A Quantified Mixed-Integer Program for a Booster Station

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1 Problem Statement

A *booster station* is a component in a fluid network which can cause a pressure increase. This component may be a single pump or rather a subnetwork of multiple pumps. If the pumps are not operated in sync and if at most one of these pumps is speed-controlled, the booster station can be regarded as a *black box* pump with a compound head curve. However, such a booster station is no improvement over a single pump with the same head curve. (A single large pump may even be more efficient due to scaling laws.)

Thus, the advantage of a booster station over a single pump consists in the freedom to deactivate individual pumps or to speed-control more than one pump, such that the active pumps may operate near their optimal working point. If the average demand of volume flow or pressure head is significantly smaller than the maximal demand, the system developer might therefore provide two or three smaller pumps in parallel or respectively serial connection.

The difficulty is that system developer has to make his decision about the system topology in advance. Therefore, the full task is: 'Find a valid flexible topology of pumps such that the investment costs plus the expected operating costs over a certain time are minimized'. Moreover, the pump system should work for nearly all possible load demands, even if the cost minimization is based on only a couple of some few predefined load scenarios.

2 Modeling Principles

The quantified model consists of two stages: First, find a low-priced investment decision in a adequate set of pumps and pipes (with optional digital valves). Second, activate a subset of these pumps and pipes for each quasistatic demand such that they satisfy the demand and operate near their optimal working point. The overall goal is to minimize the sum of investment costs and energy costs over the expected life cycle.

An interconnection of pumps can be abstracted as a directed graph $G = (V, E)$ with vertices V representing pumps and edges E representing pipes. To connect this pump subnetwork to the main network, we need two additional vertices representing plugs, namely a water supply s and a water outlet t, with corresponding pipes. Therefore, all possible topologies of a booster station can be modelled as a *complete* directed graph of all available pumps and two water plugs. Table 1 shows an example for a booster construction kit. The respective head curves are depicted in figure 1.

Table 1: Example of a booster construction kit

pump	speed-controlled	rotary speed min	rotary speed max
1	yes	730 rpm	2920 rpm
2	yes	350 rpm	1400 rpm
3	yes	725 rpm	2900 rpm

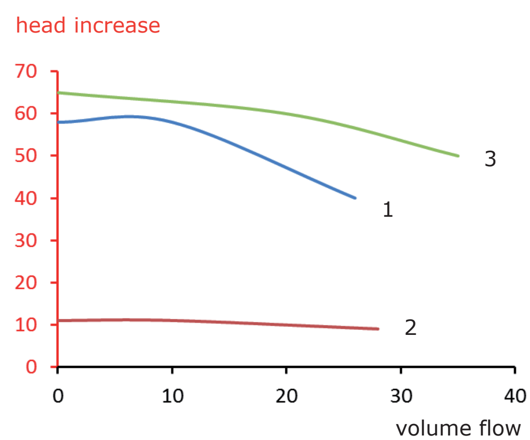

Figure 1: Head curves at maximum rotary speeds

The first-stage decision (i.e., the investment) is given by a set of binary variables (indicators) y_v for each vertex v and $y_{i,j}$ for each edge (i, j) of the graph. Of course, at least both plugs, one pump and two edges need to be selected for a reasonable booster station. In the objective function, theses indicators are weighed with the purchase costs of their respective components.

After the investment has been made, the demand of volume flow Q and head increase H^+ will change over the booster station's life time. For the sake of simplicity we present a quasistatic model, i.e., we identify similar demands to *demand scenarios* σ and give a discrete probability distribution. Table 2 shows an example demand profile. In this case, the booster station is almost always confronted with one of three representative scenarios - in arbitrary succession. The system must be able to satisfy each of these demands and the cost of operation can be estimated as a weighed sum of the cost per scenario with its respective probability. We thereby assume that switching between scenarios is fast compared to the contiguous operation phases and that it involves no significant cost.

Table 2: Example of a quasistatic demand distribution

scenario	probability	volume flow	head increase
1	50 %	25.0 m^3/h	30 m
2	25 %	50.0 m^3/h	30 m
3	25 %	25.0 m^3/h	60 m

A second-stage decision (i.e., the activation and operation) is dependent on the investment and on a specific scenario. It mainly consists of the following collection of variables:

component activation: a set of binary variables x_v for each vertex v and $x_{i,j}$ for each edge (i, j), which indicates if the respective pump is switched on or off and if the respective valve is open or closed.

fluid network: a set of three continuous variables $q_{i,j}$ for each edge (i, j) and h_v^s, h_v^t for each vertex v, indicating the volume flow through each pipe and the pressure head before and after each pump. (The absolute pressure head is arbitrary. We may fix the pressure head at the source to 0 m.)

pump operation: a set of four continuous variables q_v, h_v^+, p_v and n_v for each vertex v, indicating the volume flow through each pump or plug, its pressure increase, its power drain and its rotary speed.

Apart from these, we need several auxiliary variables, which will be introduced later in connection with their corresponding constraints.

Figure 2 shows an optimal solution to the example. The investment decision consists of 2 pumps and 5 pipes. Depending on the scenario, only one or both pumps are running. When the pumps are operated in parallel, the booster station can satisfy a high volume flow and when the pumps are configured serially, they achieve a high head increase. The optimality of the solution implies, that each cheaper investment either cannot satisfy all scenarios or has much higher expected operational costs, and that no more expensive investment can be justified by sufficiently smaller operational costs.

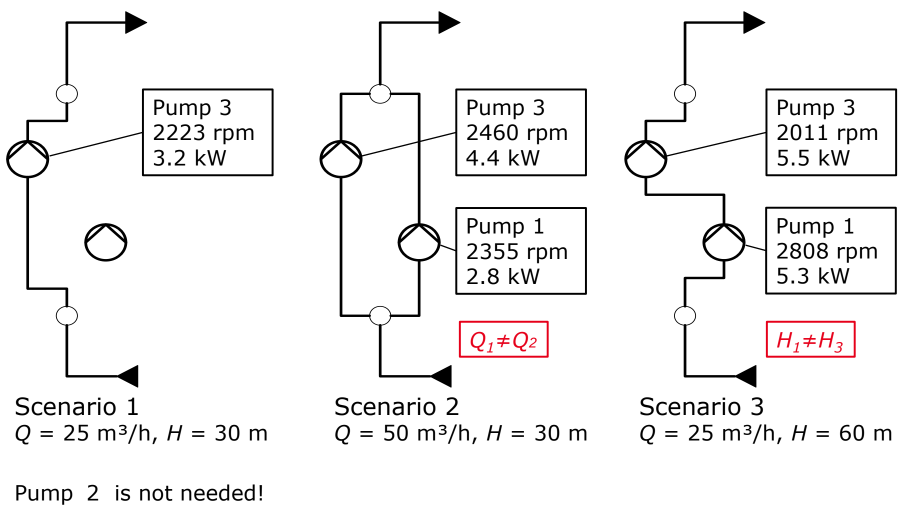

Figure 2: Depiction of an optimal solution

Investment Constraints
Water supply and water outlet are necessary.

$$y_s = 1, \; y_t = 1 \tag{1}$$

There must be at most one pipe between two pumps.

$$\forall i, j : \quad y_{i,j} + y_{j,i} <= 1 \tag{2}$$

Activation Constraints
Components may only be activated, if they have been purchased.

$$\forall v : x_v <= y_v \quad \text{and} \quad \forall i, j : \quad x_{i,j} <= y_{i,j} \tag{3}$$

Fluid Network Constraints
The demand must be satisfied in each scenario. Depending on piecewise linear

approximations in later parts of the model, it might be reasonable to soften these equations to inequalities with a small gap.

$$q_s = Q_\sigma,\ q_t = Q_\sigma \quad \text{and} \quad h_s^t = 0,\ h_t^s = H_\sigma^+ \tag{4}$$

The volume flow has to satisfy the continuity equation, i.e., volume flow is preserved.

$$\forall v: \quad \sum_i q_{i,v} = q_v \quad \text{and} \quad \forall v: \quad q_v = \sum_j q_{v,j} \tag{5}$$

Pressure head propagates through the network. Note that for simplicity, we leave out pressures losses.

$$\forall v: \quad h_v^s + h_v^+ = h_v^t \quad \text{and} \quad \forall i,j: \quad h_i^t = h_j^s \tag{6}$$

Pump Operation Constraints
The pump operation variables, i.e., volume flow, pressure increase, power drain and rotary speed, are coupled by head curves as in figure 1. In general, these are nonlinear relations which cannot be modelled exactly in a linear program. Instead, they are modeled by piecewise linear approximations, i.e., by interpolating or approximating linear splines. An overview of several possible formulations is given in[1]. Therefore, we need to introduce auxiliary variables at each vertex, e.g. some binary variables ζ to select a part of the spline and continuous variables λ to interpolate between two nodes. Depending on the chosen formulations, they are coupled by some generic constraints.

3 Adding Feasibility Robustness with the help of Quantifiers

Up to now, we discussed a two-stage optimization model with a handful of scenarios. In principle, we can approach this problem with standard *branch and cut solvers* by formulating and solving the deterministic equivalent program (DEP). That is, we duplicate the second-stage variables and constraints for each scenario, which results in a large mixed-integer linear program with a block-ladder structure. Thus, building the DEP is no reasonable solution approach if the number of scenarios becomes huge.

In practice, we need to analyse more than a couple of scenarios. In the given example, the model guarantees that all three scenarios can be satisfied - however, we do not know if every demand *in between* can also be fulfilled. Those demands may be rare, but it nevertheless is not acceptable that the booster station fails for any reasonable demand!

To our best knowledge, there are – to date – no algorithms to solve average-case two-stage mixed-integer linear programs with many scenarios exact *and*

[1] Juan Pablo Vielma, Shabbir Ahmed, George Nemhauser. Mixed-Integer Models for Non-separable Piecewise-Linear Optimization: Unifying Framework and Extensions. OPERATIONS RESEARCH Vol. 58, No. 2, March–April 2010, pp. 303–315

fast. However, there are promising solution techniques for their worst-case counterparts. We therefore propose a mixed-quantified model, which finds the best-priced booster station with respect to a handful of frequent scenarios, out of all booster stations that are simultaneously guaranteed to satisfy a huge number of infrequent scenarios α (whatever the operational costs). The proposed model takes the following schematic form:

$$\begin{aligned} &\min\,(\text{ investment costs} + E_{\text{frequent scenarios}}(\text{ operational costs })\,) \\ &\text{s.t.} \quad \exists\,\text{investment decision} \quad \forall\,\text{scenarios} \quad \exists\,\text{operational solution} \end{aligned}$$

To load this problem with a worst-case two-stage solver, we have to expand the expectation value in the objective function to its deterministic equivalent form. The size of the program grows by a factor of e.g. 5 or 10, but the huge number of infrequent scenarios can remain in the two-stage model and take full advantage of two-stage solution techniques. In the following complete example, let σ denote the frequent scenarios with probability P_σ and associated cost variable c_σ, and let the set b_k be the binary encoding and u_n be the unary encoding of the infrequent scenarios β. Then, the worst-case two-stage model can be written as follows:

$$\begin{aligned} &\min \left(C^\top(y_v,\, y_{i,j}) + \sum_\sigma (\, P_\sigma \cdot c_\sigma\,) \right) \\ &\text{s.t.} \quad \exists\, y_v,\, y_{i,j} \quad \forall\, b_k \quad \exists\, u_n,\, \tilde{q},\, \tilde{h}^+,\, x_v,\, x_{i,j},\, \ldots \end{aligned}$$

$$(1), \quad (2), \quad (3)$$

$$q_s = \tilde{q},\ q_t = \tilde{q} \quad \text{and} \quad h_s^t = 0,\ h_t^s = \tilde{h}^+ \tag{4'}$$

$$(5), \quad (6)$$

$$\sum_k 2^k \cdot b_k = \sum_n n \cdot u_n \tag{7}$$

$$1 = \sum_n u_n \tag{8}$$

$$\tilde{q} = \sum_n Q_n \cdot u_n \tag{9}$$

$$\tilde{h}^+ = \sum_n H_n^+ \cdot u_n \tag{10}$$