MAXIMAL REGULARITY OF THE SPATIALLY PERIODIC STOKES OPERATOR AND APPLICATION TO NEMATIC LIQUID CRYSTAL FLOWS

JONAS SAUER

Abstract. We consider the dynamics of spatially periodic nematic liquid crystal flows in the whole space and prove existence and uniqueness of local-in-time strong solutions using maximal L^p-regularity of the periodic Laplace and Stokes operators and a local-in-time existence theorem for quasilinear parabolic equations à la Clément-Li (1993). Maximal regularity of the Laplace and the Stokes operator is obtained using an extrapolation theorem on the locally compact abelian group $G := \mathbb{R}^{n-1} \times \mathbb{R}/L\mathbb{Z}$ to obtain an $\mathcal{R}$-bound for the resolvent estimate. Then, Weis' Theorem connecting $\mathcal{R}$-boundedness of the resolvent with maximal L^p-regularity of a sectorial operator applies.

MSC: 35B10; 35K59; 35Q35; 76A15; 76D03.

Key Words: Muckenhoupt weights; Stokes operator; maximal L^p-regularity; nematic liquid crystal flow; quasilinear parabolic equations.

1. Introduction

Consider the spatially periodic problem

$$\begin{cases} \partial_t u - \nu\Delta u + u\cdot\nabla u + \nabla\mathfrak{p} = -\kappa \operatorname{div}([\nabla d]^T[\nabla d]), & \text{in } (0,T)\times G, \\ \partial_t d + u\cdot\nabla d = \sigma(\Delta d + |\nabla d|^2 d), & \text{in } (0,T)\times G \\ \operatorname{div} u = 0, & \text{in } (0,T)\times G, \\ (u(0),d(0)) = (u_0,d_0), & \text{in } G, \end{cases} \tag{LCD}$$

where $\nu,\sigma,\kappa > 0$ and $G := \mathbb{R}^{n-1}\times\mathbb{R}/L\mathbb{Z}$, $n \geq 3$. This is the simplified Ericksen-Leslie model describing a nematic liquid crystal flow in the whole space which is periodic of length $L > 0$ in the last variable. Here, the function u denotes the velocity of the flow, $\mathfrak{p}$ the pressure and d the macroscopic molecular orientation. The constants ν, σ and κ represent viscosity, the competition between kinetic energy and potential energy and the microscopic elastic relaxation time for the molecular orientation field, respectively. This model bases on the continuum theory of liquid crystals developed by Ericksen and Leslie, see for example the survey article [10], and has been considered for the first time by [15] and [16]. In [13], problem (LCD) was treated on bounded domains using quasilinear theory based upon the maximal L^p-regularity of the Stokes operator. We will adopt this ansatz to the periodic case by showing maximal L^p-regularity of the spatially periodic Laplace and Stokes operators in weighted spaces, see Theorems 3.2 and 3.5 below. Observe that G

Fachbereich Mathematik, Technische Universität Darmstadt, 64283 Darmstadt, jsauer@mathematik.tu-darmstadt.de

The author has been supported by the International Research Training Group 1529.

is a locally compact abelian group, and hence one can define the *Haar measure* [1, 6, 12, 22] which is unique up to multiplication with a constant. We will choose the constant in such a way that

$$\int_G f \,\mathrm{d}\mu = \frac{1}{L}\int_0^L \int_{\mathbb{R}^{n-1}} f(x', x_n)\,\mathrm{d}x'\,\mathrm{d}x_n, \qquad f \in C_0(G).$$

As G satisfies Assumption 1.1 of [20], the theory developed there for Muckenhoupt weights $\omega \in A_q(G)$ applies in our case. For $1 < q < \infty$, a nonnegative function $\omega \in L^1_{\mathrm{loc}}(G)$ is said to be in $A_q(G)$ if

$$\mathcal{A}_q(\omega) := \sup_{U \subset G} \left(\frac{1}{\mu(U)}\int_U \omega \,\mathrm{d}\mu\right)\left(\frac{1}{\mu(U)}\int_U \omega^{-\frac{q'}{q}}\,\mathrm{d}\mu\right)^{\frac{q}{q'}} < \infty,$$

where the supremum runs over all base sets $U \subset G$; see [20] for details. We call a constant $c = c(\omega) > 0$ that depends on $A_q(G)$-weights $A_q(G)$-*consistent*, if for each $d > 0$ we have

$$\sup\{c(\omega) : \omega \text{ is an } A_q(G)\text{-weight with } \mathcal{A}_q(\omega) < d\} < \infty.$$

Moreover, we say that $\omega \in A_\infty(G)$ if there is some $1 < q < \infty$ such that $\omega \in A_q(G)$. For $1 < q \leq \infty$ and $\omega \in A_q(G)$, let us denote by $L^q_\omega(G)$ the space of all measurable functions that are p-integrable with respect to the measure $\omega\,\mathrm{d}\mu$. If $\omega = 1$, we will omit the index ω in all function spaces. In [21] it was shown that for $1 < q \leq \infty$ and $\omega \in A_q(G)$ the weighted Sobolev space

$$W^{m,q}_\omega(G) := \{u \in L^q_\omega(G) : D^\alpha u \in L^q_\omega(G) \text{ for all } \alpha \in \mathbb{N}_0^n \text{ with } |\alpha| \leq m\},$$
$$\|u\|_{W^{m,q}_\omega(G)} := \sum_{|\alpha| < m} \|D^\alpha u\|_{L^q_\omega(G)}.$$

yields a Banach space for all $m \in \mathbb{N}_0$ and that the space $C_0^\infty(G)$ of smooth functions with compact support is dense in $W^{m,q}_\omega(G)$ whenever $1 < q < \infty$. Here, $D^\alpha u$ is to be understood as a tempered distribution in $\mathcal{S}'(G)$, the dual of the Schwartz-Bruhat space $\mathcal{S}(G)$ [4]. Observe that using the canonical identification of G and $\mathbb{R}^{n-1} \times [0, L)$, we have $L^q_\omega(G) = L^q_\omega(\mathbb{R}^{n-1} \times [0, L))$ and

$$W^{m,q}_\omega(G) \subset W^{m,q}_\omega(\mathbb{R}^{n-1} \times [0, L)).$$

Hence, Sobolev embeddings known in the $\mathbb{R}^n$-setup carry over to our setting. In particular, for $\omega = 1$ the classical Sobolev embeddings are valid.
Moreover, for $1 < q < \infty$ and $\omega \in A_q(G)$ we introduce the space

$$L^q_{\omega,\sigma}(G) := \{u \in L^q_\omega(G)^n : \operatorname{div} u = 0\},$$
$$\|u\|_{L^q_{\omega,\sigma}(G)} := \|u\|_{L^q_\omega(G)^n},$$

where $\operatorname{div} u$ is to be understood again as a tempered distribution in $\mathcal{S}'(G)$. Note that $L^q_{\omega,\sigma}(G)$ is a closed subspace of $L^q_\omega(G)^n$ by Proposition 3.3 below.
Let $1 < p, q < \infty$ and define the spaces

$$X_0 := L^q_\sigma(G) \times L^q(G)^n,$$
$$X_1 := D(A_q) \times D(\Delta_q),$$

where $D(A_q) := W^{2,q}(G)^n \cap L^q_\sigma(G)$ and $D(\Delta_q) := W^{2,q}(G)^n$. Furthermore, we define the real interpolation space $X_\gamma := (X_0, X_1)_{1-1/p,p}$.

Theorem 1.1. *Let $n \geq 3$ and let $1 < p, q < \infty$ be such that $\frac{2}{p} + \frac{n}{q} < 1$. Assume furthermore $z_0 := (u_0, d_0) \in X_\gamma$. Then there exists $T_0 > 0$, such that (LCD) admits a unique solution $(u, d, \mathfrak{p})$ on $J = [0, T_0]$ in the regularity class*

$$\begin{aligned} (u, d) &\in W^{1,p}(J; X_0) \cap L^p(J; X_1) \hookrightarrow C(J; X_\gamma) \\ \nabla \mathfrak{p} &\in L^p(J; L^q(G)). \end{aligned}$$

The solution depends continuously on z_0, and can be extended to a maximal interval of existence $J(z_0) = [0, T^+(z_0))$.

Remark 1.2. Note that X_γ is the natural space of initial values, as it is the trace space of $W^{1,p}(J; X_0) \cap L^p(J; X_1)$ at $t = 0$, see e.g. [17, Section 1.2].

This paper is organized as follows. In section 2, we provide basic information about $\mathcal{R}$-boundedness and maximal L^p-regularity and show that the maximal regularity constant of a sectorial operator defined on a weighted L^q-space is $A_q(G)$-consistent if the $\mathcal{R}$-bound of the corresponding family of resolvent operators is $A_q(G)$-consistent. In section 3 we establish maximal L^p-regularity of the spatially periodic Laplace and Stokes operators. Finally, in section 4 we prove Theorem 1.1 using the theory of abstract quasilinear parabolic equations.

2. Maximal Regularity on UMD spaces

The notion of $\mathcal{R}$-boundedness is known to be suitable if one is interested in maximal regularity of evolution equations due to an operator-valued multiplier theorem for Bochner spaces $L^p(\mathbb{R}, X)$ by Weis [23], where X is an UMD space as defined in Definition 2.4 below. The $\mathcal{R}$-boundedness comes into play exactly at the level of this multiplier theorem, since the conditions on the multiplier symbols are given in terms of $\mathcal{R}$-bounds, see Theorem 2.10 below. We will not give proofs in full detail here, but rather refer to standard literature for maximal regularity, in particular [8] and [14], whenever they are applicable. As we will see in Remark 2.3 below, we are interested in $A_q(G)$-consistency of the maximal regularity constant in case of $X = L^q_\omega(G)^n$, where $1 < q < \infty$ and $\omega \in A_q(G)$. Therefore it will be necessary to follow the mainly well-known arguments in this section. Let us start by defining $\mathcal{R}$-boundedness and maximal L^p-regularity.

Definition 2.1. Let X be a Banach space and let $1 < p < \infty$. A set $\mathcal{T} \subset \mathcal{L}(X)$ is called $\mathcal{R}_p$*-bounded*, if there is a constant $c > 0$ such that

$$\int_0^1 \left\| \sum_{j=1}^n r_j(t) T_j x_j \right\|_X^p \mathrm{d}t \leq c \int_0^1 \left\| \sum_{j=1}^n r_j(t) x_j \right\|_X^p \mathrm{d}t \tag{1}$$

for all $T_1, \ldots, T_N \in \mathcal{T}$, $x_1, \ldots, x_n \in X$ and $n \in \mathbb{N}$. Here, $(r_j)_{j \in \mathbb{N}}$ is the sequence of the Rademacher functions

$$\begin{aligned} r_j &: [0,1] \to \{-1, 1\}, \\ r_j(t) &:= \operatorname{sgn}[\sin(2^{j-1}\pi t)]. \end{aligned}$$

The smallest constant $c > 0$ such that (1) holds is called $\mathcal{R}_p$*-bound of* $\mathcal{T}$ and is denoted by $\mathcal{R}_p(\mathcal{T})$.

It is worth noting that $\mathcal{R}_p$-boundedness is independent of $1 \leq p < \infty$, *i.e.*, there is $k_{p,q} > 0$ depending only on $1 \leq p, q < \infty$ such that for every Banach space X and every $\mathcal{T} \subset \mathcal{L}(X)$ there holds $k_{p,q}^{-1}\mathcal{R}_q(\mathcal{T}) \leq \mathcal{R}_p(\mathcal{T}) \leq k_{p,q}\mathcal{R}_q(\mathcal{T})$, see [9, Theorem 11.1]. Therefore, we will talk about $\mathcal{R}$-boundedness instead of $\mathcal{R}_p$-boundedness.

Definition 2.2. Let $1 < p < \infty$, $0 < T \leq \infty$ and let $-A$ be the generator of a bounded analytic semigroup on a Banach space X with domain $D(A) \subset X$. Then A is said to admit *maximal L^p-regularity* on $[0,T)$, if for all $f \in L^p(0,T;X)$ there is a unique solution $u \in \mathbb{E} := W^{1,p}(0,T;X) \cap L^p(0,T;D(A))$ to the abstract Cauchy problem

$$\dot{u} + Au = f, \qquad u(0) = 0. \tag{2}$$

Remark 2.3. Let $1 < p < \infty$, $0 < T \leq \infty$ and suppose that $-A$ is a generator of a bounded analytic semigroup on a Banach space X such that A admits maximal L^p-regularity on $[0,T)$.

(i) Let $u \in \mathbb{E}$ be the solution to (2) with $f \in L^p(0,T;X)$. By the closed graph theorem there is $c_A > 0$, called the *maximal regularity constant*, such that

$$\|u\|_{\mathbb{E}} := \|\dot{u}\|_{L^p(0,T;X)} + \|u\|_{L^p(0,T;D(A))} \leq c_A(\|f\|_{L^p(0,T;X)}).$$

(ii) Let $(X, D(A))_{1-1/p,p}$ denote the real interpolation space of X and $D(A)$. By [17, Section 1.2], $(X, D(A))_{1-1/p,p}$ is the trace space of $\mathbb{E}$, *i.e.*, there is a constant $c > 0$ depending only on $1 < p < \infty$ such that for all $u_0 \in (X, D(A))_{1-1/p,p}$

$$c^{-1}\|u_0\|_{(X,D(A))_{1-1/p,p}} \leq \inf_{\substack{v\in\mathbb{E}\\ v(0)=u_0}} \|v\|_{\mathbb{E}} \leq c\|u_0\|_{(X,D(A))_{1-1/p,p}}.$$

Let $u_0 \in (X, D(A))_{1-1/p,p}$, and consider the abstract Cauchy problem

$$\dot{u} + Au = f, \qquad u(0) = u_0. \tag{3}$$

By choosing an arbitrary $v \in \mathbb{E}$ with $v(0) = u_0$ and solving

$$\dot{u} - \dot{v} + A(u - v) = f - \dot{v} - Av, \qquad (u - v)(0) = 0,$$

we obtain a unique solution $u \in \mathbb{E}$ to (3) satisfying

$$\begin{aligned}\|u\|_{\mathbb{E}} &\leq \|u - v\|_{\mathbb{E}} + \|v\|_{\mathbb{E}} \\ &\leq c_A(\|f\|_{L^p(0,T;X)} + \|v\|_{\mathbb{E}}) + \|v\|_{\mathbb{E}} \\ &\leq (1 + c_A)(\|f\|_{L^p(0,T;X)} + \|v\|_{\mathbb{E}}),\end{aligned}$$

and since v was arbitrary, the same estimate holds true if we choose the infimum over all $v \in \mathbb{E}$ with $v(0) = u_0$. Thus, we obtain a constant $c'_A > 0$ such that

$$\|u\|_{\mathbb{E}} \leq c'_A(\|f\|_{L^p(0,T;X)} + \|u_0\|_{(X,D(A))_{1-1/p,p}}).$$

In the context of weighted spaces, *i.e.*, if $X = L^q_\omega(G)^n$ or $X = L^q_{\omega,\sigma}(G)$ for $1 < q < \infty$ and $\omega \in A_q(G)$, it is of interest whether the maximal regularity constant is $A_q(G)$-consistent. In hindsight of the consideration in (ii) it is clear that c'_A is $A_q(G)$-consistent if c_A is $A_q(G)$-consistent, so we may concentrate on the latter.

Definition 2.4. A Banach space X is an *UMD space* if the Hilbert transform

$$Hf(t) := \text{p.v.} \int_{\mathbb{R}} \frac{f(s)}{t - s}\,\mathrm{d}s, \qquad f \in \mathcal{S}(\mathbb{R}, X)$$

extends to a bounded linear operator in $L^p(\mathbb{R}, X)$ for some $1 < p < \infty$. Here, $\mathcal{S}(\mathbb{R}, X)$ is the Schwartz space of rapidly decreasing X-valued functions.

It is known that if X is an UMD space, then the Hilbert transform is bounded in $L^p(\mathbb{R}, X)$ for *all* exponents $1 < p < \infty$, see [18, Theorem 1.3]. Moreover, for all σ-finite measure spaces (Ω, μ_Ω), closed subspaces of $L^q(\Omega, \mu_\Omega)$ for $1 < q < \infty$ are UMD spaces, cf. [5]. In particular, the spaces $L^q_\omega(G)^n$ and $L^q_{\omega,\sigma}(G)$ are UMD spaces for all $1 < q < \infty$ and all $\omega \in A_q(G)$.

Definition 2.5. Let $\mathfrak{X}$ be a Banach space and $(x_k)_{k\in\mathbb{Z}} \subset \mathfrak{X}$. We call the series $\sum_{k=-\infty}^{\infty} x_k$ *unconditionally convergent*, if $\sum_{k=-\infty}^{\infty} x_{\pi(k)}$ is norm convergent for every permutation $\pi : \mathbb{Z} \to \mathbb{Z}$.
A sequence of projections $(\Delta_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(\mathfrak{X})$ is called a *Schauder decomposition* of $\mathfrak{X}$ if

$$\Delta_i \Delta_j = 0 \text{ for all } i \neq j, \qquad \sum_{k=-\infty}^{\infty} \Delta_k x = x \text{ for each } x \in \mathfrak{X}.$$

A Schauder decomposition is called *unconditional* if the series $\sum_{k=-\infty}^{\infty} \Delta_k x$ converges unconditionally.

Proposition 2.6. *If $(\Delta_k)_{k\in\mathbb{Z}} \subset \mathcal{L}(\mathfrak{X})$ is an unconditional Schauder decomposition of $\mathfrak{X}$, then for each $1 \leq r < \infty$ there is a constant $c_\Delta = c_\Delta(r, \mathfrak{X}) > 0$ such that for all $x \in \mathfrak{X}$ there holds the Paley-Littlewood estimate*

$$c_\Delta^{-1} \Big\| \sum_{k=i}^{j} \Delta_k x \Big\|_{\mathfrak{X}} \leq \left(\int_0^1 \Big\| \sum_{k=i}^{j} r_k(t) \Delta_k x \Big\|_{\mathfrak{X}}^r dt \right)^{\frac{1}{r}} \leq c_\Delta \Big\| \sum_{k=i}^{j} \Delta_k x \Big\|_{\mathfrak{X}}, \tag{4}$$

for all $i, j \in \mathbb{Z}$, $i \leq j$.

Proof. See [14, Proposition 3.10] or [8, Section 3]. □

It is important to note that for every Banach space $\mathfrak{X} = L^p(\mathbb{R}, X)$, where $1 < p < \infty$ and X is an UMD space, there *is* an unconditional Schauder decomposition, which is a result essentially due to Bourgain [3], who proved this for $\mathfrak{X} = L^p([0, 2\pi], X)$, and Zimmermann [24], who extended this result to the real line.

Proposition 2.7. *Let $1 < p, q < \infty$ and let X be an UMD space. Then for $\mathfrak{X} := L^p(\mathbb{R}, X)$ the Schauder decomposition $\Delta := (\Delta_k)_{k\in\mathbb{Z}}$ with $\Delta_k := \mathcal{F}_\mathbb{R}^{-1} \chi_{[2^k, 2^{k+1})} \mathcal{F}_\mathbb{R}$ is unconditional.*

This Schauder decomposition enjoys another very useful property. As long as X is an UMD space, we have that Δ is $\mathcal{R}$-bounded in $\mathfrak{X} = L^p(\mathbb{R}, X)$. We will state this in the following Lemma.

Lemma 2.8. *Let $1 < p < \infty$ and let $\mathfrak{X} = L^p(\mathbb{R}, X)$, where X is an UMD space. Then the family of operators $\{R_s := \mathcal{F}_\mathbb{R}^{-1} \chi_{[s,\infty)} \mathcal{F}_\mathbb{R} : s \in \mathbb{R}\}$ is $\mathcal{R}$-bounded in $\mathfrak{X}$, and the $\mathcal{R}$-bound depends only on $\|R_0\|_{\mathcal{L}(\mathfrak{X})}$.*
In particular, Δ as defined in Proposition 2.7 is $\mathcal{R}$-bounded.

Proof. Follows from [8, Corollary 3.7] with $\mathcal{T} := \{R_0\}$, $\phi(t) := e^{ist}$ and $\psi(t) := e^{-ist}$. For the assertion about Δ, observe that $\Delta_k = R_{2^k} - R_{2^{k+1}}$, $k \in \mathbb{Z}$. □

Remark 2.9. (i) If we choose the Schauder decomposition Δ as in Proposition 2.7, then in the particular case that $X = L^q_\omega(\mathbb{R}^n)^n$ or $X = L^q_{\omega,\sigma}(\mathbb{R}^n)$

for some $1 < q < \infty$ and some Muckenhoupt weight $\omega \in A_q(\mathbb{R}^n)$, the constant c_Δ appearing in (4) can be chosen independently of ω by an argument of Farwig and Ri [11, Remark 5.7], namely: Define

$$\iota_\omega : L^p(\mathbb{R}, L^q_\omega(\mathbb{R}^n)^n) \to L^p(\mathbb{R}, L^q(\mathbb{R}^n)^n)$$

via $\iota_\omega(f) = f\omega^{1/q}$. Then ι_ω is an isometric isomorphism. Since the Schauder projections only touch the time variable, we obtain for all $f \in L^p(\mathbb{R}, L^q_\omega(\mathbb{R}^n)^n)$ and all $k \in \mathbb{Z}$

$$\Delta_k(f(t,x)\omega(x)^{1/q}) = (\Delta_k f(t,x)) \cdot \omega(x)^{1/q}. \tag{5}$$

Hence, the operators ι_ω and Δ_k commute for all $k \in \mathbb{Z}$, whence we see from equation (4)

$$c_\Delta(r, p, L^q_\omega(\mathbb{R}^n)^n) = c_\Delta(r, p, L^q(\mathbb{R}^n)^n).$$

The same argument applies also for $L^p_{\omega,\sigma}(\mathbb{R}^n)$.

(ii) Using the isometry ι_ω, we see that for $T \in \mathcal{L}(L^p(\mathbb{R}, L^q_\omega(\mathbb{R}^n)^n)$ we have

$$\|T\|_{\mathcal{L}(L^p(\mathbb{R},L^q_\omega(\mathbb{R}^n)^n)} = \|\iota_\omega T \iota_\omega^{-1}\|_{\mathcal{L}(L^p(\mathbb{R},L^q(\mathbb{R}^n)^n)}.$$

This shows that $\mathcal{T} \subset \mathcal{L}(L^p(L^q_\omega(\mathbb{R}^n)^n)$ is $\mathcal{R}$-bounded if and only if $\tilde{\mathcal{T}} := \{\iota_\omega T \iota_\omega^{-1} : T \in \mathcal{T}\}$ is $\mathcal{R}$-bounded and that $\mathcal{R}_1(\mathcal{T}) = \mathcal{R}_1(\tilde{\mathcal{T}})$. In particular, if all $T \in \mathcal{T}$ commute with ι_ω – as is the case for the Schauder decomposition Δ chosen in Proposition 2.7 –, the $\mathcal{R}$-bound of $\mathcal{T}$ does not depend on the Muckenhoupt weight ω.

Again, the same arguments can be used in the solenoidal case.

These arguments carry over to $L^p(\mathbb{R}, L^q_\omega(G)^n)$ and $L^p(\mathbb{R}, L^q_{\omega,\sigma}(G))$ without any changes.

We can now state the operator valued Mikhlin multiplier theorem in one variable.

Theorem 2.10. *Let X be an UMD space and suppose $1 < p < \infty$. Let furthermore $M \in C^1(\mathbb{R} \setminus \{0\}, \mathcal{L}(X))$ and assume that there are constants $c_1, c_2 < \infty$ such that*

(i) *$\mathcal{R}_p(\{M(t) : t \in \mathbb{R} \setminus \{0\}\}) = c_1$ and*
(ii) *$\mathcal{R}_p(\{tM'(t) : t \in \mathbb{R} \setminus \{0\}\}) = c_2$.*

Then the operator $T_M := \mathcal{F}_\mathbb{R}^{-1} M \mathcal{F}_\mathbb{R}$ is bounded in $L^p(\mathbb{R}, X)$ with norm

$$\|T_M\|_{\mathcal{L}(L^p(\mathbb{R},X))} \le c_\Delta^2 \mathcal{R}_p(\Delta) \cdot (c_1 + c_2) =: c < \infty, \tag{6}$$

where c_Δ is the Schauder constant corresponding to the Schauder decomposition Δ chosen in Proposition 2.7.

Proof. See [8, Theorem 3.19] or [14, Theorem 3.12]. □

Remark 2.11. Let $1 < q < \infty$ and $\omega \in A_q(G)$. Suppose that in the situation of Theorem 2.10 we have $X = L^q_\omega(G)^n$ or $X = L^q_{\omega,\sigma}(G)$. Then – as a consequence of Remark 2.9 – the constant c in (6) is $A_q(G)$-consistent if the constants c_1 and c_2 are $A_q(G)$-consistent.

The following well-known result due to Weis [23] connects the notion of maximal regularity with the $\mathcal{R}$-boundedness of the corresponding resolvent operator.

Proposition 2.12. *Let $1 < p < \infty$ and $0 < T \leq \infty$. Assume furthermore that $-A$ is the generator of a bounded analytic semigroup in an UMD space X. Then A has maximal L^p-regularity on $[0, T)$ if and only if the operator family*

$$\{it(it + A)^{-1} : t \in \mathbb{R}, t \neq 0\}$$

is $\mathcal{R}$-bounded in $\mathcal{L}(X)$.

Remark 2.13. Since $\mathcal{R}$-boundedness is independent of p and T, so is the property of A admitting maximal L^p-regularity on $[0, T)$.

Theorem 2.14. *Let $1 < q < \infty$ and $\omega \in A_q(G)$ and suppose that $-A$ is the generator of a bounded analytic semigroup on $L^q_\omega(G)^n$ or $L^q_{\omega,\sigma}(G)$ such that the $\mathcal{R}$-bound of $\{it(it + A)^{-1} : t \in \mathbb{R}, t \neq 0\}$ is $A_q(G)$-consistent. Then the maximal regularity constant c_A is $A_q(G)$-consistent as well.*

Proof. Following [14, Section 1.5], one sees that the maximal regularity constant c_A is exactly $2c + 1$, where c is the constant appearing in (6), if we apply Theorem 2.10 with

$$M(t) := it(it + A)^{-1} - I,$$
$$M'(t) := (t(it + A)^{-1})^2 + it(it + A)^{-1}.$$

Now, the assertion follows from Remark 2.11. □

3. Spatially Periodic Laplace and Stokes Operator

The arguments in this section are based upon the following extrapolation theorem.

Theorem 3.1. *Let $1 < r, q < \infty$, $\upsilon \in A_r(G)$ and let $\mathcal{T}$ be a family of linear operators such that for all $\omega \in A_q(G)$ there is an $A_q(G)$-consistent constant $c_q = c_q(\omega) > 0$ with*

$$\|Tf\|_{L^q_\omega(G)} \leq c_q \|f\|_{L^q_\omega(G)}$$

for all $f \in L^q_\omega(G)$ and all $T \in \mathcal{T}$. Then every $T \in \mathcal{T}$ extends to $L^r_\upsilon(G)$ and $\mathcal{T}$ is $\mathcal{R}$-bounded in $\mathcal{L}(L^r_\upsilon(G))$ with an $A_r(G)$-consistent $\mathcal{R}$-bound c_r.

Proof. See [20, Theorem 1.5]. □

Let $1 < q < \infty$, and $\omega \in A_q(G)$. Then we may define the *spatially periodic Laplace operator* $\Delta_{q,\omega}$ by

$$\begin{aligned} D(\Delta_{q,\omega}) &:= W^{2,q}_\omega(G)^n, \\ \Delta_{q,\omega} u &:= \Delta u. \end{aligned} \tag{7}$$

Theorem 3.2. *Let $n \geq 2$, $1 < p, q < \infty$ and $\omega \in A_q(G)$. Then the spatially periodic Laplace operator $-\Delta_{q,\omega}$ admits maximal L^p-regularity in $L^q_\omega(G)^n$ and the maximal regularity constant is $A_q(G)$-consistent.*

Proof. In virtue of Theorem [21, Theorem 1.2] we know that for every $\theta \in (0, \pi)$, the sector Σ_θ is contained in the resolvent set $\rho(\Delta_{q,\omega})$ and the estimate

$$\|\lambda(\lambda - \Delta_{q,\omega})^{-1}\|_{\mathcal{L}(L^q_\omega(G)^n)} \leq c, \qquad \lambda \in \Sigma_\theta, \tag{8}$$

holds with an $A_q(G)$-consistent constant $c = c(\omega, n, q, \theta)$. Thus, since $\rho(\Delta_{q,\omega})$ is not empty, $\Delta_{q,\omega}$ is a densely defined closed operator, and (8) yields that $\Delta_{q,\omega}$ is sectorial of the angle $\phi_{\Delta_{q,\omega}} = 0$, see [8]. With the help of the Extrapolation Theorem 3.1 we see that $\lambda(\lambda - \Delta_{q,\omega})^{-1}$, $\lambda \in \Sigma_\theta$, is even $\mathcal{R}$-bounded in $\mathcal{L}(L^q_{\omega,\sigma}(G)^n)$ with an $A_q(G)$-consistent bound. Theorem 2.12 shows in turn maximal L^p-regularity of $-\Delta_{q,\omega}$ if we choose $\theta \in (\frac{\pi}{2}, \pi)$. The $A_q(G)$-consistency of the maximal regularity constant is a consequence of Theorem 2.14. □

Let us now define the *Stokes operator* $A_{q,\omega}$ via

$$\begin{aligned} D(A_{q,\omega}) &:= W^{2,q}_\omega(G)^n \cap L^q_{\omega,\sigma}(G), \\ A_{q,\omega}u &:= -P_{q,\omega}\Delta u, \end{aligned}$$

where

$$\begin{aligned} P_{q,\omega} &: \mathcal{S}'(G)^n \to \mathcal{S}'(G)^n, \\ P_{q,\omega}f &:= \mathcal{F}^{-1}\left(\left(I - \frac{\eta \otimes \eta}{|\eta|^2}\right)\hat{f}\right) \end{aligned}$$

is the Helmholtz projection on G. Here, $\mathcal{F}$ denotes the Fourier transform on the locally compact abelian group G, see [19] for details.

Proposition 3.3. *Let $1 < q < \infty$ and $\omega \in A_q(G)$. Then the Helmholtz projection $P_{q,\omega} : L^q_\omega(G)^n \to L^q_{\omega,\sigma}(G)$ is a bounded and surjective projection with an $A_q(G)$-consistent bound. In particular, $L^q_{\omega,\sigma}(G)$ is a closed subspace of $L^q_\omega(G)^n$ and hence an UMD space.*

Proof. In virtue of [21, Theorem 1.3] we obtain for every $f \in L^q_\omega(G)^n$ a unique solution $(u, \mathfrak{p}) \in W^{2,q}_\omega(G)^n \times \hat{W}^{1,q}_\omega(G)$ to the system

$$\begin{cases} u - \Delta u + \nabla \mathfrak{p} = f & \text{in } G, \\ \operatorname{div} u = 0 & \text{in } G, \end{cases} \tag{9}$$

satisfying the *a priori* estimate

$$\|u, \nabla^2 u, \nabla \mathfrak{p}\|_{L^q_\omega(G)} \le c\|f\|_{L^q_\omega(G)}, \tag{10}$$

where $c = c(\omega, n, q) > 0$ is an $A_q(G)$-consistent constant. Since $P_{q,\omega}f = f - \nabla\mathfrak{p}$, we obtain $P_{q,\omega}f \in L^q_{\omega,\sigma}(G)$ and

$$\|P_{q,\omega}f\|_{L^q_{\omega,\sigma}(G)} \le (c+1)\|f\|_{L^q_\omega(G)}.$$

Moreover, $P_{q,\omega}f = f$ for all $f \in \mathcal{S}'(G)^n$ with $\operatorname{div} f = \mathcal{F}^{-1}(i\eta \cdot \hat{f}) = 0$, and so $P_{q,\omega} : L^q_\omega(G)^n \to L^q_{\omega,\sigma}(G)$ is a surjective projection. □

Lemma 3.4. *Let $1 < q < \infty$ and $\omega \in A_q(G)$. Then $D(A_{q,\omega})$ is dense in $L^q_{\omega,\sigma}(G)$.*

Proof. Note that $P_{q,\omega}$ is given via a Fourier multiplier and hence commutes on the level of tempered distributions with differential operators. Thus, it preserves regularity, yielding

$$P_{q,\omega}(W^{2,q}_\omega(G)^n) \subset W^{2,q}_\omega(G)^n \cap L^q_{\omega,\sigma}(G).$$

Since $C_0^\infty(G)^n$ (and consequently also $W^{2,q}(G)^n$) is dense in $L^q(G)^n$, we conclude by the boundedness of the Helmholtz projection that $P_{q,\omega}(W^{2,q}_\omega(G)^n)$ is dense in $P_{q,\omega}(L^q(G)^n) = L^q_\sigma(G)$. Now $D(A_{q,\omega}) = W^{2,q}_\omega(G)^n \cap L^q_{\omega,\sigma}(G)$ shows the assertion. □

Theorem 3.5. *Let $n \geq 3$, $1 < p, q < \infty$ and $\omega \in A_q(G)$. Then the Stokes operator $A_{q,\omega}$ has maximal L^p-regularity in $L^q_{\omega,\sigma}(G)$ and the maximal regularity constant is $A_q(G)$-consistent.*

Proof. Note that $A_{q,\omega}$ is densely defined by Lemma 3.4. Theorem 1.3 in [21] and Proposition 3.3 imply that for every $\theta \in (0, \pi)$ the sector Σ_θ is contained in the resolvent set $\rho(-A_{q,\omega})$ and that the estimate

$$\|\lambda(\lambda + A_{q,\omega})^{-1}\|_{\mathcal{L}(L^q_{\omega,\sigma}(G))} \leq c, \qquad \lambda \in \Sigma_\theta,$$

holds with an $A_q(G)$-consistent constant $c = c(n, q, \theta, \omega)$. Thus, we obtain that $-A_{q,\omega}$ is sectorial of the angle $\phi_{-A_{q,\omega}} = 0$ and Theorem 3.1 yields that $\lambda(\lambda + A_{q,\omega})^{-1}$, $\lambda \in \Sigma_\theta$, is $\mathcal{R}$-bounded in $\mathcal{L}(L^q_{\omega,\sigma}(G))$ with an $A_q(G)$-consistent bound. Hence, choosing $\theta \in (\frac{\pi}{2}, \pi)$, Theorem 2.12 and Theorem 2.14 imply the assertion. □

4. Periodic Liquid Crystal Flows

In order to prove Theorem 1.1, we want to employ the theory of quasilinear evolution equations. Note that quasilinear evolution equations have been studied in more general contexts as is presented here, see Amann [2] and the references therein.

Let Y_0 and Y_1 be Banach spaces such that $Y_1 \overset{d}{\hookrightarrow} Y_0$, *i.e.*, Y_1 is continuously and densely embedded in Y_0. Assume $1 < p < \infty$ and $0 < T \leq \infty$. By a *quasilinear autonomous parabolic evolution equation* we understand an equation of the form

$$\dot{z}(t) + A(z(t))z(t) = F(z(t)), \quad t \in (0, T), \quad z(0) = z_0, \tag{QL}$$

where A is a mapping from the real interpolation space $Y_\gamma := (Y_0, Y_1)_{1-1/p,p}$ into $\mathcal{L}(Y_1, Y_0)$ and F maps from Y_γ into Y_0. Let us assume the following regularity assumptions on A and F.

(A) $A : Y_\gamma \to \mathcal{L}(Y_1, Y_0)$ locally Lipschitz,
(F) $F : Y_\gamma \to Y_0$ locally Lipschitz.

Then we have the following local-in-time existence result.

Proposition 4.1. *Let $1 < p < \infty$, $z_0 \in Y_\gamma$, and suppose that the assumptions* (A) *and* (F) *are satisfied. Furthermore assume that $A(z^*)$ has the property of maximal L^p-regularity on $[0, T)$ for all $z^* \in Y_\gamma$. Then, there exists $T_0 > 0$, such that* (QL) *admits a unique solution z on $J = [0, T_0]$ in the regularity class*

$$z \in W^{1,p}(J; Y_0) \cap L^p(J; Y_1) \hookrightarrow C(J; Y_\gamma).$$

The solution depends continuously on z_0, and can be extended to a maximal interval of existence $J(z_0) = [0, T^+(z_0))$.

Proof. See [7] for a proof in a more general non-autonomous case. □

In virtue of Proposition 4.1, we want to reformulate (LCD) equivalently as an abstract quasilinear autonomous parabolic evolution equation. Therefore, assume $1 < q, p < \infty$ and let X_0, X_1 and X_γ be defined as in the introductory part. Then by Lemma 3.4 we see that the embedding $X_1 \hookrightarrow X_0$ is dense. We define a linear operator $L \in \mathcal{L}(X_1, X_0)$ via

$$L := \begin{pmatrix} \nu A_q & 0 \\ 0 & -\sigma\Delta_q \end{pmatrix},$$

with the Laplace operator $\Delta_q := \Delta_{q,1}$ and the Stokes operator $A_q := A_{q,1}$ as defined in section 3. Observe that Theorem 3.2 and Theorem 3.5 yield maximal L^p-regularity for the operator L. Furthermore, for every $z^* = (u^*, d^*) \in X_\gamma$, we define an operator $B_q(d^*)$ on $D(\Delta_q)$ via

$$(B_q(d^*)h)_i := \partial_i d_j^* \Delta h_j + \partial_k d_j^* \partial_k \partial_i h_j, \qquad 1 \le i \le n,$$

where we have used Einstein's sum convention and summed over the indices $1 \le j, k \le n$. In particular, $B_q(d^*)d^* = \operatorname{div}([\nabla d^*]^T[\nabla d^*])$ for $z^* = (u^*, d^*) \in X_1$. We thus introduce the quasilinear part

$$S(z^*) := \begin{pmatrix} 0 & \kappa P_{q,\omega} B_q(d^*) \\ 0 & 0 \end{pmatrix}.$$

Lemma 4.2. *Let $1 < p, q < \infty$ be such that $\frac{2}{p} + \frac{n}{q} < 1$. Assume $0 < T \le \infty$ and $z^* \in X_\gamma$. The operator $L + S(z^*) : X_1 \to X_0$ admits maximal L^p-regularity on $[0, T)$.*

Proof. Recall that we have the usual Sobolev embeddings at our disposal. Hence, the result follows by the arguments given in [13, Section 3]. □

We define a right-hand side F via

$$F(z(t)) := (-P_q(u(t) \cdot \nabla u(t)), -u(t) \cdot \nabla d(t) + \sigma |\nabla d(t)|^2 d(t))^T,$$

and rewrite the system (LCD) as

$$\dot{z}(t) + (L + S(z(t))z(t) = F(z(t)), \quad t \in (0, T), \quad z(0) = z_0. \tag{11}$$

We are now in the position to prove Theorem 1.1.

Proof of Theorem 1.1. Note that for all $z^* \in X_\gamma$ we have

$$A(z^*) := L + S(z^*) \in \mathcal{L}(X_1; X_0) \text{ and } F(z^*) \in X_0,$$

since the Helmholtz projection $P_q : L^q(G) \to L^q_\sigma(G)$ is bounded and since there holds the embedding $X_\gamma \hookrightarrow W^{1,\infty}(G)^{2n}$ for $2/p + n/q < 1$. Furthermore, A and F are polynomial in z^* and consequently Fréchet differentiable. In particular, they are locally Lipschitz and hence conditions (A) and (F) are fulfilled.
Moreover, the operator $A(z^*) : X_1 \to X_0$ admits maximal L^p-regularity for all $z^* \in X_\gamma$ by Lemma 4.2. We thus can apply Proposition 4.1 to obtain a positive $T_0 > 0$ and a unique solution z of (11) on $J = [0, T_0]$ in the regularity class

$$z \in W^{1,p}(J; X_0) \cap L^p(J; X_1) \hookrightarrow C(J; X_\gamma).$$

Moreover, the solution depends continuously on z_0, and can be extended to a maximal interval of existence $J(z_0) = [0, T^+(z_0))$.
Recovering the pressure via the Helmholtz projection P_q we obtain Theorem 1.1. □

References

[1] E. M. Alfsen. A simplified constructive proof of the existence and uniqueness of Haar measure. *Math. Scand.*, 12:106–116, 1963.

[2] H. Amann. Quasilinear evolution equations and parabolic systems. *Trans. Amer. Math. Soc.*, 293(1):191–227, 1986.

[3] J. Bourgain. Vector-valued singular integrals and the H^1-BMO duality. In *Probability theory and harmonic analysis (Cleveland, Ohio, 1983)*, volume 98 of *Monogr. Textbooks Pure Appl. Math.*, pages 1–19. Dekker, New York, 1986.

[4] F. Bruhat. Distributions sur un groupe localement compact et applications à l'étude des représentations des groupes $\wp$-adiques. *Bull. Soc. Math. France*, 89:43–75, 1961.

[5] D. L. Burkholder. A geometric condition that implies the existence of certain singular integrals of Banach-space-valued functions. In *Conference on harmonic analysis in honor of Antoni Zygmund, Vol. I, II (Chicago, Ill., 1981)*, Wadsworth Math. Ser., pages 270–286. Wadsworth, Belmont, CA, 1983.
[6] H. Cartan. Sur la mesure de Haar. *C. R. Acad. Sci. Paris*, 211:759–762, 1940.
[7] P. Clément and S. Li. Abstract parabolic quasilinear equations and application to a groundwater flow problem. *Adv. Math. Sci. Appl.*, 3(Special Issue):17–32, 1993/94.
[8] R. Denk, M. Hieber, and J. Prüß. $\mathcal{R}$-boundedness, Fourier multipliers and problems of elliptic and parabolic type. *Mem. Amer. Math. Soc.*, 166(788):viii+114, 2003.
[9] J. Diestel, H. Jarchow, and A. Tonge. *Absolutely summing operators*, volume 43 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1995.
[10] J. L. Ericksen and D. Kinderlehrer, editors. *Theory and applications of liquid crystals*, volume 5 of *The IMA Volumes in Mathematics and its Applications*. Springer-Verlag, New York, 1987. Papers from the IMA workshop held in Minneapolis, Minn., January 21–25, 1985.
[11] R. Farwig and M.-H. Ri. Resolvent estimates and maximal regularity in weighted L^q-spaces of the Stokes operator in an infinite cylinder. *J. Math. Fluid Mech.*, 10(3):352–387, 2008.
[12] A. Haar. Der Maßbegriff in der Theorie der kontinuierlichen Gruppen. *Ann. of Math. (2)*, 34(1):147–169, 1933.
[13] M. Hieber, M. Nesensohn, J. Prüss, and K. Schade. Dynamics of nematic liquid crystal flows. preprint, arXiv:1302.4596.
[14] P. C. Kunstmann and L. Weis. Maximal L_p-regularity for parabolic equations, Fourier multiplier theorems and H^∞-functional calculus. In *Functional analytic methods for evolution equations*, volume 1855 of *Lecture Notes in Math.*, pages 65–311. Springer, Berlin, 2004.
[15] F.-H. Lin. Nonlinear theory of defects in nematic liquid crystals; phase transition and flow phenomena. *Comm. Pure Appl. Math.*, 42(6):789–814, 1989.
[16] F.-H. Lin and C. Liu. Nonparabolic dissipative systems modeling the flow of liquid crystals. *Comm. Pure Appl. Math.*, 48(5):501–537, 1995.
[17] A. Lunardi. *Interpolation theory*. Appunti. Scuola Normale Superiore di Pisa (Nuova Serie). [Lecture Notes. Scuola Normale Superiore di Pisa (New Series)]. Edizioni della Normale, Pisa, second edition, 2009.
[18] J. L. Rubio de Francia, F. J. Ruiz, and J. L. Torrea. Calderón-Zygmund theory for operator-valued kernels. *Adv. in Math.*, 62(1):7–48, 1986.
[19] W. Rudin. *Fourier analysis on groups*. Interscience Tracts in Pure and Applied Mathematics, No. 12. Interscience Publishers, New York-London, 1962.
[20] J. Sauer. Extrapolation Theorem on Locally Compact Abelian Groups. *Fachbereich Mathematik, TU Darmstadt*, preprint no. 2688, 2014.
[21] J. Sauer. Weighted Resolvent Estimates for the Spatially Periodic Stokes Equations. *Fachbereich Mathematik, TU Darmstadt*, preprint no. 2689, 2014.
[22] A. Weil. *L'intégration dans les groupes topologiques et ses applications*. Actual. Sci. Ind., no. 869. Hermann et Cie., Paris, 1940.
[23] L. Weis. A new approach to maximal L_p-regularity. In *Evolution equations and their applications in physical and life sciences (Bad Herrenalb, 1998)*, volume 215 of *Lecture Notes in Pure and Appl. Math.*, pages 195–214. Dekker, New York, 2001.
[24] F. Zimmermann. On vector-valued Fourier multiplier theorems. *Studia Math.*, 93(3):201–222, 1989.