Existence of solutions on the whole time axis to the Navier-Stokes equations with precompact range in L^3

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Abstract

We present an existence theorem of mild solutions on the whole time axis to the Navier-Stokes equations in unbounded domains $\Omega \subset \mathbb{R}^3$ having precompact range in $L^3(\Omega)$, if the external force is small and has precompact range in some function space. In our forthcoming paper [8] we proved the uniqueness of such solutions.

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1 Introduction

The motion of a viscous incompressible fluid in domains $\Omega \subset \mathbb{R}^3$ is governed by the Navier-Stokes equations:

$$\text{(N-S)} \begin{cases} \partial_t u - \Delta u + u \cdot \nabla u + \nabla p = f, & t \in \mathbb{R}, \quad x \in \Omega, \\ \text{div } u = 0, & t \in \mathbb{R}, \quad x \in \Omega, \\ u|_{\partial\Omega} = 0, & t \in \mathbb{R}, \end{cases}$$

where $u = (u^1(x,t), u^2(x,t), u^3(x,t))$ and $p = p(x,t)$ denote the velocity vector and the pressure, respectively, of the fluid at the point $(x,t) \in \Omega \times \mathbb{R}$. Here f is a given external force. In this paper we consider mild solutions to (N-S) in *unbounded* domains Ω which are bounded on the whole time axis. Typical examples of such solutions are periodic-in-time and almost periodic-in-time solutions.

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In case where $\Omega \subset \mathbb{R}^3$ is bounded, the existence and uniqueness of time-periodic solutions were considered by several authors; see e.g. [9] and references therein. Maremonti [26, 27] was the first to prove the existence of unique time-periodic regular solutions to (N-S) in *unbounded* domains, namely for $\Omega = \mathbb{R}^3$ and $\Omega = \mathbb{R}^3_+$. In the case of more general unbounded domains, the existence of time-periodic solutions was proven by Kozono-Nakao [22], Maremonti-Padula [28], Salvi [32], Yamazaki [38], Galdi-Sohr [16], Kubo [24], Crispo-Maremonti [5] and Kang-Miura-Tsai [21]. In [38, 21], solutions in $L^{3,\infty}$, the weak L^3 space, were dealt with. Without time-periodic condition on f, the existence of mild solutions bounded on the whole time axis was also shown in [22], [38] and [21].

Concerning the uniqueness of solutions bounded on the whole time-axis, roughly speaking, it was shown in [26, 27, 22, 28, 38, 24, 5] that a small solution in some function spaces (e.g. $BC(\mathbb{R}; L^{3,\infty}(\Omega))$) is unique within the class of solutions which are sufficiently small; i.e., if u and v are solutions for the same force f and if *both of them* are small, then $u = v$. In [16], it was shown that a small time-periodic solution is unique within the larger class of all periodic weak solutions v with $\nabla v \in L^2(0, T; L^2)$, satisfying the energy inequality $\int_0^T \|\nabla v\|_{L^2}^2 \, d\tau \le - \int_0^T (F, \nabla v) \, d\tau$ and mild integrability conditions on the corresponding pressure; here T is a period of F and $f = \nabla \cdot F$.

Another type of uniqueness theorem for $L^{3,\infty}$-solution bounded on the whole time axis was proven in our previous paper [8] without assuming the energy inequality or time-periodic condition. In the case of an exterior domain $\Omega \subset \mathbb{R}^3$, the whole space $\mathbb{R}^3$, the halfspace $\mathbb{R}^3_+$, a perturbed halfspace, or an aperture domain, it was shown in [8] that if u and v are solutions in

$$BC(\mathbb{R}; \tilde{L}^{3,\infty}) \tag{1.1}$$

for the same force f, *one of them* is small in $L^{3,\infty}$ and if the other solution has a *precompact* range in $L^{3,\infty}$, then $u = v$. Here $\tilde{L}^{3,\infty} := \overline{L^{3,\infty}_\sigma \cap L^\infty}^{\|\cdot\|_{L^{3,\infty}}}$. See also [36, 13, 14, 30, 31]. This uniqueness theorem is applicable to time-periodic and almost periodic solutions given in [22, 38], since continuous time-periodic and almost periodic-in-time $L^{3,\infty}$-solutions u have a *precompact* range $\mathcal{R}(u) = \{u(t); t \in \mathbb{R}\}$ in $L^{3,\infty}$, see [4, Theorem 6.5]. Note that there exist many functions which have a precompact range and are not almost periodic, e.g. $a \sin(t^2)$ for $a \neq 0$. Hence, the set of functions with precompact range is much larger than the set of almost periodic functions.

While in [8] we proved the uniqueness theorem of solutions on $\mathbb{R}$ with precompact range as mentioned above, the existence theorem of such solutions was not known except in the special case where f is almost periodic. In the present paper, we prove existence of continuous solutions u having precompact range $\mathcal{R}(u)$ in L^3 on the whole time axis, if the external force is small and has precompact range in some function space and if Ω satisfies the following Assumption 1.

Assumption 1 *$\Omega \subset \mathbb{R}^3$ is the half-space $\mathbb{R}^3_+$, the whole space $\mathbb{R}^3$, an infinite layer, a perturbed half-space, or an aperture domain with $\partial\Omega \in C^{2,1}$.*

For the definitions of infinite layers, perturbed half-spaces and aperture domains, see Abe-Shibata [1], Kubo-Shibata [25] and Farwig-Sohr [10, 11].

Let $BUC(\mathbb{R}; Y)$ denote the set of all bounded uniformly continuous functions on $\mathbb{R}$ with values in a Banach space Y. Now our main result reads as follows:

Theorem 1. *Let $\Omega \subset \mathbb{R}^3$ be a domain satisfying Asssumption 1. Moreover, let q, r, l satisfy $3/2 < q < 3$, $2 < r < 3$, $1/\kappa := 1/r + 1/q < 1$ and $(1/3 + 1/q)^{-1} < l < q$ and let*

$$X := \{v \in L^r_\sigma \; ; \; \nabla v \in L^q\} \quad \text{with norm} \quad \|v\|_X := \|v\|_{L^r} + \|\nabla v\|_{L^q}.$$

There exists a number $\epsilon = \epsilon(q, r, l, \Omega) > 0$ with the following property: Let f have the form $f = \nabla \cdot F$ with $F \in BUC(\mathbb{R}; L^{r/2})$ and $\nabla \cdot F \in BUC(\mathbb{R}; L^\kappa \cap L^l)$,

$$\sup_t \|\nabla \cdot F(t)\|_{L^\kappa \cap L^l} + \sup_t \|F(t)\|_{L^{r/2}} < \epsilon \tag{1.2}$$

and let the range $\mathcal{R}(F)$ be precompact in $L^{r/2}$. Then there exists a solution $u \in BUC(\mathbb{R}; X)$ of the integral equation

$$(I.E.) \qquad u(t) = \int_{-\infty}^t e^{-(t-s)A} P(-u \cdot \nabla u + \nabla \cdot F)(s)\, ds$$

such that $\mathcal{R}(u)$ is precompact in L^3_σ. Here L^3_σ, the Stokes operator A and the Helmholtz operator P are defined in the next section.

Remark 1. (i) The existence of L^3-solutions to (I.E.) was proven by Kozono-Nakao [22] and Kubo [24] with a smallness condition on f slightly different from (1.2).

(ii) Thanks to the uniqueness theorem in [8], the small solution given in Theorem 1 is unique within the class of all solutions in $BC(\mathbb{R}; L^3_\sigma)$ having a precompact range in L^3.

(iii) Since $C^\infty_{0,\sigma}$ is dense in L^3_σ and since $\mathcal{R}(u)$ is precompact in L^3_σ, we see that u has a uniform decay property in the following sense:

$$\lim_{R\to\infty} \sup_{t\in\mathbb{R}} \|u(t)\|_{L^3(\{|x|>R\})} = 0.$$

2 Notations and Key lemmata

In this section, we introduce some notation and key lemmata. Let $C^\infty_{0,\sigma}(\Omega) = C^\infty_{0,\sigma}$ denote the set of all C^∞-real vector fields $\phi = (\phi^1, \phi^2, \phi^3)$ with compact support in Ω such that $\operatorname{div}\phi = 0$. Then L^r_σ, $1 < r < \infty$, is the closure of $C^\infty_{0,\sigma}$ with respect to the L^r-norm $\|\cdot\|_r$. Concerning Sobolev spaces we use the notations $W^{k,p}(\Omega)$ and $W^{k,p}_0(\Omega)$, $k \in \mathbb{N}$, $1 \le p \le \infty$. Note that very often we will simply write L^r and $W^{k,p}$ instead of $L^r(\Omega)$ and $W^{k,p}(\Omega)$, respectively. The symbol $(\cdot,\cdot)$ denotes the L^2- inner product and the duality pairing between L^p and $L^{p'}$, where $1/p + 1/p' = 1$.

In this paper, we denote by C various constants. In particular, $C = C(*, \cdots, *)$ denotes a constant depending only on the quantities appearing in the parentheses.

Let us recall the Helmholtz decomposition: $L^r(\Omega) = L^r_\sigma \oplus G_r$ $(1 < r < \infty)$, where $G_r = \{\nabla p \in L^r; p \in L^r_{loc}(\overline{\Omega})\}$, see Fujiwara-Morimoto [15], Miyakawa [29], Simader-Sohr [35], Borchers-Miyakawa [2], Farwig-Sohr [10, 12] and Abe-Shibata [1]; then P_r denotes the projection operator from L^r onto L^r_σ along G_r. The Stokes operator on L^r_σ is defined by $A_r = -P_r\Delta$ with domain $D(A_r) = W^{2,r} \cap W^{1,r}_0 \cap L^r_\sigma$. It is known that $(L^r_\sigma)^*$ (the dual space of L^r_σ) $= L^{r'}_\sigma$ and A^*_r (the adjoint operator of A_r) $= A_{r'}$. It is shown by Giga [17], Giga-Sohr [18], Borchers-Miyakawa [2], Farwig-Sohr [10, 12] and Abe-Shibata [1] that $-A_r$ generates a uniformly bounded holomorphic semigroup $\{e^{-tA_r}; t \ge 0\}$ of class C_0 in L^r_σ. Since $P_r u = P_q u$ for all $u \in L^r \cap L^q$ $(1 < r, q < \infty)$ and since $A_r u = A_q u$ for all $u \in D(A_r) \cap D(A_q)$, for simplicity, we shall abbreviate $P_r u, P_q u$ as Pu for $u \in L^r \cap L^q$ and $A_r u, A_q u$ as Au for $u \in D(A_r) \cap D(A_q)$, respectively.

Proposition 2.1. *Let Y and Y_1 be Banach spaces, let $\mathcal{L}(Y, Y_1)$ denote the space of bounded linear operators from Y to Y_1, and let $B_r(y)$ be the open ball of radius $r > 0$ and center y in Y or Y_1.*

(i) Let (f_j) be a sequence in $BC(\mathbb{R}; Y)$. Assume that for each j the range $\mathcal{R}(f_j) := \{f_j(t); t \in \mathbb{R}\}$ is precompact in Y and f_j converges to a function f in $BC(\mathbb{R}; Y)$. Then $\mathcal{R}(f)$ is precompact in Y.

(ii) Let $A \subset Y$ be precompact, and let $T(\cdot) : [m, M] \to \mathcal{L}(Y, Y_1)$ be continuous. Then the set $T(\cdot)A = \{T(s)a : s \in [m, M], a \in A\}$ is precompact in Y_1. In particular, if $f \in BC(\mathbb{R}; Y)$ has precompact range $\mathcal{R}(f)$ in Y and $T \in \mathcal{L}(Y, Y_1)$, then $\mathcal{R}(Tf) := \{Tf(t); t \in \mathbb{R}\}$ is precompact in B_1.

(iii) Let $A \subset Y$ be precompact. Then the convex hull $\operatorname{conv}(A)$ *is precompact in Y.*

(iv) Let $1 < p < \infty$, $f = (f_1, f_2, f_3) \in BC(\mathbb{R}; L^p)$ and $\mathcal{R}(f)$ be precompact in L^p. Then, $\mathcal{R}(f \otimes f)$ is precompact in $(L^{p/2})^{3\times 3}$.

Proof. (i) Let $\epsilon > 0$ be fixed. By the assumption, we find $j_0 \in \mathbb{N}$ such that $\sup_t \|f(t) - f_{j_0}(t)\|_Y < \epsilon$. Since $\mathcal{R}(f_{j_0}) \subset Y$ is precompact, there exists a finite set $\{y_1, \dots, y_N\} \subset B$ such that $\mathcal{R}(f_{j_0}) \subset \bigcup_{k=1}^N B_\epsilon(y_k)$. Hence, we conclude that $\mathcal{R}(f) \subset \bigcup_{k=1}^N B_{2\epsilon}(y_k)$.

(ii) Given $\epsilon > 0$, there exists to $\eta := \epsilon/\big(1 + \sup_{a\in A} \|a\| + \sup_{s\in[m,M]} \|T(s)\|_{\mathcal{L}(Y,Y_1)}\big)$ finitely many $a_i \in A$ such that $A \subset \bigcup_i B_\eta(a_i)$. Moreover, there exists $n \in \mathbb{N}$ such that $\|T(s) - T(s')\|_{\mathcal{L}(Y,Y_1)} < \eta$ for all $s, s' \in [m, M]$ with $|s - s'| \le \frac{M-m}{n}$. Then, with $s_j = m + \frac{j}{n}(M - m)$, $j = 0, \dots, n-1$, we get for $s \in [s_j, s_{j+1}]$ and $a \in B_\eta(a_i)$ that

$$\|T(s)a - T(s_j)a_i\| \le \|T(s)a - T(s_j)a\| + \|T(s_j)a - T(s_j)a_i\| \tag{2.1}$$

$$\le \eta \sup_{a\in A} \|a\| + \eta \sup_{s\in[m,M]} \|T(s)\|_{\mathcal{L}(Y,Y_1)} \le \epsilon. \tag{2.2}$$

Hence $T(\cdot)A \subset \bigcup_{i,j} B_\varepsilon(T(s_j)a_i)$, proving the precompactness of $T(\cdot)A$ in Y_1. The second statement is a trivial consequence.

(iii) is well-known, see [6, Proposition 7.2 (d)]. (iv) is easy to prove. □

Lemma 2.1 ([37, 18, 2, 3, 1, 19, 25, 23])**.** *For all $t > 0$ and $\phi \in L^q_\sigma$, the following inequalities are satisfied:*

$$\|e^{-tA}\phi\|_p \le Ct^{-3/2(1/q-1/p)}\|\phi\|_q \quad \text{when } 1 < q \le p < \infty, \tag{2.3}$$

$$\|\nabla e^{-tA}\phi\|_p \le Ct^{-1/2-3/2(1/q-1/p)}\|\phi\|_q \quad \text{when } 1 < q \le p < \infty, \tag{2.4}$$

where the constant $C = C(p, q, \Omega)$ is independent of ϕ.

It is notable that in the case where Ω is an exterior domain with smooth boundary Iwashita [20] proved that (2.3) holds for $1 < q \le p < \infty$ and (2.4) holds for $1 < q \le p \le 3$. In the Lorentz spaces $L^{p,s}$, similar estimates hold, see [33, 34, 38].

Lemma 2.2. *Let $1 < p < \infty$ and $0 < m < M < \infty$.*

(i) Let $f \in BUC(\mathbb{R}; L^p)$ and let

$$g(t) := \int_m^M e^{-sA} P f(t - s)\, ds \quad \text{for } t \in \mathbb{R}.$$

If $\mathcal{R}(f)$ is precompact in L^p, then $\mathcal{R}(g)$ is precompact in L^p_σ.

(ii) Let $F = (F_{ij})_{i,j=1,2,3} \in BUC(\mathbb{R}; L^p)$, and let

$$h(t) := \int_m^M e^{-sA} P\nabla \cdot F(t - s)\, ds \quad \text{for } t \in \mathbb{R}.$$

If $\mathcal{R}(F)$ is precompact in L^p, then $\mathcal{R}(h)$ is a precompact set in L^α_σ for all $\alpha \ge p$.

Proof. (i) The main idea is to approximate integrals by Riemannian sums and to use Proposition 2.1.

Let us consider the partition $s_j = m + \frac{jL}{n}$, $j = 0, \dots, n$, where $L = M - m$, of the interval $[m, M]$, and let $f_t(s) := Pf(t-s)$, $t \in \mathbb{R}$. Then $f_t \in BUC(\mathbb{R}; L^p)$ and $\|f_t\|_{BUC(\mathbb{R};L^p)} \le c\|f\|_{BUC(\mathbb{R};L^p)}$ for each $t \in \mathbb{R}$. Moreover, we recall for any $x \in L^p_\sigma$ and $0 < t < s < \infty$ the trivial estimate

$$\begin{aligned} \|(e^{-sA} - e^{-tA})x\|_p &= \|(e^{-(s-t)A} - I)e^{-tA}x\|_p \\ &\le \int_0^{s-t} \|e^{-\tau A} A e^{-tA} x\|_p \, d\tau \\ &\le \frac{c}{t}(s-t)\|x\|_p. \end{aligned} \tag{2.5}$$

Now we get from (2.5) the convergence property

$$\begin{aligned} &\Big\| \int_m^M e^{-sA} f_t(s)\, ds - \frac{L}{n} \sum_{j=0}^{n-1} e^{-s_j A} f_t(s_j) \Big\|_p \\ &\le \sum_{j=0}^{n-1} \int_{s_j}^{s_{j+1}} \big\| e^{-sA} f_t(s) - e^{-s_j A} f_t(s_j) \big\|_p \, ds \\ &\le \sum_{j=0}^{n-1} \int_{s_j}^{s_{j+1}} \Big(\big\| \big(e^{-sA} - e^{-s_j A}\big) f_t(s)) \big\|_p + \|e^{-s_j A}\| \|f_t(s) - f_t(s_j)\|_p \Big) \, ds \\ &\le c \sum_{j=0}^{n-1} \int_{s_j}^{s_{j+1}} \Big(\frac{s - s_j}{s_j} \|f_t(s)\|_p + \|f_t(s) - f_t(s_j)\|_p \Big) \, ds \\ &\le c \frac{L^2}{nm} \|f\|_{BUC(\mathbb{R};L^p)} + cL \sup_t \sup_{|s| \le L/n} \|f(t) - f(t-s)\|_p \\ &\to 0 \end{aligned}$$

as $n \to \infty$ uniformly in $t \in \mathbb{R}$.

By Proposition 2.1 (ii), (iii), and the norm continuity of the analytic semigroup e^{-sA} in $s \in [m, M]$ the sets $E_0 = \{e^{-sA} f_t(s') : s, s' \in [m, M], t \in \mathbb{R}\}$ and also $E_1 = \{\frac{L}{n} \sum_{j=0}^{n-1} e^{-(m + \frac{jL}{n})A} f_t(m + \frac{jL}{n});\ t \in \mathbb{R}, n \in \mathbb{N}\}$ are precompact in L^p_σ. Therefore, by Proposition 2.1 (i) and the above convergence, $\mathcal{R}(g) \subset \overline{E}_1$ is precompact in L^p_σ as well.

(ii) Let $T := e^{-\frac{m}{2}A} P \nabla \cdot$. By a duality argument, it is straightforward to see that the operator T can be extended as a bounded operator from L^p to L^α_σ for all $\alpha \ge p$. Using this operator, h can be written as follows:

$$h(t) = \int_{\frac{m}{2}}^{M - \frac{m}{2}} e^{-sA} P(TF(t - s - \frac{m}{2}))\, ds.$$

Since $\mathcal{R}(F)$ is precompact in L^p and since $F \in BUC(\mathbb{R}; L^p)$, by Proposition 2.1 (ii) we see that $\mathcal{R}(TF)$ is precompact in L^α_σ and $TF \in BUC(\mathbb{R}; L^\alpha_\sigma)$ for all $\alpha \geq p$. Then, it follows immediately from part (i) that $\mathcal{R}(h)$ is precompact in L^α_σ for all $\alpha \geq p$. □

Lemma 2.3. *Let q, r, l, κ satisfy the hypotheses of Theorem 1. Then there exists a constant $C(q, r, l, \Omega) > 0$ such that*

$$\sup_t \int_M^\infty \|e^{-sA} P\nabla \cdot F(t-s)\|_X \, ds \leq CM^{\frac{1}{2}-\frac{3}{2r}}\big(\|F\|_{L^\infty(\mathbb{R};L^{r/2})} + \|\nabla \cdot F\|_{L^\infty(\mathbb{R};L^\kappa)}\big), \tag{2.6}$$

$$\sup_t \int_0^m \|e^{-sA} P\nabla \cdot F(t-s)\|_X \, ds \leq C\big(m^{1-\frac{3}{2q}}\|\nabla \cdot F\|_{L^\infty(\mathbb{R};L^\kappa)} + m^{\frac{1}{2}-\frac{3}{2l}+\frac{3}{2q}}\|\nabla \cdot F\|_{L^\infty(\mathbb{R};L^l)}\big), \tag{2.7}$$

$$\sup_t \int_M^\infty \|e^{-sA} P(u \cdot \nabla v)(t-s)\|_X \, ds \leq CM^{\frac{1}{2}-\frac{3}{2r}}\|u\|_{L^\infty(\mathbb{R};X)}\|v\|_{L^\infty(\mathbb{R};X)}, \tag{2.8}$$

$$\sup_t \int_0^m \|e^{-sA} P(u \cdot \nabla v)(t-s)\|_X \, ds \leq Cm^{1-\frac{3}{2q}}\|u\|_{L^\infty(\mathbb{R};X)}\|v\|_{L^\infty(\mathbb{R};X)} \tag{2.9}$$

for all $m, M > 0$, $u, v \in BC(\mathbb{R}; X)$ and all $F \in BC(\mathbb{R}; L^{r/2})$ with $\nabla \cdot F \in L^\infty(\mathbb{R}; L^\kappa \cap L^l)$.

Remark 2. (i) X is continuously embedded in L^3_σ.

(ii) Letting $M = 1, m = 1$, by (2.6)-(2.9) we have

$$\begin{aligned} &\sup_t \int_0^\infty \|e^{-sA} P\nabla \cdot F(t-s)\|_X \, ds \leq C(\|F\|_{L^\infty(\mathbb{R};L^{r/2})} + \|\nabla \cdot F\|_{L^\infty(\mathbb{R};L^\kappa \cap L^l)}), \\ &\sup_t \int_0^\infty \|e^{-sA} P(u \cdot \nabla v)(t-s)\|_X \, ds \leq C\|u\|_{L^\infty(\mathbb{R};X)}\|v\|_{L^\infty(\mathbb{R};X)}. \end{aligned} \tag{2.10}$$

Proof of Lemma 2.3. This lemma was essentially proven in [22]. Since

$$|(e^{-sA} P\nabla \cdot F, \phi)| = |(F, \nabla e^{-sA}\phi)| \leq C\|F\|_{r/2} s^{-\frac{1}{2}-\frac{3}{2r}}\|\phi\|_{(1-\frac{1}{r})^{-1}} \text{ for } \phi \in C^\infty_{0,\sigma},$$

it holds that

$$\|e^{-sA} P\nabla \cdot F\|_r \leq Cs^{-\frac{1}{2}-\frac{3}{2r}}\|F\|_{r/2},$$
$$\|e^{-sA} P\nabla \cdot (u \otimes v)\|_r \leq Cs^{-\frac{1}{2}-\frac{3}{2r}}\|u\|_r\|v\|_r.$$

Then, since $r < 3$,

$$\int_M^\infty \|e^{-sA} P\nabla \cdot F(t-s)\|_r \, ds \leq CM^{\frac{1}{2}-\frac{3}{2r}}\|F\|_{L^\infty(\mathbb{R};L^{r/2})}, \tag{2.11}$$

$$\int_M^\infty \|e^{-sA} P\nabla \cdot (u \otimes v)(t-s)\|_r \, ds \leq CM^{\frac{1}{2}-\frac{3}{2r}}\|u\|_{L^\infty(\mathbb{R};L^r)}\|v\|_{L^\infty(\mathbb{R};L^r)}. \tag{2.12}$$

It is straightforward to see that

$$(2.13)\qquad \int_M^\infty \|\nabla e^{-sA}P\nabla\cdot F(t-s)\|_q\,ds \;\le\; CM^{\frac12-\frac{3}{2r}}\|\nabla\cdot F\|_{L^\infty(\mathbb{R};L^\kappa)},$$

$$(2.14)\qquad \int_M^\infty \|\nabla e^{-sA}P(u\cdot\nabla v)(t-s)\|_q\,ds \;\le\; CM^{\frac12-\frac{3}{2r}}\|u\|_{L^\infty(\mathbb{R};L^r)}\|\nabla v\|_{L^\infty(\mathbb{R};L^q)}.$$

Hence, from (2.11)–(2.14) we obtain (2.6) and (2.8). Moreover, we get the estimates

$$(2.15)\qquad \begin{aligned}\int_0^m \|e^{-sA}P\nabla\cdot F(t-s)\|_r\,ds &\le C\int_0^m s^{-\frac{3}{2q}}\|\nabla\cdot F(t-s)\|_{L^\kappa}\,ds\\ &\le Cm^{1-\frac{3}{2q}}\|\nabla\cdot F\|_{L^\infty(\mathbb{R};L^\kappa)},\end{aligned}$$

$$(2.16)\qquad \int_0^m \|e^{-sA}P(u\cdot\nabla v)(t-s)\|_r\,ds \;\le\; Cm^{1-\frac{3}{2q}}\|u\|_{L^\infty(\mathbb{R};L^r)}\|\nabla v\|_{L^\infty(\mathbb{R};L^q)},$$

as well as

$$(2.17)\qquad \begin{aligned}\int_0^m \|\nabla e^{-sA}P\nabla\cdot F(t-s)\|_q\,ds &\le C\int_0^m s^{-\frac12-\frac32(\frac1l-\frac1q)}\|\nabla\cdot F(t-s)\|_l\,ds\\ &\le Cm^{\frac12-\frac{3}{2l}+\frac{3}{2q}}\|\nabla\cdot F\|_{L^\infty(\mathbb{R};L^l)},\end{aligned}$$

$$(2.18)\qquad \begin{aligned}\int_0^m \|\nabla e^{-sA}P(u\cdot\nabla v)(t-s)\|_q\,ds &\le C\int_0^m s^{-\frac12-\frac32((\frac{1}{q^*}+\frac1q)-\frac1q)}\|u\|_{q^*}\|\nabla v\|_q\,ds\\ &\le Cm^{1-\frac{3}{2q}}\|\nabla u\|_{L^\infty(\mathbb{R};L^q)}\|\nabla v\|_{L^\infty(\mathbb{R};L^q)}\end{aligned}$$

where $1/q* = 1/q - 1/3$. Here we used the Sobolev inequality $\|f\|_{q^*} \le C\|\nabla f\|_q$ for $f \in X$, see [12, Lemma 3.1]. Hence, from (2.15)–(2.18) we obtain (2.7) and (2.9). □

3 Proof of Theorem 1

Proof of Theorem 1. In the same way as in [22, 24], we can construct a solution u to (I.E.) by an iterative procedure. Indeed, let

$$(3.1)\qquad \begin{aligned}u_0(t) &\equiv 0,\\ u_{j+1}(t) &= \int_{-\infty}^t e^{-(t-s)A}P(-u_j\cdot\nabla u_j + \nabla\cdot F)(s)\,ds\\ &= -\int_0^\infty e^{-sA}P\nabla\cdot(u_j\otimes u_j)(t-s)\,ds + \int_0^\infty e^{-sA}P\nabla\cdot F(t-s)\,ds\\ &=: G_j(t) + G_0(t).\end{aligned}$$

Then, (2.10) yields that for $h \in \mathbb{R}$,

$$\begin{aligned}\sup_t \|G_0(t+h) - G_0(t)\|_X \le& C\big(\sup_t \|F(t+h) - F(t)\|_{L^{r/2}}\\ &+ \sup_t \|\nabla\cdot F(t+h) - \nabla\cdot F(t)\|_{L^\kappa\cap L^l}\big),\\ \sup_t \|G_j(t+h) - G_j(t)\|_X \le& 2C\big(\sup_t \|u_j(t)\|_X\big)\big(\sup_t \|u_j(t+h) - u_j(t)\|_X\big),\end{aligned}$$

and

$$\begin{aligned}\|u_{j+1}\|_{L^\infty(\mathbb{R};X)} \leq& C\big(\|F\|_{L^\infty(\mathbb{R};L^{r/2})} + \|\nabla\cdot F\|_{L^\infty(\mathbb{R};L^\kappa\cap L^l)}\big) + C\|u_j\|^2_{L^\infty(\mathbb{R};X)},\\ \|u_{j+1}-u_j\|_{L^\infty(\mathbb{R};X)} \leq& C\big(\|u_j\|_{L^\infty(\mathbb{R};X)} + \|u_{j-1}\|_{L^\infty(\mathbb{R};X)}\big)\|u_j-u_{j-1}\|_{L^\infty(\mathbb{R};X)}.\end{aligned}$$

By a standard argument, we observe that $G_0, G_j, u_j \in BUC(\mathbb{R};X)$, the sequence (u_j) is bounded in $BUC(\mathbb{R};X)$ and converges to a solution u of (I.E.) in $BUC(\mathbb{R};X)$.

We will show that u has a precompact range in L^3_σ. Let

$$\begin{aligned}G_{0,n}(t) &:= \int_{1/n}^n e^{-sA}P\nabla\cdot F(t-s)\,ds,\\ G_{j,n}(t) &:= \int_{1/n}^n e^{-sA}P\nabla\cdot (u_j\otimes u_j)(t-s)\,ds.\end{aligned}$$

By Lemma 2.3, we observe that

$$\begin{aligned}&\sup_t \|G_0(t)-G_{0,n}(t)\|_3 \leq C\sup_t \|G_0(t)-G_{0,n}(t)\|_X\\ \leq&\sup_t \int_0^{1/n} \|e^{-sA}P\nabla\cdot F(t-s)\|_X\,ds + \sup_t\int_n^\infty \|e^{-sA}P\nabla\cdot F(t-s)\|_X\,ds\\ \leq& C(n^{-1+\frac{3}{2q}} + n^{-\frac12+\frac{3}{2l}-\frac{3}{2q}} + n^{\frac12-\frac{3}{2r}}) \to 0 \text{ as } n\to\infty\end{aligned}$$

and

$$\begin{aligned}&\sup_t \|G_j(t)-G_{j,n}(t)\|_3 \leq C\sup_t \|G_j(t)-G_{j,n}(t)\|_X\\ \leq&\sup_t \int_0^{1/n} \|e^{-sA}P\nabla\cdot (u_j\otimes u_j)(t-s)\|_X\,ds + \sup_t\int_n^\infty \|e^{-sA}P\nabla\cdot (u_j\otimes u_j)(t-s)\|_X\,ds\\ \leq& C(n^{-1+\frac{3}{2q}} + n^{\frac12-\frac{3}{2r}}) \to 0 \text{ as } n\to\infty.\end{aligned}$$

Thus $G_{0,n}$ and $G_{j,n}$ converge to G_0 and G_j in $BC(\mathbb{R};L^3_\sigma)$ as $n\to\infty$, respectively. Since Lemma 2.2 (ii) implies that $\mathcal{R}(G_{0,n})$ is precompact in L^3_σ for each $n\in\mathbb{N}$, by Proposition 2.1 (i), we see that $\mathcal{R}(G_0)$ is precompact in L^3_σ.

If we assume that $\mathcal{R}(u_{j_0})$ is precompact in L^3 for some j_0, then, by Proposition 2.1 (iv), $\mathcal{R}(u_{j_0}\otimes u_{j_0})$ is precompact in $L^{3/2}$. Hence, in this case, by Lemma 2.2 (ii) we obtain that $\mathcal{R}(G_{j_0,n})$ is precompact in L^3_σ and consequently $\mathcal{R}(G_{j_0})$ is precompact in L^3_σ. Then we have that $\mathcal{R}(u_{j_0+1})$ is precompact in L^3_σ. Therefore, by induction, we conclude that $\mathcal{R}(u_j)$ is precompact in L^3 for each $j\in\mathbb{N}$. Since u_j converges to the mild solution u in $BC(\mathbb{R};L^3_\sigma)$, from Proposition 2.1 (i) it follows that $\mathcal{R}(u)$ is precompact in L^3_σ. This proves Theorem 1. □

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