EXTRAPOLATION THEOREM ON LOCALLY COMPACT ABELIAN GROUPS

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Abstract. We prove a generalization of an extrapolation theorem in the fashion of García-Cuerva and Rubio de Francia (1985) towards $\mathcal{R}$-boundedness on weighted $L^q_\omega(G)$-spaces with G being a locally compact abelian group and ω being a Muckenhoupt weight. As a main tool, we generalize the classical Muckenhoupt theorem, which states that the maximal operator is bounded in the weighted space $L^q_\omega(\mathbb{R}^n)$ whenever $1 < q < \infty$ and the weight ω is in the Muckenhoupt class A_q, to locally compact abelian groups. This result is achieved without making use of a Reversed Hölder Inequality.

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1. Introduction

In the setup of $\mathbb{R}^n$ the concept of Muckenhoupt weights has been studied extensively throughout the last four decades or so, with many remarkable results in fields of harmonic analysis, weighted inequalities and partial differential equations (cf. [2, 8, 9, 10, 11, 15, 20]). For $1 < q < \infty$, a nonnegative weight function $\omega \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is said to be in the Muckenhoupt class $A_q(\mathbb{R}^n)$ if

$$\mathcal{A}_q(\omega) := \sup_{r>0} \sup_{y\in\mathbb{R}^n} \left(\frac{1}{|B_r(y)|} \int_{B_r(y)} \omega \,\mathrm{d}x \right) \left(\frac{1}{|B_r(y)|} \int_{B_r(y)} \omega^{-\frac{q'}{q}} \,\mathrm{d}x \right)^{\frac{q}{q'}} < \infty,$$

where $B_r(x)$ denotes the open ball of radius r around the center x, and where q' is the Hölder conjugate of q. The weight ω is said to be in $A_1(\mathbb{R}^n)$ if there is a constant $c > 0$ such that $\mathcal{M}_{\mathbb{R}^n}\omega(x) \leq c\omega(x)$ for almost all $x \in \mathbb{R}^n$. Here, $\mathcal{M}_{\mathbb{R}^n}$ denotes the usual (centered) maximal operator on $\mathbb{R}^n$. These classes of weights have been introduced by Benjamin Muckenhoupt, who considered such weights for bounded intervals and products of intervals [16]. Muckenhoupt weights are known to possess several interesting properties. In particular, the maximal operator is bounded on weighted L^q-spaces for $1 < q < \infty$, [20, Theorem 5.3.1]. This result was used by García-Cuerva and Rubio de Francia to show their Extrapolation Theorem [10, Section IV.5], which states that if a family of operators is uniformly bounded in $L^q_\omega(\mathbb{R}^n)$ for one $1 \leq q < \infty$ but all $\omega \in A_q(\mathbb{R}^n)$, then it is already bounded in $L^p_\nu(\mathbb{R}^n)$ for *all* $1 \leq p < \infty$ and all $\nu \in A_q(\mathbb{R}^n)$. Strengthening this result towards $\mathcal{R}$-boundedness of family of operators as defined in Section 5, Fröhlich [9] proved maximal L^p-regularity of the Stokes operator on weighted spaces $L^q_\omega(\Omega)$, where Ω is

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the whole space, the half space or a bounded domain of class $C^{1,1}$. For details about maximal L^p-regularity see e.g. [5], [14]. In this paper, we wish to generalize the theory of Muckenhoupt weights and extrapolation towards locally compact abelian groups G. In two forthcoming papers [18], [19] we apply the abstract methods obtained here to obtain maximal regularity of the spatially periodic Stokes operator and to treat a spatially periodic nonlinear model describing the dynamics of nematic liquid crystal flows.

Note that on locally compact abelian groups one can define a nontrivial, translation-invariant, regular measure μ, called *Haar measure* [1, 3, 12, 22], with $\mu(K) < \infty$ for all compact $K \subset G$. Furthermore, such a measure is unique up to multiplication with a constant. However, we often deal with the measure $\mathrm{d}\mu_\omega := \omega \,\mathrm{d}\mu$, which is not translation-invariant anymore. Therefore, if not stated otherwise, we shall drop the translation-invariance condition on μ. For $1 \leq q \leq \infty$ one can thus introduce the space $L^q(G)$ of q-integrable functions $f : G \to \mathbb{R}$, which turns into a Banach space if equipped with the usual norm

$$\|f\|_q := \left(\int_G |f|^q \,\mathrm{d}\mu\right)^{\frac{1}{q}}, \qquad 1 \leq q < \infty,$$
$$\|f\|_\infty := \mu\text{-}\operatorname*{ess\,sup}_G |f|.$$

Further we introduce the notion $L^{q,\infty}(G)$ for the weak $L^q(G)$-space, as introduced e.g. in [21]. Note that the space of continuous functions with compact support $C_0(G)$ is dense in $L^q(G)$ for all $1 \leq q < \infty$, see [17, Appendix E.8] for details.

As we wish to carry over as much concepts known from classical harmonic analysis as possible to the general setting, we will have to assume that the group G is furnished with something that resembles the concept of balls and that the measure μ enjoys a doubling property with respect to these balls. We therefore make the following assumption, using the notation $U - U' := \{x \in G : x = y - z \text{ with } y \in U, z \in U'\}$ for $U, U' \subset G$.

Assumption 1.1. Suppose that G is a locally compact abelian group equipped with a nontrivial and regular measure μ, such that $\mu(K) < \infty$ for all compact $K \subset G$. Furthermore, assume that there is a local base of $0 \in G$ consisting of relatively compact measurable neighbourhoods U_k, $k \in \mathbb{Z}$, such that

(i) $\bigcup_{k \in \mathbb{Z}} U_k = G$,
(ii) $U_k \subset U_m$, if $k \leq m$,
(iii) there exist a positive constant A and a mapping $\theta : \mathbb{Z} \to \mathbb{Z}$ such that for all $k \in \mathbb{Z}$ and all $x \in G$

$$k < \theta(k),$$
$$U_k - U_k \subset U_{\theta(k)},$$
$$\mu(x + U_{\theta(k)}) \leq A\mu(x + U_k).$$

Observe that necessarily $A \geq 1$ because $U_k \subset U_{\theta(k)}$.

Remark 1.2. From now on, we will always assume that the locally compact abelian group G admits a family of sets $(U_k)_{k\in\mathbb{Z}}$ satisfying Assumption 1.1. We will call any set of the form $x + U_k$, $x \in G$, $k \in \mathbb{Z}$ a *base set*. It is instructive to think of such base sets as an equivalent of balls in the $\mathbb{R}^n$ with center in x and radius 2^k. Observe that by the following considerations we can assume without loss of generality the base sets to be symmetric and of the function θ to be increasing.

(i) Replacing θ by $\tilde{\theta}$ defined via

$$\tilde{\theta}(k) := \min\{l \in \mathbb{Z} : l > k \text{ with } U_k - U_k \subset U_l\},$$

we may assume that θ is non-decreasing. Indeed, the thusly defined function satisfies $\tilde{\theta}(k) \leq \theta(k)$ for all $k \in \mathbb{Z}$. Therefore, for all $x \in G$ and $k \in \mathbb{Z}$,

$$\mu(U_{\tilde{\theta}(k)}) \leq \mu(U_{\theta(k)}) \leq A\mu(U_k).$$

(ii) We call a set $U \subset G$ *symmetric* if $U = -U$. Since G is abelian, the set $V := U - U$ is symmetric for any $U \subset G$. Replacing the base sets U_k by the symmetric sets $V_k := U_k - U_k$ and replacing the doubling constant A by A^2, we may assume that all of our base sets are symmetric. Indeed, the V_k still form a local base of $0 \in G$ consisting of relatively compact neighbourhoods, as seen in Proposition 2.1 (iii) below. The inclusion $V_k \subset V_m$ for $k \leq m$ is obvious and the union of the $V_k (\supset U_k)$ covers the whole group.
Concerning condition (iii) of Assumption 1.1, we see

$$V_k - V_k \subset U_{\theta(k)} - U_{\theta(k)} = V_{\theta(k)}.$$

Moreover, the doubling property will be fulfilled with constant A^2, since for all $x \in G$ and all $k \in \mathbb{Z}$

$$\mu(x + V_{\theta(k)}) \leq \mu(x + U_{\theta^2(k)}) \leq A^2\mu(x + U_k) \leq A^2\mu(x + V_k).$$

Thus, from now on we will assume the base sets U_k to be symmetric and we will write $U_k - U_k = U_k + U_k =: 2U_k$.

Remark 1.3. Among the most prominent groups satisfying Assumption 1.1 are the groups $\mathbb{R}$, $\mathbb{Z}$, the torus T and finite products of these groups.

(i) In the case of the real numbers $\mathbb{R}$ equipped with the Lebesgue measure, define $U_k := (-2^{k-1}, 2^{k-1})$, $A = 2$ and $\theta(k) = k + 1$.
(ii) For integers, an analogous construction to (i) corresponding to the counting measure satisfies Assumption 1.1. Namely, choose $U_k := (-2^{k-1}, 2^{k-1}) \cap \mathbb{Z}$, $A = 3$ and $\theta(k) = k + 1$.
(iii) If one chooses the arc length as a measure on the torus, possible choices are $U_k := \{z \in \mathbb{C} : |\arg z| < 2^k\}$, $A = 2$ and $\theta = k + 1$.

Let us define the (centered) *maximal operator* on G. Suppose that $f \in L^1_{\mathrm{loc}}(G)$ and define the sublinear operator

$$\mathcal{M}_G f(x) := \sup_{k \in \mathbb{Z}} \frac{1}{\mu(x + U_k)} \int_{x+U_k} |f| \, \mathrm{d}\mu. \tag{1}$$

Note that $\mathcal{M}_G f$ is lower semi-continuous by Lemma 2.3 below and therefore measurable.
Our two main theorems can be viewed as direct generalizations of their equivalents in the classical $\mathbb{R}^n$-setup. For the definition of $A_q(G)$-consistency see Section 3.

Theorem 1.4. *Let G be a locally compact abelian group satisfying Assumption 1.1 and assume $1 < q < \infty$ and $\omega \in A_q(G)$. Then $\mathcal{M}_G$ is bounded in $L^q_\omega(G)$ with an $A_q(G)$-consistent bound.*

Theorem 1.5. *Let G be a locally compact abelian group satisfying Assumption 1.1. Suppose that $1 < r, q < \infty$, $\upsilon \in A_r(G)$ and that $\mathcal{T}$ is a family of linear operators such that for all $\omega \in A_q(G)$ there is an $A_q(G)$-consistent constant $c_q = c_q(\omega) > 0$ with*

$$\|Tf\|_{L^q_\omega(G)} \leq c_q \|f\|_{L^q_\omega(G)}$$

for all $f \in L^q_\omega(G)$ and all $T \in \mathcal{T}$. Then every $T \in \mathcal{T}$ can be extended to $L^r_\upsilon(G)$ and $\mathcal{T}$ is $\mathcal{R}$-bounded in $\mathcal{L}(L^r_\upsilon(G))$ with an $A_r(G)$-consistent $\mathcal{R}$-bound c_r.

This paper is organized as follows. In Section 2 we provide further properties of the group G subject to Assumption 1.1 and the maximal operator $\mathcal{M}_G$. In the case of a translation-invariant measure μ, most of the results in this section are known and can be found in [7, Chapter 2]. Section 3 is devoted to establishing Theorem 1.4. In Section 4 we prove an equivalent to the classical extrapolation theorem due to García-Cuerva and Rubio de Francia, both in the scalar-valued and in the vector-valued case. Theorem 1.5 will be proven in Section 5.

2. Harmonic Analysis on Locally Compact Abelian Groups

We first state two propositions, providing some basic properties that follow directly from Assumption 1.1.

Proposition 2.1. *Suppose Assumption 1.1 is satisfied.*

(i) *For every $x \in G$ and $k \in \mathbb{Z}$ it holds $\mu(x + U_k) > 0$.*
(ii) *G enjoys the engulfing property,* i.e., *if $x, y \in G$ are such that $(x + U_m) \cap (y + U_k) \neq \emptyset$ for some $m, k \in \mathbb{Z}$ with $k \leq m$, then $y + U_k \subset x + U_{\theta^2(m)}$.*
(iii) *The family of sets $\{U_k - U_k\}_{k\in\mathbb{Z}}$ forms a local base of relatively compact neighbourhoods of $0 \in G$.*
(iv) *The interiors of the base sets U_k cover G,* i.e., *$\bigcup_{k\in\mathbb{Z}} \mathring{U}_k = G$. In particular, for every compact $K \subset G$ there is $k \in \mathbb{Z}$ such that $K \subset U_k$.*

Proof. (i) Let $x \in G$ and $k \in \mathbb{Z}$. Since the sets $\{U_k\}_{k\in\mathbb{Z}}$ cover G by Assumption 1.1 (i), we can write every $z \in U_k$ in the form $z = x+(-x+z) \in x+U_{k'}$ for some $k' \in \mathbb{Z}$ such that $-x + z \in U_{k'}$. Therefore also

$$\bigcup_{k\in\mathbb{Z}} x + U_k = G,$$

and because the measure μ is regular, we have $\mu(x+U_k) \to \mu(G)$ as $k \to G$. Since μ is nontrivial, we have $\mu(G) > 0$ and hence there exists $K \in \mathbb{Z}$ with $\mu(x+U_K) > 0$. Then for $k \in \mathbb{Z}$, Assumption 1.1 (iii) gives $k < \theta(k)$, which shows that for all $k \in \mathbb{Z}$ there exists $N \in \mathbb{N}$ with $\theta^N(k) \geq K$. Hence

$$0 < \mu(x + U_K) \leq \mu(x + U_{\theta^N(k)}) \leq A^n \mu(x + U_k),$$

proving the assertion.

(ii) By hypothesis, there are $x' \in U_m$ and $y' \in U_k$ such that $x+x' = y+y'$ and hence $y \in x + (U_m - U_k) \subset x + (U_m - U_m) \subset x + U_{\theta(m)}$. By Assumption 1.1 (iii) we see $-U_k \subset -U_m \subset U_{\theta(m)}$ and consequently $y + U_k \subset x + U_{\theta(m)} - U_{\theta(m)} \subset x + U_{\theta^2(m)}$.

(iii) See [17, Appendix B.4].

(iv) It suffices to show that for every $k \in \mathbb{Z}$ we have $U_k \subset \mathring{U}_{\theta(k)}$ and then use property (i) of Assumption 1.1. So fix $k \in \mathbb{Z}$ and choose an open neighbourhood O of $0 \in G$ such that $O \subset U_k$. Then we have

$$U_k \subset O' := \bigcup_{x \in U_k} (x - O) \subset U_k - U_k.$$

Observe that O' is open, since it is the union of the open sets $x - O$. It follows $U_k \subset O' \subset U_{\theta(k)}$ and by definition of the interior even $U_k \subset O' \subset \mathring{U}_{\theta(k)}$, which is what we wanted to show.

For the assertion about the compact set K we simply note that $\{\mathring{U}_k\}_{k \in \mathbb{Z}}$ is an open cover of K and we thus find a finite subcover by compactness. But since the base sets U_k are nested, so are their interiors, and so the finite subcover consists only of the largest element. Hence there is $k \in \mathbb{Z}$ with $K \subset \mathring{U}_k \subset U_k$.

□

Proposition 2.2. *Given Assumption 1.1, the following statements are true for the locally compact abelian group G.*

(i) *G is first countable,* i.e., *each point $x \in G$ has a countable local base.*
(ii) *G is σ-compact,* i.e., *it is a countable union of compact subspaces.*
(iii) *G is a Lindelöf-space,* i.e., *every open cover of G has a countable subcover.*
(iv) *G is a seperable space,* i.e., *it contains a countable dense subset D.*

Proof. (i) The base sets $\{U_k\}_{k \in \mathbb{Z}}$ form a local base of relatively compact neighbourhoods of $0 \in G$. Hence, since addition is a continuous operation in G, the sets $\{x + U_k\}_{k \in \mathbb{Z}}$ form a local base of relatively compact neighbourhoods of $x \in G$.

(ii) Clearly, the closures of the (countably many) relatively compact sets $\{U_k\}_{k \in \mathbb{Z}}$ are compact and cover G.

(iii) On the compact subspaces $\overline{U}_k$ we can extract a finite subcover and the σ-compactness of G then yields a countable subcover.

(iv) Consider for fixed $k \in \mathbb{Z}$ the open cover $\{x - \mathring{U}_k\}_{x \in G}$. Since G is Lindelöf, we can extract a countable subcover $\{x_n^{(k)} - \mathring{U}_k\}_{n \in \mathbb{N}}$. Define D_k to be the countable set of the centers $x_n^{(k)}$ at height k. Doing so for every k, the countable union $D := \bigcup_{k \in \mathbb{Z}} D_k$ is dense in G, because for every $x \in G$ and $k \in \mathbb{Z}$, there is an $x_n^{(k)} \in D_k$ such that $x \in x_n^{(k)} - U_k$, and hence we obtain $x_n^{(k)} \in x + U_k$. Since the sets $\{x + U_k\}_{k \in \mathbb{Z}}$ form a local base of x, we see that for every open neighbourhood O of x, there is $k \in \mathbb{Z}$ and $x_n^{(k)} \in D_k$ with $x_n^{(k)} \in x + U_k \subset O$.

□

Lemma 2.3. *Let $f \in L^1_{loc}(G)$. Then $\mathcal{M}_G f$ is lower semi-continuous.*

Proof. For each $k \in \mathbb{Z}$ the map $I_k : G \to \mathbb{R}$ defined via $I_k(x) = \frac{1}{\mu(x+U_k)} \int_{x+U_k} |f| \, \mathrm{d}\mu$ is continuous by Lebesgue's Theorem on Dominated Convergence. Therefore, for every $r \in \mathbb{R}$ the set

$$\{x \in G : \mathcal{M}_G f(x) > r\} = \bigcup_{k \in \mathbb{Z}} \{x \in G : I_k(x) > r\} \tag{2}$$

is a union of open sets and hence open itself. This is exactly the lower semi-continuity of $\mathcal{M}_G f$. $\square$

One can define the uncentered maximal operator M_G in an analogous way, if one takes the supremum in (1) not only over all $k \in \mathbb{Z}$, but also over all $y \in G$ such that $x \in y + U_k$. By a similar reasoning as for the centered maximal operator, $M_G f$ is measurable. In fact, the uncentered maximal operator is comparable to the centered maximal operator.

Lemma 2.4. *Let $f \in L^1_{loc}(G)$. Then*

$$\mathcal{M}_G f \leq M_G f \leq A^2 \mathcal{M}_G f. \tag{3}$$

Proof. The first inequality is obvious. For the second inequality, let $x, y \in G$ and $k \in \mathbb{Z}$ be such that $x \in y + U_k$. Hence, we obtain $x + U_k \subset y + 2U_k \subset y + U_{\theta(k)}$, and the doubling property yields

$$\mu(x + U_{\theta(k)}) \leq A\mu(x + U_k) \leq A\mu(y + U_{\theta(k)}) \leq A^2 \mu(y + U_k). \tag{4}$$

On the other hand $y + U_k \subset x + 2U_k \subset x + U_{\theta(k)}$, and thus

$$\frac{1}{\mu(y + U_k)} \int_{y+U_k} |f| \, \mathrm{d}\mu \leq \frac{1}{\mu(y + U_k)} \int_{x+U_{\theta(k)}} |f| \, \mathrm{d}\mu \leq \frac{A^2}{\mu(x + U_{\theta(k)})} \int_{x+U_{\theta(k)}} |f| \, \mathrm{d}\mu.$$

Taking now the supremum first on the right-hand side and then on the left-hand side yields the assertion. $\square$

As the measure μ possesses the doubling property, we expect the weak estimate

$$\mu(\{x \in G : \mathcal{M}_G f(x) > t\}) \leq \frac{A}{t} \|f\|_1, \qquad t > 0, \tag{5}$$

and even the stronger form

$$\mu(\{x \in G : \mathcal{M}_G f(x) > t\}) \leq \frac{2A}{t} \int_{\{|f| > t/2\}} |f| \, \mathrm{d}\mu, \qquad t > 0. \tag{6}$$

In order to show this, we need the following covering lemma due to Edwards and Gaudry [7] to apply the known technique from the $\mathbb{R}^n$-setting.

Lemma 2.5. *Let E be a subset of G and $k : E \to \mathbb{Z}$ a mapping bounded from above such that for every $k_0 \in \mathbb{Z}$ the set $\{x \in E : k(x) \geq k_0\}$ is relatively compact in G. Then there is a sequence $(x_n) \subset E$, finite or infinite, such that*

(i) *the sequence $(k_n) := (k(x_n))$ is decreasing,*
(ii) *the sets $x_n + U_{k_n}$ are pairwise disjoint and*
(iii) *$E \subset \bigcup (x_n + 2U_{k_n})$.*

Proof. The lemma has been proven in [7, Lemma 2.2.1] in the case of an translation-invariant measure μ. We include here the whole proof in the more general case.
If there is a finite sequence $x_1, \ldots, x_m$ of points of E such that the base sets $x_j + U_{k(x_j)}$ are pairwise disjoint and

$$E \subset \bigcup_{j=1}^{m} \left(x_j + 2U_{k(x_j)} \right),$$

we may always rename the x_j in such a way that the $k(x_j)$ increase with j and there is nothing further to prove. Hence, assume that there is no such finite sequence.
Begin by defining $k_1 := \max\{k(x) : x \in E\}$, which is a finite number since the

mapping k is bounded from above, and choose $x_1 \in E$ with $k_1 = k(x_1)$.
Suppose $m \in \mathbb{N}$ and the points $x_1, \ldots, x_m \in E$ have been chosen in such a way that the sets $x_j + U_{k_j}$, $1 \leq j \leq m$, are pairwise disjoint and such that $k_j = \max\{k(x) : x \in A_{j-1}\}$, where

$$A_j := E \setminus \left(\bigcup_{1 \leq l \leq j} x_n + 2U_{k_l} \right)$$

for each $1 \leq j \leq m$. Note that this is satisfied, if $m = 1$.
Continue the process by defining x_{m+1} as follows: by hypothesis, A_m is nonempty. Therefore $k_{m+1} := \max\{k(x) : x \in A_m\}$ is well-defined. Choose $x_{m+1} \in A_m$ such that $k_{m+1} = k(x_{m+1})$. Let us verify that

$$\big(x_j + U_{k_j}\big) \cap \big(x_{m+1} + U_{k_{m+1}}\big) = \emptyset, \qquad 1 \leq j \leq m. \tag{7}$$

Assume that (7) does not hold, *i.e.*, $x_{m+1} \in x_j + \big(U_{k_j} + U_{k_{m+1}}\big)$ for some $1 \leq j \leq m$. From the definition of the A_j it is clear that $A_m \subset A_{j-1}$, and so $k_{m+1} \leq k_j$. This yields

$$x_{m+1} \in x_j + \big(U_{k_j} + U_{k_{m+1}}\big) \subset x_j + 2U_{k_j},$$

contradicting $x_{m+1} \in A_m$.
Thus, proceeding in such a way for all $n \in \mathbb{N}$, we find a sequence of points $(x_n)_{n \in \mathbb{N}} \subset E$ such that (i) is satisfied. Also, we have $x_n \in A_{n-1}$ for $n \in \mathbb{N}$, and since $A_n \subset A_{n-1}$, the sequence $(k_n)_{n \in \mathbb{N}}$ is decreasing by the definition of the $k_n = \max\{k(x) : x \in A_{n-1}\}$.
It remains to proof (ii), *i.e*, that the intersection over all A_n is empty. Were this not the case, there would exist a point $x \in E$ belonging to every A_n, yielding $k_n \geq k(x)$ for all $n \in \mathbb{N}$. Therefore, by assumption, the set $M := \{x_n : n \in \mathbb{N}\}$ is relatively compact in G. Since $U_{k_n} \subset U_{k_1}$ and U_{k_1} is relatively compact, it follows that

$$F := \bigcup_{n \in \mathbb{N}} (x_n + U_{k_n}) \subset M + U_{k_1}$$

is relatively compact and so $\mu(F) \leq \mu(\overline{F}) < \infty$. On the other hand, the compact set $\overline{M}$ is contained in a base set U_K, $K \in \mathbb{Z}$, by Proposition 2.1 (v). Hence $x_n \in U_K$ for all $n \in \mathbb{N}$. Furthermore, by the monotonicity of θ, we find $N \in \mathbb{N}$ with $\theta^N(k(x)) \geq K$, and the consideration $x_n \in U_K \subset U_{\theta^N(k(x))} \Rightarrow 0 \in x_n + U_{\theta^N(k(x))}$ shows

$$U_K = 0 + U_K \subset \big(x_n + U_{\theta^N(k(x))}\big) + U_K \subset x_n + 2U_{\theta^N(k(x))} \subset x_n + U_{\theta^{N+1}(k(x))}.$$

Since the $x_n + U_{k_n}$ are disjoint, this finally yields

$$\begin{aligned} \mu(F) &= \sum_{n \in \mathbb{N}} \mu(x_n + U_{k_n}) \geq \sum_{n \in \mathbb{N}} \mu(x_n + U_{k(x)}) \\ &\geq A^{-(N+1)} \sum_{n \in \mathbb{N}} \mu(x_n + U_{\theta^{N+1}(k(x))}) \geq A^{-(N+1)} \sum_{n \in \mathbb{N}} \mu(U_K) = \infty, \end{aligned}$$

since $\mu(U_K) > 0$ by Proposition 2.1 (i). This contradicts the finiteness of $\mu(F)$. Hence $\bigcap_{n \in \mathbb{N}} A_n$ is empty, finishing the proof. □

Theorem 2.6. *Let $1 < q \leq \infty$. Then the maximal operator $\mathcal{M}_G$ is bounded in $L^q(G)$. Furthermore, $\mathcal{M}_G$ is weakly bounded in $L^1(G)$,* i.e.*, estimate (5) (and even (6)) holds true.*

Proof. Since $\mathcal{M}_G f$ is lower semi-continuous for $f \in L^1_{\mathrm{loc}}(G)$ and since obviously $\mathcal{M}_G f(x) \le \|f\|_\infty$ almost everywhere, the maximal operator extends to a bounded operator in $L^\infty(G)$.

Let us now establish (5). Assume that $t > 0$ is such that $\mu(G) > \frac{A}{t}\|f\|_1$, since otherwise the assertion is trivial. As we want to apply Lemma 2.5, consider the set $E_t := \{x \in G : \mathcal{M}_G f(x) > t\}$. If E_t is empty, there is nothing to prove. Otherwise, choose a compact subset $E_t' \subset E_t$ and define a function $k : E_t' \to \mathbb{Z}$ via

$$k(x) := \max\left\{k \in \mathbb{Z} : \frac{1}{\mu(x+U_k)} \int_{x+U_k} |f| \,\mathrm{d}\mu > t\right\}.$$

This mapping is certainly well-defined. Indeed, if there was no maximal $k \in \mathbb{Z}$, then we would find a sequence $(k_n) \subset \mathbb{Z}$ with $k_n \to \infty$ as $n \to \infty$ such that for all $n \in \mathbb{N}$ it holds

$$\frac{A}{t}\|f\|_1 \ge \frac{1}{t}\|f\|_1 \ge \frac{1}{t}\int_{x+U_{k_n}} |f|\,\mathrm{d}\mu \ge \mu(x+U_{k_n}) \to \mu(G), \qquad \text{as } n \to \infty, \tag{8}$$

contradicting our assumption.

We have to show that the mapping k is bounded from above. Assume again otherwise. Then there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset E_t'$ such that $k_n := k(x_n) \to \infty$ as $n \to \infty$. Since E_t' is compact, there is a $K \in \mathbb{Z}$ with $\bigcup_{n\in\mathbb{N}}\{x_n\} \subset E_t' \subset U_K$ by Proposition 2.1 (v). Taking sufficiently large $n \in \mathbb{N}$, we obtain $k_n \ge K$. Therefore $0 \in x_n + U_{k_n}$ and consequently $U_{k_n} \subset x_n + 2U_{k_n} \subset x_n + U_{\theta(k_n)}$. Therefore, we see

$$\mu(U_{k_n}) \le \mu(x_n + U_{\theta(k_n)}) \le A\mu(x_n + U_{k_n}).$$

But then

$$\frac{A}{t}\|f\|_1 \ge \frac{A}{t}\int_{x_n+U_{k_n}} |f|\,\mathrm{d}\mu \ge A\mu(x_n+U_{k_n}) \ge \mu(U_{k_n}) \to \mu(G), \qquad \text{as } n \to \infty, \tag{9}$$

yielding again a contradiction.

Since for every $k_0 \in \mathbb{Z}$ the set $\{x \in E_t' : k(x) \ge k_0\}$ is a subset of the compact E_t' and therefore relatively compact in G, we can invoke Lemma 2.5 to obtain a finite or infinite sequence of points x_n, such that $E_t' \subset \bigcup (x_n + 2U_{k_n})$, but the sets $x_n + U_{k_n}$ are pairwise disjoint and $\mu(x_n + U_{k_n}) < \frac{1}{t}\int_{x_n+U_{k_n}} |f|\,\mathrm{d}\mu$. Assume the obtained sequence to be infinite, the finite case being even easier. This yields

$$\begin{aligned}\mu(E_t') &\le \sum_{n=1}^\infty \mu(x_n + 2U_{k_n}) \le A\sum_{n=1}^\infty \mu(x_n + U_{k_n})\\ &\le \frac{A}{t}\sum_{n=1}^\infty \int_{x_n+U_{k_n}} |f|\,\mathrm{d}\mu \le \frac{A}{t}\|f\|_1.\end{aligned}$$

Observe that this estimate is independent of the particular compact subset $E_t' \subset E_t$. Since E_t is open by (2) and the measure μ is inner regular, we may take the supremum over all compact subsets of E_t to obtain (5). Therefore $\mathcal{M}_G$ is continuous from $L^1(G)$ to $L^{1,\infty}(G)$. Inequality (6) can be verified by considering $g := f\chi_{\{|f|>t/2\}}$, where $\chi_{\{|f|>t/2\}}$ is the characteristic function on the set $\{x \in G : |f(x)| > t/2\}$. Since $\mathcal{M}_G$ is sublinear and obviously $\mathcal{M}_G(c) = c$ for all constant functions, we obtain $\mathcal{M}_G f \le \mathcal{M}_G g + t/2$. Therefore, $\{x \in G : \mathcal{M}_G f(x) > t\} \subset$

$\{x \in G : \mathcal{M}_G g(x) > t/2\}$, and so

$$\begin{aligned}\mu(\{x \in G :&\mathcal{M}_G f(x) > t\}) \\ &\leq \mu(\{x \in G : \mathcal{M}_G g(x) > t/2\}) \leq \frac{2A}{t}\|g\|_1 = \frac{2A}{t}\int_{\{|f|>t/2\}} |f| \,\mathrm{d}\mu,\end{aligned}$$

which is exactly (6).

Since $\mathcal{M}_G$ is weakly bounded in $L^1(G)$ and bounded in $L^\infty(G)$, it is also bounded in $L^p(G)$ for $1 < p < \infty$ by the Marcinkiewicz interpolation theorem [7, Appendix A]. ☐

3. Muckenhoupt Weights

Assume that G is a locally compact abelian group with a measure μ satisfying Assumption 1.1.

Definition 3.1. Let $1 < q < \infty$. A function $0 \leq \omega \in L^1_{loc}(G)$ is called an $A_q(G)$-*weight* if

$$\mathcal{A}_q(\omega) := \sup_{U \subset G} \left(\frac{1}{\mu(U)}\int_U \omega \,\mathrm{d}\mu\right)\left(\frac{1}{\mu(U)}\int_U \omega^{-\frac{q'}{q}} \,\mathrm{d}\mu\right)^{\frac{q}{q'}} < \infty, \tag{10}$$

where the supremum runs over all base sets $U \in G$. In that case, $\mathcal{A}_q(\omega)$ is called the $A_q(G)$-constant of ω. We say that ω belongs to the Muckenhoupt class $A_q(G)$ or even $\omega \in A_q(G)$ if it is an $A_q(G)$-weight.

Furthermore, we call a locally integrable, nonnegative function ω an $A_1(G)$-weight if there exists a constant $c \geq 0$ such that

$$\mathcal{M}_G\omega(x) \leq c\omega(x), \qquad \text{a.a. } x \in G. \tag{11}$$

The infimum over all these constants is called the $A_1(G)$-constant of ω and is denoted by $\mathcal{A}_1(\omega)$.

We call a constant $c = c(\omega) > 0$ that depends on $A_q(G)$-weights $A_q(G)$-*consistent*, if for each $d > 0$ we have

$$\sup\{c(\omega) : \omega \text{ is an } A_q(G)\text{-weight with } \mathcal{A}_q(\omega) < d\} < \infty.$$

Let us note some important observations on basic properties of the Muckenhoupt classes, in particular, that they are nested.

Proposition 3.2. *Let $\omega \in A_q(G)$ for $1 < q < \infty$. Then the following hold true.*

(i) *$\omega \in A_p(G)$ for $q < p < \infty$ and $\mathcal{A}_p(\omega)$ is $A_q(G)$-consistent. Here, even $1 \leq q < \infty$ is allowed.*

(ii) *$\omega^{-\frac{q'}{q}} \in A_{q'}(G)$, where q' is the Hölder conjugate of q. Moreover, $\mathcal{A}_{q'}(\omega^{-\frac{q'}{q}})$ is $A_q(G)$-consistent.*

(iii) *For $0 < \varepsilon < 1$ we have $\omega^\varepsilon \in A_r(G)$ with $r := 1 + \varepsilon(q-1) < q$ and $\mathcal{A}_r(\omega^\varepsilon)$ is $A_q(G)$-consistent.*

Proof. Fix a base set $U = x_0 + U_k$ and let first $q > 1$. Then by Hölder's inequality we see that

$$\begin{aligned}\left(\frac{1}{\mu(U)}\int_U \omega^{-\frac{p'}{p}} \,\mathrm{d}\mu\right)^{\frac{p}{p'}} &\leq \left(\frac{1}{\mu(U)}\mu(U)^{1-\frac{p'q}{pq'}}\right)^{\frac{p}{p'}}\left(\int_U \omega^{-\frac{q'}{q}} \,\mathrm{d}\mu\right)^{\frac{q}{q'}} \\ &= \left(\frac{1}{\mu(U)}\int_U \omega^{-\frac{q'}{q}} \,\mathrm{d}\mu\right)^{\frac{q}{q'}},\end{aligned}$$

and since U was chosen arbitrarily, (10) is fulfilled with exponent p and constant $\mathcal{A}_q(\omega)$ as well.
If $q = 1$ observe that

$$\left(\frac{1}{\mu(U)}\int_U \omega^{-\frac{p'}{p}}\,\mathrm{d}\mu\right)^{\frac{p}{p'}} \leq \|\omega^{-1}\|_{L^\infty(U)}, \tag{12}$$

Moreover, we obtain by (11) for almost all $x \in U$ that

$$\left(\frac{1}{\mu(U)}\int_U \omega\,\mathrm{d}\mu\right) \leq M_G\omega(x) \leq A^2\mathcal{M}_G\omega(x) \leq A^2\mathcal{A}_1\omega(x), \tag{13}$$

where we have used the relation (3) comparing the centered and uncentered maximal operators. Passing to the essential infimum on the right-hand side of (13) shows that

$$\left(\frac{1}{\mu(U)}\int_U \omega\,\mathrm{d}\mu\right)\|\omega^{-1}\|_{L^\infty(U)} \leq A^2\mathcal{A}_1. \tag{14}$$

Again, since U was chosen arbitrarily, putting inequalities (12) and (14) together yields the first assertion.
For the second assertion we simply calculate

$$\begin{aligned}\left(\frac{1}{\mu(U)}\int_U \omega^{-\frac{q'}{q}}\,\mathrm{d}\mu\right)&\left(\frac{1}{\mu(U)}\int_U \left(\omega^{-\frac{q'}{q}}\right)^{\frac{q}{q'}}\mathrm{d}\mu\right)^{\frac{q'}{q}}\\ &= \left(\left(\frac{1}{\mu(U)}\int_U \omega^{-\frac{q'}{q}}\,\mathrm{d}\mu\right)^{\frac{q}{q'}}\left(\frac{1}{\mu(U)}\int_U \omega\,\mathrm{d}\mu\right)\right)^{\frac{q'}{q}} \leq \mathcal{A}_q^{q'/q}(\omega).\end{aligned}$$

The third assertion follows from Jensen's inequality and the fact that $\varepsilon r'/r = q'/q$, which immediately follows from $r - 1 = \varepsilon(q-1)$ and the calculation

$$q - 1 = \frac{q'}{q'-1} - 1 = \frac{1}{q'-1} = \frac{q}{q'}. \tag{15}$$

Then it follows

$$\begin{aligned}\left(\frac{1}{\mu(U)}\int_U \omega^{\varepsilon}\,\mathrm{d}\mu\right)&\left(\frac{1}{\mu(U)}\int_U (\omega^{\varepsilon})^{-\frac{r'}{r}}\,\mathrm{d}\mu\right)^{\frac{r}{r'}}\\ &\leq \left(\frac{1}{\mu(U)}\int_U \omega\,\mathrm{d}\mu\right)^{\varepsilon}\left(\frac{1}{\mu(U)}\int_U \omega^{-\frac{q'}{q}}\,\mathrm{d}\mu\right)^{\frac{\varepsilon q}{q'}} \leq \mathcal{A}_q^{\varepsilon}(\omega).\end{aligned}$$

□

It is interesting to see that if we have two Muckenhoupt weights of class $A_1(G)$, say ω_0 and ω_1, then we can construct weights in $A_p(G)$. This will be of great use later on proving the Extrapolation Theorem in Section 4.

Proposition 3.3. *Let $\omega_0, \omega_1 \in A_1(G)$. Then for $1 < q < \infty$, $\omega_0 \cdot \omega_1^{-q/q'} \in A_q(G)$.*

Proof. Fix a base set $U \in G$. Observe that for $\omega \in A_1(G)$

$$\omega(x)^{-1} \leq (c\mathcal{M}_G\omega(x))^{-1} \leq \left(\frac{c}{\mu(U)}\int_U \omega\,\mathrm{d}\mu\right)^{-1},$$

with $c := \mathcal{A}_1^{-1}(\omega)$ for almost all $x \in U$. Hence

$$\begin{aligned}
&\left(\frac{1}{\mu(U)}\int_U \omega_0\omega_1^{-q/q'}\,\mathrm{d}\mu\right)\left(\frac{1}{\mu(U)}\int_U \left(\omega_0\omega_1^{-q/q'}\right)^{-\frac{q'}{q}}\,\mathrm{d}\mu\right)^{\frac{q}{q'}}\\
&\leq c'\left(\frac{1}{\mu(U)}\int_U \omega_1\,\mathrm{d}\mu\right)^{-\frac{q}{q'}}\left(\frac{1}{\mu(U)}\int_U \omega_0\,\mathrm{d}\mu\right)\left(\frac{1}{\mu(U)}\int_U \omega_0\,\mathrm{d}\mu\right)^{-1}\left(\frac{1}{\mu(U)}\int_U \omega_1\,\mathrm{d}\mu\right)^{\frac{q}{q'}}\\
&= c',
\end{aligned}$$

where $c' = \mathcal{A}_1(\omega_0)\mathcal{A}_1^{q/q'}(\omega_1)$. □

The Muckenhoupt weights can be characterized as those weight functions such that the maximal operator is weakly bounded in $L^q(G)$. In fact, we will see later on, that for $1 < q < \infty$, the maximal operator is bounded in $L^q(G)$ even in the strong sense. However, we first focus on the weak boundedness, which is true also for $q = 1$.
In order to state the theorem, let us fix the notation.

Definition 3.4. Given a weight function $\omega \in L^1_{\mathrm{loc}}(G)$, we denote by μ_ω the measure defined via $\mu_\omega(E) := \int_E \omega\,\mathrm{d}\mu$ and by $L^q_\omega(G)$ the space of all measurable functions such that the q-norm with respect to the measure μ_ω is finite. Furthermore we denote by $\mathcal{M}_{G,\omega}$ the maximal operator defined as in (1) with respect to the measure μ_ω.

Then we have the following two propositions.

Proposition 3.5. *Let $0 \leq \omega \in L^1_{loc}(G)$ and let $1 \leq q < \infty$. Then $\omega \in A_q(G)$ if and only if there is an $A_q(G)$-consistent constant $c > 0$ such that for every nonnegative measurable function $f : G \to \mathbb{R}$ and every base set $U \subset G$ it holds*

$$\left(\frac{1}{\mu(U)}\int_U f\,d\mu\right)^q \leq \frac{c}{\mu_\omega(U)}\int_U f^q\omega\,d\mu. \tag{16}$$

Proof. We first show the assertion for $1 < q < \infty$. Assume $\omega \in A_q(G)$. Then writing $f = f\omega^{1/q}\omega^{-1/q}$ and applying Hölder's inequality with exponents q and q' to the expression on the left-hand side of (16) yields

$$\begin{aligned}
\left(\frac{1}{\mu(U)}\int_U f\,\mathrm{d}\mu\right)^q &= \frac{1}{\mu(U)^q}\left(\int_U f\omega^{1/q}\omega^{-1/q}\,\mathrm{d}\mu\right)^q\\
&\leq \frac{1}{\mu(U)^q}\left(\int_U f^q\omega\,\mathrm{d}\mu\right)\left(\int_U \omega^{-\frac{q'}{q}}\,\mathrm{d}\mu\right)^{\frac{q}{q'}}\\
&= \frac{1}{\mu_\omega(U)}\left(\int_U f^q\omega\,\mathrm{d}\mu\right)\frac{\mu_\omega(U)}{\mu(U)}\left(\frac{1}{\mu(U)}\int_U \omega^{-\frac{q'}{q}}\,\mathrm{d}\mu\right)^{\frac{q}{q'}}\\
&\leq \frac{1}{\mu_\omega(U)}\left(\int_U f^q\omega\,\mathrm{d}\mu\right)\mathcal{A}_q(\omega),
\end{aligned}$$

and choosing $c := \mathcal{A}_q(\omega)$ we obtain the inequality (16).
Conversely, assume that (16) holds for all nonnegative measurable functions $f : G \to \mathbb{R}$. If the left-hand side of (16) is finite, we obtain equivalently

$$\mu_\omega(U)\left(\frac{1}{\mu(U)}\int_U f\,\mathrm{d}\mu\right)^q\left(\int_U f^q\omega\,\mathrm{d}\mu\right)^{-1} \leq c. \tag{17}$$

Therefore, we define $f_n := (\omega + 1/n)^{-q'/q}$ for $n \in \mathbb{N}$. It follows

$$\begin{aligned}\left(\frac{1}{\mu(U)}\int_U \omega \,\mathrm{d}\mu\right)&\left(\frac{1}{\mu(U)}\int_U (\omega+\frac{1}{n})^{-q'}\omega\,\mathrm{d}\mu\right)^{\frac{q}{q'}}\\ &= \frac{\mu_\omega(U)}{\mu(U)}\left(\frac{1}{\mu(U)}\int_U (\omega+\frac{1}{n})^{-q'}\omega\,\mathrm{d}\mu\right)^{q-1}\\ &= \frac{\mu_\omega(U)}{\mu(U)}\left(\frac{1}{\mu(U)}\int_U f_n^q\,\omega\,\mathrm{d}\mu\right)^{q-1}.\\ &= \mu_\omega(U)\left(\frac{1}{\mu(U)}\int_U f_n^q\,\omega\,\mathrm{d}\mu\right)^{q}\left(\int_U f_n^q\,\omega\,\mathrm{d}\mu\right)^{-1}\end{aligned}$$

Hence, we can estimate

$$\begin{aligned}\left(\frac{1}{\mu(U)}\int_U \omega \,\mathrm{d}\mu\right)&\left(\frac{1}{\mu(U)}\int_U (\omega+\frac{1}{n})^{-q'}\omega\,\mathrm{d}\mu\right)^{\frac{q}{q'}}\\ &\le \mu_\omega(U)\left(\frac{1}{\mu(U)}\int_U f_n^q(\omega+\frac{1}{n})\,\mathrm{d}\mu\right)^{q}\left(\int_U f_n^q\,\omega\,\mathrm{d}\mu\right)^{-1}\\ &= \mu_\omega(U)\left(\frac{1}{\mu(U)}\int_U f_n\,\mathrm{d}\mu\right)^{q}\left(\int_U f_n^q\omega\,\mathrm{d}\mu\right)^{-1} \le c.\end{aligned}$$

Observe that the right-hand side is independent of $n \in \mathbb{N}$. If we denote the integrand $(\omega + 1/n)^{-q'}\omega$ in the second term of the left-hand side by g_n, we see that $g_n \le g_m$ if $n \le m$ and $g_n \to \omega^{-q'/q}$ as $n \to \infty$ in the pointwise sense. Hence, Lebesgue's Theorem on Monotone Convergence yields the result for $1 < q < \infty$.
For $q = 1$, the proof is similar to [10] and will be omitted here. □

Proposition 3.6. *Let $1 \le q < \infty$ and $\omega \in A_q(G)$. Then*

(i) *the measure μ_ω is regular and has the doubling property, i.e., $\mu_\omega(x + U_{\theta(k)}) \le c_\omega \mu_\omega(x + U_k)$ for all $x \in G$ and $k \in \mathbb{Z}$, where $c_\omega > 0$ is an $A_q(G)$-consistent constant,*

(ii) *slightly more general, for any base set U and any measurable subset $S \subset U$ we have*

$$\left(\frac{\mu(S)}{\mu(U)}\right)^q \le c\frac{\mu_\omega(S)}{\mu_\omega(U)}, \tag{18}$$

where $c > 0$ is the bound appearing in (16),

(iii) *it holds $L^\infty(G) = L^\infty_\omega(G)$ with equal norms,*

(iv) *$\mathcal{M}_{G,\omega}$ is bounded in $L^p_\omega(G)$ for all $1 < p \le \infty$ and weakly bounded in $L^1_\omega(G)$ with an $A_q(G)$-consistent bound and*

(v) *$C_0(G)$ is dense in $L^q_\omega(G)$.*

Proof. Regularity follows by Lebesgue's Theorem on Dominated Convergence. To verify the doubling property, simply use (16) with $U = x + U_{\theta(k)}$ and $f = \chi_{x+U_k}$. Since μ has the doubling property with doubling constant A, we obtain

$$A^{-q} \le \left(\frac{\mu(x+U_k)}{\mu(x+U_{\theta(k)})}\right)^q \le \frac{c\mu_\omega(x+U_k)}{\mu_\omega(x+U_{\theta(k)})},$$

which shows (i) with $c_\omega = cA^q$.
For (ii), we argue analogously, using (16) with $f = \chi_S$.
To show (iii), recall that the norm on $L^\infty_\omega(G)$ can be represented via

$$\|f\|_{L^\infty_\omega(G)} = \sup\{r \in \mathbb{R} : \mu_\omega(\{x \in G : f(x) > r\}) > 0\}, \tag{19}$$

and a similar expression for the norm on $L^\infty(G)$, if we replace the measure μ_ω by the measure μ. Since μ_ω is absolutely continuous with respect to μ, we clearly have $\|f\|_{L^\infty_\omega(G)} \le \|f\|_{L^\infty(G)}$. Moreover, $\omega > 0$ almost everywhere, excepting the trivial case $\omega = 0$. Indeed, if $\omega = 0$ on a set S such that $\mu(S) > 0$, we get in virtue of (18) that $\mu_\omega(U) = 0$ for every base set U containing S. If S is not contained in any base set, then consider the set $\tilde{S} := S \cap U$ for some base set U large enough such that $\mu(\tilde{S}) > 0$, which certainly exists, since otherwise

$$\mu(S) = \mu(S \cap G) = \mu(S \cap \bigcup_{k\in\mathbb{Z}} U_k) = \mu\left(\bigcup_{k\in\mathbb{Z}} (S \cap U_k)\right) \le \sum_{k\in\mathbb{Z}} \mu(S \cap U_k) = 0$$

Hence, $\omega = 0$ almost everywhere on every base set containing $\tilde{S}$ and thus on the whole group G. This shows that for every nontrivial Muckenhoupt weight ω we have $\omega > 0$ almost everywhere. Thus, μ is absolutely continuous with respect to μ_ω. Consequently $\|f\|_{L^\infty(G)} = \|f\|_{L^\infty_\omega(G)}$.
The boundedness of the maximal operators follows by Theorem 2.6. The $A_q(G)$-consistency of the bound is clear, since we apply Marcinkiewicz' interpolation theorem and the bound for the weak estimate from $L^1_\omega(G)$ to $L^1_\omega(G)$ is the doubling constant of the measure μ_ω, which is $A_q(G)$-consistent by part (i), and the bound of the maximal operator in $L^\infty_\omega(G) = L^\infty(G)$ is 1 and therefore trivially $A_q(G)$-consistent.
Since μ_ω is regular, (v) follows from [17, Appendix E.8]. □

Theorem 3.7. *Let $0 \le \omega \in L^1_{loc}(G)$ and let $1 \le q < \infty$. Then $\omega \in A_q(G)$ if and only if $\mathcal{M}_G$ is bounded from $L^q_\omega(G)$ to $L^{q,\infty}_\omega(G)$ with an $A_q(G)$-consistent bound.*

Proof. Assume $\omega \in A_q(G)$. We can apply Proposition 3.6 to obtain that $\mathcal{M}_{G,\omega}$ is weakly bounded in $L^1_\omega(G)$ with an $A_q(G)$-consistent bound. Taking the supremum first on the right-hand side and then on the left-hand side of (16), we see that

$$(\mathcal{M}_G f(x))^q \le c\mathcal{M}_{G,\omega}(|f|^q)(x),$$

for almost all $x \in G$. Then we may use the weak boundedness of $\mathcal{M}_{G,\omega}$ to obtain

$$\mu_\omega\{x \in G : \mathcal{M}_G f(x) > t\} \le \mu_\omega\{x \in G : \mathcal{M}_{G,\omega}(|f|^q)(x) > t^q/c\} \le \frac{c'}{t^q}\int_G |f|^q \, \mathrm{d}\mu_\omega,$$

which is what we wanted to show.
Conversely, assume that $\mathcal{M}_G$ is bounded from L^q_ω to $L^{q,\infty}_\omega$. Let $f \ge 0$ be measurable and let $U \subset G$ be a base set. If

$$(f_U) := \frac{1}{\mu(U)}\int_U f \, \mathrm{d}\mu = 0,$$

there is nothing left to prove. Hence, assume $(f_U) > 0$ and observe that $(f_U) \le M_G f(x) \le A^2\mathcal{M}_G f(x)$ for every $x \in U$. Hence, fixing $0 < t < (f_U)$, we obtain

$$\begin{aligned} U &= \{x \in U : \mathcal{M}_G f(x) \ge (f_U)/A^2\} \\ &\subset \{x \in U : \mathcal{M}_G f(x) > t/A^2\} \subset \{x \in G : \mathcal{M}_G f(x) > t/A^2\}, \end{aligned}$$

and by the weak boundedness of the maximal operator we obtain

$$\mu_\omega(U) \leq \frac{cA^{2q}}{t^q} \int_U |f|^q \, \mathrm{d}\mu_\omega.$$

Using once again $t < (f_U)$, we finally see

$$(f_U)^q \mu_\omega(U) \leq (f_U)^q \frac{cA^{2q}}{t^q} \int_U |f|^q \, \mathrm{d}\mu_\omega \leq cA^{2q} \int_U |f|^q \, \mathrm{d}\mu_\omega,$$

and in virtue of Proposition 3.5 we obtain $\omega \in A_q(G)$.

□

Using the Marcinkiewicz interpolation theorem, we deduce the following corollary.

Corollary 3.8. *Let $\omega \in A_q$, $1 \leq q < \infty$. Then for $q < p < \infty$ there exists an $A_q(G)$-consistent constant $c > 0$ such that the strong estimate*

$$\int_G |\mathcal{M}_G f|^p \omega \, d\mu \leq c \int_G |f|^p \omega \, d\mu, \qquad f \in L^1_{loc}(G), \tag{20}$$

holds. In other words, $\mathcal{M}_G$ extends to a bounded operator from $L^p_\omega(G)$ to $L^p_\omega(G)$.

Proof. By Proposition 3.6 (iii) we have $L^\infty_\omega(G) = L^\infty(G)$ and thus we obtain the estimate

$$\|\mathcal{M}_G f\|_{L^\infty_\omega(G)} = \|\mathcal{M}_G f\|_{L^\infty(G)} \leq \|f\|_{L^\infty(G)} = \|f\|_{L^\infty_\omega(G)}. \tag{21}$$

Hence, the Marcinkiewicz interpolation theorem yields the boundedness of $\mathcal{M}_G$ in $L^\infty_\omega(G)$. The $A_q(G)$ consistency follows, since the by the Marcinkiewicz interpolation theorem the bound depends directly on the $A_q(G)$-consistent weak bound of $\mathcal{M}_G : L^q_\omega(G) \to L^{q,\infty}_\omega(G)$ and the bound of $\mathcal{M}_G : L^\infty_\omega(G) \to L^\infty_\omega(G)$, which is 1 and hence trivially $A_q(G)$-consistent. □

Observe that Corollary 3.8 is only stated for $q < p$. Indeed, if we want to strengthen this result towards $q = p$, we will necessarily have to exclude the case $q = 1$: There are counterexamples in this case even for the group $G := \mathbb{R}^n$. Take for example $\omega = 1$. It is easy to see that applying the maximal operator to a nontrivial integrable function never yields an integrable function.
However, if $1 < q = p$, then we do obtain such a strong estimate. In the classical setting $G = \mathbb{R}^n$, this is called the Muckenhoupt theorem. It is usually proven via the so-called Reversed Hölder Inequality (cf. [11, 20]), which in turn shows for $1 < q < \infty$ that $\omega \in A_q(G)$ implies $\omega \in A_p(G)$ for some smaller $p < q$. Then Corollary 3.8 may be applied to this new, smaller exponent to show the assertion. Unfortunately, the proof of the Reversed Hölder Inequality heavily relies on the existence of dyadic cubes. In our situation, we lack of such a concept. Therefore we use a different approach, which is mainly due to Jawerth [13].

Proof of Theorem 1.4. Let $f \in L^q_\omega(G)$. For every $m \in \mathbb{Z}$ define the set

$$S_m := \left\{ x \in G : 2^m < \mathcal{M}_G f(x) \leq 2^{m+1} \right\},$$

which implies $S_m \subset \bigcup_{j \in \mathbb{N}} U^{m,j}$, where the right-hand side consists of a countable union of sets of the form $U^{m,j} = x^{m,j} + U^{m,j}_k$ satisfying

$$\frac{1}{\mu(U^{m,j})} \int_{U^{m,j}} |f| \, \mathrm{d}\mu > \frac{2^m}{A^2}. \tag{22}$$

Let us explain, why we have include the extra factor A^2 here, which makes sure that the union can be assumed to be countable. Suppose $x \in S_m$. Then by the comparability of the centered and the uncentered maximal operator in (3) we have $M_G f(x) \geq \mathcal{M}_G f(x) > 2^m$, and by the definition of the uncentered maximal operator we find a base set $U(x)$ such that $x \in U(x)$ and such that

$$\frac{1}{\mu(U(x))} \int_{U(x)} |f| \, \mathrm{d}\mu > 2^m. \tag{23}$$

Clearly, S_m would be contained in the union over all these $U(x)$, but this union might not be countable. Therefore, we take an element $x^{m,n}$ of the dense subset $D \subset G$ obtained in Proposition 2.2 (iv) such that $x^{m,n} \in U(x)$. Assume that $U(x) = y + U_k$ for some $y \in G$ and $k \in \mathbb{Z}$. Then $x^{m,n} + U_k \subset y + 2U_k \subset y + U_{\theta(k)}$ and so

$$\mu(x^{m,n} + U_{\theta(k)}) \leq A\mu(x^{m,n} + U_k) \leq A\mu(y + U_{\theta(k)}) \leq A^2\mu(y + U_k) = A^2\mu(U(x)).$$

Furthermore $y \in x^{m,n} + U_k$ by symmetry of U_k, and so $x \in y + U_k \subset x^{m,n} + 2U_k \subset x^{m,n} + U_{\theta(k)}$. Hence, using (23), we obtain

$$\begin{aligned} \frac{1}{\mu(x^{m,n} + U_{\theta_k})} \int_{x^{m,n} + U_{\theta_k}} \omega \, \mathrm{d}\mu &\geq \frac{1}{\mu(x^{m,n} + U_{\theta_k})} \int_{U(x)} \omega \, \mathrm{d}\mu \\ &\geq \frac{1}{A^2\mu(U(x))} \int_{U(x)} \omega \, \mathrm{d}\mu > \frac{2^m}{A^2}. \end{aligned}$$

To summarize, for every $x \in S_m$ we have found a base set $U^{m,(n,k)} = x^{m,n} + U_{\theta(k)}$ with center $x^{m,n}$ in the countable dense set $D \subset G$ such that $x \in U^{m,(n,k)}$ and such that the inequality (22) holds. Since there are only countably many base sets $U^{m,(n,k)}$, $n \in \mathbb{N}$, $k \in \mathbb{Z}$, we can relabel them as $U^{m,j}$. This shows that we can assume the collection of base sets covering the set S_m to be countable.

We therefore can define the sets

$$E^{m,j} := \left(U^{m,j} \setminus \bigcup_{s<j} U^{m,s} \right) \cap S_m, \qquad j \in \mathbb{N}, m \in \mathbb{Z}. \tag{24}$$

Since the S_m are clearly a disjoint decomposition of G and each S_m itself decomposes into the disjoint subsets $E^{m,j}$, we may write in virtue of (22)

$$\begin{aligned} \int_G |\mathcal{M}_G f|^q \omega \, \mathrm{d}\mu &= \sum_{m,j} \int_{E^{m,j}} |\mathcal{M}_G f|^q \omega \, \mathrm{d}\mu \leq \sum_{m,j} 2^{(m+1)q} \mu_\omega(E^{m,j}) \\ &\leq 2^q A^{2q} \sum_{k,j} \mu_\omega(E^{m,j}) \left(\frac{1}{\mu(U^{m,j})} \int_{U^{m,j}} |f| \, \mathrm{d}\mu \right)^q \\ &= c \sum_{m,j} \mu^{m,j} g^{m,j}, \end{aligned} \tag{25}$$

with the $A_q(G)$-consistent constant $c := 2^q A^{2q}$ and

$$\begin{aligned} \mu^{m,j} &:= \mu_\omega(E^{m,j}) \left(\frac{\mu_\upsilon(U^{m,j})}{\mu(U^{m,j})} \right)^q, \\ g^{m,j} &:= \left(\frac{1}{\mu_\upsilon(U^{m,j})} \int_{U^{m,j}} \left(|f| \upsilon^{-1} \right) \upsilon \, \mathrm{d}\mu \right)^q. \end{aligned}$$

Here we used the notation $\upsilon := \omega^{-q'/q}$, so that $\upsilon \in A_{q'}(G)$ by Proposition 3.2 (ii). Consider now the set $X := \{(m,j) : m \in \mathbb{Z}, j \in \mathbb{N}\}$. Obviously, X turns into a measure space $(X, \mathcal{P}(X), \mu_X)$ if we define the measure μ_X via

$$\mu_X(E) := \sum_{(m,j)\in E} \mu^{m,j}, \qquad E \in \mathcal{P}(X). \tag{26}$$

With this measure space at hand and defining $g : X \to \mathbb{R}_+$ via $g(m,j) := g^{m,j}$, we can now write the last sum appearing in (25) as an integral via

$$\sum_{m,j} \mu^{m,j} g^{m,j} = \int_X g \,\mathrm{d}\mu_X = \int_0^\infty d_g \,\mathrm{d}\lambda \tag{27}$$

where λ is the Lebesgue measure and the function d_g is the distribution function of g defined via $d_g(r) := \mu_X\{(m,j) \in X : g(m,j) > r\}$. Since $\omega \in A_q(G)$, we have

$$\left(\frac{\mu_\upsilon(U^{m,j})}{\mu(U^{m,j})}\right)^q \le \left(\mathcal{A}_q(\omega)\frac{\mu(U^{m,j})}{\mu_\omega(U^{m,j})}\right)^{q'},$$

which is just the Muckenhoupt condition (10) rewritten, and thus

$$\begin{aligned}\left(\frac{\mu_\upsilon(U^{m,j})}{\mu(U^{m,j})}\right)^q &\le \left(\mathcal{A}_q(\omega)\frac{\mu(U^{m,j})}{\mu_\omega(U^{m,j})}\right)^{q'} \\ &= \left(\mathcal{A}_q(\omega)\frac{1}{\mu_\omega(U^{m,j})}\int_{U^{m,j}} \chi_{U^{m,j}}\omega^{-1}\omega \,\mathrm{d}\mu\right)^{q'} \\ &\le \left(\mathcal{A}_q(\omega)\inf_{x\in U^{m,j}} M_{G,\omega}(\chi_{U^{m,j}}\omega^{-1})(x)\right)^{q'} \\ &\le \left(c\mathcal{A}_q(\omega)\inf_{x\in U^{m,j}} \mathcal{M}_{G,\omega}(\chi_{U^{m,j}}\omega^{-1})(x)\right)^{q'},\end{aligned} \tag{28}$$

where the last inequality follows from the fact that the centered and uncentered maximal operators are comparable with a doubling constant $c = c_\omega > 0$ by (3), and c_ω is $A_q(G)$-consistent by Proposition 3.6 (i). Hence

$$\begin{aligned}\mu^{m,j} := \mu_\omega(E^{m,j})\left(\frac{\mu_\upsilon(U^{m,j})}{\mu(U^{m,j})}\right)^q &\le c\mu_\omega(E^{m,j})\left(\inf_{x\in U^{m,j}} \mathcal{M}_{G,\omega}(\chi_{U^{m,j}}\omega^{-1})(x)\right)^{q'} \\ &\le c\int_{E^{m,j}} |\mathcal{M}_{G,\omega}(\chi_{U^{m,j}}\omega^{-1})|^{q'}\omega \,\mathrm{d}\mu,\end{aligned}$$

with an $A_q(G)$-consistent constant $c > 0$. We may now estimate the integrand in (27). With the notation $G(r) := \bigcup_{\{(m,j)\in X : g(m,j)>r\}} U^{m,j}$ we obtain

$$\begin{aligned}G(r) &\subset \{x \in G : \frac{1}{\mu_\upsilon(U)}\int_U (|f|\upsilon^{-1})\upsilon \,\mathrm{d}\mu > r \text{ for some base set } U \subset G \text{ with } x \in U\} \\ &\subset \{x \in G : (M_{G,\upsilon}(|f|\upsilon^{-1})(x))^q > r\} \subset \{x \in G : (A^2\mathcal{M}_{G,\upsilon}(|f|\upsilon^{-1})(x))^q > r\}.\end{aligned}$$

Hence, by the boundedness of $\mathcal{M}_{G,\omega}$ in $L^{q'}_\omega(G)$

$$\begin{aligned} d_g(r) = \sum_{\{(m,j)\in X : g>r\}} \mu^{m,j} &\le c \sum_{\{(m,j)\in X : g>r\}} \int_{E^{m,j}} |\mathcal{M}_{G,\omega}(\chi_{U^{m,j}}\omega^{-1})|^{q'} \omega \,\mathrm{d}\mu \\ &\le c \int_G |\mathcal{M}_{G,\omega}(\chi_{G(r)}\omega^{-1})|^{q'} \omega \,\mathrm{d}\mu \\ &\le c \int_{G(r)} \omega^{1-q'} \,\mathrm{d}\mu = c\mu_\upsilon(G(r)) \\ &\le c\mu_\upsilon(\{x \in G : (A^2 \mathcal{M}_{G,\upsilon}(|f|\upsilon^{-1})(x))^q > r\}), \end{aligned} \tag{29}$$

where $c > 0$ is still $A_q(G)$-consistent, since the bound of $\mathcal{M}_{G,\omega} : L^{q'}_\omega(G) \to L^{q'}_\omega(G)$ is $A_q(G)$-consistent by Proposition 3.6 (iv). Now we are almost finished. Recall that also μ_υ is a doubling measure (since υ is a Muckenhoupt weight as well) and therefore the maximal operator $\mathcal{M}_{G,\upsilon}$ is bounded in $L^q_\upsilon(G)$ with an $A_q(G)$-consistent bound by Proposition 3.6 (iv) and Proposition 3.2 (ii).
Therefore we can calculate with (25) and (29)

$$\begin{aligned} \int_G |\mathcal{M}_G f|^q \omega \,\mathrm{d}\mu \le c \sum_{m,j} \mu^{m,j} g^{m,j} &= c \int_0^\infty d_g \,\mathrm{d}\lambda \\ &\le c \int_0^\infty \mu_\upsilon(\{x \in G : (c\mathcal{M}_{G,\upsilon}(|f|\upsilon^{-1})(x))^q > r\}) \,\mathrm{d}\lambda(r) \\ &= c \int_G (c\mathcal{M}_{G,\upsilon}(|f|\upsilon^{-1}))^q \,\mathrm{d}\mu_\upsilon \\ &\le c \int_G (|f|\upsilon^{-1})^q \upsilon \,\mathrm{d}\mu = c \int_G |f|^q \omega \,\mathrm{d}\mu, \end{aligned}$$

with an $A_q(G)$-consistent constant $c > 0$. This finishes the proof. □

4. Extrapolation Theorem for Locally Compact Abelian Groups

This section is devoted to establishing an extrapolation theorem generalizing the classical Extrapolation Theorem due to García-Cuerva and Rubio de Francia. We will not follow the arguments given in their book [10], but rather use a more modern and direct way due to Cruz-Uribe, Martell and Pérez [4], which enables us to deal with the vector-valued case corresponding to Theorem 6.4 in chapter V of [10], simultaneously.
A cornerstone of the arguments used here are the following three operators. Let $1 < q < \infty$. Then the *conjugate maximal operator* $\mathcal{M}'_G$ defined via $\mathcal{M}'_G f := \mathcal{M}_G(f\omega)/\omega$ extends to a bounded operator on $L^{q'}_\omega(G)$ with $A_q(G)$-consistent bound. Indeed, $\nu := \omega^{1-q'} \in A_{q'}(G)$ by Proposition 3.2 and $1 - q' = -q'/q$, which has been calculated in (15). Therefore, $\mathcal{M}_G$ is bounded in $L^{q'}_\nu(G)$, which implies that

$$\begin{aligned} \|\mathcal{M}'_G f\|^{q'}_{L^{q'}_\omega} &= \int_G \left(\frac{\mathcal{M}_G(f\omega)}{\omega}\right)^{q'} \omega \,\mathrm{d}\mu = \int_G (\mathcal{M}_G(f\omega))^{q'} \omega^{1-q'} \,\mathrm{d}\mu \\ &\le c \int_G |f\omega|^{q'} \omega^{1-q'} \,\mathrm{d}\mu = c \int_G |f|^{q'} \omega \,\mathrm{d}\mu = c\|f\|^{q'}_{L^{q'}_\omega}. \end{aligned}$$

Moreover, we will consider the two corresponding operators

$$Rf := \sum_{k=0}^{\infty} \frac{\mathcal{M}_G^k f}{2^k \|\mathcal{M}_G\|^k_{\mathcal{L}(L^q_\omega(G))}} \text{ and } R'f := \sum_{k=0}^{\infty} \frac{(\mathcal{M}_G)^k f}{2^k \|\mathcal{M}'{}_G\|^k_{L^{q'}_\omega(G)}}.$$

These operators enjoy the following properties.

Proposition 4.1. *Let $1 < q < \infty$, $f \in L^q_\omega(G)$ and $h \in L^{q'}_\omega(G)$. Then*

(i) (a) $0 \le |f| \le Rf$,
 (b) $\|Rf\|_{L^q_\omega(G)} \le 2\|f\|_{L^q_\omega(G)}$,
 (c) $Rf \in A_1(G)$ *with* $\mathcal{A}_1(Rf) \le 2\|\mathcal{M}_G\|_{\mathcal{L}(L^q_\omega(G))}$,

(ii) (a) $0 \le |h| \le R'h$,
 (b) $\|R'h\|_{L^{q'}_\omega(G)} \le 2\|h\|_{L^{q'}_\omega(G)}$ *and*
 (c) $R'h \cdot \omega \in A_1(G)$ *with* $\mathcal{A}_1(Rh\cdot\omega) \le 2\|\mathcal{M}'_G\|_{\mathcal{L}(L^q_\omega(G))}$.

Proof. We start with the operator R. For the first assertion, we observe that every summand in the definition of R is nonnegative. Therefore every partial sum is greater or equal $M^0 f = |f|$. The second assertion follows simply by observing

$$\begin{aligned}\|Rf\|_{L^q_\omega(G)} &= \left\| \sum_{k=0}^{\infty} \frac{\mathcal{M}_G^k f}{2^k \|\mathcal{M}_G\|^k_{\mathcal{L}(L^q_\omega(G))}} \right\|_{L^q_\omega(G)} \\ &\le \sum_{k=0}^{\infty} \frac{\|\mathcal{M}_G\|^k_{\mathcal{L}(L^q_\omega(G))} \|f\|_{L^q_\omega(G)}}{2^k \|\mathcal{M}_G\|^k_{\mathcal{L}(L^q_\omega(G))}} = 2\|f\|_{L^q_\omega(G)}.\end{aligned}$$

The last assertion can be seen by considering

$$\begin{aligned}\mathcal{M}_G(Rf) &= \mathcal{M}_G\left(\sum_{k=0}^{\infty} \frac{\mathcal{M}_G^k f}{2^k \|\mathcal{M}_G\|^k_{\mathcal{L}(L^q_\omega(G))}} \right) = \sum_{k=0}^{\infty} \frac{\mathcal{M}_G^{k+1} f}{2^k \|\mathcal{M}_G\|^k_{\mathcal{L}(L^q_\omega(G))}} \\ &= 2\|\mathcal{M}_G\|_{\mathcal{L}(L^q_\omega(G))} \sum_{k=0}^{\infty} \frac{\mathcal{M}_G^{k+1} f}{2^{k+1} \|\mathcal{M}_G\|^{k+1}_{\mathcal{L}(L^q_\omega(G))}} \\ &\le 2\|\mathcal{M}_G\|_{\mathcal{L}(L^q_\omega(G))} \sum_{k=0}^{\infty} \frac{\mathcal{M}_G^k f}{2^k \|\mathcal{M}_G\|^k_{\mathcal{L}(L^q_\omega(G))}} = 2\|\mathcal{M}_G\|_{\mathcal{L}(L^q_\omega(G))} Rf.\end{aligned}$$

The proof of the first two assertions concerning operator R' follow as in the case of the operator R. The last assertion follows by

$$\begin{aligned}\mathcal{M}_G(R'h\cdot\omega) &= \mathcal{M}_G\left(\sum_{k=0}^{\infty} \frac{(\mathcal{M}'_G)^k h\cdot\omega}{2^k \|\mathcal{M}'_G\|^k_{\mathcal{L}(L^q_\omega(G))}} \right) = \sum_{k=0}^{\infty} \frac{\mathcal{M}_G((\mathcal{M}'_G)^k h\cdot\omega)\omega/\omega}{2^k \|\mathcal{M}'_G\|^k_{\mathcal{L}(L^q_\omega(G))}} \\ &= \sum_{k=0}^{\infty} \frac{(\mathcal{M}'_G)^{k+1} h\cdot\omega}{2^k \|\mathcal{M}'_G\|^k_{\mathcal{L}(L^q_\omega(G))}} = 2\|\mathcal{M}'_G\|_{\mathcal{L}(L^q_\omega(G))} \sum_{k=0}^{\infty} \frac{(\mathcal{M}'_G)^{k+1} h\cdot\omega}{2^{k+1} \|\mathcal{M}'_G\|^{k+1}_{\mathcal{L}(L^q_\omega(G))}} \\ &\le 2\|\mathcal{M}'_G\|_{\mathcal{L}(L^q_\omega(G))} \sum_{k=0}^{\infty} \frac{(\mathcal{M}'_G)^k h\cdot\omega}{2^k \|\mathcal{M}'_G\|^k_{\mathcal{L}(L^q_\omega(G))}} = 2\|\mathcal{M}'_G\|_{\mathcal{L}(L^q_\omega(G))} R'h\cdot\omega.\end{aligned}$$

□

Now we can prove an abstract version of the Extrapolation Theorem for locally compact abelian groups.

Theorem 4.2. *Let G be a locally compact abelian group satisfying Assumption 1.1. Moreover, let $1 < r < \infty$ and assume that there is*

$$\mathcal{F} \subset \{(f,g) : f,g : G \to \mathbb{R} \text{ are nonnegative, measurable functions}\},$$

such that for every $\upsilon \in A_r(G)$,

$$\|g\|_{L^r_\upsilon(G)} \le c\|f\|_{L^r_\upsilon(G)}, \qquad (f,g) \in \mathcal{F}, \tag{30}$$

with an $A_r(G)$-consistent constant $c > 0$. Then for every $1 < q < \infty$ and every $\omega \in A_q(G)$,

$$\|g\|_{L^q_\omega(G)} \le c\|f\|_{L^q_\omega(G)}, \qquad (f,g) \in \mathcal{F}, \tag{31}$$

with an $A_q(G)$-consistent constant $c = c(q) > 0$.

Proof. Fix $(f,g) \in \mathcal{F}$, $1 < q < \infty$ and $\omega \in A_q(G)$. Let us also assume that $f \in L^q_\omega(G)$, since otherwise there is nothing to prove. By the Theorem of Hahn-Banach we may write

$$\|g\|_{L^q_\omega(G)} = \int_G gh \,\mathrm{d}\mu_\omega,$$

where $h \in L^{q'}_\omega(G)$ with norm $\|h\|_{L^{q'}_\omega(G)} = 1$. Since μ_ω is a positive measure and $g \ge 0$, also $h \ge 0$. Therefore, by Proposition 4.1 we obtain the pointwise estimate $h \le R'h$, and Hölder's inequality with exponents r and r' and with respect to the measure $R'h\,\mathrm{d}\mu$ yields

$$\begin{aligned}
\|g\|_{L^q_\omega(G)} &= \int_G gh\,\mathrm{d}\mu_\omega = \int_G g\,(Rf)^{-1/r'}\,(Rf)^{1/r'}\,h\,\mathrm{d}\mu_\omega \\
&\le \int_G g\,(Rf)^{-1/r'}\,(Rf)^{1/r'}\,R'h\,\mathrm{d}\mu_\omega \\
&\le \left(\int_G g^r\,(Rf)^{-r/r'}\,R'h\,\mathrm{d}\mu_\omega\right)^{1/r}\left(\int_G RfR'h\,\mathrm{d}\mu_\omega\right)^{1/r'} \\
&= \|g\|_{L^r_\upsilon(G)}\|\tilde{R}^{1/r'}\|_{L^{r'}_\omega(G)},
\end{aligned} \tag{32}$$

with $\upsilon := (Rf)^{-r/r'}\,R'h\cdot\omega$ and $\tilde{R} := RfR'h$. Since $f \in L^q_\omega(G)$ and $h \in L^{q'}_\omega$, Proposition 4.1 yields both $Rf \in A_1(G)$ and $R'h\cdot\omega \in A_1(G)$ with Muckenhoupt norms independent of f and h. Therefore $\upsilon \in A_r(G)$ by Proposition 3.3 with $\mathcal{A}_r(\upsilon)$ depending only on $\mathcal{A}_q(\omega)$. Hence, by assumption,

$$\|g\|_{L^r_\upsilon(G)} \le c\|f\|_{L^r_\upsilon(G)},$$

with an $A_q(G)$-consistent constant $c = c(\omega) > 0$. Plugging this into (32), we obtain with the pointwise estimate $|f| \le Rf$

$$\|g\|_{q,\omega} \le c\|f\|_{L^r_\upsilon(G)}\|\tilde{R}^{1/r'}\|_{L^{r'}_\omega(G)} \le c\|Rf\|_{L^r_\upsilon(G)}\|\tilde{R}^{1/r'}\|_{L^{r'}_\omega(G)}.$$

Taking a closer look at the first norm of the right-hand side, we realize that by the definition of $\upsilon = (Rf)^{-r/r'}\,R'h\cdot\omega$ and the calculation $r - r/r' = 1$,

$$\begin{aligned}
\|Rf\|_{L^r_\upsilon(G)} &= \left(\int_G (Rf)^r\,\mathrm{d}\mu_\upsilon\right)^{1/r} = \left(\int_G (Rf)^r\,(Rf)^{-r/r'}\,R'h\,\mathrm{d}\mu_\omega\right)^{1/r} \\
&= \left(\int_G RfR'h\,\mathrm{d}\mu_\omega\right)^{1/r} = \|\tilde{R}^{1/r}\|_{L^r_\omega(G)}.
\end{aligned}$$

It follows

$$\begin{aligned}\|g\|_{q,\omega} &\le c\|\tilde{R}^{1/r}\|_{L^r_\omega(G)}\|\tilde{R}^{1/r'}\|_{L^{r'}_\omega(G)} = c\|\tilde{R}\|_{L^1_\omega(G)} = c\|RfR'h\|_{L^1_\omega(G)}\\ &\le c\|Rf\|_{L^q_\omega(G)}\|R'h\|_{L^{q'}_\omega(G)} \le c\cdot 2\|f\|_{L^q_\omega(G)}\cdot 2\|h\|_{L^{q'}_\omega(G)}\\ &= 4c\|f\|_{L^q_\omega(G)}.\end{aligned}$$

□

Remark 4.3. An extrapolation theorem on locally compact abelian groups in the style of García-Cuerva and Rubio de Francia is contained in the assertion of Theorem 4.2.

(i) Choose

$$\mathcal{F}_{cl} := \{(|f|, |Tf|) : f : G \to \mathbb{R} \text{ measurable }\}.$$

If $T : L^r_\upsilon(G) \to L^r_\upsilon(G)$ is bounded with an $A_r(G)$-consistent bound, then we always have

$$\|g\|_{L^r_\upsilon(G)} = \|Tf\|_{L^r_\upsilon(G)} \le c\|f\|_{L^r_\upsilon(G)}, \qquad (f,g) \in \mathcal{F}_{cl},$$

and thus Theorem 4.2 gives us

$$\|Tf\|_{L^q_\omega(G)} = \|g\|_{L^r_\upsilon(G)} \le c\|f\|_{L^q_\omega(G)}, \qquad (f,g) \in \mathcal{F}_{cl},$$

with an $A_q(G)$-consistent constant $c = c(q) > 0$.

(ii) We also get a vector-valued version of Theorem 4.2, *i.e.*, under the assumption of the theorem we have for all $1 < p, q < \infty$ and for all $\omega \in A_q(G)$

$$\left\|\left(\sum_{j=1}^n g_j^p\right)^{1/p}\right\|_{L^q_\omega(G)} \le c\left\|\left(\sum_{j=1}^n f_j^p\right)^{1/p}\right\|_{L^q_\omega(G)},$$

for all finite sequences $\{(f_j, g_j)\}_{j=1}^n \subset \mathcal{F}$, where $c = c(q,p) > 0$ is $A_q(G)$-consistent. To see this, consider

$$\mathcal{F}_p := \left\{(F,G) = \left(\left(\sum_{j=1}^n f_j^p\right)^{1/p}, \left(\sum_{j=1}^n g_j^p\right)^{1/p}\right) : \{(f_j,g_j)\}_{j=1}^n \subset \mathcal{F}\right\},$$

and observe that Theorem 4.2 applied with q replaced by p gives for all $\nu \in A_p(G)$ and $(F,G) \in \mathcal{F}_p$

$$\|G\|^p_{L^p_\nu(G)} = \sum_{j=1}^n \int_G g_j^p \,\mathrm{d}\mu_\nu \le c\sum_{j=1}^n \int_G f_j^p \,\mathrm{d}\mu_\nu \le c\|F\|^p_{L^p_\nu(G)},$$

with an $A_p(G)$-consistent constant $c = c(p) > 0$. Thus, taking the pth-root, we obtain $\|G\|_{L^p_\nu(G)} \le c\|F\|_{L^p_\nu(G)}$ for all $(F,G) \in \mathcal{F}_p$. If we apply now Theorem 4.2 again, but this time with exponents $r = p$, $q = q$ and $\mathcal{F} = \mathcal{F}_p$, we obtain

$$\left\|\left(\sum_{j=1}^n g_j^p\right)^{1/p}\right\|_{L^q_\omega(G)} = \|G\|_{L^q_\omega(G)} \le c\|F\|_{L^q_\omega(G)} = c\left\|\left(\sum_{j=1}^n f_j^p\right)^{1/p}\right\|_{L^q_\omega(G)},$$

with an $A_q(G)$ consistent constant $c = c(q,p) > 0$.

5. $\mathcal{R}$-boundedness and Muckenhoupt weights

Definition 5.1. We call the sequence of functions $(r_j)_{j\in\mathbb{N}}$ defined via

$$r_j : [0,1] \to \{-1,1\},$$
$$r_j(t) := \operatorname{sgn}\,[\sin(2^{j-1}\pi t)],$$

the sequence of *Rademacher functions*.

Remark 5.2. Observe that the Rademacher functions are symmetric, independent, $\{-1,1\}$-valued random variables on the probability space $([0,1],\mathcal{B},\lambda)$. In fact, all arguments used in this section can be be transferred from Rademacher functions to symmetric, independent, $\{-1,1\}$-valued random variables on $[0,1]$ without any changes.

Definition 5.3. Let $\mathcal{X}$ be a Banach space. A subset $\mathcal{T} \subset \mathcal{L}(\mathcal{X})$ is called $\mathcal{R}$-bounded, if there exists a constant $c > 0$ such that

$$\int_0^1 \Big\| \sum_{j=1}^n r_j(t) T_j x_j \Big\|_{\mathcal{X}} \,\mathrm{d}t \le c \int_0^1 \Big\| \sum_{j=1}^n r_j(t) x_j \Big\|_{\mathcal{X}} \,\mathrm{d}t \tag{33}$$

for all $T_1,\dots,T_N \in \mathcal{T}$, $x_1,\dots,x_n \in \mathcal{X}$ and $n \in \mathbb{N}$. Here, $(r_j)_{j\in\mathbb{N}}$ is the sequence of Rademacher functions.
The smallest constant $c > 0$ such that (33) holds is called $\mathcal{R}$*-bound of* $\mathcal{T}$ and is denoted by $\mathcal{R}_1(\mathcal{T})$.

For $1 \le p < \infty$, we can replace the condition (33) in Definition 5.3 by

$$\int_0^1 \Big\| \sum_{j=1}^n r_j(t) T_j x_j \Big\|_{\mathcal{X}}^p \,\mathrm{d}t \le \mathcal{R}_p(\mathcal{T}) \int_0^1 \Big\| \sum_{j=1}^n r_j(t) x_j \Big\|_{\mathcal{X}}^p \,\mathrm{d}t, \tag{34}$$

due to the following lemma, which is known as *Kahane's inequality.*

Lemma 5.4. *Let $(r_j)_{j\in\mathbb{N}}$ be the sequence of Rademacher functions. Then there is a constant $k_p > 0$ such that for every Banach space $\mathcal{X}$ and for all $x_1,\dots x_n \in \mathcal{X}$*

$$\int_0^1 \Big\| \sum_{j=1}^n r_j(t) x_j \Big\|_{\mathcal{X}} \,\mathrm{d}t \le \left(\int_0^1 \Big\| \sum_{j=1}^n r_j(t) x_j \Big\|_{\mathcal{X}}^p \,\mathrm{d}t \right)^{\frac{1}{p}} \le k_p \int_0^1 \Big\| \sum_{j=1}^n r_j(t) x_j \Big\|_{\mathcal{X}} \,\mathrm{d}t.$$

Hence, (33) holds with a bound $\mathcal{R}_1(\mathcal{T}) := k_p \mathcal{R}_p(\mathcal{T})^{\frac{1}{p}}$ if (34) holds with a bound $\mathcal{R}_p(\mathcal{T})$, and (34) holds with a bound $\mathcal{R}_p(\mathcal{T}) := (k_p \mathcal{R}_1(\mathcal{T}))^p$ if (33) holds with a bound $\mathcal{R}_1(\mathcal{T})$.

Proof. See [6, Theorem 11.1]. □

In the particular case that $\mathcal{X}$ is an $L^q(X,\mu_X)$-space, where (X,μ_X) is a measure space, we can give a characterization of $\mathcal{R}$-boundedness that is much easier to handle. It relies on the following *Khinchin's inequality.*

Lemma 5.5. *Let $0 < q < \infty$ and $(r_j)_{j\in\mathbb{N}}$ be the sequence of Rademacher functions. Then there is a constant $c_q > 0$ such that*

$$c_q^{-1} \left(\sum_{j=1}^n |a_j|^2 \right)^{\frac{1}{2}} \le \left(\int_0^1 \Big| \sum_{j=1}^n r_j(t) a_j \Big|^q dt \right)^{\frac{1}{q}} \le c_q \left(\sum_{j=1}^n |a_j|^2 \right)^{\frac{1}{2}}, \tag{35}$$

for all $a_1,\dots,a_n \in \mathbb{C}$ and all $n \in \mathbb{N}$.

Proof. See [6, Theorem 1.10]. □

Proposition 5.6. *Let $(X, \mathcal{A}, \mu_X)$ be a measure space, $1 < q < \infty$ and write $\mathcal{X} := L^q(X, \mu_X)$. Then $\mathcal{T} \subset \mathcal{L}(\mathcal{X})$ is $\mathcal{R}$-bounded if and only if there is a constant $c > 0$ such that*

$$\Big\| \Big(\sum_{j=1}^n |T_j f_j|^2 \Big)^{\frac{1}{2}} \Big\|_{\mathcal{X}} \leq c \cdot \Big\| \Big(\sum_{j=1}^n |f_j|^2 \Big)^{\frac{1}{2}} \Big\|_{\mathcal{X}}, \tag{36}$$

for all $T_1, \ldots, T_n \in \mathcal{T}$, $f_1, \ldots, f_n \in \mathcal{X}$ and $n \in \mathbb{N}$.

Proof. See e.g. [9, Lemma 4.2]. □

Remark 5.7. If in the situation of Proposition 5.6 the constant c appearing in (36) is $A_q(G)$-consistent, then also the $\mathcal{R}$-bound of $\mathcal{T}$ is $A_q(G)$-consistent. Indeed, from the proof of Lemma 4.2 in [9] it is apparent that $\mathcal{R}_q(\mathcal{T}) = c_q^2 c$ is $A_q(G)$-consistent; here, c_q is the constant from Khinchin's inequality (35) which is independent of ω. But since $\mathcal{R}_1(\mathcal{T}) = k_q \mathcal{R}_q(\mathcal{T})^{1/q}$ by Lemma 5.4, and since k_q is independent of the underlying Banach space and therefore in particular $A_q(G)$-consistent, we see that $\mathcal{R}_1(\mathcal{T})$ is $A_q(G)$-consistent.

We can finally give the proof of our main theorem.

Proof of Theorem 1.5. We will choose

$$\mathcal{F} := \{(|f|, |Tf|) : f : G \to \mathbb{R} \text{ measurable }, T \in \mathcal{T}\}.$$

Then using the vector-valued extrapolation estimate in Remark 4.3 (ii) with $p = 2$, we obtain

$$\Big\| \Big(\sum_{j=1}^n |T_j f_j|^2 \Big)^{\frac{1}{2}} \Big\|_{L^q_\omega(G)} \leq c \Big\| \Big(\sum_{j=1}^n |f_j|^2 \Big)^{\frac{1}{2}} \Big\|_{L^q_\omega(G)},$$

for all $T_1, \ldots T_n \in \mathcal{T}$, $f_1, \ldots, f_n$ and all $n \in \mathbb{N}$. Hence, Proposition 5.6 yields the $\mathcal{R}$-boundedness of $\mathcal{T}$ and Remark 5.7 shows that the $\mathcal{R}$-bound is $A_r(G)$-consistent. □

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