Very weak solutions and the Fujita-Kato approach to the Navier-Stokes system in general unbounded domains

Reinhard Farwig* and Paul Felix Riechwald†

We consider the instationary Navier-Stokes system in general unbounded domains $\Omega \subset \mathbb{R}^n$, $n \geq 3$, with smooth boundary and construct by the Fujita-Kato method mild solutions $u \in L^\infty(0,T;\tilde{L}^n(\Omega))$ with initial value $u_0 \in \tilde{L}^n(\Omega)$. Here the classical $L^n(\Omega)$–space is replaced by $\tilde{L}^n(\Omega)$ defined as $L^q \cap L^2$ when $q \geq 2$ but as $L^q + L^2$ when $1 < q < 2$. Moreover, for suitable initial values we identify mild solutions in $L^\infty(0,T;\tilde{L}^n(\Omega))$ with very weak solutions in Serrin's class $L^r(0,T;\tilde{L}^q(\Omega))$ where $\frac{2}{r} + \frac{n}{q} = 1$, $2 < r < \infty$.

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1 Introduction

We consider the instationary Navier-Stokes system

$$\begin{aligned} u_t - \Delta u + \operatorname{div}(u \otimes u) + \nabla p &= f && \text{in} \quad (0,T) \times \Omega, \\ \operatorname{div} u &= 0 && \text{in} \quad (0,T) \times \Omega, \\ u &= 0 && \text{on} \quad (0,T) \times \partial\Omega, \\ u(0) &= u_0 && \text{at} \quad t = 0, \end{aligned} \tag{1.1}$$

in a general unbounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 3$, with uniform C^2-boundary and a finite time interval $(0,T)$. Here $u = (u_1, \dots, u_n)$ denotes the unknown velocity field, p an associated pressure, f a given external force, and u_0 the initial value of

*R. Farwig, Fachbereich Mathematik, Technische Universität Darmstadt *and* International Research Training Group (IRTG 1529) Darmstadt-Tokyo, 64289 Darmstadt, Germany, farwig@mathematik.tu-darmstadt.de

†P.F. Riechwald, Fachbereich Mathematik, Technische Universität Darmstadt, 64289 Darmstadt, Germany, felix.riechwald@gmx.de

u at time $t = 0$. For simplicity, the viscosity is set to $\nu = 1$. A precise definition of domains with uniform C^k-boundary can be found in Definition 2.1 below.

A problem in this setting is the unboundedness of the underlying domain Ω. Due to counter-examples by M.E. Bogovskij and V.N. Maslennikova [6, 7] the Helmholtz decomposition of vector fields in $L^q(\Omega)$, $1 < q < \infty$, on an unbounded smooth domain may fail unless $q = 2$. By analogy, a bounded Helmholtz projection P_q with the properties required to define the Stokes operator $A_q = -P_q\Delta$ when $q \neq 2$ may not exist. Therefore, in [9, 11, 12, 13, 14] H. Kozono, H. Sohr and the first author of this article introduced the spaces

$$\tilde{L}^q(\Omega) := \begin{cases} L^q(\Omega) + L^2(\Omega), & \text{if } 1 \le q < 2, \\ L^q(\Omega) \cap L^2(\Omega), & \text{if } 2 \le q \le \infty. \end{cases} \tag{1.2}$$

The corresponding norm is defined as $\|u\|_{\tilde{L}^q} = \max\{\|u\|_q, \|u\|_2\}$ when $q \ge 2$, and as $\inf\{\|u_1\|_q + \|u_2\|_2 : u = u_1 + u_2, u_1 \in L^q(\Omega), u_2 \in L^2(\Omega)\}$. For bounded domains we have that $\tilde{L}^q(\Omega) = L^q(\Omega)$ with equivalent norms. We note that functions in $\tilde{L}^q(\Omega)$ locally behave like L^q-functions, but globally exploit L^2-properties. By analogy, function spaces like $\tilde{L}^q_\sigma(\Omega)$ of solenoidal vector fields and $\tilde{W}^{k,q}(\Omega)$ of weakly differentiable functions will be defined.

As shown in [11] a Helmholtz projection $\tilde{P}_q : \tilde{L}^q(\Omega)^n \to \tilde{L}^q_\sigma(\Omega)$ is well defined, allowing to define a closed Stokes operator $\tilde{A}_q = -\tilde{P}_q\Delta$ with domain $\tilde{D}^1_q = \tilde{W}^{2,q}(\Omega) \cap \tilde{W}^{1,q}_0(\Omega) \cap \tilde{L}^q_\sigma(\Omega)$ dense in $\tilde{L}^q_\sigma(\Omega)$. The operator $\tilde{A}_q$ has similar properties as the usual Stokes operator A_q, generates an analytic semigroup $e^{-t\tilde{A}_q}$, $t \ge 0$, enjoys the property of bounded imaginary powers and maximal regularity; for details and further properties of these function spaces and operators we refer to [9, 11, 12, 13, 14] and [22], [24] as well as to Sect. 2.

In this article we are looking for mild solutions $u \in L^\infty(0, T; \tilde{L}^n_\sigma(\Omega))$ to be constructed by the well-known Fujita-Kato method of successive approximation, see H. Fujita and T. Kato [17] working in $\mathcal{D}(A_2^{1/4}) \subset L^3_\sigma(\Omega)$ when Ω is bounded, T. Kato [21] for the case $\Omega = \mathbb{R}^3$ and Y. Giga [19] for a more general and abstract approach. An exposition of the method for the whole space $\mathbb{R}^n$ based on harmonic analysis can be found in the monograph [5]. The Fujita-Kato method is strongly based on the variation of constants formula (1.3) in the following definition.

Definition 1.1. *Let $0 < T < \infty$ and $\Omega \subseteq \mathbb{R}^n$, $n \ge 3$, be a uniform C^2-domain. Assume that an initial velocity $u_0 \in \tilde{L}^n_\sigma(\Omega)$ is given. Then $u \in L^\infty(0, T; \tilde{L}^n_\sigma(\Omega))$ with $t^{1/2}\nabla u(t) \in L^\infty(0, T; \tilde{L}^n(\Omega))$ is called mild solution to the Navier-Stokes system with initial velocity u_0 if it solves the integral equation*

$$u(t) = e^{-t\tilde{A}_n}u_0 - \int_0^t e^{-(t-s)\tilde{A}_{n/2}}\tilde{P}_{n/2}(u(s)\cdot\nabla u(s))\,ds \tag{1.3}$$

for almost all $0 \le t < T$.

The integral equation is to be read as equation in $\tilde{L}^n_\sigma(\Omega)$. By $\tilde{L}^r$–$\tilde{L}^q$-estimates of the Stokes semigroup, see (2.8), (2.9) below, it can be verified that the integral term on the right hand side of (1.3) is well defined and contained in $\tilde{L}^n_\sigma(\Omega)$.

The main result of this article reads as follows:

Theorem 1.2 (Existence of Mild Solutions)**.** *Let $0 < T < \infty$, a uniform C^2-domain $\Omega \subseteq \mathbb{R}^n$ and an initial velocity $u_0 \in \tilde{L}^n_\sigma(\Omega)$ be given. Choose $n < q < \infty$. Then there is a constant $\gamma = \gamma(\tau(\Omega), q) > 0$ such that the condition*

$$\sup_{0\le t<T} t^{(1-n/q)/2}\|e^{-t\tilde{A}_n}u_0\|_{\tilde{L}^q(\Omega)} + \sup_{0\le t<T} t^{1/2}\|\nabla e^{-t\tilde{A}_n}u_0\|_{\tilde{L}^n(\Omega)} \le \gamma \tag{1.4}$$

implies the existence of a mild solution u to the Navier-Stokes system with initial velocity u_0. For the term $\tau(\Omega)$ we refer to Definition 2.1.

Remark 1.3. *Note that the smallness condition in Theorem 1.2 is satisfied if $\|u_0\|_{\tilde{L}^n}$ is small. Moreover, even for arbitrarily large initial values, we can achieve the smallness condition by choosing a smaller time interval, i.e. choosing a new $T > 0$. This follows directly from Lemma 3.1 below.*

Concerning uniqueness we have the following result:

Theorem 1.4 (Uniqueness)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be a uniform C^2-domain, let $0 < T < \infty$ and $u_0 \in \tilde{L}^n_\sigma(\Omega)$. Let v and w be mild solutions to the Navier-Stokes equations with the initial datum u_0. Furthermore, assume that*

- *$t^{(1-n/q)/2}\|v(t)\|_{\tilde{L}^q(\Omega)} \to 0$, as $t \to 0+$, for some $n < q < \infty$, and*
- *$t^{1/2}\|\nabla w(t)\|_{\tilde{L}^n(\Omega)} \to 0$, as $t \to 0+$.*

Then there exists $0 < T_* \le T$ such that $v = w$ on $[0, T_*)$.

Mild solutions $u \in L^\infty(0,T;\tilde{L}^n_\sigma(\Omega))$ are solutions in the sense of distributions which are not necessarily *weak solutions* in the sense of Leray-Hopf with finite kinetic energy ($u \in L^\infty(0,T;L^2(\Omega))$) and finite dissipation integral ($\nabla u \in L^2(0,T;L^2(\Omega))$), see [20, 23, 28]. These mild solutions determine a limiting case of the so-called *very weak solutions* contained in a Serrin class $L^r(0,T;\tilde{L}^q_\sigma(\Omega))$ where $2 < r < \infty$, $n < q < \infty$ and $\frac{2}{r} + \frac{n}{q} = 1$.

The concept of very weak solutions was discussed e.g. by H. Amann [2, 3, 4] in the setting of Besov spaces, by G.P. Galdi, H. Kozono, C. Simader, H. Sohr and the first author of this paper in classical L^q-spaces ([8, 10] and [18]), and in weighted Lebesgue and Bessel potential spaces using arbitrary Muckenhoupt weights, see the work of K. Schumacher ([25, 26, 27]). The adaptation to smooth unbounded domains was performed by the authors in [15]. For a precise definition and more properties especially for general unbounded domains we refer to [15] and to Sect. 4 below.

The theory of very weak solutions is strongly based on duality arguments which are not feasible in L^∞–spaces and for mild solutions. This is one of the reasons of introducing the Fujita-Kato iteration method. Nevertheless, mild and and very weak solutions are strongly related to each other and do coincide under slightly stronger conditions on the initial datum.

Theorem 1.5. *Let $\Omega \subseteq \mathbb{R}^n$ be a C^2-domain, $n \geq 3$, and $0 < T < \infty$. Assume Serrin exponents $n \leq r \leq 2n$ and $n < q < 2n$ are given. Let $u_0 \in \tilde{L}^n_\sigma(\Omega)$ be an initial velocity.*

(i) A very weak solution $u \in L^r(0,T;\tilde{L}^q_\sigma(\Omega))$ to the Navier-Stokes system with $u(0) = u_0$ also belongs to the space $L^\infty(0,T;\tilde{L}^n_\sigma(\Omega))$.

(ii) Let $u \in L^\infty(0,T;\tilde{L}^n_\sigma(\Omega))$ be the solution constructed in Theorem 1.2. Then, for some $0 < T_* \leq T$, the solution u is contained in $L^r(0,T_*;\tilde{L}^q_\sigma(\Omega))$ and is a very weak solution to the Navier-Stokes equations on $[0,T_*)$.

For statements more precise than in Theorem 1.5 we refer to Theorems 4.1 and 4.2 in Sect. 4. In the following Sect. 2 we discuss the spaces $\tilde{L}^q(\Omega)$ and related spaces of Lorentz, Sobolev and Bochner type as well as properties of the Stokes operator $\tilde{A}_q$. Moreover, several results on very weak solutions are summarized in Sect. 2. Complete proofs of Theorems 1.2 and 1.4 as well as further results on mild solutions can be found in Sect. 3.

2 Preliminaries

Definition 2.1. *A domain $\Omega \subset \mathbb{R}^n$ is called* uniform C^k-domain, *$k \in \mathbb{N}_0$, if there are constants $\alpha, \beta, K > 0$ such that for all $x_0 \in \partial\Omega$ there exist - after an orthogonal and an affine coordinate transform - a function h on the closed ball $\overline{B'_\alpha(0)} \subseteq \mathbb{R}^{n-1}$ of class C^k and a neighborhood $U_{\alpha,\beta,h}(x_0)$ of x_0 with the following properties: $\|h\|_{C^k} \leq K$ and $h(0) = 0$ and, if $k \geq 1$, $h'(0) = 0$; moreover,*

$$\begin{aligned}
U_{\alpha,\beta,h}(x_0) &:= \{(y', y_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \colon |y'| < \alpha, |h(y') - y_n| < \beta\}, \\
U^-_{\alpha,\beta,h}(x_0) &:= \{(y', y_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \colon |y'| < \alpha, h(y') - \beta < y_n < h(y')\} \\
&= \Omega \cap U_{\alpha,\beta,h}(x_0), \\
\partial\Omega \cap U_{\alpha,\beta,h}(x_0) &= \{(y', y_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \colon h(y') = y_n\}.
\end{aligned}$$

The triple (α, β, K) is called the type *of Ω and will be denoted by $\tau(\Omega) = (\alpha, \beta, K)$.*

For a constant C in some estimate we will write $C = C(\tau(\Omega))$ if it does depend only on α, β and K, but in no other way on Ω.

Note that bounded and exterior domains are uniform C^k-domains as long as the boundary is smooth enough.

For spaces of Sobolev-type we proceed analogously to the definition in (1.2): For $k \in \mathbb{N}$ and $1 \leq q \leq \infty$ we let

$$\tilde{W}^{k,q}(\Omega) := \begin{cases} W^{k,2}(\Omega) + W^{k,q}(\Omega), & 1 \leq q < 2, \\ W^{k,2}(\Omega) \cap W^{k,q}(\Omega), & 2 \leq q \leq \infty. \end{cases} \tag{2.1}$$

Similarly, we define the spaces $\tilde{W}_0^{1,q}(\Omega)$, $1 < q < 2$ and $2 \leq q < \infty$, based on the classical Sobolev spaces $W_0^{1,q}(\Omega)$ and $W_0^{1,2}(\Omega)$.

The $\tilde{L}^q$- and $\tilde{W}^{k,q}(\Omega)$-spaces have the following properties; for a proof see [24]:

- Let $1 \leq q < r \leq \infty$. Then $(\tilde{L}^q(\Omega))^* = \tilde{L}^{q'}(\Omega)$ and $\|u\|_{\tilde{L}^q} \leq \|u\|_{\tilde{L}^r}$.
- Let $1 \leq r, p, q \leq \infty$, $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ and let $u \in \tilde{L}^p$, $v \in \tilde{L}^q$. Then $uv \in \tilde{L}^r$ and $\|uv\|_{\tilde{L}^r} \leq \|u\|_{\tilde{L}^p} \|v\|_{\tilde{L}^q}$.
- Let $m \in \mathbb{N}$, $1 \leq q < \infty$ and $\Omega \subseteq \mathbb{R}^n$ be a uniform C^2-domain. Then

  $$\tilde{W}^{m,q}(\Omega) \hookrightarrow \tilde{L}^r(\Omega)$$

  if either $q \leq r \leq \infty$ and $mq > n$, or $q \leq r < \infty$ and $mq = n$, or $q \leq r \leq \frac{nq}{n-mq}$ and $mq < n$.

Concerning the Helmholtz projection on $\tilde{L}^q(\Omega)$ for a domain $\Omega \subseteq \mathbb{R}^n$ of uniform type C^1 we have the following result, see [11]. We define

$$\tilde{L}^q_\sigma(\Omega) := \begin{cases} L^q_\sigma(\Omega) + L^2_\sigma(\Omega), & 1 < q < 2 \\ L^q_\sigma(\Omega) \cap L^2_\sigma(\Omega), & 2 \leq q < \infty \end{cases}, \tag{2.2}$$

equipped with the norm of $\tilde{L}^q(\Omega)$, and gradient spaces by

$$\tilde{G}_q(\Omega) := \begin{cases} G_q(\Omega) + G_2(\Omega), & 1 < q < 2, \\ G_q(\Omega) \cap G_2(\Omega), & 2 \leq q < \infty, \end{cases} \tag{2.3}$$

which are based on the gradient spaces $G^r(\Omega) = \{\nabla p \in L^r(\Omega) : p \in L^r_{\text{loc}}(\Omega)\}$. The norm in $\tilde{G}_q(\Omega)$ is denoted by $\|\cdot\|_{\tilde{G}_q(\Omega)} := \|\cdot\|_{\tilde{L}^q(\Omega)}$. The dual space of $\tilde{G}_{q'}(\Omega)$ is denoted by $\tilde{G}_q^{-1}(\Omega)$.

The space $\tilde{L}^q(\Omega)$ admits the direct algebraic and topological decomposition

$$\tilde{L}^q(\Omega) = \tilde{L}^q_\sigma(\Omega) \oplus \tilde{G}_q(\Omega).$$

The corresponding projection $\tilde{P}_q$ from $\tilde{L}^q(\Omega)$ onto its range $\tilde{L}^q_\sigma(\Omega)$ and with kernel $\tilde{G}_q(\Omega)$ has a norm bounded by a constant $c = c(q, \tau(\Omega))$. We have the duality relations $(\tilde{P}_q)^* = \tilde{P}_{q'}$ and $\tilde{L}^q(\Omega)^* = \tilde{L}^{q'}(\Omega)$. Using the Helmholtz projection $\tilde{P}_q$

we define the Stokes operator $\tilde{A}_q$, $1 < q < \infty$, for a uniform C^2-domain $\Omega \subseteq \mathbb{R}^n$ with domain

$$\mathcal{D}(\tilde{A}_q) := \begin{cases} \mathcal{D}_q + \mathcal{D}_2, & 1 < q < 2, \\ \mathcal{D}_q \cap \mathcal{D}_2, & 2 \le q < \infty, \end{cases} \tag{2.4}$$

where $\mathcal{D}_q := L^q_\sigma(\Omega) \cap W^{1,q}_0(\Omega) \cap W^{2,q}(\Omega)$. Then $\tilde{A}_q : \mathcal{D}(\tilde{A}_q) \subseteq \tilde{L}^q_\sigma(\Omega) \to \tilde{L}^q_\sigma(\Omega)$ is defined by $\tilde{A}_q u := -\tilde{P}_q \Delta u$ and has the following properties, see [14]:

- $\tilde{A}_q$ is a densely defined closed operator in $\tilde{L}^q_\sigma(\Omega)$ satisfying $\big(\tilde{A}_q\big)^* = \tilde{A}_{q'}$.
- $\tilde{A}_q$ generates an analytic semigroup $e^{-t\tilde{A}_q}$, $t \ge 0$, having the bound

  $$\|e^{-t\tilde{A}_q} f\|_{\tilde{L}^q(\Omega)} \le C e^{\delta t} \|f\|_{\tilde{L}^q(\Omega)}$$

  for all $f \in \tilde{L}^q_\sigma(\Omega)$ and $t \ge 0$ with a constant $C = C(q, \delta, \tau(\Omega))$, $\delta > 0$.

It is unknown whether the usual resolvent estimate for the infinitesimal generator $\tilde{A}_q$ of the analytic semigroup $e^{-t\tilde{A}_q}$ holds uniformly in the resolvent parameter $\lambda \in \mathbb{C}$ as $|\lambda| \to 0$. Therefore, the semigroup may increase exponentially fast and the maximal regularity estimate in Theorem 2.2 below is stated only for finite time intervals. For the same reason, the operator $\tilde{A}_q$ has often to be replaced by $I + \tilde{A}_q$ in the following.

Note that from time to time we will omit the symbols Ω and $0, T$ for domain and time interval, respectively, when this data is known from the context.

Theorem 2.2. ([13, Theorem 1.4]) *Let $\Omega \subseteq \mathbb{R}^n$ be a uniform C^2-domain and let $1 < r, q < \infty$, $0 < T < \infty$. Given an external force $f \in L^r(0,T; \tilde{L}^q_\sigma(\Omega))$ and an initial value $u_0 \in \mathcal{D}(\tilde{A}_q)$ (for simplicity) there exists a unique vector field $u \in L^r(0,T; \mathcal{D}(\tilde{A}_q)) \cap W^{1,r}(0,T; \tilde{L}^q_\sigma(\Omega))$ solving the Cauchy problem*

$$u_t + \tilde{A}_q u = f, \quad u(0) = u_0.$$

It can be represented by the variation of constants formula

$$u(t) = e^{-t\tilde{A}_q} u_0 + \int_0^t e^{-(t-\tau)\tilde{A}_q} f(\tau)\, d\tau \quad \text{for a.a. } 0 \le t \le T$$

and satisfies the maximal regularity estimate

$$\|u\|_{L^r(0,T;\mathcal{D}(\tilde{A}_q))} + \|u_t\|_{L^r(0,T;\tilde{L}^q)} \le C\big(\|u_0\|_{\mathcal{D}(\tilde{A}_q)} + \|f\|_{L^r(0,T;\tilde{L}^q)}\big)$$

with a constant $C = C(q, r, T, \tau(\Omega)) > 0$.

A further crucial property of the Stokes operator is the fact that $1+\tilde{A}_q$ admits bounded imaginary powers, see [22]. Hence complex interpolation methods can be used to describe domains of fractional powers $(1 + \tilde{A}_q)^\alpha$, $-1 \le \alpha \le 1$. To be more precise, for $0 \le \alpha \le 1$ let the domain of the fractional power $(1+\tilde{A}_q)^\alpha$ be denoted by

$$\tilde{D}_q^\alpha = \tilde{D}_q^\alpha(\Omega) = \mathcal{D}((1+\tilde{A}_q)^\alpha), \tag{2.5}$$

equipped with the norm $\|(1+\tilde{A}_q)^\alpha \cdot \|_{\tilde{L}^q}$. If $-1 \le \alpha < 0$ define $\tilde{D}_q^\alpha$ as the completion of $\tilde{L}^q_\sigma(\Omega)$ in the norm $\|(1+\tilde{A}_q)^\alpha \cdot \|_{\tilde{L}^q}$. These spaces are reflexive and satisfy the duality relation $(\tilde{D}_q^\alpha)^* \cong \tilde{D}_{q'}^{-\alpha}$. As special case we get that

$$\tilde{D}_q^{1/2} = \tilde{W}_0^{1,q}(\Omega) \cap \tilde{L}^q_\sigma(\Omega) \quad \text{with norm } \ \|(1+\tilde{A}_q)^{1/2} \cdot \|_{\tilde{L}^q} \sim \| \cdot \|_{\tilde{W}^{1,q}(\Omega)}.$$

Moreover, for $-1 \le \alpha \le \beta \le 1$, the operator $(1+\tilde{A}_q)^{\beta-\alpha}$ is an isomorphism between $\tilde{D}_q^\beta$ and $\tilde{D}_q^\alpha$. Finally,

$$\left[\tilde{D}_q^\alpha, \tilde{D}_q^\beta\right]_\theta = \tilde{D}_q^\gamma, \tag{2.6}$$

when $-1 \le \alpha \le \beta \le 1$ and $(1-\theta)\alpha + \theta\beta = \gamma$, $\theta \in (0,1)$. This result implies the following embedding and decay estimates ([24, Proposition 3, Theorem 1]):

Let $n \ge 3$, $1 < q \le r < \infty$, and $\alpha := \frac{n}{2}\left(\frac{1}{q} - \frac{1}{r}\right) \ge 0$. Then

$$\|u\|_{\tilde{L}^r(\Omega)} \le C\|(1+\tilde{A}_q)^\alpha u\|_{\tilde{L}^q(\Omega)}, \quad 0 \le \alpha \le 1, \tag{2.7}$$

for all $u \in \tilde{D}_q^\alpha$ and a constant $C = C(\tau(\Omega), q, \alpha)$. Moreover,

$$\left\|e^{-t\tilde{A}_r} f\right\|_{\tilde{L}^r(\Omega)} \le Ce^{\delta t}(1+t)^\alpha t^{-\alpha} \|f\|_{\tilde{L}^q(\Omega)}, \tag{2.8}$$

$$\left\|\nabla e^{-t\tilde{A}_r} f\right\|_{\tilde{L}^r(\Omega)} \le Ce^{\delta t}(1+t)^{\alpha+\frac{1}{2}} t^{-\alpha-\frac{1}{2}} \|f\|_{\tilde{L}^q(\Omega)} \tag{2.9}$$

$$\left\|e^{-t\tilde{A}_r} \tilde{P}_r \operatorname{div} F\right\|_{\tilde{L}^r(\Omega)} \le Ce^{\delta t}(1+t)^{\alpha+\frac{1}{2}} t^{-\alpha-\frac{1}{2}} \|F\|_{\tilde{L}^q(\Omega)} \tag{2.10}$$

for every $f \in \tilde{L}^q_\sigma(\Omega)$ and matrix-valued field $F \in \tilde{L}^q(\Omega)$, respectively, for any $t > 0$ and $\delta > 0$; here $C = C(\tau(\Omega), r, q, \delta) > 0$. Note that in (2.10) the operator $e^{-t\tilde{A}_r}\tilde{P}_r \operatorname{div}$ must be defined by duality to the operator $\nabla e^{-t\tilde{A}_{r'}}$ in (2.9).

For later use we need Lorentz spaces over $\tilde{L}^q$ and their solenoidal subspaces. Fist we define for $1 \le q, \rho \le \infty$ the Lorentz spaces

$$\tilde{L}^{q,\rho}(\Omega) := \begin{cases} L^{q,\rho}(\Omega) + L^2(\Omega), & q < 2, \\ L^{q,\rho}(\Omega) \cap L^2(\Omega), & q > 2, \end{cases}$$

letting the case $q = 2$ undefined; here $L^{q,\rho}(\Omega)$ denotes a usual Lorentz space, cf. [29, Ch. 1.18.6]. From [24, Theorem 3] we recall that for $1 \le q, r, s \le \infty$ with $r \ne q$, $s \ne 2$, satisfying $\frac{1}{s} = \frac{1-\theta}{q} + \frac{\theta}{r}$ with some $0 < \theta < 1$, and for $1 \le \rho \le \infty$

$$(\tilde{L}^q(\Omega), \tilde{L}^r(\Omega))_{\theta,\rho} = \tilde{L}^{s,\rho}(\Omega).$$

Next we define for $1 < q < \infty$, $q \neq 2$, and $1 \leq \rho < \infty$ the spaces

$$\tilde{L}^{q,\rho}_\sigma(\Omega) := \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_{\tilde{L}^{q,\rho}(\Omega)}}.$$

Then, by [24, Corollary 1], $1 < q, r, s < \infty$ with $r \neq q$, $s \neq 2$, satisfying $\frac{1}{s} = \frac{1-\theta}{q} + \frac{\theta}{r}$ with some $0 < \theta < 1$, and for $1 \leq \rho < \infty$ we get that

$$(\tilde{L}^q_\sigma(\Omega), \tilde{L}^r_\sigma(\Omega))_{\theta,\rho} = \tilde{L}^{s,\rho}_\sigma(\Omega). \tag{2.11}$$

Finally, we provide some details on the theory of very weak solutions in the context of general unbounded smooth domains using the spaces $\tilde{L}^q(\Omega)$. For more details we refer to [15] where a generalized Navier-Stokes system with external force $\mathcal{F}$ (and even nonzero divergence and nonzero boundary values) has been considered. The abstract external force field $\mathcal{F}$ in [15] combines the initial value u_0 and an external force f as follows:

$$\langle \mathcal{F}, \phi \rangle = (u_0, \phi(0)) + (f, \phi)_{T,\Omega}. \tag{2.12}$$

This point of view is advantageous to interpret the theory of very weak solutions as a problem dual to the theory of strong (regular) solutions. In (2.12) the brackets $(\cdot,\cdot)_{T,\Omega}$ denote the usual duality product for functions on $\Omega \times (0,T)$ whereas $(\cdot,\cdot)$ denotes the corresponding duality product on Ω.

Definition 2.3. *Let $\Omega \subseteq \mathbb{R}^n$ be a uniform C^2-domain, $0 < T < \infty$ and $2 < r < \infty$, $n < q < \infty$ and $2/r + n/q = 1$.*

(i) The test function space of very weak solutions is defined as

$$\mathcal{T}^{1,r',q'}(T,\Omega) := \{\phi \in L^{r'}(0,T;\tilde{D}^1_{q'}) \cap W^{1,r'}(0,T;\tilde{L}^{q'}(\Omega)) \colon \phi(T) = 0\}$$

and equipped with the norm

$$\|\phi\|_{\mathcal{T}^{1,r',q'}(T,\Omega)} := \|\phi_t\|_{L^{r'}(0,T;\tilde{L}^{q'}(\Omega))} + \|\phi\|_{L^{r'}(0,T;\tilde{D}^1_{q'})}.$$

The set of bounded functionals on $\mathcal{T}^{1,r',q'}(T,\Omega)$ is denoted by $\mathcal{T}^{-1,r,q}(T,\Omega)$.

(ii) For an external force $\mathcal{F} \in \mathcal{T}^{-1,r,q}(T,\Omega)$ we call $u \in L^r(0,T;\tilde{L}^q(\Omega))$ a very weak solution to the Navier-Stokes system *with data $\mathcal{F}$ if the conditions*

$$\begin{aligned} -(u,\phi_t)_{T,\Omega} - (u,\Delta\phi)_{T,\Omega} - (u \otimes u, \nabla\phi)_{T,\Omega} &= \langle \mathcal{F}, \phi \rangle, \\ (u, \nabla\psi)_{T,\Omega} &= 0 \end{aligned} \tag{2.13}$$

hold for all test functions $\phi \in \mathcal{T}^{1,r',q'}(T,\Omega)$ and $\nabla\psi \in L^{r'}(0,T;\tilde{L}^{q'}(\Omega))$.

(iii) Concerning the linear nonstationary Stokes system *the nonlinear term $(u \otimes u, \nabla\phi)_{T,\Omega}$ in (2.13) is omitted. In that case $1 < r, q < \infty$ can be chosen arbitrarily, ignoring the Serrin condition $2/r + n/q = 1$.*

Let us have a close look at the test function space $\mathcal{T}^{1,r',q'} = \mathcal{T}^{1,r',q'}(T,\Omega)$ in Definition 2.3 and the functional $\mathcal{F}$, see (2.12).

Lemma 2.4. *Let $1 < r,q < \infty$, $0 < T < \infty$ and $\Omega \subseteq \mathbb{R}^n$ be a uniform C^2-domain. For every $v \in L^{r'}(0,T;\tilde{L}^{q'}_\sigma(\Omega))$ there exists a unique solution $\phi = \phi(v) \in \mathcal{T}^{1,r',q'}$ to the backward Stokes equation*

$$-\phi_t + \tilde{A}_q\phi = v \text{ on } (0,T), \quad \phi(T) = 0.$$

It can be represented by the formula

$$\phi(v)(T-t) = \int_0^t e^{-(t-\tau)\tilde{A}_{q'}} v(T-\tau)\,d\tau. \tag{2.14}$$

The map $v \mapsto \phi(v)$ is linear and satisfies, with a constant $C = C(q,r,T,\tau(\Omega)) > 0$, the bound

$$\|\phi(v)\|_{\mathcal{T}^{1,r',q'}(T,\Omega)} \le C\|v\|_{L^{r'}(0,T;\tilde{L}^{q'}(\Omega))}.$$

Proof. The assertions follow from the maximal regularity of the Stokes equation, cf. Theorem 2.2, and a variable transformation $\tilde{t} := T - t$. □

Proposition 2.5. *Let $\Omega \subset \mathbb{R}^n$ be a uniform C^2-domain, $0 < T < \infty$, and let Serrin exponents $2 < r < \infty$, $n < q < \infty$, $\frac{2}{r} + \frac{n}{q} = 1$, $n \ge 3$, be given. Then the following conditions on u_0 are sufficient to imply that the functional $\phi \mapsto \langle \mathcal{F}, \phi\rangle := (u_0, \phi(0))$ is contained in the data space $\mathcal{T}^{-1,r,q}(T,\Omega)$.*

(i) The optimal condition on u_0 in terms of real interpolation theory is

$$u_0 \in \big(\tilde{D}_q^{-1}, \tilde{L}^q_\sigma(\Omega)\big)_{1/r',r},$$

i.e. $u_0 \in \tilde{D}_q^{-1}$ and $\int_0^T \|e^{-t\tilde{A}_q}u_0\|^r_{\tilde{L}^q}\,dt < \infty$.

(ii) In particular, the conditions $u_0 \in \tilde{L}^\rho_\sigma(\Omega)$ and $\int_0^T \|e^{-t\tilde{A}_\rho}u_0\|^r_{\tilde{L}^q}\,dt < \infty$ for some $1 < \rho < \infty$ imply that $u_0 \in \big(\tilde{D}_q^{-1}, \tilde{L}^q_\sigma(\Omega)\big)_{1/r',r}$.

(iii) The conditions $u_0 \in \tilde{L}^{n,r}_\sigma(\Omega)$ and, if even $r \ge n \ge 3$, also $u_0 \in \tilde{L}^n_\sigma(\Omega)$ are sufficient.

Proof. For the convenience of the reader we repeat the proofs of (i) and (ii) from [15].

(i) We must show that $\langle \mathcal{F}, \phi\rangle = (u_0, \phi(0))$ is bounded in $\phi \in \mathcal{T}^{1,r',q'}$. The optimality condition is determined by the optimal space for the trace $\phi(0)$, i.e., by the real interpolation space $(\tilde{L}^{q'}_\sigma, \tilde{D}^1_{q'})_{1/r,r'}$, cf. [1, Theorem III.4.10.2], and duality. By the duality theorem for real interpolation ([29, Theorem 1.11.2]), this is the space $(\tilde{L}^q_\sigma, (\tilde{D}^1_{q'})^*)_{1/r,r} = (\tilde{D}^{-1}_q, \tilde{L}^q_\sigma)_{1/r',r}$ using the duality relation $(\tilde{D}^1_{q'})^* = \tilde{D}^{-1}_q$. Since $\tilde{D}^1_q = \tilde{D}(I+\tilde{A}_q)$, $(I+\tilde{A}_q)^{-1}\tilde{D}^{-1}_q = \tilde{L}^q_\sigma$, and $I+\tilde{A}_q$ generates the exponentially decreasing analytic semigroup $e^{-t}e^{-t\tilde{A}_q}$, the space $(\tilde{D}_q^{-1}, \tilde{L}^q_\sigma)_{1/r',r}$ is characterized by the condition $u_0 \in \tilde{D}_q^{-1}$ such that

$$\|(I+\tilde{A}_q)^{-1}u_0\|_{\tilde{L}^q} + \left(\int_0^\infty \left\|(I+\tilde{A}_q)e^{-t(I+\tilde{A}_q)}(I+\tilde{A}_q)^{-1}u_0\right\|^r_{\tilde{L}^q} dt\right)^{1/r}$$
$$\sim \left(\int_0^T \|e^{-t\tilde{A}_q}u_0\|^r_{\tilde{L}^q} dt\right)^{1/r} < \infty,$$

cf. [29, Theorem 1.14.5].

(ii) A direct proof for $u_0 \in L^\rho_\sigma(\Omega)$ with finite integral $\int_0^T \|e^{-t\tilde{A}_\rho}u_0\|^r_{\tilde{L}^q} dt$ uses Lemma 2.4. Let $\phi \in \mathcal{T}^{1,r',q'}(T,\Omega)$ and $v = -\phi_t + \tilde{A}_{q'}\phi$. Then

$$|(u_0, \phi(0))| = \left|\int_0^T \left(e^{-t\tilde{A}_\rho}u_0, v(t)\right) dt\right| \le \|e^{-t\tilde{A}_\rho}u_0\|_{L^r(0,T;\tilde{L}^q)}\|v\|_{L^{r'}(0,T;\tilde{L}^{q'})}$$

where $\|v\|_{L^{r'}(0,T;\tilde{L}^{q'})} \le C\|\phi\|_{\mathcal{T}^{1,r',q'}}$.

(iii) The next two conditions are consequences of [24, Theorem 2]. If $u_0 \in \tilde{L}^{n,r}_\sigma(\Omega)$ or $u_0 \in \tilde{L}^n_\sigma(\Omega)$ and $r \ge n \ge 3$, then $\int_0^T \|e^{-t\tilde{A}_n}u_0\|^r_{\tilde{L}^q} dt$ is finite. □

Theorem 2.6 (Very Weak Solutions [15])**.** *Let $\Omega \subseteq \mathbb{R}^n$ be a uniform C^2-domain and let $0 < T < \infty$. Assume that $\mathcal{F} \in \mathcal{T}^{-1,r,q}(T,\Omega)$ where $2 < r < \infty$, $n < q < \infty$ and Serrin's condition $\frac{2}{r} + \frac{n}{q} = 1$ is satisfied.*

(i) There exists an $\eta = \eta(\tau(\Omega), q, T) > 0$ with the following property: if

$$\|\mathcal{F}\|_{\mathcal{T}^{-1,r,q}(T,\Omega)} \le \eta,$$

then there exists a very weak solution $u \in L^r(0,T;\tilde{L}^q(\Omega))$ to the Navier-Stokes system with datum $\mathcal{F}$ in the sense of Definition 2.3. The a priori estimate

$$\|u\|_{L^r(0,T;\tilde{L}^q(\Omega))} \le C\|\mathcal{F}\|_{\mathcal{T}^{-1,r,q}(T,\Omega)}$$

holds with a constant $C = C(\tau(\Omega), q, T)$.

(ii) There exists a $T' \in (0,T)$ such that there is a very weak solution $u \in L^r(0,T';\tilde{L}^q(\Omega))$ to the Navier-Stokes system with data $\mathcal{F}|_{[0,T']} \in \mathcal{T}^{-1,r,q}(T',\Omega)$.

3 Mild Solutions

We will need the following norms (*Kato norms*):

$$\|u\|_{K_0^q} := \|u\|_{K_0^q(T,\Omega)} := \sup_{0\le t<T} t^{(1-n/q)/2}\|u(t)\|_{\tilde{L}^q(\Omega)}, \quad n \le q < \infty,$$
$$\|u\|_{K_1^n} := \|u\|_{K_1^n(T,\Omega)} := \sup_{0\le t<T} t^{1/2}\|\nabla u(t)\|_{\tilde{L}^n(\Omega)}, \quad \text{and}$$
$$\|u\|_{\mathcal{K}^q} := \|u\|_{\mathcal{K}^q(T,\Omega)} := \max\{\|u\|_{K_0^q(T,\Omega)}\ ,\ \|u\|_{K_1^n(T,\Omega)}\}, \quad n \le q < \infty.$$

To prove Theorem 1.2 we need some preparations. First we recall that

$$\int_0^t (t-s)^\alpha s^\beta ds = Bt^{1+\alpha+\beta}, \quad \alpha, \beta > -1,\ 0 < t < \infty,$$

where the constant B equals the Eulerian Beta function $B_{1+\alpha,1+\beta}$.

Lemma 3.1. *Let $\Omega \subseteq \mathbb{R}^n$ be a uniform C^2-domain and let $u_0 \in \tilde{L}^n_\sigma(\Omega)$. Then for any exponent $n < q < \infty$, $2 < r < \infty$ satisfying $\frac{2}{r} + \frac{n}{q} = 1$ it holds that*

$$\lim_{t\to 0+} t^{1/r} \| e^{-t\tilde{A}_n} u_0 \|_{\tilde{L}^q(\Omega)} = 0,$$
$$\lim_{t\to 0+} t^{1/2} \| \nabla e^{-t\tilde{A}_n} u_0 \|_{\tilde{L}^n(\Omega)} = 0.$$

Proof. First note that, by (2.8), for all $v \in \tilde{L}^n_\sigma(\Omega)$

$$t^{1/r} \| e^{-t\tilde{A}_n} v \|_{\tilde{L}^q(\Omega)} \le C \|v\|_{\tilde{L}^n(\Omega)}, \quad 0 \le t \le 1,$$

with some constant $C \ge 1$. Moreover, we use that $\| e^{-t\tilde{A}_q} v \|_{\tilde{L}^q(\Omega)} \le C \|v\|_{\tilde{L}^q(\Omega)}$ for every $v \in \tilde{L}^q_\sigma(\Omega)$ and $0 \le t \le 1$. Next we approximate $u_0 \in \tilde{L}^n_\sigma(\Omega)$ by $u_\varepsilon \in C^\infty_{0,\sigma}(\Omega)$ to get the estimate

$$\begin{aligned} t^{1/r} \| e^{-t\tilde{A}_n} u_0 \|_{\tilde{L}^q} &\le t^{1/r} \| e^{-t\tilde{A}_n} (u_\varepsilon - u_0) \|_{\tilde{L}^q} + t^{1/r} \| e^{-t\tilde{A}_n} u_\varepsilon \|_{\tilde{L}^q} \\ &\le C_0 \|u_\varepsilon - u_0\|_{\tilde{L}^n} + C_1 t^{1/r} \|u_\varepsilon\|_{\tilde{L}^q}. \end{aligned}$$

This proves the first assertion.

For the second assertion we find a constant $C \ge 1$ such that the following estimates hold: By (2.8), (2.9)

$$\| e^{-t\tilde{A}_q} v \|_{\tilde{L}^n} + t^{1/2} \| \nabla e^{-t\tilde{A}_n} v \|_{\tilde{L}^n} \le C_2 \|v\|_{\tilde{L}^n}, \quad 0 \le t \le 1,$$

and $\|\nabla v\|_{\tilde{L}^n} \le C_4 \|(1+\tilde{A}_n)^{1/2} v\|_{\tilde{L}^n}$ for all $v \in \tilde{D}^{1/2}_n$, cf. Sect. 2. Approximating $u_0 \in \tilde{L}^n_\sigma(\Omega)$ by $u_\varepsilon \in C^\infty_{0,\sigma}(\Omega)$, the estimate

$$\begin{aligned} t^{1/2} \| \nabla e^{-t\tilde{A}_n} u_0 \|_{\tilde{L}^n} &\le t^{1/2} \| \nabla e^{-t\tilde{A}_n} (u_\varepsilon - u_0) \|_{\tilde{L}^n} + t^{1/2} \| \nabla e^{-t\tilde{A}_n} u_\varepsilon \|_{\tilde{L}^n} \\ &\le C \|u_\varepsilon - u_0\|_{\tilde{L}^n} + C t^{1/2} \| e^{-tA_n} (1+\tilde{A}_n)^{1/2} u_\varepsilon \|_{\tilde{L}^n} \\ &\le C \|u_\varepsilon - u_0\|_{\tilde{L}^n} + C^2 t^{1/2} \|(1+\tilde{A}_n)^{1/2} u_\varepsilon\|_{\tilde{L}^n} \end{aligned}$$

easily proves the second claim. □

Remark 3.2. *A similar lemma holds for u_0 in the Lorentz space $\tilde{L}^{n,\rho}_\sigma(\Omega)$ if $n \le \rho < \infty$. However, this argument does not work for the space $\tilde{L}^{n,\infty}_\sigma(\Omega)$, which can be defined by real interpolation as well.*

Proof of Theorem 1.2. We define the iteration procedure $u^{(0)}(t) := e^{-t\tilde{A}_n}u_0$, and

$$u^{(j+1)}(t) := e^{-t\tilde{A}}u_0 - \int_0^t e^{-(t-s)\tilde{A}}\tilde{P}(u^{(j)}(s)\cdot\nabla u^{(j)}(s))\,ds, \tag{3.1}$$

for $j \geq 0$ and $0 \leq t < T$; here e.g. $\tilde{A} = \tilde{A}_{n/2}$, $\tilde{P} = \tilde{P}_{n/2}$. We will show convergence of the sequence $(u^{(j)})_{j\in\mathbb{N}}$ in the norm $\|\cdot\|_{\mathcal{K}^q}$ where we fixed $n < q < \infty$. Related to q there exists $2 < r < \infty$ such that $\frac{2}{r} + \frac{n}{q} = 1$. By assumption $\|u^{(0)}\|_{K_0^q} \leq \|u^{(0)}\|_{\mathcal{K}^q} =: I < \infty$.

Claim 1 The sequence $\left(u^{(j)}\right)$ is bounded with respect to the norm $\|\cdot\|_{\mathcal{K}^q}$.

Proof of Claim 1 From (2.8) we conclude that for all $j \geq 0$

$$\begin{aligned}
\|u^{(j+1)}\|_{K_0^q} &\leq \|u^{(0)}\|_{K_0^q} + \sup_{0\leq t<T} t^{1/r}\int_0^t \left\|e^{-(t-s)\tilde{A}}\tilde{P}(u^{(j)}(s)\cdot\nabla u^{(j)}(s))\right\|_{\tilde{L}^q} ds \\
&\leq I + C\sup_{0\leq t<T} t^{1/r}\int_0^t (t-s)^{-1/2}\|u^{(j)}(s)\cdot\nabla u^{(j)}(s)\|_{\tilde{L}^{nq/(n+q)}}\,ds \\
&\leq I + C\sup_{0\leq t<T} t^{1/r}\int_0^t (t-s)^{-1/2}\|u^{(j)}(s)\|_{\tilde{L}^q}\|\nabla u^{(j)}(s)\|_{\tilde{L}^n}\,ds \\
&\leq I + C\|u^{(j)}\|^2_{\mathcal{K}^q}\sup_{0\leq t<T} t^{1/r}\int_0^t (t-s)^{-1/2}s^{-1/2-1/r}\,ds \\
&= I + C\|u^{(j)}\|^2_{\mathcal{K}^q}\cdot B_{1/2,1/2-1/r}.
\end{aligned}$$

For the other part of the norm $\|\cdot\|_{\mathcal{K}^q}$ we get due to (2.9) that

$$\begin{aligned}
\|u^{(j+1)}\|_{K_1^n} &\leq \|u^{(0)}\|_{K_1^n} + \sup_{0\leq t<T} t^{1/2}\int_0^t \|\nabla e^{-(t-s)\tilde{A}}\tilde{P}(u^{(j)}(s)\cdot\nabla u^{(j)}(s))\|_{\tilde{L}^n}\,ds \\
&\leq I + C\sup_{0\leq t<T} t^{1/2}\int_0^t (t-s)^{-1+1/r}\|u^{(j)}(s)\cdot\nabla u^{(j)}(s)\|_{\tilde{L}^{nq/(n+q)}}\,ds \\
&\leq I + C\sup_{0\leq t<T} t^{1/2}\int_0^t (t-s)^{-1+1/r}\|u^{(j)}(s)\|_{\tilde{L}^q}\|\nabla u^{(j)}(s)\|_{\tilde{L}^n}\,ds \\
&\leq I + C\|u^{(j)}\|^2_{\mathcal{K}^q}\sup_{0\leq t<T} t^{1/2}\int_0^t (t-s)^{-1+1/r}s^{-1/r-1/2}\,ds \\
&= I + C\|u^{(j)}\|^2_{\mathcal{K}^q}\cdot B_{1/r,1/2-1/r}.
\end{aligned}$$

Combining these estimates we find that

$$\|u^{(j+1)}\|_{\mathcal{K}^q} \leq \|u^{(0)}\|_{\mathcal{K}^q} + C_1\|u^{(j)}\|^2_{\mathcal{K}^q} \tag{3.2}$$

with a constant C_1. Now we fix γ in (1.4) by

$$\gamma := \frac{1}{6C_1}.$$

Since $\|u^{(0)}\|_{\mathcal{K}^q} \le \gamma$ it is seen by induction that $\|u^{(j)}\|_{\mathcal{K}^q} \le \frac{1}{3C_1}$. (■)

Claim 2 The sequence $\big(u^{(j)}\big)$ converges to a limit $u \in L^\infty(0,T;\tilde{L}^n_\sigma(\Omega))$ and $\|u\|_{\mathcal{K}^q} < \infty$.

Proof of Claim 2 We write $u^{(j)}$ as telescoping sum $u^{(j)} = \sum_{k=0}^{j} w^{(k)}$, where $w^{(0)} := u^{(0)}$ and $w^{(k)} := u^{(k)} - u^{(k-1)}$. Note that this implies that

$$w^{(j+1)}(t) = -\int_0^t e^{-(t-s)\tilde{A}}\tilde{P}\big(w^{(j)}(s)\cdot\nabla u^{(j)}(s) + u^{(j-1)}\cdot\nabla w^{(j)}(s)\big)\,ds$$

for all t and all $j \in \mathbb{N}$.

Repeating the arguments from above we find, with C_1 as in (3.2), that

$$\|w^{(0)}\|_{\mathcal{K}^q} \le \|u^{(0)}\|_{\mathcal{K}^q}, \quad \|w^{(j+1)}\|_{\mathcal{K}^q} \le C_1\|w^{(j)}\|_{\mathcal{K}^q}\big(\|u^{(j)}\|_{\mathcal{K}^q} + \|u^{(j-1)}\|_{\mathcal{K}^q}\big)$$

for all j. Since $\|u^{(j)}\|_{\mathcal{K}^q} \le \frac{1}{3C_1}$ we get the estimate $\|w^{(j+1)}\|_{\mathcal{K}^q} \le \frac{2}{3}\|w^{(j)}\|_{\mathcal{K}^q}$. Consequently,

$$\|u^{(j)}\|_{\mathcal{K}^q} \le \sum_{k=0}^{j}\|w^{(k)}\|_{\mathcal{K}^q} \le \sum_{k=0}^{j}\left(\frac{2}{3}\right)^k\|u^{(0)}\|_{\mathcal{K}^q}.$$

Hence the infinite sum $\sum_{k=0}^{\infty} w^{(k)}$ is absolutely convergent in the norm $\|\cdot\|_{\mathcal{K}^q}$, and the sequence $(u^{(j)})_{j\in\mathbb{N}}$ converges in this norm to some element u. This implies also pointwise convergence for every $t \in (0,T)$. Moreover, we get the bound

$$\|u\|_{\mathcal{K}^q} \le 3\|u^{(0)}\|_{\mathcal{K}^q}.$$

To prove the convergence of $(u^{(j)})$ to u in $L^\infty(0,T;\tilde{L}^n_\sigma(\Omega))$ we estimate for any $0 \le t < T$ and $j \in \mathbb{N}$ the iterate $w^{(j+1)}$ by

$$\begin{aligned}
\|w^{(j+1)}(t)\|_{\tilde{L}^n} &\le \int_0^t \big\|e^{-(t-s)\tilde{A}}\tilde{P}[w^{(j)}(s)\cdot\nabla u^{(j)}(s) + u^{(j-1)}(s)\cdot\nabla w^{(j)}(s)]\big\|_{\tilde{L}^n}\,ds \\
&\le C_1\int_0^t (t-s)^{-n/(2q)}\big\|w^{(j)}(s)\cdot\nabla u^{(j)}(s) + u^{(j-1)}(s)\cdot\nabla w^{(j)}(s)\big\|_{\tilde{L}^{nq/(n+q)}}\,ds \\
&\le 2C_1\|w^{(j)}\|_{\mathcal{K}^q}\frac{1}{3C_1}\int_0^t (t-s)^{-n/(2q)}s^{-1/r}s^{-1/2}\,ds \\
&= \frac{2}{3}\|w^{(j)}\|_{\mathcal{K}^q}\cdot B' \le \left(\frac{2}{3}\right)^j\|u^{(0)}\|_{\mathcal{K}^q}B' \qquad (3.3)
\end{aligned}$$

where $B' = B_{1-n/(2q),1/2-1/r}$. We conclude that the sequence $u^{(j)} = \sum_{k=0}^{j} w^{(k)}$ converges to u in the norm of $L^\infty(0,T;\tilde{L}^n(\Omega))$, too. Hence $u \in L^\infty(0,T;\tilde{L}^n_\sigma(\Omega))$ and $\|u\|_{L^\infty(0,T;\tilde{L}^n)} \le 3B'\|u^{(0)}\|_{\mathcal{K}^q}$. (■)

Claim 3 u is a solution of the integral equation.

Proof of Claim 3 We start from the integral equation (3.1) and pass to the limit $j \to \infty$. Since $u^{(j)}$ tends pointwise in $\tilde{L}^n(\Omega)$ to u, the left hand side tends to $u(t)$ for a.a. $t \in [0,T)$ in $\tilde{L}^n(\Omega)$. Concerning the integral term we estimate, using the techniques as before,

$$\begin{aligned}
&\left\| \int_0^t e^{-(t-s)\tilde{A}} \tilde{P}(u^{(j)} \cdot \nabla u^{(j)})(s)\, ds - \int_0^t e^{-(t-s)\tilde{A}} \tilde{P}(u \cdot \nabla u)(s)\, ds \right\|_{\tilde{L}^n} \\
&\quad \le C \int_0^t (t-s)^{-1/2} \big(\|u^{(j)} - u\|_{\tilde{L}^n} \|\nabla u^{(j)}\|_{\tilde{L}^n} + \|u\|_{\tilde{L}^n} \|\nabla (u^{(j)} - u)\|_{\tilde{L}^n} \big)\, ds \\
&\quad \le C \big(\|u^{(j)}\|_{K_1^n} \|u^{(j)} - u\|_{L^\infty(0,T;\tilde{L}^n)} + \|u\|_{L^\infty(0,T;\tilde{L}^n)} \|u^{(j)} - u\|_{K_1^n} \big) B_{1/2,1/2} \\
&\quad \le C B_{1/2,1/2} \|u^{(0)}\|_{\mathcal{K}^q} \big(\|u^{(j)} - u\|_{L^\infty(0,T;\tilde{L}^n)} + \|u^{(j)} - u\|_{K_1^n} \big),
\end{aligned}$$

which tends to 0 as $j \to \infty$. This implies that u satisfies the integral equation. (■)

Hence u is a mild solution in the sense of Definition 1.1. □

Theorem 3.3. *The mild solution $u \in L^\infty(0,T;\tilde{L}^n_\sigma(\Omega))$ constructed in Theorem 1.2 is contained in $C([0,T);\tilde{L}^n_\sigma(\Omega))$ and satisfies $u(0) = u_0$.*

Moreover, it shares the following properties:

- $[t \mapsto t^{1/r} u(t)] \in C([0,T);\tilde{L}^q_\sigma(\Omega))$ *for all $n < q < \infty$ with function value* 0 *at $t = 0$. Here r is defined by $\frac{2}{r} + \frac{n}{q} = 1$.*
- $[t \mapsto t^{1/2} \nabla u(t)] \in C([0,T);\tilde{L}^n(\Omega))$ *with value* 0 *at $t = 0$.*

Proof. By mathematical induction on $j \in \mathbb{N}$ we will show for all $n \le q < \infty$ (with $1/r = 0$ when $q = n$) the continuity properties of the functions

$$t^{1/r} u^{(j)}(t) \in C([0,T);\tilde{L}^q_\sigma(\Omega)), \quad t^{1/2} \nabla u^{(j)}(t) \in C([0,T);\tilde{L}^n(\Omega)), \tag{3.4}$$

for any $j \in \mathbb{N}_0$. All functions in (3.4) are understood to equal zero at $t = 0$, except in case $q = n$, where $u^{(j)}(0) = u_0$.

The initial step for $u^{(0)}$ is easy. For $0 < \tau \le t < T$ we have that

$$\begin{aligned}
t^{1/r} u^{(0)}(t) - \tau^{1/r} u^{(0)}(\tau) &= \tau^{1/r} e^{-\tau \tilde{A}_n} \big((t/\tau)^{1/r} e^{-(t-\tau)\tilde{A}_n} - 1 \big) u_0, \\
t^{1/2} \nabla u^{(0)}(t) - \tau^{1/2} \nabla u^{(0)}(\tau) &= \tau^{1/2} \nabla e^{-\tau \tilde{A}_n} \big((t/\tau)^{1/2} e^{-(t-\tau)\tilde{A}_n} - 1 \big) u_0,
\end{aligned}$$

and get, as long as τ stays bounded away from 0, that

$$\begin{aligned}
\|t^{1/r} u^{(0)}(t) - \tau^{1/r} u^{(0)}(\tau)\|_{\tilde{L}^q} &\le C \big\| \big((t/\tau)^{1/r} e^{-(t-\tau)\tilde{A}_n} - 1 \big) u_0 \big\|_{\tilde{L}^q} \to 0, \\
\|t^{1/2} \nabla u^{(0)}(t) - \tau^{1/2} \nabla u^{(0)}(\tau)\|_{\tilde{L}^n} &\le C \big\| \big((t/\tau)^{1/2} e^{-(t-\tau)\tilde{A}_n} - 1 \big) u_0 \big\|_{\tilde{L}^n} \to 0,
\end{aligned}$$

as $|t-\tau| \to 0$. This proves continuity of the functions in (3.4) for $j = 0$ in the open interval $(0, T)$. The continuity in $t = 0$ follows from Lemma 3.1.

The inductive step will be split into two parts, one on continuity in $(0, T)$, the other one on continuity at $t = 0$.

Claim 1 The functions in (3.4) are continuous in the open interval $(0, T)$.

Proof Assume that the assertion is true for a fixed but arbitrary $j \in \mathbb{N}$. For $0 < \tau \le t < T$ we have

$$\begin{aligned}
&\big\|t^{1/r}u^{(j+1)}(t) - \tau^{1/r}u^{(j+1)}(\tau)\big\|_{\tilde{L}^q} \le \big\|t^{1/r}e^{-t\tilde{A}_n}u_0 - \tau^{1/r}e^{-\tau\tilde{A}_n}u_0\big\|_{\tilde{L}^q} \\
&\quad + t^{1/r}\int_\tau^t \big\|e^{-(t-s)\tilde{A}_{n/2}}\tilde{P}_{n/2}(u^{(j)}(s)\cdot\nabla u^{(j)}(s))\big\|_{\tilde{L}^q}\, ds \\
&\quad + \int_0^\tau \big\|\big(t^{1/r}e^{-(t-s)\tilde{A}_{n/2}} - \tau^{1/r}e^{-(\tau-s)\tilde{A}_{n/2}}\big)\tilde{P}_{n/2}(u^{(j)}(s)\cdot\nabla u^{(j)}(s))\big\|_{\tilde{L}^q}\, ds \\
&\quad =: I_1 + I_2 + I_3.
\end{aligned}$$

We have to show that the terms I_i tend to zero as $|t-\tau| \to 0$. From the initial step we know that I_1 tends to zero. The term I_2 is treated as follows:

$$\begin{aligned}
I_2 &\le Ct^{1/r}\int_\tau^t (t-s)^{-1/2}\|u^{(j)}(s)\|_{\tilde{L}^q}\|\nabla u^{(j)}(s)\|_{\tilde{L}^n} ds \\
&\le Ct^{1/r}K^2\int_\tau^t (t-s)^{-1/2}s^{-1/r-1/2}ds \\
&= CK^2\int_{\tau/t}^1 (1-s)^{-1/2}s^{-1/r-1/2}ds
\end{aligned}$$

where $K = \frac{1}{3C_1}$ is the constant bounding the sequence $\|u^{(j)}\|_{\mathcal{K}^q}$ as in the proof of Theorem 1.2. Of course the above bound for I_2 tends to zero as $|t-\tau| \to 0$ as long as τ stays bounded away from 0.

Now we discuss the term I_3. For technical reasons we change variables to get

$$I_3 = \int_0^1 \tau\big\|\big(t^{1/r}e^{-(t-\tau)\tilde{A}} - \tau^{1/r}\big)e^{-\tau(1-s)\tilde{A}}\tilde{P}(u^{(j)}(\tau s)\cdot\nabla u^{(j)}(\tau s))\big\|_{\tilde{L}^q}ds.$$

Interested in continuity on the open interval $(0, T)$, we assume that $0 < \varepsilon \le \tau \le t \le T - \varepsilon$ with some $\varepsilon > 0$. Then a uniform continuity argument in $(s, t, \tau) \in (0, 1) \times [\varepsilon, T-\varepsilon]^2$ proves that for fixed $s \in (0, 1)$ the integrand in I_3,

$$B_{\tau,t}(s) := \tau\big\|\big(t^{1/r}e^{-(t-\tau)\tilde{A}} - \tau^{1/r}\big)e^{-\tau(1-s)\tilde{A}}\tilde{P}(u^{(j)}(\tau s)\cdot\nabla u^{(j)}(\tau s))\big\|_{\tilde{L}^q},$$

satisfies $B_{\tau,t}(s) \to 0$ as $|t-\tau| \to 0$. We want to use Lebesgue's theorem on dominated convergence to show that I_3 tends to zero. To find a bound for the integrand $B_{\tau,t}$, independent of $\tau, t \in [\varepsilon, T-\varepsilon]$, we estimate as follows:

$$\begin{aligned}
B_{\tau,t}(s) &\le C_\varepsilon\big\|e^{-\tau(1-s)\tilde{A}_{n/2}}\tilde{P}_{n/2}(u^{(j)}(\tau s)\cdot\nabla u^{(j)}(\tau s))\big\|_{\tilde{L}^q} \\
&\le C_\varepsilon(\tau(1-s))^{-1/2}\|u^{(j)}(\tau s)\|_{\tilde{L}^q}\|\nabla u^{(j)}(\tau s)\|_{\tilde{L}^n} \\
&\le C_\varepsilon K^2(1-s)^{-1/2}s^{-1/r-1/2}.
\end{aligned}$$

The integrability of the latter term is obvious. Hence $I_3 \to 0$ as $|t-\tau| \to 0$.

Next we consider the continuity of the function $[t \mapsto t^{1/2}\nabla u^{(j+1)}(t)]$ in $\tilde{L}^n(\Omega)$ on $[\varepsilon, T-\varepsilon]$, $\varepsilon > 0$. Since the induction hypothesis gives the properties in (3.4) for all $n \le q < \infty$ at the same time, we choose $q_0 = 3n$ with corresponding Serrin exponent $r_0 = 3$. We calculate for $0 < \tau \le t < T$

$$\begin{aligned}
\|t^{1/2}\nabla u^{(j+1)}(t) - \tau^{1/2}\nabla u^{(j+1)}(\tau)\|_{\tilde{L}^n} &\le \big\|t^{1/2}\nabla e^{-t\tilde{A}_n}u_0 - \tau^{1/2}\nabla e^{-\tau\tilde{A}_n}u_0\big\|_{\tilde{L}^n} \\
&+ t^{1/2}\int_\tau^t \big\|\nabla e^{-(t-s)\tilde{A}_{n/2}}\tilde{P}_{n/2}(u^{(j)}(s)\cdot\nabla u^{(j)}(s))\big\|_{\tilde{L}^n}\,ds \\
&+ \int_0^\tau \big\|\nabla\big(t^{1/2}e^{-(t-s)\tilde{A}_{n/2}} - \tau^{1/2}e^{-(\tau-s)\tilde{A}_{n/2}}\big)\tilde{P}_{n/2}(u^{(j)}(s)\cdot\nabla u^{(j)}(s))\big\|_{\tilde{L}^n}\,ds \\
&=: J_1 + J_2 + J_3.
\end{aligned}$$

It has already been shown that $J_1 \to 0$ as $|t-\tau| \to 0$. For the next term we find similarly as above

$$\begin{aligned}
J_2 &\le Ct^{1/2}\int_\tau^t (t-s)^{-2/3}\|u^{(j)}(s)\|_{\tilde{L}^{3n}}\|\nabla u^{(j)}(s)\|_{\tilde{L}^n}\,ds \\
&\le Ct^{1/2}K^2\int_\tau^t (t-s)^{-2/3}s^{-5/6}\,ds \\
&\le CK^2\int_{\tau/t}^1 (1-s)^{-2/3}s^{-5/6}\,ds,
\end{aligned}$$

tending to 0 as $|t-\tau| \to 0$, as long as τ stays away from 0.

The last integral J_3 can be rewritten and estimated by

$$\begin{aligned}
&\int_0^1 \tau\left\|\nabla\left(t^{\frac{1}{2}}e^{-(t-\tau)\tilde{A}_{n/2}} - \tau^{\frac{1}{2}}\right)e^{-\tau(1-s)\tilde{A}_{n/2}}\tilde{P}_{n/2}(u^{(j)}(\tau s)\cdot\nabla u^{(j)}(\tau s))\right\|_{\tilde{L}^n}ds \\
\le{}& C\int_0^1 \left\|\left(t^{\frac{1}{2}}e^{-(t-\tau)\tilde{A}} - \tau^{\frac{1}{2}}\right)(1+\tilde{A})^{\frac{1}{2}}e^{-\tau(1-s)\tilde{A}}\tilde{P}(u^{(j)}(\tau s)\cdot\nabla u^{(j)}(\tau s))\right\|_{\tilde{L}^n}ds.
\end{aligned}$$

Again, an argument using uniform continuity shows that the integrand

$$B'_{\tau,t}(s) := \big\|\big(t^{\frac{1}{2}}e^{-(t-\tau)\tilde{A}} - \tau^{\frac{1}{2}}\big)(1+\tilde{A})^{\frac{1}{2}}e^{-\tau(1-s)\tilde{A}}\tilde{P}(u^{(j)}(\tau s)\cdot\nabla u^{(j)}(\tau s))\big\|_{\tilde{L}^n}$$

tends to zero pointwise for every $s \in (0,1)$ as $|t-\tau| \to 0$. Moreover, it can be estimated by the integrable pointwise upper bound

$$\begin{aligned}
B'_{\tau,t}(s) &\le C_\varepsilon\big\|(1+\tilde{A}_{n/2})^{1/2}e^{-\tau(1-s)\tilde{A}}\tilde{P}(u^{(j)}(\tau s)\cdot\nabla u^{(j)}(\tau s))\big\|_{\tilde{L}^n} \\
&\le C_\varepsilon K^2(1-s)^{-2/3}s^{-5/6}
\end{aligned}$$

uniformly for $t, \tau \in [\varepsilon, T-\varepsilon]$. Now Lebesgue's theorem on dominated convergence shows that $J_3 \to 0$ as $|t-\tau| \to 0$. This finishes the proof of the continuity of

$t \mapsto t^{1/2}\nabla u^{(j+1)}(t)$ in $(0,T)$ with values in the space $\tilde{L}^n_\sigma(\Omega)$. (■)

Claim 2 The functions in (3.4) are continuous at $t=0$.
Proof As in (3.2) we calculate, replacing T by an arbitrary $0 < t \le T$,

$$\|u^{(j+1)}\|_{\mathcal{K}^q(t,\Omega)} \le \|u^{(0)}\|_{\mathcal{K}^q(t,\Omega)} + C_1\|u^{(j)}\|^2_{\mathcal{K}^q(t,\Omega)}.$$

By Lemma 3.1, the first term on the right-hand side tends to zero as $t \to 0$ when $q > n$, and the second term tends to zero as well, by induction hypothesis.

For $q = n$, the above inequality needs a small modification since $u^0(t) \to u_0$ in $\tilde{L}^n(\Omega)$. Using (3.3)

$$\|w^{(j+1)}(t)\|_{\tilde{L}^n} \le \frac{2}{3}\|w^{(j)}\|_{\mathcal{K}^q(t,\Omega)} \cdot B' \le \left(\frac{2}{3}\right)^j \|w^{(1)}\|_{\mathcal{K}^q(t,\Omega)} \cdot B'.$$

Hence $u^{(j+1)}(t) - u^{(0)}(t) \to 0$ in $\tilde{L}^n(\Omega)$ as $t \to 0$.

The continuity at $t = 0$ of both functions in (3.4) is thus shown. (■)

We know from the proof of Theorem 1.2 that $\|u^{(j)} - u\|_{\mathcal{K}^q} \to 0$ for all $q > n$ and also that $\|u^{(j)} - u\|_{L^\infty(0,T;\tilde{L}^n(\Omega))} \to 0$ for $j \to \infty$. This means that $t^{1/r}u^{(j)}(t)$ converges to $t^{1/r}u(t)$ in $\tilde{L}^q(\Omega)$, $n \le q < \infty$, and that $t^{1/2}\nabla u^{(j)}(t)$ converges to $t^{1/2}\nabla u^{(j)}(t)$ in $\tilde{L}^n(\Omega)$ uniformly in t. So the continuity properties for all j can be carried over to the limit function u. Now the proof is finished. □

Proof of Theorem 1.4. We estimate as above for $0 < T' \le T$ to get that

$$\|v - w\|_{\mathcal{K}^q(T',\Omega)} \le C\big(\|v\|_{K^q_0(T',\Omega)} + \|w\|_{K^n_1(T',\Omega)}\big)\|v - w\|_{\mathcal{K}^q(T',\Omega)}.$$

By the assumptions we choose $T' > 0$ such that $\|v\|_{K^q_0(T',\Omega)} + \|w\|_{K^n_1(T',\Omega)} < \frac{1}{C}$, yielding $\|v - w\|_{\mathcal{K}^q(T',\Omega)} = 0$. Hence $v = w$ on $[0, T')$. □

Remark 3.4. *Note that the solution u constructed in Theorem 1.2 satisfies either of the requirements of Theorem 1.4. Hence, any further mild solution $\tilde{u}$ for which $\|\tilde{u}\|_{K^q_0(t,\Omega)}$ or (!) $\|\tilde{u}\|_{K^n_1(t,\Omega)}$ tends to zero as $t \to 0$ coincides with u, at least on a small interval. In particular, $\tilde{u}$ has the continuity properties from Theorem 3.3.*

4 Mild vs. Very Weak Solutions

Theorem 4.1. *Let $u \in L^\infty(0,T;\tilde{L}^n_\sigma(\Omega))$ be the solution constructed in Theorem 1.2 and let $n \le r < \infty$, $n < q < \infty$, $\frac{2}{r} + \frac{n}{q} = 1$ be given Serrin exponents. Then, for some $0 < T_* \le T$, the solution u is a very weak solution to the Navier-Stokes equations contained in $L^r(0,T_*;\tilde{L}^q_\sigma(\Omega))$.*

Proof. First of all note that $u^{(0)}$ defined by $u^{(0)}(t) = e^{-t\tilde{A}}u_0$ is contained in $L^r(0,T;\tilde{L}^q(\Omega))$ and that the functional $\mathcal{F}$ defined by $\langle \mathcal{F}, \phi\rangle = \langle u_0, \phi(0)\rangle$ is contained in $\mathcal{T}^{-1,r,q}(T,\Omega)$ by Proposition 2.5 (iii). Here we need the additional assumption $r \geq n$.

To show that $u \in L^r(0,T_*;\tilde{L}^q_\sigma(\Omega))$ for a sufficiently small T_* we have to estimate the nonlinear term. To this end we write $u^{(j)}\cdot\nabla u^{(j)} = \operatorname{div}(u^{(j)}\otimes u^{(j)})$ and estimate $u^{(j+1)}$ with the help of (2.10) by

$$\begin{aligned}\|u^{(j+1)}(t)\|_{\tilde{L}^q} &\leq \|u^{(0)}(t)\|_{\tilde{L}^q} + \int_0^t \|e^{-(t-s)\tilde{A}_{n/2}}\tilde{P}\operatorname{div}(u^{(j)}(s)\otimes u^{(j)}(s))\|_{\tilde{L}^q}\,ds \\ &\leq \|u^{(0)}(t)\|_{\tilde{L}^q} + C\int_0^t (t-s)^{-1/r'}\|u^{(j)}(s)\|^2_{\tilde{L}^q}\,ds\end{aligned}$$

for a.a. $t \in [0,T)$. Then the Hardy-Littlewood-Sobolev inequality implies for every $0 < T' \leq T$ that

$$\|u^{(j+1)}\|_{L^r(0,T';\tilde{L}^q)} \leq \|u^{(0)}\|_{L^r(0,T';\tilde{L}^q)} + C_2\|u^{(j)}\|^2_{L^r(0,T';\tilde{L}^q)}.$$

Choose $T_* > 0$ such that $\|u^{(0)}\|_{L^r(0,T_*;\tilde{L}^q)} \leq 1/(6C_2)$. Then it is easily seen by induction that the sequence $\big(\|u^{(j)}\|_{L^r(0,T_*;\tilde{L}^q)}\big)_{j\in\mathbb{N}}$ stays bounded by $1/(2C_2)$. Since we have $u^{(j)} \to u$ in $\tilde{L}^q(\Omega)$ uniformly on any interval $[\varepsilon, T_*)$, as long as $\varepsilon > 0$ (cf. the proof of Theorem 1.2), we conclude $\|u\|_{L^r(\varepsilon,T_*;\tilde{L}^q)} \leq 1/(2C_2)$. Letting $\varepsilon \to 0$ we get from Fatou's Lemma that $u \in L^r(0,T_*;\tilde{L}^q)$ with norm bounded by $1/(2C_2)$. This finishes the first part of the proof.

We still have to show that the mild solution u satisfies the variational equality

$$-(u,\phi_t)_{T_*,\Omega} + (u, \tilde{A}_{q'}\phi)_{T_*,\Omega} = (u_0, \phi(0)) + (u\otimes u, \nabla\phi)_{T_*,\Omega} \tag{4.1}$$

for every $\phi \in \mathcal{T}^{1,r',q'}(T_*,\Omega)$. From the integral representations of u, cf. (1.3), and of ϕ in terms of $v = -\phi_t + \tilde{A}_{q'}\phi$, cf. (2.14) in Lemma 2.4, the left hand side of (4.1) reads

$$\begin{aligned}&-(u,\phi_t)_{T_*,\Omega} + (u,\tilde{A}_{q'}\phi)_{T_*,\Omega} = (u,v)_{T_*,\Omega} \\ &= \int_0^{T_*}\left\{\Big(e^{-t\tilde{A}_n}u_0\,,\ v(t)\Big) - \left(\int_0^t e^{-(t-s)\tilde{A}_{n/2}}\tilde{P}_{n/2}\operatorname{div}(u(s)\otimes u(s))\,ds\,,\ v(t)\right)\right\}dt \\ &= (u_0,\phi(0)) - \int_0^{T_*}\int_0^t \Big(e^{-(t-s)\tilde{A}_{n/2}}\tilde{P}_{n/2}\operatorname{div}(u(s)\otimes u(s))\,,\ v(t)\Big)\,ds\,dt \\ &= (u_0,\phi(0)) + \int_0^{T_*}\int_0^t \Big(u(s)\otimes u(s)\,,\ \nabla e^{-(t-s)\tilde{A}_{q'}}v(t)\Big)\,ds\,dt.\end{aligned}$$

In the second term on the right-hand side we change the order of integration and get the integral

$$\begin{aligned}
&\int_0^{T_*}\int_s^{T_*}\Big(u(s)\otimes u(s),\nabla e^{-(t-s)\tilde{A}_{q'}}v(t)\Big)\,dt\,ds\\
&\quad=\int_0^{T_*}\left(u(s)\otimes u(s),\nabla\phi(s)\right)\,ds\\
&\quad=(u\otimes u,\nabla\phi)_{T_*,\Omega}.
\end{aligned}$$

Summarizing, we have proved that the mild solution satisfies (4.1) and is hence a very weak solution. □

A partial converse of Theorem 4.1 will be described in the following theorem. The result states that a very weak solution is contained in $L^\infty(0,T;\tilde{L}^n_\sigma(\Omega))$, but it is not necessarily a mild solution as constructed in Theorem 1.2. In particular, the continuity property $C([0,T);\tilde{L}^n(\Omega))$ is missing. The main problem in the proof compared to similar results proved in [15, Theorem 3.2, Theorem 3.3 and Proposition 3.4] is the fact that with $r=\infty$ the Hardy-Littlewood-Sobolev inequality does *not* hold for $r'=1$.

Theorem 4.2. *Let $\Omega\subseteq\mathbb{R}^n$ be a C^2-domain, $n\geq 3$, and $0<T<\infty$. Assume Serrin exponents $n\leq r\leq 2n$ and $n<q<2n$ are given, and let $u_0\in\tilde{L}^n_\sigma(\Omega)$.*

Then a very weak solution $u\in L^r(0,T;\tilde{L}^q_\sigma(\Omega))$ to the Navier-Stokes system with data $\mathcal{F}\in\mathcal{T}^{-1,r,q}(T,\Omega)$ defined by $\langle\mathcal{F},\phi\rangle=(u_0,\phi(0))$ also belongs to the space $L^\infty(0,T;\tilde{L}^n_\sigma(\Omega))$. It satisfies the estimate

$$\|u(t)-e^{-t\tilde{A}_n}u_0\|_{\tilde{L}^n(\Omega)}\leq C\|u\|^2_{L^r(0,t;\tilde{L}^q(\Omega))},\tag{4.2}$$

for a.a. $0\leq t<T$, with a constant $C=C(q,T,\tau(\Omega))>0$. In particular, $u(t)$ converges to u_0 in $\tilde{L}^n(\Omega)$ as $t\to 0$ on a dense subset of $(0,T)$. Moreover,

$$\left\|\frac{1}{\varepsilon}\int_0^\varepsilon u(s)\,ds-u_0\right\|_{\tilde{L}^n_\sigma(\Omega)}\leq\|(u_0)_\varepsilon-u_0\|_{\tilde{L}^n_\sigma(\Omega)}+C\|u\|^2_{L^r(0,\varepsilon;\tilde{L}^q(\Omega))}\tag{4.3}$$

where $(u_0)_\varepsilon:=\frac{1}{\varepsilon}\int_0^\varepsilon e^{-\tau\tilde{A}_n}u_0\,d\tau\to u_0$ in $\tilde{L}^n_\sigma(\Omega)$ as $\varepsilon\to 0$.

Note that the set of exponents r,q as in Theorem 4.2 is nonempty for all n; e.g., $r=2n$, $q=nn'$ is a possible choice. For the proof of this theorem we need a technical lemma based on real interpolation.

Lemma 4.3. *Let $\Omega\subseteq\mathbb{R}^n$, $n\geq 3$, be a uniform C^2-domain and let $t\in(0,T)$ and $\varepsilon>0$ satisfy $0<t-\varepsilon<t+\varepsilon<T$. Choose some Serrin exponents*

$$2<r\leq 2n,\quad n<q<2n,\quad \frac{2}{r}+\frac{n}{q}=1.$$

Define the linear map $B_\varepsilon : C^\infty_{0,\sigma}(\Omega) \to \mathbb{R}^n$ *by*

$$B_\varepsilon(\psi)(s) = \frac{1}{2\varepsilon}\int_{t-\varepsilon}^{t+\varepsilon} \nabla e^{-(\tau-s)\tilde{A}_{n'}}\psi\, d\tau, \quad s\in(0,t-\varepsilon).$$

Then the inequality

$$\|B_\varepsilon(\psi)\|_{L^{(r/2)'}(0,t-\varepsilon;\tilde{L}^{(q/2)'}(\Omega))} \le C\|\psi\|_{\tilde{L}^{n'}(\Omega)}$$

holds with a constant C *only depending on* r, $\tau(\Omega)$, *but neither on* ψ, ε *nor on* t.

Proof. Define $\rho_1 := (q/2)'$ and $\rho_2 := q'$ so that $1<\rho_2<n'<\rho_1<\infty$.

Now we derive weak type estimates and use real interpolation. Using (2.9) we obtain for almost every $s\in(0,t-\varepsilon)$ that

$$\begin{aligned}\|B_\varepsilon(\psi)(s)\|_{\tilde{L}^{(q/2)'}} &\le \frac{C}{2\varepsilon}\int_{t-\varepsilon}^{t+\varepsilon}(\tau-s)^{-1/2}d\tau\|\psi\|_{\tilde{L}^{\rho_1}}\\ &\le C\|\psi\|_{\tilde{L}^{\rho_1}}(t-\varepsilon-s)^{-1/2}.\end{aligned}$$

Consequently,

$$B_\varepsilon\colon \tilde{L}^{\rho_1}_\sigma(\Omega)\to L^{2,\infty}\big(0,t-\varepsilon;\tilde{L}^{(q/2)'}_\sigma(\Omega)\big)$$

satisfying $\|B_\varepsilon\psi\|_{L^{2,\infty}(0,t-\varepsilon;\tilde{L}^{(q/2)'}_\sigma)} \le C\|\psi\|_{\tilde{L}^{\rho_1}}$ since $\|(t-\varepsilon-\cdot)^{-1/2}\|_{L^{2,\infty}(0,t-\varepsilon)}=1$. The second estimate is similar: We get for every $s\in(0,t-\varepsilon)$ that

$$\begin{aligned}\|B(\psi)(s)\|_{\tilde{L}^{(q/2)'}} &\le \frac{C}{2\varepsilon}\|\psi\|_{\tilde{L}^{\rho_2}}\int_{t-\varepsilon}^{t+\varepsilon}(\tau-s)^{-1/r'}d\tau\\ &\le C\|\psi\|_{\tilde{L}^{\rho_2}}(t-\varepsilon-s)^{-1/r'}.\end{aligned}$$

Hence $B_\varepsilon\colon \tilde{L}^{\rho_2}_\sigma(\Omega)\to L^{r',\infty}\big(0,t-\varepsilon;\tilde{L}^{(q/2)'}_\sigma(\Omega)\big)$ with norm bounded by C independent of t and ε. Thus real interpolation, with $\theta = 2-q/n$, yields

$$B\colon (\tilde{L}^{\rho_1}_\sigma,\tilde{L}^{\rho_2}_\sigma)_{\theta,(r/2)'} \to \big(L^{2,\infty}(0,t-\varepsilon;\tilde{L}^{(q/2)'}_\sigma), L^{r',\infty}(0,t-\varepsilon;\tilde{L}^{(q/2)'}_\sigma)\big)_{\theta,(r/2)'}.$$

Note that $n'\le(r/2)'$ since $r\le 2n$. Hence we get with (2.11) that

$$\tilde{L}^{n'}_\sigma = \tilde{L}^{n',n'}_\sigma = (\tilde{L}^{\rho_1}_\sigma,\tilde{L}^{\rho_2}_\sigma)_{\theta,n'} \hookrightarrow (\tilde{L}^{\rho_1}_\sigma,\tilde{L}^{\rho_2}_\sigma)_{\theta,(r/2)'}.$$

Finally we also have that

$$\begin{aligned}&\big(L^{2,\infty}(0,t-\varepsilon;\tilde{L}^{(q/2)'}_\sigma), L^{r',\infty}(0,t-\varepsilon;\tilde{L}^{(q/2)'}_\sigma)\big)_{\theta,(r/2)'}\\ &\quad = L^{(r/2)',(r/2)'}\big(0,t-\varepsilon;\tilde{L}^{(q/2)'}_\sigma\big) = L^{(r/2)'}\big(0,t-\varepsilon;\tilde{L}^{(q/2)'}_\sigma\big)\end{aligned}$$

by [29, Theorem 1.18.6.2]. Thus we conclude that

$$B_\varepsilon\colon \tilde{L}^{n'}_\sigma \to L^{(r/2)'}(0,t-\varepsilon;\tilde{L}^{(q/2)'}_\sigma).$$

Since the constants in the weak type estimates depend neither on ε nor on t, the same holds for the constant in this estimate. This finishes the proof. □

Proof of Theorem 4.2. For $\psi \in C_{0,\sigma}^{\infty}(\Omega)$ and every Lebesgue point t of u we will prove the estimate

$$|(u(t) - e^{-t\tilde{A}_n} u_0, \psi)| \le C\|u\|^2_{L^r(0,t;\tilde{L}^q)} \|\psi\|_{\tilde{L}^{n'}}. \tag{4.4}$$

Let $\varepsilon > 0$ satisfy $0 < t-\varepsilon < t+\varepsilon < T$. Then we put $v_\varepsilon(s) := \frac{1}{2\varepsilon}\chi_{(t-\varepsilon,t+\varepsilon)}(s)\psi$. Note that $v_\varepsilon \in L^{r'}(0,T;\tilde{L}^{q'}_\sigma(\Omega))$. Using Lemma 2.4 we define $\phi_\varepsilon \in \mathcal{T}^{1,r',q'}(T,\Omega)$ by $-(\phi_\varepsilon)_t + \tilde{A}_{q'}\phi_\varepsilon = v_\varepsilon$. Then we consider the identity

$$\begin{aligned} &\frac{1}{2\varepsilon}\int_{t-\varepsilon}^{t+\varepsilon} (u(s),\psi)\,ds = (u, v_\varepsilon)_{T,\Omega} \\ &\quad = -(u,\phi_{\varepsilon t})_{T,\Omega} + (u, \tilde{A}_{q'}\phi_\varepsilon)_{T,\Omega} = (u_0, \phi_\varepsilon(0)) + (u\otimes u, \nabla\phi_\varepsilon)_{T,\Omega}. \end{aligned} \tag{4.5}$$

Since

$$\phi_\varepsilon(0) = \int_0^T e^{-\tau\tilde{A}_{n'}} v_\varepsilon(\tau)\,d\tau = \frac{1}{2\varepsilon}\int_{t-\varepsilon}^{t+\varepsilon} e^{-\tau\tilde{A}_{n'}}\psi\,d\tau \;\to\; e^{-t\tilde{A}_{n'}}\psi$$

in $\tilde{L}^{n'}(\Omega)$ as $\varepsilon \to 0$, and since t is a Lebesgue point of u, by (4.5)

$$(u(t) - e^{-t\tilde{A}_n}u_0, \psi) = \lim_{\varepsilon\to 0+} (u\otimes u, \nabla\phi_\varepsilon)_{T,\Omega}.$$

To estimate $\nabla\phi_\varepsilon$ we calculate

$$\begin{aligned} \phi_\varepsilon(s) &= \int_0^{T-s} e^{-(T-s-\tau)\tilde{A}_{n'}} v_\varepsilon(T-\tau)\,d\tau \\ &= \frac{1}{2\varepsilon}\int_s^T e^{-(\tau-s)\tilde{A}_{n'}} \chi_{(t-\varepsilon,t+\varepsilon)}(\tau)\,\psi\,d\tau. \end{aligned}$$

In particular, $\phi_\varepsilon(s) = 0$ for $t+\varepsilon < s < T$. Observe that $(r/2)' = q/n$. Hence

$$\begin{aligned} |(u\otimes u, \nabla\phi_\varepsilon)_{T,\Omega}| &\le \int_0^{t+\varepsilon} |(u(s)\otimes u(s), \nabla\phi_\varepsilon(s))|\,ds \\ &\le \|u\|^2_{L^r(0,t-\varepsilon;\tilde{L}^q)} \|\nabla\phi_\varepsilon\|_{L^{(r/2)'}(0,t-\varepsilon;\tilde{L}^{(q/2)'})} \\ &\quad + \|u\|^2_{L^r(t-\varepsilon,t+\varepsilon;\tilde{L}^q)} \|\nabla\phi_\varepsilon\|_{L^{(r/2)'}(t-\varepsilon,t+\varepsilon;\tilde{L}^{(q/2)'})}. \end{aligned} \tag{4.6}$$

Now Lemma 4.3 will be applied to the term $\nabla\phi_\varepsilon$ in $(4.6)_2$ yielding

$$\|\nabla\phi_\varepsilon\|_{L^{(r/2)'}(0,t-\varepsilon;\tilde{L}^{(q/2)'})} \le C\|\psi\|_{\tilde{L}^{n'}}$$

with a constant independent of ε and t. Finally, let us consider the term in $(4.6)_3$. Since for $t-\varepsilon < s < t+\varepsilon$ we have $\phi_\varepsilon(s) = \frac{1}{2\varepsilon}\int_s^{t+\varepsilon} e^{-(\tau-s)\tilde{A}_{n'}}\psi\,d\tau$ we get that

$$\begin{aligned} \|\nabla\phi_\varepsilon\|_{L^{(r/2)'}(t-\varepsilon,t+\varepsilon;\tilde{L}^{(q/2)'})} &\le \frac{1}{2\varepsilon}\left(\int_{t-\varepsilon}^{t+\varepsilon}\left(\int_s^{t+\varepsilon} \|\nabla e^{-(\tau-s)\tilde{A}_{n'}}\psi\|_{\tilde{L}^{(q/2)'}}\,d\tau\right)^{q/n} ds\right)^{n/q} \\ &\le C\frac{\|\psi\|_{\tilde{L}^{n'}}}{2\varepsilon}\left(\int_{t-\varepsilon}^{t+\varepsilon}\left(\int_s^{t+\varepsilon}(\tau-s)^{-n/q}d\tau\right)^{q/n} ds\right)^{n/q} \\ &\le C\|\psi\|_{\tilde{L}^{n'}}. \end{aligned}$$

This leads to the estimate

$$
\begin{aligned}
|(u \otimes u, \nabla \phi_\varepsilon)_{T,\Omega}| &\le C\big(\|u\|^2_{L^r(0,t-\varepsilon;\tilde{L}^q)} + \|u\|^2_{L^r(t-\varepsilon,t+\varepsilon;\tilde{L}^q)}\big)\|\psi\|_{\tilde{L}^{n'}} \\
&\le C\|u\|^2_{L^r(0,t+\varepsilon;\tilde{L}^q)}\|\psi\|_{\tilde{L}^{n'}}
\end{aligned}
$$

for every $\psi \in C^\infty_{0,\sigma}(\Omega)$. Passing to the limit $\varepsilon \to 0$ we arrive at the estimate (4.4) for every Lebesgue point $t \in (0,T)$ of u.

For the estimate at $t = 0$ we exploit (4.2) for a.a. $t \in (0,\varepsilon)$, take the mean value over $(0,\varepsilon)$ and get (4.3) by the triangle inequality. A more direct argument copies the previous proof more or less by using $v_\varepsilon = \frac{1}{\varepsilon}\chi_{(0,\varepsilon)}\psi$ and even avoids Lemma 4.3. □

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