Resolvent Estimates and Maximal Regularity in Weighted Lebesgue Spaces of the Stokes Operator in Unbounded Cylinders

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Abstract

We study resolvent estimate and maximal regularity of the Stokes operator in L^q-spaces with exponential weights in the axial directions of unbounded cylinders of $\mathbb{R}^n, n \geq 3$. For straights cylinders we obtain these results in Lebesgue spaces with exponential weights in the axial direction and Muckenhoupt weights in the cross-section. Next, for general cylinders with several exits to infinity we prove that the Stokes operator in L^q-spaces with exponential weight along the axial directions generates an exponentially decaying analytic semigroup and has maximal regularity.

The proofs for straight cylinders use an operator-valued Fourier multiplier theorem and techniques of unconditional Schauder decompositions based on the $\mathcal{R}$-boundedness of the family of solution operators for a system in the cross-section of the cylinder parametrized by the phase variable of the one-dimensional partial Fourier transform. For general cylinders we use cut-off techniques based on the result for straight cylinders and the result for the case without exponential weight.

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1 Introduction

Let

$$\Omega = \bigcup_{i=0}^{m} \Omega_i \tag{1.1}$$

be a cylindrical domain of $C^{1,1}$-class where Ω_0 is a bounded domain and $\Omega_i, i = 1, \ldots, m$, are disjoint semi-infinite straight cylinders, that is, in possibly different

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coordinates,

$$\Omega_i = \{x^i = (x_1^i, \dots, x_n^i) \in \mathbb{R}^n : \, x_n^i > 0, \, (x_1^i, \dots, x_{n-1}^i) \in \Sigma^i\},$$

where the cross sections $\Sigma^i \subset \mathbb{R}^{n-1}$, $i = 1, \dots, m$, are bounded domains and $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$.

Given $\beta_i \geq 0, i = 1, \dots, m$, introduce the space

$$\begin{aligned} L^q_{\mathbf{b}}(\Omega) &= \{U \in L^q(\Omega) : e^{\beta_i x_n^i} U|_{\Omega_i} \in L^q(\Omega_i)\}, \\ \|U\|_{L^q_{\mathbf{b}}(\Omega)} &:= \left(\|U\|^q_{L^q(\Omega_0)} + \textstyle\sum_{i=1}^m \|e^{\beta_i x_n^i} U\|^q_{L^q(\Omega_i)}\right)^{1/q} \end{aligned}$$

for $1 < q < \infty$. Moreover, let $W^{k,q}_{\mathbf{b}}(\Omega)$, $k \in \mathbb{N}$, be the space of functions whose derivatives up to k-th order belong to $L^q_{\mathbf{b}}(\Omega)$, where a norm is endowed in the standard way. Let $W^{1,q}_{0,\mathbf{b}}(\Omega) = \{u \in W^{1,q}_{\mathbf{b}}(\Omega) : u|_{\partial\Omega} = 0\}$. Let $L^q_\sigma(\Omega)$ and $L^q_{\mathbf{b},\sigma}(\Omega)$ be the completion of the set $C^\infty_{0,\sigma}(\Omega) = \{u \in C^\infty_0(\Omega)^n : \operatorname{div} u = 0\}$ in the norm of $L^q(\Omega)$ and $L^q_{\mathbf{b}}(\Omega)$, respectively. Then we consider the Stokes operator $A = A_{q,\mathbf{b}} = -P_q\Delta$ in $L^q_{\mathbf{b},\sigma}(\Omega)$ with domain

$$\mathcal{D}(A) = W^{2,q}_{\mathbf{b}}(\Omega)^n \cap W^{1,q}_{0,\mathbf{b}}(\Omega)^n \cap L^q_\sigma(\Omega),$$

where P_q is the Helmholtz projection of $L^q(\Omega)$ onto $L^q_\sigma(\Omega)$.

The goal of this paper is to study resolvent estimates and maximal L^p-regularity of the Stokes operator in Lebesgue spaces with exponential weights in the axial direction. The semigroup approach to instationary Navier-Stokes equations is a very convenient tool to prove existence, uniqueness and stability of solutions; to this end, resolvent estimates of the Stokes operator must be obtained. Moreover, maximal regularity of the Stokes operator helps to deal with the nonlinearity of the Navier-Stokes equations.

There are many papers dealing with resolvent estimates ([7], [8], [15], [16], [20]; see Introduction of [10] for more details) or maximal regularity (see e.g. [1], [14], [16]) of Stokes operators for domains with compact as well as noncompact boundaries. General unbounded domains are considered in [6] by replacing the space L^q by $L^q \cap L^2$ or $L^q + L^2$. For resolvent estimates and maximal regularity in unbounded cylinders without exponential weights in the axial direction we refer the reader e.g. to [10]-[13] and [31]. For partial results in the Bloch space of uniformly square integrable functions on a cylinder we refer to [33].

Further results on stationary Stokes and instationary Stokes and Navier-Stokes systems in unbounded cylindrical domains can be found e.g. in [2], [3], [17], [18], [21]-[30], [33]-[35].

Despite of some references showing the existence of stationary flows in L^q-setting (e.g. [25], [26], [28]) and instationary flows in L^2-setting (e.g. [29], [30]) that converge at $|x| \to \infty$ to some limit states (Poiseuille flow or zero flow) in unbounded cylinders, resolvent estimates and maximal regularity of the Stokes operator in L^q-spaces with exponential weights on unbounded cylinders do not seem to have been obtained yet.

We start our work with consideration of the Stokes operator in straight cylinders; we get resolvent estimate and maximal regularity of the Stokes operator even in

$L^q_\beta(\mathbb{R}; L^r_\omega(\Sigma))$, $1 < q, r < \infty$, with exponential weight $e^{\beta x_n}$, $\beta > 0$, and arbitrary Muckenhoupt weight $\omega \in A_r(\mathbb{R}^{n-1})$ with respect to $x' \in \Sigma$ (see Section 2 for the definition). We note that our resolvent estimate gives, in particular when $\lambda = 0$, a new result on the existence of a unique flow with zero flux for the stationary Stokes system in $L^q_\beta(\mathbb{R}, L^r_\omega(\Sigma))$. Next, based on the results for straight cylinders, we get resolvent estimates and maximal L^p-regularity of the Stokes operator in $L^q_{\mathbf{b}}(\Omega)$, $1 < q < \infty$, for general cylinders Ω using a cut-off technique.

The proofs for straight cylinders are mainly based on the theory of Fourier analysis. By the application of the partial Fourier transform along the axis of the cylinder $\Sigma \times \mathbb{R}$ the *generalized Stokes resolvent system*

$$
(R_\lambda) \qquad \begin{array}{rcll} \lambda U - \Delta U + \nabla P &=& F & \text{in } \Sigma \times \mathbb{R}, \\ \operatorname{div} U &=& G & \text{in } \Sigma \times \mathbb{R}, \\ u &=& 0 & \text{on } \partial\Sigma \times \mathbb{R}, \end{array}
$$

is reduced to the *parametrized Stokes system* in the cross-section Σ:

$$
(R_{\lambda,\eta}) \qquad \begin{array}{rcll} (\lambda + \eta^2 - \Delta')\hat{U}' + \nabla' \hat{P} &=& \hat{F}' & \text{in } \Sigma, \\ (\lambda + \eta^2 - \Delta')\hat{U}_n + i\eta \hat{P} &=& \hat{F}_n & \text{in } \Sigma, \\ \operatorname{div}' \hat{U}' + i\eta \hat{U}_n &=& \hat{G} & \text{in } \Sigma, \\ \hat{U}' = 0, \quad \hat{U}_n &=& 0 & \text{on } \partial\Sigma, \end{array}
$$

which involves the Fourier phase variable $\eta \in \mathbb{C}$ as parameter. Now, for fixed $\beta \geq 0$ let

$$
(\hat{u}, \hat{p}, \hat{f}, \hat{g})(\xi) := (\hat{U}, \hat{P}, \hat{F}, \hat{G})(\xi + i\beta).
$$

Then $(R_{\lambda,\eta})$ is reduced to the system

$$
(R_{\lambda,\xi,\beta}) \qquad \begin{array}{rcll} (\lambda + (\xi + i\beta)^2 - \Delta')\hat{u}'(\xi) + \nabla' \hat{p}(\xi) &=& \hat{f}'(\xi) & \text{in } \Sigma, \\ (\lambda + (\xi + i\beta)^2 - \Delta')\hat{u}_n(\xi) + i(\xi + i\beta)\hat{p}(\xi) &=& \hat{f}_n(\xi) & \text{in } \Sigma, \\ \operatorname{div}' \hat{u}'(\xi) + i(\xi + i\beta)\hat{u}_n(\xi) &=& \hat{g}(\xi) & \text{in } \Sigma, \\ \hat{u}'(\xi) = 0, \quad \hat{u}_n(\xi) &=& 0 & \text{on } \partial\Sigma. \end{array}
$$

We will get estimates of solutions to $(R_{\lambda,\xi,\beta})$ independent of $\xi \in \mathbb{R}^* := \mathbb{R} \setminus \{0\}$ and λ in L^r-spaces with Muckenhoupt weights, which yield $\mathcal{R}$-boundedness of the family of solution operators $a(\xi)$ for $(R_{\lambda,\xi,\beta})$ with $g = 0$ due to an extrapolation property of operators defined on L^r-spaces with Muckenhoupt weights, see Theorem 4.8. Then, an operator-valued Fourier multiplier theorem ([36]) implies the estimate of $e^{\beta x_n} U = \mathcal{F}^{-1}(a(\xi)\mathcal{F}f)$ for the solution U to (R_λ) with $G = 0$ in the straight cylinder $\Sigma \times \mathbb{R}$. In order to prove maximal regularity of the Stokes operator in straight cylinders we use that maximal regularity of an operator A in a *UMD* space X is implied by the $\mathcal{R}$-boundedness of the operator family

$$
\{\lambda(\lambda + A)^{-1} : \ \lambda \in i\,\mathbb{R}\} \tag{1.2}
$$

in $\mathcal{L}(X)$, see [36]. We show the $\mathcal{R}$-boundedness of (1.2) for the Stokes operator $A := A_{q,r;\beta,\omega}$ in $L^q_\beta(\mathbb{R} : L^r_\omega(\Sigma))$ by virtue of Schauder decomposition techniques; to be more precise, we use the Schauder decomposition $\{\Delta_j\}_{j\in\mathbb{Z}}$ where $\Delta_j = \mathcal{F}^{-1}\chi_{[2^j,2^{j+1})}\mathcal{F}$ to get R-boundedness of the family (1.2).

The proof for general cylinders, Theorem 2.4 and Theorem 2.5, uses a cut-off technique based on the result for resolvent estimates and maximal regularity without exponential weights in [13] and the result (Theorem 2.3) for straight cylinders.

This paper is organized as follows. In Section 2 the main results of this paper (Theorem 2.1, Corollary 2.2, Theorem 2.3 – Theorem 2.5) and preliminaries are given. In Section 3 we obtain the estimate for $(R_{\lambda,\xi,\beta})$ on bounded domains, see Theorem 3.8. In Section 4 proofs of the main results are given.

2 Main Results and Preliminaries

Let $\Sigma \times \mathbb{R}$ be an infinite cylinder of $\mathbb{R}^n$ with bounded cross section $\Sigma \subset \mathbb{R}^{n-1}$ and with generic point $x \in \Sigma \times \mathbb{R}$ written in the form $x = (x', x_n) \in \Sigma \times \mathbb{R}$, where $x' \in \Sigma$ and $x_n \in \mathbb{R}$. Similarly, differential operators in $\mathbb{R}^n$ are split, in particular, $\Delta = \Delta' + \partial_n^2$ and $\nabla = (\nabla', \partial_n)$.

For $q \in (1,\infty)$ we use the standard notation $L^q(\Sigma \times \mathbb{R}) = L^q(\mathbb{R}; L^q(\Sigma))$ for classical Lebesgue spaces with norm $\|\cdot\|_q = \|\cdot\|_{q;\Sigma\times\mathbb{R}}$ and $W^{k,q}(\Sigma \times \mathbb{R}), k \in \mathbb{N}$, for the usual Sobolev spaces with norm $\|\cdot\|_{k,q;\Sigma\times\mathbb{R}}$. We do not distinguish between spaces of scalar functions and vector-valued functions as long as no confusion arises. In particular, we use the short notation $\|u, v\|_X$ for $\|u\|_X + \|v\|_X$, even if u and v are tensors of different order.

Let $1 < r < \infty$. A function $0 \le \omega \in L^1_{\mathrm{loc}}(\mathbb{R}^{n-1})$ is called A_r-*weight* (*Muckenhoupt weight*) on $\mathbb{R}^{n-1}$ iff

$$\mathcal{A}_r(\omega) := \sup_Q \left(\frac{1}{|Q|}\int_Q \omega\, dx'\right)\cdot\left(\frac{1}{|Q|}\int_Q \omega^{-1/(r-1)}\, dx'\right)^{r-1} < \infty$$

where the supremum is taken over all cubes of $\mathbb{R}^{n-1}$ and $|Q|$ denotes the $(n-1)$-dimensional Lebesgue measure of Q. We call $\mathcal{A}_r(\omega)$ the A_r-constant of ω and denote the set of all A_r-weights on $\mathbb{R}^{n-1}$ by $A_r = A_r(\mathbb{R}^{n-1})$. Note that

$$\omega \in A_r \quad \text{iff} \quad \omega' := \omega^{-1/(r-1)} \in A_{r'}, \quad r' = r/(r-1),$$

and $\mathcal{A}_{r'}(\omega') = \mathcal{A}_r(\omega)^{r'/r}$. A constant $C = C(\omega)$ is called A_r-*consistent* if for every $d > 0$

$$\sup\{C(\omega) :\ \omega \in A_r,\, \mathcal{A}_r(\omega) < d\} < \infty.$$

We write $\omega(Q)$ for $\int_Q \omega\, dx'$.

Typical Muckenhoupt weights are the radial functions $\omega(x) = |x|^\alpha$: it is well-known that $\omega \in A_r(\mathbb{R}^{n-1})$ if and only if $-(n-1) < \alpha < (r-1)(n-1)$; the same bounds for α hold when $\omega(x) = (1+|x|)^\alpha$ and $\omega(x) = |x|^\alpha(\log(e+|x|)^\beta$ for all $\beta \in \mathbb{R}$. For further examples we refer to [8].

Given $\omega \in A_r, r \in (1,\infty)$, and an arbitrary domain $\Sigma \subset \mathbb{R}^{n-1}$ let

$$L^r_\omega(\Sigma) = \Big\{ u \in L^1_{\text{loc}}(\bar{\Sigma}) : \|u\|_{r,\omega} = \|u\|_{r,\omega;\Sigma} = \Big(\int_\Sigma |u|^r \omega \, dx' \Big)^{1/r} < \infty \Big\}.$$

For short we will write L^r_ω for $L^r_\omega(\Sigma)$ provided that the underlying domain Σ is known from the context. It is well-known that L^r_ω is a separable reflexive Banach space with dense subspace $C_0^\infty(\Sigma)$. In particular $(L^r_\omega)^* = L^{r'}_{\omega'}$. As usual, $W^{k,r}_\omega(\Sigma)$, $k \in \mathbb{N}$, denotes the weighted Sobolev space with norm

$$\|u\|_{k,r,\omega} = \Big(\sum_{|\alpha| \le k} \|D^\alpha u\|^r_{r,\omega} \Big)^{1/r},$$

where $|\alpha| = \alpha_1 + \cdots + \alpha_{n-1}$ is the length of the multi-index $\alpha = (\alpha_1, \ldots, \alpha_{n-1}) \in \mathbb{N}_0^{n-1}$ and $D^\alpha = \partial_1^{\alpha_1} \cdot \ldots \cdot \partial_{n-1}^{\alpha_{n-1}}$; moreover, $W^{k,r}_{0,\omega}(\Sigma) := \overline{C_0^\infty(\Sigma)}^{\|\cdot\|_{k,r,\omega}}$ and $W^{-k,r}_{0,\omega}(\Sigma) := (W^{k,r'}_{0,\omega'}(\Sigma))^*$, where $r' = r/(r-1)$. We introduce the weighted homogeneous Sobolev space

$$\widehat{W}^{1,r}_\omega(\Sigma) = \big\{ u \in L^1_{\text{loc}}(\bar{\Sigma})/\mathbb{R} : \; \nabla' u \in L^r_\omega(\Sigma) \big\}$$

with norm $\|\nabla' u\|_{r,\omega}$ and its dual space $\widehat{W}^{-1,r'}_{\omega'} := (\widehat{W}^{1,r}_\omega)^*$ with norm $\|\cdot\|_{-1,r',\omega'} = \|\cdot\|_{-1,r',\omega';\Sigma}$.

Let $q, r \in (1,\infty)$. On an infinite cylinder $\Sigma \times \mathbb{R}$, where Σ is a bounded $C^{1,1}$-domain of $\mathbb{R}^{n-1}$, we introduce the function space $L^q(L^r_\omega) := L^q(\mathbb{R}; L^r_\omega(\Sigma))$ with norm

$$\|u\|_{L^q(L^r_\omega)} = \Big(\int_\mathbb{R} \Big(\int_\Sigma |u(x', x_n)|^r \omega(x') \, dx' \Big)^{q/r} dx_n \Big)^{1/q}.$$

Furthermore, $W^{k;q,r}_\omega(\Sigma \times \mathbb{R}), k \in \mathbb{N}$, denotes the Banach space of all functions in $\Sigma \times \mathbb{R}$ whose derivatives of order up to k belong to $L^q(L^r_\omega)$ with norm $\|u\|_{W^{k;q,r}_\omega} = (\sum_{|\alpha| \le k} \|D^\alpha u\|^2_{L^q(L^r_\omega)})^{1/2}$, where $\alpha \in \mathbb{N}_0^n$, and let $W^{1;q,r}_{0,\omega}(\Omega)$ be the completion of the set $C_0^\infty(\Omega)$ in $W^{1;q,r}_\omega(\Omega)$. Given $\beta > 0$, we denote by

$$L^q_\beta(L^r_\omega) := \{u : e^{\beta x_n} u \in L^q(L^r_\omega)\}$$

with norm $\|e^{\beta x_n} \cdot \|_{L^q(L^r_\omega)}$ and for $k \in \mathbb{N}$

$$W^{k;q,r}_{\beta,\omega}(\Sigma \times \mathbb{R}) := \{u : e^{\beta x_n} u \in W^{k;q,r}_\omega(\Sigma \times \mathbb{R})\}$$

with norm $\|e^{\beta x_n} \cdot \|_{W^{k;q,r}_\omega(\Sigma \times \mathbb{R})}$. Finally, $L^q(L^r_\omega)_\sigma$ and $L^q_\beta(L^r_\omega)_\sigma$ are completions in the space $L^q(L^r_\omega)$ and $L^q_\beta(L^r_\omega)$ of the set

$$C^\infty_{0,\sigma}(\Sigma \times \mathbb{R}) = \{u \in C_0^\infty(\Sigma \times \mathbb{R})^n; \quad \operatorname{div} u = 0\},$$

respectively.

The Fourier transform in the variable x_n is denoted by $\mathcal{F}$ or $\widehat{}$ and the inverse Fourier transform by $\mathcal{F}^{-1}$ or $^\vee$. For $\varepsilon \in (0, \frac{\pi}{2})$ we define the complex sector

$$S_\varepsilon = \{\lambda \in \mathbb{C}; \lambda \neq 0, |\text{arg}\lambda| < \frac{\pi}{2} + \varepsilon\}.$$

The first main theorem of this paper is as follows.

Theorem 2.1 (Weighted Resolvent Estimates) *Let Σ be a bounded domain of $C^{1,1}$-class with $\alpha_0 > 0$ and $\alpha_1 > 0$ being the least eigenvalue of the Dirichlet and Neumann Laplacian in Σ, and let $\bar{\alpha} := \min\{\alpha_0, \alpha_1\}$, $\beta \in (0, \sqrt{\bar{\alpha}})$, $\alpha \in (0, \bar{\alpha} - \beta^2)$, $0 < \varepsilon < \varepsilon^* := \arctan\left(\frac{1}{\beta}\sqrt{\bar{\alpha} - \beta^2 - \alpha}\right)$, $1 < q, r < \infty$ and $\omega \in A_r$. Then for every $f \in L^q_\beta(\mathbb{R}; L^r_\omega(\Sigma))$, and $\lambda \in -\alpha + S_\varepsilon$ there exists a unique solution $(u, \nabla p)$ to (R_λ) (with $g = 0$) such that*

$$(\lambda + \alpha)u, \nabla^2 u, \nabla p \in L^q_\beta(L^r_\omega)$$

and

$$\|(\lambda + \alpha)u, \nabla^2 u, \nabla p\|_{L^q_\beta(L^r_\omega)} \leq C\|f\|_{L^q_\beta(L^r_\omega)} \tag{2.1}$$

with an A_r-consistent constant $C = C(q, r, \alpha, \beta, \varepsilon, \Sigma, \mathcal{A}_r(\omega))$ independent of λ.

In particular we obtain from Theorem 2.1 the following corollary on resolvent estimates of the Stokes operator in the cylinder Ω.

Corollary 2.2 (Stokes Semigroup in Straight Cylinders) *Let $1 < q, r < \infty$, $\omega \in A_r(\mathbb{R}^{n-1})$ and define the Stokes operator $A = A_{q,r;\beta,\omega}$ on $\Sigma \times \mathbb{R}$ by*

$$D(A) = W^{2;q,r}_{\beta,\omega}(\Sigma \times \mathbb{R}) \cap W^{1;q,r}_{0,\beta,\omega}(\Sigma \times \mathbb{R}) \cap L^q_\beta(L^r_\omega)_\sigma \subset L^q_\beta(L^r_\omega)_\sigma,\ Au = -P_{q,r;\beta,\omega}\Delta u, \tag{2.2}$$

where $P_{q,r;\beta,\omega}$ is the Helmholtz projection in $L^q_\beta(L^r_\omega)$ (see [9]). Then, for every $\varepsilon \in (0, \varepsilon^*)$ and $\alpha \in (0, \bar{\alpha} - \beta^2)$, $\beta \in (0, \sqrt{\bar{\alpha}})$, $-\alpha + S_\varepsilon$ is contained in the resolvent set of $-A$, and the estimate

$$\|(\lambda + A)^{-1}\|_{\mathcal{L}(L^q(L^r_\omega)_\sigma)} \leq \frac{C}{|\lambda + \alpha|}, \quad \forall \lambda \in -\alpha + S_\varepsilon, \tag{2.3}$$

holds with an A_r-consistent constant $C = C(\Sigma, q, r, \alpha, \beta, \varepsilon, \mathcal{A}_r(\omega))$.

As a consequence, the Stokes operator generates a bounded analytic semigroup $\{e^{-tA_{q,r;\beta,\omega}}; t \geq 0\}$ on $L^q_\beta(L^r_\omega)_\sigma$ satisfying the estimate

$$\|e^{-tA_{q,r;\beta,\omega}}\|_{\mathcal{L}(L^q_\beta(L^r_\omega)_\sigma)} \leq C\, e^{-\alpha t} \quad \forall \alpha \in (0, \bar{\alpha} - \beta^2), \forall t > 0, \tag{2.4}$$

with a constant $C = C(q, r, \alpha, \beta, \varepsilon, \Sigma, \mathcal{A}_r(\omega))$.

The second important result of this paper is the *maximal regularity* of the Stokes operator in an infinite straight cylinder.

Theorem 2.3 (Maximal Regularity in Straight Cylinders) *Let $1 < p, q, r < \infty$, $\omega \in A_r(\mathbb{R}^{n-1})$ and $\beta \in (0, \sqrt{\bar{\alpha}})$. Then the Stokes operator $A = A_{q,r;\beta,\omega}$ has maximal regularity in $L^q_\beta(L^r_\omega)_\sigma$. To be more precise, for each $F \in L^p(\mathbb{R}_+; L^q_\beta(L^r_\omega)_\sigma)$ the instationary problem*

$$U_t + AU = F, \quad U(0) = 0, \tag{2.5}$$

has a unique solution $U \in W^{1,p}(\mathbb{R}_+; L^q_\beta(L^r_\omega)_\sigma) \cap L^p(\mathbb{R}_+; D(A))$ such that

$$\|U, U_t, AU\|_{L^p(\mathbb{R}_+; L^q_\beta(L^r_\omega)_\sigma)} \leq C\|F\|_{L^p(\mathbb{R}_+; L^q_\beta(L^r_\omega)_\sigma)}. \tag{2.6}$$

Analogously, for every $F \in L^p(\mathbb{R}_+; L^q_\beta(L^r_\omega))$*, the instationary system*

$$U_t - \Delta U + \nabla P = F, \quad \operatorname{div} U = 0, \quad U(0) = 0,$$

has a unique solution

$$(U, \nabla P) \in \big(W^{1,p}(\mathbb{R}_+; L^q_\beta(L^r_\omega)_\sigma) \cap L^p(\mathbb{R}_+; D(A))\big) \times L^p(\mathbb{R}_+; L^q_\beta(L^r_\omega))$$

satisfying the a priori estimate

$$\|U_t, U, \nabla U, \nabla^2 U, \nabla P\|_{L^p(\mathbb{R}_+; L^q_\beta(L^r_\omega))} \leq C\|F\|_{L^p(\mathbb{R}_+; L^q_\beta(L^r_\omega))} \tag{2.7}$$

with $C = C(\Sigma, q, r, \beta, \mathcal{A}_r(\omega))$*. Moreover, if* $e^{\alpha t}F \in L^p(\mathbb{R}_+; L^q_\beta(L^r_\omega))$ *for some* $\alpha \in (0, \bar{\alpha} - \beta^2)$*, then the solution* u *satisfies the estimate*

$$\|e^{\alpha t}U, e^{\alpha t}U_t, e^{\alpha t}\nabla^2 U\|_{L^p(\mathbb{R}_+; L^q_\beta(L^r_\omega))} \leq C\|e^{\alpha t}F\|_{L^p(\mathbb{R}_+; L^q_\beta(L^r_\omega))} \tag{2.8}$$

with $C = C(\Sigma, q, r, \alpha, \beta, \mathcal{A}_r(\omega))$*.*

As a corollary of Theorem 2.3 we get the maximal regularity result for general cylinder Ω with several exits to infinity given by (1.1).

Theorem 2.4 (Stokes Semigroup in General Cylinders) *Let a* $C^{1,1}$*-domain* Ω *be given by* (1.1) *and* $\beta_i > 0$ *for* $i = 1, \ldots, m$ *satisfy the same assumptions on* β *with* Σ^i *in place of* Σ*. Then, the Stokes operator* $A_{q,\boldsymbol{b}}(\Omega)$ *generates an exponentially decaying analytic semigroup* $\{e^{-tA_{q,\boldsymbol{b}}}\}_{t\geq 0}$ *in* $L^q_{\boldsymbol{b},\sigma}(\Omega)$*.*

Theorem 2.5 (Maximal Regularity in General Cylinders) *Let a* $C^{1,1}$*-domain* Ω *be given by* (1.1) *and* $\beta_i > 0$ *for* $i = 1, \ldots, m$ *satisfy the same assumptions on* β *with* Σ^i *in place of* Σ*. Then, the Stokes operator* $A_{q,\boldsymbol{b}}$ *has maximal regularity in* $L^q_{\boldsymbol{b},\sigma}(\Omega)$*; to be more precise, for any* $F \in L^p(\mathbb{R}_+; L^q_{\boldsymbol{b},\sigma}(\Omega))$ *the Cauchy problem*

$$U_t + A_{q,\boldsymbol{b}}U = F,\ U(0) = 0, \quad \text{in } L^q_{\boldsymbol{b},\sigma}(\Omega), \tag{2.9}$$

has a unique solution U *such that*

$$\|U, U_t, A_{q,\boldsymbol{b}}U\|_{L^p(\mathbb{R}+; L^q_{b,\sigma}(\Omega))} \leq C\|F\|_{L^p(\mathbb{R}+; L^q_{b,\sigma}(\Omega))} \tag{2.10}$$

with some constant $C = C(q, \Omega)$*.*

Equivalently, if $F \in L^p(\mathbb{R}_+; L^q_{\boldsymbol{b}}(\Omega))$*, then the instationary Stokes system*

$$\begin{aligned} U_t - \Delta U + \nabla P &= F \quad \text{in } \mathbb{R}_+ \times \Omega, \\ \operatorname{div} U &= 0 \quad \text{in } \mathbb{R}_+ \times \Omega, \\ U(0) &= 0 \quad \text{in } \Omega, \\ U &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{2.11}$$

has a unique solution $(U, \nabla P)$ *such that*

$$\begin{aligned} &(U, \nabla P) \in (L^p(\mathbb{R}_+; W^{2,q}_{\boldsymbol{b}}(\Omega) \cap W^{1,q}_0(\Omega)) \cap L^q_\sigma(\Omega)) \times L^p(\mathbb{R}_+; L^q_{\boldsymbol{b}}(\Omega)), \\ &U_t \in L^p(\mathbb{R}_+; L^q_{\boldsymbol{b}}(\Omega)), \\ &\|U\|_{L^p(\mathbb{R}_+; W^{2,q}_{\boldsymbol{b}}(\Omega) \cap W^{1,q}_0(\Omega))} + \|U_t, \nabla P\|_{L^p(\mathbb{R}_+; L^q_{\boldsymbol{b}}(\Omega))} \leq C\|F\|_{L^p(\mathbb{R}_+; L^q_{\boldsymbol{b}}(\Omega))}. \end{aligned} \tag{2.12}$$

Remark 2.6 We note that in (2.5) and in (2.11) we may take nonzero initial values $u(0) = u_0$ in the interpolation space $(L^q_\beta(L^r_\omega)_\sigma, D(A_{q,r;\beta,\omega}))_{1-1/p,p}$ and $U(0) = U_0 \in (L^q_{\mathbf{b}}(\Omega), W^{2,q}_{\mathbf{b}}(\Omega) \cap W^{1,q}_{0,\mathbf{b}}(\Omega))_{1-1/p,p}$, respectively.

For the proofs in Section 3 and Section 4, we need some preliminary results for Muckenhoupt weights.

Proposition 2.7 ([9], Lemma 2.4) *Let $1 < r < \infty$ and $\omega \in A_r(\mathbb{R}^{n-1})$.*

(1) Let $T : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ be a bijective, bi-Lipschitz vector field. Then, it holds that $\omega \circ T \in A_r(\mathbb{R}^{n-1})$ and $\mathcal{A}_r(\omega \circ T) \le c\, \mathcal{A}_r(\omega)$ with a constant $c = c(T, r) > 0$ independent of ω.

(2) Define the weight $\tilde{\omega}(x') = \omega(|x_1|, x'')$ for $x' = (x_1, x'') \in \mathbb{R}^{n-1}$. Then $\tilde{\omega} \in A_r$ and $\mathcal{A}_r(\tilde{\omega}) \le 2^r\, \mathcal{A}_r(\omega)$.

(3) Let $\Sigma \subset \mathbb{R}^{n-1}$ be a bounded domain. Then there exist $\tilde{s}, s \in (1, \infty)$ satisfying

$$L^{\tilde{s}}(\Sigma) \hookrightarrow L^r_\omega(\Sigma) \hookrightarrow L^s(\Sigma).$$

Here $\tilde{s}$ and $\frac{1}{s}$ are A_r-consistent. Moreover, the embedding constants can be chosen uniformly on a set $W \subset A_r$ provided that

$$\sup_{\omega \in W} \mathcal{A}_r(\omega) < \infty, \qquad \int_Q \omega\, dx' = 1 \quad \text{for all } \omega \in W, \tag{2.13}$$

for a cube $Q \subset \mathbb{R}^{n-1}$ with $\bar{\Sigma} \subset Q$.

Proposition 2.8 ([9], Proposition 2.5) *Let $\Sigma \subset \mathbb{R}^{n-1}$ be a bounded Lipschitz domain and let $1 < r < \infty$.*

(1) For every $\omega \in A_r$ the continuous embedding $W^{1,r}_\omega(\Sigma) \hookrightarrow L^r_\omega(\Sigma)$ is compact.

(2) Consider a sequence of weights $(\omega_j) \subset A_r$ satisfying (2.13) for $W = \{\omega_j : j \in \mathbb{N}\}$ and a fixed cube $Q \subset \mathbb{R}^{n-1}$ with $\bar{\Sigma} \subset Q$. Further let (u_j) be a sequence of functions on Σ satisfying

$$\sup_j \|u_j\|_{1,r,\omega_j} < \infty \quad \text{and} \quad u_j \rightharpoonup 0 \quad \text{in } W^{1,s}(\Sigma)$$

for $j \to \infty$ where s is given by Proposition 2.7 (3). Then

$$\|u_j\|_{r,\omega_j} \to 0 \quad \text{for } j \to \infty.$$

(3) Under the same assumptions on $(\omega_j) \subset A_r$ as in (2) consider a sequence of functions (v_j) on Σ satisfying

$$\sup_j \|v_j\|_{r,\omega_j} < \infty \quad \text{and} \quad v_j \rightharpoonup 0 \quad \text{in } L^s(\Sigma)$$

for $j \to \infty$. Then considering v_j as functionals on $W^{1,r'}_{\omega'_j}(\Sigma)$

$$\|v_j\|_{(W^{1,r'}_{\omega'_j}(\Sigma))^*} \to 0 \quad \text{for } j \to \infty.$$

Proposition 2.9 *Let $r \in (1,\infty)$, $\omega \in A_r$ and $\Sigma \subset \mathbb{R}^{n-1}$ be a bounded Lipschitz domain. Then there exists an A_r-consistent constant $c = c(r,\Sigma,\mathcal{A}_r(\omega)) > 0$ such that*

$$\|u\|_{r,\omega} \le c\|\nabla' u\|_{r,\omega}$$

for all $u \in W^{1,r}_\omega(\Sigma)$ with vanishing integral mean $\int_\Sigma u\,dx' = 0$.

Proof: See the proof of [16], Corollary 2.1 and its conclusions; checking the proof, one sees that the constant $c = c(r,\Sigma,\mathcal{A}_r(\omega))$ is A_r-consistent. ■

Finally we cite the Fourier multiplier theorem in weighted spaces.

Theorem 2.10 ([19], Ch. IV, Theorem 3.9) *Let $m \in C^k(\mathbb{R}^k \setminus \{0\}), k \in \mathbb{N}$, admit a constant $M \in \mathbb{R}$ such that*

$$|\eta|^\gamma |D^\gamma m(\eta)| \le M \quad \text{for all} \quad \eta \in \mathbb{R}^k \setminus \{0\}$$

and multi-indices $\gamma \in \mathbb{N}_0^k$ with $|\gamma| \le k$. Then for all $1 < r < \infty$ and $\omega \in A_r(\mathbb{R}^k)$ the multiplier operator $Tf = \mathcal{F}^{-1}m(\cdot)\mathcal{F}f$ defined for all rapidly decreasing functions $f \in \mathcal{S}(\mathbb{R}^k)$ can be uniquely extended to a bounded linear operator from $L^r_\omega(\mathbb{R}^k)$ to $L^r_\omega(\mathbb{R}^k)$. Moreover, there exists an A_r-consistent constant $C = C(r,\mathcal{A}_r(\omega))$ such that

$$\|Tf\|_{r,\omega} \le CM\|f\|_{r,\omega}, \quad f \in L^r_\omega(\mathbb{R}^k).$$

3 Resolvent estimate of the Stokes operator in weighted spaces on infinite straight cylinders

In this section we obtain the resolvent estimate of the Stokes operator in Lebesgue spaces with exponential weight with respect to the axial variable and Muckenhoupt weight for cross-sectional variables in an infinite straight cylinder $\Sigma \times \mathbb{R}$, where the cross-section Σ is a $C^{1,1}$-bounded domain.

3.1 Estimate for the problem $(R_{\lambda,\xi,\beta})$

In this subsection we get estimates for $(R_{\lambda,\xi,\beta})$ independent of λ and $\xi \in \mathbb{R}^*$ in L^r-spaces with Muckenhoupt weights. To this aim we rely partly on cut-off techniques using the results for $(R_{\lambda,\xi})$ (i.e., the case $\beta = 0$) in the whole and bent half spaces in [12] (Theorem 3.1 below). The main existence and uniqueness result in weighted L^r-spaces for $(R_{\lambda,\xi,\beta})$ is described in Theorem 3.8.

For whole or bent half spaces Σ, $g \in \widehat{W}^{-1,r}_\omega(\Sigma) + L^r_\omega(\Sigma)$ and $\eta = \xi + i\beta, \xi \in \mathbb{R}^*, \beta \ge 0$, we use notation

$$\|g; \widehat{W}^{-1,r}_\omega + L^r_{\omega,1/\eta}\| = \inf\{\|g_0\|_{-1,r,\omega} + \|g_1/\eta\|_{r,\omega} : g = g_0 + g_1, g_0 \in \widehat{W}^{-1,r}_\omega, g_1 \in L^r_\omega\}.$$

In the following we put $R_{\lambda,\xi} \equiv R_{\lambda,\xi,0}$.

Theorem 3.1 *Let $n \ge 3$, $1 < r < \infty$, $\omega \in A_r(\mathbb{R}^{n-1})$, $0 < \varepsilon < \frac{\pi}{2}$, $\xi \in \mathbb{R}^*$,$\lambda \in S_\varepsilon$, $0 < \varepsilon < \pi/2$ and $\mu = |\lambda + \xi^2|^{1/2}$.*

(i) ([12], Theorem 3.1) Let $\Sigma = \mathbb{R}^{n-1}$. *If* $f \in L^r_\omega(\Sigma)$ *and* $g \in W^{1,r}_\omega(\Sigma)$, *then the problem* $(R_{\lambda,\xi})$ *has a unique solution* $(u,p) \in W^{2,r}_\omega(\Sigma) \times W^{1,r}_\omega(\Sigma)$ *satisfying*

$$\|\mu^2 u, \mu\nabla' u, \nabla'^2 u, \nabla' p, \xi p\|_{r,\omega} \le c\big(\|f, \nabla' g, \xi g\|_{r,\omega} + \|\lambda g; \widehat{W}^{-1,r}_\omega + L^r_{\omega,1/\xi}\|\big) \quad (3.1)$$

with an A_r*-consistent constant* $c = c(\varepsilon, r, \mathcal{A}_r(\omega))$ *independent of* λ *and* ξ.

(ii) ([12], Theorem 3.5) Let

$$\Sigma = H_\sigma = \{x' = (x_1, x''); \, x_1 > \sigma(x''), x'' \in \mathbb{R}^{n-2}\}$$

for a given function $\sigma \in C^{1,1}(\mathbb{R}^{n-2})$. *Then there are* A_r*-consistent constants* $K_0 = K_0(r, \varepsilon, \mathcal{A}_r(\omega)) > 0$ *and* $\lambda_0 = \lambda_0(r, \varepsilon, \mathcal{A}_r(\omega)) > 0$ *independent of* λ *and* ξ *such that, if* $\|\nabla'\sigma\|_\infty \le K_0$, *for every* $f \in L^r_\omega(\Sigma)$ *and* $g \in W^{1,r}_\omega(\Sigma)$ *the problem* $(R_{\lambda,\xi})$ *has a unique solution* $(u,p) \in (W^{2,r}_\omega(\Sigma) \cap W^{1,r}_{0,\omega}(\Sigma)) \times W^{1,r}_\omega(\Sigma)$. *This solution satisfies the estimate*

$$\begin{aligned} &\|\mu^2 u, \mu\nabla' u, \nabla'^2 u, \nabla' p, \xi p\|_{r,\omega} \\ &\qquad \le c\big(\|f, \nabla' g, \xi g\|_{r,\omega} + \|\lambda g; \widehat{W}^{-1,r}_\omega(\Sigma) + L^r_{\omega,1/\xi}(\Sigma)\|\big) \end{aligned} \quad (3.2)$$

with an A_r*-consistent constant* $c = c(r, \varepsilon, \mathcal{A}_r(\omega))$.

On the bounded domain $\Sigma \subset \mathbb{R}^{n-1}$ of $C^{1,1}$-class let α_0 and α_1 denote the smallest eigenvalue of the Dirichlet and Neumann Laplacian, respectively, i.e.,

$$\begin{aligned} &\alpha_0 := \inf\{\|\nabla u\|_2^2 : \, u \in W^{1,2}_0(\Sigma), \|u\|_2 = 1\} > 0, \\ &\alpha_1 := \inf\{\|\nabla u\|_2^2 : \, u \in W^{1,2}(\Sigma), \tfrac{\partial u}{\partial n}|_{\partial\Sigma} = 0, \|u\|_2 = 1\} > 0, \\ &\bar{\alpha} := \min\{\alpha_0, \alpha_1\}. \end{aligned} \quad (3.3)$$

For fixed $\lambda \in \mathbb{C} \setminus (-\infty, -\alpha_0]$, $\eta = \xi + i\beta$, $\xi \in \mathbb{R}^*$, $\beta \ge 0$, and $\omega \in A_r$ we introduce the *parametrized Stokes operator* $S = S^\omega_{r,\lambda,\eta}$ by

$$S(u,p) = \begin{pmatrix} (\lambda + \eta^2 - \Delta')u' + \nabla' p \\ (\lambda + \eta^2 - \Delta')u_n + i\eta p \\ -\operatorname{div}_\eta u \end{pmatrix}$$

defined on $\mathcal{D}(S) = \mathcal{D}(\Delta'_{r,\omega}) \times W^{1,r}_\omega(\Sigma)$, where $\mathcal{D}(\Delta'_{r,\omega}) = W^{2,r}_\omega(\Sigma) \cap W^{1,r}_{0,\omega}(\Sigma)$ and

$$\operatorname{div}_\eta u = \operatorname{div}' u' + i\eta u_n.$$

For $\omega \equiv 1$ the operator $S^\omega_{r,\lambda,\eta}$ will be denoted by $S_{r,\lambda,\eta}$. Note that the image of $\mathcal{D}(S)$ by div_η is included in $W^{1,r}_\omega(\Sigma)$ and $W^{1,r}_\omega(\Sigma) \subset L^r_{0,\omega}(\Sigma) + L^r_\omega(\Sigma)$, where

$$L^r_{0,\omega}(\Sigma) := \big\{u \in L^r_\omega(\Sigma) : \int_\Sigma u \, dx' = 0\big\}.$$

Using Poincaré's inequality in weighted spaces, see Proposition 2.9, one can easily check the continuous embedding $L^r_{0,\omega}(\Sigma) \hookrightarrow \widehat{W}^{-1,r}_\omega(\Sigma)$; more precisely,

$$\|u\|_{-1,r,\omega} \le c\|u\|_{r,\omega}, \quad u \in L^r_{0,\omega}(\Sigma),$$

with an A_r-consistent constant $c > 0$. For bounded domain Σ we use the notation

$$\|g; L^r_{0,\omega} + L^r_{\omega,1/\eta}\|_0 := \inf\{\|g_0\|_{-1,r,\omega} + \|g_1/\eta\|_{r,\omega} : \ g = g_0 + g_1, g_0 \in L^r_{0,\omega}, g_1 \in L^r_\omega\};$$

note that this norm is equivalent to the norm $\|\cdot\|_{(W^{1,r'}_{\omega',\eta})^*}$ where $W^{1,r'}_{\omega',\eta}$ is the usual weighted Sobolev space on Σ with norm $\|\nabla' u, \eta u\|_{r',\omega'}$.

First we consider Hilbert space setting of $(R_{\lambda,\xi,\beta})$. For $\eta = \xi + i\beta$, $\xi \in \mathbb{R}^*, \beta \geq 0$, let us introduce a closed subspace of $W^{1,r}_0(\Sigma)$ as

$$V_\eta := \{u \in W^{1,r}_0(\Sigma) : \operatorname{div}_\eta u = 0\}.$$

Lemma 3.2 *Let $\phi = (\phi', \phi_n) \in W^{-1,2}(\Sigma)$ be such that $\langle \phi, v\rangle_{W^{-1,2}(\Sigma), W^{1,2}_0(\Sigma)} = 0$ for all $v \in W^{1,2}_0(\Sigma)$. Then, there is some $p \in L^2(\Sigma)$ with $\phi = (\nabla p, i\eta p)$.*

Proof: This lemma can be proved just by copy of the proof of [10], Lemma 3.1 with $\xi \in \mathbb{R}^*$ replaced by $\eta = \xi + i\beta$. ■

Lemma 3.3 *(i) For any $g \in W^{1,2}(\Sigma)$, $\eta = \xi + i\beta$, $x \in \mathbb{R}^*, \beta \geq 0$, the equation $\operatorname{div}_\eta u = g$ has at least one solution $u \in W^{2,2}(\Sigma) \cap W^{1,2}_0(\Sigma)$ and*

$$\|u\|_{2,2} \leq c(\|g\|_{1,2} + \frac{1}{\eta}\int_\Sigma g\, dx'),$$

where c is independent of g.

(ii) Let $\varepsilon \in (0, \pi/2)$, $\beta \in (0, \sqrt{\alpha_0})$ and

$$\lambda \in \{-\alpha_0 + \beta^2 + S_\varepsilon\} \cap \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > -\frac{(\operatorname{Im}\lambda)^2}{4\beta^2} - \alpha_0 + \beta^2\}. \tag{3.4}$$

Then, for any $f \in L^2(\Sigma)$, $g \in W^{1,2}(\Sigma)$ the system $(R_{\lambda,\xi,\beta})$ has a unique solution $(u, p) \in (W^{2,2}(\Sigma) \cap W^{1,2}(\Sigma)) \times W^{1,2}(\Sigma)$.

Proof: – *Proof of (i)*: Let a scalar function $w \in C^\infty_0(\Sigma)$ be such that $\int_\Sigma w\, dx' = 0$. Given $g \in W^{1,2}(\Sigma)$, let $\bar g = \int_\Sigma g\, dx$ and consider a divergence problem in Σ, that is,

$$\operatorname{div}' u' = g - \bar g w, \quad u'|_{\partial\Sigma} = 0,$$

which has a solution $u' \in W^{2,2}(\Sigma) \cap W^{1,2}_0(\Sigma)$ with $\|u'\|_{2,2} \leq c\|\nabla(g - \bar g w)\|_2 \leq c\|g\|_{1,2}$, by [7], Theorem 1.2. Then, $u := (u', \frac{\bar g w}{i\eta})$ satisfies $\operatorname{div}_\eta u = g$ and required estimate.

– *Proof of (ii)*: By the assertion (i) of the lemma, we may assume w.l.o.g. that $g \equiv 0$. Now, for fixed $\lambda \in -\alpha_0 + \beta^2 + S_\varepsilon$ define the bilinear form $b : V_\eta \times V_\eta \mapsto \mathbb{R}$ by

$$b(u, v) := \int_\Sigma \big((\lambda + \eta^2) u \cdot \bar v + \nabla' u \cdot \nabla' \bar v\big)\, dx'.$$

Obviously, b is continuous in $V_\eta \times V_\eta$. Moreover, b is coercive, that is,

$$|b(u, u)| \geq l(\lambda, \xi, \beta)\|u\|^2_{1,2} \tag{3.5}$$

with some $l(\lambda,\xi,\beta)>0$. In fact,

$$b(u,u)=\int_\Sigma((\mathrm{Re}\,\lambda+\xi^2-\beta^2)|u|^2+|\nabla' u|^2)\,dx'+i\int_\Sigma(\mathrm{Im}\,\lambda+2\xi\beta)|u|^2\,dx'.$$

Note that, due to Poincaré's inequality,

$$(\xi^2-\alpha_0)\|u\|_2^2+\|\nabla' u\|_2^2>0,\ \forall\xi\in\mathbb{R}^*.$$

Hence, if $\mathrm{Re}\,\lambda+\alpha_0-\beta^2\geq 0$, then

$$|b(u,u)|\geq|\int_\Sigma((\mathrm{Re}\,\lambda+\xi^2-\beta^2)|u|^2+|\nabla' u|^2)\,dx'|\geq(\xi^2-\alpha_0)\|u\|_2^2+\|\nabla' u\|_2^2,$$

where

$$(\xi^2-\alpha_0)\|u\|_2^2+\|\nabla' u\|_2^2\geq\|\nabla' u\|_2^2$$

if $\xi^2-\alpha_0\geq 0$ and

$$(\xi^2-\alpha_0)\|u\|_2^2+\|\nabla' u\|_2^2\geq(\xi^2/\alpha_0-1)\|\nabla' u\|_2^2+\|\nabla' u\|_2^2\geq\frac{\xi^2}{\alpha_0}\|\nabla' u\|_2^2$$

if $\xi^2-\alpha_0<0$.

Therefore, it remained to prove (3.5) for the case $\mathrm{Re}\,\lambda+\alpha_0-\beta^2<0$.

Note that if $\mathrm{Im}\,\lambda+2\xi\beta\neq 0$ then $(R_{\lambda,\xi,\beta})$ coincides with $(R_{\lambda_1,\xi})$ where $\lambda_1=\lambda-\beta^2+2i\xi\beta\in-\alpha_0+S_{\varepsilon_1}$ with $\varepsilon_1=\max\{\varepsilon,\arctan\frac{|\mathrm{Re}\,\lambda+\alpha_0-\beta^2|}{|\mathrm{Im}\,\lambda+2\xi\beta|}\}\in(0,\pi/2)$. Hence, (3.5) can be proved in the same way as the proof of [10], Lemma 3.2 (ii).

Now, suppose that

$$\mathrm{Im}\,\lambda+2\xi\beta=0,\quad i.e.,\quad \xi=-\frac{\mathrm{Im}\,\lambda}{2\beta}.$$

Since (3.5) is trivial for the case $\mathrm{Re}\,\lambda+\xi^2-\beta^2\geq 0$, we assume that

$$\mathrm{Re}\,\lambda+\xi^2-\beta^2<0.$$

In this case, note that due to the condition $\mathrm{Re}\,\lambda+\frac{(\mathrm{Im}\,\lambda)^2}{4\beta^2}-\beta^2>-\alpha_0$ there is some $c(\lambda,\beta)>0$ such that

$$0>\mathrm{Re}\,\lambda+\frac{(\mathrm{Im}\,\lambda)^2}{4\beta^2}-\beta^2>c(\lambda,\beta)-\alpha_0,\quad c(\lambda,\beta)-\alpha_0<0.$$

Then,

$$\begin{aligned}|b(u,u)| &\geq \int_\Sigma((\mathrm{Re}\,\lambda+\tfrac{(\mathrm{Im}\,\lambda)^2}{4\beta^2}-\beta^2)|u|^2+|\nabla' u|^2)\,dx'\\ &\geq \int_\Sigma(c(\lambda,\beta)-\alpha_0)|u|^2+|\nabla' u|^2)\,dx'\\ &\geq \tfrac{c(\lambda,\beta)}{\alpha_0}\|\nabla' u\|_2^2.\end{aligned}$$

Finally, (3.5) is proved.

By Lax-Milgram's lemma in view of (3.5), the variational problem

$$b(u,v)=\int_\Sigma f\cdot\bar v\,dx',\forall v\in V_\eta,$$

has a unique solution u in V_η. Then, by Lemma 3.2, there is some $p\in L^2(\Sigma)$ such that

$$(\lambda+\eta^2-\Delta')u'+\nabla'p=f',(\lambda+\eta^2-\Delta')u_n+i\eta p=f_n.$$

Now, applying the well-known regularity theory for Stokes system and Poisson's equation in Σ to

$$-\Delta'u'+\nabla'p=f'-(\lambda+\eta^2)u',\operatorname{div}'u'=-i\eta u_n,u'|_{\partial\Sigma}=0$$

and

$$-\Delta'u_n=f_n-(\lambda+\eta^2)u_n-i\eta p,u_n|_{\partial\Sigma}=0,$$

respectively, we have $(u,p)\in(W^{2,2}(\Sigma)\cap W_0^{1,2}(\Sigma))\times W^{1,2}(\Sigma)$. Thus, the assertion (ii) of the lemma is proved. ■

Remark 3.4 *It is seen by elementary calculation that the assumption* (3.4) *on* λ *of Lemma 3.3 is satisfied for all* $\lambda\in-\alpha+S_\varepsilon$ *if either* $\alpha\in(0,\alpha_0-\beta^2)$ *and* $\varepsilon\in(0,\arctan\frac{\sqrt{\alpha_0-\beta^2-\alpha}}{\beta})$ *or if* $\alpha\in(0,\bar\alpha-\beta^2)$ *and* $\varepsilon\in(0,\arctan\frac{\sqrt{\bar\alpha-\beta^2-\alpha}}{\beta})$*. Note that* $\bar\alpha<\alpha_0$*, see* (3.3)*.*

Now, we turn in considering $(R_{\lambda,\xi,\beta})$ in weighted spaces with weights w.r.t. cross-section as well.

Lemma 3.5 *Let* $\xi\in\mathbb{R}^*$*,* $\beta\in(0,\sqrt{\alpha_0})$*,* $\alpha\in(0,\alpha_0-\beta^2)$*,* $\varepsilon\in(0,\arctan\frac{\sqrt{\alpha_0-\beta^2-\alpha}}{\beta})$*,* $\lambda\in-\alpha+S_\varepsilon$*, and* $\omega\in A_r$*,* $1<r<\infty$*. Then the operator* $S=S^\omega_{r,\lambda,\eta}$ *is injective and the range* $\mathcal{R}(S)$ *of* S *is dense in* $L^r_\omega(\Sigma)\times W^{1,r}_\omega(\Sigma)$*.*

Proof: Since, by Proposition 2.7 (3), there is an $s\in(1,r)$ such that $L^r_\omega(\Sigma)\subset L^s(\Sigma)$, one sees immediately that $\mathcal{D}(S^\omega_{r,\lambda,\eta})\subset\mathcal{D}(S_{s,\lambda,\eta})$. Therefore, $S^\omega_{r,\lambda,\eta}(u,p)=0$ for some $(u,p)\in\mathcal{D}(S^\omega_{r,\lambda,\eta})$ yields $(u,p)\in\mathcal{D}(S_{s,\lambda,\eta})$ and $S_{s,\lambda,\eta}(u,p)=0$. Here note that $S_{s,\lambda,\eta}(u,p)=0$ implies

$$S_{s,\lambda,\eta}(u,p)=((\beta^2-2i\xi\beta)u',(\beta^2-2i\xi\beta)u_n+\beta p,\beta u_n)^T.$$

Hence, by applying [10], Theorem 3.4 finite number of times and the Sobolev embedding theorem, we get that $(u,p)\in(W^{2,2}(\Sigma)\cap W_0^{1,2}(\Sigma))\times W^{1,2}(\Sigma)$. Therefore, by Lemma 3.3 we get that $(u,p)=0$, i.e., $S^\omega_{r,\lambda,\eta}$ is injective.

On the other hand, by Proposition 2.7 (3), there is an $\tilde s\in(r,\infty)$ such that $S_{\tilde s,\lambda,\eta}\subset S^\omega_{r,\lambda,\eta}$. Moreover, by Lemma 3.3, for every $(f,g)\in C_0^\infty(\Sigma)\times C^\infty(\bar\Sigma)$, there is some $(u,p)\in D(S_{2,\lambda,\eta})$ with $S_{2,\lambda,\eta}(u,p)=(f,-g)$. Applying the regularity result of [7], Theorem 1,2 for the Stokes resolvent system in Σ finite number of times using the Sobolev embedding theorem, it can be seen that $(u,p)\in D(S_{q,\lambda,\eta})$ for all $q\in(1,\infty)$, in particular, for $q=\tilde s$. Therefore,

$$C_0^\infty(\Sigma)\times C^\infty(\bar\Sigma)\subset\mathcal{R}(S_{\tilde s,\lambda,\eta})\subset\mathcal{R}(S^\omega_{r,\lambda,\eta})\subset L^r_\omega(\Sigma)\times W^{1,r}_\omega(\Sigma),$$

which proves the assertion on the denseness of $\mathcal{R}(S)$. ■

The following lemma gives a preliminary *a priori* estimate for a solution (u,p) of $S(u,p) = (f,-g)$.

Lemma 3.6 *Assume the same for r, ω, α, β and λ as in Lemma 3.5. Then there exists an A_r-consistent constant $c = c(\varepsilon, r, \beta, \Sigma, \mathcal{A}_r(\omega)) > 0$ such that for every $(u,p) \in \mathcal{D}(S^{\omega}_{r,\lambda,\eta})$,*

$$\begin{aligned}\|\mu_+^2 u, \mu_+\nabla' u, \nabla'^2 u, \nabla' p, \eta p\|_{r,\omega} \le c\big(&\|f, \nabla' g, g, \xi g\|_{r,\omega} + |\lambda| \|g; L^r_{0,\omega} + L^r_{\omega,1/\eta}\|_0 \\ &+\|\nabla' u, \xi u, p\|_{r,\omega} + |\lambda| \|u\|_{(W^{1,r'}_{\omega'})^*}\big),\end{aligned} \tag{3.6}$$

where $\mu_+ = |\lambda + \alpha + \xi^2|^{1/2}$, $(f,-g) = S(u,p)$ and $(W^{1,r'}_{\omega'})^*$ denotes the dual space of $W^{1,r'}_{\omega'}(\Sigma)$.

Proof: The proof is devided into two parts, i.e, the case $\xi^2 > \beta^2$ and the other case $\xi^2 \le \beta^2$.

The proof of the case $\xi^2 > \beta^2$ is based on a partition of unity in Σ and on the localization procedure reducing the problem to a finite number of problems of type $(R_{\lambda,\xi})$ in bent half spaces and in the whole space $\mathbb{R}^{n-1}$. Since $\partial\Sigma \in C^{1,1}$, we can cover $\partial\Sigma$ by a finite number of balls $B_j, j \ge 1$, such that, after a translation and rotation of coordinates, $\Sigma \cap B_j$ locally coincides with a bent half space $\Sigma_j = \Sigma_{\sigma_j}$ where $\sigma_j \in C^{1,1}(\mathbb{R}^{n-1})$ has a compact support, $\sigma_j(0) = 0$ and $\nabla''\sigma_j(0) = 0$. Choosing the balls B_j small enough (and its number large enough) we may assume that $\|\nabla''\sigma_j\|_\infty \le K_0(\varepsilon, r, \Sigma, \mathcal{A}_r(\omega))$ for all $j \ge 1$ where K_0 was introduced in Theorem 3.1 (ii).

According to the covering $\partial\Sigma \subset \bigcup_{j\ge1} B_j$ there are cut-off functions $(\varphi_j)_{j=0}^m$ such that such that

$$0 \le \varphi_0, \varphi_j \in C^\infty(\mathbb{R}^{n-1}), \ \sum_{j\ge0} \varphi_j \equiv 1 \text{ in } \Sigma, \ \operatorname{supp} \varphi_0 \subset \Sigma, \operatorname{supp} \varphi_j \subset B_j, j \ge 1. \tag{3.7}$$

Given $(u,p) \in \mathcal{D}(S)$ and $(f,-g) = S(u,p)$, we get for each φ_j, $j \ge 0$, the local $(R_{\lambda,\xi})$-problems

$$\begin{aligned}(\lambda + \xi^2 - \Delta')(\varphi_j u') + \nabla'(\varphi_j p) &= f_j' \\ (\lambda + \xi^2 - \Delta')(\varphi_j u_n) + i\xi(\varphi_j p) &= f_{jn} \\ \operatorname{div}_\xi(\varphi_j u) &= g_j\end{aligned} \tag{3.8}$$

for $(\varphi_j u, \varphi_j p), j \ge 0$, in $\mathbb{R}^{n-1}$ or Σ_j; here

$$\begin{aligned}f_j' &= \varphi_j f' - 2\nabla'\varphi_j \cdot \nabla' u' - (\Delta'\varphi_j)u' + (\beta^2 - 2i\xi)(\varphi_j u') + (\nabla'\varphi_j)p \\ f_{jn} &= \varphi_j f_n - 2\nabla'\varphi_j \cdot \nabla' u_n - (\Delta'\varphi_j)u_n + (\beta^2 - 2i\xi)(\varphi_j u_n) + \beta(\varphi_j p) \\ g_j &= \varphi_j g + \nabla'\varphi_j \cdot u' + \beta\varphi_j u_n.\end{aligned} \tag{3.9}$$

To control f_j and g_j note that $u = 0$ on $\partial\Sigma$; hence Poincaré's inequality for Muckenhoupt weighted space yields for all $j \ge 0$ the estimate

$$\|f_j, \nabla' g_j, \xi g_j\|_{r,\omega;\Sigma_j} \le c(\|f, \nabla' g, g, \xi g\|_{r,\omega;\Sigma} + \|\nabla' u, \xi u, p\|_{r,\omega;\Sigma}), \tag{3.10}$$

where $\Sigma_0 \equiv \mathbb{R}^{n-1}$ and $c > 0$ is A_r-consistent. Moreover, let $g = g_0 + g_1$ denote any splitting of $g \in L^r_{0,\omega} + L^r_{\omega,1/\eta}$. Defining the characteristic function χ_j of $\Sigma \cap \Sigma_j$ and the scalar

$$\begin{aligned} m_j &= \frac{1}{|\Sigma \cap \Sigma_j|} \int_{\Sigma\cap\Sigma_j} (\varphi_j g_0 + u' \cdot \nabla' \varphi_j + \beta \varphi_j u_n) dx' \\ &= \frac{1}{|\Sigma \cap \Sigma_j|} \int_{\Sigma\cap\Sigma_j} (i\xi u_n - g_1)\varphi_j dx', \end{aligned}$$

we split g_j in the form

$$g_j = g_{j0} + g_{j1} := (\varphi_j g_0 + u' \cdot \nabla' \varphi_j + \beta \varphi_j u_n - m_j \chi_j) + (\varphi_j g_1 + m_j \chi_j).$$

Concerning g_{j1} we get

$$\begin{aligned} \|g_{j1}\|^r_{r,\omega;\Sigma_j} &= \int_{\Sigma\cap\Sigma_j} |\varphi_j g_1 + m_j|^r \omega \, dx' \\ &\le c(r)\big(\|g_1\|^r_{r,\omega;\Sigma} + |m_j|^r \omega(\Sigma \cap \Sigma_j)\big) \\ &\le c(r)\Big(\|g_1\|^r_{r,\omega;\Sigma} + \frac{\omega(\Sigma\cap\Sigma_j)\cdot\omega'(\Sigma\cap\Sigma_j)^{r/r'}}{|\Sigma\cap\Sigma_j|^r}\big(\|\xi u_n\|^r_{(W^{1,r'}_{\omega'})^*} + \|g_1\|^r_{r,\omega;\Sigma}\big)\Big) \end{aligned}$$

with $c(r) > 0$ independent of ω. Since we chose the balls B_j for $j \ge 1$ small enough, for each $j \ge 0$ there is a cube Q_j with $\Sigma \cap \Sigma_j \subset Q_j$ and $|Q_j| < c(n)|\Sigma \cap \Sigma_j|$ where the constant $c(n) > 0$ is independent of j. Therefore

$$\begin{aligned} \|g_{j1}\|_{r,\omega;\Sigma_j} &\le c(r)\Big(\|g_1\|_{r,\omega} + \tfrac{c(n)\omega(Q_j)^{1/r}\cdot\omega'(Q_j)^{1/r'}}{|Q_j|}\big(\|\xi u_n\|_{(W^{1,r'}_{\omega'})^*} + \|g_1\|_{r,\omega}\big)\Big) \\ &\le c(r)(1+\mathcal{A}_r(\omega)^{1/r})\big(\|\xi u_n\|_{(W^{1,r'}_{\omega'})^*} + \|g_1\|_{r,\omega;\Sigma}\big) \end{aligned} \tag{3.11}$$

for $j \ge 0$. Furthermore, for every test function $\Psi \in C_0^\infty(\bar{\Sigma}_j)$ let

$$\tilde{\Psi} = \Psi - \frac{1}{|\Sigma\cap\Sigma_j|}\int_{\Sigma\cap\Sigma_j} \Psi dx'.$$

By the definition of $m_j\chi_j$ we have $\int_{\Sigma_j} g_{j0}\, dx' = 0$; hence by Poincaré's inequality (see Proposition 2.9)

$$\begin{aligned} &|\textstyle\int_{\Sigma_j} g_{j0}\Psi dx'| = |\int_{\Sigma_j} g_{j0}\tilde{\Psi} dx'| \\ &= |\textstyle\int_\Sigma g_0(\varphi_j\tilde{\Psi})dx' + \int_\Sigma u'\cdot(\nabla'\varphi_j)\tilde{\Psi}dx' + \int_\Sigma \beta u_n \varphi_j \tilde{\Psi}\, dx'| \\ &\le \|g_0\|_{-1,r,\omega}\|\nabla'(\varphi_j\tilde{\Psi})\|_{r',\omega'} + \|u'\|_{(W^{1,r'}_{\omega'})^*}\|(\nabla'\varphi_j)\tilde{\Psi}\|_{1,r',\omega'} + \|\beta u_n\|_{(W^{1,r'}_{\omega'})'}\|\varphi_j\tilde{\Psi}\|_{1,r,\omega'} \\ &\le c(\|g_0\|_{-1,r,\omega} + \|u\|_{(W^{1,r'}_{\omega'})^*})\|\nabla'\Psi\|_{r',\omega';\Sigma_j}, \end{aligned}$$

where $c > 0$ is A_r-consistent. Thus

$$\|g_{j0}\|_{-1,r,\omega;\Sigma_j} \le c\big(\|g_0\|_{-1,r,\omega} + \|u\|_{(W^{1,r'}_{\omega'})^*}\big) \quad \text{for } j \ge 0. \tag{3.12}$$

Summarizing (3.11) and (3.12), we get for $j \ge 0$

$$\|g_j; \widehat{W}^{-1,r}_\omega(\Sigma_j) + L^r_{\omega,1/\xi}(\Sigma_j)\| \le c\big(\|u\|_{(W^{1,r'}_{\omega'})^*} + \|g; L^r_{0,\omega} + L^r_{\omega,1/\xi}\|_0\big)$$

with an A_r-consistent $c = c(r, \mathcal{A}_r(\omega)) > 0$, which yields in view of $\xi^2 > \beta^2$ that

$$\|g_j; \widehat{W}_\omega^{-1,r}(\Sigma_j) + L^r_{\omega,1/\xi}(\Sigma_j)\| \le c\big(\|u\|_{(W^{1,r'}_{\omega'})^*} + \|g; L^r_{0,\omega} + L^r_{\omega,1/\eta}\|_0\big) \tag{3.13}$$

with an A_r-consistent $c = c(r, \mathcal{A}_r(\omega)) > 0$.

To complete the proof, apply Theorem 3.1 (i) to (3.8), (3.9) when $j = 0$. Further use Theorem 3.1 (ii) in (3.8), (3.9) for $j \ge 1$, but with λ replaced by $\lambda + M$ with $M = \lambda_0 + \alpha_0$, where $\lambda_0 = \lambda_0(\varepsilon, r, \mathcal{A}_r(\omega))$ is the A_r-consistent constant indicated in Theorem 3.1 (ii). This shift in λ implies that f_j has to be replaced by $f_j + M\varphi_j u$ and that (3.2) will be used with λ replaced by $\lambda + M$. Summarizing (3.1), (3.2) as well as (3.10), (3.13) and summing over all j we arrive at (3.6) with the additional terms

$$I = \|Mu\|_{r,\omega} + \|Mu\|_{(W^{1,r'}_{\omega'})^*} + \|Mg; L^r_{0,\omega} + L^r_{\omega,1/\eta}\|_0$$

on the right-hand side of the inequality. Note that $M = M(\varepsilon, r, \mathcal{A}_r(\omega))$ is A_r-consistent, $|\eta| \le \max\{\sqrt{2}|\xi|, \sqrt{2}\beta\}$ and that $g = \operatorname{div}'u' + i\eta u_n$ defines a natural splitting of $g \in L^r_{0,\omega}(\Sigma) + L^r_\omega(\Sigma)$. Hence Poincaré's inequality yields

$$\begin{aligned} I &\le M\big(\|u\|_{r,\omega;\Sigma} + \|\operatorname{div}'u'\|_{-1,r,\omega} + \|u_n\|_{r,\omega;\Sigma}\big) \\ &\le c_1\|u\|_{r,\omega;\Sigma} \le c_2\|\nabla' u\|_{r,\omega;\Sigma} \end{aligned}$$

with A_r-consistent constants $c_i = c_i(\varepsilon, r, \Sigma, \mathcal{A}_r(\omega)) > 0$, $i = 1, 2$.

Thus (3.6) is proved.

Next, consider the case $\xi^2 \le \beta^2$. Since $S(u,p) = (f, -g)$, we have

$$\begin{aligned} &(\lambda - \Delta')u' + \nabla' p = f' - \eta^2 u', \ \operatorname{div}'u' = g - i\eta u_n, \quad \text{in } \Sigma, \\ &u'|_{\partial\Sigma} = 0, \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} &(\lambda - \Delta')u_n = f_n - \eta^2 u_n - i\eta p, \quad \text{in } \Sigma, \\ &u_n|_{\partial\Sigma} = 0. \end{aligned} \tag{3.15}$$

Now, apply [16], Lemma 3.2 to (3.14). Then, in view of $|\eta| \le \sqrt{2}\beta$ and Poincaré's inequality, for all $\lambda \in -\alpha + S_\varepsilon$, $\alpha \in (0, \alpha_0 - \beta^2)$ we have

$$\begin{aligned} &\|(\lambda+\alpha)u', \nabla'^2 u', \nabla' p\|_{r,\omega;\Sigma} \\ &\le c\big(\|f, \eta^2 u\|_{r,\omega;\Sigma} + |\lambda|\|g - i\eta u_n\|_{\hat{W}^{-1,r}_\omega(\Sigma)} + \|g - i\eta u_n\|_{W^{1,r}_\omega(\Sigma)} + |\lambda|\|u'\|_{(W^{1,r'}_{\omega'}(\Sigma))'}\big) \\ &\le c\big(\|f, \nabla' u, p\|_{r,\omega;\Sigma} + \|g\|_{W^{1,r}_\omega(\Sigma)} + |\lambda|\|g - i\eta u_n\|_{\hat{W}^{-1,r}_\omega(\Sigma)} + |\lambda|\|u\|_{(W^{1,r'}_{\omega'}(\Sigma))'}\big) \end{aligned}$$

with A_r-consistent constant $c = c(r, \varepsilon, \alpha, \beta, \Sigma, \mathcal{A}_r(\Omega))$.

In order to control $\|g - i\eta u_n\|_{\hat{W}^{-1,r}_\omega(\Sigma)}$, let us split g as $g = g_0 + g_1$, $g_0 \in L^r_{0,\omega}(\Sigma)$, $g_1 \in L^r_{\omega,1/\eta}(\Sigma)$. Since $g_1 - i\eta u_n$ has mean value zero in Σ, we get by poincaré's inequality that

$$\begin{aligned} &|\langle g_1 - i\eta u_n, \psi\rangle| = |\langle g_1 - i\eta u_n, \bar\psi\rangle| \\ &|\eta|\|g_1/\eta\|_{r,\omega}\|\|\bar\psi\|_{r',\omega'} + |\eta|\|u_n\|_{(W^{1,r'}_{\omega'}(\Sigma))'}\|\bar\psi\|_{W^{1,r'}_{\omega'}(\Sigma)} \\ &\le c(r,\Sigma)(\|g_1/\eta\|_{r,\omega}\| + \|u_n\|_{(W^{1,r'}_{\omega'}(\Sigma))'})\|\nabla'\psi\|_{r',\omega';\Sigma}, \end{aligned}$$

for all $\psi \in C^\infty(\bar{\Sigma})$, where $\bar{\psi} = \psi - \frac{1}{|\Sigma|}\int_\Sigma \psi\, dx'$. Therefore,

$$\|g - i\eta u_n\|_{\hat{W}_\Omega^{-1,r}(\Sigma)} \le \|g_0\|_{\hat{W}_\Omega^{-1,r}(\Sigma)} + c(\|g_1/\eta\|_{r,\omega}\| + \|u_n\|_{(W^{1,r'}_{\omega'}(\Sigma))'}).$$

Thus, for all $\lambda \in -\alpha + S_\varepsilon$, $\alpha \in (0, \alpha_0 - \beta^2)$ we have

$$\begin{aligned}&\|(\lambda+\alpha)u', \nabla'^2 u', \nabla' p\|_{r,\omega;\Sigma}\\ &\le c\big(\|f, \nabla' u, p\|_{r,\omega;\Sigma} + \|g\|_{W^{1,r}_\omega(\Sigma)} + |\lambda|\|u\|_{(W^{1,r'}_{\omega'}(\Sigma))'} + |\lambda|\|g : L^r_{0,\omega} + L^r_{\omega,1/\eta}\|_0\big)\end{aligned} \tag{3.16}$$

with A_r-consistent constant $c = c(r, \varepsilon, \alpha, \beta, \Sigma, \mathcal{A}_r(\Omega))$.

On the other hand, applying well-known results for the Laplace resolvent equations (cf. [16]) to (3.15), we get that

$$\|(\lambda+\alpha)u_n, \nabla'^2 u_n\|_{r,\omega;\Sigma} \le c(\|f_n, u, p\|_{r,\omega;\Sigma} \tag{3.17}$$

with $c = c(r, \varepsilon, \alpha, \beta, \Sigma, \mathcal{A}_r(\Omega))$. Thus, from (3.16) and (3.17) the assertion of the lemma for the case $\xi^2 \le \beta^2$ is proved.

The proof of the lemma is complete. ∎

Lemma 3.7 *Let $1 < r < \infty$, $\omega \in A_r$ and $\xi \in \mathbb{R}^*$, $\beta \in (0, \sqrt{\bar\alpha})$, $\alpha \in (0, \bar\alpha - \beta^2)$, $\varepsilon \in (0, \arctan \frac{\sqrt{\bar\alpha - \beta^2 - \alpha}}{\beta})$, $\lambda \in -\alpha + S_\varepsilon$. Then there is an A_r-consistent constant $c = c(\alpha, \varepsilon, r, \beta, \Sigma, \mathcal{A}_r(\omega))$ such that for every $(u, p) \in \mathcal{D}(S)$ and $(f, -g) = S(u, p)$ the estimate*

$$\begin{aligned}&\|\mu_+^2 u, \mu_+ \nabla' u, \nabla'^2 u, \nabla' p, \eta p\|_{r,\omega}\\ &\qquad \le c\big(\|f, \nabla' g, g, \xi g\|_{r,\omega} + (|\lambda|+1)\|g; L^r_{0,\omega} + L^r_{\omega,1/\xi}\|_0\big)\end{aligned} \tag{3.18}$$

holds; here $\mu_+ = |\lambda + \alpha + \xi^2|^{1/2}$.

Proof: Assume that this lemma is wrong. Then there is a constant $c_0 > 0$, a sequence $\{\omega_j\}_{j=1}^\infty \subset A_r$ with $\mathcal{A}_r(\omega_j) \le c_0$ for all j, sequences $\{\lambda_j\}_{j=1}^\infty \subset -\alpha + S_\varepsilon, \{\xi_j\}_{j=1}^\infty \subset \mathbb{R}^*$ and $(u_j, p_j) \in \mathcal{D}(S^{\omega_j}_{r,\lambda_j,\xi_j})$ for all $j \in \mathbb{N}$ such that

$$\begin{aligned}&\|(\lambda_j + \alpha + \xi_j^2)u_j, (\lambda_j + \alpha + \xi_j^2)^{1/2}\nabla' u_j, \nabla'^2 u_j, \nabla' p_j, \eta_j p_j\|_{r,\omega_j}\\ &\qquad \ge j\big(\|f_j, \nabla' g_j, g_j, \xi_j g_j\|_{r,\omega_j} + (|\lambda_j|+1)\|g_j; L^r_{m,\omega_j} + L^r_{\omega_j,1/\eta_j}\|_0\end{aligned} \tag{3.19}$$

where $\eta_j = \xi_j + i\beta$, $(f_j, -g_j) = S^{\omega_j}_{r,\lambda_j,\eta_j}(u_j, p_j)$. Fix an arbitrary cube Q containing Σ. We may assume without loss of generality that

$$\mathcal{A}_r(\omega_j) \le c_0, \quad \omega_j(Q) = 1 \quad \forall j \in \mathbb{N}, \tag{3.20}$$

by using the A_r-weight $\tilde\omega_j := \omega_j(Q)^{-1}\omega_j$ instead of ω_j if necessary. Note that (3.20) also holds for $r', \{\omega_j'\}$ in the following form: $\mathcal{A}_r(\omega_j) \le c_0^{r'/r}$, $\omega_j'(Q) \le c_0^{r'/r}|Q|^{r'}$. Therefore, by a minor modification of Proposition 2.7 (3), there exist numbers s, s_1 such that

$$L^r_{\omega_j}(\Sigma) \hookrightarrow L^s(\Sigma), \quad L^{s_1}(\Sigma) \hookrightarrow L^{r'}_{\omega_j'}(\Sigma), \quad j \in \mathbb{N}, \tag{3.21}$$

with embedding constants independent of $j \in \mathbb{N}$. Furthermore, we may assume without loss of generality that

$$\|(\lambda_j + \alpha + \xi_j^2)u_j, (\lambda_j + \alpha + \xi_j^2)^{1/2}\nabla' u_j, \nabla'^2 u_j, \nabla' p_j, \eta_j p_j\|_{r,\omega_j} = 1 \tag{3.22}$$

and consequently that

$$\|f_j, \nabla' g_j, g_j, \xi_j g_j\|_{r,\omega_j} + (|\lambda_j| + 1)\|g_j; L^r_{m,\omega_j} + L^r_{\omega_j,1/\xi_j}\|_0 \to 0 \quad \text{as} \quad j \to \infty. \tag{3.23}$$

From (3.21), (3.22) we have

$$\|(\lambda_j + \alpha + \xi_j^2)u_j, (\lambda_j + \alpha + \xi_j^2)^{1/2}\nabla' u_j, \nabla'^2 u_j, \nabla' p_j, \eta_j p_j\|_s \le K, \tag{3.24}$$

with some $K > 0$ for all $j \in \mathbb{N}$ and

$$\|f_j, \nabla' g_j, g_j, \eta_j g_j\|_s \to 0 \quad \text{as} \quad j \to \infty. \tag{3.25}$$

Without loss of generality let us suppose that as $j \to \infty$,

$$\begin{array}{l} \lambda_j \to \lambda \in -\alpha + \bar{S}_\varepsilon \quad \text{or} \quad |\lambda_j| \to \infty \\ \xi_j \to 0 \quad \text{or} \quad \xi_j \to \xi \neq 0 \quad \text{or} \quad |\xi_j| \to \infty. \end{array}$$

Thus we have to consider six possibilities.

(i) The case $\lambda_j \to \lambda \in -\alpha + \bar{S}_\varepsilon, \quad \xi_j \to \xi < \infty.$
Due to (3.24) $\{u_j\} \subset W^{2,s}$ and $\{p_j\} \subset W^{1,s}$ are bounded sequences. In virtue of the compactness of the embedding $W^{1,s}(\Sigma) \hookrightarrow L^s(\Sigma)$ for the bounded domain Σ, we may assume (suppressing indices for subsequences) that

$$\begin{array}{lll} u_j \to u, \nabla' u_j \to \nabla' u & \text{in } L^s & \text{(strong convergence)} \\ \nabla'^2 u_j \rightharpoonup \nabla'^2 u & \text{in } L^s & \text{(weak convergence)} \\ p_j \to p & \text{in } L^s & \text{(strong convergence)} \\ \nabla' p_j \rightharpoonup \nabla' p & \text{in } L^s & \text{(weak convergence)} \end{array} \tag{3.26}$$

for some $(u,p) \in \mathcal{D}(S_{s,\lambda,\xi})$ as $j \to \infty$. Therefore, $S_{s,\lambda,\xi}(u,p) = 0$ and, consequently, $u = 0$, $p = 0$ by Lemma 3.5. On the other hand we get from (3.22) that $\sup_{j\in\mathbb{N}} \|u_j\|_{2,r,\omega_j} < \infty$ and $\sup_{j\in\mathbb{N}} \|p_j\|_{1,r,\omega_j} < \infty$ which, together with the weak convergences $u_j \rightharpoonup 0$ in $W^{2,s}(\Sigma)$, $p_j \rightharpoonup 0$ in $W^{1,s}(\Sigma)$, yields

$$\|u_j\|_{1,r,\omega_j} \to 0, \quad \|p_j\|_{r,\omega_j} \to 0$$

due to Proposition 2.8 (2). Moreover, since $\sup_{j\in\mathbb{N}} \|\lambda_j u_j\|_{r,\omega_j} < \infty$ and $\lambda_j u_j \rightharpoonup \lambda u = 0$ in $L^s(\Sigma)$, Proposition 2.8 (3) implies that

$$\|\lambda_j u_j\|_{(W^{1,r'}_{\omega'_j})^*} \to 0. \tag{3.27}$$

Thus (3.6), (3.22) and (3.23) yield the contradiction $1 \le 0$.

(ii) The case $\lambda_j \to \lambda \in -\alpha + \bar{S}_\varepsilon, \quad |\xi_j| \to \infty$.
From (3.22) we get $\|\nabla' u_j, \xi_j u_j, p_j\|_{r,\omega_j} \to 0$. On the other hand, since $\|u_j\|_{r,\omega_j} \to 0$ and $u_j \to 0$ in L^s as $j \to \infty$, Proposition 2.8 (3) implies (3.27). Thus, from (3.6), (3.22) and (3.23) we get the contradiction $1 \le 0$.

(iii) The case $|\lambda_j| \to \infty, \quad \xi_j \to \xi < \infty$.
By (3.22)

$$\|\nabla' u_j, \xi_j u_j\|_{r,\omega_j} \to 0 \quad \text{as} \quad j \to \infty. \tag{3.28}$$

Further, (3.24) yields the convergence

$$\begin{array}{lll} u_j \to 0, \nabla' u_j \to 0 & \text{and} & \nabla'^2 u_j \rightharpoonup 0, \lambda_j u_j \rightharpoonup v, \\ p_j \to p & \text{and} & \nabla' p_j \rightharpoonup \nabla' p, \end{array}$$

in L^s, which, together with (3.25), leads to

$$v' + \nabla' p = 0, \quad v_n + i\eta p = 0. \tag{3.29}$$

Let $g_j := g_{j0} + g_{j1}$, $g_{j0} \in L^r_{0,\omega_j}$, $g_{j1} \in L^r_{\omega_j}$. Then, by (3.24) we have

$$\|\lambda_j g_{j0}\|_{-1,r,\omega_j} + \|\lambda_j g_{j1}/\eta_j\|_{r,\omega_j} \to 0 \quad (j \to \infty). \tag{3.30}$$

From (3.21), (3.30) we see that

$$\begin{aligned} |\langle \lambda_j g_j, \varphi\rangle| &= |\langle \lambda_j g_{j0}, \varphi\rangle + \langle \lambda_j g_{j1}, \varphi\rangle| \\ &\le \|\lambda_j g_{j0}\|_{-1,r,\omega_j} \|\nabla'\varphi\|_{r',\omega_j'} + \|\lambda_j g_{j1}\|_{r,\omega_j} \|\varphi\|_{r',\omega_j'} \\ &\le c\big(\|\lambda_j g_{j0}\|_{-1,r,\omega_j}\| + \|\lambda_j g_{j1}/\eta_j\|_{r,\omega_j}\big)\|\varphi\|_{W^{1,s_1}(\Sigma)}. \end{aligned}$$

Consequently,

$$\lambda_j g_j \in (W^{1,s_1}(\Sigma))^* \quad \text{and} \quad \|\lambda_j g_j\|_{(W^{1,s_1}(\Sigma))^*} \to 0 \quad \text{as } j \to \infty. \tag{3.31}$$

Therefore, it follows from the divergence equation $\text{div}'_{\eta_j} u_j = g_j$ that for all $\varphi \in C^\infty(\bar{\Sigma})$

$$\begin{aligned} \langle v', -\nabla'\varphi\rangle + \langle i\eta v_n, \varphi\rangle &= \lim_{j\to\infty} \langle \text{div}'\lambda_j u_j' + i\lambda_j \xi_j u_{jn}, \varphi\rangle \\ &= \lim_{j\to\infty} \langle \lambda_j g_j, \varphi\rangle = 0, \end{aligned}$$

yielding $\text{div}' v' = -i\eta v_n,\ v' \cdot N|_{\partial\Sigma} = 0$. Therefore (3.29) implies

$$-\Delta' p + \eta^2 p = 0 \text{ in } \Sigma, \quad \frac{\partial p}{\partial N} = 0 \text{ on } \partial\Sigma. \tag{3.32}$$

Here note that $\eta^2 = \xi^2 - \beta^2 + 2i\xi\beta$. Hence, if $\xi \ne 0$ then $p \equiv 0$ since the all eigenvalues of the Neumann Laplacian in Σ is real; if $\xi = 0$, then $\eta^2 = -\beta^2$ and hance $p \equiv 0$ due to the condition $\beta^2 < \bar{\alpha} \le \alpha_1$. That is, we have $p \equiv 0$, and and also $v \equiv 0$. Now, due to Proposition 2.8 (2), (3), we get (3.27) and the convergence $\|p_j\|_{r,\omega_j} \to 0$, since $\lambda_j u_j \rightharpoonup 0$ in L^s, $p_j \rightharpoonup 0$ in $W^{1,s}$ and $\sup_{j\in\mathbb{N}} \|\lambda_j u_j\|_{r,\omega_j} < \infty$,

$\sup_{j\in\mathbb{N}} \|p_j\|_{1,r,\omega_j} < \infty$. Thus (3.6), (3.22), (3.23) and (3.28) lead to the contradiction $1 \le 0$.

(iv) The case $|\lambda_j| \to \infty, \quad |\xi_j| \to \infty$.
To come to a contradiction, it is enough to prove (3.27) since $\|\nabla' u_j, \xi_j u_j, p_j\|_{r,\omega_j} \to 0$ as $j \to \infty$. From (3.22) we get the convergence

$$\begin{array}{lll} u_j \to 0, \nabla' u_j \to 0 & \text{and} & \nabla'^2 u_j \rightharpoonup 0, (\lambda_j + \eta_j^2) u_j \rightharpoonup v, \\ p_j \to 0 & \text{and} & \nabla' p_j \rightharpoonup 0, \quad \eta_j p_j \rightharpoonup q \end{array}$$

in L^s with some $v, q \in L^s$. Therefore, (3.25) and (R_{λ_j,ξ_j}) yield

$$v' = 0, \quad v_n + iq = 0.$$

Since $\|\lambda_j u_j\|_s \le c_\varepsilon \|(\lambda_j + \eta_j^2) u_j\|_s$, there exists $w = (w', w_n) \in L^s$ such that, for a suitable subsequence, $\lambda_j u_j \rightharpoonup w$. Let $g_j = g_{j0} + g_{j1}$, $j \in \mathbb{N}$, be a sequence of splittings satisfying (3.30). By (3.21) we get for all $\varphi \in C^\infty(\bar{\Sigma})$

$$|\langle \lambda_j g_{j0}, \varphi\rangle| + \left|\langle \frac{\lambda_j g_{j1}}{\eta_j}, \varphi\rangle\right| \to 0 \quad \text{as} \quad j \to \infty,$$

cf. (3.31) and (3.31). Hence, the divergence equation implies that for $j \to \infty$

$$\langle \lambda_j u_{jn}, \varphi\rangle = \frac{1}{i\eta_j}\langle \lambda_j g_{j0}, \varphi\rangle + \langle \frac{\lambda_j g_{j1}}{i\eta_j}, \varphi\rangle + \frac{1}{i\eta_j}\langle \lambda_j u_j', \nabla'\varphi\rangle \to 0$$

for all $\varphi \in C^\infty(\bar{\Sigma})$ yielding $\langle w_n, \varphi\rangle = 0$ and consequently $w_n = 0$.

Obviously, $\eta_j u_j \to 0$ in L^s as $j \to \infty$. Therefore, by (3.25) and the boundedness of the sequence $\{\|\eta_j \nabla u_j\|_{r,\omega_j}\}$, we get from the identity $\operatorname{div}'(\eta_j u_j') + i\eta_j^2 u_{jn} = \eta_j g_j$ that

$$\eta_j^2 u_{jn} \rightharpoonup 0 \;\text{ and hance }\; \xi_j^2 u_{jn} \rightharpoonup 0 \quad \text{in } L^s \text{ as } j \to \infty.$$

Thus we proved $v_n = 0$. Now $v = 0$ together with the estimate $\|(\lambda_j + \xi_j^2) u_j\|_{r,\omega_j} \le 1$ imply due to Proposition 2.8 (3) that $\|(\lambda_j + \xi_j^2) u_j\| \to 0$ in $(W^{1,r'}_{\omega_j'})^*$ as $j \to \infty$. Hence also (3.27) is proved.

Now the proof of this lemma is complete. ■

Theorem 3.8 *Let* $1 < r < \infty$, $\omega \in A_r$ *and* $\xi \in \mathbb{R}^*$, $\beta \in (0, \sqrt{\bar{\alpha}})$, $\alpha \in (0, \bar{\alpha} - \beta^2)$, $\varepsilon \in (0, \arctan \frac{\sqrt{\bar{\alpha} - \beta^2 - \alpha}}{\beta})$. *Then for every* $\lambda \in -\alpha + S_\varepsilon$, $\xi \in \mathbb{R}^*$ *and* $f \in L^r_\omega(\Sigma)$, $g \in W^{1,r}_\omega(\Sigma)$ *the parametrized resolvent problem* $(R_{\lambda,\xi,\beta})$ *has a unique solution* $(u, p) \in \big(W^{2,r}_\omega(\Sigma) \cap W^{1,r}_{0,\omega}(\Sigma)\big) \times W^{1,r}_\omega(\Sigma)$. *Moreover, this solution satisfies the estimate* (3.18) *with an* A_r*-consistent constant* $c = c(\alpha, \beta, \varepsilon, r, \Sigma, \mathcal{A}_r(\omega)) > 0$.

Proof: The existence is obvious since, for every $\lambda \in -\alpha + S_\varepsilon, \xi \in \mathbb{R}^*$ and $\omega \in A_r(\mathbb{R}^{n-1})$, the range $\mathcal{R}(S^\omega_{r,\lambda,\xi})$ is closed and dense in $L^r_\omega(\Sigma) \times W^{1,r}_\omega(\Sigma)$ by Lemma 3.6 and by Lemma 3.5, respectively. Here note that for fixed $\lambda \in \mathbb{C}$, $\xi \in \mathbb{R}^*$ the norm

$\|\nabla' g, g, \xi g\|_{r,\omega} + (1 + |\lambda|)\|g; L^r_{m,\omega} + L^r_{\omega,1/\xi}\|_0$ is equivalent to the norm of $W^{1,r}_\omega(\Sigma)$. The uniqueness of solutions is obvious from Lemma 3.5. ■

Now, for fixed $\omega \in A_r, 1 < r < \infty$, define the operator-valued functions

$$\begin{aligned} a_1 &: \mathbb{R}^* \to \mathcal{L}(L^r_\omega(\Sigma); W^{2,r}_{0,\omega}(\Sigma) \cap W^{1,r}_\omega(\Sigma)), \\ b_1 &: \mathbb{R}^* \to \mathcal{L}(L^r_\omega(\Sigma); W^{1,r}_\omega(\Sigma)) \end{aligned}$$

by

$$a_1(\xi)f := u_1(\xi), \quad b_1(\xi)f := p_1(\xi), \tag{3.33}$$

where $(u_1(\xi), p_1(\xi))$ is the solution to $(R_{\lambda,\xi,\beta})$ corresponding to $f \in L^r_\omega(\Sigma)$ and $g = 0$. Further, define

$$\begin{aligned} a_2 &: \mathbb{R}^* \to \mathcal{L}(W^{1,r}_\omega(\Sigma); W^{2,r}_{0,\omega}(\Sigma) \cap W^{1,r}_\omega(\Sigma)), \\ b_2 &: \mathbb{R}^* \to \mathcal{L}(W^{1,r}_\omega(\Sigma); W^{1,r}_\omega(\Sigma)) \end{aligned}$$

by

$$a_2(\xi)g := u_2(\xi), \quad b_2(\xi)g := p_2(\xi). \tag{3.34}$$

with $(u_2(\xi), p_2(\xi))$ the solution to $(R_{\lambda,\xi,\beta})$ corresponding to $f = 0$ and $g \in W^{1,r}_\omega(\Sigma)$.

Corollary 3.9 *Assume the same for $\alpha, \beta, \xi, \lambda$ as in Theorem 3.8. Then, the operator-valued functions a_1, b_1 and a_2, b_2 defined by* (3.33), (3.34) *are Fréchet differentiable in $\xi \in \mathbb{R}^*$. Furthermore, their derivatives $w_1 = \frac{d}{d\xi}a_1(\xi)f$, $q_1 = \frac{d}{d\xi}b_1(\xi)f$ for fixed $f \in L^r_\omega(\Sigma)$ and $w_2 = \frac{d}{d\xi}a_2(\xi)g$, $q_2 = \frac{d}{d\xi}b_2(\xi)g$ for fixed $g \in W^{1,r}_\omega(\Sigma)$ satisfy the estimates*

$$\|(\lambda + \alpha)\xi w_1, \xi\nabla'^2 w_1, \xi^3 w_1, \xi\nabla' q_1, \xi\eta q_1\|_{r,\omega} \le c\|f\|_{r,\omega} \tag{3.35}$$

and

$$\begin{aligned} &\|(\lambda + \alpha)\xi w_2, \xi\nabla'^2 w_2, \xi^3 w_2, \xi\nabla' q_2, \xi\eta q_2\|_{r,\omega} \\ &\qquad \le c\big(\|\nabla' g, g, \xi g\|_{r,\omega} + (|\lambda| + 1)\|g; L^r_{0,\omega} + L^r_{\omega,1/\eta}\|_0\big), \end{aligned} \tag{3.36}$$

with an A_r-consistent constant $c = c(\alpha, \beta, r, \varepsilon, \Sigma, \mathcal{A}_r(\omega))$ independent of $\lambda \in -\alpha + S_\varepsilon$ and $\xi \in \mathbb{R}^*$.

Proof: Since ξ enters in $(R_{\lambda,\xi})$ in a polynomial way, it is easy to prove that $a_j(\xi), b_j(\xi), j = 1, 2$, are Fréchet differentiable and their derivatives w_j, q_j solve the system

$$\begin{aligned} (\lambda + \eta^2 - \Delta')w'_j + \nabla' q_j &= -2\eta u'_j \\ (\lambda + \eta^2 - \Delta')w_{jn} + i\eta q_j &= -2\eta u_{jn} - ip_j \\ \operatorname{div}' w'_j + i\eta w_{jn} &= -iu_{jn}, \end{aligned} \tag{3.37}$$

where $(u_1, p_1), (u_2, p_2)$ are the solutions to $(R_{\lambda,\xi,\beta})$ for $f \in L^r_\omega(\Sigma)$, $g = 0$ and $f = 0$, $g \in W^{1,r}_\omega(\Sigma)$, respectively.

We get from (3.37) and Theorem 3.8 for $j = 1, 2$,

$$\begin{aligned}
&\|(\lambda+\alpha)\xi w_j, \xi\nabla'^2 w_j, \xi^3 w_j, \xi\nabla' q_j, \xi\eta q_j\|_{r,\omega} \\
&\quad \le c\big(\|\xi\eta u_j', \xi p_j, \xi\nabla' u_{jn}, \xi^2 u_{jn}\|_{r,\omega} + (|\lambda|+1)\|i\eta u_{jn}; L^r_{0,\omega} + L^r_{\omega,1/\eta}\|_0\big) \\
&\quad \le c\big(\|\xi^2 u_j, \xi p_j, \xi\nabla' u_j\|_{r,\omega} + (|\lambda|+1)\|u_j\|_{r,\omega}\big) \qquad (3.38) \\
&\quad \le c\|u_j, (\lambda+\alpha+\xi^2)u_j, \sqrt{\lambda+\alpha+\xi^2}\nabla' u_j, \xi p_j\|_{r,\omega} \\
&\quad \le c\|(\lambda+\alpha+\xi^2)u_j, \sqrt{\lambda+\alpha+\xi^2}\nabla' u_j, \nabla'^2 u_j, \xi p_j\|_{r,\omega},
\end{aligned}$$

with an A_r-consistent constant $c = c(\alpha, r, \varepsilon, \Sigma, \mathcal{A}_r(\omega))$; here we used the fact that $\xi^2 + |\lambda+\alpha| \le c(\varepsilon, \alpha)|\lambda+\alpha+\xi^2|$ for all $\lambda \in -\alpha + S_\varepsilon, \xi \in \mathbb{R}$ and $\|u_j\|_{r,\omega} \le c(\mathcal{A}_r(\omega))\|\nabla'^2 u_j\|_{r,\omega}$ (see [16], Corollary 2.2). Thus Theorem 3.8 and (3.38) prove (3.35), (3.36). ■

4 Proof of the Main Results

4.1 Proof of Theorem 2.1 – Theorem 2.3

The proof of Theorem 2.1 is based on the theory of operator-valued Fourier multipliers. The classical Hörmander-Michlin theorem for scalar-valued multipliers for $L^q(\mathbb{R}^k)$, $q \in (1,\infty)$, $k \in \mathbb{N}$, extends to an operator-valued version for Bochner spaces $L^q(\mathbb{R}^k; X)$ provided that X is a *UMD space* and that the boundedness condition for the derivatives of the multipliers is strengthened to *$\mathcal{R}$-boundedness.*

Definition 4.1 *A Banach space X is called a UMD space if the Hilbert transform*

$$Hf(t) = -\frac{1}{\pi}\ PV \int \frac{f(s)}{t-s}\, ds \qquad \text{for } f \in \mathcal{S}(\mathbb{R}; X),$$

where $\mathcal{S}(\mathbb{R}; X)$ is the Schwartz space of all rapidly decreasing X-valued functions, extends to a bounded linear operator in $L^q(\mathbb{R}; X)$ for some $q \in (1,\infty)$.

It is well known that, if X is a *UMD* space, then the Hilbert transform is bounded in $L^q(\mathbb{R}; X)$ *for all* $q \in (1,\infty)$ (see e.g. [32], Theorem 1.3) and that weighted Lebesgue spaces $L^r_\omega(\Sigma), 1 < r < \infty, \omega \in A_r$, are *UMD* spaces.

Definition 4.2 *Let X, Y be Banach spaces. An operator family $\mathcal{T} \subset \mathcal{L}(X; Y)$ is called $\mathcal{R}$-bounded if there is a constant $c > 0$ such that for all $T_1, \dots, T_N \in \mathcal{T}$, $x_1, \dots, x_N \in X$ and $N \in \mathbb{N}$*

$$\Big\| \sum_{j=1}^N \varepsilon_j(s) T_j x_j \Big\|_{L^q(0,1;Y)} \le c \Big\| \sum_{j=1}^N \varepsilon_j(s) x_j \Big\|_{L^q(0,1;X)} \qquad (4.1)$$

for some $q \in [1,\infty)$, where (ε_j) is any sequence of independent, symmetric $\{-1,1\}$-valued random variables on $[0,1]$. The smallest constant c for which (4.1) holds is denoted by $R_q(\mathcal{T})$, the $\mathcal{R}$-bound of $\mathcal{T}$.

Remark 4.3 (1) Due to *Kahane's inequality* ([5])

$$\Big\|\sum_{j=1}^{N}\varepsilon_j(s)x_j\Big\|_{L^{q_1}(0,1;X)} \le c(q_1,q_2,X)\Big\|\sum_{j=1}^{N}\varepsilon_j(s)x_j\Big\|_{L^{q_2}(0,1;X)},\ \ 1\le q_1,q_2<\infty, \quad (4.2)$$

the inequality (4.1) holds *for all* $q\in[1,\infty)$ if it holds for some $q\in[1,\infty)$.

(2) If an operator family $\mathcal{T}\subset\mathcal{L}(L^r_\omega(\Sigma))$, $1<r<\infty$, $\omega\in A_r(\mathbb{R}^{n-1})$, is $\mathcal{R}$-bounded, then $\mathcal{R}_{q_1}(\mathcal{T})\le C\mathcal{R}_{q_2}(\mathcal{T})$ for all $q_1,q_2\in[1,\infty)$ with a constant $C=C(q_1,q_2)>0$ independent of ω. In fact, introducing the isometric isomorphism

$$I_\omega : L^r_\omega(\Sigma)\to L^r(\Sigma), \quad I_\omega f = f\omega^{1/r},$$

for all $T\in\mathcal{L}(L^r_\omega(\Sigma))$ we have $\tilde{T}_\omega = I_\omega T I_\omega^{-1}\in\mathcal{L}(L^r(\Sigma))$ and $\|T\|_{\mathcal{L}(L^r_\omega(\Sigma))}=\|\tilde{T}_\omega\|_{\mathcal{L}(L^r(\Sigma))}$. Then it is easily seen that $\tilde{\mathcal{T}}_\omega := \{I_\omega T I_\omega^{-1} : T\in\mathcal{T}\}\subset\mathcal{L}(L^r(\Sigma))$ is $\mathcal{R}$-bounded and $\mathcal{R}_q(\tilde{\mathcal{T}}_\omega)=\mathcal{R}_q(\mathcal{T})$ for all $q\in[1,\infty)$. Thus the assertion follows.

Definition 4.4 *(1) Let X be a Banach space and $(x_n)_{n=1}^\infty\subset X$. A series $\sum_{n=1}^\infty x_n$ is called unconditionally convergent if $\sum_{n=1}^\infty x_{\sigma(n)}$ is convergent in norm for every permutation $\sigma:\mathbb{N}\to\mathbb{N}$.*

(2) A sequence of projections $(\Delta_j)_{j\in\mathbb{N}}\subset\mathcal{L}(X)$ is called a Schauder decomposition of a Banach space X if

$$\Delta_i\Delta_j = 0 \ \textit{for all}\ i\ne j, \quad \sum_{j=1}^{\infty}\Delta_j x = x \ \textit{for each}\ x\in X.$$

A Schauder decomposition $(\Delta_j)_{j\in\mathbb{N}}$ is called unconditional if the series $\sum_{j=1}^\infty \Delta_j x$ converges unconditionally for each $x\in X$.

Remark 4.5 (1) If $(\Delta_j)_{j\in\mathbb{N}}$ is an unconditional Schauder decomposition of a Banach space Y, then for each $p\in[1,\infty)$ there is a constant $c_\Delta=c_\Delta(p)>0$ such that for all x_j in the range $\mathcal{R}(\Delta_j)$ of Δ_j the inequalities

$$c_\Delta^{-1}\Big\|\sum_{j=l}^{k}x_j\Big\|_Y \quad\le\quad \Big\|\sum_{j=l}^{k}\varepsilon_j(s)x_j\Big\|_{L^p(0,1;Y)} \le c_\Delta\Big\|\sum_{j=l}^{k}x_j\Big\|_Y \quad (4.3)$$

are valid for any sequence $(\varepsilon_j(s))$ of independent, symmetric $\{-1,1\}$-valued random variables defined on $(0,1)$ and for all $l\le k\in\mathbb{Z}$, see e.g. [4], (3.8).

(2) Let $Y=L^q(\mathbb{R};L^r_\omega(\Sigma))$ and assume that each Δ_j commutes with the isomorphism I_ω introduced in Remark 4.3 (2). Then the constant c_Δ is easily seen to be independent of the weight ω.

(3) In the previous definitions and results the set of indices $\mathbb{N}$ may be replaced by $\mathbb{Z}$ without any further changes.

(4) Let X be a *UMD* space and $\chi_{[a,b)}$ denote the characteristic function for the interval $[a,b)$. Let $R_s=\mathcal{F}^{-1}\chi_{[s,\infty)}\mathcal{F}$ and

$$\Delta_j := R_{2^j} - R_{2^{j+1}},\ j\in\mathbb{Z}.$$

It is well known that the Riesz projection R_0 is bounded in $L^q(\mathbb{R};X)$ and that the set $\{R_s - R_t: \ s,t \in \mathbb{R}\}$ is $\mathcal{R}$-bounded in $\mathcal{L}(L^q(\mathbb{R};X))$ for each $q \in (1,\infty)$. In particular, $\{\Delta_j: \ j \in \mathbb{Z}\}$ is $\mathcal{R}$-bounded in $\mathcal{L}(L^q(\mathbb{R};X))$ and an unconditional Schauder decomposition of $R_0 L^q(\mathbb{R};X)$, the image of $L^q(\mathbb{R};X)$ by the Riesz projection R_0, see [4], proof of Theorem 3.19.

We recall an operator-valued Fourier multiplier theorem in Banach spaces. Let $\mathcal{D}_0(\mathbb{R};X)$ denote the set of C^∞-functions $f: \mathbb{R} \to X$ with compact support in $\mathbb{R}^*$.

Theorem 4.6 ([4], Theorem 3.19, [36], Theorem 3.4) *Let X and Y be UMD spaces and $1 < q < \infty$. Let $M: \mathbb{R}^* \to \mathcal{L}(X,Y)$ be a differentiable function such that*

$$\mathcal{R}_q\big(\{M(t),\, tM'(t): \ t \in \mathbb{R}^*\}\big) \le A.$$

Then the operator

$$Tf = \big(M(\cdot)\hat{f}(\cdot)\big)^\vee, \quad f \in \mathcal{D}_0(X),$$

extends to a bounded operator $T: L^q(\mathbb{R};X) \to L^q(\mathbb{R};Y)$ with operator norm $\|T\|_{\mathcal{L}(L^q(\mathbb{R};X);L^q(\mathbb{R};Y))} \le CA$ where $C > 0$ depends only on q, X and Y.

Remark 4.7 *Checking the proof of [4], Theorem 3.19, one can see that the constant C in Theorem 4.6 equals*

$$C = \mathcal{R}(\mathcal{P}) \cdot (c_\Delta)^2$$

where $\mathcal{R}(\mathcal{P})$ is the $\mathcal{R}$-bound of the operator family $\mathcal{P} = \{R_s - R_t: \ s,t \in \mathbb{R}\}$ in $\mathcal{L}(L^q(\mathbb{R};X))$ and c_Δ is the unconditional constant of the Schauder decomposition $\{\Delta_j: \ j \in \mathbb{Z}\}$ of the space $R_0 L^q(\mathbb{R};X)$; see [4], Section 3, for details. In particular, for $X = L^r_\omega(\Sigma)$, $1 < r < \infty$, $\omega \in A_r$, using the isometry I_ω of Remark 4.3 (2), we get that the constants $\mathcal{R}(\mathcal{P})$, see Remark 4.3 (2), and c_Δ do not depend on the weight ω; concerning c_Δ we again use that I_ω commutes with each Δ_j.

Theorem 4.8 (Extrapolation Theorem) *Let $1 < r, s < \infty$, $\omega \in A_r(\mathbb{R}^{n-1})$ and $\Sigma \subset \mathbb{R}^{n-1}$ be an open set. Moreover let $\mathcal{T}$ be a family of linear operators with the property that there exists an A_s-consistent constant $C_\mathcal{T} = C_\mathcal{T}(\mathcal{A}_s(\nu)) > 0$ such that for all $\nu \in A_s$*

$$\|Tf\|_{s,\nu} \le C_\mathcal{T} \|f\|_{s,\nu}$$

for all $T \in \mathcal{T}$ and all $f \in L^s_\nu(\Sigma)$. Then every $T \in \mathcal{T}$ can be extended to $L^r_\omega(\Sigma)$ and $\mathcal{T}$ is $\mathcal{R}$-bounded in $\mathcal{L}(L^r_\omega(\Sigma))$ with an A_r-consistent $\mathcal{R}$-bound $c_\mathcal{T}(q,r,\mathcal{A}_r(\omega))$, i.e.,

$$\mathcal{R}_q(\mathcal{T}) \le c_\mathcal{T}(q,r,\mathcal{A}_r(\omega)) \quad \text{for all} \quad q \in (1,\infty). \tag{4.4}$$

Proof: From the proof of [16], Theorem 4.3, it can be deduced that $\mathcal{T}$ is $\mathcal{R}$-bounded in $\mathcal{L}(L^r_\omega(\Sigma))$ and that (4.4) is satisfied for $q = r$. Then, Remark 4.3 yields (4.4) for every $1 < q < \infty$. ■

Now we are in a position to prove Theorem 2.1.

Proof of Theorem 2.1: Let $f(x', x_n) := e^{\beta x_n} F(x', x_n)$ for $(x', x_n) \in \Sigma \times \mathbb{R}$ and let us define u, p in the cylinder $\Omega = \Sigma \times \mathbb{R}$ by

$$u(x) = \mathcal{F}^{-1}(a_1 \hat{f})(x), \quad p(x) = \mathcal{F}^{-1}(b_1 \hat{f})(x),$$

where a_1, b_1 are the operator-valued multiplier functions defined in (3.33). We will show that $(U, P) = (e^{\beta x_n} u, e^{\beta x_n} p)$ is the unique solution to (R_λ) with $g = 0$ such that

$$(u, p) \in \big(W^{2;q,r}_\omega(\Omega) \cap W^{1;q,r}_{0,\omega}(\Omega)\big) \times \widehat{W}^{1;q,r}_\omega(\Omega) \tag{4.5}$$

and the estimate (2.1) holds. Obviously, (U, P) solves the resolvent problem (R_λ) with $g = 0$. For $\xi \in \mathbb{R}^*$ define $m_\lambda(\xi) : L^r_\omega(\Sigma) \to L^r_\omega(\Sigma)$ by

$$m_\lambda(\xi) f := \big((\lambda + \alpha) a_1(\xi) \hat{f}, \xi \nabla' a_1(\xi) \hat{f}, \nabla'^2 a_1(\xi) \hat{f}, \xi^2 a_1(\xi) \hat{f}, \nabla' b_1(\xi) \hat{f}, \xi b_1(\xi) \hat{f}\big).$$

Theorem 3.8 and Corollary 3.9 show that the operator family $\{m_\lambda(\xi), \xi m'_\lambda(\xi) : \xi \in \mathbb{R}^*\}$ satisfies the assumptions of Theorem 4.8, e.g., with $s = r$. Therefore, this operator family is $\mathcal{R}$-bounded in $\mathcal{L}(L^r_\omega(\Sigma))$; to be more precise,

$$\mathcal{R}_q\big(\{m_\lambda(\xi), \xi m'_\lambda(\xi) : \xi \in \mathbb{R}^*\}\big) \le c(q, r, \alpha, \beta, \varepsilon, \Sigma, \mathcal{A}_r(\omega)) < \infty.$$

Hence Theorem 4.6 and Remark 4.7 imply that

$$\|(m_\lambda \hat{f})^\vee\|_{L^q(L^r_\omega)} \le C \|f\|_{L^q(L^r_\omega)}$$

with an A_r-consistent constant $C = C(q, r, \alpha, \beta, \varepsilon, \Sigma, \mathcal{A}_r(\omega)) > 0$ independent of the resolvent parameter $\lambda \in -\alpha + S_\varepsilon$. Note that, due to the definition of the multiplier $m_\lambda(\xi)$, we have $(\lambda + \alpha) u, \nabla^2 u, \nabla p \in L^q(L^r_\omega)$ and

$$\|(\lambda + \alpha) u, \nabla^2 u, \nabla p\|_{L^q(L^r_\omega)} \le \|(m_\lambda \hat{f})^\vee\|_{L^q(L^r_\omega)}.$$

Thus the existence of a solution satisfying (2.1) is proved.

The uniqueness of solutions is obvious by the uniqueness result for $\beta = 0$ of [12], Theorem 2.1. Now the proof of Theorem 2.1 is complete. ■

Proof of Corollary 2.2: Defining the Stokes operator $A = A_{q,r;\beta,\omega}$ by (2.2), due to the Helmholtz decomposition of the space $L^q_\beta(L^r_\omega)$ on the cylinder Ω, see [9], we get that for $F \in L^q_\beta(L^r_\omega)_\sigma$ the solvability of the equation

$$(\lambda + A) U = F \quad \text{in} \quad L^q_\beta(L^r_\omega)_\sigma \tag{4.6}$$

is equivalent to the solvability of (R_λ) with right-hand side $G \equiv 0$. By virtue of Theorem 2.1 for every $\lambda \in -\alpha + S_\varepsilon$ there exists a unique solution $U = (\lambda + A)^{-1} F \in D(A)$ to (4.6) satisfying the estimate

$$\|(\lambda + \alpha) U\|_{L^q_\beta(L^r_\omega)_\sigma} = \|(\lambda + \alpha) u\|_{L^q(L^r_\omega)} \le C \|f\|_{L^q(L^r_\omega)} = C \|F\|_{L^q_\beta(L^r_\omega)_\sigma}$$

with $C = C(q, r, \alpha, \beta, \varepsilon, \Sigma, \mathcal{A}_r(\omega))$ independent of λ, where $u = e^{\beta x_n} U$, $f = e^{\beta x_n} F$. Hence (2.3) is proved. Then (2.4) is a direct consequence of (2.3) using semigroup theory. ■

Proof of Theorem 2.3: The proof will be done if we show that the operator family

$$\mathcal{T} = \{\lambda(\lambda + A_{q,r;\beta,\omega})^{-1} :\ \lambda \in \mathrm{i}\mathbb{R}\}$$

is $\mathcal{R}$-bounded in $\mathcal{L}(L^q_\beta(L^r_\omega)_\sigma)$. By the way, since $L^q_\beta(L^r_\omega)_\sigma$ is isomorphic to a closed subspace X of $L^q(L^r_\omega)$ with isomorphism $I_\beta F := e^{\beta x_n}F$, it is enough to show $\mathcal{R}$-boundedness of

$$\tilde{\mathcal{T}} = \{I_\beta \lambda(\lambda + A_{q,r;\beta,\omega})^{-1} I_\beta^{-1} :\ \lambda \in \mathrm{i}\mathbb{R}\} \subset \mathcal{L}(X).$$

In the following we write shortly

$$H_\beta \equiv I_\beta \lambda(\lambda + A_{q,r;\beta,\omega})^{-1} I_\beta^{-1}.$$

For $\xi \in \mathbb{R}^*$ and $\lambda \in S_\varepsilon$, let $m_\lambda(\xi) := \lambda a_1(\xi)$ where $a_1(\xi)$ is the solution operator for $(R_{\lambda,\xi,\beta})$ with $g = 0$ defined by (3.33). Then, we have

$$H_\beta f = I_\beta \lambda U = \lambda I_\beta U = (m_\lambda(\xi)\hat{f})^\vee, \quad \forall f \in \mathcal{S}(\mathbb{R}; L^r_\omega(\Sigma)) \cap X,$$

with U the solution to (R_λ) with $F = I_\beta^{-1} f$, $G = 0$. Note that $\mathcal{S}(\mathbb{R}; L^r_\omega(\Sigma))$ is dense in $L^q(\mathbb{R}; L^r_\omega(\Sigma))$ and hence $\mathcal{S}(\mathbb{R}; L^r_\omega(\Sigma)) \cap X$ is dense in X. Hence, in view of Definition 4.2 and Remark 4.3, $\mathcal{R}$-boundedness of $\tilde{\mathcal{T}}$ in $\mathcal{L}(X)$ is proved if there is a constant $C > 0$ such that

$$\Big\| \sum_{i=1}^N \varepsilon_i (m_{\lambda_i}\hat{f}_i)^\vee \Big\|_{L^q(0,1;L^q(\mathbb{R}:L^r_\omega(\Sigma)))} \le C \Big\| \sum_{i=1}^N \varepsilon_i f_i \Big\|_{L^q(0,1;L^q(\mathbb{R}:L^r_\omega(\Sigma)))} \tag{4.7}$$

for any independent, symmetric and $\{-1,1\}$-valued random variables $(\varepsilon_i(s))$ defined on $(0,1)$, for all $(\lambda_i) \subset \mathrm{i}\mathbb{R}$ and $(f_i) \subset \mathcal{S}(\mathbb{R}; L^r_\omega(\Sigma)) \cap X$. Without loss of generality we may assume that $\mathrm{supp}\hat{f}_i \subset [0,\infty)$, $i = 1, \dots, N$, since $R_0 f := (\chi_{[0,\infty)}(\xi)\hat{f})^\vee$ is continuous in $L^q(\mathbb{R}; L^r_\omega(\Sigma))$ and

$$f_i(x', x_n) = (\chi_{[0,\infty)}\hat{f}_i(\xi))^\vee(x', x_n) + (\chi_{[0,\infty)}\hat{f}_i(-\xi))^\vee(x', -x_n).$$

Note that, if $\mathrm{supp}\hat{f} \subset [0,\infty)$, then $\mathrm{supp}(m_\lambda \hat{f}) \subset [0,\infty)$ as well. Therefore, instead of (4.7) we shall prove the estimate

$$\Big\| \sum_{i=1}^N \varepsilon_i (m_{\lambda_i}\hat{f}_i)^\vee \Big\|_{L^q(0,1;Y)} \le C \Big\| \sum_{i=1}^N \varepsilon_i f_i \Big\|_{L^q(0,1;Y)} \tag{4.8}$$

for $(f_i) \subset \mathcal{S}(\mathbb{R}; L^r_\omega(\Sigma)) \cap X \cap Y$.

Obviously $m_\lambda(\xi) = m_\lambda(2^j) + \int_{2^j}^\xi m'_\lambda(\tau)\,d\tau$ for $\xi \in [2^j, 2^{j+1})$, $j \in \mathbb{Z}$, and $\big(m_\lambda(2^j)\widehat{\Delta_j f}\big)^\vee = m_\lambda(2^j)\Delta_j f$ for $f \in \mathcal{S}(\mathbb{R}; L^r_\omega(\Sigma)) \cap X \cap Y$. Furthermore,

$$\begin{aligned}
\Big(\int_{2^j}^\xi m'_\lambda(\tau)\,d\tau\, \widehat{\Delta_j f}(\xi)\Big)^\vee &= \Big(\int_{2^j}^{2^{j+1}} m'_\lambda(\tau)\chi_{[2^j,\xi)}(\tau)\widehat{\Delta_j f}(\xi)\,d\tau\Big)^\vee \\
&= \Big(\int_0^1 2^j m'_\lambda(2^j(1+t))\chi_{[2^j,\xi)}(2^j(1+t))\chi_{[2^j,2^{j+1})}(\xi)\hat{f}(\xi)\,dt\Big)^\vee \\
&= \int_0^1 2^j m'_\lambda(2^j(1+t)) B_{j,t}\Delta_j f\,dt.
\end{aligned}$$

where $B_{j,t} = R_{2^j(1+t)} - R_{2^{j+1}}$. Thus we get

$$\begin{aligned} \big(m_\lambda(\xi)\hat{f}(\xi)\big)^\vee &= \sum_{j\in\mathbb{Z}} \Big(\big(m_\lambda(2^j) + \int_{2^j}^{\xi} m_\lambda'(\tau)\,d\tau\big)\,\widehat{\Delta_j f}\Big)^\vee \\ &= \sum_{j\in\mathbb{Z}} \big(m_\lambda(2^j)\widehat{\Delta_j f}\big)^\vee + \sum_{j\in\mathbb{Z}} \Big(\int_{2^j}^{\xi} m_\lambda'(\tau)\,d\tau\,\widehat{\Delta_j f}\Big)^\vee \\ &= \sum_{j\in\mathbb{Z}} m_\lambda(2^j)\Delta_j f + \sum_{j\in\mathbb{Z}} \int_0^1 2^j m_\lambda'(2^j(1+t))B_{j,t}\Delta_j f\,dt. \end{aligned} \tag{4.9}$$

First let us prove

$$\Big\|\sum_{i=1}^N \varepsilon_i(s)\sum_{j\in\mathbb{Z}} m_{\lambda_i}(2^j)\Delta_j f_i\Big\|_{L^q(0,1;Y)} \le C\Big\|\sum_{i=1}^N \varepsilon_i(s) f_i\Big\|_{L^q(0,1;Y)}. \tag{4.10}$$

Note that the operator $m_{\lambda_i}(2^j)$ commutes with Δ_j, $j\in\mathbb{Z}$; hence, for almost all $s\in(0,1)$, the sum $\sum_{i=1}^N \varepsilon_i(s)m_{\lambda_i}(2^j)\Delta_j f_i$ belongs to the range of Δ_j. Therefore, for any $l,k\in\mathbb{Z}$ we get by (4.3) that

$$\begin{aligned} &\Big\|\sum_{i=1}^N \varepsilon_i \sum_{j=l}^k m_{\lambda_i}(2^j)\Delta_j f_i\Big\|_{L^q(0,1;Y)} \\ &\qquad = \Big(\int_0^1 \Big\|\sum_{j=l}^k\sum_{i=1}^N \varepsilon_i(s)m_{\lambda_i}(2^j)\Delta_j f_i\Big\|_Y^q\,ds\Big)^{1/q} \\ &\qquad \le c_\Delta\Big(\int_0^1\int_0^1 \Big\|\sum_{j=l}^k \varepsilon_j(\tau)\sum_{i=1}^N \varepsilon_i(s)m_{\lambda_i}(2^j)\Delta_j f_i\Big\|_Y^q\,d\tau\,ds\Big)^{1/q} \\ &\qquad = c_\Delta\Big\|\sum_{i=1}^N\sum_{j=l}^k \varepsilon_{ij}(s,\tau)m_{\lambda_i}(2^j)\Delta_j f_i\Big\|_{L^q((0,1)^2;Y)} \end{aligned} \tag{4.11}$$

where $\varepsilon_{ij}(s,\tau) = \varepsilon_i(s)\varepsilon_j(\tau)$; note that $(\varepsilon_{ij})_{i,j\in\mathbb{Z}}$ is a sequence of independent, symmetric and $\{-1,1\}$-valued random variables defined on $(0,1)\times(0,1)$. Furthermore, due to Theorem 3.8, the operator family $\{m_\lambda(\xi) : \lambda\in i\mathbb{R},\ \xi\in\mathbb{R}^*\}\subset\mathcal{L}(L^r_\omega(\Sigma))$ is uniformly bounded by an A_r-consistent constant, and hence it is $\mathcal{R}$-bounded by Theorem 4.8. Therefore, using Fubini's theorem and (4.3), we proceed in (4.11) as follows:

$$\begin{aligned} &= c_\Delta\Big\|\sum_{i=1}^N\sum_{j=l}^k \varepsilon_{ij}(s,\tau)m_{\lambda_i}(2^j)\Delta_j f_i\Big\|_{L^q(\mathbb{R};L^q((0,1)^2;L^r_\omega(\Sigma)))} \\ &\le Cc_\Delta\Big\|\sum_{i=1}^N\sum_{j=l}^k \varepsilon_{ij}(s,\tau)\Delta_j f_i\Big\|_{L^q(\mathbb{R};L^q((0,1)^2;L^r_\omega(\Sigma)))} \\ &= Cc_\Delta\Big\|\sum_{i=1}^N\sum_{j=l}^k \varepsilon_{ij}(s,\tau)\Delta_j f_i\Big\|_{L^q((0,1)^2;Y)} \le Cc_\Delta^2\Big\|\sum_{i=1}^N \varepsilon_i\sum_{j=l}^k \Delta_j f_i\Big\|_{L^q(0,1;Y)}. \end{aligned} \tag{4.12}$$

Since $\{\sum_{j=l}^k \Delta_j : l,k \in \mathbb{Z}\}$ is $\mathcal{R}$-bounded in $\mathcal{L}(Y)$ and (Δ_j) is a Schauder decomposition of Y, we see by Lebesgue's theorem that the right-hand side of (4.12) converges to 0 as either $l,k \to \infty$ or $l,k \to -\infty$. Thus, by (4.11), (4.12), the series $\sum_{i=1}^N \varepsilon_i(s) \sum_{j\in\mathbb{Z}} m_{\lambda_i}(2^j)\Delta_j f_i$ converges in $L^q(0,1;Y)$, and (4.10) holds.

Next let us show that

$$\Big\| \sum_{i=1}^N \varepsilon_i(s) \sum_{j\in\mathbb{Z}} \int_0^1 2^j m'_{\lambda_i}(2^j(1+t))B_{j,t}\Delta_j f_i \, dt \Big\|_{L^q(0,1;Y)} \le C \Big\| \sum_{i=1}^N \varepsilon_i(s) f_i \Big\|_{L^q(0,1;Y)}. \tag{4.13}$$

Using the same argument as in the proof of (4.10) and the $\mathcal{R}$-boundedness of the operator families $\{B_{j,t} : j \in \mathbb{Z}, t \in (0,1)\} \subset \mathcal{L}(Y)$ and $\{2^j(1+t)m'_\lambda(2^j(1+t)) : \lambda \in \mathrm{i}\mathbb{R}, j \in \mathbb{Z}, t \in (0,1)\} \subset \mathcal{L}(L^r_\omega(\Sigma))$, see Corollary 3.9, we have

$$\begin{aligned}
&\Big\| \sum_{i=1}^N \varepsilon_i(s) \sum_{j=l}^k \int_0^1 2^j m'_{\lambda_i}(2^j(1+t))B_{j,t}\Delta_j f_i \, dt \Big\|_{L^q(0,1;Y)} \\
&\qquad \le \int_0^1 \Big\| \sum_{i=1}^N \varepsilon_i(s) \sum_{j=l}^k 2^j m'_{\lambda_i}(2^j(1+t))B_{j,t}\Delta_j f_i \Big\|_{L^q(0,1;Y)} dt \\
&\qquad \le c_\Delta \int_0^1 \Big\| \sum_{i=1}^N \sum_{j=l}^k \varepsilon_{ij}(s,\tau) 2^j m'_{\lambda_i}(2^j(1+t))B_{j,t}\Delta_j f_i \Big\|_{L^q((0,1)^2;Y)} dt \\
&\qquad \le c_\Delta \int_0^1 \Big\| \sum_{i=1}^N \sum_{j=l}^k \varepsilon_{ij}(s,\tau) 2^j(1+t) m'_{\lambda_i}(2^j(1+t))\Delta_j f_i \Big\|_{L^q((0,1)^2;Y)} dt \\
&\qquad \le C c_\Delta^2 \Big\| \sum_{i=1}^N \varepsilon_i(s) \sum_{j=l}^k \Delta_j f_i \Big\|_{L^q((0,1);Y)}
\end{aligned}$$

for all $l,k \in \mathbb{Z}$. Thus (4.13) is proved.

By (4.10), (4.13) we conclude that the operator family $\mathcal{T} = \{\lambda(\lambda + A_{q,r;\beta,\omega})^{-1} : \lambda \in \mathrm{i}\mathbb{R}\}$ is $\mathcal{R}$-bounded in $\mathcal{L}(L^q_\beta(L^r_\omega))$. Then, by [36], Corollary 4.4, for each $f \in L^p(\mathbb{R}_+; L^q_\beta(L^r_\omega)_\sigma), 1 < p < \infty$, the mild solution U to the system

$$U_t + A_{q,r;\beta,\omega}U = F, \quad u(0) = 0 \tag{4.14}$$

belongs to $L^p(\mathbb{R}_+; L^q_\beta(L^r_\omega)_\sigma) \cap L^p(\mathbb{R}_+; D(A_{q,r;\beta,\omega}))$ and satisfies the estimate

$$\|U_t, A_{q,r;\beta,\omega}U\|_{L^p(\mathbb{R}_+;L^q_\beta(L^r_\omega)_\sigma)} \le C\|F\|_{L^p(\mathbb{R}_+;L^q_\beta(L^r_\omega)_\sigma)}.$$

Furthermore, (2.3) with $\lambda = 0$ implies that U also satisfies this inequality. If $F \in L^p(\mathbb{R}_+; L^q_\beta(L^r_\omega))$, let U be the solution of (4.14) with F replaced by $P_{q,r;\beta,\omega}F$, where $P_{q,r;\beta,\omega}$ denotes the Helmholtz projection in $L^q_\beta(L^r_\omega))$, and define P by $\nabla P = (I - P_{q,r;\beta,\omega})(f - u_t + \Delta u)$. By (2.1) with $\lambda = 0$ and the boundedness of $P_{q,r;\beta,\omega}$ we get (2.7). Finally, assume $e^{\alpha t}F \in L^p(\mathbb{R}_+; L^q_\beta(L^r_\omega))$ for some $\alpha \in (0, \bar\alpha - \beta^2)$ and let V be the solution of the system $V_t + (A - \alpha)V = e^{\alpha t}P_{q,r;\beta,\omega}F$, $V(0) = 0$. Obviously, replacing A by $A - \alpha$ in the previous arguments, v is easily seen to satisfy estimate (2.6). Then $U(t) = e^{-\alpha t}V(t)$ solves (4.14) and satisfies (2.8). In each case the constant C depends only on $\mathcal{A}_r(\omega)$ due to Remark 4.7.

The proof of Theorem 2.3 is complete. ∎

4.2 Proof of Theorem 2.4 and Theorem 2.5

Proof of Theorem 2.4: Let $1 < q < \infty$ and $\xi \in \mathbb{R}^*$, $\beta \in (0, \sqrt{\alpha^*})$, $\alpha^* = \min_{1\le i\le m} \bar{\alpha}_i$, $\alpha \in (0, \alpha^* - \beta^2)$, $\varepsilon \in (0, \arctan \frac{\sqrt{\alpha^*-\beta^2-\alpha}}{\beta})$. Fix $\lambda \in -\alpha + S_\varepsilon$ and $\xi \in \mathbb{R}^*$. Note that $\lambda + A_{q,\mathbf{b}}$ with $\beta_i = 0$ for all $i = 1, \ldots, m$ is injective and surjective, see [13], Theorem 1.2. Hence, given any $F \in L^q_{\mathbf{b},\sigma}(\Omega)$, for all $\lambda \in -\alpha + S_\varepsilon$ there is a unique $(U, \nabla P) \in D(A_q) \times L^q(\Omega)$ such that

$$\begin{aligned} \lambda U - \Delta U + \nabla P &= F \quad \text{in } \Omega, \\ \operatorname{div} U &= 0 \quad \text{in } \Omega, \\ U &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{4.15}$$

Without loss of generality we may assume that there exist cut-off functions $\{\varphi_i\}_{i=0}^m$ such that

$$\begin{aligned} &\textstyle\sum_{i=0}^m \varphi_i(x) = 1, \quad 0 \le \varphi_i(x) \le 1 \quad \text{for } x \in \Omega, \\ &\varphi_i \in C^\infty(\bar{\Omega}_i), \quad \operatorname{dist}(\operatorname{supp}\varphi_i,\, \partial\Omega_i \cap \Omega) \ge \delta > 0,\ i = 0, \ldots, m, \end{aligned} \tag{4.16}$$

where 'dist' means the distance. In what follows, for $i = 1, \ldots, m$ let $\widetilde{\Omega}_i$ be the infinite straight cylinder extending the semi-infinite cylinder Ω_i, and denote the zero extension of $\varphi_i v$ to $\widetilde{\Omega}_i$ by $\widetilde{\varphi_i v}$.

Let

$$(u^0, p^0) := (\varphi_0 U, \varphi_0 P),\ (u^i, p^i) := (\widetilde{\varphi_i U}, \widetilde{\varphi_i P}) \ \text{ for } i = 1, \ldots, m.$$

Then (u^0, p^0) on Ω_0 satisfies

$$\begin{aligned} \lambda u^0 - \Delta u^0 + \nabla p^0 &= f^0 \quad \text{in } \Omega_0, \\ \operatorname{div} u^0 &= g^0 \quad \text{in } \Omega_0, \\ u^0 &= 0 \quad \text{on } \partial\Omega_0, \end{aligned}$$

and (u^i, p^i) on $\widetilde{\Omega}_i$, $i = 1, \ldots, m$, satisfy

$$\begin{aligned} \lambda u^i - \Delta u^i + \nabla p^i &= \tilde{f}^i \quad \text{in } \widetilde{\Omega}_i, \\ \operatorname{div} u^i &= \tilde{g}^i \quad \text{in } \widetilde{\Omega}_i, \\ u^i &= 0 \quad \text{on } \partial\widetilde{\Omega}_i, \end{aligned}$$

where

$$f^i := \varphi_i F + (\nabla\varphi_i)P - (\Delta\varphi_i)U - 2\nabla\varphi_i \cdot \nabla U, \quad g^i := \nabla\varphi_i \cdot U, \quad i = 0, \ldots, m.$$

Since $\operatorname{supp} g^i \subset \Omega_0$, $g^i \in W_0^{1,q}(\Omega_0)$ and $\int_{\Omega_0} g^i\, dx = 0$ for $i = 0, \ldots, m$, we get by the well-known theory of the divergence problem for $i = 0, \ldots, m$ that there is some $w_i \in W_0^{2,q}(\Omega_0)$ satisfying $\operatorname{div} w_i = g^i$ in Ω_0 and

$$\begin{aligned} &\|\nabla^2 w_i\|_{L^q(\Omega_0))} \le c\|\nabla g^i\|_{L^q(\Omega_0)} \le c\|\nabla U\|_{L^q_0(\Omega_0)}, \\ &\|w_i\|_{L^q(\Omega_0))} \le c\|g^i\|_{(W^{1,q}(\Omega_0))^*} \le c\|U\|_{(W^{1,q}(\Omega_0))^*}, \end{aligned} \tag{4.17}$$

where $c = c(\Omega_0, q)$, cf. [17]. Then $\tilde{w}_i$, the extension by 0 of w_i to $\widetilde{\Omega}_i$, $i = 1\dots, m$, satisfies

$$e^{\beta_i x_n^i}\nabla^2\tilde{w}_i \in L^q(\tilde{\Omega}_i),\ \|e^{\beta_i x_n^i}\nabla^2\tilde{w}_i\|_{L^q(\tilde{\Omega}_i)} \le c\|\nabla U\|_{L^q(\Omega_0)}. \tag{4.18}$$

Now, $v^0 := u^0 - w_0$ and $v^i := u^i - \tilde{w}_i$, $i = 1, \dots, m$, solve, respectively,

$$\begin{aligned} \lambda v^0 - \Delta v^0 + \nabla p^0 &= f^0 - (\lambda w_0 - \Delta w_0) && \text{in } \Omega_0,\\ \operatorname{div} v^0 &= 0 && \text{in } \Omega_0,\\ v^0 &= 0 && \text{on } \partial\Omega_0, \end{aligned}$$

and

$$\begin{aligned} \lambda v^i - \Delta v^i + \nabla p^i &= \tilde{f}^i - (\lambda \tilde{w}_i - \Delta \tilde{w}_i) && \text{in } \widetilde{\Omega}_i,\\ \operatorname{div} v^i &= 0 && \text{in } \widetilde{\Omega}_i,\\ v^i &= 0 && \text{on } \partial\widetilde{\Omega}_i. \end{aligned}$$

Then, using the fact that the Stokes operator in L^q-spaces on bounded domains is injective and surjective we get that

$$\|v^0, \lambda v^0, \nabla^2 v^0, \nabla p^0\|_{L^q(\Omega_0)} \le c\|F, \nabla U, P\|_{L^q(\Omega_0)} + (|\lambda| + 1)\|U\|_{(W^{1,q}(\Omega_0))^*} \tag{4.19}$$

with c independent of λ. Moreover, by Theorem 2.1 we have

$$\begin{aligned} \|v^i, \lambda v^i, \nabla^2 v^i, \nabla p^i\|_{L^q_{\beta_i}(\mathbb{R};L^q(\Sigma^i))} \le c(\|F\|_{L^q_{\beta_i}(\tilde{\Omega}_i)} + \|\nabla U, P\|_{L^q(\Omega_0)}\\ +(|\lambda| + 1)\|U\|_{(W^{1,q}(\Omega_0))^*}),\ i = 1, \dots, m, \end{aligned} \tag{4.20}$$

with c independent of λ. Due to $U = \sum_{i=0}^m u^i$, $P = \sum_{i=0}^m p^i$ in Ω and (4.18), we get $\nabla^2 U, \nabla P \in L^q_{\mathbf{b}}(\Omega)$ and

$$\begin{aligned} \|U, \lambda U, \nabla^2 U, \nabla P\|_{L^q_{\mathbf{b}}(\Omega)} \le c(\|F\|_{L^q_{\mathbf{b}}(\Omega)} + \|\nabla U, P\|_{L^q(\Omega_0)})\\ +(|\lambda| + 1)\|U\|_{(W^{1,q}(\Omega_0))^*}. \end{aligned} \tag{4.21}$$

Indeed, by a contradiction argument (4.21) yields

$$\|U, \lambda U, \nabla^2 U, \nabla P\|_{L^q_{\mathbf{b}}(\Omega)} \le c(\|F\|_{L^q_{\mathbf{b}}(\Omega)} + \|\nabla U, P\|_{L^q(\Omega_0)}) \tag{4.22}$$

with c independent of λ.

Assume that (4.22) does not hold. Then there are sequences $\{\lambda_j\} \subset -\alpha + S_\varepsilon$, $\{(U_j, P_j)\} \in$ such that

$$\|U_j, \lambda_j U_j, \nabla^2 U_j, \nabla P_j\|_{L^q_{\mathbf{b}}(\Omega)} = 1, \quad \|F_j\|_{L^q_{\mathbf{b}}(\Omega)} \to 0 \quad \text{as } j \to \infty, \tag{4.23}$$

where $F_j = \lambda U_j - \Delta U_j + \nabla P_j$. Without loss of generality we may assume that

$$\lambda_j U_j \rightharpoonup V,\ U_j \rightharpoonup U,\ \nabla^2 u_j \rightharpoonup \nabla^2 U,\ \nabla P_j \rightharpoonup \nabla P \quad \text{as } j \to \infty \tag{4.24}$$

with some $V \in L^q_{\mathbf{b}}(\Omega)$, $U \in W^{2,q}_{\mathbf{b}}(\Omega) \cap W^{1,q}_{0,\mathbf{b}}(\Omega)$ and $P \in \widehat{W}^{1,q}_{\mathbf{b}}(\Omega)$. Moreover, we may assume $\int_{\Omega_0} P_j\,dx = 0$, $\int_{\Omega_0} P\,dx = 0$ and that $\lambda_j \to \lambda \in \{-\alpha + \bar{S}_\varepsilon\} \cup \{\infty\}$.

(i) Let $\lambda_j \to \lambda \in -\alpha + \bar{S}_\varepsilon$.

Then, $V = \lambda U$ and it follows that (U, P) solves (4.15) with $F = 0$ yielding $(U, P) = 0$. On the other hand, we have the strong convergence

$$U_j \to 0 \text{ in } W^{1,q}(\Omega_0), \;\; P_j \to 0 \text{ in } L^q(\Omega_0), \;\; (|\lambda_j| + 1)U_j \to 0 \text{ in } (W^{1,q'}(\Omega_0))^* \quad (4.25)$$

due to the compact embeddings $W^{2,q}(\Omega_0) \subset\subset W^{1,q}(\Omega_0) \subset\subset L^q(\Omega_0) \subset\subset (W^{1,q'}(\Omega_0))^*$, Poincaré's inequality on Ω_0. Thus (4.22) yields the contradiction $1 \le 0$.

(ii) Let $|\lambda_j| \to \infty$. Then, we conclude that $U = 0$, and consequently $V + \nabla P = 0$ where $V \in L^q_\sigma(\Omega)$. Note that this is the L^q-Helmholtz decomposition of the null vector field on Ω. Therefore, $V = 0$, $\nabla P = 0$. Again we get (4.25) and finally the contradiction $1 \le 0$.

The proof of the theorem is complete. ■

Proof of Theorem 2.5: The idea of the proof is also to use a cut-off technique. Note that any $F \in L^p(\mathbb{R}_+; L^q_{\mathbf{b}}(\Omega))$ also belongs to $L^p(\mathbb{R}_+; L^q(\Omega))$ for $1 < p, q < \infty$. Hence, by maximal L^p-regularity of the Stokes operator in $L^q(\Omega)$, which follows by [13], Theorem 1.2, we get that the problem (2.11) has a unique solution $(U, \nabla P)$ such that

$$(U, \nabla P) \in L^p(\mathbb{R}_+; W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega) \cap L^q_\sigma(\Omega)) \times L^p(\mathbb{R}_+; L^q(\Omega)), U_t \in L^p(\mathbb{R}_+; L^q(\Omega)).$$

We shall prove that this solution $(U, \nabla P)$, furthermore, satisfies

$$(U, \nabla P) \in L^p(\mathbb{R}_+; W^{2,q}_{\mathbf{b}}(\Omega)) \times L^p(\mathbb{R}_+; L^q_{\mathbf{b}}(\Omega)), U_t \in L^p(\mathbb{R}_+; L^q_{\mathbf{b}}(\Omega)). \quad (4.26)$$

Once (4.26) is proved, the (linear) solution operator

$$L^p(\mathbb{R}_+; L^q_{\mathbf{b}}(\Omega)) \ni F \mapsto (U, \nabla P) \in L^p(\mathbb{R}_+; W^{2,q}_{\mathbf{b}}(\Omega) \cap W^{1,q}_0(\Omega) \cap L^q_\sigma(\Omega)) \times L^p(\mathbb{R}_+; L^q_{\mathbf{b}}(\Omega))$$

is obviously closed and hence bounded by the closed graph theorem thus implying (2.12).

The proof of (4.26) is based on cut-off technique using Theorem 2.3. Let $\{\varphi_i\}_{i=0}^m$ be cut-off functions given by (4.16) and let

$$(u^0, p^0) := (\varphi_0 U, \varphi_0 P), \; (u^i, p^i) := (\widetilde{\varphi_i U}, \widetilde{\varphi_i P}) \;\text{ for } i = 1, \dots, m.$$

Then (u^0, p^0) on Ω_0 satisfies

$$\begin{aligned} u^0_t - \Delta u^0 + \nabla p^0 &= f^0 && \text{in } \mathbb{R}_+ \times \Omega_0 \\ \operatorname{div} u^0 &= g^0 && \text{in } \mathbb{R}_+ \times \Omega_0 \\ u^0(0, x) &= 0 && \text{in } \Omega_0, \\ u^0 &= 0 && \text{on } \partial\Omega_0, \end{aligned}$$

and (u^i, p^i) on $\widetilde{\Omega}_i$, $i = 1, \dots, m$, satisfy

$$\begin{aligned} u^i_t - \Delta u^i + \nabla p^i &= \tilde{f}^i && \text{in } \mathbb{R}_+ \times \widetilde{\Omega}_i \\ \operatorname{div} u^i &= \tilde{g}^i && \text{in } \mathbb{R}_+ \times \widetilde{\Omega}_i \\ u^i(0, x) &= 0 && \text{in } \widetilde{\Omega}_i, \\ u^i &= 0 && \text{on } \partial\widetilde{\Omega}_i, \end{aligned}$$

where

$$f^i := \varphi_i F + (\nabla\varphi_i)P - (\Delta\varphi_i)U - 2\nabla\varphi_i\cdot\nabla U, \quad g^i := \nabla\varphi_i\cdot U, \quad i = 0,\dots,m.$$

Note that one has $\operatorname{supp} g^i \subset \Omega_0$ hence $g^i \in L^p(\mathbb{R}_+; W_0^{1,q}(\Omega_0))$ and $\int_{\Omega_0} g^i\,dx = 0$ for $i = 0,\dots,m$. Therefore, by the well-known theory of the divergence problem for $i = 0,\dots,m$ there is some $w_i \in L^p(\mathbb{R}_+; W_0^{2,q}(\Omega_0))$ such that $\operatorname{div} w_i(t) = g^i(t)$ in Ω_0 for almost all $t \in \mathbb{R}_+$, $w_{it} \in L^p(\mathbb{R}_+; L^q(\Omega_0))$ and

$$\begin{aligned} &\|\nabla^2 w_i\|_{L^p(\mathbb{R}_+;L^q(\Omega_0))} \le c\|\nabla g^i\|_{L^p(\mathbb{R}_+;L^q(\Omega_0))} \le c\|\nabla U\|_{L^p(\mathbb{R}_+;L^q_0(\Omega_0))},\\ &\|w_{it}\|_{L^p(\mathbb{R}_+;L^q(\Omega_0))} \le c\|g^i_t\|_{L^p(\mathbb{R}_+;(W^{1,q'}(\Omega_0))')} \le c\|U_t\|_{L^p(\mathbb{R}_+;(W^{1,q'}(\Omega_0))')}, \end{aligned} \tag{4.27}$$

where $c = c(\Omega_0, q)$, cf. [17]. Then $\tilde w_i$, the extension by 0 of w_i to $\widetilde\Omega_i$, $i = 1\dots,m$, satisfies

$$\begin{aligned} &e^{\beta_i x_n^i}\tilde w_{it}, e^{\beta_i x_n^i}\nabla^2\tilde w_i \in L^p(\mathbb{R}_+; L^q(\tilde\Omega_i)),\\ &\|e^{\beta_i x_n^i}\tilde w_{it}, e^{\beta_i x_n^i}\nabla^2\tilde w_i\|_{L^p(\mathbb{R}_+;L^q(\tilde\Omega_i))} \le c(\|\nabla U\|_{L^p(\mathbb{R}_+;L^q(\Omega_0))} + \|U_t\|_{L^p(\mathbb{R}_+;(W^{1,q'}(\Omega_0))')}). \end{aligned} \tag{4.28}$$

Moreover, note that $w_i(0,x) = 0$ due to $g^i(0,x) = 0$ for $x \in \Omega$.

Now, $v^0 := u^0 - w_0$ and $v^i := u^i - \tilde w_i$, $i = 1,\dots,m$, solve, respectively,

$$\begin{aligned} v^0_t - \Delta v^0 + \nabla p^0 &= f^0 - w_{0t} + \Delta w_0 && \text{in } \mathbb{R}_+\times\Omega_0,\\ \operatorname{div} v^0 &= 0 && \text{in } \mathbb{R}_+\times\Omega_0,\\ v^0(0,x) &= 0 && \text{in } \Omega_0,\\ v^0 &= 0 && \text{on } \partial\Omega_0, \end{aligned}$$

and

$$\begin{aligned} v^i_t - \Delta v^i + \nabla p^i &= \tilde f^i - \tilde w_{it} + \Delta\tilde w_i && \text{in } \mathbb{R}_+\times\widetilde\Omega_i,\\ \operatorname{div} v^i &= 0 && \text{in } \mathbb{R}_+\times\widetilde\Omega_i,\\ v^i(0,x) &= 0 && \text{in } \widetilde\Omega_i,\\ v^i &= 0 && \text{on } \partial\widetilde\Omega_i. \end{aligned}$$

Then, by the maximal regularity of Stokes operator in bounded domains in view of (4.27) we obtain that

$$\|v^0, v^0_t, \nabla^2 v^0, \nabla p^0\|_{L^p(\mathbb{R}_+;L^q(\Omega_0))} \le c(\|F, \nabla U, P\|_{L^p(\mathbb{R}_+;L^q(\Omega_0))} + \|U_t\|_{L^p(\mathbb{R}_+;(W^{1,q'}(\Omega_0))')}), \tag{4.29}$$

and, by Theorem 2.3 in view of (4.28), that

$$\begin{aligned} \|v^i, v^i_t, \nabla^2 v^i, \nabla p^i\|_{L^p(\mathbb{R}_+;L^q_{\beta_i}(\mathbb{R};L^q(\Sigma^i)))} &\le c(\|F\|_{L^p(\mathbb{R}_+;L^q_{\beta_i}(\tilde\Omega_i))}\\ &\quad + \|\nabla U, P\|_{L^p(\mathbb{R}_+;L^q_0(\Omega_0))} + \|U_t\|_{L^p(\mathbb{R}_+;(W^{1,q'}(\Omega_0))')}),\ i = 1,\dots,m. \end{aligned} \tag{4.30}$$

Thus, from (4.27)-(4.30) we get that

$$\begin{aligned} &\|u_0, u^0_t, \nabla^2 u^0, \nabla p^0\|_{L^p(\mathbb{R}_+;L^q(\Omega_0))}\\ &\le c(\|F, \nabla U, P\|_{L^p(\mathbb{R}_+;L^q(\Omega_0))} + \|U_t\|_{L^p(\mathbb{R}_+;(W^{1,q'}(\Omega_0))')}),\\ &\|u^i_t, \nabla^2 u^i, \nabla p^i\|_{L^p(\mathbb{R}_+;L^q_{\beta_i}(\mathbb{R};L^q(\Sigma^i)))} \le c(\|F\|_{L^p(\mathbb{R}_+;L^q_{\beta_i}(\tilde\Omega_i))}\\ &\qquad + \|\nabla U, P\|_{L^p(\mathbb{R}_+;L^q(\Omega_0))} + \|U_t\|_{L^p(\mathbb{R}_+;(W^{1,q'}(\Omega_0))')}),\ i = 1,\dots,m. \end{aligned} \tag{4.31}$$

Note that $U=\sum_{i=0}^m u^i$, $P=\sum_{i=0}^m p^i$ in Ω. Therefore, by (4.31) we have (4.26) and

$$\begin{aligned}\|U,U_t,\nabla^2 U,\nabla P\|_{L^p(\mathbb{R}_+;L^q_{\mathbf{b}}(\Omega))} &\le c(\|F\|_{L^p(\mathbb{R}_+;L^q_{\mathbf{b}}(\Omega))}\\ &+\|\nabla U,P\|_{L^p(\mathbb{R}_+;L^q(\Omega_0))}+\|U_t\|_{L^p(\mathbb{R}_+;(W^{1,q'}(\Omega_0))')}).\end{aligned} \tag{4.32}$$

Consequently, it follows that the Stokes operator $A_{q,\mathbf{b}}$ in $L^q_{\mathbf{b},\sigma}(\Omega)$ has maximal L^p-regularity for $1<p<\infty$ satisfying (2.10).

Thus, the proof of the Theorem 2.5 is complete, ∎

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