# WEIGHTED RESOLVENT ESTIMATES FOR THE SPATIALLY PERIODIC STOKES EQUATIONS

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Abstract. We consider the spatially periodic Laplace and Stokes resolvent problem and show corresponding weighted resolvent estimates on the locally compact abelian group $G := \mathbb{R}^{n-1} \times \mathbb{R}/L\mathbb{Z}$, where the weight is in the Muckenhoupt class $A_q(G)$ for $1 < q < \infty$. A main tool is the use of Fourier techniques on the Schwarz-Bruhat space $\mathcal{S}(G)$ and on the tempered distributions $\mathcal{S}'(G)$ together with a weighted transference principle à la Anderson and Mohanty (2009) and a splitting of the function spaces into a mean-value free part and a lower-dimensional, nonperiodic part, which then enables us to make use of a weighted Mikhlin multiplier theorem.

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*Key Words:* Muckenhoupt weights; Stokes equation; spatially periodic; resolvent estimates.

## 1. Introduction

Let us consider the periodic linear Stokes resolvent problem

$$\begin{cases} \lambda u - \Delta u + \nabla \mathfrak{p} = f & \text{in } \mathbb{R}^n, \\ \operatorname{div} u = g & \text{in } \mathbb{R}^n, \\ u(x', x_n + L) = u(x', x_n), & \\ \lim_{|x'| \to \infty} u(x', x_n) = 0, & \end{cases} \tag{1}$$

with periodic external force $f(x', x_n) = f(x', x_n + L)$. Here, $L > 0$ is fixed and $\lambda \in \Sigma_\vartheta := \{\lambda \in \mathbb{C} : |\arg \lambda| < \vartheta, \lambda \neq 0\}$ for some $\vartheta \in (0, \pi)$. We want to study this problem in an $L^q$-setting on the periodic whole space using Fourier multiplier techniques, following an ansatz by Kyed [10], who was considering time-periodic systems. Our setup calls for different methods, as the problems investigated here are spatially periodic. Nevertheless, the crucial cornerstones will be a weighted transference principle of Fourier multipliers, which enables us to switch between multipliers in different group settings, and a weighted version of the Mikhlin multiplier theorem. The transference principle is important, since Mikhlin's theorem only works in an $\mathbb{R}^n$-setting.

Let us consider $G := \mathbb{R}^{n-1} \times \mathbb{R}/L\mathbb{Z}$, $L > 0$, which together with addition as group operation and the canonical quotient topology inherited from $\mathbb{R}^n$ yields a locally compact abelian group as seen in [6, Example 2.1.3]. Thus, under the canonical identification of $G$ with $\mathbb{R}^{n-1} \times [0, L)$ the Haar measure $\mu$ on $G$ is given up to

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a normalization factor by the product of the Lebesgue measure on $\mathbb{R}^{n-1}$ and the Lebesgue measure on $[0, L)$, that is

$$\int_G f \,\mathrm{d}\mu = \frac{1}{L}\int_0^L \int_{\mathbb{R}^{n-1}} f(x', x_n)\,\mathrm{d}x'\,\mathrm{d}x_n, \qquad f \in C_0(G).$$

We will choose base sets $U_k$ of the form $\prod_{j=1}^n I_j^k$, where the $I_j^k$, $1 \le j \le n-1$ are open intervals of length $2^k L$ and $I_n^k$ is an open arc of length $\min\{2^k L, L\}$. These sets obviously enjoy the doubling order $2^n$. In the following we will refer to such sets as *$G$-cubes of length* $2^k L$.
As the structure of the group $G$ is of very concrete nature, it will be possible to define a differentiable structure on $G$ and consequently also Sobolev spaces. This yields a promising starting point for investigating the linear Stokes resolvent problem (1), which in terms of the group $G$ may be equivalently rewritten as

$$\begin{cases} \lambda u - \Delta u + \nabla \mathfrak{p} = f & \text{in } G, \\ \operatorname{div} u = g & \text{in } G, \end{cases} \tag{2}$$

with $\lambda \in \Sigma_\vartheta$ for some $\vartheta \in (0, \pi)$.
The following two theorems are our main theorems. Their proofs are postponed until Section 5. For the definition of the projection $\mathcal{P}$ and the divergence space $W^{1,q}_{\omega,\mathrm{div}}(G)$, see (17) and (24) below, respectively. Firstly, we deal with the spatially periodic Laplace equation, that is we will look at the problem

$$\lambda u - \Delta u = f \qquad \text{in } G, \tag{3}$$

with $\lambda \in \Sigma_\vartheta := \{\lambda \in \mathbb{C} : |\arg \lambda| < \vartheta, \lambda \neq 0\}$ for some $\vartheta \in (0, \pi)$.

**Theorem 1.1.** *Let $n \ge 2$, $1 < q < \infty$ and $\omega \in A_q(G)$ and $0 < \vartheta < \pi$. Then for each $f \in L^q_\omega(G)$ and $\lambda \in \Sigma_\vartheta$ there is a unique solution $u \in W^{2,q}_\omega(G)$ to (3). Moreover, $u$ satisfies the estimate*

$$\|\lambda u, \nabla^2 u\|_{L^q_\omega(G)} \le c\|f\|_{L^q_\omega(G)},$$

*where $c = c(\omega, n, q, \vartheta, L) > 0$ is an $A_q(G)$-consistent constant.*

Similarly, we obtain existence and uniqueness of a solution to (2) in weighted spaces.

**Theorem 1.2.** *Let $n \ge 3$, $1 < q < \infty$, $\omega \in A_q(G)$ and $0 < \vartheta < \pi$. Then to each $f \in L^q_\omega(G)$, $g \in W^{1,q}_{\omega,\mathrm{div}}(G)$ and $\lambda \in \Sigma_\vartheta$ there is a unique solution $(u, \mathfrak{p}) \in W^{2,q}_\omega(G) \times \hat{W}^{1,q}_\omega(G)$ to (2) satisfying the* a priori *estimate*

$$\begin{aligned} &\|\lambda u, \nabla^2 u, \nabla \mathfrak{p}\|_{L^q_\omega(G)} \\ &\qquad \le c(\|f\|_{L^q_\omega(G)} + \|\nabla g\|_{L^q_\omega(G)} + |\lambda|(\|\nabla \mathcal{P}_\perp g\|_{L^q_\omega(G)} + \|\mathcal{P} g\|_{\hat{W}^{-1,q}_{\tilde{\omega}}(\mathbb{R}^{n-1})})), \end{aligned} \tag{4}$$

*where $c = c(\omega, n, q, \vartheta, L) > 0$ is an $A_q(G)$-consistent constant.*
*The same conclusion holds true for $n = 2$ if $\mathcal{P} g = 0$.*

In the context of the classical whole space $\mathbb{R}^n$, a corresponding result has been obtained by Farwig and Sohr in [7] using the following multiplier theorem.

**Proposition 1.3** (Weighted Mikhlin)**.** *Suppose that $m \in L^\infty(\mathbb{R}^n) \cap C^n(\mathbb{R}^n \setminus \{0\})$ and there is a constant $c > 0$ such that for all $0 < R < \infty$ and all multi-indices $\alpha$*

*with* $|\alpha| \le n$

$$R^{|\alpha|-\frac{1}{2}} \left( \int_{R<|\xi|<2R} |D^\alpha m(\xi)|^2 \, d\xi \right)^{\frac{1}{2}} \le c. \tag{5}$$

*Then for every* $\omega \in A_q(\mathbb{R}^n)$, $1 < q < \infty$, $m$ *is an* $L^q_\omega(\mathbb{R}^n)$*-multiplier with an* $A_q(\mathbb{R}^n)$*-consistent bound.*
*A sufficient condition for* (5) *to hold is* $|\xi|^{|\alpha|}|D^\alpha m(\xi)| < c$ *for all* $\xi \in \mathbb{R}^n \setminus \{0\}$ *and all multi-indices* $\alpha$ *with* $|\alpha| \le n$.

*Proof.* See Theorem 3.9 of Chapter IV in [8]. ☐

In our case, the trick is to employ a transference principle of Fourier multipliers. The idea of transference has been used for the first time by de Leeuw [5]. We will state a weighted version due to Anderson and Mohanty [2], see Proposition 5.3 below.
This paper is organized as follows. In Section 2 we establish a differentiable structure on the group $G$ and its dual group $\hat{G}$. This will enable us to introduce the concept of spatially periodic Sobolev spaces in Section 3. In Section 4 we will derive a solution formula for the Stokes resolvent problem using Fourier techniques. Finally, Theorem 1.1 and 1.2 will be proven in Section 5.

## 2. Differentiable Structure on $G$ and $\hat{G}$

Let us denote by $\pi_G$ the canonical quotient mapping

$$\pi_G : \mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R} \to G, \qquad \pi_G(x', x_n) := (x', [x_n]),$$

where $[x_n] \in \mathbb{R}/L\mathbb{Z}$ is the equivalence class of $x_n \in \mathbb{R}$. For $m \in \mathbb{N}_0 \cup \{\infty\}$, we call

$$C^m(G) := \{u : G \to \mathbb{C} : \exists \tilde{u} \in C^m(\mathbb{R}^n) \text{ such that } \tilde{u} = u \circ \pi_G\}$$

the space of *m-times differentiable functions on* $G$ and define the derivatives via

$$D^\alpha u = D^\alpha \tilde{u}|_{\mathbb{R}^{n-1}\times[0,L)},$$

where $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le m$ and where we have identified $G$ with $\mathbb{R}^{n-1} \times [0, L)$ in virtue of the canonical bijection. Observe that for $u \in C^m(G)$, $0 \le m \le \infty$, every corresponding $\tilde{u} \in C^m(\mathbb{R}^n)$ is necessarily periodic of length $L$ in the last variable, and hence so is $D^\alpha \tilde{u}$ for any multi-index $\alpha$ with $|\alpha| \le m$. Therefore, we may write $D^\alpha \tilde{u} = (D^\alpha u) \circ \pi$, and thus $D^\alpha u \in C^{m-|\alpha|}(G)$. Moreover, let us introduce

$$C_0^m(G) := \{u \in C^m(G) : \operatorname{supp} u \text{ compact}\}.$$

It is clear that $C^\infty(G) \subset C^{m_1}(G) \subset C^{m_2}(G)$ for $m_1, m_2 \in \mathbb{N}_0$ with $m_2 \le m_1$ and that a similar chain of inclusions holds for the spaces with compact support. Moreover, since the topology of $G$ is inherited from $\mathbb{R}^n$, we see that $C^0(G) = C(G)$ and $C_0^0(G) = C_0(G)$, where $C(G)$ is the space of continuous functions and

$$C_0(G) := \{u \in C(G) : \operatorname{supp} u \text{ compact}\}.$$

We introduce the Schwartz-Bruhat space (see [3, 11]) for the locally compact abelian group $G$ as follows. Let $u \in C^\infty(G)$ and define for $j \in \mathbb{N}_0$

$$\rho_j(u) := \sup_{x=(x',x_n)\in G} (1 + |x'|)^j |D^j u(x)|,$$

where $D^j u := (D^\alpha u)_{|\alpha|\le j}$. Then we denote the Schwartz-Bruhat space by

$$\mathcal{S}(G) := \{u \in C^\infty(G) : \rho_j(u) < \infty \text{ for all } j \in \mathbb{N}_0\}.$$

Clearly, $\rho_j$ is a semi-norm on $\mathcal{S}(G)$ and so we can endow the space with the semi-norm topology induced by the family $\{\rho_j : j \in \mathbb{N}_0\}$. Notice that the semi-norms are in fact norms and that they are increasing in the sense that $\rho_j(u) \le \rho_k(u)$ if $j \le k$. Having set the topology of $\mathcal{S}(G)$, we can denote its dual space by $\mathcal{S}'(G)$ and equip it with the weak-* topology. In analogy to the classical $\mathbb{R}^n$-setup, $\mathcal{S}'(G)$ is called the *space of tempered distributions* on $G$. Tempered distributions in $\mathcal{S}'(G)$ are of finite type. More precisely, we have the following lemma.

**Lemma 2.1.** *A linear functional $T$ on the Schwartz-Bruhat space $\mathcal{S}(G)$ is in $\mathcal{S}'(G)$ if and only if there are $c > 0$ and $j \in \mathbb{N}_0$ such that $|\langle T, u\rangle| \le c\rho_j(u)$ for all $\mathcal{S}(G)$, where $\langle\cdot,\cdot\rangle$ is the duality paring of $\mathcal{S}'(G)$ and $\mathcal{S}(G)$.*

*Proof.* Assume there are $c > 0$ and $j \in \mathbb{N}_0$ such that $|\langle T, u\rangle| \le c\rho_j(u)$ for all $\mathcal{S}(G)$. Let $(u_n)_{n\in\mathbb{N}} \subset \mathcal{S}(G)$ be a null sequence, that is $u_n \to 0$ in $\mathcal{S}(G)$ as $n \to \infty$. In particular, $\rho_j(u_n) \to 0$ as $n \to \infty$. This gives the continuity of $T$ in the weak-* topology of $\mathcal{S}'(G)$.
Conversely assume that $T \in \mathcal{S}'(G)$ and that for each $j \in \mathbb{N}$ we have $u_j \in \mathcal{S}(G)$ such that

$$|\langle T, u_j\rangle| > j\rho_j(u_j).$$

This shows that $u_j \ne 0$ for all $j \in \mathbb{N}$. Since $\rho_j$ is a norm for every $j \in \mathbb{N}_0$, we can define $\tilde{u}_j := u_j/(j\rho_j(u_j)) \in \mathcal{S}(G)$. Then we have

$$|\langle T, \tilde{u}_j\rangle| > 1, \qquad \rho_j(\tilde{u}_j) = \frac{1}{j}.$$

Thus, fixing $j \in \mathbb{N}_0$, we see by the monotonicity of the seminorms that $\rho_j(\tilde{u}_m) \le \rho_m(\tilde{u}_m) = \frac{1}{m}$ for all $m \ge j$. Hence $\rho_j(\tilde{u}_m) \to 0$ as $m \to \infty$ for each $j \in \mathbb{N}_0$. This yields $\tilde{u}_j \to 0$ in $\mathcal{S}(G)$, contradicting $|\langle T, \tilde{u}_j\rangle| > 1$. □

With the help of the lemma it is easy to see that derivatives of a tempered distribution $\psi \in \mathcal{S}'(G)$ defined via

$$\langle D^\alpha T, u\rangle := (-1)^{|\alpha|}\langle T, D^\alpha u\rangle, \qquad u \in \mathcal{S}(G), \alpha \in \mathbb{N}_0^n$$

yield again tempered distributions.
It is well-known that the Pontryagin dual of $G$ is $\hat{G} = \mathbb{R}^{n-1} \times \frac{2\pi}{L}\mathbb{Z}$, see for example [10, Section 3.2.3]. We will introduce a differentiable structure in a similar way as for the group $G$. That is, we define

$$C^\infty(\hat{G}) := \{u : \hat{G} \to \mathbb{C} : u(\cdot, k) \in C^\infty(\mathbb{R}^{n-1}) \text{ for all } k \in \frac{2\pi}{L}\mathbb{Z}\}.$$

Furthermore, for $\alpha \in \mathbb{N}_0^{n-1}$ and $u \in C^\infty(\hat{G})$ we define the derivatives $D^\alpha u(\xi, k) = D_\xi^\alpha u(\xi, k)$. Note that for $\eta \in \hat{G}$ we write $|\eta|$ for the Euclidean norm of the vector $\eta = (\xi, k)$. With the seminorms

$$\hat{\rho}_j(u) := \sup_{\eta\in\hat{G}} (1 + |\eta|)^j |D^j u(\eta)|,$$

for $u \in C^\infty(\hat{G})$, $j \in \mathbb{N}_0$ and $D^j u := (D^\alpha u)_{|\alpha|\leq j}$ we may introduce the Schwartz-Bruhat space on $\hat{G}$ via

$$\mathcal{S}(\hat{G}) := \{u \in C^\infty(\hat{G}) : \hat{\rho}_j(u) < \infty \text{ for all } j \in \mathbb{N}_0\}.$$

The topology on $\mathcal{S}(\hat{G})$ will be given by the semi-norm topology induced by the seminorms $\hat{\rho}_j$, $j \in \mathbb{N}_0$. Then we may introduce the dual space $\mathcal{S}'(\hat{G})$ and equip it with the weak-* topology.

**Remark 2.2.** Let $p$ be a polynomial in $G$, *i.e.*, $p(x) = \sum_{\beta\in\mathbb{N}_0^{n-1},|\beta|<m} a_\beta \cdot (x')^\beta$ for some $m \in \mathbb{N}_0$ and $a_\beta \in \mathbb{C}$. Then for $u \in \mathcal{S}(G)$ we have $\psi \cdot u \in \mathcal{S}(G)$ for all $\psi \in C^\infty(G)$ with $|\psi(x)| \leq c|p(x)|$ for some $c > 0$. That is, the Schwartz-Bruhat space is closed under multiplication with smooth functions of at most polynomial growth and in particular under multiplication with itself. Moreover, also $\mathcal{S}'(G)$ is closed under multiplication with smooth functions of at most polynomial growth. In fact, considering such $\psi \in \mathcal{S}(G)$ and $T \in \mathcal{S}'(G)$ we see that $v \cdot T$ is well-defined in $\mathcal{S}'(G)$ via

$$\langle v \cdot T, u\rangle := \langle T, v \cdot u\rangle < \infty, \qquad u \in \mathcal{S}(G).$$

One readily checks that for any $\alpha \in \mathbb{N}_0^n$ we have the Leibniz product rule

$$D^\alpha(v \cdot T) = \sum_{|\gamma|\leq|\alpha|} \binom{\alpha}{\gamma} (D^\gamma v) \cdot (D^{\alpha-\gamma} T). \tag{6}$$

Similar arguments also apply in the case of the group $\hat{G}$.

**Remark 2.3.** The construction of the spaces $\mathcal{S}(G)$ and $\mathcal{S}(\hat{G})$ is the original construction due to F. Bruhat [3]. Later, Osborne [11] gave a different construction, which turns out to be equivalent. For a proof we refer to [11, Theorem 1].

## 3. Function Spaces

With $G$ being a locally compact abelian group with Haar measure $\mu$, one can define a nontrivial, translation-invariant, regular measure $\mu$, called *Haar measure* [1, 4, 9, 16], with $\mu(K) < \infty$ for all compact $K \subset G$. Furthermore, such a measure is unique up to multiplication with a constant. For $1 \leq q \leq \infty$, one can thus introduce the space $L^q(G)$ of $q$-integrable functions $f : G \to \mathbb{R}$, which turns into a Banach space if equipped with the usual norm

$$\|f\|_q := \left(\int_G |f|^q \,\mathrm{d}\mu\right)^{\frac{1}{q}}, \qquad 1 \leq q < \infty,$$
$$\|f\|_\infty := \mu\text{-}\operatorname*{ess\,sup}_G |f|.$$

Moreover, we consider the spaces $L^q_\omega(G)$ for $1 \leq q < \infty$ and Muckenhoupt weights $\omega \in A_q(G)$, which have been introduced in [14] in the context of locally compact abelian groups. As $G$ satisfies Assumption 1.1 of [14], the theory developed there applies in our case. One noteworthy point is that we can additionaly show that smooth functions with compact support are dense in $L^q_\omega(G)$.

**Lemma 3.1.** *Let $1 \leq q < \infty$ and $\omega \in A_q(G)$. Then $C_0^\infty(G)$ is dense in $L^q_\omega(G)$.*

*Proof.* We want to use a mollifier argument involving the boundedness of the maximal operator on $L^q_\omega(G)$. The first main effort will be to construct a suitable approximate identity $\psi_k \in C_0^\infty(G)$. Consider a real-valued nonnegative function $\tilde\psi \in C_0^\infty(\mathbb{R}^n)$ with $\operatorname{supp}\tilde\psi \subset B_{L/4}(0)$ and $\int_{\mathbb{R}^n}\tilde\psi \,\mathrm{d}x = L$ and define for $k \in \mathbb{Z}$ the function $\tilde\psi_k \in C_0^\infty(\mathbb{R}^n)$ via $\tilde\psi_k(x) := 2^{-kn}\tilde\psi(2^{-k}x)$. Furthermore define

$$\Psi_k : \mathbb{R}^n \to \mathbb{R}, \qquad \Psi_k(x', x_n) = \sum_{j=-\infty}^{\infty} \tilde\psi_k(x', x_n + jL).$$

Since $\tilde\psi_k$ is compactly supported, for each $k \in \mathbb{Z}$ and for each $(x', x_n) \in \mathbb{R}^n$ at most a finite number of summands is nonzero and hence $\Psi_k$ is periodic of length $L$ in the last variable and $\Psi_k \in C^\infty(\mathbb{R})$. Furthermore,

$$\psi_k : G \to \mathbb{R}, \qquad \psi_k(x', [x_n]) := \Psi_k(x', x_n),$$

is well-defined and we obtain that $\psi_k$ is a nonnegative smooth function with $\operatorname{supp}\psi_k$ contained in $U_{k-2}$, where $U_k$ is a $G$-cube of length $2^k L$ centered at $0 \in G$. Moreover, if $k \le 0$, then in the definition of $\Psi_k$ there is at most one summand nonzero. Therefore we obtain

$$\int_G \psi_k \,\mathrm{d}\mu = \frac{1}{L}\int_0^L \int_{\mathbb{R}^{n-1}} \Psi_k \,\mathrm{d}x' \,\mathrm{d}x_n = \frac{1}{L}\int_{\mathbb{R}^n} \tilde\psi_k \,\mathrm{d}x = 1, \qquad k \le 0.$$

Let now $f \in L^q_\omega(G)$. The outline of the rest of the proof is as follows: Considering $f_k := f \cdot \chi_{U_k}$, $k \in \mathbb{Z}$, we see by the Lebesgue Dominated Convergence Theorem that functions with compact support are dense in $L^q_\omega(G)$. Hence we can assume $\operatorname{supp} f$ to be compact. If we can show that $|\psi_k * f - f| \to 0$ almost everywhere in $G$ as $k \to -\infty$ and $\sup_{k\le 0}\|\psi_k * f\|_{L^q_\omega(G)} \le c\|f\|_{L^q_\omega(G)}$, then we obtain $\|\psi_k * f - f\|_{L^q_\omega(G)} \to 0$ as $k \to -\infty$ by the Lebesgue Dominated Convergence Theorem. Here, the convolution $g * h$ for two measurable functions $g, h : G \to \mathbb{C}$ is defined as

$$(g * h)(x) = \int_G g(x-y)h(y)\,\mathrm{d}\mu(y),$$

and it is easy to see that with $\operatorname{supp} g$ and $\operatorname{supp} h$ compact also $\operatorname{supp}(g * h)$ is compact. In our case, this yields $\operatorname{supp}(\psi_k * f)$ compact for all $k \le 0$. Furthermore, since $D^\alpha(\psi_k * f) = (D^\alpha\psi_k) * f$ whenever $\alpha \in \mathbb{N}_0^n$, we also have $\psi_k * f \in C_0^\infty(G)$. Having this in mind, let us first show that $|\psi_k * f - f| \to 0$ almost everywhere in $G$ as $k \to -\infty$. Since $\int_G \psi_k \,\mathrm{d}\mu = 1$ and $\operatorname{supp}\psi_k \subset U_{k-2}$, it suffices to show that

$$\int_{U_{k-2}} \psi_\varepsilon(y)|f(x-y) - f(x)|\,\mathrm{d}\mu(y) = \int_G \psi_k(y)|f(x-y) - f(x)|\,\mathrm{d}\mu(y) \to 0,$$

as $k \to -\infty$ for almost all $x \in G$. By Lebesgue's differentiation theorem, see [6, Chapter 2.2], there is $N \subset G$ with $\mu(N) = 0$ such that we can find for every $\delta > 0$ and every $x \in G \setminus N$ an integer $K(\delta, x) \in \mathbb{Z}$ such that

$$\int_{x+U_k} |f - f(x)|\,\mathrm{d}\mu \le \delta \cdot \mu(x + U_k),$$

for all $k \le K(\delta, x)$. Therefore, fix $\delta > 0$ and $x \in G \setminus N$. Without loss of generality $K(\delta, x) \le 0$ and consequently the $G$-cubes $x + U_k$ are actual cubes for $k \le K(\delta, x)$, whence we see $\mu(x+U_k) = 2^{nk}L^n$. Observe that $\|\psi_k\|_{L^\infty(G)} = 2^{-kn}\|\psi_0\|_{L^\infty(G)}$. Hence, for $k \in \mathbb{Z}$ with $k \leq K$, we obtain

$$\begin{aligned}\int_{x+U_{k-2}} \psi_k(y)|f(x-y)-f(x)|\,\mathrm{d}\mu(y) &\leq \|\psi_k\|_{L^\infty(G)} \int_{x+U_{k-2}} |f(y)-f(x)|\,\mathrm{d}\mu(y)\\ &\leq \delta \cdot \mu(x+U_{k-2})\|\psi_k\|_{L^\infty(G)}\\ &= \delta \cdot 2^{n(k-2)}L^n 2^{-kn}\|\psi_0\|_{L^\infty(G)}\\ &= \delta \cdot 2^{-2n}L^n\|\psi_0\|_{L^\infty(G)}.\end{aligned}$$

Since $\delta > 0$ was chosen arbitrarily, we obtain the almost everywhere convergence $|\psi_k * f - f| \to 0$ in $G$ as $\tilde{k} \to -\infty$.
Next we want to prove $\sup_{k\leq 0} \|\psi_k * f\|_{L^q_\omega(G)} \leq c\|f\|_{L^q_\omega(G)}$. We notice

$$\begin{aligned}\sup_{k\leq 0} |(\psi_k * f)(x)| &\leq \sup_{k\leq 0} \int_G \psi_k(x-y)|f(y)|\,\mathrm{d}\mu(y) = \sup_{k\leq 0} \int_{x+U_{k-2}} \psi_k(x-y)|f(y)|\,\mathrm{d}\mu(y)\\ &\leq \sup_{k\leq 0} \|\psi_k\|_{L^\infty(G)} \int_{x+U_{k-2}} |f|\,\mathrm{d}\mu = \|\psi_0\|_{L^\infty(G)} \sup_{k\leq 0} 2^{-nk} \int_{x+U_{k-2}} |f|\,\mathrm{d}\mu\\ &= \|\psi_0\|_{L^\infty(G)} \sup_{k\leq 0} \frac{2^{-2n}L^n}{\mu(x+U_{k-2})} \int_{x+U_{k-2}} |f|\,\mathrm{d}\mu\\ &\leq 2^{-2n}L^n\|\psi_0\|_{L^\infty(G)}\mathcal{M}_G f(x),\end{aligned}$$

where $\mathcal{M}_G$ is the maximal operator on $G$, see [14] for details about the maximal operator, the class $A_q(G)$ and $A_q(G)$-consistency. By [14, Theorem 1.4] we know that the maximal operator $\mathcal{M}(G)$ is $A_q(G)$-consistently bounded on $L^q_\omega(G)$. It follows

$$\sup_{k\leq 0} \|\psi_k * f\|_{L^q_\omega(G)} \leq c\|\mathcal{M}_G f\|_{L^q_\omega(G)} \leq c\|f\|_{L^q_\omega(G)},$$

where $c = c(\omega) > 0$ is an $A_q(G)$-consistent constant. As mentioned above, the convergence $|\psi_\varepsilon * f - f| \to 0$ almost everywhere in $G$ as $k \to -\infty$ together with $\|\psi_k * f - f\|_{L^q_\omega(G)} \leq (c+1)\|f\|_{L^q_\omega(G)} < \infty$ for all $k \leq 0$ yields $\|\psi_k * f - f\|_{L^q_\omega(G)} \to 0$ as $k \to -\infty$ in virtue of the Lebesgue Dominated Convergence Theorem. □

As we deal with partial differential equations, there is the need to set up a notion of differentiability in the context of Lebesgue spaces. As is well known in the classical $\mathbb{R}^n$-setting, a suitable concept is given by Sobolev spaces. In order to introduce a corresponding concept in our setup, we need to discuss some more properties of tempered distributions on $G$. We will write $\omega \in A_\infty(G)$ if $\omega \in A_q(G)$ for some $1 < q < \infty$.

**Lemma 3.2.** *Let $1 < q \leq \infty$ and $\omega \in A_q(G)$. Then for every $u \in \mathcal{S}(G)$ it holds $\|u\|_{L^q_\omega(G)} < \infty$. Moreover, the continuous embeddings $\mathcal{S}(G) \hookrightarrow L^q_\omega(G) \hookrightarrow \mathcal{S}'(G)$ hold true, where we identify $u \in L^q_\omega(G)$ with $T_u \in \mathcal{S}'(G)$ via*

$$\langle T_u, \psi\rangle = \int_G u\psi\, d\mu.$$

*Proof.* Note that the assertion $\|u\|_{L^\infty_\omega(G)} < \infty$, $\omega \in A_\infty(G)$ is trivial, because as shown in [14, Proposition 3.6 (iii)] it holds $L^\infty_\omega(G) = L^\infty(G)$ with equal norms and for every $u \in \mathcal{S}(G)$ we certainly have $\|u\|_{L^\infty(G)} = \rho_0(u) < \infty$. We thus concentrate

on showing $\|u\|_{L^q_\omega(G)} < \infty$ for $\omega \in A_q(G)$ with $1 < q < \infty$.
Fix some $\psi_0 \in C_0^\infty(G)$ with $\operatorname{supp} \psi_0 \subset U_0$. We claim that for all $x = (x', x_n) \in G$

$$(L + |x'|)^{1-n} \le \frac{2^{3n} L}{\|\psi_0\|_{L^1(G)}} \mathcal{M}_G \psi_0(x).$$

Indeed, let $x \in G$ and set $k \in \mathbb{Z}$, as the largest integer such that $2^k L \le L + |x'|$. Observe that necessarily $k \ge 0$. Thus for the $G$-cube of length $2^k L$ we have $\mu(U_k) = L \cdot (2^k L)^{n-1}$. Furthermore, $U_0 \subset x + U_{k+3}$. Indeed, since $k \ge 0$, we have $|x'| < L + |x'| < 2^{k+1} L$ and so $x \in U_{k+1}$ (note carefully that the choice of $x_n \in \mathbb{R}/L\mathbb{Z}$ does not enter the argument here, as the $G$-cubes $U_k$ for $k \ge 0$ already stretch across the entire period $L$). Since obviously $0 \in U_{k+1}$, we obtain by the engulfing property of $G$, see [14, Proposition 2.1 (ii)] the inclusion $U_0 \subset x + U_{k+3}$.
We can now calculate

$$\begin{aligned}
\frac{1}{(L + |x'|)^{n-1}} \int_G |\psi_0| \,\mathrm{d}\mu &\le \frac{1}{(2^k L)^{n-1}} \int_G |\psi_0| \,\mathrm{d}\mu = \frac{L}{L (2^k L)^{n-1}} \int_G |\psi_0| \,\mathrm{d}\mu \\
&= \frac{L}{\mu(x + U_k)} \int_G |\psi_0| \,\mathrm{d}\mu = \frac{L}{\mu(x + U_k)} \int_{x + U_{k+3}} |\psi_0| \,\mathrm{d}\mu \\
&= \frac{2^{3n} L}{\mu(x + U_{k+3})} \int_{x + U_{k+3}} |\psi_0| \,\mathrm{d}\mu \le 2^{3n} L \cdot \mathcal{M}_G \psi_0(x),
\end{aligned}$$

which proves the claim. Since the maximal operator is bounded on $L^q_\omega(G)$, we obtain by introducing the function $P(x) = (L + |x'|)^{n-1}$

$$\int_G P^{-q} \,\mathrm{d}\mu_\omega \le c \int_G (\mathcal{M}\psi_0)^q \,\mathrm{d}\mu_\omega \le c \int_G |\psi_0|^q \,\mathrm{d}\mu_\omega < \infty.$$

Now let $\psi \in \mathcal{S}(G)$. Then

$$\|\psi\|_{L^q_\omega(G)} \le \|P \cdot \psi\|_{L^\infty(G)} \|P^{-1}\|_{L^q_\omega(G)} \le \rho_{n-1}(\psi) \|P^{-1}\|_{L^q_\omega(G)} < \infty.$$

This gives the first assertion.
For the assertion concerning the continuous embeddings, we calculate with Hölder's inequality for $u \in L^q_\omega(G)$ and $\psi \in \mathcal{S}(G)$

$$\begin{aligned}
|\langle u, \psi \rangle| &\le \int_G |u\psi| \,\mathrm{d}\mu \le \|u\|_{L^q_\omega(G)} \|\psi\|_{L^{q'}_{\omega'}(G)} \\
&\le \|u\|_{L^q_\omega(G)} \|P \cdot \psi\|_{L^\infty(G)} \|P^{-1}\|_{L^{q'}_{\omega'}(G)} \le c \|u\|_{L^q_\omega(G)} \rho_{n-1}(\psi),
\end{aligned}$$

where $\omega' := \omega^{-\frac{q'}{q}} \in A_{q'}$ by [14, Proposition 3.2 (ii)] and therefore $\|P^{-1}\|_{L^{q'}_{\omega'}(G)} \le c$ by what we have just proven. So Lemma 2.1 yields $L^q_\omega(G) \hookrightarrow \mathcal{S}'(G)$. We note that the dual space of $L^q_\omega(G)$ can be identified with $L^{q'}_{\omega'}(G)$ via the dual pairing $\langle u, v \rangle = \int_G uv \,\mathrm{d}\mu$. Therefore, by duality we arrive at

$$\mathcal{S}(G) \hookrightarrow L^q_\omega(G) \hookrightarrow \mathcal{S}'(G).$$

Moreover, in the unweighted case we obtain

$$\begin{aligned}
|\langle u, \psi \rangle| &\le \|u\|_{L^\infty(G)} \|\psi\|_{L^1(G)} \le \|u\|_{L^\infty(G)} \|P^2 \cdot \psi\|_{L^\infty(G)} \|P^{-2}\|_{L^1(G)} \\
&\le c \|u\|_{L^\infty(G)} \rho_{2(n-1)}(\psi),
\end{aligned}$$

since $P^{-2}$ is easily seen to be integrable. Therefore $L^\infty_\omega(G) = L^\infty(G) \hookrightarrow \mathcal{S}'(G)$ for all $\omega \in A_\infty(G)$. Since $\|\psi\|_{L^\infty(G)} = \rho_0(\psi)$ for $\psi \in \mathcal{S}(G)$, also $\mathcal{S}(G) \hookrightarrow L^\infty(G) = L^\infty_\omega(G)$ is valid and the lemma is proven. $\square$

We are now in the position to define weighted Sobolev spaces on $G$. Since for $1 < q \le \infty$ and $\omega \in A_q(G)$ we know $L^q_\omega(G) \hookrightarrow \mathcal{S}'(G)$, given $u \in L^q_\omega(G)$ the derivative $D^\alpha u$ is well-defined in $\mathcal{S}'(G)$ for any $\alpha \in \mathbb{N}_0^n$ and we say that $D^\alpha u \in L^q_\omega(G)$ if there is $u_\alpha \in L^q_\omega(G)$ with $T_{u_\alpha} = D^\alpha u$ as an identity in $\mathcal{S}'(G)$.

**Definition 3.3.** Let $m \in \mathbb{N}_0$, $1 < q \le \infty$ and $\omega \in A_q(G)$. Then we denote the *weighted Sobolev space of $m$-th order of $q$-integrable functions* by

$$W^{m,q}_\omega(G) := \{u \in L^q_\omega(G) : D^\alpha u \in L^q_\omega(G) \text{ for all } \alpha \in \mathbb{N}_0^n \text{ with } |\alpha| \le m\},$$
$$\|u\|_{W^{m,q}_\omega(G)} := \sum_{|\alpha|<m} \|D^\alpha u\|_{L^q_\omega(G)}.$$

**Remark 3.4.** It should be noted that for all $1 < q \le \infty$ and $\omega \in A_q(G)$, $W^{0,q}_\omega(G) = L^q(G)$. Furthermore, for all $m \in \mathbb{N}_0$ we have that $W^{m,q}_\omega(G)$ equipped with its respective norm yields Banach spaces. This follows from the continuous embedding $L^q_\omega(G) \hookrightarrow \mathcal{S}'(G)$. Indeed, let $(u_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $W^{m,q}_\omega(G)$. Then by definition of the norm of $W^{m,q}_\omega(G)$ the sequences $(u_n)_{n\in\mathbb{N}}$ and $(D^\alpha u_n)_{n\in\mathbb{N}}$ are Cauchy sequences in $L^q_\omega(G)$ for all multi-indices $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le m$. Since $L^q_\omega(G)$ is a Banach space, $u_n \to u$ and $D^\alpha u_n \to u_\alpha$ in $L^q_\omega(G)$ as $n \to \infty$ for some $u, u_\alpha \in L^q_\omega(G)$. $L^q_\omega(G) \hookrightarrow \mathcal{S}'(G)$ is continuous, and so the same convergences hold true in $\mathcal{S}'(G)$. Hence

$$\langle u_\alpha, \psi\rangle = \lim_{n\to\infty} \langle D^\alpha u_n, \psi\rangle = \lim_{n\to\infty} (-1)^{|\alpha|}\langle u_n, D^\alpha \psi\rangle = (-1)^{|\alpha|}\langle u, D^\alpha\psi\rangle$$

for all $\psi \in \mathcal{S}(G)$. This shows $u_\alpha = D^\alpha u$ in $\mathcal{S}'(G)$ and thus also in $L^q_\omega(G)$. Therefore, $u_n \to u$ in $W^{m,q}_\omega(G)$, showing that $W^{m,q}_\omega(G)$ is a Banach space.

**Lemma 3.5.** *Let $k \in \mathbb{N}_0$, $1 \le q < \infty$ and $\omega \in A_q(G)$. Then $C_0^\infty(G)$ is dense in $W^{k,q}_\omega(G)$.*

*Proof.* Consider the approximate identity $(\psi_k)_{k\le 0}$ from the proof of Lemma 3.1. In the proof we have seen that $\|\psi_k * f - f\|_{L^q_\omega(G)} \to 0$ as $k \to -\infty$ for all $f \in L^q_\omega(G)$. If $f \in W^{m,q}_\omega(G)$, then $D^\alpha(\psi_k * f) = \psi_k * D^\alpha f$ for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le m$ and all $k \in \mathbb{Z}$ with $k \le 0$. Hence we see $\|\psi_k * f - f\|_{W^{m,q}_\omega(G)} \to 0$ as $k \to -\infty$ for all $f \in W^{m,q}_\omega(G)$. Therefore, the assertion is proven if we can show that the set of all $f \in W^{m,q}_\omega(G)$ with compact support is dense in $W^{m,q}_\omega(G)$, since $\psi_k * f \in C_0^\infty(G)$ for compactly supported $f \in L^q_\omega(G)$.
In contrast to the proof of Lemma 3.1 we choose smooth cut-off functions instead of looking at the truncated functions $f\chi_{U_k}$, $k \in \mathbb{Z}$. Write $Q_k$ for the projection of the $G$-cube $U_k$ to $\mathbb{R}^{n-1}$, *i.e.*, $Q_k$ is the $n-1$-dimensional cube of length $2^k L$ and center in 0. Then choose for each $k \in \mathbb{Z}$ with $k \ge 0$ a smooth $\tilde\chi_k \in C_0^\infty(\mathbb{R}^{n-1})$ such that

$$\tilde\chi_k = 1 \text{ in } Q_k, \qquad \operatorname{supp} \tilde\chi_k \subset Q_{k+1}, \qquad \|D^\alpha \chi_k\|_{L^\infty(\mathbb{R}^{n-1})} \le c/k,$$

for all $\alpha \in \mathbb{N}_0^{n-1}$ with $|\alpha| \le m$ for some $c > 0$ independent of $k \in \mathbb{Z}$. The function $\chi_k : G \to \mathbb{C}$ defined via $\chi_k(x', [x_n]) := \tilde\chi_k(x')$ is in $C_0^\infty(G)$ and hence $f\chi_k \in W^{m,q}_\omega(G)$ by the Leibnitz product rule (6) for products of Schwartz-Bruhat functions with tempered distributions. Because we have for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le m$ the pointwise convergence $|D^\alpha(f\chi_k)(x) - D^\alpha f(x)| \to 0$ almost everywhere as $k \to \infty$ and moreover $\|D^\alpha(f\chi_k)\|_{L^q_\omega(G)} \le c'\|D^\alpha f\|_{L^q_\omega(G)}$, where $c'$ is independent of $k \in \mathbb{Z}$, the Lebesgue Dominated Convergence Theorem yields the assertion. $\square$

## 4. Fourier Transform and Solution Formula

On the group $G := \mathbb{R}^n \times \mathbb{R}/L\mathbb{Z}$ we introduce the Fourier transform $\mathcal{F}_G$ via

$$\mathcal{F}_G : L^1(G) \to C(\hat{G}),$$
$$\mathcal{F}_G(u)(\xi, k) := \hat{u}(\xi, k) := \frac{1}{L} \int_0^L \int_{\mathbb{R}^{n-1}} u(x', x_n) e^{-ix'\cdot\xi - ikx_n} \,\mathrm{d}x' \,\mathrm{d}x_n.$$

Let us clarify the notation in order to avoid confusion. When we write $\hat{G}$, we jump freely between the actual Pontryagin dual of $G$, which consists of mappings of the form $(x, x_n) \mapsto e^{-ix'\cdot\xi - ikx_n}$, and its homeomorphic identification $\mathbb{R}^{n-1} \times \frac{2\pi}{L}\mathbb{Z}$, which consists of elements of the form $\eta = (\xi, k)$. Observe that $\mathcal{F}_G$ maps into $C(\hat{G})$ by Lebesgue's differentiation theorem. Furthermore, $\mathcal{F}_G : \mathcal{S}(G) \to \mathcal{S}(G)$ is a homeomorphism by [3], where $\mathcal{S}(G)$ and $\mathcal{S}(\hat{G})$ are the Schwartz-Bruhat spaces of rapidly decaying functions on $G$ and $\hat{G}$, respectively, as introduced in Section 2. The inverse Fourier transform is given by

$$\mathcal{F}_G^{-1} : L^1(\hat{G}) \to C(G),$$
$$\mathcal{F}_G^{-1}(u)(x', x_n) := \check{u}(x', x_n) := \sum_{k \in \frac{2\pi}{L}\mathbb{Z}} \int_{\mathbb{R}^{n-1}} u(\xi, k) e^{ix'\cdot\xi + ikx_n} \,\mathrm{d}\xi.$$

By the Pontryagin duality theorem in [12, Theorem 39, p.259], we can also introduce the Fourier transform $\mathcal{F}_{\hat{G}} : \mathcal{S}(\hat{G}) \to \mathcal{S}(G)$, which again yields a homeomorphism with inverse $\mathcal{F}_{\hat{G}}^{-1} : \mathcal{S}(G) \to \mathcal{S}(\hat{G})$. In fact, by inversion of the Fourier transform, see [13, Theorem 1.5.1], we have $\mathcal{F}_G^{-1}(\hat{f})(x) = f(x) = \mathcal{F}_{\hat{G}}(\hat{f})(-x)$ for all $x \in G$ and all $f \in \mathcal{S}(G)$. We can now introduce the Fourier transform on tempered distributions via

$$\mathcal{F}_G : \mathcal{S}'(G) \to \mathcal{S}'(\hat{G}),$$
$$\langle \mathcal{F}_G T, \psi \rangle := \langle T, \mathcal{F}_{\hat{G}} \psi \rangle, \qquad \psi \in \mathcal{S}(\hat{G}),$$

and in an analogous way we may introduce

$$\mathcal{F}_G^{-1} : \mathcal{S}'(\hat{G}) \to \mathcal{S}'(G),$$
$$\langle \mathcal{F}_G^{-1} T, \psi \rangle := \langle T, \mathcal{F}_{\hat{G}}^{-1} \psi \rangle, \qquad \psi \in \mathcal{S}(G).$$

Since $\mathcal{F}_G : \mathcal{S}(G) \to \mathcal{S}(\hat{G})$ is a homeomorphism, so is $\mathcal{F}_G : \mathcal{S}'(G) \to \mathcal{S}'(\hat{G})$.
One main reason to use Fourier techniques in partial differential equations is the correspondence between differential operators on $G$ and polynomials on $\hat{G}$. That is, we obtain for all $T \in \mathcal{S}'(G)$ and all multi-indeces $\alpha \in \mathbb{N}_0^n$ the relation

$$\mathcal{F}_G(D^\alpha T) = i^{|\alpha|} \eta^\alpha \mathcal{F}(T),$$

and $i^{|\alpha|} \eta^\alpha \mathcal{F}(T)$ is well-defined in $\mathcal{S}'(\hat{G})$ by Remark 2.2.
If we apply the Fourier transform on $\mathcal{S}'(G)$ to the linear Stokes resolvent system (2), we are thus led to the system

$$(7) \qquad \begin{cases} \lambda\hat{u} + |\eta|^2\hat{u} + i\eta\hat{\mathfrak{p}} = \hat{f} & \text{in } \hat{G}, \\ \eta \cdot \hat{u} = \hat{g} & \text{in } \hat{G}. \end{cases}$$

From this we see that $-|\eta|^2\hat{\mathfrak{p}} = i\eta\cdot\hat{f} - (\lambda+|\eta|^2)\hat{g}$ and hence

$$(\lambda+|\eta|^2)\hat{u} = \left(I - \frac{\eta\otimes\eta}{|\eta|^2}\right)\hat{f} - \frac{\lambda+|\eta|^2}{|\eta|^2} i\eta\hat{g}.$$

This gives formally a representation formula for $u$ reading

$$u = \mathcal{F}_G^{-1}\left(\frac{1}{\lambda+|\eta|^2}\left(I - \frac{\eta\otimes\eta}{|\eta|^2}\right)\hat{f} + \frac{i\eta}{|\eta|^2}\hat{g}\right), \tag{8}$$

valid as an identity in $\mathcal{S}'(G)$.
A word about notation in the Fourier spaces is in order. As already introduced, the variable in the Fourier space $\hat{G} = \mathbb{R}^{n-1}\times\frac{2\pi}{L}\mathbb{Z}$ will be called $\eta$ and split into $\eta = (\xi, k)$. Note that $k$ is not an integer, but $k \in \frac{2\pi}{L}\mathbb{Z}$. A variable in the Fourier space $\hat{\mathbb{R}^n} = \mathbb{R}^n = \mathbb{R}^{n-1}\times\frac{2\pi}{L}\mathbb{R}$ will be called $\zeta$ and split into $\zeta := (\xi, \kappa)$.

## 5. Weighted Resolvent Estimates

We will have to investigate the structure of Muckenhoupt weights on the group $G$ in more detail.

**Proposition 5.1.** *For all $1 < q < \infty$, the Muckenhoupt weights in $A_q(G)$ can be identified with those Muckenhoupt weights in $A_q(\mathbb{R}^n)$ which are periodic of length $L$ with respect to the variable $x_n$.*

*Proof.* Let $\omega \in A_q(\mathbb{R}^n)$ be periodic of length $L$ with respect to $x_n$ and let $U$ be a $G$-cube of length $2^k L$. If $k \le 0$, $U$ is an actual cube and thus the Muckenhoupt condition for $\omega$ immediately yields

$$\left(\frac{1}{\mu(U)}\int_U \omega \,\mathrm{d}\mu\right)\left(\frac{1}{\mu(U)}\int_U \omega^{-\frac{q'}{q}}\,\mathrm{d}\mu\right)^{\frac{q}{q'}} \le \mathcal{A}_q(\omega).$$

If $k > 0$, we notice that $U$ is a cuboid. Therefore, we define the cube $Q := \bigcup_{1\le l\le 2^k} U^l$ of edge length $2^k L$, where $U^l$ is $U$ translated by $(l-1)L$ in the direction of the $x_n$-axis. Strictly speaking, $Q$ is not really a cube, as it is not connected at the interfaces of the $U^l$. However, it is a cube up to a set of measure zero. Using the translation invariance of $\omega$, we can now calculate

$$\begin{aligned}
&\left(\frac{1}{\mu(U)}\int_U \omega \,\mathrm{d}\mu\right)\left(\frac{1}{\mu(U)}\int_U \omega^{-\frac{q'}{q}}\,\mathrm{d}\mu\right)^{\frac{q}{q'}} \\
&\qquad = \left(\frac{1}{2^k\mu(U)}\int_{\bigcup_{0\le l\le 2^k-1} U^l} \omega \,\mathrm{d}\mu\right)\left(\frac{1}{2^k\mu(U)}\int_{\bigcup_{1\le l\le 2^k} U^l} \omega^{-\frac{q'}{q}}\,\mathrm{d}\mu\right)^{\frac{q}{q'}} \\
&\qquad = \left(\frac{1}{\mu(Q)}\int_Q \omega \,\mathrm{d}\mu\right)\left(\frac{1}{\mu(Q)}\int_Q \omega^{-\frac{q'}{q}}\,\mathrm{d}\mu\right)^{\frac{q}{q'}} \le \mathcal{A}_q(\omega).
\end{aligned}$$

The converse direction can be proven similarly. □

Next we show that taking the average over one period of a Muckenhoupt weight $\omega \in A_q(G)$ yields a Muckenhoupt weight in $A_q(\mathbb{R}^{n-1})$. This will be crucial in identifying the correct function space for the divergence.

**Proposition 5.2.** *Let $1 < q < \infty$, $n \ge 2$ and $\omega \in A_q(G)$. Then*

$$\bar{\omega}(x_1,\dots,x_{n-1}) := \int_0^L \omega(x_1,\dots,x_n)\,dx_n \in A_q(\mathbb{R}^{n-1})$$

*and* $\mathcal{A}_q(\bar{\omega}) \leq \mathcal{A}_q(\omega)$.

*Proof.* We will show that

$$\left(\frac{1}{\lambda(Q)}\int_Q f\,\mathrm{d}\mu\right)^q \leq \frac{c}{\lambda_{\bar{\omega}}(Q)}\int_Q f^q\bar{\omega}\,\mathrm{d}\mu. \tag{9}$$

for any cube $Q \subset \mathbb{R}^{n-1}$ of length $2^kL$, $k \in \mathbb{Z}$. Note that this suffices, since a condition of type (9) is equivalent to the Muckenhoupt condition by by [14, Proposition 3.5]. Hence, let $f : \mathbb{R}^{n-1} \to \mathbb{R}$ be a nonnegative function that is measurable with respect to the $(n-1)$-dimensional Lebesgue measure $\lambda$ and define $\tilde{f} : G \to \mathbb{R}$ via $\tilde{f}(x', x_n) := f(x')$ for all $x = (x', x_n) \in G$. Obviously, $\tilde{f}$ is nonnegative and measurable with respect to the Haar measure $\mu$. Moreover, for any cube $Q \subset \mathbb{R}^{n-1}$ of length $2^kL$, $k \in \mathbb{Z}$, and any set $U \in G$ of the form $U := Q \times I_n$ with $I_n$ being an arc of length $\min\{2^kL, L\}$

$$\frac{1}{\lambda(Q)}\int_Q f\,\mathrm{d}\lambda = \frac{1}{\mu(U)}\int_U \tilde{f}\,\mathrm{d}\mu. \tag{10}$$

If $k \geq 0$, $U$ is a $G$-cube of length $2^kL$ such that

$$\lambda_{\bar{\omega}}(Q) = \int_Q \bar{\omega}\,\mathrm{d}\lambda = \int_Q\int_0^L \omega(x', x_n)\,\mathrm{d}x_n\,\mathrm{d}x' = \int_U \omega\,\mathrm{d}\mu = \mu_\omega(U). \tag{11}$$

Thus, we obtain

$$\begin{aligned}\left(\frac{1}{\lambda(Q)}\int_Q f\,\mathrm{d}\lambda\right)^q &= \left(\frac{1}{\mu(U)}\int_U \tilde{f}\,\mathrm{d}\mu\right)^q \leq \frac{c}{\mu_\omega(U)}\int_U \tilde{f}^q\omega\,\mathrm{d}\mu \\ &= \frac{c}{\lambda_{\bar{\omega}}(Q)}\int_Q f^q\bar{\omega}\,\mathrm{d}\lambda,\end{aligned} \tag{12}$$

where $c = \mathcal{A}_q(\omega) > 0$ is the Muckenhoupt constant of $\omega$. If $k < 0$, relation (11) does not hold anymore, since the edge length of $U$ is less the $L$. However, it does hold with $U$ being replaced by the cuboid $R := \bigcup_{1\leq l\leq 2^k} U^l$, where $U_l$ is $U$ translated by $(l-1)L$ in the direction of the $x_n$-axis. Clearly, also (10) holds with $U$ replaced by $R$. Unfortunately, $R$ is no $G$-cube anymore. Nevertheless, we still can calculate for every $1 \leq l \leq 2^k$

$$\left(\frac{1}{\mu(U^l)}\int_{U^l} \tilde{f}\,\mathrm{d}\mu\right)^q \leq \frac{c}{\mu_\omega(U^l)}\int_{U^l} \tilde{f}^q\omega\,\mathrm{d}\mu,$$

which can be written in the more accessible way

$$\left(\int_{U^l} \tilde{f}\,\mathrm{d}\mu\right)^q \mu_\omega(U^l) \leq c\mu(U^l)^q\int_{U^l} \tilde{f}^q\omega\,\mathrm{d}\mu.$$

Because $\mu(U^l) = \mu(U)$ and $\int_{U^l} \tilde{f}\,\mathrm{d}\mu = \int_U \tilde{f}\,\mathrm{d}\mu$ for all $1 \leq l \leq 2^k$, this implies

$$\begin{aligned}\left(\int_R \tilde{f}\,\mathrm{d}\mu\right)^q \mu_\omega(R) &= 2^{kq}\left(\int_U \tilde{f}\,\mathrm{d}\mu\right)^q\sum_{l=1}^{2^k}\mu_\omega(U^l) = 2^{kq}\sum_{l=1}^{2^k}\left(\int_{U^l}\tilde{f}\,\mathrm{d}\mu\right)^q\mu_\omega(U^l) \\ &\leq c2^{kq}\sum_{l=1}^{2^k}\mu(U^l)^q\int_{U^l}\tilde{f}^q\omega\,\mathrm{d}\mu = c\left(2^k\mu(U)\right)^q\sum_{l=1}^{2^k}\int_{U^l}\tilde{f}^q\omega\,\mathrm{d}\mu \\ &= c\mu(R)^q\int_R \tilde{f}^q\omega\,\mathrm{d}\mu.\end{aligned}$$

Thus the same calculation as in (12) with $U$ replaced by $R$ yields the claim. □

Now we can turn our focus back to the transference principle. In the context of $\mathbb{R}^n$ and the $n$-dimensional torus $T^n$, the following weighted restriction theorem has been shown by Anderson and Mohanty [2, Theorem 1.1].

**Proposition 5.3.** *Let $1 < q < \infty$ and $0 \le \omega \in L^q_{\text{loc}}(\mathbb{R}^n)$ be periodic of length 1. Suppose furthermore that $M \in L^\infty(\mathbb{R}^n)$ is continuous and an $L^q_\omega(\mathbb{R}^n)$-multiplier, i.e., there is a constant $c > 0$ with*

$$\left\| \mathcal{F}^{-1}\left[M \cdot \hat{f}\right] \right\|_{L^q_\omega(\mathbb{R}^n)} \le c \|f\|_{L^q_\omega(\mathbb{R}^n)}, \qquad f \in L^q_\omega(\mathbb{R}^n). \tag{13}$$

*Then $m := M|_{\mathbb{Z}^n} \in L^\infty(\mathbb{Z}^n)$ is an $L^q_\omega(T^n)$-multiplier with*

$$\left\| \mathcal{F}^{-1}\left[m \cdot \hat{f}\right] \right\|_{L^q_\omega(T^n)} \le c \|f\|_{L^q_\omega(T^n)}, \qquad f \in L^q_\omega(T^n). \tag{14}$$

*with the same constant $c > 0$.*

**Remark 5.4.** The continuity condition on $M$ may be weakened, see [15, Lemma 3.16, p.263]. Also, by revising the proof of [2, Theorem 1.1], Proposition 5.3 can be generalized to arbitrary periods $L > 0$ and also to our setting of the group $G$.

We will tackle the two terms involving $f$ and $g$ on the right hand side of the representation formula (8) seperately. If we denote the multiplier appearing in the first term by

$$m_f : \hat{G} \to \mathbb{C}, \qquad m_f(\xi, k) = \frac{1}{\lambda + |\eta|^2}\left(I - \frac{\eta \otimes \eta}{|\eta|^2}\right) \tag{15}$$

and define

$$M_f : \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{C}, \qquad M_f(\xi, \kappa) = \frac{1}{\lambda + |\zeta|^2}\left(I - \frac{\zeta \otimes \zeta}{|\zeta|^2}\right), \qquad \zeta = (\xi, \frac{2\pi}{L}\kappa), \tag{16}$$

we see that $m_f = M_f|_{\hat{G}}$. Furthermore, in the formula of $M_f$ we recognize the usual symbols of the resolvent problem on the whole space and the Helmholtz projection. It is well-known that these symbols satisfy Mikhlin's condition and therefore extend to continuous $L^q_\omega(\mathbb{R}^n)$-multipliers for all $\omega \in A_q(\mathbb{R}^n)$, in particular for all $\omega \in A_q(\mathbb{R}^n)$ that are periodic of length $L$ in the direction of the $x_n$-axis. Therefore, we would like to use Propositions 5.1 and 5.3 to obtain that $m_f$ is an $L^q_\omega(G)$-multiplier for all $\omega \in A_q(G)$, and use a similar argument for $\eta^\alpha m_f(\eta)$ with any multi-index $|\alpha| \le 2$. Unfortunately, the multiplier $M_f$ is not continuous at the origin, so Proposition 5.3 can not be applied directly. Also, it is not immediately clear how to deal with the second term involving the divergence function $g$. If we simply transferred the symbol $\frac{i\eta}{|\eta|^2}$ to the $\mathbb{R}^n$-setting, the singularity at the origin would prevent us from concluding a corresponding *a priori* estimate.
To overcome these difficulties appearing at the origin of the Fourier space, we first perform an averaging procedure and exploit that the respective functions will be split into a familiar part on $\mathbb{R}^{n-1}$ and a well-behaved part on $G$ whose average vanishes over one period.

**Definition 5.5.** Let

$$\begin{aligned} \mathcal{P} : C_0^\infty(G) \to C_0^\infty(G), \qquad & \mathcal{P}f(x') := \frac{1}{L}\int_0^L f(x', x_n)\,\mathrm{d}x_n, \\ \mathcal{P}_\perp : C_0^\infty(G) \to C_0^\infty(G), \qquad & \mathcal{P}_\perp = \mathsf{id} - \mathcal{P}. \end{aligned} \tag{17}$$

Then $\mathcal{P}$ induces a decomposition of $L^q_\omega(G)$ and, more generally, $W^{k,q}_\omega(G)$ into direct sums of average-free functions with respect to $x_n$ and functions that are independent of the variable $x_n$.

**Lemma 5.6.** *Let $1 < q < \infty$, $\omega \in A_q(G)$ and $k \in \mathbb{N}$. Then*

$$\begin{aligned} L^q_\omega(G) &= L^q_{\bar\omega}(\mathbb{R}^{n-1}) \oplus \mathcal{P}_\perp L^q_\omega(G), \\ W^{k,q}_\omega(G) &= W^{k,q}_{\bar\omega}(\mathbb{R}^{n-1}) \oplus \mathcal{P}_\perp W^{k,q}_\omega(G). \end{aligned} \tag{18}$$

*where $\bar\omega = \frac{1}{L}\int_0^L \omega(x', x_n)\,dx_n$. Moreover, the decompositions are $A_q(G)$-consistent, that is the bound of projection $\mathcal{P}$ on the respective spaces is $A_q(G)$-consistent.*

*Proof.* First of all notice that $\mathcal{P}^2 = \mathcal{P}$. Moreover, by Fubini we obtain for $f \in \mathcal{S}(G)$

$$\begin{aligned} \mathcal{F}_G\left(\mathcal{P}(f)\right)(\xi, k) &= \frac{1}{L}\int_0^L \int_{\mathbb{R}^{n-1}} \left(\frac{1}{L}\int_0^L f(x', s)\,\mathrm{d}s\right) e^{-ix'\cdot\xi - ik\frac{2\pi}{L}x_n}\,\mathrm{d}x'\,\mathrm{d}x_n \\ &= m_{\mathcal{P}}(k)\int_{\mathbb{R}^{n-1}} \left(\frac{1}{L}\int_0^L f(x', s)\,\mathrm{d}s\right) e^{-ix'\cdot\xi}\,\mathrm{d}x' \\ &= m_{\mathcal{P}}(k)\mathcal{F}_G f(\xi, 0) = m_{\mathcal{P}}(k)\mathcal{F}_G f(\xi, k), \end{aligned} \tag{19}$$

where $m_{\mathcal{P}} := \chi_{\{0\}}$ is the characteristic function concentrated in 0. Let us write $\tilde m_{\mathcal{P}} := \mathbb{1}_{\mathbb{R}^{n-1}} \otimes m_{\mathcal{P}}$, where $\mathbb{1}_{\mathbb{R}^{n-1}}$ is the constant 1-function on $\mathbb{R}^{n-1}$. Then $\tilde m_{\mathcal{P}}$ is the Fourier symbol of $\mathcal{P}$ and in virtue of Proposition 5.3 we see that $m_{\mathcal{P}}$ is an $L^q_\omega(G)$-multiplier: Indeed, take a cut-off function $\varphi \in C_0^\infty(\mathbb{R})$ with $\varphi(0) = 1$ and $\operatorname{supp}\varphi \subset (-\frac{1}{2}, \frac{1}{2})$. Writing $\tilde\varphi := \mathbb{1}_{\mathbb{R}^{n-1}} \otimes \varphi$, we have $\tilde m_{\mathcal{P}} = \tilde\varphi|_{\hat G}$. As $\varphi$ has compact support, it obviously satisfies Mikhlin's condition and hence $\tilde\varphi$ extends to an $L^q_\omega(\mathbb{R}^n)$-multiplier. Therefore Proposition 5.3 yields the claim. Thus, $\mathcal{P} : L^q_\omega(G) \to L^q_\omega(G)$ is continuous and we obtain the decomposition

$$L^q_\omega(G) = \mathcal{P}L^q_\omega(G) \oplus \mathcal{P}_\perp L^q_\omega(G).$$

But since elements in $\mathcal{P}L^q_\omega(G)$ do not depend on $x_n$ anymore, we have obviously the norm equality $\|\mathcal{P}f\|_{L^q_{\bar\omega}(\mathbb{R}^{n-1})} = \|\mathcal{P}f\|_{L^q_\omega(G)}$ and thus $L^q_{\bar\omega}(\mathbb{R}^{n-1}) = \mathcal{P}L^q_\omega(G)$. This shows

$$L^q_\omega(G) = L^q_{\bar\omega}(\mathbb{R}^{n-1}) \oplus \mathcal{P}_\perp L^q_\omega(G).$$

Concerning higher derivatives, let $k \in \mathbb{N}$ and let $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le k$. Then for $f \in W^{k,q}_\omega(G)$ it holds $\mathcal{P}D^\alpha f = D^\alpha\mathcal{P}f$, since by (19) both $\mathcal{P}$ and $D^\alpha$ can be viewed as Fourier multiplier functions. Hence, a similar argument as above yields the assertion for $W^{k,q}_\omega(G)$. □

With this decomposition at hand, we can immediately prove Theorem 1.1.

*Proof of Theorem 1.1.* Applying the Fourier transform we can read of the solution formula

$$\lambda u = \mathcal{F}_G^{-1}\frac{\lambda}{\lambda + |\eta|^2}\hat f,$$

valid in $\mathcal{S}'(G)$. If we denote the multiplier appearing in this expression by

$$m : \hat{G} \to \mathbb{C}, \qquad m(\eta) = \frac{\lambda}{\lambda + |\eta|^2} \tag{20}$$

and define

$$M : \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{C}, \qquad M(\zeta) = \frac{\lambda}{\lambda + |\zeta|^2}, \tag{21}$$

we see that $m = M|_{\hat{G}}$ and that $M$ is a smooth function. It is well-known that $M$ is a multiplier satisfying the Mikhlin condition in Proposition 1.3. Hence, by Proposition 5.3 we obtain that $\lambda u \in L^q_\omega(G)$ and that $\|\lambda u\|_{L^q_\omega(G)} \leq c\|f\|_{L^q_\omega(G)}$. But then also $-\Delta u = f - \lambda u \in L^q_\omega(G)$ and we have an estimate

$$\|\Delta u\|_{L^q_\omega(G)} \leq \|f\|_{L^q_\omega(G)} + \|\lambda u\|_{L^q_\omega(G)} \leq c\|f\|_{L^q_\omega(G)}.$$

In order to obtain the full *a priori* estimate, we consider for $1 \leq i, j \leq n$ the symbol

$$m_{ij} : \hat{G} \to \mathbb{C}, \qquad m_{ij}(\eta) = \frac{\eta_i \eta_j}{|\eta|^2},$$

and define

$$M_{ij} : \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{C}, \qquad M_{ij}(\zeta) = \frac{\zeta_i \zeta_j}{|\zeta|^2}.$$

Again we note that $m_{ij} = M_{ij}|_{\hat{G}}$ and that $M_{ij}$ is a Mikhlin multiplier, but this time it is not smooth at the origin. Therefore, we recall the multiplier $m_{\mathcal{P}}$ of the projection $\mathcal{P}$ from (19). We have seen that $m_{\mathcal{P}} = \chi_{\{k=0\}}$. Thus, writing $\eta = (\xi, k)$ as usual, we split

$$-\Delta u = \mathcal{P}\Delta u + \mathcal{P}_\perp \Delta u = \Delta' \mathcal{P} u + \Delta \mathcal{P}_\perp u,$$

and similarly $\partial_{ij} u = \partial_{ij}\mathcal{P} u + \partial_{ij}\mathcal{P}_\perp u$. Certainly,

$$\|\partial_{ij}\mathcal{P} u\|_{L^q_{\bar{\omega}}(\mathbb{R}^{n-1})} \leq c\|\Delta' \mathcal{P} u\|_{L^q_{\bar{\omega}}(\mathbb{R}^{n-1})}.$$

In fact, if $i = n$ or $j = n$, then $\partial_{ij}\mathcal{P} u = 0$ and there is nothing to prove. Otherwise, this follows by the weighted Mikhlin theorem and the constant is $A_q(G)$-consistent since $\mathcal{A}_q(\bar{\omega}) \leq \mathcal{A}_q(\omega)$ as shown in Proposition 5.2. For the complement projection we have

$$\partial_{ij}\mathcal{P}_\perp u = \partial_{ij}\mathcal{P}_\perp^2 u = \mathcal{P}_\perp \partial_{ij}\mathcal{P}_\perp u = \mathcal{F}_G^{-1}\big((1 - m_{\mathcal{P}})\frac{\eta_i\eta_j}{|\eta|^2}\mathcal{F}_G \Delta \mathcal{P}_\perp u\big),$$

again valid as an identity in $\mathcal{S}'(G)$. Now we introduce

$$m_{ij,\perp} : \hat{G} \to \mathbb{C}, \qquad m_{ij,\perp}(\eta) = (1 - m_{\mathcal{P}}(\eta))\frac{\eta_i\eta_j}{|\eta|^2},$$

and define

$$M_{ij,\perp} : \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{C}, \qquad M_{ij,\perp}(\zeta) = M_{ij,\perp}(\xi, \kappa) = (1 - \varphi(\kappa))\frac{\zeta_i\zeta_j}{|\zeta|^2},$$

where $\varphi \in C_0^\infty(\mathbb{R})$ with $\varphi(0) = 1$ and $\operatorname{supp}\varphi \subset (-\frac{1}{2}, \frac{1}{2})$. Notice that we could not use the transference principle of Proposition 5.3, because $M_{ij}$ was not continuous at the origin. Here lies the key in introducing the symbol $M_{ij,\perp}$: in a neighbourhood of $\kappa = 0$, this smoothened symbol $M_{ij,\perp}$ vanishes. Since the behaviour at infinity does not change, $M_{ij,\perp}$ is still a Mikhlin multiplier *that is also continuous*. Thus, we can use 5.3 now to obtain

$$\|\partial_{ij}\mathcal{P}_\perp u\|_{L^q_\omega(G)} \leq c\|\Delta \mathcal{P}_\perp u\|_{L^q_\omega(G)},$$

where $c = c(\omega, n, q, L) > 0$ is $A_q(G)$-consistent. Thus, in total we obtain

$$\begin{aligned}\|\partial_{ij} u\|_{L^q_\omega(G)} &\le \|\partial_{ij}\mathcal{P}u\|_{L^q_{\bar\omega}(\mathbb{R}^{n-1})} + \|\partial_{ij}\mathcal{P}_\perp u\|_{L^q_\omega(G)} \\ &\le c(\|\Delta'\mathcal{P}u\|_{L^q_{\bar\omega}(\mathbb{R}^{n-1})} + \|\Delta\mathcal{P}_\perp u\|_{L^q_\omega(G)}) \le c\|\Delta u\|_{L^q_\omega(G)},\end{aligned}$$

and the constant is still $A_q(G)$-consistent, since the decomposition

$$L^q_\omega(G) = L^q_{\bar\omega}(\mathbb{R}^{n-1}) \oplus \mathcal{P}_\perp L^q_\omega(G)$$

is $A_q(G)$-consistent. □

We can now consider the Stokes resolvent problem again and use the decomposition to split the original problem into the two problems

$$\begin{cases} \lambda u_{\mathcal{P}} - \Delta' u_{\mathcal{P}} + \nabla'\mathfrak{p}_{\mathcal{P}} = \mathcal{P}f & \text{in } \mathbb{R}^{n-1}, \\ \operatorname{div}' u_{\mathcal{P}} = \mathcal{P}g & \text{in } \mathbb{R}^{n-1}, \end{cases} \tag{22}$$

and

$$\begin{cases} \lambda u_\perp - \Delta u_\perp + \nabla\mathfrak{p}_\perp = \mathcal{P}_\perp f & \text{in } G, \\ \operatorname{div} u_\perp = \mathcal{P}_\perp g & \text{in } G. \end{cases} \tag{23}$$

The solution of the original problem will then be given as $u = u_{\mathcal{P}} + u_\perp$, and due to Lemma 5.6 this decomposition is unique. Observe that if $u \in W^{2,q}_\omega(G)$ is a solution to (2), then $\operatorname{div} u_\perp = \operatorname{div}\mathcal{P}_\perp u = \mathcal{P}_\perp \operatorname{div} u \in \mathcal{P}_\perp W^{1,q}_\omega(G)$. Together with Lemmas 5.7 and 5.8 below, this justifies the notion

$$W^{1,q}_{\omega,\mathrm{div}}(G) = \Big(W^{1,q}_{\bar\omega}(\mathbb{R}^{n-1}) \cap \hat{W}^{-1,q}_{\bar\omega}(\mathbb{R}^{n-1})\Big) \oplus \mathcal{P}_\perp W^{1,q}_\omega(G). \tag{24}$$

Concerning system (22), we have the following statement.

**Proposition 5.7.** *Let $n \ge 3$, $1 < q < \infty$, $L > 0$, $\omega \in A_q(G)$ and $0 < \theta < \frac{\pi}{2}$. Then to each $f \in L^q_\omega(G)$, $g \in W^{1,q}_{\omega,\mathrm{div}}(G)$ and $\lambda \in \Sigma_{\theta+\frac{\pi}{2}}$ there is a unique solution $(u_{\mathcal{P}}, \mathfrak{p}_{\mathcal{P}}) \in W^{2,q}_{\bar\omega}(\mathbb{R}^{n-1}) \times \hat{W}^{1,q}_{\bar\omega}(\mathbb{R}^{n-1})$ to (22) satisfying the* a priori *estimate*

$$\begin{aligned}&\|\lambda u_{\mathcal{P}}, \nabla^2 u_{\mathcal{P}}, \nabla\mathfrak{p}_{\mathcal{P}}\|_{L^q_{\bar\omega}(\mathbb{R}^{n-1})} \\ &\qquad \le c\left(\|\mathcal{P}f\|_{L^q_{\bar\omega}(\mathbb{R}^{n-1})} + \|\nabla\mathcal{P}g\|_{L^q_{\bar\omega}(\mathbb{R}^{n-1})} + \|\lambda\mathcal{P}g\|_{\hat{W}^{-1,q}_{\bar\omega}(\mathbb{R}^{n-1})}\right),\end{aligned} \tag{25}$$

*where $c = c(\omega, n, q, \theta) > 0$ is an $A_q(G)$-consistent constant.*
*The same conclusion holds true for $n = 2$ if $\mathcal{P}g = 0$.*

*Proof.* It suffices to observe that $\mathcal{P}f \in L^q_{\bar\omega}(\mathbb{R}^{n-1})$ and $\mathcal{P}g \in W^{1,q}_{\bar\omega}(\mathbb{R}^{n-1}) \cap \hat{W}^{-1,q}_{\bar\omega}(\mathbb{R}^{n-1})$ and that Proposition 5.2 yields $\bar\omega \in A_q(\mathbb{R}^{n-1})$. Then we may invoke Theorem 4.5 of [7] to obtain the assertion. Note that Theorem 4.5 of [7] is stated only for dimension at least 2. Hence, we need $n \ge 3$ in order to employ this result, since the projected spaces are of dimension $n-1$. □

We are now in the position to prove a dual assertion dealing with problem (23) using the properties of the projection $\mathcal{P}$ and Proposition 5.3.

**Proposition 5.8.** *Let $n \ge 2$, $1 < q < \infty$, $\omega \in A_q(G)$, and $0 < \theta < \frac{\pi}{2}$. Then to each $f \in L^q_\omega(G)$, $g \in W^{1,q}_{\omega,\mathrm{div}}(G)$ and $\lambda \in \Sigma_{\theta+\frac{\pi}{2}}$ there is a unique solution $(u_\perp, \mathfrak{p}_\perp) \in \mathcal{P}_\perp W^{2,q}_\omega(G) \times \mathcal{P}_\perp \hat{W}^{1,q}_\omega(G)$ to (23) satisfying the* a priori *estimate*

$$\|\lambda u_\perp, \nabla^2 u_\perp, \nabla\mathfrak{p}_\perp\|_{L^q_\omega(G)} \le c(\|\mathcal{P}_\perp f\|_{L^q_\omega(G)} + (1 + |\lambda|)\|\nabla\mathcal{P}_\perp g\|_{L^q_\omega(G)}), \tag{26}$$

*where $c = c(\omega, n, q, \theta, L) > 0$ is an $A_q(G)$-consistent constant.*

*Proof.* Concerning the uniqueness, let $u_\perp, \mathfrak{p}_\perp \in \mathcal{S}'(G)$ with $\mathcal{P}u_\perp = 0$ and $\mathcal{P}\mathfrak{p}_\perp = 0$ satisfy (23) with homogeneous data. Then in the Fourier space we have by chapter 4 it holds $(\lambda + |\eta|^2)\hat{u}_\perp = 0$ and $-|\eta|^2\hat{\mathfrak{p}}_\perp = 0$ for $\eta \neq 0$. On the other hand, if $\eta = 0$, we observe $\hat{u}_\perp = (1 - m_\mathcal{P})\hat{u}_\perp$ due to $\mathcal{P}u_\perp = 0$, where $m_\mathcal{P} = \chi_{k=0}$ is the Fourier multiplier of the projection $\mathcal{P}$. Hence $\hat{u}_\perp(0) = (1-1)\hat{u}_\perp = 0$. A similar argument shows $\hat{\mathfrak{p}}_\perp(0) = 0$. Hence, we see that any pair $(u_\perp, \mathfrak{p}_\perp) \in \mathcal{P}_\perp W^{2,q}_\omega(G) \times \mathcal{P}_\perp \hat{W}^{1,q}_\omega(G)$ solving (23) with homogeneous data has to satisfy $(u_\perp, \mathfrak{p}_\perp) = (0,0)$.
As we have seen above, the transference principle of Fourier multipliers seems to be a promising tool in order to furnish us with an *a priori* estimate in terms of the force term $f$ and the divergence $g$. To simplify things for the moment, assume first that the solution we are looking for is assumed to be solenoidal. That is, let us investigate the case $f \in L^q_\omega(G)$ and $g = 0$. Consider the (potential) multipliers $m_f$ and $M_f$ as defined in (15) and (16), respectively. Now observe that for $P_\perp f := f_\perp \in \mathcal{P}_\perp L^q_\omega(G)$ we have $f_\perp = \mathcal{P}_\perp f_\perp$. In view of (19) we can rewrite this as $f_\perp = \mathcal{F}_G^{-1}(1-m_\mathcal{P})\mathcal{F}_G f_\perp$. Therefore, we introduce $\tilde{M}_f(\xi,\kappa) := (1-\varphi(\kappa))M_f(\xi,\kappa)$ with $\varphi \in C_0^\infty(\mathbb{R})$ with $\varphi(0) = 1$ and $\operatorname{supp}\varphi \subset (-\frac{1}{2}, \frac{1}{2})$. Just as in the proof of the Laplace resolvent problem, we notice that we could not use the transference principle of Proposition 5.3, because $M_f$ was not continuous at the origin, but that $\tilde{M}_f$ is the correct substitute for $M_f$. We can repeat this procedure for $\zeta^\alpha M_f(\zeta)$ with any multi-index $|\alpha| \leq 2$, that is we look at $\zeta^\alpha \tilde{M}_f(\zeta)$ instead. Hence, we can now apply the transference principle of Proposition 5.3 to obtain, that for the solution

$$u_\perp = \mathcal{F}_G^{-1}\left(\frac{1}{\lambda + |\eta|^2}\left(I - \frac{\eta \otimes \eta}{|\eta|^2}\right)\mathcal{F}_G f_\perp\right) = \mathcal{F}_G^{-1} m_f \mathcal{F}_G f_\perp = \mathcal{F}_G^{-1}(1-m_\mathcal{P})m_f\mathcal{F}_G f_\perp$$

obtained in (8) we have $u_\perp \in W^{2,q}_\omega(G)$ and the *a priori* estimate

$$\|\lambda u_\perp, \nabla^2 u_\perp\|_{L^q_\omega(G)} \leq c\|f_\perp\|_{L^q_\omega(G)}.$$

Therefore also $\nabla\mathfrak{p}_\perp := f_\perp - \lambda u_\perp + \Delta u_\perp \in L^q_\omega(G)$ enjoys this estimate and we finally arrive at

$$\|\lambda u_\perp, \nabla^2 u_\perp, \nabla\mathfrak{p}_\perp\|_{L^q_\omega(G)} \leq c\|f_\perp\|_{L^q_\omega(G)}.$$

Next we want to consider general $g \in W^{1,q}_{\omega,\mathrm{div}}(G)$. Without loss of generality we may also assume $f = 0$, as we have just proven the corresponding estimate for general external forces $f \in L^q_\omega(G)$ and our problem at hand is linear.
Suppose first that $\mathcal{P}_\perp g =: g_\perp \in \mathcal{P}_\perp C_0^\infty(G)$. Then $g_\perp = \mathcal{P}_\perp g_\perp$ and in view of (8) and (19) we put

$$u_\perp := \mathcal{F}_G^{-1}\left(m_1 \mathcal{F}_G(\nabla g_\perp)\right), \qquad m_1(\eta) := \frac{1 - m_\mathcal{P}(k)}{|\eta|^2} \tag{27}$$

$$\mathfrak{p}_\perp := \mathcal{F}_G^{-1}\left(m_2 \mathcal{F}_G(g_\perp)\right), \qquad m_2(\eta) := \frac{1 - m_\mathcal{P}(k)}{|\eta|^2}(\lambda + |\eta|^2). \tag{28}$$

One readily checks that $(u_\perp, \mathfrak{p}_\perp)$ solve (23) in the sense of $\mathcal{S}'(G)$ and that both $\mathcal{P}_\perp u_\perp = u_\perp$ and $\mathcal{P}_\perp \nabla\mathfrak{p}_\perp = \nabla\mathfrak{p}_\perp$. Using again the cut-off function $\varphi \in C_0^\infty(\mathbb{R})$ with $\varphi(0) = 1$ and $\operatorname{supp}\varphi \subset (-\frac{1}{2}, \frac{1}{2})$, we define the Fourier multipliers $M_1, M_2$ :

$\mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{C}$ via

$$M_1(\xi, \kappa) := \frac{1 - \varphi(\kappa)}{|\zeta|^2}, \tag{29}$$

$$M_2(\xi, \kappa) := \frac{1 - \varphi(\kappa)}{|\zeta|^2}(\lambda + |\zeta|^2), \tag{30}$$

Observe that the denominators of these functions vanish if and only if both $\xi$ and $\kappa$ vanish. But since the nominators vanish in a neighbourhood of $\kappa = 0$, $M_1$ and $M_2$ are smooth and thus bounded near the origin. It is standard to check that they fulfill Mikhlin's condition with a constant $C_1$ and $C_2(1 + |\lambda|)$, respectively. Employing Theorem 5.3, we see that both $m_1$ and $m_2$ are $L^q_\omega(G)$-multiplier functions and that we have the estimate

$$\|\lambda u_\perp, \nabla \mathfrak{p}_\perp\|_{L^q_\omega(G)} \leq C(1 + |\lambda|)\|\nabla g_\perp\|_{L^q_\omega(G)}. \tag{31}$$

But then also $\Delta u_\perp = \lambda u_\perp + \nabla \mathfrak{p}_\perp$ fulfills the estimate. Using again Proposition 5.3 and the fact that $\zeta \mapsto (1 - \varphi(\kappa))\frac{\zeta_i \zeta_j}{|\zeta|^2}$ satisfies Mikhlin's condition, we also get

$$\|\nabla^2 u_\perp\|_{L^q_\omega(G)} \leq c\|\Delta u_\perp\|_{L^q_\omega(G)} \leq C(1 + |\lambda|)\|\nabla g_\perp\|_{L^q_\omega(G)}. \tag{32}$$

Putting these results together, we constructed to each smooth $g_\perp \in \mathcal{P}_\perp C_0^\infty(G)$ a solution $(u_\perp, \mathfrak{p}_\perp) \in \mathcal{P}_\perp W^{2,q}_\omega(G) \times \mathcal{P}_\perp \hat{W}^{1,q}_\omega(G)$ satisfying the estimate (26) whenever $f = 0$. As $\mathcal{P}_\perp C_0^\infty(G)$ is dense in $\mathcal{P}_\perp W^{1,q}_\omega(G)$, the result for general $g \in W^{1,q}_{\omega,\mathrm{div}}(G)$ then follows by a standard approximation procedure.

□

Theorem 1.2 follows now easily.

*Proof of Theorem 1.2.* The $A_q(G)$-consistent decomposition of $L^q(G)$ proved in Lemma 5.6 together with Propositions 5.7 and 5.8 yield the assertion. □

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