), \quad L^q_0(\Omega) := \Big\{u \in L^q(\Omega) : \int_\Omega u = 0\Big\}.$$

Lemma 2.2 (Poincaré and Friedrichs inequalities)**.** *Let $1 < q < \infty$ and H be a bounded domain as in (2.4). Let either $u \in W_0^{1,q}(H)$ or $u \in W^{1,q}(H)$, $\int_H u = 0$. Then*

$$\|u\|_{L^q(H)} \le C(q, \tau_\Omega)\|\nabla u\|_{L^q(H)}. \tag{2.6}$$

In the case of vector fields $\boldsymbol{u} \in W^{1,q}(H)$ satisfying $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on ∂H a similar estimate holds.

Proof. The result for $u \in W_0^{1,q}(H)$ is well known. For $u \in W^{1,q}(H)$, $\int_H u = 0$, the inequality is known to hold with a constant $C = C(q, \Omega)$, see [17, Theorem II.5.4]. The more concrete dependence $C = C(q, \tau_\Omega)$ uses the condition (2.5) and will be proved in a forthcoming paper [12].

Concerning $\boldsymbol{u} \in W^{1,q}(H)$ satisfying $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on ∂H we apply [17, Exercise II.4.5] where the inequality is proven with a constant $C \le \operatorname{diam} H(|q-2|+n+1)$. Here $\operatorname{diam} H \le 2r(\tau_\Omega)$. □

Lemma 2.3 (Divergence equation, [5], Lemma 2.1 in [8])**.** *Let $1 < q < \infty$.*

(i) *There is a bounded linear operator $R : L^q_0(H) \to W_0^{1,q}(H)$ such that* $\operatorname{div} Rf = f$ *for all $f \in L^q_0(H)$. Moreover, there exists a constant $C = C(q, \tau_\Omega) > 0$ such that*

$$\|Rf\|_{W^{1,q}(H)} \le C\|f\|_{L^q(H)} \quad \text{for all } f \in L^q_0(H).$$

(ii) *There exists $C = C(q, \tau_\Omega) > 0$ such that for every $p \in L^q_0(H)$*

$$\|p\|_{L^q(H)} \le C\|\nabla p\|_{W^{-1,q}(H)} = C \sup\left\{\frac{|\langle p, \operatorname{div} \boldsymbol{v}\rangle|}{\|\nabla \boldsymbol{v}\|_{L^{q'}(H)}} : \boldsymbol{0} \ne \boldsymbol{v} \in W_0^{1,q'}(H)\right\}. \tag{2.7}$$

In the following, we consider the Stokes resolvent system with inhomogeneous divergence and non-zero boundary data. We will apply these results to prove resolvent estimates for a bounded uniformly smooth domain with constant independent of its size. We start with an auxiliary result in bent half spaces.

Given $\omega \in C^3(\mathbb{R}^{n-1})$ we define the bent half space

$$\tilde{H}_\omega = \{y \in \mathbb{R}^n : y_n > \omega(y')\}.$$

For the control of ω we use the definition $\|\nabla'\omega\|_{C^k} = \sum\limits_{1\le|\alpha'|\le k+1} \|\partial_{x'}^{\alpha'}\omega\|_{L^\infty}$, $k = 1, 2$.

Proposition 2.4 ([26])**.** *Let $1 < q < \infty$ and let $0 < \varepsilon < \frac{\pi}{2}$. Let $\lambda \in \mathcal{S}_\varepsilon$, $\boldsymbol{f} \in L^q(\tilde{H}_\omega)$, $g \in \hat{W}^{1,q}(\tilde{H}_\omega) \cap \hat{W}^{-1,q}(\tilde{H}_\omega)$, $\operatorname{supp} g$ compact and $\boldsymbol{h} \in W^{1,q}(\tilde{H}_\omega)$, $\boldsymbol{h} \cdot \boldsymbol{n} = 0$ on $\partial\tilde{H}_\omega$. Then there exist constants $\lambda_0 = \lambda_0(q, \varepsilon, \|\nabla'\omega\|_{C^2}) \ge 1$ and $K_0 = K_0(q, \varepsilon)$, $0 < K_0 \le 1$, with the following property: If $\|\nabla'\omega\|_{L^\infty(\mathbb{R}^{n-1})} \le K_0$ and $|\lambda| \ge \lambda_0$, then the resolvent problem (with $H = \tilde{H}_\omega$)*

$$\begin{aligned} \lambda\boldsymbol{u} - \operatorname{div}\mathbf{S}(\boldsymbol{u}) + \nabla p = \boldsymbol{f} \qquad & \text{in } H \\ \operatorname{div}\boldsymbol{u} = g \qquad & \text{in } H \\ \boldsymbol{u}\cdot\boldsymbol{n} = 0 \qquad & \text{on } \partial H \\ B_{\alpha,\beta}(\boldsymbol{u}) = \boldsymbol{h} \qquad & \text{on } \partial H. \end{aligned} \tag{2.8}$$

admits a unique solution $(\boldsymbol{u}, p) \in W^{2,q}(\tilde{H}_\omega) \times \hat{W}^{1,q}(\tilde{H}_\omega)$ which satisfies the estimate (with $H = \tilde{H}_\omega$)

$$\begin{aligned} &\|\lambda\boldsymbol{u}\|_{L^q(H)} + |\lambda|^{1/2}\|\nabla\boldsymbol{u}\|_{L^q(H)} + \|\nabla^2\boldsymbol{u}\|_{L^q(H)} + \|\nabla p\|_{L^q(H)} \\ &\quad \le C\big(\|\boldsymbol{f}\|_{L^q(H)} + \|\lambda g\|_{\hat{W}^{-1,q}(H)} + \|\lambda^{1/2}g,\ \lambda^{1/2}\boldsymbol{h},\ \nabla g,\ \nabla\boldsymbol{h}\|_{L^q(H)}\big) \end{aligned} \tag{2.9}$$

with some constant $C = C(q, \varepsilon, \|\nabla'\omega\|_{C^2}) > 0$.

Lemma 2.5 (Stokes resolvent system in H as in (2.4))**.** *Let $1 < q < \infty$ and $0 < \varepsilon < \frac{\pi}{2}$. For given $\boldsymbol{f} \in L^q(H)$, $g \in \hat{W}^{1,q}(H) \cap \hat{W}^{-1,q}(H)$ and $\boldsymbol{h} \in W^{1,q}(H)$ with $\boldsymbol{h} \cdot \boldsymbol{n} = 0$ on ∂H let $\boldsymbol{u} \in W^{2,q}(H)$, $p \in \hat{W}^{1,q}(H)$ satisfy the Stokes resolvent system* (2.8) *with $\lambda \in \mathcal{S}_\varepsilon$. Moreover, assume that*

$$\operatorname{supp}\boldsymbol{u} \cup \operatorname{supp} p \subset B_r \quad \text{and} \quad \operatorname{dist}(\operatorname{supp}\boldsymbol{u} \cup \operatorname{supp} p, \partial_2 H) > 0. \tag{2.10}$$

Then there are constants $\lambda_0 = \lambda_0(q, \varepsilon, \tau_\Omega) \ge 1$, $C = C(q, \varepsilon, \tau_\Omega) > 0$ such that for all $|\lambda| \ge \lambda_0$ the estimate (2.9) *holds for (u, p).*

Proof. Due to (2.10) we extend $\boldsymbol{u}$, p by zero so that $(\boldsymbol{u}, \nabla p)$ may be considered as a solution of the Stokes resolvent system in a bent half-space $\tilde{H}_\omega$ and use Proposition 2.4. The smallness assumption on $\nabla'\omega$ is satisfied thanks to (2.1), where we choose $M \le 1$. Moreover, checking all constants appearing in the proof of [26], we see that they depend on the parameters $q, \varepsilon, \tau_\Omega$ only. □

Proposition 2.6 (Stokes operator in bounded domains)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of uniform C^3-type.*

(i) Let $1 < q < \infty$. The Stokes operator $A_q = -P_q\Delta : D(A_q) \to L^q_\sigma(\Omega)$ is a densely defined closed operator on $L^q_\sigma(\Omega)$, and the resolvent problem

$$\lambda \boldsymbol{u} + A_q \boldsymbol{u} = \boldsymbol{f} \in L^q_\sigma(\Omega), \quad \lambda \in \mathcal{S}_\varepsilon, |\lambda| > \delta > 0, \, 0 < \varepsilon < \frac{\pi}{2}, \tag{2.11}$$

has a unique solution $\boldsymbol{u} \in D(A_q)$ satisfying the resolvent estimate

$$\|\lambda \boldsymbol{u}\|_{L^q(\Omega)} + \|A_q \boldsymbol{u}\|_{L^q(\Omega)} \le C\|\boldsymbol{f}\|_{L^q(\Omega)}, \quad C = C(\varepsilon, q, \delta, \Omega) > 0. \tag{2.12}$$

Hence $-A_q$ generates an analytic semigroup $\{e^{-tA_q}\}_{t\ge 0}$ on $L^q_\sigma(\Omega)$. Moreover, the duality relation $A'_q = A_{q'}$ is satisfied.

The resolvent estimate (2.12) *also holds in a neighborhood of $\lambda = 0$ when $\alpha > 0$ or in case that $\alpha = 0$ and Ω is not rotationally symmetric. In particular,*

$$\|\boldsymbol{u}\|_{W^{2,q}(\Omega)} \le C(q, \Omega)\|A_q \boldsymbol{u}\|_{L^q(\Omega)}.$$

In the latter cases the semigroup is even uniformly bounded in $t > 0$.

(ii) For $q = 2$ the resolvent problem (2.11) *has a unique solution $\boldsymbol{u} \in D(A_2)$ satisfying*

$$\|\lambda \boldsymbol{u}\|_{L^2(\Omega)} + \|A_2 \boldsymbol{u}\|_{L^2(\Omega)} \le C\|\boldsymbol{f}\|_{L^2(\Omega)} \tag{2.13}$$

with $C = C(\varepsilon) > 0$ independent of Ω and δ. The operator A_2 is selfadjoint.

Proof. (i) With help of the Helmholtz projection P_q the resolvent estimate (2.12) follows from [26, Theorem 1.3] where a resolvent estimate such as (2.9) is proved. The other properties of A_q are standard results based on the resolvent estimate.

For the proof of the duality relations, let $\boldsymbol{u} \in D(A_q)$ and $\boldsymbol{v} \in D(A_{q'})$. Since $\boldsymbol{u}$ and $\boldsymbol{v}$ are solenoidal, tangential to $\partial\Omega$, and (1.2) holds for u and v,

$$\begin{aligned}
\langle -\Delta \boldsymbol{u}, \boldsymbol{v}\rangle &= \langle -\operatorname{div} \mathbf{S}(\boldsymbol{u}), \boldsymbol{v}\rangle \\
&= \langle \mathbf{S}(\boldsymbol{u}), \nabla \boldsymbol{v}\rangle - \int_{\partial\Omega} \boldsymbol{v} \cdot \mathbf{S}(\boldsymbol{u})\boldsymbol{n} \, \mathrm{d}\sigma \\
&= \langle \nabla \boldsymbol{u}, \mathbf{S}(\boldsymbol{v})\rangle - \int_{\partial\Omega} \boldsymbol{v} \cdot (\mathbf{S}(\boldsymbol{u})\boldsymbol{n})_\tau \, \mathrm{d}\sigma \\
&= \langle \boldsymbol{u}, -\operatorname{div} \mathbf{S}(\boldsymbol{v})\rangle - \int_{\partial\Omega} \big(\boldsymbol{v} \cdot (\mathbf{S}(\boldsymbol{u})\boldsymbol{n})_\tau - \boldsymbol{u} \cdot (\mathbf{S}(\boldsymbol{v})\boldsymbol{n})_\tau\big) \, \mathrm{d}\sigma \\
&= \langle \boldsymbol{u}, -\Delta \boldsymbol{v}\rangle.
\end{aligned}$$

Hence even $\langle A_q \boldsymbol{u}, \boldsymbol{v}\rangle = \langle \boldsymbol{u}, A_{q'} \boldsymbol{v}\rangle$, and $D(A_{q'}) \subset D(A'_q)$. Since e.g. $1 \in \rho(-A_q) \cap \rho(-A_{q'})$, standard arguments prove that $D(A_{q'}) = D(A'_q)$, i.e., $A_{q'} = A'_q$, cf. [13, Proof of Corollary 1.6].

(ii) In view of (i) it remains to prove the estimate (2.13) with $C = C(\varepsilon)$ independent of Ω and δ. Let $\lambda \boldsymbol{u} + A_2 \boldsymbol{u} = \boldsymbol{f}$, $\boldsymbol{f} \in L^2_\sigma(\Omega)$ and $\lambda \in \mathcal{S}_\varepsilon$. Since A_2 is selfadjoint, it holds the estimate

$$\|\boldsymbol{f}\|^2_{L^2(\Omega)} = \|\lambda \boldsymbol{u} + A_2 \boldsymbol{u}\|^2_{L^2(\Omega)} \geq |\operatorname{Im} \lambda|^2 \|\boldsymbol{u}\|^2_{L^2(\Omega)}.$$

For $\operatorname{Re} \lambda \leq 0$ we have $\operatorname{Im} \lambda = \pm|\lambda| \cos \varepsilon'$ and $|\operatorname{Im} \lambda| = |\lambda| \cos \varepsilon' \geq |\lambda| \cos \varepsilon$, $0 \leq \varepsilon' < \varepsilon < \pi/2$. Thus $C(\varepsilon)|\lambda| \|\boldsymbol{u}\|_{L^2} \leq \|\boldsymbol{f}\|_{L^2}$ for $\operatorname{Re} \lambda \leq 0$. Note that we get a similar estimate for $\operatorname{Re} \lambda > 0$ and $\frac{\pi}{2} - \varepsilon < |\arg \lambda| < \frac{\pi}{2}$.

It remains to consider the case $\operatorname{Re} \lambda > 0$ when $|\arg \lambda| \leq \frac{\pi}{2} - \varepsilon$. There holds

$$\langle \boldsymbol{f}, \overline{\boldsymbol{u}} \rangle = \lambda \|\boldsymbol{u}\|^2_{L^2(\Omega)} + \|\nabla \boldsymbol{u}\|^2_{L^2(\Omega)} + \int_\Omega (\nabla \boldsymbol{u})^T \cdot \nabla \overline{\boldsymbol{u}} \, \mathrm{d}x + \frac{\alpha}{\beta} \|\boldsymbol{u}\|^2_{L^2(\partial\Omega)}.$$

Since $\left| \operatorname{Re} \int_\Omega (\nabla \boldsymbol{u})^T \cdot \nabla \overline{\boldsymbol{u}} \, \mathrm{d}x \right| \leq \|\nabla \boldsymbol{u}\|^2_{L^2}$, we obtain the estimate $\operatorname{Re} \lambda \|\boldsymbol{u}\|^2_{L^2} \leq \|\boldsymbol{f}\|_{L^2} \|\boldsymbol{u}\|_{L^2}$ which implies $\|\boldsymbol{f}\|_{L^2} \geq \operatorname{Re} \lambda \|\boldsymbol{u}\|_{L^2} \geq C(\varepsilon)|\lambda| \|\boldsymbol{u}\|_{L^2}$.

Now the proof of (2.13) is complete. □

Lemma 2.7. *Let $1 < q < \infty$ and let $\Omega \subset \mathbb{R}^n$ be a bounded domain of uniform C^3-type. Then there exists a constant $C = C(q, \tau_\Omega) > 0$ such that*

$$C \|\boldsymbol{u}\|_{W^{2,q}(\Omega)} \leq \|\boldsymbol{u}\|_{D(A_q)} = \|\boldsymbol{u}\|_{L^q(\Omega)} + \|A_q \boldsymbol{u}\|_{L^q(\Omega)}, \ \boldsymbol{u} \in D(A_q). \tag{2.14}$$

Proof. Consider a parametrization $\{h_j\}$ of $\partial\Omega$, the covering of Ω with balls $\{B_j\}$ and the corresponding partition of unity $\{\varphi_j\}$, $1 \leq j \leq N$, as described above. We define

$$U'_j := U^-_{\alpha,\beta,h_j}(x_j) \cap B_j \text{ for } x_j \in \partial\Omega, \quad U'_j := B_j \text{ for } x_j \in \Omega, \ 1 \leq j \leq N.$$

Hence we may work in domains $U_j \subset U'_j$, assume that each U_j has the form as the set H in (2.4), (2.5), and apply the results of Lemma 2.5 for H.

Given $\boldsymbol{u} \in D(A_q)$ let $\boldsymbol{f} \in L^q_\sigma(\Omega)$ satisfy $\lambda_0 \boldsymbol{u} + A_q \boldsymbol{u} = \boldsymbol{f}$ where $\lambda_0 \geq 1$ is the constant appearing in Lemma 2.5, i.e.,

$$\lambda_0 \boldsymbol{u} - \Delta \boldsymbol{u} + \nabla p = \boldsymbol{f}, \ \operatorname{div} \boldsymbol{u} = 0 \text{ in } \Omega, \quad \boldsymbol{u} \cdot \boldsymbol{n} = 0, \ B_{\alpha,\beta} \boldsymbol{u} = \boldsymbol{0} \text{ on } \partial\Omega$$

with $\nabla p = (I - P_q)\Delta \boldsymbol{u}$. Our aim is to prove the estimate

$$\|\nabla^2 \boldsymbol{u}\|_{L^q(\Omega)} \leq C(q, \tau_\Omega) \big(\|\boldsymbol{f}\|_{L^q(\Omega)} + \|\boldsymbol{u}\|_{L^q(\Omega)} \big).$$

Let $M_j = M_j(p)$ be the constant such that $p - M_j \in L^q_0(U_j)$, $j = 1, \ldots, N$. Then $\boldsymbol{u}\varphi_j$ and $(p - M_j)\varphi_j$ satisfy the local resolvent system

$$\begin{aligned} \lambda_0 \boldsymbol{u}\varphi_j - \operatorname{div} \mathbf{S}(\boldsymbol{u}\varphi_j) + \nabla(\varphi_j(p - M_j)) &= \boldsymbol{f}\varphi_j + (p - M_j)\nabla\varphi_j \\ -2\nabla \boldsymbol{u} \nabla \varphi_j - \Delta\varphi_j \boldsymbol{u} - (\nabla \boldsymbol{u})^T \nabla\varphi_j - \nabla^2 \varphi_j \boldsymbol{u} & \qquad \text{in } U_j \\ \operatorname{div}(\boldsymbol{u}\varphi_j) &= \nabla\varphi_j \cdot \boldsymbol{u} \qquad \text{in } U_j \\ \varphi_j \boldsymbol{u} \cdot \boldsymbol{n} &= 0 \qquad \text{on } \partial U_j \\ B_{\alpha,\beta}(\boldsymbol{u}\varphi_j) &= \beta \boldsymbol{u}(\nabla\varphi_j \cdot \boldsymbol{n}) =: \boldsymbol{h}_\partial \qquad \text{on } \partial U_j. \end{aligned} \tag{2.15}$$

Since $\nabla\varphi_j \in C_0^\infty(B_j)$ and U_j is a bounded C^3-domain of uniform type τ_Ω, we know that $\boldsymbol{n} \in C^2(\partial U_j)$ and $\sup_j \|\boldsymbol{n}\|_{C^2(\partial U_j)} < \infty$. Hence in the sequel we assume that $\nabla\varphi_j \cdot \boldsymbol{n}$ is extended to a function $\Phi_j \in C^2(\overline{U_j})$ satisfying $\|\Phi_j\|_{C^2} \le C(\tau_\Omega)$. Moreover, as $\boldsymbol{n} = \boldsymbol{n}_{U_j}$ on $\operatorname{supp} \boldsymbol{h}_\partial \subset \partial\Omega \cap B_j$, the condition $\boldsymbol{h}_\partial \cdot \boldsymbol{n} = 0$ is satisfied. Applying the estimate (2.9) to the local resolvent system with $\lambda = \lambda_0$ we gain

$$\begin{aligned}
&\|\nabla^2(\boldsymbol{u}\varphi_j)\|_{L^q(U_j)} + \|\nabla(\varphi_j(p - M_j))\|_{L^q(U_j)} \\
&\qquad \le C\big(\|\boldsymbol{f}\varphi_j\|_{L^q(U_j)} + \|(p - M_j)\nabla\varphi_j\|_{L^q(U_j)} + 2\|\nabla\boldsymbol{u}\nabla\varphi_j\|_{L^q(U_j)} \\
&\qquad\quad + \|\Delta\varphi_j\boldsymbol{u}\|_{L^q(U_j)} + \|(\nabla\boldsymbol{u})^T\nabla\varphi_j\|_{L^q(U_j)} + \|\nabla^2\varphi_j\boldsymbol{u}\|_{L^q(U_j)} \\
&\qquad\quad + \lambda_0\|\nabla\varphi_j \cdot \boldsymbol{u}\|_{\hat{W}^{-1,q}(U_j)} + \lambda_0^{1/2}\|\nabla\varphi_j \cdot \boldsymbol{u}\|_{L^q(U_j)} \\
&\qquad\quad + \|\nabla(\nabla\varphi_j \cdot \boldsymbol{u})\|_{L^q(U_j)} + \lambda_0^{1/2}\|\beta\boldsymbol{u}\Phi_j\|_{L^q(U_j)} + \|\nabla(\beta\boldsymbol{u}\Phi_j)\|_{L^q(U_j)}\big)
\end{aligned} \tag{2.16}$$

where here and in the remaining part of the proof $C = C(q, \tau_\Omega) > 0$. Thanks to (2.3), estimate (2.16) simplifies to the inequality

$$\begin{aligned}
\|\varphi_j\nabla^2\boldsymbol{u}\|_{L^q(U_j)} + \|\varphi_j\nabla p\|_{L^q(U_j)} \ \le\ & C\big(\|\boldsymbol{f}\|_{L^q(U_j)} + \|\boldsymbol{u}\|_{L^q(U_j)} \\
&+ \|\nabla\boldsymbol{u}\|_{L^q(U_j)} + \|p - M_j\|_{L^q(U_j)} + \|\nabla\varphi_j \cdot \boldsymbol{u}\|_{\hat{W}^{-1,q}(U_j)}\big).
\end{aligned} \tag{2.17}$$

In the following we estimate the last two terms on the right-hand side of (2.17). Since $p - M_j \in L_0^q(U_j)$ we have, due to (2.7), $\|p - M_j\|_{L^q(U_j)} \le C\|\nabla p\|_{W^{-1,q}(U_j)}$. Let $\psi \in W_0^{1,q'}(U_j)$. Then, using the Poincaré inequality (2.6) for ψ, we obtain

$$\begin{aligned}
|\langle p, \operatorname{div}\psi\rangle_{U_j}| = |\langle\nabla p, \psi\rangle_{U_j}| &= |\langle\boldsymbol{f}, \psi\rangle_{U_j} - \langle\nabla\boldsymbol{u}, \nabla\psi\rangle_{U_j} - \langle\lambda_0\boldsymbol{u}, \psi\rangle_{U_j}| \\
&\le C\big(\|\boldsymbol{f}\|_{L^q(U_j)} + \|\boldsymbol{u}\|_{L^q(U_j)} + \|\nabla\boldsymbol{u}\|_{L^q(U_j)}\big)\|\nabla\psi\|_{L^{q'}(U_j)}.
\end{aligned} \tag{2.18}$$

Hence we conclude the estimate

$$\|p - M_j\|_{L^q(U_j)} \le C\big(\|\boldsymbol{f}\|_{L^q(U_j)} + \|\boldsymbol{u}\|_{L^q(U_j)} + \|\nabla\boldsymbol{u}\|_{L^q(U_j)}\big). \tag{2.19}$$

Now let $v \in \hat{W}^{1,q'}(U_j)$, i.e. $v \in W^{1,q'}(U_j)$, $\int_{U_j} v = 0$. Using (2.6) again we conclude from

$$|\langle\nabla\varphi_j \cdot \boldsymbol{u}, v\rangle_{U_j}| \le \|\nabla\varphi_j \cdot \boldsymbol{u}\|_{L^q(U_j)}\|v\|_{L^{q'}(U_j)} \le C\|\boldsymbol{u}\|_{L^q(U_j)}\|\nabla v\|_{L^{q'}(U_j)}$$

that

$$\|\nabla\varphi_j \cdot \boldsymbol{u}\|_{\hat{W}^{-1,q}(U_j)} \le C\|\boldsymbol{u}\|_{L^q(U_j)}.$$

Altogether we get from (2.17), when raising all terms to the power q, the local inequalities

$$\|\varphi_j\nabla^2\boldsymbol{u}\|^q_{L^q(U_j)} + \|\varphi_j\nabla p\|^q_{L^q(U_j)} \le C\big(\|\boldsymbol{f}\|^q_{L^q(U_j)} + \|\boldsymbol{u}\|^q_{L^q(U_j)} + \|\nabla\boldsymbol{u}\|^q_{L^q(U_j)}\big). \tag{2.20}$$

Now we take the sum of these inequalities over $j = 1, \dots, N$ and use the crucial property that at most $N_0 = N_0(\tau_\Omega)$ of the neighborhoods $U_1, \dots, U_N$ intersect. We obtain by Hölder's inequality and (2.20) that

$$\begin{aligned}
\|\nabla^2 \boldsymbol{u}\|^q_{L^q(\Omega)} + \|\nabla p\|^q_{L^q(\Omega)} &= \int_\Omega \Big(\Big(\sum_j \varphi_j |\nabla^2 \boldsymbol{u}| \Big)^q + \Big(\sum_j \varphi_j |\nabla p| \Big)^q \Big) \,\mathrm{d}x \\
&\le N_0^{q/q'} \sum_j \big(\|\varphi_j \nabla^2 \boldsymbol{u}\|^q_{L^q(U_j)} + \|\varphi_j \nabla p\|^q_{L^q(U_j)} \big) \\
&\le C N_0^{q/q'} \sum_j \big(\|\boldsymbol{f}\|^q_{L^q(U_j)} + \|\boldsymbol{u}\|^q_{L^q(U_j)} + \|\nabla \boldsymbol{u}\|^q_{L^q(U_j)} \big) \\
&\le C N_0^{q/q'} N_0 \big(\|\boldsymbol{f}\|^q_{L^q(\Omega)} + \|\boldsymbol{u}\|^q_{L^q(\Omega)} + \|\nabla \boldsymbol{u}\|^q_{L^q(\Omega)} \big).
\end{aligned}$$

To estimate the term $\|\nabla \boldsymbol{u}\|_{L^q(\Omega)}$ we use that for $0 < M < 1$ there exists a constant $C_M = C(M, q, \tau_\Omega) > 0$ such that for $u \in W^{2,q}(\Omega)$ the interpolation estimate

$$\|\nabla u\|_{L^q(\Omega)} \le M \|\nabla^2 u\|_{L^q(\Omega)} + C_M \|u\|_{L^q(\Omega)}. \tag{2.21}$$

holds; cf. [5]. Choosing $M \in (0,1)$ sufficiently small we get that

$$\begin{aligned}
\|\nabla^2 \boldsymbol{u}\|_{L^q(\Omega)} &\le C \big(\|\boldsymbol{f}\|_{L^q(\Omega)} + \|\boldsymbol{u}\|_{L^q(\Omega)} \big) \\
&= C \big(\|\lambda_0 \boldsymbol{u} + A_q \boldsymbol{u}\|_{L^q(\Omega)} + \|\boldsymbol{u}\|_{L^q(\Omega)} \big) \le C \|\boldsymbol{u}\|_{D(A_q)}.
\end{aligned}$$

Since $\|\boldsymbol{u}\|_{L^q(\Omega)} \le \|\boldsymbol{u}\|_{D(A_q)}$, (2.21) completes the proof of (2.14). $\square$

3 Proofs

It is our aim to show the Stokes resolvent estimate in the $\tilde{L}^q(\Omega)$-norm with a constant depending on Ω only through its type $\tau_\Omega = (\alpha, \beta, K)$. We start with the bounded domain case, first when $2 < q < \infty$ and then, by duality arguments, for $q \in (1,2)$. Finally, we consider unbounded domains.

3.1 Resolvent estimates in bounded domains

Case $2 \le q < \infty$**.** For $\lambda \in \mathcal{S}_\varepsilon$, $0 < \varepsilon < \frac{\pi}{2}$, we consider the resolvent equation

$$\lambda \boldsymbol{u} + A_q \boldsymbol{u} = \lambda \boldsymbol{u} - \Delta \boldsymbol{u} + \nabla p = \boldsymbol{f}, \quad \boldsymbol{f} \in L^q_\sigma(\Omega). \tag{3.1}$$

For its solution $\boldsymbol{u} \in D(A_q)$ and $\nabla p = (I - P_q)\Delta \boldsymbol{u}$, given by Proposition 2.6 (i) we will prove the estimate

$$\|\lambda \boldsymbol{u}\|_{\tilde{L}^q(\Omega)} + \|\boldsymbol{u}\|_{\tilde{W}^{2,q}(\Omega)} + \|\nabla p\|_{\tilde{L}^q(\Omega)} \le C \|\boldsymbol{f}\|_{\tilde{L}^q(\Omega)}, \tag{3.2}$$

with $|\lambda| \geq \delta > 0$ and a constant $C = C(q, \varepsilon, \delta, \tau_\Omega) > 0$.

We use the localization procedure as in Lemma 2.7, let $M_j = M_j(p)$ be the constant such that $p - M_j \in L^q_0(U_j)$, and obtain the local system (2.15), $j = 1, \ldots, N$, with λ_0 replaced by a general $\lambda \in \mathcal{S}_\varepsilon$. We again assume that $\nabla\varphi_j \cdot \boldsymbol{n}$ is extended to a function $\Phi_j \in C^2(\overline{U_j})$ satisfying $\|\Phi_j\|_{C^2} \leq C(\tau_\Omega)$.

However, in order to apply the resolvent estimate (2.9) we replace λ by $\lambda + \lambda_0' \in \mathcal{S}_\varepsilon$ with $\lambda_0' \geq 0$ sufficiently large such that $|\lambda + \lambda_0'| \geq \lambda_0 \geq 1$ for $|\lambda| \geq \delta$, λ_0 as in (2.9) (e.g., λ_0' can be chosen as $\frac{\lambda_0}{\cos\varepsilon}$). Hence the term $\lambda_0' \boldsymbol{u}\varphi_j$ appears on the right-hand side of $(2.15)_1$.

Then from the resolvent estimate (2.16) in U_j including the lower order terms $|\lambda + \lambda_0'| \|\boldsymbol{u}\varphi_j\|_q$, $|\lambda + \lambda_0'|^{1/2} \|\nabla(\boldsymbol{u}\varphi_j)\|_q$ as in (2.9) we get (with $\|\cdot\|_q = \|\cdot\|_{L^q(U_j)}$)

$$\begin{aligned}
&|\lambda + \lambda_0'| \|\boldsymbol{u}\varphi_j\|_q + |\lambda + \lambda_0'|^{1/2} \|\nabla(\boldsymbol{u}\varphi_j)\|_q + \|\nabla^2(\boldsymbol{u}\varphi_j)\|_q + \|\nabla(\varphi_j(p - M_j))\|_q \\
&\leq C\big(\|\boldsymbol{f}\varphi_j\|_q + \lambda_0' \|\boldsymbol{u}\varphi_j\|_q + \|(p - M_j)\nabla\varphi_j\|_q + 2\|\nabla\boldsymbol{u}\nabla\varphi_j\|_q + \|\Delta\varphi_j \boldsymbol{u}\|_q \\
&\quad + \|(\nabla\boldsymbol{u})^T \nabla\varphi_j\|_q + \|\nabla^2\varphi_j \boldsymbol{u}\|_q + |\lambda + \lambda_0'| \|\nabla\varphi_j \cdot \boldsymbol{u}\|_{\hat{W}^{-1,q}(U_j)} \\
&\quad + \|\nabla(\nabla\varphi_j \cdot \boldsymbol{u})\|_q + |\lambda + \lambda_0'|^{1/2} \big(\|\nabla\varphi_j \cdot \boldsymbol{u}\|_q + \|\boldsymbol{u}\Phi_j\|_q\big) + \|\nabla(\boldsymbol{u}\Phi_j)\|_q\big).
\end{aligned}$$

Thanks to the property (2.3) of φ_j and $|\lambda + \lambda_0'| \geq |\lambda| \cos\varepsilon$ we obtain the estimate

$$\begin{aligned}
\|\lambda\varphi_j \boldsymbol{u}\|_q + |\lambda|^{1/2} \|\varphi_j \nabla\boldsymbol{u}\|_q + \|\varphi_j \nabla^2 \boldsymbol{u}\|_q + \|\varphi_j \nabla p\|_q& \\
\leq C\big(\|\boldsymbol{f}\|_q + \|\boldsymbol{u}\|_q + \|\nabla\boldsymbol{u}\|_q + \|p - M_j\|_q& \\
+ \|\lambda\nabla\varphi_j \cdot \boldsymbol{u}\|_{\hat{W}^{-1,q}(U_j)} + |\lambda|^{1/2} \|\boldsymbol{u}\|_q\big)&
\end{aligned}$$

with $C = C(q, \varepsilon, \delta, \tau_\Omega)$ as λ_0' depends on $\lambda_0 = \lambda_0(q, \varepsilon, \tau_\Omega)$ and ε only.

To estimate the pressure term, we have by (2.7) and (2.6) (cf. (2.18), (2.19))

$$\begin{aligned}
\|p - M_j\|_q \leq C(q, \tau_\Omega)\Bigg(&\|\boldsymbol{f}\|_q + \|\nabla\boldsymbol{u}\|_q \\
&+ \sup\left\{\frac{|\langle\lambda\boldsymbol{u}, \psi\rangle_{U_j}|}{\|\nabla\psi\|_{q'}} : \boldsymbol{0} \neq \psi \in W^{1,q'}_0(U_j)\right\}\Bigg).
\end{aligned}$$

$$\|p - M_j\|_q \leq C(q, \tau_\Omega)\left(\|\boldsymbol{f}\|_q + \|\nabla\boldsymbol{u}\|_q + \sup\left\{\frac{|\langle\lambda\boldsymbol{u}, \psi\rangle_{U_j}|}{\|\nabla\psi\|_{q'}} : \boldsymbol{0} \neq \psi \in W^{1,q'}_0(U_j)\right\}\right).$$

It remains to estimate $|\langle\lambda\boldsymbol{u}, \psi\rangle_{U_j}|$ for $\psi \in W^{1,q'}_0(U_j)$. We use the usual interpolation inequality

$$\|v\|_r \leq \theta(1/\epsilon)^{1/\theta} \|v\|_2 + (1 - \theta)\epsilon^{1/(1-\theta)} \|v\|_q, \tag{3.3}$$

with $r \in [2, q]$, $\theta \in [0, 1]$, $\frac{1}{r} = \frac{\theta}{2} + \frac{1-\theta}{q}$, $\epsilon > 0$. Let $r \in [2, q)$ be such that the embedding $W^{1,q'}(U_j) \hookrightarrow L^{r'}(U_j)$ holds. Then the estimate

$$\begin{aligned}
|\langle\lambda\boldsymbol{u}, \psi\rangle_{U_j}| &\leq \|\lambda\boldsymbol{u}\|_r \|\psi\|_{r'} \\
&\leq \big(C(M, q)\|\lambda\boldsymbol{u}\|_2 + M\|\lambda\boldsymbol{u}\|_q\big)\|\nabla\psi\|_{q'}
\end{aligned}$$

implies that

$$\|p - M_j\|_q \le C\big(\|\boldsymbol{f}\|_q + \|\nabla \boldsymbol{u}\|_q + \|\lambda \boldsymbol{u}\|_2\big) + M\|\lambda \boldsymbol{u}\|_q \tag{3.4}$$

with $C = C(q, M, \tau_\Omega)$ and $M \in (0,1)$ to be chosen later. Similarly, for $v \in \hat{W}^{1,q'}(U_j) = W^{1,q'}(U_j) \cap L_0^{q'}(U_j)$ we have due to (3.3), (2.3) and (2.6)

$$|\langle \nabla\varphi_j \cdot \boldsymbol{u}, v\rangle_{U_j}| \le \big(C(q, M, \tau_\Omega)\|\boldsymbol{u}\|_2 + M\|\boldsymbol{u}\|_q\big)\|\nabla v\|_{q'},$$

which yields

$$\|\lambda \nabla\varphi_j \cdot \boldsymbol{u}\|_{\hat{W}^{-1,q}(U_j)} \le C\|\lambda \boldsymbol{u}\|_2 + M\|\lambda \boldsymbol{u}\|_q \tag{3.5}$$

with $C = C(q, M, \tau_\Omega)$, $M \in (0,1)$. Finally,

$$|\lambda|^{1/2}\|\boldsymbol{u}\|_q \le C(M)\|\boldsymbol{u}\|_q + M\|\lambda \boldsymbol{u}\|_q. \tag{3.6}$$

Considering the estimates (3.4), (3.5) and (3.6) we obtain - with all norms evaluated on U_j, $j = 1, \dots, N$ -

$$\begin{aligned}&\|\lambda\varphi_j \boldsymbol{u}\|_q + |\lambda|^{1/2}\|\varphi_j \nabla \boldsymbol{u}\|_q + \|\varphi_j \nabla^2 \boldsymbol{u}\|_q + \|\varphi_j \nabla p\|_q \\ &\qquad \le C\big(\|\boldsymbol{f}\|_q + \|\boldsymbol{u}\|_q + \|\nabla \boldsymbol{u}\|_q + \|\lambda \boldsymbol{u}\|_2\big) + M\|\lambda \boldsymbol{u}\|_q\end{aligned}$$

where $C = C(q, \varepsilon, \delta, M, \tau_\Omega)$.

We raise this inequality to its qth power, sum over $j = 1, \dots, N$ and use the important property of the number N_0 as well as the reverse Hölder inequality $(\sum_j a_j^q)^{1/q} \le (\sum_j a_j^2)^{1/2}$, $2 \le q$, for $a_j = \|\lambda \boldsymbol{u}\|_{L^2(U_j)}$. Now we obtain that

$$\begin{aligned}&\|\lambda \boldsymbol{u}\|^q_{L^q(\Omega)} + \||\lambda|^{1/2}\nabla \boldsymbol{u}\|^q_{L^q(\Omega)} + \|\nabla^2 \boldsymbol{u}\|^q_{L^q(\Omega)} + \|\nabla p\|^q_{L^q(\Omega)} \\ &\le N_0^{q/q'} \sum_j \big(\|\lambda\varphi_j \boldsymbol{u}\|^q_{L^q(U_j)} + \||\lambda|^{1/2}\varphi_j \nabla \boldsymbol{u}\|^q_{L^q(U_j)} + \|\varphi_j \nabla^2 \boldsymbol{u}\|^q_{L^q(U_j)} + \|\varphi_j \nabla p\|^q_{L^q(U_j)}\big) \\ &\le N_0^{q/q'} \sum_j \big(C\big(\|\boldsymbol{f}\|^q_{L^q(U_j)} + \|\boldsymbol{u}\|^q_{L^q(U_j)} + \|\nabla \boldsymbol{u}\|^q_{L^q(U_j)} + \|\lambda \boldsymbol{u}\|^q_{L^2(U_j)}\big) + M\|\lambda \boldsymbol{u}\|^q_{L^q(U_j)}\big) \\ &\le N_0^q\big(C\|\boldsymbol{f}\|^q_{L^q(\Omega)} + \|\boldsymbol{u}\|^q_{L^q(\Omega)} + \|\nabla \boldsymbol{u}\|^q_{L^q(\Omega)} + M\|\lambda \boldsymbol{u}\|^q_{L^q(\Omega)}\big) + N_0^{q/q'+q/2}C\|\lambda \boldsymbol{u}\|^q_{L^2(\Omega)},\end{aligned}$$

and therefore the estimate

$$\begin{aligned}&\|\lambda \boldsymbol{u}\|_{L^q(\Omega)} + |\lambda|^{1/2}\|\nabla \boldsymbol{u}\|_{L^q(\Omega)} + \|\nabla^2 \boldsymbol{u}\|_{L^q(\Omega)} + \|\nabla p\|_{L^q(\Omega)} \\ &\qquad \le C\big(\|\boldsymbol{f}\|_{L^q(\Omega)} + \|\boldsymbol{u}\|_{L^q(\Omega)} + \|\nabla \boldsymbol{u}\|_{L^q(\Omega)} + \|\lambda \boldsymbol{u}\|_{L^2(\Omega)}\big) + M\|\lambda \boldsymbol{u}\|_{L^q(\Omega)}\end{aligned} \tag{3.7}$$

with $C = C(q, \varepsilon, M, \delta, \tau_\Omega)$, $|\lambda| \ge \delta$.

Employing (2.21) and the estimate

$$\|\boldsymbol{u}\|_{L^q(\Omega)} \le M\|\nabla^2 \boldsymbol{u}\|_{L^q(\Omega)} + C(M, q, \tau_\Omega)\big(\|\boldsymbol{u}\|_{L^2(\Omega)} + \|\nabla^2 \boldsymbol{u}\|_{L^2(\Omega)}\big),$$

cf. [5], [8, Lemma 2.2], we eliminate by absorption arguments for sufficiently small M the terms $\|\boldsymbol{u}\|_{L^q(\Omega)}, \|\nabla \boldsymbol{u}\|_{L^q(\Omega)}, \|\lambda \boldsymbol{u}\|_{L^q(\Omega)}$ from the right-hand side of (3.7). Thus we get that

$$\begin{aligned}\|\lambda \boldsymbol{u}\|_{L^q(\Omega)}+&|\lambda|^{1/2}\|\nabla \boldsymbol{u}\|_{L^q(\Omega)} + \|\nabla^2 \boldsymbol{u}\|_{L^q(\Omega)} + \|\nabla p\|_{L^q(\Omega)} \\ &\leq C\big(\|\boldsymbol{f}\|_{L^q(\Omega)} + \|\lambda \boldsymbol{u}\|_{L^2(\Omega)} + \|\boldsymbol{u}\|_{L^2(\Omega)} + \|\nabla^2 \boldsymbol{u}\|_{L^2(\Omega)}\big).\end{aligned}$$

With the help of (2.14) for $q = 2$ and (2.13) for $|\lambda| \geq \delta$ we know that all L^2-norms above can be estimated by

$$C(q, \varepsilon, \tau_\Omega)\big(\|\boldsymbol{u}\|_{L^2(\Omega)} + \|\boldsymbol{f}\|_{L^2(\Omega)}\big) \leq C(q, \varepsilon, \delta, \tau_\Omega)\|\boldsymbol{f}\|_{L^2(\Omega)}.$$

Thus we have

$$\|\lambda \boldsymbol{u}\|_{L^q(\Omega)}+|\lambda|^{1/2}\|\nabla \boldsymbol{u}\|_{L^q(\Omega)}+\|\nabla^2 \boldsymbol{u}\|_{L^q(\Omega)}+\|\nabla p\|_{L^q(\Omega)} \leq C\big(\|\boldsymbol{f}\|_{L^q(\Omega)}+\|\boldsymbol{f}\|_{L^2(\Omega)}\big),$$

$|\lambda| \geq \delta$, $C = C(q, \varepsilon, \delta, \tau_\Omega)$. Moreover, since $|\lambda| \geq \delta$ we even have

$$\|\lambda \boldsymbol{u}\|_{L^q(\Omega)} + \|\boldsymbol{u}\|_{W^{2,q}(\Omega)} + \|\nabla p\|_{L^q(\Omega)} \leq C(\delta)\big(\|\boldsymbol{f}\|_{L^q(\Omega)} + \|\boldsymbol{f}\|_{L^2(\Omega)}\big). \tag{3.8}$$

Finally adding (3.8) with q replaced by 2 we obtain the estimate (3.2). □

Case $1 < q < 2$. For $\boldsymbol{f} \in L^q_\sigma(\Omega) + L^2_\sigma(\Omega) = L^q_\sigma(\Omega)$ and $\lambda \in \mathcal{S}_\varepsilon$ we consider the resolvent equation (3.1) and its unique solution $\boldsymbol{u} \in D(A_q) + D(A_2) = D(A_q) = D(\tilde{A}_q)$, $\nabla p = (I - \tilde{P}_q)\Delta \boldsymbol{u}$. We show the estimate

$$\|\lambda \boldsymbol{u}\|_{\tilde{L}^q(\Omega)} + \|\boldsymbol{u}\|_{\tilde{W}^{2,q}(\Omega)} + \|\nabla p\|_{\tilde{L}^q(\Omega)} \leq C\|\boldsymbol{f}\|_{\tilde{L}^q(\Omega)}, \tag{3.9}$$

where here and in the rest of this subsection $C = C(q, \varepsilon, \delta, \tau_\Omega) > 0$, $|\lambda| \geq \delta > 0$.

Note the following facts: $\tilde{P}_q = P_q$, $\tilde{A}_q = A_q$ and $\tilde{L}^q_\sigma(\Omega)' = \tilde{L}^{q'}_\sigma(\Omega)$; moreover, $D(A_{q'}) \cap D(A_2) = D(A_{q'}) = D(\tilde{A}_{q'})$ is dense in $\tilde{L}^{q'}_\sigma(\Omega) = L^{q'}_\sigma(\Omega)$ and $\lambda + \tilde{A}_{q'} : D(\tilde{A}_{q'}) \to \tilde{L}^{q'}_\sigma(\Omega)$ is surjective. Hence it follows from the duality $\tilde{A}_{q'} = \tilde{A}'_q$ and the resolvent estimate in $L^{q'}_\sigma(\Omega)$ by setting $\boldsymbol{g} = \lambda \boldsymbol{v} + \tilde{A}_{q'}\boldsymbol{v}$ for $\boldsymbol{v} \in D(\tilde{A}_{q'})$ that

$$\begin{aligned}
\|\boldsymbol{f}\|_{\tilde{L}^q(\Omega)} &\geq \sup\left\{\frac{|\langle \boldsymbol{f}, \boldsymbol{v}\rangle|}{\|\boldsymbol{v}\|_{\tilde{L}^{q'}_\sigma(\Omega)}} : \boldsymbol{0} \neq \boldsymbol{v} \in \tilde{L}^{q'}_\sigma(\Omega)\right\} \\
&= \sup\left\{\frac{|\langle \lambda\boldsymbol{u} + \tilde{A}_q\boldsymbol{u}, \boldsymbol{v}\rangle|}{\|\boldsymbol{v}\|_{\tilde{L}^{q'}_\sigma(\Omega)}} : \boldsymbol{0} \neq \boldsymbol{v} \in D(\tilde{A}_{q'})\right\} \\
&= \sup\left\{\frac{|\langle \boldsymbol{u}, \lambda\boldsymbol{v} + \tilde{A}_{q'}\boldsymbol{v}\rangle|}{\|\boldsymbol{v}\|_{\tilde{L}^{q'}_\sigma(\Omega)}} : \boldsymbol{0} \neq \boldsymbol{v} \in D(\tilde{A}_{q'})\right\} \\
&= \sup\left\{\frac{|\langle \boldsymbol{u}, \boldsymbol{g}\rangle|}{\|(\lambda + \tilde{A}_{q'})^{-1}\boldsymbol{g}\|_{\tilde{L}^{q'}_\sigma(\Omega)}} : \boldsymbol{0} \neq \boldsymbol{g} \in \tilde{L}^{q'}_\sigma(\Omega)\right\} \\
&\geq C^{-1}\sup\left\{\frac{|\langle \lambda\boldsymbol{u}, \boldsymbol{g}\rangle|}{\|\boldsymbol{g}\|_{\tilde{L}^{q'}_\sigma(\Omega)}} : \boldsymbol{0} \neq \boldsymbol{g} \in \tilde{L}^{q'}_\sigma(\Omega)\right\} \\
&= C^{-1}\|\lambda\boldsymbol{u}\|_{\tilde{L}^q(\Omega)}
\end{aligned}$$

for all $\lambda \in \mathcal{S}_\varepsilon$, $|\lambda| \geq \delta > 0$.

In particular, we have $\delta\|\boldsymbol{u}\|_{\tilde{L}^q(\Omega)} \leq C\|\boldsymbol{f}\|_{\tilde{L}^q(\Omega)}$ and, thanks to $\lambda\boldsymbol{u} + A_q\boldsymbol{u} = \boldsymbol{f}$, we even get that

$$\|\lambda\boldsymbol{u}\|_{\tilde{L}^q(\Omega)} + \|\boldsymbol{u}\|_{\tilde{L}^q(\Omega)} + \|A_q\boldsymbol{u}\|_{\tilde{L}^q(\Omega)} \leq C\|\boldsymbol{f}\|_{\tilde{L}^q(\Omega)}. \tag{3.10}$$

From the properties of the sum spaces and from Lemma 2.7 we have that

$$\|\boldsymbol{u}\|_{\tilde{W}^{2,q}(\Omega)} \leq C\|\boldsymbol{u}\|_{D(\tilde{A}_q)} \leq C\big(\|\boldsymbol{u}\|_{\tilde{L}^q(\Omega)} + \|A_q\boldsymbol{u}\|_{\tilde{L}^q(\Omega)}\big)$$

where the last estimate follows from [4, (2.2)]. Then we conclude from (3.10) the resolvent estimate (3.9) and a similar estimate for $\boldsymbol{f} \in \tilde{L}^q(\Omega)$ by applying $\tilde{P}_q = P_q$. □

3.2 Resolvent estimates in unbounded domains

Case $2 \leq q < \infty$**.** Let $\boldsymbol{f} \in \tilde{L}^q_\sigma(\Omega)$, and $\lambda \in \mathcal{S}_\varepsilon$. We set $\boldsymbol{f}_j := \boldsymbol{f}|_{\Omega_j} \in \tilde{L}^q(\Omega_j)$, where $\{\Omega_j\}_{j\in\mathbb{N}}$ is a sequence of bounded smooth subdomains as in Assumption 1.1. Then we consider the solution $(\boldsymbol{u}_j, \nabla p_j) \in \tilde{D}^q(\Omega_j) \times \tilde{L}^q(\Omega_j)$ of the Stokes resolvent system in the domain Ω_j,

$$\begin{aligned}
\lambda\boldsymbol{u}_j - \Delta\boldsymbol{u}_j + \nabla p_j = \boldsymbol{f}_j, \qquad & \operatorname{div}\boldsymbol{u}_j = 0 \qquad \text{in } \Omega_j \\
\boldsymbol{u}_j \cdot \boldsymbol{n}_j = 0, \qquad & B_{\alpha,\beta}(\boldsymbol{u}_j) = \boldsymbol{0} \qquad \text{on } \partial\Omega_j,
\end{aligned} \tag{3.11}$$

satisfying the estimate

$$\|\lambda\boldsymbol{u}_j\|_{\tilde{L}^q(\Omega_j)} + \|\boldsymbol{u}_j\|_{\tilde{W}^{2,q}(\Omega_j)} + \|\nabla p_j\|_{\tilde{L}^q(\Omega_j)} \leq C\|\boldsymbol{f}_j\|_{\tilde{L}^q(\Omega_j)} \leq C\|\boldsymbol{f}\|_{\tilde{L}^q(\Omega)} \tag{3.12}$$

with $|\lambda| \geq \delta > 0$ and $C = C(q, \delta, \varepsilon, \tau_\Omega) > 0$ independent of $j \in \mathbb{N}$.

We denote by $\widetilde{\boldsymbol{f}_j}$ the extension of $\boldsymbol{f}_j$ by zero to the whole of Ω; for vector- or matrix-valued functions this definition is understood componentwise. Obviously, $\widetilde{\boldsymbol{f}_j} \to \boldsymbol{f}$ strongly in $\tilde{L}^q(\Omega)$ as $j \to \infty$.

We consider the pressure first. Then $\nabla p_j \in \tilde{L}^q(\Omega_j)$, and the extensions $\widetilde{\nabla p_j} \in \tilde{L}^q(\Omega)$ satisfy the estimate

$$\|\widetilde{\nabla p_j}\|_{\tilde{L}^q(\Omega)} = \|\nabla p_j\|_{\tilde{L}^q(\Omega_j)} \leq C\|\boldsymbol{f}\|_{\tilde{L}^q(\Omega)}$$

uniformly in $j \in \mathbb{N}$ with the same constant as in (3.12); note that $\widetilde{\nabla p_j}$ is not necessarily a gradient field on $\mathbb{R}^n$. By a reflexivity argument we obtain (at least for a not relabelled subsequence) that $\widetilde{\nabla p_j} \rightharpoonup Q$ weakly in $\tilde{L}^q(\Omega)$. The weak lower semicontinuity of norms implies the estimate $\|Q\|_{\tilde{L}^q(\Omega)} \leq C\|\boldsymbol{f}\|_{\tilde{L}^q(\Omega)}$. Furthermore, a de Rham argument yields the existence of a gradient ∇p such that $Q = \nabla p \in G^q(\Omega) \cap G^2(\Omega)$. Hence ∇p satisfies $\|\nabla p\|_{\tilde{L}^q(\Omega)} \leq \|f\|_{\tilde{L}^q(\Omega)}$.

Concerning the velocity field we have $\boldsymbol{u}_j \in D^q(\Omega_j) \cap D^2(\Omega_j)$ where $D^q(\Omega_j) = L^q_\sigma(\Omega_j) \cap W^{2,q}_B(\Omega_j)$. The componentwise extensions of $\boldsymbol{u}_j, \nabla\boldsymbol{u}_j, \nabla^2\boldsymbol{u}_j$ from Ω to $\mathbb{R}^n$ satisfy $\widetilde{\boldsymbol{u}_j}$, $\widetilde{\nabla\boldsymbol{u}_j}$, $\widetilde{\nabla^2\boldsymbol{u}_j} \in L^q(\Omega) \cap L^2(\Omega)$ and even $\widetilde{\boldsymbol{u}_j} \in \tilde{L}^q_\sigma(\Omega)$. Moreover, (3.12) implies that

$$\|\lambda\widetilde{\boldsymbol{u}_j}\|_{\tilde{L}^q(\Omega)} + \|\widetilde{\boldsymbol{u}_j}\|_{\tilde{L}^q(\Omega)} + \|\widetilde{\nabla\boldsymbol{u}_j}\|_{\tilde{L}^q(\Omega)} + \|\widetilde{\nabla^2\boldsymbol{u}_j}\|_{\tilde{L}^q(\Omega)} \leq C\|\boldsymbol{f}\|_{\tilde{L}^q(\Omega)}.$$

From this uniform estimate we easily get the existence of $\boldsymbol{u} \in \tilde{L}^q(\Omega)$ such that (at least for not relabelled subsequences) $\widetilde{\boldsymbol{u}_j} \rightharpoonup \boldsymbol{u}$, $\widetilde{\nabla\boldsymbol{u}_j} \rightharpoonup \nabla\boldsymbol{u}$, $\widetilde{\nabla^2\boldsymbol{u}_j} \rightharpoonup \nabla^2\boldsymbol{u}$ weakly in $\tilde{L}^q(\Omega)$. Moreover, since $\tilde{L}^q_\sigma(\Omega)$ is a closed and hence weakly closed subspace of $\tilde{L}^q(\Omega)$, we see that $\boldsymbol{u} \in \tilde{L}^q_\sigma(\Omega)$. From these weak convergences it follows that

$$\|\lambda\boldsymbol{u}\|_{\tilde{L}^q(\Omega)} + \|\boldsymbol{u}\|_{\tilde{W}^{2,q}(\Omega)} \leq C\|\boldsymbol{f}\|_{\tilde{L}^q(\Omega)} \tag{3.13}$$

for $|\lambda| \geq \delta > 0$.

To show that $\boldsymbol{u}$ satisfies the Navier boundary condition let us fix $j_0 \in \mathbb{N}$. Thanks to trace theorems and compact embeddings we conclude from the weak convergence $\boldsymbol{u}_j \rightharpoonup \boldsymbol{u}$ in $\tilde{W}^{2,q}(\Omega_{j_0})$ that (at least for a subsequence) $\boldsymbol{u}_j \to \boldsymbol{u}$ and $\nabla\boldsymbol{u}_j \to \nabla\boldsymbol{u}$ in $L^q(\partial\Omega_{j_0})$ as $j \to \infty$. Since $B_{\alpha,\beta}\boldsymbol{u}_j = \boldsymbol{0}$ on $\partial\Omega_j$ we get that $B_{\alpha,\beta}\boldsymbol{u} = \boldsymbol{0}$ on $\partial\Omega_{j_0}$. With the help of Assumption 1.1 we conclude that $B_{\alpha,\beta}\boldsymbol{u} = \boldsymbol{0}$ on $\partial\Omega$.

Finally note that the weak convergences prove that $\boldsymbol{u}, \nabla p$ is a solution of the Stokes resolvent system in Ω satisfying the resolvent estimates (1.6), (1.7).

Case $1 < q < 2$**.** Looking at (3.11) we find pressure gradients $\nabla p_j = \nabla p_j^1 + \nabla p_j^2 \in L^q(\Omega_j) + L^2(\Omega_j) = \tilde{L}^q(\Omega_j)$ such that $\|\nabla p_j\|_{\tilde{L}^q(\Omega_j)} = \|\nabla p_j^1\|_{L^q(\Omega_j)} + \|\nabla p_j^2\|_{L^2(\Omega_j)}$. Their extensions $\widetilde{\nabla p_j^1}$ and $\widetilde{\nabla p_j^2}$ satisfy with the same constant $C = C(q, \delta, \varepsilon, \tau_\Omega)$ as in (3.12) the estimates $\|\widetilde{\nabla p_j^1}\|_{L^q(\Omega)} + \|\widetilde{\nabla p_j^2}\|_{L^2(\Omega)} \le C\|\boldsymbol{f}\|_{\tilde{L}^q(\Omega)}$. This implies that (at least for a subsequence) $\widetilde{\nabla p_j^1} \rightharpoonup Q^1 = \nabla p^1$ weakly in $L^q(\Omega)$ and $\widetilde{\nabla p_j^2} \rightharpoonup Q^2 = \nabla p^2$ weakly in $L^2(\Omega)$, where we also used de Rham's argument. Consequently, we obtain the inequalities $\|\nabla p^1\|_{L^q(\Omega)} + \|\nabla p^2\|_{L^2(\Omega)} \le C\|\boldsymbol{f}\|_{\tilde{L}^q(\Omega)}$, and $\nabla p = \nabla p^1 + \nabla p^2$ satisfies $\nabla p \in \tilde{G}^q(\Omega)$ and

$$\|\nabla p\|_{\tilde{L}^q(\Omega)} \le \|\nabla p^1\|_{L^q(\Omega)} + \|\nabla p^2\|_{L^2(\Omega)} \le C\|\boldsymbol{f}\|_{\tilde{L}^q(\Omega)}. \tag{3.14}$$

Let us now concentrate on the velocity $\boldsymbol{u}_j \in D^q(\Omega_j) + D^2(\Omega_j)$. We choose $\boldsymbol{u}_j = \boldsymbol{u}_j^1 + \boldsymbol{u}_j^2$ such that $\boldsymbol{u}_j^1 \in D^q(\Omega_j)$, $\boldsymbol{u}_j^2 \in D^2(\Omega_j)$ and

$$\|\boldsymbol{u}_j\|_{\tilde{W}^{2,q}(\Omega_j)} = \|\boldsymbol{u}_j^1\|_{W^{2,q}(\Omega_j)} + \|\boldsymbol{u}_j^2\|_{W^{2,2}(\Omega_j)}.$$

For $\boldsymbol{u}_j^1, \boldsymbol{u}_j^2$ we define the extensions $\widetilde{\boldsymbol{u}_j^1} \in L^q_\sigma(\Omega)$, $\widetilde{\nabla \boldsymbol{u}_j^1}$, $\widetilde{\nabla^2 \boldsymbol{u}_j^1} \in L^q(\Omega)$, and $\widetilde{\boldsymbol{u}_j^2} \in L^2_\sigma(\Omega)$, $\widetilde{\nabla \boldsymbol{u}_j^2}$, $\widetilde{\nabla^2 \boldsymbol{u}_j^2} \in L^2(\Omega)$. By estimate (3.12) we have for $|\lambda| \ge \delta > 0$.

$$\begin{aligned}
\|\widetilde{\boldsymbol{u}_j^1} + \widetilde{\boldsymbol{u}_j^2}\|_{\tilde{L}^q(\Omega)} &= \sup\left\{\frac{|\langle \widetilde{\boldsymbol{u}_j^1} + \widetilde{\boldsymbol{u}_j^2}, \varphi\rangle_\Omega|}{\|\varphi\|_{\tilde{L}^{q'}(\Omega)}} : \boldsymbol{0} \ne \varphi \in \tilde{L}^{q'}(\Omega)\right\} \\
&= \sup\left\{\frac{|\langle \boldsymbol{u}_j^1 + \boldsymbol{u}_j^2, \varphi\rangle_{\Omega_j}|}{\|\varphi\|_{\tilde{L}^{q'}(\Omega)}} : \boldsymbol{0} \ne \varphi \in \tilde{L}^{q'}(\Omega)\right\} \\
&\le \sup\left\{\frac{|\langle \boldsymbol{u}_j^1 + \boldsymbol{u}_j^2, \varphi\rangle_{\Omega_j}|}{\|\varphi\|_{\tilde{L}^{q'}(\Omega_j)}} : \boldsymbol{0} \ne \varphi \in \tilde{L}^{q'}(\Omega_j)\right\} \\
&= \|\boldsymbol{u}_j\|_{L^q(\Omega_j)+L^2(\Omega_j)} \ \le \ C|\lambda|^{-1}\|\boldsymbol{f}\|_{\tilde{L}^q(\Omega)}
\end{aligned} \tag{3.15}$$

as well as for $k = 0, 1, 2$

$$\begin{aligned}
\|\widetilde{\nabla^k \boldsymbol{u}_j^1}\|_{L^q(\Omega)} + \|\widetilde{\nabla^k \boldsymbol{u}_j^2}\|_{L^2(\Omega)} &\le \|\boldsymbol{u}_j^1\|_{W^{2,q}(\Omega_j)} + \|\boldsymbol{u}_j^2\|_{W^{2,2}(\Omega_j)} \\
&= \|\boldsymbol{u}_j\|_{\tilde{W}^{2,q}(\Omega_j)} \le C\|\boldsymbol{f}\|_{\tilde{L}^q(\Omega)}.
\end{aligned} \tag{3.16}$$

From the uniform estimate (3.16) we obtain for $k = 0, 1, 2$ the (componentwise) weak convergences $\widetilde{\nabla^k \boldsymbol{u}_j^1} \rightharpoonup \nabla^k \boldsymbol{u}^1$ weakly in $L^q(\Omega)$ and $\widetilde{\nabla^k \boldsymbol{u}_j^2} \rightharpoonup \nabla^k \boldsymbol{u}^2$ weakly in $L^2(\Omega)$. Defining $\boldsymbol{u} := \boldsymbol{u}_1 + \boldsymbol{u}_2$ we get that $\widetilde{\boldsymbol{u}_j} := \widetilde{\boldsymbol{u}_j^1} + \widetilde{\boldsymbol{u}_j^2} \rightharpoonup \boldsymbol{u}^1 + \boldsymbol{u}^2 =: \boldsymbol{u}$ weakly in $L^q_\sigma(\Omega) + L^2_\sigma(\Omega)$ and $\boldsymbol{u} \in (L^q_\sigma(\Omega) \cap W^{2,q}(\Omega)) + (L^2_\sigma(\Omega) \cap W^{2,2}(\Omega))$. Moreover, for $|\lambda| \ge \delta > 0$,

$$\|\lambda \boldsymbol{u}\|_{\tilde{L}^q(\Omega)} + \|\boldsymbol{u}\|_{\tilde{W}^{2,q}(\Omega)} \le C\|\boldsymbol{f}\|_{\tilde{L}^q(\Omega)}. \tag{3.17}$$

Concerning the Navier boundary condition for $\boldsymbol{u} = \boldsymbol{u}_1 + \boldsymbol{u}_2$ we argue as in the previous case. To be more precise, on Ω_{j_0}, $j_0 \in \mathbb{N}$ fixed, we use trace theorems and compactness arguments to conclude that $\boldsymbol{u}_j^1 \to \boldsymbol{u}^1$ in $W^{1,q}(\partial\Omega_{j_0})$ and $\boldsymbol{u}_j^2 \to \boldsymbol{u}^2$ in $W^{1,2}(\partial\Omega_{j_0})$ as $j \to \infty$, $j \geq j_0$. Hence $B_{\alpha,\beta}\boldsymbol{u} = \mathbf{0}$ on $\partial\Omega$. Obviously, the weak convergences also imply that $\boldsymbol{u}, \nabla p$ is a solution of the Stokes resolvent problem satsifying the resolvent estimates.

3.3 Further proofs

The next step is the proof of uniqueness based on the duality relation

$$\langle \tilde{A}_q \boldsymbol{u}, \boldsymbol{v} \rangle = \langle \boldsymbol{u}, \tilde{A}_{q'} \boldsymbol{v} \rangle \quad \text{for all } \boldsymbol{u} \in D(\tilde{A}_q), \boldsymbol{v} \in D(\tilde{A}_{q'}). \tag{3.18}$$

Indeed, let us assume (3.18) and that $\boldsymbol{u} \in D(\tilde{A}_q)$ and $\lambda \in \mathcal{S}_\varepsilon$ satisfy the resolvent equation $\lambda\boldsymbol{u} + \tilde{A}_q\boldsymbol{u} = \mathbf{0}$. Then for any $\boldsymbol{f} \in \tilde{L}^{q'}_\sigma(\Omega)$ there is a solution $\boldsymbol{v} \in D(\tilde{A}_{q'})$ (constructed as above) of $\lambda\boldsymbol{v} + \tilde{A}_{q'}\boldsymbol{v} = \boldsymbol{f}$. So we have

$$\langle \boldsymbol{u}, \boldsymbol{f} \rangle = \langle \boldsymbol{u}, \lambda\boldsymbol{v} + \tilde{A}_{q'}\boldsymbol{v} \rangle = \langle \lambda\boldsymbol{u} + \tilde{A}_q\boldsymbol{u}, \boldsymbol{v} \rangle = 0$$

for all $\boldsymbol{f} \in \tilde{L}^{q'}_\sigma(\Omega)$ and thus $\boldsymbol{u} = \mathbf{0}$.

Hence it remains to show (3.18). Actually, since $\tilde{P}_q' = \tilde{P}_{q'}$, we only have to verify the relation $\langle \Delta\boldsymbol{u}, \boldsymbol{v} \rangle = \langle \boldsymbol{u}, \Delta\boldsymbol{v} \rangle$ for all $\boldsymbol{u} \in D(\tilde{A}_q)$, $\boldsymbol{v} \in D(\tilde{A}_{q'})$. Moreover, it is enough to show this relation for functions from $D^q(\Omega)$ and $D^{q'}(\Omega)$. Indeed, for $\boldsymbol{u} \in D(\tilde{A}_q) = D^q(\Omega) \cap D^2(\Omega)$, $q \geq 2$, and for $\boldsymbol{v} \in D(\tilde{A}_{q'})$, $\boldsymbol{v} = \boldsymbol{v}_1 + \boldsymbol{v}_2 \in D^{q'}(\Omega) + D^2(\Omega)$, $q' < 2$, we obtain

$$\langle \Delta\boldsymbol{u}, \boldsymbol{v} \rangle = \langle \Delta\boldsymbol{u}, \boldsymbol{v}_1 \rangle + \langle \Delta\boldsymbol{u}, \boldsymbol{v}_2 \rangle = \langle \boldsymbol{u}, \Delta\boldsymbol{v}_1 \rangle + \langle \boldsymbol{u}, \Delta\boldsymbol{v}_2 \rangle = \langle \boldsymbol{u}, \Delta\boldsymbol{v} \rangle.$$

Let $\boldsymbol{u} \in D^q(\Omega)$, $\boldsymbol{v} \in D^{q'}(\Omega)$. Since the domain Ω is in general unbounded we cannot apply integration by parts directly. Therefore, let $\psi \in C_0^\infty(B_2)$ be a cut-off function such that $\psi = 1$ in $\overline{B_1}$, and let $\psi_j(x) := \psi\big(\frac{x}{j}\big)$, $j \in \mathbb{N}$. Then there holds $\psi_j = 1$ in $\overline{B_j}$, $\operatorname{supp}\psi_j \subset B_{2j}$ and $\|\nabla^k\psi_j\|_\infty \leq \frac{c}{j^k}$, $k = 0, 1, 2$; here $B_r \subset \mathbb{R}^n$ denotes the ball with center 0 and radius $r > 0$.

Now, for $\boldsymbol{u} \in W^{2,q}(\Omega)$, $\boldsymbol{v} \in W^{2,q'}(\Omega)$, by Lebesgue's theorem on dominated convergence $\int_\Omega \Delta\boldsymbol{u} \cdot \boldsymbol{v} \,\mathrm{d}x = \lim_{j\to\infty} \int_\Omega \Delta(\boldsymbol{u}\psi_j) \cdot \boldsymbol{v}\psi_j \,\mathrm{d}x$. Here, after lengthy calculations and two integrations by parts, we get for each $j \in \mathbb{N}$ that

$$\begin{aligned}\int_\Omega \Delta(\boldsymbol{u}\psi_j) \cdot \boldsymbol{v}\psi_j \,\mathrm{d}x &= \int_{\Omega\cap B_{2j}} \operatorname{div}\mathbf{S}(\boldsymbol{u}\psi_j) \cdot \boldsymbol{v}\psi_j \,\mathrm{d}x - \int_{\Omega\cap B_{2j}} \nabla\operatorname{div}(\boldsymbol{u}\psi_j) \cdot \boldsymbol{v}\psi_j \,\mathrm{d}x \\ &= \int_\Omega \psi_j\boldsymbol{u} \cdot \Delta(\boldsymbol{v}\psi_j) \,\mathrm{d}x + J_j^1 - J_j^2 + J_j^3 - J_j^4\end{aligned}$$

with the terms $J_j^1 - J_j^2 + J_j^3 - J_j^4$ standing for the boundary integrals

$$\int_{\partial\Omega\cap B_{2j}} \mathsf{S}(\boldsymbol{u}\psi_j)\boldsymbol{n}\cdot\boldsymbol{v}\psi_j \,\mathrm{d}\sigma - \int_{\partial\Omega\cap B_{2j}} \psi_j\boldsymbol{u}\cdot\mathsf{S}(\boldsymbol{v}\psi_j)\boldsymbol{n}\,\mathrm{d}\sigma$$
$$+ \int_{\partial\Omega\cap B_{2j}} \psi_j\boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{div}(\boldsymbol{v}\psi_j)\,\mathrm{d}\sigma - \int_{\partial\Omega\cap B_{2j}} \mathrm{div}(\boldsymbol{u}\psi_j)\psi_j\boldsymbol{v}\cdot\boldsymbol{n}\,\mathrm{d}\sigma.$$

As $\boldsymbol{u}\in D^q(\Omega)$, $\boldsymbol{v}\in D^{q'}(\Omega)$ and $\partial\Omega\cap B_{2j}\subset B_{2j}$ we have even in the trace sense that $\boldsymbol{u}\cdot\boldsymbol{n} = \boldsymbol{v}\cdot\boldsymbol{n} = 0$ and $B_{\alpha,\beta}(\boldsymbol{u}) = B_{\alpha,\beta}(\boldsymbol{v}) = \boldsymbol{0}$ on $\partial\Omega\cap B_{2j}$. Consequently, $J_j^3 = J_j^4 = 0$. Moreover, it holds the identity

$$\mathsf{S}(\boldsymbol{u}\psi_j)\boldsymbol{n}\cdot\boldsymbol{v}\psi_j = \mathsf{S}(\boldsymbol{u})\boldsymbol{n}\cdot\boldsymbol{v}\psi_j^2 + \boldsymbol{u}\cdot\boldsymbol{n}(\nabla\psi_j\cdot\boldsymbol{v})\psi_j + \boldsymbol{u}\cdot\boldsymbol{v}(\nabla\psi_j\cdot\boldsymbol{n})\psi_j,$$

and a similar one holds for $\mathsf{S}(\boldsymbol{v}\psi_j)\boldsymbol{n}\cdot\boldsymbol{u}\psi_j$. Here the boundary integrals over the terms involving $\boldsymbol{u}\cdot\boldsymbol{n}$ and $\boldsymbol{v}\cdot\boldsymbol{n}$ vanish. Next, the difference of the boundary integrals over the right-hand side terms (involving $\boldsymbol{u}\cdot\boldsymbol{v}$) vanishes. Finally, $\mathsf{S}(\boldsymbol{u})\boldsymbol{n}\cdot\boldsymbol{v}\psi_j^2 = (\mathsf{S}(\boldsymbol{u})\boldsymbol{n})_\tau\cdot\boldsymbol{v}\psi_j^2$, so that due to the Navier boundary condition

$$\int_{\partial\Omega\cap B_{2j}} \mathsf{S}(\boldsymbol{u})\boldsymbol{n}\cdot\boldsymbol{v}\psi_j^2\,\mathrm{d}\sigma = -\frac{\alpha}{\beta}\int_{\partial\Omega\cap B_{2j}} \boldsymbol{u}\cdot\boldsymbol{v}\psi_j^2\,\mathrm{d}\sigma.$$

Now we conclude that $J_j^1 - J_j^2 = 0$.

Altogether, by the dominated convergence theorem, we arrive at the duality relation $\langle\Delta\boldsymbol{u},\boldsymbol{v}\rangle = \langle\boldsymbol{u},\Delta\boldsymbol{v}\rangle$. Now the proof of uniqueness is complete.

Finally, we prove parts (iii) and (iv) of Theorem 1.2. Clearly, $\tilde{A}_q$ is densely defined since $C_{0,\sigma}^\infty(\Omega)$ is contained in $D(\tilde{A}_q)$.

By part (iii) the resolvent $(\lambda+\tilde{A}_q)^{-1}$ is well defined for all $\lambda\in\mathcal{S}_\varepsilon$ and satisfies $\|(\lambda+\tilde{A}_q)^{-1}\| \le \frac{C}{|\lambda|}$ for $\lambda\in\mathcal{S}_\varepsilon$ with $|\lambda|\ge\delta>0$. Then standard semigroup theory implies that $-\tilde{A}_q$ generates an analytic semigroup satisfying the estimate $\|e^{-t\tilde{A}_q}\|\le Ce^{\delta t}$, $t\ge 0$, $\delta>0$, where $C = C(q,\delta,\tau_\Omega)>0$.

To prove that $\tilde{A}_q' = \tilde{A}_{q'}$, in view of (3.18) it suffices to show that $D(\tilde{A}_q')\subset D(\tilde{A}_{q'})$. However, since $\rho(\tilde{A}_q)\cap\rho(\tilde{A}_{q'})\neq\emptyset$, this inclusion is obvious.

Finally, statement (iv) is an easy consequence of the previous assertions. Theorem 1.2 is thus completely proven. □

Proof of Corollary 1.3 Let $\boldsymbol{u}\in D(\tilde{A}_q)$. Then the statement of the corollary follows from the chain of inequalities

$$\begin{aligned}\|\boldsymbol{u}\|_{\tilde{L}^q(\Omega)} + \|\tilde{A}_q\boldsymbol{u}\|_{\tilde{L}^q(\Omega)} &\le C\|\boldsymbol{u}\|_{\tilde{W}^{2,q}(\Omega)} \le C\|(1+\tilde{A}_q)\boldsymbol{u}\|_{\tilde{L}^q(\Omega)}\\ &\le C\big(\|\boldsymbol{u}\|_{\tilde{L}^q(\Omega)} + \|(1+\tilde{A}_q)\boldsymbol{u}\|_{\tilde{L}^q(\Omega)}\big)\\ &\le C\big(\|\boldsymbol{u}\|_{\tilde{L}^q(\Omega)} + \|\tilde{A}_q\boldsymbol{u}\|_{\tilde{L}^q(\Omega)}\big)\end{aligned}$$

with a constant $C = C(q, \tau_\Omega)$. The second estimate is due to the resolvent estimate (1.6) with $\lambda = 1$. $\square$

For the proof of the $\tilde{L}^r$-$\tilde{L}^q$-estimates, see Corollary 1.4, we need Sobolev embeddings for spaces of type $\tilde{W}^{k,q}$.

Lemma 3.1 (Sobolev embeddings, [24]). *Let $m \in \mathbb{N}$, $1 \le q < \infty$ and let $\Omega \subset \mathbb{R}^n$ be a uniform C^k-domain, $k \ge 1$. Then the embedding*

$$\tilde{W}^{m,q}(\Omega) \hookrightarrow \tilde{L}^r(\Omega)$$

holds either if $mq < n$ and $q \le r \le \frac{nq}{n-mq}$, or if $mq = n$ and $q \le r < \infty$, or if $mq > n$ and $q \le r \le \infty$. The embedding constant depends on m, q, r, n and τ_Ω.

Proof of Corollary 1.4. Let $\boldsymbol{u} \in \tilde{L}^q_\sigma(\Omega)$. Notice that in both cases the estimate holds for $q = r$ with $\gamma = 0$. Thus let $1 < q < r < \infty$ and, moreover, let $s \in (r, \infty)$ be such that $\frac{1}{r} = \frac{1-\gamma}{q} + \frac{\gamma}{s}$ with $0 < \gamma < 1$ and $\tilde{W}^{2,q}(\Omega) \hookrightarrow \tilde{L}^s(\Omega)$. Then by the interpolation inequality $\|v\|_{\tilde{L}^r} \le \|v\|_{\tilde{L}^q}^{1-\gamma} \|v\|_{\tilde{L}^s}^{\gamma}$ based on the complex interpolation $[\tilde{L}^q(\Omega), \tilde{L}^s(\Omega)]_\gamma = \tilde{L}^r(\Omega)$, see [24, Corollary 1], Lemma 3.1 and Corollary 1.3 we obtain

$$\begin{aligned}
\|e^{-t\tilde{A}_q}\boldsymbol{u}\|_{\tilde{L}^r(\Omega)} &\le \|e^{-t\tilde{A}_q}\boldsymbol{u}\|_{\tilde{L}^q(\Omega)}^{1-\gamma} \|e^{-t\tilde{A}_q}\boldsymbol{u}\|_{\tilde{W}^{2,q}(\Omega)}^{\gamma} \\
&\le Ce^{\delta t(1-\gamma)} \|\boldsymbol{u}\|_{\tilde{L}^q(\Omega)}^{1-\gamma} \|(1+\tilde{A}_q)e^{-t\tilde{A}_q}\boldsymbol{u}\|_{\tilde{L}^q(\Omega)}^{\gamma} \\
&\le Ce^{\delta t(1-\gamma)} \|\boldsymbol{u}\|_{\tilde{L}^q(\Omega)}^{1-\gamma} \Big(\frac{1+t}{t}\Big)^{\gamma} e^{\delta t\gamma} \|\boldsymbol{u}\|_{\tilde{L}^q(\Omega)}^{\gamma} \\
&= C\Big(\frac{1+t}{t}\Big)^{\gamma} e^{\delta t} \|\boldsymbol{u}\|_{\tilde{L}^q(\Omega)},
\end{aligned}$$

where $C = C(q, \tau_\Omega) > 0$. It remains to discuss the cases (i) and (ii).

For $q < \frac{n}{2}$ the embedding $\tilde{W}^{2,q}(\Omega) \hookrightarrow \tilde{L}^s(\Omega)$ holds for $q \le s \le \frac{nq}{n-2q}$. Thus choosing $s = \frac{nq}{n-2q}$, i.e. $\frac{1}{s} = \frac{1}{q} - \frac{2}{n}$, we obtain the desired estimate for $q < r < \frac{nq}{n-2q}$ with $0 < \gamma = \frac{n}{2}\big(\frac{1}{q} - \frac{1}{r}\big) < 1$. If $r = \frac{nq}{n-2q}$ then $\tilde{W}^{2,q}(\Omega) \hookrightarrow \tilde{L}^r(\Omega)$ and the estimate holds with $\gamma = 1$.

Now let $\frac{n}{2} \le q < r$. Then $\tilde{W}^{2,q}(\Omega) \hookrightarrow \tilde{L}^s(\Omega)$ for all $q \le s < \infty$. If, in particular, $\tilde{W}^{2,q}(\Omega) \hookrightarrow \tilde{L}^r(\Omega)$ then the estimate holds with $\gamma = 1$. Otherwise, with some $s \in (r, \infty)$, we have $0 < \gamma < 1$. Choosing s as large as possible and considering the identity $\frac{1}{r} = \frac{1-\gamma}{q} + \frac{\gamma}{s}$ for such s we get the condition $\gamma > 1 - \frac{q}{r}$. $\square$

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