STRONG SOLUTIONS OF THE BOUSSINESQ SYSTEM IN EXTERIOR DOMAINS

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ABSTRACT. We consider the instationary Boussinesq equations in a smooth three-dimensional exterior domain. A strong solution is a weak solution such that the velocity field additionally satisfies Serrin's condition. The crucial point in this concept of a strong solution is the fact that we have required no additional integrability condition for the temperature. We present a sufficient criterion for the existence of such a strong solution. Further we will characterize the class of initial values that allow the existence of such a strong solution in a sufficiently small interval. Finally we will obtain an uniqueness criterion for weak solutions of the Boussinesq equations which is based on the identification of a weak solution with a strong solution.

1. Introduction and main results

Let $\Omega \subseteq \mathbb{R}^3$ be a domain, and let $[0,T[\,, 0 < T \leq \infty$, be a time interval. We consider the Boussinesq equations

$$\begin{aligned} u_t - \Delta u + u \cdot \nabla u + \nabla p &= \theta g + f_1 && \text{in }]0,T[\times\Omega\,,\\ \operatorname{div} u &= 0 && \text{in }]0,T[\times\Omega\,,\\ \theta_t - \Delta\theta + u\cdot\nabla\theta &= f_2 && \text{in }]0,T[\times\Omega\,,\\ u = 0\,,\quad \theta &= 0 && \text{on }]0,T[\times\partial\Omega\,,\\ u = u_0\,,\quad \theta &= \theta_0 && \text{at } t=0\,, \end{aligned} \tag{1.1}$$

where $u : [0,T[\times\Omega \to \mathbb{R}^3$ denotes the velocity of the fluid, $\theta : [0,T[\times\Omega \to \mathbb{R}$ the difference of the temperature to a fixed reference temperature and $p : [0,T[\times\Omega \to \mathbb{R}$ denotes the pressure. We consider the following data: $f_1 : [0,T[\to \mathbb{R}^3$ is the external force per unit mass, $f_2 : [0,T[\times\Omega \to \mathbb{R}$ the external thermal radiation per unit mass, $u_0 : \Omega \to \mathbb{R}^3, \theta_0 : \Omega \to \mathbb{R}$ are the initial values and $g : [0,T[\times\Omega \to \mathbb{R}^3$ denotes the gravitational force. We remark that in most applications the gravitational force is a constant vector field in time. The Boussinesq equations constitute a model of motion of a viscous, incompressible buoyancy-driven fluid flow coupled with heat convection. The Boussinesq system has been investigated by many researchers, see e.g. [1, 2, 4, 14, 15, 18, 23, 24, 27] and papers cited there. We introduce the following space of test functions:

$$C_0^\infty([0,T[; C_{0,\sigma}^\infty(\Omega)) := \{\, w|_{[0,T[\times\Omega}\ ; w \in C_0^\infty(]-1,T[\times\Omega)\ ; \operatorname{div} w = 0\,\}.$$

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Motivated by the concept of a weak solution (in the sense of Leray-Hopf) of the instationary Navier-Stokes equations we start with the following

Definition 1.1. Let $\Omega \subseteq \mathbb{R}^3$ be a general domain, let $0 < T \leq \infty$, let $g :]0,T[\times\Omega \to \mathbb{R}^3$ be a measurable vector field, and $f_1, f_2 \in L^1_{\mathrm{loc}}([0,T[; L^2(\Omega))$. Further assume $u_0 \in L^2_\sigma(\Omega)$ and $\theta_0 \in L^2(\Omega)$. A pair

$$u \in L^\infty_{\mathrm{loc}}([0,T[; L^2_\sigma(\Omega)) \cap L^2_{\mathrm{loc}}([0,T[; W^{1,2}_{0,\sigma}(\Omega))\,, \tag{1.2}$$

$$\theta \in L^\infty_{\mathrm{loc}}([0,T[; L^2(\Omega)) \cap L^2_{\mathrm{loc}}([0,T[; H^1_0(\Omega))\,, \tag{1.3}$$

with $\theta g \in L^1_{\mathrm{loc}}([0,T[; L^2(\Omega))$ is called a weak solution of the Boussinesq system (1.1) if the following identities are satisfied for all $w \in C^\infty_0([0,T[; C^\infty_{0,\sigma}(\Omega))$ and all $\phi \in C^\infty_0([0,T[\times\Omega)$:

$$\begin{aligned}
&-\langle u, w_t\rangle_{\Omega,T} + \langle \nabla u, \nabla w\rangle_{\Omega,T} + \langle u\cdot\nabla u, w\rangle_{\Omega,T} \\
&\quad = \langle \theta g, w\rangle_{\Omega,T} + \langle f_1, w\rangle_{\Omega,T} + \langle u_0, w\rangle_\Omega\,, \\
&-\langle \theta, \phi_t\rangle_{\Omega,T} + \langle \nabla\theta, \nabla\phi\rangle_{\Omega,T} + \langle u\cdot\nabla\theta, \phi\rangle_{\Omega,T} = \langle f_2, \phi\rangle_{\Omega,T} + \langle \theta_0, \phi(0)\rangle_\Omega.
\end{aligned}$$

In the identities above $\langle\cdot,\cdot\rangle_\Omega\,, \langle\cdot,\cdot\rangle_{\Omega,T}$ denotes the usual L^2-scalar product in Ω and in $]0,T[\times\Omega$, respectively.

Given a weak solution (u,θ) of (1.1) we may assume, after a possible redefinition on a set of vanishing Lebesgue measure, that $u : [0,T[\to L^2_\sigma(\Omega)$ and $\theta : [0,T[\to L^2(\Omega)$ are both weakly continuous functions and the initial values u_0, θ_0 are attained in the following sense:

$$\lim_{t\searrow 0}\langle u(t), w\rangle_\Omega = \langle u_0, w\rangle_\Omega\,, \quad \lim_{t\searrow 0}\langle \theta(t), \phi\rangle_\Omega = \langle \theta_0, \phi\rangle_\Omega$$

for all $w \in L^2_\sigma(\Omega)$ and all $\phi \in L^2(\Omega)$. If $g \in L^\infty(]0,T[\times\Omega)$ we can show with the Faedo-Galerkin method analogously as in [23, Theorem 1] that there exists a weak solution of (1.1) in $[0,T[\times\Omega$. Moreover, there exists a distribution p, called an associated pressure, such that

$$u_t - \Delta u + u\cdot\nabla u + \nabla p = \theta g + f_1$$

holds in the sense of distributions in $]0,T[\times\Omega$, see [25, V.1.7]. For exponents s,q with $1 < q,\ s < \infty$ we define the *Serrin number* by

$$\mathcal{S}(s,q) := \frac{2}{s} + \frac{3}{q}.$$

Up to now, uniqueness and regularity of weak solutions of the three-dimensional Boussinesq equations is an unsolved problem. To motivate Definition 1.2 below let us consider the well known instationary Navier-Stokes equations. By definition, u is called a strong solution of the Navier-Stokes equations if u is a weak solution satisfying additionally *Serrin's condition* $u \in L^s(0,T; L^q(\Omega))$ where $1 < s,q < \infty$ with $\mathcal{S}(s,q) = 1$. It is well known that such a strong solution is uniquely determined and regular (see [25, Section V.4]). Motivated by this result we give the following

Definition 1.2. Consider data as in Definition 1.1. We say that (u,θ) is a *strong solution* of (1.1) if (u,θ) is a weak solution of (1.1) and if there are exponents $1 < s,q < \infty$ with $\mathcal{S}(s,q) = 1$ such that $u \in L^s(0,T; L^q(\Omega))$.

The crucial point in the definition above is the fact that we have required no additional integrability condition for θ. The paper [19] deals with strong solutions of the Boussinesq system in a smooth bounded domain. It follows from [19, Theorem 1.6] that strong solutions of the Boussinesq equations are smooth if the data are smooth. Further (see Theorem 1.7 below) strong solutions are uniquely determined. The goal of the present paper is to investigate existence of strong solutions of (1.1) in an exterior domain and to apply these results to obtain an uniqueness criterion for weak solutions which is based on the identification of weak solutions with strong solutions. Our first main result is a sufficient criterion for the existence of a strong solution of (1.1). We denote by $\Delta = \Delta_2\,, A = A_2$ the Laplace and Stokes operator, respectively. For further information about these operators we refer to the preliminaries.

Theorem 1.3. *Let $\Omega \subseteq \mathbb{R}^3$ be an exterior domain with $\partial\Omega \in C^{2,1}$, let $0 < T < \infty$ and $1 < s, q < \infty$ with $\mathcal{S}(s,q) = 1$. Consider $g \in L^{8/5}(0,T;L^4(\Omega)) \cap L^{\mu}(0,T;L^p(\Omega))$ where $1 < \mu, p < \infty$ satisfy $\mathcal{S}(\mu,p) = \frac{3}{2}$ and $\frac{1}{12} > \frac{1}{p} - \frac{1}{q}$. Let $f_1 \in L^{s_*}(0,T;L^{q_*}(\Omega)) \cap L^1(0,T;L^2(\Omega))$ where $1 < s_*, q_* < \infty$ with $\mathcal{S}(s_*,q_*) = 3$ satisfy $\frac{1}{3} + \frac{1}{q} \geq \frac{1}{q_*} \geq \frac{1}{q}$. Further assume $f_2 \in L^1(0,T;L^2(\Omega))$ and $u_0 \in L^2_\sigma(\Omega)\,, \theta_0 \in L^2(\Omega)$. Introduce*

$$E_1(t) := e^{-tA}u_0 + \int_0^t e^{-(t-\tau)A} P f_1(\tau)\, d\tau\,, \quad t \in [0,T[\,, \tag{1.4}$$

$$E_2(t) := e^{t\Delta}\theta_0 + \int_0^t e^{(t-\tau)\Delta} f_2(\tau)\, d\tau\,, \quad t \in [0,T[. \tag{1.5}$$

Then there exists a constant $\epsilon_* = \epsilon_*(\Omega, q, p) > 0$ such that if the conditions

$$\|E_1\|_{q,s;T} + \|E_1\|_{2,2;T} + \|E_2\|_{4,\frac{8}{3};T} \leq \frac{\epsilon_*}{1 + T^{\frac{1}{2}+\frac{3}{2q}}}\,, \tag{1.6}$$

$$\|g\|_{p,\mu;T} + T^{1/2}\|g\|_{4,\frac{8}{5};T} \leq \epsilon_* \tag{1.7}$$

are fulfilled, then there exists a strong solution (u,θ) of the Boussinesq equations (1.1).

Remark. Since for all $q > 3$, there exists $2 < p < 12$ satisfying $\frac{1}{12} > \frac{1}{p} - \frac{1}{q}$, the requirements on μ, p can be fulfilled for all possible exponents s, q.

For a proof of this result we refer to Section 4. The idea is to construct (u,θ) as a solution of a suitable non-linear problem, see (3.18) below. Afterwards we have to show that (u,θ) fulfils (1.2), (1.3). Due to the missing imbedding $L^q(\Omega) \hookrightarrow L^p(\Omega)\,, q > p$, in an unbounded domain and also more restrictive imbedding properties of fractional powers of the Stokes and Laplace operator in an exterior domain we have to modify the proof of [19, Theorem 1.3] where the corresponding result is shown for a bounded domain. Especially due to the application of (2.6) the existence result above can only be shown for $0 < T < \infty$.

The next corollary presents a smallness condition on $u_0, \theta_0, g, f_1, f_2$ implying the existence of a strong solution (u,θ) of (1.1) with $u \in L^s(0,T;L^q(\Omega))$ where $1 < s, q$ with $\mathcal{S}(s,q) = 1$.

Corollary 1.4. *Consider data as in Theorem 1.3. There exists a constant* $\epsilon_* = \epsilon_*(\Omega, q, q_*, p) > 0$ *with the following property: If the conditions*

$$\Big(\int_0^T \|e^{-tA}u_0\|_2^2\,dt\Big)^{1/2} + \int_0^T \|e^{-tA}u_0\|_q^s\,dt\Big)^{1/s} \le \frac{\epsilon_*}{1+T^{\frac{1}{2}+\frac{3}{2q}}}\,, \tag{1.8}$$

$$\Big(\int_0^T \|e^{t\Delta}\theta_0\|_4^{8/3}\,dt\Big)^{3/8} \le \frac{\epsilon_*}{1+T^{\frac{1}{2}+\frac{3}{2q}}}\,, \tag{1.9}$$

$$\|f_1\|_{q_*,s_*;T} + T^{1/2}\|f_1\|_{2,1;T} + \|f_2\|_{2,1;T} \le \frac{\epsilon_*}{1+T^{\frac{1}{2}+\frac{3}{2q}}}\,, \tag{1.10}$$

$$\|g\|_{p,\mu;T} + T^{1/2}\|g\|_{4,\frac{8}{5};T} \le \epsilon_*\,, \tag{1.11}$$

are fulfilled, then there exists a strong solution (u, θ) *of the Boussinesq equations (1.1).*

A proof can be found in Section 5.1. It follows from (2.2), (2.4) below that $e^{-tA}u_0 \in L^q(\Omega)$ for a.a. $t \in [0, T[$, yielding that the left hand side of (1.8) is well defined. In the following theorem we will show the condition (1.12) below on $u_0 \in L^2_\sigma(\Omega)\,, \theta_0 \in L^2(\Omega)$ defines the largest possible class of initial values to obtain a strong solution (u, θ) of (1.1) with $u \in L^s(0, T; L^q(\Omega))\,, 0 < T < \infty$, in an exterior domain for all s, q with $\mathcal{S}(s, q) = 1$. Especially no additional integrability condition for θ_0 is required. For optimal initial value conditions of the Boussinesq system in a bounded domain we refer to [19, Theorem 1.4], for a completely general domain to [21, Theorem 1.3 (ii)].

Theorem 1.5. *Consider data as in Theorem 1.3. Then the condition*

$$\int_0^\infty \|e^{-tA}u_0\|_q^s\,dt < \infty \tag{1.12}$$

is necessary and sufficient for the existence of $0 < T' \le T$ *and a strong solution* (u, θ) *with* $u \in L^s(0, T'; L^q(\Omega))$ *of the Boussinesq equations (1.1).*

The proof of the theorem above is the content of Section 5.2. Before we are ready to present the uniqueness criterion we need the following

Definition 1.6. Consider data as in Definition 1.1, assume additionally $g \in L^{8/3}_{\mathrm{loc}}([0, T[; L^4(\Omega))$, and let (u, θ) be a weak solution of (1.1). We say that (u, θ) fulfils the *strong energy inequality* if there is a null set $N \subseteq]0, T[$ such that

$$\frac{1}{2}\|u(t)\|_2^2 + \int_s^t \|\nabla u\|_2^2\,d\tau \le \frac{1}{2}\|u(s)\|_2^2 + \int_s^t \langle \theta g, u\rangle_\Omega\,d\tau + \int_s^t \langle f_1, u\rangle_\Omega\,d\tau \tag{1.13}$$

for all $s \in (]0, T[\backslash N) \cup \{0\}$ and all $t \in [s, T[$.

For a proof of this result we refer to Section 5.3. We need the additional assumption $g \in L^{8/3}_{\mathrm{loc}}([0, T[; L^4(\Omega))$ compared to Definition 1.1 to guarantee that $\int_s^t \langle \theta g, u\rangle_\Omega\,d\tau$ exists. Now we have all ingredients at handto formulate Theorem 1.7 which is a uniqueness theorem for weak solutions of (1.1). This result is based on the construction of a strong solution of (1.1) and the identification of this solution with the given weak solutions. For uniqueness and regularity results for the Navier-Stokes equations which are based on the method of the identification of a strong solution with a weak solution we refer to [6, 7, 10] and papers cited there.

Theorem 1.7. *Consider data as in Theorem 1.3. Let (u,θ) and (v,Θ) be weak solutions of the Boussinesq system (1.1). Assume $u \in L^s_{loc}([0,T[;L^q(\Omega))$ and that (v,Θ) satisfies the strong energy inequality (1.13). Then $u(t)=v(t)$ and $\theta(t)=\Theta(t)$ for almost all $t\in[0,T[$.*

The present paper is organized as follows. After presenting some preliminaries in Section 2, we deal with the construction of a suitable fixed point needed for proving Theorem 1.3 in Section 3. The topic of the following section is the proof of Theorem 1.3. Finally, the last section is dedicated to the proof of Corollary 1.4, Theorem 1.5 and Theorem 1.7.

2. Preliminaries

Given a domain $\Omega \subseteq \mathbb{R}^n, n\in\mathbb{N}$, and $1\le q\le\infty, k\in\mathbb{N}$, we need the usual Lebesgue and Sobolev spaces, $L^q(\Omega), W^{k,q}(\Omega)$ with norm $\|\cdot\|_{L^q(\Omega)} = \|\cdot\|_q$ and $\|\cdot\|_{W^{k,q}(\Omega)}$, respectively. For two measurable functions f,g with $f\cdot g\in L^1(\Omega)$, where $f\cdot g$ means the usual scalar product of vector or matrix fields, we set $\langle f,g\rangle_\Omega := \int_\Omega f(x)\cdot g(x)\,dx$. Note that the same symbol $L^q(\Omega)$ etc. will be used for spaces of scalar-, vector- or matrix-valued functions. Let $C^m(\Omega), m=0,1,\dots,\infty$, denote the space of functions for which all partial derivatives of order $|\alpha|\le m$ ($|\alpha|<\infty$ when $m=\infty$) exist and are continuous. As usual, $C_0^m(\Omega)$ is the set of all functions from $C^m(\Omega)$ with compact support in Ω. Further $C^\infty_{0,\sigma}(\Omega) := \{\, v\in C_0^\infty(\Omega);\ \operatorname{div} v = 0\,\}$. For $1<q<\infty$ we define $L^q_\sigma(\Omega) := \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_q}$ and $W^{1,2}_{0,\sigma}(\Omega) := \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_{W^{1,2}}}$. For $1\le q\le\infty$ let q' be the dual exponent such that $\frac{1}{q}+\frac{1}{q'}=1$. It is well known that $L^q_\sigma(\Omega)'\cong L^{q'}_\sigma(\Omega), 1<q<\infty$, using the standard pairing $\langle\cdot,\cdot\rangle_\Omega$.

Given a Banach space X, $1\le p\le\infty$, and an interval $]0,T[$ we denote by $L^p(0,T;X)$ the space of (equivalence classes of) strongly measurable functions $f:]0,T[\to X$ such that $\|f\|_p := \left(\int_0^T \|f(t)\|_X^p\,dt\right)^{1/p} < \infty$ if $1\le p<\infty$ and $\|f\|_\infty := \operatorname{ess\,sup}_{t\in]0,T[}\|f(t)\|_X$ if $p=\infty$. Moreover

$$L^p_{\mathrm{loc}}([0,T[;X) := \{\, u : [0,T[\to X \text{ strongly measurable},\\ u\in L^p(0,T';X) \text{ for all } 0<T'<T\}.$$

If $X=L^q(\Omega)$, $1\le q\le\infty$, the norm in $L^p(0,T;L^q(\Omega))$ will be denoted by $\|f\|_{q,p;T}$. Fix an exterior domain $\Omega\subseteq\mathbb{R}^3$ with $\partial\Omega\in C^{2,1}$ and $1<q<\infty$. Let $P_q : L^q(\Omega)\to L^q_\sigma(\Omega)$, be the *Helmholtz projection* and let Δ_q denote the *Laplace operator* with domain $\mathcal{D}(\Delta_q) := W_0^{1,q}(\Omega)\cap W^{2,q}(\Omega)$. We introduce the *Stokes operator* by

$$\mathcal{D}(A_q) = L^q_\sigma(\Omega)\cap W_0^{1,q}(\Omega)\cap W^{2,q}(\Omega), \quad A_q u := -P_q\Delta_q u\,, \quad u\in\mathcal{D}(A_q).$$

The Stokes operator is *consistent* in the sense that for $1<q,r<\infty$

$$A_q u = A_r u \quad \forall u\in\mathcal{D}(A_q)\cap\mathcal{D}(A_r). \tag{2.1}$$

Throughout this paper we will write $A=A_2$ and $\Delta=\Delta_2$. For $\alpha\in[-1,1]$ the fractional power $A_q^\alpha : \mathcal{D}(A_q^\alpha)\to L^q_\sigma(\Omega)$ with dense domain $\mathcal{D}(A_q^\alpha)\subseteq L^q_\sigma(\Omega)$ is a well defined, injective, closed operator such that

$$(A_q^\alpha)^{-1} = A_q^{-\alpha}, \quad \mathcal{R}(A_q^\alpha) = \mathcal{D}(A_q^{-\alpha}) \quad \text{and } (A_q^\alpha)' = A_{q'}^\alpha.$$

In general, $\mathcal{D}(A_q^\alpha)$, will be equipped with the graph norm $\|u\|_{\mathcal{D}(A_q^\alpha)} := \|u\|_q + \|A_q^\alpha u\|_q$ which makes $D(A_q^\alpha)$ to a Banach space since A_q^α is closed. Analogous properties hold for fractional powers $(-\Delta_q)^\alpha : \mathcal{D}((-\Delta_q)^\alpha) \subseteq L^q(\Omega) \to L^q(\Omega)$ of $-\Delta_q$.

It is well known that $-A_q$ generates a uniformly bounded analytic semigroup $\{ e^{-tA_q}; t \geq 0 \}$ on $L^q_\sigma(\Omega)$ and that Δ_q generates a uniformly bounded analytic semigroup $\{ e^{t\Delta_q}; t \geq 0\}$ on $L^q(\Omega)$. The decay estimates

$$\|A_q^\alpha e^{-tA_q} u\|_q \leq c\, t^{-\alpha}, \quad \forall u \in L^q_\sigma(\Omega)\,, t > 0\,, \tag{2.2}$$

$$\|(-\Delta_q)^\alpha e^{t\Delta_q} \phi\|_q \leq c\, t^{-\alpha} \quad \forall \phi \in L^q(\Omega)\,, t > 0\,, \tag{2.3}$$

are satisfied where $\alpha \geq 0\,, q > 1$, and $c = c(\Omega, q, \alpha) > 0$. There holds

$$\|u\|_r \leq c\|A_q^\alpha u\|_q \qquad \forall u \in \mathcal{D}(A_q^\alpha)\,, \tag{2.4}$$

$$\|\phi\|_r \leq c\|(-\Delta_q)^\alpha \phi\|_q \quad \forall \phi \in \mathcal{D}((-\Delta_q)^\alpha) \tag{2.5}$$

with a constant $c = c(\Omega, q, \alpha) > 0$ where $0 \leq \alpha \leq \frac{1}{2}\,, 1 < q < 3$, with $2\alpha + \frac{3}{r} = \frac{3}{q}$. We refer to [11, 12, 13] for the results above and further properties.

We also need the perturbed Stokes operator $(I + A_q)$. For each $\alpha \in [0, 1]$ we have that $(I + A_q)^\alpha$ with dense domain $\mathcal{D}((I + A_q)^\alpha) \subseteq L^q_\sigma(\Omega)$ is a well defined bijective, closed operator. There holds (see [22, Lemma 4.11]) that $\mathcal{D}((I + A_q)^\alpha) = \mathcal{D}(A_q^\alpha)$. For each $1 < q < \infty\,, 0 \leq \alpha \leq 1$ the estimate

$$\|(I + A_q)^\alpha e^{-tA_q} u\|_q \leq c(\Omega, q, \alpha) \frac{(t + 1)^\alpha}{t^\alpha} \|u\|_q \tag{2.6}$$

holds for all $t > 0$ and all $u \in L^q_\sigma(\Omega)$. Further for $1 < q < 3$ and $0 \leq \alpha \leq \frac{1}{2}$ let $q < r < \infty$ be defined by $2\alpha + \frac{3}{r} = \frac{3}{q}$. Then

$$\|u\|_r \leq c(\Omega, q, r)\|(I + A_q)^\alpha u\|_q \tag{2.7}$$

for all $u \in \mathcal{D}((I + A_q)^\alpha)$. For a proof of (2.6) and (2.7) we refer to [8, Lemma 3.3]. We remark that analogous estimates of (2.6), (2.7) hold where A_q is replaced by $-\Delta_q$.

3. Construction of a suitable fixed Point

The proof of Theorem 1.3 is essentially based on the existence of a suitable solution of (3.18) below. To solve this system we need the estimates presented in Lemma 3.3. To begin with, let us cite the following lemma which will be frequently used in the progress of this paper without referring back to it every time we use it.

Lemma 3.1. *Let $\Omega \subseteq \mathbb{R}^3$ be an exterior domain with $\partial\Omega \in C^{2,1}$.*

(i) Let $q > \frac{3}{2}\,, F \in L^q(\Omega)$. Choose $r, \sigma \geq 0$ with

$$2\sigma + \frac{3}{r} = \frac{3}{q}, \quad 0 \leq \sigma \leq \frac{1}{2}. \tag{3.1}$$

There exists a unique element in $L^r_\sigma(\Omega)$ denoted by $A_r^{-1/2-\sigma} P_r \mathrm{div} F \in L^r_\sigma(\Omega)$ with

$$\langle A_r^{-1/2-\sigma} P_r \mathrm{div} F, A_{r'}^{1/2+\sigma} w\rangle_\Omega = -\langle F, \nabla w\rangle_\Omega \tag{3.2}$$

for all $w \in \mathcal{D}(A_{r'}^{1/2+\sigma})$. *There holds*

$$\|A_r^{-1/2-\sigma} P_r \mathrm{div} F\|_r \leq c\|F\|_p \tag{3.3}$$

with a constant $c = c(\Omega, q, r) > 0$.

(ii) Let $1 < q < \infty$, *and let* $F \in L^q(\Omega)$. *There exists a unique element* $(I + A_q)^{-1/2} P_q \mathrm{div} F \in L^q_\sigma(\Omega)$ *with*

$$\langle (I + A_q)^{-1/2} P_q \mathrm{div} F, (I + A_{q'})^{1/2} w\rangle_\Omega = -\langle F, \nabla w\rangle_\Omega \tag{3.4}$$

for all $w \in \mathcal{D}((I + A_{q'})^{1/2})$. *The estimate*

$$\|(I + A_q)^{-1/2} P_q \mathrm{div} F\|_q \leq c\|F\|_q \tag{3.5}$$

is satisfied for all $F \in L^q(\Omega)$ *with a constant* $c = c(\Omega, q) > 0$.

Proof. See [8, Lemma 3.1] and [8, Lemma 3.4]. □

Remark. Since $\|\nabla w\|_q \leq c(\Omega, q)\|(I + A_q)^{1/2} w\|_q$, $w \in \mathcal{D}((I + A_q)^{1/2})$ holds for all $1 < q < \infty$ (see [8, Lemma 3.3 (1)]) the expression $(I+A_q)^{-1/2} P_q \mathrm{div} F$ can be defined for all $1 < q < \infty$, in contrast to $A_q^{-1/2} P_q \mathrm{div} F$, which needs the restriction $q > \frac{3}{2}$ in an exterior domain.

We need the following system of integral equations characterizing weak solutions of the Boussinesq system (1.1).

Lemma 3.2. *Consider data as in Definition 1.1. Then* (u, θ) *with (1.2), (1.3) and* $\theta g \in L^1_{loc}([0, T[; L^2(\Omega))$ *is a weak solution of (1.1) if and only if the system of integral equations*

$$\begin{aligned} u(t) = e^{-tA} u_0 &+ \int_0^t e^{-(t-\tau)A} P\big(\theta(\tau) g(\tau)\big)\, d\tau + \int_0^t e^{-(t-\tau)A} P f_1(\tau)\, d\tau \\ &- A^{1/2} \int_0^t e^{-(t-\tau)A} A^{-1/2} P \mathrm{div}\big(u(\tau) \otimes u(\tau)\big)\, d\tau\,, \end{aligned} \tag{3.6}$$

$$\begin{aligned} \theta(t) = e^{t\Delta} \theta_0 &+ \int_0^t e^{(t-\tau)\Delta} f_2(\tau)\, d\tau \\ &- (-\Delta)^{1/2} \int_0^t e^{(t-\tau)\Delta} (-\Delta)^{-1/2} \mathrm{div}\big(\theta(\tau) u(\tau)\big)\, d\tau \end{aligned} \tag{3.7}$$

is fulfilled for almost all $t \in [0, T[$.

Proof. The representation formula (3.6) follows from [25, Chapter IV, Section 2.4] with $f := \theta g \in L^1_{\mathrm{loc}}([0, T[; L^2(\Omega))$. To prove (3.7) we replace $-A$ by Δ and use the same argumentation as in the proof of (3.6). □

We proceed with the following lemma:

Lemma 3.3. *Let* $\Omega \subseteq \mathbb{R}^3$ *be an exterior domain with* $\partial\Omega \in C^{2,1}$, *let* $0 < T < \infty$. *Consider* $1 < s, q < \infty$ *with* $\mathcal{S}(s, q) = 1$. *Consider* $g \in L^{8/5}(0, T; L^4(\Omega)) \cap L^\mu(0, T; L^p(\Omega))$ *where* $1 < \mu < \infty$, $1 < p < 12$, *satisfy* $\mathcal{S}(\mu, p) = \frac{3}{2}$ *and* $\frac{1}{12} > \frac{1}{p} - \frac{1}{q}$. *Define* $\alpha := \frac{1}{2} + \frac{3}{2q}$ *and the Banach spaces*

$$X := L^s(0, T; L^q_\sigma(\Omega)) \cap L^2(0, T; L^2_\sigma(\Omega))\,, \quad Y := L^{8/3}(0, T; L^4(\Omega)).$$

(i) Define the bilinear form

$$\mathcal{F}_1 : X \times X \to X\,,$$
$$\mathcal{F}_1(u,v)(t) := -\int_0^t A_q^{\alpha} e^{-(t-\tau)A_q} A_q^{-\alpha} P_q \mathrm{div}\big(u(\tau)\otimes v(\tau)\big)\, d\tau$$

for a.a. $t \in [0,T[$. *Then*

$$\|\mathcal{F}_1(u,v)\|_X \le c(1+T^{\alpha})\|u\otimes v\|_{\frac{q}{2},\frac{s}{2};T} \le c(1+T^{\alpha})\|u\|_X\|v\|_X \tag{3.8}$$

for all $u, v \in X$ *where* $c = c(\Omega, q) > 0$ *is a constant.*

(ii) Define the bilinear form

$$\mathcal{F}_2 : X \times Y \to Y\,,$$
$$\mathcal{F}_2(u,\theta)(t) := -\int_0^t (-\Delta_4)^{\alpha} e^{(t-\tau)\Delta_4}(-\Delta_4)^{-\alpha}\mathrm{div}\big(\theta(\tau)u(\tau)\big)\, d\tau$$

for a.a. $t \in [0,T[$. *Then*

$$\|\mathcal{F}_2(u,\theta)\|_Y \le c\|u\|_X\|\theta\|_Y \tag{3.9}$$

for all $u \in X\,, \theta \in Y$ *with* $c = c(\Omega, q) > 0$.

(iii) Define the linear map

$$\mathcal{L} : Y \to X\,,$$
$$\big(\mathcal{L}\theta\big)(t) := \int_0^t e^{-(t-\tau)A} P\big(\theta(\tau)g(\tau)\big)\, d\tau$$

for a.a. $t \in [0,T[$. *Then*

$$\|\mathcal{L}\theta\|_X \le c\big(\|g\|_{p,\mu;T} + T^{1/2}\|g\|_{4,\frac{8}{5};T}\big)\,\|\theta\|_Y \tag{3.10}$$

for all $\theta \in Y$ *with* $c = c(\Omega, q, p) > 0$.

Proof. Fix $u\,, v \in X$ and $\theta \in Y$. Define $1 < q_2, s_2 < \infty$ by

$$\frac{1}{q_2} = \frac{1}{2} + \frac{1}{q}\,, \quad \frac{1}{s_2} = \frac{1}{2} + \frac{1}{s}.$$

Proof of (i), step 1. Estimates (2.2), (2.4) and (3.3) with $r = \frac{3}{2q}$ imply

$$\begin{aligned}\|\mathcal{F}(u,v)(t)\|_q &\le c\int_0^t |t-\tau|^{-\alpha}\|A_q^{-\alpha}P_q\mathrm{div}\big(u\otimes v\big)(\tau)\|_q\, d\tau\\ &\le c\int_0^T |t-\tau|^{-\alpha}\|(u\otimes v)(\tau)\|_{\frac{q}{2}}\, d\tau\end{aligned}$$

for a.a. $t \in [0,T[$. Due to the Hardy-Littlewood inequality (see [26, V, 1.2]) with $(1-\alpha) + \frac{1}{s} = \frac{1}{s/2}$ we obtain

$$\|\mathcal{F}(u,v)\|_{q,s;T} \le c\|u\otimes v\|_{\frac{q}{2},\frac{s}{2};T} \le c\|u\|_{q,s;T}\|v\|_{q,s;T} \tag{3.11}$$

with $c = c(\Omega, q) > 0$.

Proof of (i), step 2. Due to $u\otimes v \in L^{\frac{s}{2}}(0,T;L^{\frac{q}{2}}(\Omega)) \cap L^{s_2}(0,T;L^{q_2}(\Omega))$ and (3.2), (3.4) we can show with the consistence of the Stokes operator and duality arguments analogously as in [8, Lemma 3.2], that

$$\mathcal{F}_1(u,v)(t) = -\int_0^t (I+A_{q_2})^{1/2}e^{-(t-\tau)A_{q_2}}(I+A_{q_2})^{-1/2}P_{q_2}\mathrm{div}\big(u(\tau)\otimes v(\tau)\big) \tag{3.12}$$

for a.a. $t \in [0,T[$. Since $2 \cdot \frac{3}{2q} + \frac{3}{2} = \frac{3}{q_2}$ and $0 < T < \infty$ we obtain with (2.6), (2.7) and the closedness of $I + A_q$ in combination with [16, Theorem 3.7.12],

$$\begin{aligned}
&\|\mathcal{F}_1(u,v)(t)\|_2 \\
&\le c\Big\| - \int_0^t (I + A_{q_2})^{\alpha} e^{-(t-\tau)A_{q_2}} (I + A_{q_2})^{-1/2} P_{q_2} \mathrm{div}\big(u(\tau) \otimes v(\tau)\big)\, d\tau \Big\|_{q_2} \\
&\le c \int_0^t \Big(\frac{1 + (t-\tau)}{(t-\tau)}\Big)^{\alpha} \|(I + A_{q_2})^{-1/2} P_{q_2} \mathrm{div}\big(u(\tau) \otimes v(\tau)\big)\|_{q_2}\, d\tau \\
&\le c(1+T)^{\alpha} \int_0^T |t-\tau|^{-\alpha} \|(I + A_{q_2})^{-1/2} P_{q_2} \mathrm{div}\big(u(\tau) \otimes v(\tau)\big)\|_{q_2}\, d\tau \\
&\le c(1+T^{\alpha}) \int_0^T |t-\tau|^{-\alpha} \|u(\tau) \otimes v(\tau)\|_{q_2}\, d\tau
\end{aligned}$$

for a.a. $t \in [0,T[$ with $c = c(\Omega, q) > 0$. Since $(1-\alpha) + \frac{1}{2} = \frac{1}{s_2}$ we get with the Hardy Littlewood inequality

$$\|\mathcal{F}(u,v)\|_{2,2;T} \le c(1+T^{\alpha}) \|u \otimes v\|_{q_2,s_2;T} \le c(1+T^{\alpha}) \|u\|_{2,2;T} \|v\|_{q,s;T} \tag{3.13}$$

with $c = c(\Omega, q) > 0$. Combining (3.11), (3.13) yields

$$\|\mathcal{F}(u,v)\|_X \le c(1+T^{\alpha}) \|u\|_X \|v\|_X$$

for all $u, v \in X$ with a constant $c = c(\Omega, q) > 0$

Proof of (ii). Introduce $1 < x_1, x_2 < \infty$ by

$$\frac{1}{x_1} = \frac{1}{8/3} + \frac{1}{s}, \quad \frac{1}{x_2} = \frac{1}{4} + \frac{1}{q}.$$

Especially $x_2 > \frac{3}{2}$. We get with (2.3), (2.5) and (3.3) (where A_4 is replaced by $(-\Delta_4)$)

$$\begin{aligned}
\|(\mathcal{F}_2(u,\theta))(t)\|_4 &\le c \int_0^t |t-\tau|^{-\alpha} \|(-\Delta_4)^{-\alpha} \mathrm{div}\big(\theta(\tau) u(\tau)\big)\|_4\, d\tau \\
&\le c \int_0^T |t-\tau|^{-\alpha} \|\theta(\tau) u(\tau)\|_{x_2}\, d\tau
\end{aligned}$$

for a.a. $t \in [0,T[$ with $c = c(\Omega, q) > 0$. The Hardy-Littlewood inequality with $(1-\alpha) + \frac{1}{8/3} = \frac{1}{x_1}$, combined with Hölder's inequality, yields

$$\|\mathcal{F}_2(u,\theta)\|_{4,\frac{8}{3};T} \le c \|\theta u\|_{x_2,x_1;T} \le c \|u\|_{q,s;T} \|\theta\|_{4,\frac{8}{3};T} \tag{3.14}$$

with $c = c(\Omega, q) > 0$.

Proof of (iii), step 1. Define $\sigma := \frac{3}{2}(\frac{1}{4} - \frac{1}{q} + \frac{1}{p})$. By construction $0 \le \sigma < \frac{1}{2}$ and $2\sigma + \frac{3}{q} = \frac{3}{4} + \frac{3}{p}$. Define $1 < x_1, x_2 < \infty$ by

$$\frac{1}{x_1} = \frac{1}{8/3} + \frac{1}{\mu}, \quad \frac{1}{x_2} = \frac{1}{4} + \frac{1}{p}.$$

Due to $p < 12$ it follows $1 < x_2 < 3$. From (2.2), (2.4) we obtain with the closedness of A_q in combination with [16, Theorem 3.7.12] that

$$\begin{aligned} \|(\mathcal{L}\theta)(t)\|_q &\le c \Big\| \int_0^t A_{x_2}^\sigma e^{-(t-\tau)A_{x_2}} P_{x_2}(\theta g)\, d\tau \Big\|_{x_2} \\ &\le c \int_0^T |t-\tau|^{-\sigma} \|\theta(\tau) g(\tau)\|_{x_2}\, d\tau \end{aligned} \tag{3.15}$$

for almost all $t \in [0, T[$ with $c = c(\Omega, q, p) > 0$. Since $(1-\sigma) + \frac{1}{s} = \frac{1}{x_1}$ we can apply the Hardy-Littlewood estimate to (3.15) and get

$$\|\mathcal{L}\theta\|_{q,s;T} \le c\|\theta g\|_{x_2,x_1;T} \le c(\Omega, q, p)\|g\|_{p,\mu;T}\|\theta\|_{4,\frac{8}{3};T}. \tag{3.16}$$

Proof of (iii), step 2. By [25, Chapter IV, Lemma 2.4.2] there holds

$$\frac{1}{2}\|\mathcal{L}\theta\|_{2,\infty;T}^2 + \|\nabla\mathcal{L}\theta\|_{2,2;T}^2 \le 8\|\theta g\|_{2,1;T}^2.$$

It follows by interpolation

$$\|\mathcal{L}\theta\|_{2,2;T} \le cT^{1/2}\|\theta g\|_{2,1;T} \le cT^{1/2}\|\theta\|_{4,\frac{8}{3};T}\|g\|_{4,\frac{8}{5};T}. \tag{3.17}$$

Proof of (iii), step 3. Combining (3.16), (3.17) yields (3.10). □

The final result of this section reads as follows:

Theorem 3.4. *Consider data as in Lemma 3.3, let $\mathcal{F}_1, \mathcal{F}_2, \mathcal{L}, X, Y$ be defined as in Lemma 3.3, and let E_1, E_2 be as in (1.4), (1.5). Then there exists a constant $\epsilon_* = \epsilon_*(\Omega, q, p) > 0$ such that if the conditions (1.6), (1.7) are fulfilled, then there exists $u \in X$, $\theta \in Y$ with*

$$\begin{aligned} u &= E_1 + \mathcal{F}_1(u,u) + \mathcal{L}\theta\,, \\ \theta &= E_2 + \mathcal{F}_2(u,\theta) \end{aligned} \tag{3.18}$$

and

$$\|u\|_X + \|\theta\|_Y \le 4\big(\|E_1\|_X + \|E_2\|_Y\big). \tag{3.19}$$

Proof. Fix a constant $K > 0$ such that the estimates (3.8), (3.9), (3.10) are satisfied where c is replaced by K. By construction $K = K(\Omega, q, p)$. Define $\alpha := \frac{1}{2} + \frac{3}{2q}$. In the following we will show that if

$$12K(1+T^\alpha)(\|E_1\|_X + \|E_2\|_Y) < \frac{1}{2}\,, \tag{3.20}$$

$$K(\|g\|_{p,\mu;T} + T^{1/2}\|g\|_{4,\frac{8}{5};T}) < \frac{1}{2} \tag{3.21}$$

are fulfilled, then there exists $u \in X$, $\theta \in Y$ fulfilling (3.18), (3.19). With no loss of generality assume $A := \|E_1\|_X + \|E_2\|_Y > 0$. We endow $X \times Y$ with the norm $\|(u,\theta)\|_{X\times Y} := \|u\|_X + \|\theta\|_Y$ and obtain that $X \times Y$ is a Banach space. Define the nonlinear map

$$T : X \times Y \to X \times Y\,, \quad T(u,\theta) := \big(E_1 + \mathcal{F}_1(u,u) + \mathcal{L}\theta, E_2 + \mathcal{F}_2(u,\theta)\big).$$

Since $4KA < 1$ we can introduce R as the smallest positive root of the polynomial $Kx^2 - \frac{1}{2}x + A$, i.e.

$$R := \frac{\frac{1}{2} - \sqrt{\frac{1}{4} - 4KA}}{2K} = \frac{2A}{\frac{1}{2} + \sqrt{\frac{1}{4} - 4KA}}.$$

Introduce the closed ball $\mathcal{B} := \{(u,\theta) \in X \times Y; \|(u,\theta)\|_{X\times Y} \le R\}$. We obtain with (3.10), (3.21)

$$\|\mathcal{L}\theta\|_X \le \frac{1}{2}\|\theta\|_Y \tag{3.22}$$

for all $\theta \in Y$. There holds

$$\|T(u,\theta)\|_{X\times Y} \le K\|u\|_X(\|u\|_X + \|\theta\|_Y) + \tfrac{1}{2}\|\theta\|_Y + A \le KR^2 + \tfrac{1}{2}R + A = R.$$

Thus $T(\mathcal{B}) \subseteq \mathcal{B}$. From (3.8), (3.9), (3.10) and (3.22) it follows

$$\begin{aligned}
&\|T(u,\theta) - T(\tilde{u},\tilde{\theta})\|_{X\times Y}\\
&= \big(\mathcal{F}_1(u, u-\tilde{u}) + \mathcal{F}_1(u-\tilde{u},\tilde{u}) + \mathcal{L}(\theta-\tilde{\theta})\,, \mathcal{F}_2(u,\theta-\tilde{\theta}) + \mathcal{F}_2(u-\tilde{u},\tilde{\theta})\big)\\
&\le K(1+T^\alpha)(\|u\|_X + \|\tilde{u}\|_X)\|u-\tilde{u}\|_X + \tfrac{1}{2}\|\theta-\tilde{\theta}\|_Y\\
&\quad + K\|u\|_X\|\theta-\tilde{\theta}\|_Y + K\|u-\tilde{u}\|_X\|\tilde{\theta}\|_Y\\
&\le 3K(1+T^\alpha)R\|u-\tilde{u}\|_X + KR\|\theta-\tilde{\theta}\|_Y + \tfrac{1}{2}\|\theta-\tilde{\theta}\|_Y\\
&\le \Big(3K(1+T^\alpha)R + \tfrac{1}{2}\Big)\|(u,\theta) - (\tilde{u},\tilde{\theta})\|_{X\times Y}
\end{aligned}$$

for all $(u,\theta)\,,(\tilde{u},\tilde{\theta}) \in \mathcal{B}$. From (3.20) and $R < 4A$ we get the inequality $3K(1+T^\alpha)R + \frac{1}{2} < 1$. Altogether $T : \mathcal{B} \to \mathcal{B}$ is a strict contraction. By Banach's fixed point theorem there exists $(u,\theta) \in \mathcal{B}$ with $T(u,\theta) = (u,\theta)$. Furthermore (u,θ) is the unique fixed point of T in $\mathcal{B}$. Especially we get (3.19).

Consequently, there exists a constant $\epsilon_* = \epsilon_*(\Omega, q, p) > 0$ with the following property: If the conditions (1.6), (1.7) are fulfilled, then (3.20), (3.21) hold. The proof is complete. □

4. Proof of Thoerem 1.3

To begin with, let us present a sketch of proof. In the first step we use Theorem 3.4 to construct $u \in L^s(0,T; L^q(\Omega)) \cap L^2(0,T; L^2(\Omega))$ and $\theta \in L^{8/3}(0,T; L^4(\Omega))$ fulfilling (3.18). The proof of $\nabla u\,, \nabla\theta \in L^2(0,T; L^2(\Omega))$ differs from the proof for a bounded domain and is based on the application of Yosida's smoothing procedure. We remark that the additional $L^2(L^2)$-integrability for u is needed for the proof of $\nabla u \in L^2(0,T; L^2(\Omega))$.

Proof. **Step 1.** Let E_1, E_2 be as in (1.4), (1.5). Introduce the Banach spaces $X := L^s(0,T; L^q_\sigma(\Omega)) \cap L^2(0,T; L^2_\sigma(\Omega))$ and $Y := L^{8/3}(0,T; L^4(\Omega))$ and let $\mathcal{F}_1, \mathcal{F}_2$, and $\mathcal{L}$ as in Lemma 3.3. By Theorem 3.4 there exists a constant $\epsilon_* = \epsilon_*(\Omega, q, p) > 0$ with the following property: If the conditions (1.6), (1.7) are fulfilled, then there exists $u \in L^s(0,T; L^q_\sigma(\Omega)) \cap L^2(0,T; L^2_\sigma(\Omega))$ and $\theta \in L^{8/3}(0,T; L^4(\Omega))$ satisfying (3.18) and (3.19). To finish the proof we have to show that, after a possible reduction of ϵ_* (see the discussion following (4.5)), that such a solution (u,θ) fulfils (1.2), (1.3).

Step 2. Introduce $\alpha := \frac{1}{2} + \frac{3}{2q}$ and $1 < s_2, q_2 < \infty$ by

$$\frac{1}{s_2} = \frac{1}{2} + \frac{1}{s}\,, \quad \frac{1}{q_2} = \frac{1}{2} + \frac{1}{q}.$$

Further, define

$$\begin{aligned} E(t) &:= E_1(t) + \mathcal{L}(\theta)(t) \\ &= e^{-tA}u_0 + \int_0^t e^{-(t-\tau)A} P(\theta g)(\tau)\,d\tau + \int_0^t e^{-(t-\tau)A} P f_1(\tau)\,, \end{aligned} \tag{4.1}$$

$$\tilde{u}(t) := -\int_0^t A_q^{\alpha} e^{-(t-\tau)A_q} A_q^{-\alpha} P_q \operatorname{div}\big(u\otimes u\big)(\tau)\,d\tau \tag{4.2}$$

for almost all $t \in [0,T[$, so that $u = \tilde{u} + E$. We remark that u is constructed as a very weak solution of the instationary Navier-Stokes equations with the additional property $u \in L^2(0,T;L^2(\Omega))$. For further information about very weak solutions of the Navier-Stokes equations we refer to [5, 7]. Since $u\otimes u \in L^{\frac{s}{2}}(0,T;L^{\frac{q}{2}}(\Omega)) \cap L^{s_2}(0,T;L^{q_2}(\Omega))$ we obtain as in (3.12)

$$\widetilde{u}(t) = -\int_0^t (I + A_{q_2})^{1/2} e^{-(t-\tau)A_{q_2}} (I + A_{q_2})^{-1/2} P_{q_2} \operatorname{div}\big(u \otimes (\tilde{u} + E)\big)(\tau)\,d\tau \tag{4.3}$$

for almost all $t \in [0,T[$.

To prove $\nabla u \in L^2(0,T;L^2(\Omega))$ let $J_n := (I + \frac{1}{n}A_{q_2}^{1/2})^{-1}\,, n \in \mathbb{N}$, be the Yosida approximation of I in $L^q_\sigma(\Omega)$, so that $\tilde{u} = J_n\tilde{u} + \frac{1}{n}A_{q_2}^{1/2}J_n\tilde{u}$. Since $A_q^{1/2}$ generates a bounded semigroup, the sequences $(\|J_n\|)_{n\in\mathbb{N}}$ and $(\|\frac{1}{n}A_q^{1/2}J_n\|)_{n\in\mathbb{N}}$ are bounded, see [3, II, Sections 3,4]. We can rewrite equality (4.3) in the form

$$\begin{aligned} \tilde{u}(t) = &-\int_0^t e^{-(t-\tau)A_{q_2}}\Big(u\cdot\nabla J_n\tilde{u} + u\cdot\nabla E\Big)\,d\tau \\ &-\int_0^t (I + A_{q_2})^{1/2} e^{-(t-\tau)A_{q_2}} (I + A_{q_2})^{-1/2} P_{q_2}\operatorname{div}\big(u \otimes \frac{1}{n}A_{q_2}^{1/2}J_n u\big)\,d\tau \end{aligned}$$

for a.a. $t \in [0,T[$. We apply $A_{q_2}^{1/2}J_n$ to the identity above and get

$$\begin{aligned} &A_{q_2}^{1/2}J_n\tilde{u}(t) \\ &= -\int_0^t A_{q_2}^{1/2} e^{-(t-\tau)A_{q_2}}\Big(J_n(u\cdot\nabla J_n\tilde{u}) + J_n(u\cdot\nabla E)\Big)\,d\tau \\ &\quad -\int_0^t (I + A_{q_2})^{1/2} e^{-(t-\tau)A_{q_2}} \tfrac{1}{n}A_{q_2}^{1/2}J_n(I + A_{q_2})^{-1/2}P_{q_2}\operatorname{div}\big(u\otimes A_{q_2}^{1/2}J_n\tilde{u}\big)\,d\tau \\ &=: \mathcal{T}_1(t) + \mathcal{T}_2(t) \end{aligned} \tag{4.4}$$

for almost all $t \in [0,T[$. With $2\cdot\frac{3}{2q} + \frac{3}{2} = \frac{3}{q_2}$ and (2.2), (2.4) it follows

$$\begin{aligned} \|\mathcal{T}_1(t)\|_2 &\le c\Big\|\int_0^t A_{q_2}^{\alpha} e^{-(t-\tau)A_{q_2}}\Big(J_n(u\cdot\nabla J_n\tilde{u}) + J_n(u\cdot\nabla E)\Big)\,d\tau\,\Big\|_{q_2} \\ &\le c\int_0^T |t-\tau|^{-\alpha}\Big(\|u\cdot\nabla J_n\tilde{u}\|_{q_2} + \|u\cdot\nabla E\|_{q_2}\Big)\,d\tau \end{aligned}$$

for a.a. $t \in [0,T[$ with a constant $c = c(\Omega, q) > 0$. Moreover, with (2.6), (2.7) and (3.5) we get, since T is finite, that

$$\begin{aligned}
&\|\mathcal{T}_2(t)\|_2 \\
&\le c\Big\| \int_0^t (I+A_{q_2})^\alpha e^{-(t-\tau)A_{q_2}} \tfrac{1}{n} A_{q_2}^{1/2} J_n (I+A_{q_2})^{-1/2} P_{q_2} \mathrm{div}\big(u \otimes A_{q_2}^{1/2} J_n \tilde{u}\big)\, d\tau \Big\|_{q_2} \\
&\le c \int_0^t \frac{(1+|t-\tau|)^\alpha}{|t-\tau|^\alpha} \big\| \tfrac{1}{n} A_{q_2}^{1/2} J_n (I+A_{q_2})^{-1/2} P_{q_2} \mathrm{div}\big(u \otimes A_{q_2}^{1/2} J_n \tilde{u}\big)\big\|_{q_2}\, d\tau \\
&\le c \int_0^T |t-\tau|^{-\alpha} \|u \otimes A_{q_2}^{1/2} J_n \tilde{u}\|_{q_2} d\tau
\end{aligned}$$

with a constant $c = c(\Omega, q, T) > 0$. The Hardy-Littlewood inequality with $(1-\alpha) + \frac{1}{2} = \frac{1}{s_2}$ applied to (4.4), Hölder's inequality and the identity $\|\nabla J_n \tilde{u}\|_{2,2;T} = \|A_{q_2}^{1/2} J_n \tilde{u}\|_{2,2;T}$ combined with (2.1) yield

$$\begin{aligned}
&\|A_{q_2}^{1/2} J_n \tilde{u}\|_{2,2;T} \\
&\le c\Big(\|u \cdot \nabla J_n \tilde{u}\|_{q_2,s_2;T} + \|u \cdot \nabla E\|_{q_2,s_2;T} + \|u \otimes A_{q_2}^{1/2} J_n \tilde{u}\|_{q_2,s_2;T} \Big) \qquad (4.5)\\
&\le c_* \|u\|_{q,s;T} \big(\|A_{q_2}^{1/2} J_n \tilde{u}\|_{2,2;T} + \|\nabla E\|_{2,2;T} \big)
\end{aligned}$$

with a fixed constant $c_* = c_*(\Omega, q, T) > 0$. Replacing ϵ_* by $\min\{\epsilon_*, \frac{1}{8c_*}\}$ it follows from (1.6), (3.19) that

$$c_* \|u\|_{q,s;T} \le \frac{4c_*\epsilon_*}{1 + T^{\frac{1}{2}+\frac{3}{2q}}} \le 4c_*\epsilon_* \le \frac{1}{2}. \qquad (4.6)$$

Therefore, the absorption principle can be applied to (4.6) and yields

$$\|A_{q_2}^{1/2} J_n \tilde{u}\|_{2,2;T} \le c \|u\|_{q,s;T} \|\nabla E\|_{2,2;T} \qquad (4.7)$$

with a constant $c = c(\Omega, q, T) > 0$ independent of $n \in \mathbb{N}$. By a functional analytic argument (see [25, II.(3.18)]) and (2.1) we get $\tilde{u}(t) \in \mathcal{D}(A^{1/2})$ for a.a. $t \in [0,T[$ and $A^{1/2}\tilde{u} \in L^2(0,T;L^2(\Omega))$. Consequently $\nabla \tilde{u} \in L^2(0,T;L^2(\Omega))$. It follows

$$\tilde{u}(t) = -\int_0^t e^{-(t-\tau)A_{q_2}} P_{q_2}\big(u \cdot \nabla u\big)\, d\tau \qquad (4.8)$$

for a.a. $t \in [0,T[$. The same argumentation as in [9, page 102] shows that (4.8) implies $u \otimes u \in L^2(0,T;L^2(\Omega))$. A careful inspection shows that this proof remains true although we consider an exterior domain instead of a bounded domain. Consequently, we can write

$$\tilde{u}(t) = -\int_0^t A^{1/2} e^{-(t-\tau)A} A^{-1/2} P \mathrm{div}\big(u \otimes u\big)(\tau).$$

Altogether, $\tilde{u}$ can be considered as a weak solution of the (linear) Stokes system with initial value 0 and external force $f = -\mathrm{div}(u \otimes u)$ where $u \otimes u \in L^2(0,T;L^2(\Omega))$. By linear theory (see [25, IV, Theorems 2.3.1 and 2.4.1]) it follows that $\tilde{u}$ satisfies (1.2). Thus, $u = \tilde{u} + E$ also satisfies (1.2).

Step 3. Introduce

$$\tilde{\theta}(t) := -\int_0^t (-\Delta_4)^\alpha e^{(t-\tau)\Delta_4} (-\Delta_4)^{-\alpha} \mathrm{div}\big(\theta(\tau) u(\tau)\big)\, d\tau \qquad (4.9)$$

for a.a. $t \in [0, T[$. Thus $\theta = \tilde{\theta} + E_2$. Due to step 2 we know that (1.2) holds. It follows by interpolation $u \in L^{8/3}(0, T; L^4(\Omega))$. Consequently $\theta u \in L^{4/3}(0, T; L^2(\Omega))$.

Therefore, (see (4.3)) we can rewrite (4.9) as

$$\tilde{\theta}(t) = -\int_0^t (-\Delta)^{1/2} e^{(t-\tau)\Delta} (-\Delta)^{-1/2} \mathrm{div}\big(\theta(\tau) u(\tau)\big)\, d\tau$$

for a.a. $t \in [0, T[$. Consequently, (2.3), (2.5) imply

$$\|\tilde{\theta}(t)\|_2 \le c(\Omega) \int_0^T |t - \tau|^{-1/2} \|(-\Delta)^{-1/2} \mathrm{div} \theta(\tau) u(\tau)\|_2 \, d\tau$$

for a.a $t \in [0, T[$. The Hardy-Littlewood inequality in the form $\frac{1}{2} + \frac{1}{4} = \frac{1}{4/3}$ yields

$$\|\tilde{\theta}\|_{2,4;T} \le c \|(-\Delta)^{-1/2} \mathrm{div}(\theta u)\|_{2,\frac{4}{3};T} \le c \|\theta u\|_{2,\frac{4}{3};T}.$$

Especially $\tilde{\theta} \in L^2(0, T; L^2(\Omega))$. The consistence of the Laplace operator and $\theta u \in L^{s_2}(0, T; L^{q_2}(\Omega))$ yield

$$\tilde{\theta}(t) = -\int_0^t (I - \Delta_{q_2})^{1/2} e^{(t-\tau)\Delta_{q_2}} (I - \Delta_{q_2})^{-1/2} \mathrm{div}\big(\theta(\tau) u(\tau)\big)\, d\tau \tag{4.10}$$

for almost all $t \in [0, T[$. In the proof that θ fulfils (1.3) we will proceed as in step 2. Let $J_n := (I + \frac{1}{n}(-\Delta_{q_2})^{1/2})^{-1}$, $n \in \mathbb{N}$, be the Yosida approximation of I in $L^{q_2}(\Omega)$. Then $\tilde{\theta} = J_n \tilde{\theta} + \frac{1}{n}(-\Delta_{q_2})^{1/2} J_n \tilde{\theta}$. An analogous argumentation as in step 2 that leads from (4.3) to (4.6) shows that

$$\|(-\Delta_{q_2})^{1/2} J_n \tilde{\theta}\|_{2,2;T} \le c \|u\|_{q,s;T} \|\nabla E_2\|_{2,2;T}$$

holds with a constant $c = c(\Omega, q, T) > 0$ independent of $n \in \mathbb{N}$. Consequently $\nabla\tilde{\theta} \in L^2(0, T; L^2(\Omega))$. It follows from (4.9) that

$$\tilde{\theta}(t) = -\int_0^t e^{(t-\tau)\Delta_{q_2}} \big(u \cdot \nabla\theta\big)(\tau)\, d\tau \quad \text{for a.a. } t \in [0, T[. \tag{4.11}$$

Proceeding analogously as in step 2 we can prove that θ fulfils (1.3). The proof is complete. □

5. Proof of the remaining results

5.1. **Proof of Corollary 1.4.** Let ϵ_* be the constant constructed in Theorem 1.3, let E_1, E_2 be defined as in (1.4), (1.5). Further, introduce

$$(\mathcal{G} f_1)(t) := \int_0^t e^{-(t-\tau)A} P f_1(\tau)\, d\tau\,, \quad \text{a.a. } t \in [0, T[\,, \tag{5.1}$$

$$(\mathcal{H} f_2)(t) := \int_0^t e^{(t-\tau)\Delta} f_2(\tau)\, d\tau\,, \quad \text{a.a. } t \in [0, T[. \tag{5.2}$$

With the help of [25, IV, Lemma 2.4.2 d)] it follows

$$\|\mathcal{G} f_1\|_{2,2;T} \le T^{1/2} \|\mathcal{G} f_1\|_{2,\infty;T} \le 4\, T^{1/2} \|f_1\|_{2,1;T}. \tag{5.3}$$

Due to $\frac{1}{3} + \frac{1}{q} \geq \frac{1}{q_*} \geq \frac{1}{q}$ we can choose $0 \leq \sigma \leq \frac{1}{2}$ such that $2\sigma + \frac{3}{q} = \frac{3}{q_*}$. Further, since $\mathcal{S}(s_*, q_*) = 3$ it follows $1 < q_* < 3$. Consequently, (2.2), (2.4) can be applied to get

$$\begin{aligned} \|(\mathcal{G}f_1)(t)\|_q &\leq c\Big\| \int_0^t A_{q_*}^{\sigma} e^{-(t-\tau)A_{q_*}} Pf_1(\tau)\, d\tau \Big\|_{q_*} \\ &\leq c \int_0^T |t-\tau|^{-\sigma} \|f_1(\tau)\|_{q_*}\, d\tau \end{aligned} \tag{5.4}$$

for a.a. $t \in [0, T[$. We use $(1-\sigma) + \frac{1}{s} = \frac{1}{s_*}$ and the Hardy-Littlewood inequality to get

$$\|\mathcal{G}f_1\|_{q,s;T} \leq c\|f_1\|_{q_*,s_*;T} \tag{5.5}$$

with a constant $c = c(\Omega, q, q_*) > 0$. Combining (5.3), (5.5) yields

$$\|\mathcal{G}f_1\|_{2,2;T} + \|\mathcal{G}f_1\|_{q,s;T} \leq c(\Omega, q, q_*)\big(\|f_1\|_{q_*,s_*;T} + T^{1/2}\|f_1\|_{2,1;T}\big).$$

From [21, Lemma 3.2 (ii)] we get the estimate $\|\mathcal{H}f_2\|_{4,\frac{8}{3};T} \leq c\|f_2\|_{2,1;T}$ with an absolute constant $c > 0$. Altogether

$$\begin{aligned} &\|E_1\|_{q,s;T} + \|E_1\|_{2,2;T} + \|E_2\|_{4,\frac{8}{3};T} \\ &\leq \Big(\int_0^T \|e^{-tA}u_0\|_2^2\, dt\Big)^{1/2} + \int_0^T \Big(\|e^{-tA}u_0\|_q^s\, dt\Big)^{1/s} + \Big(\int_0^T \|e^{t\Delta}\theta_0\|_4^{8/3}\, dt\Big)^{3/8} \\ &\quad + c\|f_1\|_{q_*,s_*;T} + c\,T^{1/2}\|f_1\|_{2,1;T} + c\|f_2\|_{2,1;T} \end{aligned} \tag{5.6}$$

holds with a constant $c = c(\Omega, q, q_*) > 0$. Looking at (5.6) it follows that there exists a constant $c_* = c_*(\Omega, q, q_*, p) > 0$ such that if the conditions (1.8), (1.9), (1.10), (1.11) are satisfied where ϵ_* is replaced by c_*, then (1.6), (1.7) hold. Consequently, Theorem 1.3 implies the existence of a strong solution in this case.

5.2. **Proof of Theorem 1.5.** Define $\mathcal{T}(t) := e^{t\Delta}\theta_0\,, t \in [0, T[$. By linear theory (see [25, Lemma IV.2.4.2]) if follows $\mathcal{T} \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$. Due to the continuous imbedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$ it follows $\mathcal{T} \in L^{8/3}(0, T; L^4(\Omega))$. Therefore, the sufficiency of (1.12) for the existence of $0 < T' \leq T$ and strong solution (u, θ) of (1.1) in $[0, T'[\times\Omega$ follows from Corollary 1.4.

For the proof of the converse direction let (u, θ) be a strong solution of (1.1) with $u \in L^s(0, T'; L^q(\Omega))$ where $0 < T' \leq T$. Let $\mathcal{L}\theta$ be as in Lemma 3.3 and $\mathcal{G}f_1$ as in (5.1). From (3.6) we get

$$e^{-tA} = u(t) - \tilde{u}(t) - (\mathcal{L}\theta)(t) - (\mathcal{G}f_1)(t)\,, \quad \text{a.a. } t \in [0, T'[\,, \tag{5.7}$$

where

$$\tilde{u}(t) := -\int_0^t A^{1/2} e^{-(t-\tau)A} A^{-1/2} P\mathrm{div}\big(u(\tau) \otimes u(\tau)\big)\, d\tau$$

for a.a. $t \in [0, T'[$. From [8, (3.11)] with $r_1 := \frac{q}{2}\,, r_2 := q$ it follows

$$\tilde{u}(t) = -\int_0^t A_q^{\alpha} e^{-(t-\tau)A_q} A_q^{-\alpha} P_q\mathrm{div}\big(u(\tau) \otimes u(\tau)\big)\, d\tau$$

for a.a. $t \in [0, T'[$ where $\alpha := \frac{1}{2} + \frac{3}{2q}$. By (3.11), (3.16) and (5.5) we get $e^{-tA} \in L^s(0, T'; L^q(\Omega))$. With the help of [17, Theorem 1.2 (ii)] we obtain $\int_{T'}^{\infty} \|e^{-tA}u_0\|_q^s \, dt \le c \int_{T'}^{\infty} t^{-\frac{3}{2}s(\frac{1}{2}-\frac{1}{q})} \|u_0\|_2^s \, dt < \infty$.

5.3. **Proof of Theorem 1.7.** First, let us remark that the following theorem holds:

Theorem 5.1. *Consider data as in Theorem 1.3. Assume that* (u, θ) *and* (v, Θ) *are weak solutions of (1.1) with* $u, v \in L^s_{loc}([0, T[; L^q(\Omega))$. *Then* $u(t) = v(t)$ *and* $\theta(t) = \Theta(t)$ *for a.a.* $t \in [0, T[$.

Proof. The proof of [19, Theorem 1.5] is based on [19, (3.5), (3.6), (3.7)]. Since we can replace these estimates by (3.11) (3.14), (3.16) the proof of the result above is the same as [19, Theorem 1.5] with $s_1 = \frac{8}{3}, q_1 = 4$. □

Let (u, θ) and (v, Θ) be as in Theorem 1.7 and let $1 < x_1, x_2 < \infty$ be defined by $\frac{1}{x_1} = \frac{1}{2} - \frac{1}{s}$ and $\frac{1}{x_2} = \frac{1}{2} - \frac{1}{q}$. Since $\mathcal{S}(x_1, x_2) = \frac{3}{2}$ it follows $u \in L^{x_1}(0, T; L^{x_2}(\Omega))$. Therefore

$$\|u \otimes u\|_{2,2;T} \le \|u\|_{x_2,x_1;T} \|u\|_{q,s;T} < \infty.$$

By [25, Theorem IV.2.3.1] we obtain that $u : [0, T[\to L^2_\sigma(\Omega)$ is strongly continuous and that (u, θ) satisfy (1.13). Considering (u, θ) as a weak solution of the Boussinesq equations (1.1) in $[t_0, T - t_0[$ we obtain from Theorem 1.5

$$\int_0^\infty \|e^{-tA}u(t_0)\|_q^s \, dt < \infty \quad \text{for all } t_0 \in [0, T[.$$

Now all requirements of [21, Corollary 1.7] are fulfilled. Using Theorem 1.5 and Theorem 5.1 we can follow the proof of [21, Corollary 1.7] to show that $u(t) = v(t)$ and $\theta(t) = \Theta(t)$ for a.a. $t \in [0, T[$.

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