A Linearized Model for Compressible Flow past a Rotating Obstacle: Analysis via Modified Bochner–Riesz Multipliers

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Abstract

Consider the flow of a compressible Newtonian fluid around or past a rotating rigid obstacle in $\mathbb{R}^3$. After a coordinate transform to get a problem in a time-independent domain we assume the new system to be stationary, then linearize and use Fourier transform to prove the existence of a unique solution in L^q-spaces. However, in contrast to the incompressible case with multipliers based on the heat kernel the new multiplier functions are related to Bochner-Riesz multipliers and require the restriction $6/5 < q < 6$.

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1 Introduction

We consider the flow of a compressible Newtonian fluid past a rotating rigid compact obstacle $\mathcal{K}(t) \subset \mathbb{R}^3$, $t > 0$, of constant nonzero angular velocity $\vec{\omega} \in \mathbb{R}^3$. Without loss of generality we assume that $\vec{\omega} = \omega(0,0,1)^\top$, $\omega = |\vec{\omega}| > 0$. In the time-dependent exterior domain $\Omega(t) = \mathbb{R}^3 \setminus \mathcal{K}(t)$ the flow is described by the nonlinear system

$$\begin{aligned} \rho\frac{\partial u}{\partial t} + \rho u\cdot\nabla u - \mu\Delta u - (\mu+\nu)\nabla \operatorname{div} u + \nabla p(\rho) &= \rho f \\ \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho u) &= 0 \end{aligned} \tag{1.1}$$

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together with the boundary conditions

$$
\begin{aligned}
u(x,t) &= \vec{\omega} \wedge x \quad \text{at} \quad \bigcup_{t\in(0,\infty)} \partial\Omega(t) \times \{t\} \\
u(x,t) &\to u_\infty \qquad \text{as} \quad |x| \to \infty \\
\rho(x,t) &\to \rho_\infty \qquad \text{as} \quad |x| \to \infty.
\end{aligned}
\tag{1.2}
$$

Here $\rho_\infty \geq 0$ is a constant and $u_\infty \in \mathbb{R}^3$ is a zero or nonzero constant vector. We will assume that $\rho_\infty \neq 0$ and, for the sake of simplicity, we take $\rho_\infty = 1$.

Following ideas from [2, 4, 10, 11, 12] we introduce a t-dependent coordinate transform via orthogonal matrices and reduce the problem to a new one in a t-independent domain, see (2.3) below. Under the assumptions that the transformed solution is stationary which corresponds to a time-periodic solution of the original problem and that the axis of rotation is parallel to u_∞ we linearize with respect to a basic state to get a linear stationary system in a new velocity field v and density function σ coupling an elliptic PDE system with a hyperbolic equation, see (1.3) below. This system considered on the whole space $\mathbb{R}^3$ is solved explicitly for $g = \operatorname{div} v$ in terms of Fourier transforms. In the incompressible case the corresponding solution v can be treated by using Littlewood-Paley decomposition and classical multipliers theorems, see [2, 4]; for a more recent and simpler proof we refer to [7]. However, in the compressible case, the solution formula for g is much more complicated and defined by a multiplier function which is non-differentiable on a sphere in the phase plane with radius $\frac{2}{2\mu+\nu}$ and related to the *modified Bochner-Riesz multiplier function*

$$
(|\zeta|^2 - 1)_+^{1/2}.
$$

Following the proof on the usual Bochner-Riesz multipliers $(1-|\zeta|^2)_+^\lambda$, $\lambda \geq 0$, the range of admissible exponents q to allow for L^q-estimates is restricted to values of q close to 2; to be more precise, we need that $\frac{6}{5} < q < 6$, see the Main Theorem 1.1 below. For a different approach avoiding Fourier transformation and to prove the existence of weak solutions we refer to [13]; here, however, the fully evolutionary system has been considered. Other results concerning the flow of rigid bodies in compressible Newtonian fluid can be found in [1] or [6].

Our main result concerns solutions (v, σ) of the linear system

$$
\begin{aligned}
-\mu\Delta v - (\mu+\nu)\nabla \operatorname{div} v + (u_\infty - \vec{\omega} \wedge y) \cdot \nabla v + \vec{\omega} \wedge v + \nabla\sigma &= F \\
\operatorname{div} v + \operatorname{div} \big(\sigma(u_\infty - \vec{\omega} \wedge y)\big) &= G.
\end{aligned}
\tag{1.3}
$$

Actually, (v, σ) will be obtained from a stationary Oseen problem when $0 \neq u_\infty \parallel \vec{\omega}$ (or Stokes problem when $u_\infty = 0$) with rotation term and prescribed divergence g, see (2.6) below. On the other hand, the divergence g is obtained from the system

$$
\begin{aligned}
-(2\mu+\nu)\Delta g + (u_\infty - \vec{\omega} \wedge y) \cdot \nabla g + \Delta\sigma &= \operatorname{div} F \\
g + (u_\infty - \vec{\omega} \wedge y) \cdot \nabla\sigma &= G
\end{aligned}
\tag{1.4}
$$

to be analyzed in Sect. 3.

Note that F is related to the external force f given in (1.1) by the coordinate transform; we included a general right-hand side G in $(1.3)_2$, $(1.4)_2$ for possible future application. Now the Main Theorem reads as follows:

Theorem 1.1. *Let $\frac{6}{5} < q < 6$ and $|u_\infty| \leq \frac{1}{2}$. Furthermore, let $G \in L^q(\mathbb{R}^3)$ and assume that $F = \operatorname{div} \Phi$ satisfies $\Phi, (\vec{\omega} \wedge y) \cdot \nabla\Phi,\ u_\infty \cdot \nabla\Phi \in L^q(\mathbb{R}^3)$.*

Then problem (1.4) has a unique solution $g \in L^q(\mathbb{R}^3)$ which satisfies the estimate

$$\|g\|_q \leq c\Big(\big(\omega(2\mu+\nu)\big)^2 + \frac{1}{\big(\omega(2\mu+\nu)\big)^4}\Big)\cdot \\ \cdot \big(\|G\|_q + \|\omega\Phi\|_q + \|(\vec{\omega}\wedge y)\cdot\nabla\Phi\|_q + \|u_\infty\cdot\nabla\Phi\|_q\big) \tag{1.5}$$

where the constant $c = c_q > 0$ is independent of ω, u_∞, and μ, ν.

By [2], [3, Theorem 1.1 (1)] the solution g of problem (1.4) yields a solution (v, σ) of (1.3) (or (2.6)) satisfying the estimate

$$\|\mu\nabla^2 v\|_q + \|\nabla\sigma\|_q \leq c_q\big(\|F + (\mu+\nu)\nabla g\|_q + \|\mu\nabla g + (\vec{\omega}\wedge y)g - u_\infty g\|_q\big) \tag{1.6}$$

with a constant $c_q > 0$ independent of ω and u_∞, provided $\mu\nabla g + (\vec{\omega}\wedge y)g \in L^q(\mathbb{R}^3)$; for estimates of lower order derivatives of v we refer to [2].

Note that Theorem 1.1 does not yield the assumptions on g needed in (1.6), i.e. $\mu\nabla g + (\vec{\omega}\wedge y)g \in L^q(\mathbb{R}^3)$. On the other hand, (1.6) holds for any $1 < q < \infty$. Actually, the main aim of this paper is to prove the *a priori estimate* (1.5) which results in rather elaborate estimates of the multiplier functions involved in the explicit solution (see (2.10)–(2.18) in Sect. 3) of (1.4).

In the following section we describe in more details the procedure of the coordinate transform and of calculating an explicit solution of the linearized problem (1.4) in Fourier space. An analysis of the corresponding multiplier functions will be performed in Sect. 3. The crucial estimates for the modified Bochner-Riesz multipliers are found in Lemmata 3.5 and 3.7. We remark that this analysis is performed not only for $(|\zeta|^2-1)_+^{1/2}$, but simultaneously for the more general multiplier function $(|\zeta|^2-1)_+^{\lambda}$ with $\operatorname{Re}\lambda > \frac{1}{4}$, cf. Theorem 3.9.

2 The coordinate transform and the solution formula

As already mentioned a main disadvantage of the system (1.1)–(1.2) is the problem that the spatial domain $\Omega(t)$ is time-dependent. In order to work in a fixed domain $\Omega \subset \mathbb{R}^3$ we introduce the orthogonal matrix

$$O(t) = O_\omega(t) = \begin{pmatrix} \cos\omega t & -\sin\omega t & 0 \\ \sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.1}$$

Obviously, $\dot{O} := \frac{d}{dt}O(t) = \tilde{\Omega}O(t)$ where $\tilde{\Omega} \in \mathbb{R}^{3,3}$ is the skew symmetric matrix such that $\tilde{\Omega}a = \vec{\omega} \wedge a$ for $a \in \mathbb{R}^3$. Moreover, we define the new variable and unknowns

$$y = O(t)^\top x, \quad v(y,t) = O(t)^\top \big(u(x,t) - u_\infty\big), \quad \tilde{\rho}(y,t) = \rho(x,t) \tag{2.2}$$

as well as $F(y,t) = O(t)^\top f(x,t)$. Then we get with the fixed domain $\Omega = O(t)^\top \Omega(t)$ the following problem:

$$\begin{aligned} \tilde{\rho}\frac{\partial v}{\partial t} - \mu\Delta v - (\mu+\nu)\nabla \mathrm{div}\, v + \rho(\dot{O}^\top O y)\cdot\nabla v + (\dot{O}^\top O)^\top v \\ +\tilde{\rho}(O^\top u_\infty)\cdot\nabla v + \tilde{\rho}v\cdot\nabla v + \nabla p(\tilde{\rho}) = \tilde{\rho}F, \\ \frac{\partial\tilde{\rho}}{\partial t} + \mathrm{div}\,\big(\tilde{\rho}v + (\dot{O}^\top O y)\tilde{\rho} + (O^\top u_\infty)\tilde{\rho}\big) = 0 \end{aligned} \tag{2.3}$$

in $\Omega \times (0,\infty)$, together with the conditions

$$\begin{aligned} & v(y,t) = (\dot{O}^\top O)^\top y - O^\top u_\infty \quad \text{on } \partial\Omega\times(0,\infty) \\ & v(y,t) \to 0 \quad \text{as } |y| \to \infty \\ & \tilde{\rho}(y,t) \to 1 \quad \text{as } |y| \to \infty. \end{aligned} \tag{2.4}$$

The computations how to get from (1.1)–(1.2) to (2.3)–(2.4) are relatively standard in the case of incompressible flows, see e.g. [5]. Hence we only describe some ideas to get $(2.3)_3$. By (2.1)–(2.2) and the simple calculation

$$\dot{O}^\top O = -\tilde{\Omega}, \quad \dot{O}^\top O a = -\omega \wedge a \quad \text{ for } \ a \in \mathbb{R}^3$$

we get that

$$\begin{aligned} \frac{\partial}{\partial t}\rho(x,t) &= \frac{\mathrm{d}}{\mathrm{d}t}\tilde{\rho}(O^\top x,t) = \frac{\partial\tilde{\rho}}{\partial t}(y,t) + \sum_{j,k,l=1}^{3} \frac{\partial\tilde{\rho}}{\partial y_j}\ \dot{O}^\top_{jk}O_{kl}\ y_l \\ &= \frac{\partial\tilde{\rho}}{\partial t} + (\dot{O}^\top O y)\cdot\nabla_y\tilde{\rho} = \frac{\partial\tilde{\rho}}{\partial t} + \mathrm{div}_y\big((\dot{O}^\top O y)\rho\big) \end{aligned}$$

and

$$\begin{aligned} \mathrm{div}_x\big(\rho(x,t)u(x,t)\big) &= \sum_{j,k=1}^{3}\frac{\partial}{\partial x_j}\Big(\tilde{\rho}(O^\top x,t)\big(O_{jk}v_k(O^\top x,t) + (u_\infty)_j\big)\Big) \\ &= \sum_{j,k,l=1}^{3}\frac{\partial}{\partial y_l}\Big(O^\top_{lj}\tilde{\rho}(y,t)\big(O_{jk}v_k(y,t) + (u_\infty)_j\big)\Big) \\ &= \mathrm{div}_y\big((O^\top u_\infty)\tilde{\rho}\big) + \mathrm{div}_y(\tilde{\rho}v). \end{aligned}$$

Summarizing the last two identities we are led to $(2.3)_3$.

We further simplify (2.3) and assume that the flow in Ω is time-independent, i.e. $\frac{\partial v}{\partial t} = 0$ and $\frac{\partial \tilde{\rho}}{\partial t} = 0$; this assumption is related to a time-periodic flow in the original problem. To this aim the terms $\dot{O}^\top O$ and $O^\top u_\infty$ must be constant in t. For the latter condition we assume that ω is parallel to u_∞ so that

$$O(t)^\top u_\infty = u_\infty \quad \text{for all} \;\; t > 0.$$

Now, writing $\tilde{\rho} = \tilde{\rho}(y)$ in the form

$$\tilde{\rho} = 1 + \sigma,$$

and linearizing (2.3) - (2.4) in (v, σ) around $(0, 0)$ we get the coupled linear system

$$\begin{aligned} -\mu\Delta v - (\mu+\nu)\nabla \mathrm{div}\, v + (u_\infty - \vec{\omega}\wedge y)\cdot\nabla v + \vec{\omega}\wedge v + \nabla\sigma &= F \\ \mathrm{div}\, v + \mathrm{div}\,\big(\sigma(u_\infty - \vec{\omega}\wedge y)\big) &= G. \end{aligned} \tag{2.5}$$

In view of [2, 3, 4] it suffices to find a solution formula for the unknown function

$$g = \mathrm{div}\, v,$$

since then (v, σ) can be considered as a solution of the generalized Oseen system with rotation effect

$$\begin{aligned} -\mu\Delta v + (u_\infty - \vec{\omega}\wedge y)\cdot\nabla v + \vec{\omega}\wedge v + \nabla\sigma &= F + (\mu+\nu)\nabla g \\ \mathrm{div}\, v &= g \end{aligned} \tag{2.6}$$

in $\mathbb{R}^3$. For this reason we apply div to $(2.5)_1$ and get, since $\mathrm{div}\,\big((\vec{\omega}\wedge y)\cdot\nabla v - \vec{\omega}\wedge v\big) = (\vec{\omega}\wedge y)\cdot\nabla g$, that (g, σ) solves the system

$$\begin{aligned} -(2\mu+\nu)\Delta g + (u_\infty - \vec{\omega}\wedge y)\cdot\nabla g + \Delta\sigma &= \mathrm{div}\, F \\ g + (u_\infty - \vec{\omega}\wedge y)\cdot\nabla\sigma &= G. \end{aligned} \tag{2.7}$$

Since the operators Δ and $(\vec{\omega}\wedge y)\cdot\nabla$ commute, $(2.7)_2$ yields

$$\Delta g + (u_\infty - \vec{\omega}\wedge y)\cdot\nabla(\Delta\sigma) = \Delta G. \tag{2.8}$$

Now we may insert $\Delta\sigma$ from $(2.7)_1$ into (2.8) to get that

$$\begin{aligned} &\Delta g - \big((u_\infty - \vec{\omega}\wedge y)\cdot\nabla\big)^2 g + (u_\infty - \vec{\omega}\wedge y)\cdot\nabla(2\mu+\nu)\Delta g \\ &\quad = \Delta G - (u_\infty - \vec{\omega}\wedge y)\cdot\nabla(\mathrm{div}\, F). \end{aligned}$$

Due to the geometry of the problem we introduce cylindrical coordinates for $y \in \mathbb{R}^3$, i.e., let $y \triangleq (r, \theta, y_3)$, $r = |(y_1, y_2)| \geq 0$, $\theta \in [0, 2\pi)$, $y_3 \in \mathbb{R}$. Then it is easily seen that $(\vec{\omega}\wedge y)\cdot\nabla = \omega\partial_\theta$ where ∂_θ denotes the angular derivative with respect to θ. Exploiting the commutator identity $[u_\infty \cdot \nabla, (\vec{\omega} \wedge y) \cdot \nabla] = 0$ we are led to the equation

$$\begin{aligned} &\omega^2 \partial_\theta^2 g + \omega\big((2\mu+\nu)\Delta - 2u_\infty \cdot \nabla\big)\partial_\theta g \\ &\qquad + \big(-\Delta - (2\mu+\nu)u_\infty \cdot \nabla\Delta + (u_\infty \cdot \nabla)^2\big)g = H \end{aligned} \tag{2.9}$$

where

$$H = -\Delta G + (u_\infty \cdot \nabla - \omega\partial_\theta)\mathrm{div}\, F. \tag{2.10}$$

For the subsequent computation we write that $u_\infty = ke_3$, $k \geq 0$, so that

$$u_\infty \cdot \nabla = k\partial_3, \quad k = |u_\infty|,$$

and use the Fourier transform, formally defined by

$$\mathcal{F}u(\xi) = \hat{u}(\xi) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{-ix\cdot\xi} u(x)\, \mathrm{d}x, \quad \xi \in \mathbb{R}^3.$$

Since in cylindrical coordinates in ξ-space, say $\xi \mathrel{\hat{=}} (s, \varphi, \xi_3)$,

$$\widehat{\partial_\theta u}(\xi) = \partial_\varphi \hat{u}(\xi),$$

(2.9), (2.10) can be written in Fourier space in the form

$$\begin{aligned} &\omega^2 \partial_\varphi^2 \hat{g} + \omega\partial_\varphi \hat{g}\big(-(2\mu+\nu)|\xi|^2 - 2ik\xi_3\big) \\ &\qquad + \hat{g}\big(|\xi|^2 + (2\mu+\nu)ik\xi_3|\xi|^2 - k^2\xi_3^2\big) = \hat{H} \end{aligned} \tag{2.11}$$

where

$$\hat{H}(\xi) = |\xi|^2\hat{G} + (-i\omega\,\partial_\varphi - k\xi_3)(\xi \cdot \hat{F}). \tag{2.12}$$

Note that (2.11) may be considered as a second order differential equation for g with respect to $\varphi \in [0, 2\pi]$, and that we are looking for a 2π-periodic solution $\hat{g}(\xi) \mathrel{\hat{=}} \hat{g}(\varphi)$. The characteristic polynomial

$$\chi(\lambda) = \lambda^2 - \frac{1}{\omega}\big((2\mu+\nu)|\xi|^2 + 2ik\xi_3\big)\lambda + \frac{1}{\omega^2}\big(|\xi|^2 + (2\mu+\nu)ik\xi_3|\xi|^2 - k^2\xi_3^2\big)$$

of (2.11) has the zeros

$$\lambda_{1,2} = \lambda_{1,2}(\xi) = \frac{1}{2\omega}\Big((2\mu+\nu)|\xi|^2 + 2ik\xi_3 \pm \sqrt{(2\mu+\nu)^2|\xi|^4 - 4|\xi|^2}\Big). \tag{2.13}$$

Then the general solution $\hat{g}(\varphi)$ of (2.11) has the form

$$\hat{g}(\varphi) = c_1 e^{\lambda_1\varphi} + c_2 e^{\lambda_2\varphi} + \frac{1/\omega^2}{\lambda_1 - \lambda_2}\int_0^\varphi \big(e^{-\lambda_1(t-\varphi)} - e^{-\lambda_2(t-\varphi)}\big)\hat{H}(t)\,\mathrm{d}t,$$

where, with a slight abuse of notation, $\hat{H}(t) = \hat{H}(\xi)$ for $\xi \triangleq (s, t, \xi_3)$ in cylindrical coordinates and where $c_1, c_2 \in \mathbb{C}$ are arbitrary. In order to get a 2π-periodic solution of class C^2 with respect to φ, $\hat{g}$ must satisfy $\hat{g}(0) = \hat{g}(2\pi)$ and $\partial_\varphi \hat{g}(0) = \partial_\varphi \hat{g}(2\pi)$, i.e.

$$\begin{aligned} c_1 + c_2 =& c_1 e^{2\pi\lambda_1} + c_2 e^{2\pi\lambda_2} + \frac{1/\omega^2}{\lambda_1 - \lambda_2} \int_0^{2\pi} \big(e^{-\lambda_1(t-2\pi)} - e^{-\lambda_2(t-2\pi)}\big) \hat{H}(t)\,\mathrm{d}t, \\ \lambda_1 c_1 + \lambda_2 c_2 =& \lambda_1 e^{2\pi\lambda_1} c_1 + \lambda_2 e^{2\pi\lambda_2} c_2 + \\ &+ \frac{1/\omega^2}{\lambda_1 - \lambda_2} \int_0^{2\pi} \big(e^{-\lambda_1(t-2\pi)}\lambda_1 - e^{-\lambda_2(t-2\pi)}\lambda_2\big) \hat{H}(t)\,\mathrm{d}t. \end{aligned}$$

The previous identities define a (2×2)-linear system for c_1, c_2 with unique solution

$$\begin{aligned} c_1 &= \frac{1/\omega^2}{(\lambda_1 - \lambda_2)(1 - e^{2\pi\lambda_1})} \int_0^{2\pi} e^{-\lambda_1(t-2\pi)} \hat{H}(t)\,\mathrm{d}t\,, \\ c_2 &= \frac{-1/\omega^2}{(\lambda_1 - \lambda_2)(1 - e^{2\pi\lambda_2})} \int_0^{2\pi} e^{-\lambda_2(t-2\pi)} \hat{H}(t)\,\mathrm{d}t. \end{aligned}$$

Hence the unique 2π-periodic solution $\hat{g}(\varphi)$ of (2.11) reads

$$\begin{aligned} \hat{g}(\varphi) =& \frac{1/\omega^2}{\lambda_1 - \lambda_2} \left\{ \frac{-1}{1 - e^{-2\pi\lambda_1}} \int_0^{2\pi} e^{-\lambda_1(t-\varphi)} \hat{H}(t)\,\mathrm{d}t + \int_0^{\varphi} e^{-\lambda_1(t-\varphi)} \hat{H}(t)\,\mathrm{d}t \right. \\ &\left. + \frac{1}{1 - e^{-2\pi\lambda_2}} \int_0^{2\pi} e^{-\lambda_2(t-\varphi)} \hat{H}(t)\,\mathrm{d}t - \int_0^{\varphi} e^{-\lambda_2(t-\varphi)} \hat{H}(t)\,\mathrm{d}t \right\}. \end{aligned} \tag{2.14}$$

From (2.14) we deduce two different representation formula which will be used for $|\xi|$ small and for $|\xi|$ large, see (2.17) for small $|\xi|$ and (2.18) for large $|\xi|$, respectively. Consider the first two terms involving λ_1 in (2.14): since $\hat{H}(t)$ is 2π-periodic a shift of coordinates implies that

$$\begin{aligned} &\frac{-1}{1 - e^{-2\pi\lambda_1}} \int_0^{2\pi} e^{-\lambda_1(t-\varphi)} \hat{H}(t)\,\mathrm{d}t + \int_0^{\varphi} e^{-\lambda_1(t-\varphi)} \hat{H}(t)\,\mathrm{d}t \\ &= \frac{1}{1 - e^{-2\pi\lambda_1}} \left(\int_0^{\varphi} \big(e^{-\lambda_1(t-\varphi)} - e^{-\lambda_1(t+2\pi-\varphi)}\big) \hat{H}(t)\,\mathrm{d}t - \int_0^{2\pi} e^{-\lambda_1(t-\varphi)} \hat{H}(t)\,\mathrm{d}t \right) \\ &= \frac{-1}{1 - e^{-2\pi\lambda_1}} \int_0^{2\pi} e^{-\lambda_1 t} \hat{H}(t+\varphi)\,\mathrm{d}t. \end{aligned}$$

A similar result holds for the last two terms in (2.14) involving λ_2.

At this point let us introduce the orthogonal matrix $O_1(t)$, cf. the definition in (2.5), modeling rotation around the x_3- or ξ_3-axis by the angle $t \in \mathbb{R}$. We note that $\mathcal{F}H\big(O_1(t)\,\cdot\,\big)(\xi) = \hat{H}\big(O_1(t)\xi\big)$ and $\hat{H}(t+\varphi) = \hat{H}\big(O_1(t)\xi\big)$ for $\xi \triangleq (s, \varphi, \xi_3)$ in polar coordinates. Hence we get from (2.14) that

$$\hat{g}(\xi) = \frac{1/\omega^2}{\lambda_1 - \lambda_2} \int_0^{2\pi} \left(\frac{e^{-\lambda_2 t}}{1 - e^{-2\pi\lambda_2}} - \frac{e^{-\lambda_1 t}}{1 - e^{-2\pi\lambda_1}} \right) \hat{H}\big(O_1(t)\xi\big)\,\mathrm{d}t. \tag{2.15}$$

In order to "get rid of" the term $2ik\xi_3$ in the definition of λ_j, see (2.13), recall that the multiplication in Fourier space by $-\frac{ik\xi_3}{\omega}t$ equals the change of variables $x \mapsto x - \frac{k}{\omega}te_3$ in x-space. Hence, defining the modified zeros

$$\mu_{1,2} = \frac{1}{2\omega}\big((2\mu+\nu)|\xi|^2 \pm \sqrt{(2\mu+\nu)^2|\xi|^4 - 4|\xi|^2}\big), \tag{2.16}$$

we may rewrite (2.15) in the form

$$\hat{g}(\xi) = \frac{1/\omega^2}{\mu_1-\mu_2}\int_0^{2\pi}\left(\frac{e^{-\mu_2 t}}{1-e^{-2\pi\lambda_2}} - \frac{e^{-\mu_1 t}}{1-e^{-2\pi\lambda_1}}\right)\mathcal{F}H\big(O_1(t)\cdot - \frac{k}{\omega}te_3\big)(\xi)\,\mathrm{d}t. \tag{2.17}$$

On the other hand, we may write the term $(1-e^{-2\pi\lambda_j})^{-1}$ as a geometric series, since Re $\lambda_j > 0$ when $\xi \neq 0$. Then, using the 2π-periodicity of $\hat{H}(O_1(t)\xi)$ in t, the first term involving λ_2 in (2.15) can be written in the form

$$\sum_{k=0}^{\infty}\int_0^{2\pi} e^{-\lambda_2(t+2\pi k)}\hat{H}\big(O_1(t)\xi\big)\,\mathrm{d}t = \int_0^{\infty} e^{-\lambda_2 t}\hat{H}\big(O_1(t)\xi\big)\,\mathrm{d}t.$$

Hence (2.14) simplifies to the formula

$$\hat{g}(\xi) = \frac{1/\omega^2}{\mu_1-\mu_2}\int_0^{\infty}(e^{-\mu_2 t} - e^{-\mu_1 t})\mathcal{F}\big(H(O_1(t)\cdot - \frac{k}{\omega}te_3)\big)(\xi)\,\mathrm{d}t. \tag{2.18}$$

For later use we assume that F has the form

$$F = \operatorname{div}\Phi = (\operatorname{div}\Phi_1, \operatorname{div}\Phi_2, \operatorname{div}\Phi_3),\ \ \Phi = (\Phi_{kj})_{k,j=1}^3,$$

with $\Phi \in L^q(\mathbb{R}^3)$, or in Fourier space that

$$\hat{F} = i\xi\cdot\hat{\Phi} = i(\xi\cdot\hat{\Phi}_1, \xi\cdot\hat{\Phi}_2, \xi\cdot\hat{\Phi}_3), \quad \hat{\Phi}_k = (\hat{\Phi}_{kj})_{j=1}^3.$$

Since $\partial_\varphi = (e_3 \wedge \xi)\cdot\nabla$, we have

$$\partial_\varphi(\xi\cdot\hat{F}) = (\xi\cdot\partial_\varphi\hat{F}) + (\xi\wedge\hat{F})_3$$

and finally

$$\partial_\varphi(\xi\cdot\hat{F}) = i\{\xi\cdot\partial_\varphi\hat{\Phi}\cdot\xi + \xi\cdot(\xi\wedge\hat{\Phi})_3 + \big(\xi\wedge(\xi\cdot\hat{\Phi})\big)_3\}$$

where the vector and scalar product of ξ with $\hat{\Phi}$ are calculated as vector and scalar products of ξ with the vectors $\hat{\Phi}_1, \hat{\Phi}_2$ and $\hat{\Phi}_3$. In particular, we see that $\partial_\varphi(\xi\cdot\hat{F}) = \xi\cdot\hat{A}_0(\xi)\cdot\xi$ where the coefficients of $\hat{A}_0(\xi)$ consist of terms like $\hat{\Phi}_{ij}(\xi)$ and $\partial_\varphi\hat{\Phi}_{ij}(\xi)$. Moreover, $\xi_3\xi\cdot\hat{F} = i\xi_3\xi\cdot\hat{\Phi}\cdot\xi$. Hence $\hat{H}$, see (2.12), has the form

$$\hat{H}(\xi) = \xi\cdot\hat{A}(\xi)\cdot\xi \ \text{ with } \ \hat{A}(\xi) = \hat{G}(\xi)I_3 + \hat{A}_1(\xi) \tag{2.19}$$

where $\hat{A}_1(\xi)$ is a (3×3)-matrix with coefficients like $\omega\hat{\Phi}_{ij}$, $\omega\partial_\varphi\hat{\Phi}_{ij}$ and $k\xi_3\hat{\Phi}_{ij}$, and where $I_3 = (\delta_{ij}) \in \mathbb{R}^{3,3}$. From the Hörmander-Mikhlin Multiplier Theorem 3.1 below we conclude that A satisfies for every $1 < q < \infty$ the L^q-estimate

$$\|A\|_q \leq c_q\big(\|G\|_q + \|\omega\Phi\|_q + \|(\vec{\omega}\wedge y)\cdot\nabla\Phi\|_q + \|u_\infty\cdot\nabla\Phi\|_q\big) \tag{2.20}$$

with a constant $c_q > 0$ independent of ω and u_∞.

3 Proofs

For the estimate of g in $L^q(\mathbb{R}^3)$ we use multiplier theory and have to distinguish three cases concerning the behavior of $\lambda_j = \lambda_j(\xi)$, $j = 1, 2$, as functions of $\xi \in \mathbb{R}^3$.

3.1 Preliminaries

Let $\eta_0, \eta_1, \eta_2 \in C^\infty\big(\mathbb{R}^3; [0,1]\big)$ be a partition of unity of $\mathbb{R}^3$ such that

$$\begin{aligned}
\eta_0 &= 1 \quad \text{for } |\xi| \le \frac{1}{4}\,\frac{2}{2\mu+\nu}, \qquad \eta_0 = 0 \text{ for } |\xi| \ge \frac{1}{2}\,\frac{2}{2\mu+\nu},\\
\eta_1 &= 1 \quad \text{for } \frac{1}{2}\,\frac{2}{2\mu+\nu} \le |\xi| \le 2\,\frac{2}{2\mu+\nu},\\
\eta_2 &= 1 \quad \text{for } 4\,\frac{2}{2\mu+\nu} \le |\xi|, \qquad \eta_2 = 0 \text{ for } |\xi| \le 2\,\frac{2}{2\mu+\nu}.
\end{aligned}$$

Looking at this distinction of cases it will be advantageous to define the new variable

$$\zeta = \frac{2\mu+\nu}{2}\,\xi \tag{3.1}$$

and the new parameter

$$\omega' = \frac{2\mu+\nu}{2}\,\omega.$$

Using ζ, ω' the terms η_0, η_1, η_2 and $\lambda_{1,2}$, $\mu_{1,2}$ have the following properties:

$$\eta_0 = \begin{cases} 1, & |\zeta| \le \frac{1}{4} \\ 0, & |\zeta| \ge \frac{1}{2} \end{cases}, \qquad \eta_1 = 1 \quad \text{for } \frac{1}{2} \le |\zeta| \le 2, \quad \eta_2 = \begin{cases} 1, & |\zeta| \ge 4 \\ 0, & |\zeta| \le 2 \end{cases},$$

and

$$\lambda_{1,2} = \frac{1}{\omega'}\big(|\zeta|^2 + ik\zeta_3 \pm \sqrt{|\zeta|^4 - |\zeta|^2}\,\big)$$

with a similar formula for $\mu_{1,2}$ when omitting the term $ik\zeta_3$ in $\lambda_{1,2}$. Moreover,

$$\begin{aligned}
\mu_1 \sim \frac{2}{\omega'}|\zeta|^2, \quad \mu_2 \sim \frac{1}{2\omega'}, \quad \mu_1 - \mu_2 \sim \frac{2}{\omega'}|\zeta|^2 \quad \text{as } |\zeta| \to \infty,\\
\mu_{1,2} \sim \frac{1}{\omega'}\,|\zeta|^2 \pm \frac{i}{\omega'}|\zeta|\big(1 + o(1)\big), \quad \mu_1 - \mu_2 \sim \frac{2i}{\omega'}|\zeta| \quad \text{as } |\zeta| \to 0.
\end{aligned}$$

Since $\eta_0 + \eta_1 + \eta_2 = 1$ on $\mathbb{R}^3$, we write

$$g = g_0 + g_1 + g_2, \quad g_k = \mathcal{F}^{-1}(\eta_k \hat{g}), \quad k = 0, 1, 2,$$

and consider each term separately. However, we do need further cut-off functions η_0', η_1', $\eta_2' \in C^\infty\big(\mathbb{R}^3; [0,1]\big)$ just as η_0, η_1, η_2 such that e.g. $\eta_0' = 1$ for $|\zeta| \le \frac{1}{2}$ and

$\eta_0' = 0$ for $|\zeta| \ge \frac{3}{4}$; then $\eta_0 = \eta_0\eta_0' = \eta_0(\eta_0')^2$. The functions η_1', η_2' are defined in a similar way so that $\eta_j = \eta_j\eta_j' = \eta_j(\eta_j')^2$ for $j = 1, 2$ as well. Hence

$$g = \mathcal{F}^{-1}(\eta_0\eta_0'\hat{g} + \eta_1\eta_1'\hat{g} + \eta_2\eta_2'\hat{g}).$$

The main tool in Subsections 3.2 and 3.3 below will be the multiplier theorem of Hörmander-Mikhlin, see [8, Theorem 5.2.7].

Theorem 3.1. *Let $1 < q < \infty$ and let $m \in C^2\big(\mathbb{R}^3 \setminus \{0\}\big)$ be a multiplier function satisfying the pointwise Hörmander-Mikhlin condition*

$$\|m\|_{\mathcal{M}} := \sup\big\{|\xi|^{|\alpha|}\,|D^\alpha m(\xi)| : 0 \ne \xi \in \mathbb{R}^3, |\alpha| \le 2\big\} < \infty \tag{3.2}$$

where α runs through the set of multi-indices $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \le 2$. Then the multiplier operator $T : u \mapsto \mathcal{F}^{-1}(m\hat{u})$, $u \in \mathcal{S}(\mathbb{R}^3)$, can be extended to a bounded linear operator from $L^q(\mathbb{R}^3)$ to $L^q(\mathbb{R}^3)$. To be more precise, there exists a constant $c(q) > 0$ such that

$$\|Tu\|_q \le c(q)\|m\|_{\mathcal{M}}\,\|u\|_q.$$

We note that $\|m(\beta\,\cdot)\|_{\mathcal{M}} = \|m\|_{\mathcal{M}}$ for every $\beta > 0$. Hence it suffices to consider multiplier functions below as functions of ζ instead of ξ. In particular, the product of two multiplier functions m_1, m_2 satisfying (3.2) yields a multiplier function $\xi \mapsto m_1(\xi)m_2(\beta\xi)$ satisfying (3.2) as well and $\|m_1(\cdot)m_2(\beta\cdot)\|_{\mathcal{M}} \le c\|m_1\|_{\mathcal{M}}\|m_2\|_{\mathcal{M}}$ with an absolute constant $c > 0$ independent of m_1, m_2 and $\beta > 0$.

3.2 The term g_2

Concerning $\eta_2\hat{g}$ we use the representation (2.18), (2.19) and note that

$$m_1(\xi) := \frac{\xi_j\xi_k/|\omega|^2}{\mu_1 - \mu_2}\eta_2(\xi) = \frac{1}{\omega'}\cdot\frac{\zeta_j\zeta_k}{|\zeta|^2}\cdot\frac{\eta_2(2\zeta/(2\mu+\nu))}{\sqrt{1-|\zeta|^{-2}}}, \quad 1 \le j, k \le 3,$$

satisfies the pointwise Hörmander-Mikhlin multiplier condition (3.2) with bound $\|m_1\|_{\mathcal{M}} \le c\,\frac{1}{\omega'}$ where c is independent of k, μ, ν, ω. For the term $e^{-\mu_2(\xi)t}$ to be multiplied with $\eta_2'(\xi)$ we note that

$$\mu_2(\xi) = \frac{1}{\omega'}\,\frac{1}{1+\sqrt{1-|\zeta|^{-2}}} = \frac{1}{2\omega'}\,(1 + m_2(\xi)), \quad m_2(\xi) = \frac{1-\sqrt{1-|\zeta|^{-2}}}{1+\sqrt{1+|\zeta|^{-2}}}\,,$$

where m_2 satisfies $0 \le m_2(\xi) \le 1$ for all $\xi \in \operatorname{supp}\eta_2' \subseteq \{\xi \in \mathbb{R}^3 : |\zeta| \ge \frac{3}{2}\}$ and the pointwise Hörmander-Mikhlin bound (independent of the parameter $2\mu+\nu$)

$$\sup_{\xi \in \operatorname{supp}\eta_2'} |\xi|^{|\alpha|}|D^\alpha m_2(\xi)| \le c_\alpha, \quad |\alpha| \le 2.$$

Consequently, the multiplier $\eta_2'(\xi)e^{-\mu_2(\xi)t}$ satisfies (3.2) with bound $ce^{-t/(4\omega')}$ where c is independent of k, μ, ν, ω.

Consider $g_{2,2}$ defined by

$$\hat{g}_{2,2}(\xi) = \eta_2(\xi)\frac{\xi_j\xi_k/\omega^2}{\mu_1 - \mu_2}\int_0^\infty \eta_2'(\xi)e^{-\mu_2(\xi)t}\hat{A}_{t,jk}(\xi)\,\mathrm{d}t \tag{3.3}$$

$1 \le j,\ k \le 3$, cf. (2.18), (2.19), where

$$A_t(x) = A\big(O(t)x - \frac{k}{\omega}te_3\big).$$

By the previous arguments and since $\|A_t\|_q = \|A\|_q$ for all $t > 0$, $g_{2,2}$ can be estimated for $1 < q < \infty$ as follows:

$$\|g_{2,2}\|_q \le c\frac{1}{\omega'}\int_0^\infty e^{-t/(4\omega')}\|A_t\|_q\,\mathrm{d}t \le c\|A\|_q \tag{3.4}$$

with a constant $c = c_q > 0$ independent of k, ω and μ, ν.

Next we consider $g_{2,1}$ similarly defined as $g_{2,2}$ in (3.3) but with μ_2 replaced by μ_1. Since

$$\mu_1(\xi) = \frac{1}{\omega'}|\zeta|^2\Big(1 + \sqrt{1 - |\zeta|^{-2}}\Big) \ge \frac{5|\zeta|^2}{3\omega'}, \quad \xi \in \operatorname{supp}\eta_2',$$

and similar *upper* bounds hold for $|\xi|^{|\alpha|}|D^\alpha\mu_1|$, $\xi \in \operatorname{supp}\eta_2'$, we see that $\eta_2'(\xi)e^{-\mu_1(\xi)t}$ has a pointwise multiplier bound $\exp(-t/\omega')$ uniformly in $t > 0$. In particular $g_{2,1}$ satisfies an estimate similar to (3.4) with $c > 0$ independent of k, ω, μ, ν, i.e., for $1 < q < \infty$

$$\|g_{2,1}\|_q \le c_q\|A\|_q. \tag{3.5}$$

3.3 The term g_0

Next we analyze $g_0 = \mathcal{F}^{-1}(\eta_0\hat{g})$ in $L^q(\mathbb{R}^3)$, $1 < q < \infty$, using the representation (2.17), (2.19), i.e.,

$$\begin{aligned}
\hat{g}_0(\xi) &= \frac{1}{\omega^2}\eta_0(\xi)\frac{\xi_j\xi_k}{\mu_1 - \mu_2}\int_0^{2\pi}\left(\frac{e^{-\mu_2 t}}{1 - e^{-2\pi\lambda_2}} - \frac{e^{-\mu_1 t}}{1 - e^{-2\pi\lambda_1}}\right)\hat{A}_{t,jk}(\xi)\,\mathrm{d}t \\
&= \frac{\xi_j/\omega}{\mu_1 - \mu_2}\eta_0(\xi)\cdot\frac{\xi_k/\omega}{1 - e^{-2\pi\lambda_2(\xi)}}\eta_0'(\xi)\cdot\int_0^{2\pi}\eta_0'(\xi)e^{-\mu_2(\xi)t}\hat{A}_{t,jk}(\xi)\,\mathrm{d}t \\
&\quad - \text{ similar terms with } \lambda_2,\ \mu_2 \text{ replaced by } \lambda_1,\ \mu_1,
\end{aligned} \tag{3.6}$$

$1 \le j,\ k \le 3$. Due to the properties of the cut-off function η_0 and its derivatives, the function

$$m_3(\xi) := \frac{\xi_j/\omega}{\mu_1 - \mu_2}\eta_0(\xi) = \frac{\zeta_j}{2i|\zeta|\sqrt{1 - |\zeta|^2}}\eta_0\big(\frac{2\zeta}{2\mu + \nu}\big) \tag{3.7}$$

is easily shown to be a multiplier with bound $\|m_3\|_{\mathcal{M}} \le c$. Moreover,

$$m_t(\xi) = e^{-\mu_{1,2}(\xi)t}\,\eta_0'(\xi) = \exp\Big(-\frac{t}{\omega'}\big(|\zeta|^2 \pm i|\zeta|\sqrt{1-|\zeta|^2}\big)\Big)\,\eta_0'\big(\frac{2\zeta}{2\mu+\nu}\big)$$

is a multiplier function with multiplier bound $\|m_t\|_{\mathcal{M}} \le \big(1+\frac{1}{\omega'}\big)$ uniformly in $t \in (0, 2\pi)$; note that the term $\frac{1}{\omega'}$ comes into play when differentiating the purely imaginary term $i|\zeta|\sqrt{1-|\zeta|^2}$ in the exponential function.

Finally we have to consider the functions

$$m_{1,2}(\xi) = \frac{\xi_k/\omega}{1-e^{-2\pi\lambda_{1,2}}}\eta_0'(\xi) = \frac{1}{\omega'}\,\frac{\zeta_k\eta_0'\big(2\zeta/(2\mu+\nu)\big)}{D_{1,2}(\zeta)}$$

with the denominator

$$D_{1,2}(\zeta) = 1-\exp\Big\{-\frac{2\pi}{\omega'}\big(|\zeta|^2 + ik\zeta_3 \pm i|\zeta|\sqrt{1-|\zeta|^2}\big)\Big\}\,.$$

For $\zeta \in \overline{B_{3/4}(0)} \cap \overline{B_{\omega'/2}(0)}$ Taylor's expansion of the complex exponential function implies that

$$D_{1,2}(\zeta) = \frac{2\pi i}{\omega'}\big(\pm|\zeta| + k\zeta_3\big) + O(|\zeta|^2) \quad \text{as } |\zeta| \to 0\,.$$

Under the assumption $|k| \le \frac{1}{2}$ we conclude that $m_{1,2}$ is uniformly bounded for $\zeta \in \overline{B_{3/4}(0)} \cap \overline{B_{\omega'/2}(0)}$. By analogy, we can estimate derivatives of $m_{1,2}$ and finally get a constant $c > 0$ independent of k, μ, ν, ω such that

$$|m_{1,2}| + |\xi|\,|\nabla_\xi m_{1,2}| + |\xi|^2\,|\nabla_\xi^2 m_{1,2}| \le c, \quad \zeta \in \overline{B_{3/4}(0)} \cap \overline{B_{\omega'/2}(0)}\,.$$

However, for the remaining $\zeta \in \overline{B_{3/4}(0)} \setminus \overline{B_{\omega'/2}(0)}$, i.e., $\frac{3}{4} \ge |\zeta| > \frac{\omega'}{2}$ (provided that $\omega' < \frac{3}{2}$),

$$|D_{1,2}(\zeta)| \ge 1-\exp\Big(-\frac{2\pi|\zeta|^2}{\omega'}\Big) \ge 1-\exp\big(-\pi|\zeta|\big) \ge c|\zeta|$$

with a suitable constant $c > 0$. Hence $|m_{1,2}|$ is bounded by $\frac{c}{\omega'}$. Analogously, we get the estimates

$$\begin{aligned} |\xi|\,|\nabla_\xi m_{1,2}| &\le c\big(1+\tfrac{1}{\omega'^2}\big), \quad \zeta \in \overline{B_{3/4}(0)} \setminus \overline{B_{\omega'/2}(0)}\,,\\ |\xi|^2\,|\nabla_\xi^2 m_{1,2}| &\le c\big(1+\tfrac{1}{\omega'^3}\big), \quad \zeta \in \overline{B_{3/4}(0)} \setminus \overline{B_{\omega'/2}(0)}\,. \end{aligned}$$

Summarizing the previous estimates we see that $m_{1,2}$ are multipliers with multiplier bound $\|m_{1,2}\|_{\mathcal{M}} \le c\big(1+\omega'^{-3}\big)$.

Hence the product of multipliers m_3, m_t and $m_{1,2}$ as discussed above and used to define $\hat{g}_0$ in (3.6) yields a multiplier with bound $c\big(1+\frac{1}{\omega'^4}\big)$. Now Theorem 3.1 leads for each $1<q<\infty$ to the estimate

$$\|g_0\|_q \le c\Big(1+\frac{1}{\omega(2\mu+\nu)}\Big)^4 \|A\|_q \tag{3.8}$$

where $c>0$ is independent of k,μ,ν,ω. Here the assumption $k=|u_\infty|\le\frac{1}{2}$ from Theorem 1.1 has been used.

Remark 3.2. (i) By (3.8) the *a priori* estimate of $\|g_0\|_q$ by $\|A\|_q$ indicates a possible blow-up when the angular speed ω tends to zero. This seems to contradict better estimates in the case when $\omega=0$. However, this phenomenon is known also for the first order derivative $u_\infty\cdot\nabla v$ in the Oseen system of incompressible flow with rotation effect and even in the L^2- case, cf. [3, Theorem 2].

(ii) Formula (3.6) of $\hat{g}_0$ looks like a problem for second order derivatives, but due to the behaviour of $\mu_1-\mu_2$ for small $|\zeta|$ in the denominator in (3.6) one order of $|\xi|$ has been cancelled, see (3.7).

3.4 The term g_1

Finally we consider the term $g_1=\mathcal{F}^{-1}(\eta_1\hat{g})$ which will pose most difficulties since the functions $\lambda_{1,2}(\xi)$ are non-differentiable on the manifold $|\xi|=\frac{2}{2\mu+\nu}$ (i.e. $|\zeta|=1$) where additionally the denominator $\mu_1-\mu_2$ in (2.18) will vanish.

First we observe that

$$m(\xi)=\frac{1}{\omega^2}\xi_j\xi_k\eta_1'(\xi) \tag{3.9}$$

is a multiplier on $L^q(\mathbb{R}^3)$, $1<q<\infty$, with multiplier bound $\|m\|_{\mathcal{M}}\le\frac{c}{\omega'^2}$. Looking at the representation formula (2.18) and (2.19) we still have to analyze

$$\hat{g}_1(\xi)=\eta_1(\xi)\int_0^\infty \frac{e^{-\mu_2 t}-e^{-\mu_1 t}}{\mu_1-\mu_2}\,\hat{A}_t\,\mathrm{d}t \tag{3.10}$$

where, as before, $A_t(x)=A\big(O(t)x-\frac{k}{\omega}te_3\big)$. For this reason we write

$$\frac{e^{-\mu_2 t}-e^{-\mu_1 t}}{\mu_1-\mu_2}=t\int_0^1 e^{-(\mu_2 s+\mu_1(1-s))t}\,\mathrm{d}s. \tag{3.11}$$

Hence the problem is reduced to the analysis of the family of multipliers

$$m_{t,s}(\xi)=\eta_1(\xi)\,e^{-(\mu_2 s+\mu_1(1-s))t},\quad t\in(0,\infty),\ s\in[0,1]. \tag{3.12}$$

Note that in (3.12) we get an exponent

$$\mu_2 s+\mu_1(1-s)=\frac{1}{\omega'}\big(|\zeta|^2+i(1-2s)|\zeta|\sqrt{1-|\zeta|^2}\big)$$

which has to be considered for $\frac{1}{4} \le |\zeta| \le 4$ and $s \in [0,1]$.

However, the classical Hörmander-Mikhlin multiplier theorem does not apply, since $m_{t,s}(\xi)$ is not differentiable for $|\zeta| = 1$. Even the integral version of the multiplier condition ([8, (5.2.11) in Theorem 5.2.7]) is not applicable due to singularities of the type $(1-|\zeta|^2)^{-1/2}$ and $(1-|\zeta|^2)^{-3/2}$. Therefore, we refer to the analysis of Bochner-Riesz multipliers of the type $(1-|\zeta|^2)_+^\lambda$, [8, 9], and perform a much more careful analysis of the multiplier $m_{t,s}$. Using a "partial" Taylor expansion of the exponential functions and the notation

$$t_* = \frac{t}{\omega'}, \quad h_s(\zeta) = -i(1-2s)\cdot|\zeta|\sqrt{1-|\zeta|^2} \tag{3.13}$$

$m_{t,s}$ will be written in the form

$$\begin{aligned} m_{t,s}(\xi) &= \eta_1(\xi)e^{-|\zeta|^2 t_*}\Big[\eta_1'(\xi)\frac{-i(1-2s)|\zeta|}{h_s(\zeta)}\Big(e^{h_s t_*} - 1 - \frac{h_s^2 t_*^2}{2} - \frac{h_s^4 t_*^4}{4!}\Big)\Big] & (3.14)\\ &\times \big\{\eta_1'(\xi)\sqrt{1-|\zeta|^2}\,\chi_{B_1}(\zeta) + i\eta_1'(\xi)\sqrt{|\zeta|^2-1}\,\chi_{B_4\setminus B_1}(\zeta)\big\} & (3.15)\\ &+ \eta_1(\xi)e^{-|\zeta|^2 t_*}\eta_1'(\xi)\Big(1+\frac{h_s^2 t_*^2}{2}+\frac{h_s^4 t_*^4}{4!}\Big) & (3.16)\\ &=: m_{t,s}^1(\xi)\,[m_{t,s}^2(\xi)] \times \{M_1(\xi)+M_2(\xi)\} + m_{t,s}^1(\xi)\,m_{t,s}^3(\xi). & (3.17)\end{aligned}$$

The crucial terms are the multipliers M_1 and M_2 in (3.15) due to the non-differentiable factor $\sqrt{1-|\zeta|^2}$ well-known from Bochner-Riesz multipliers. Note that for $m_{t,s}^2(\xi)$ in (3.14) the singularity $\sqrt{1-|\zeta|^2}$ cancels due to the partial Taylor expansion, at least for small $h_s t_*$; hence $m_{t,s}^2$ will be a multiplier function of class C^2 on $\mathbb{R}^3$.

Lemma 3.3. *There exists a constant $c > 0$ independent of $t \in (0,\infty)$, $s \in [0,1]$ and $\omega > 0$ such that*

$$\|m_{t,s}^1 \cdot m_{t,s}^2\|_{\mathcal{M}} \le Ce^{-t/(16\omega)}\Big(1+\Big(\frac{t}{\omega}\Big)^7\Big), \tag{3.18}$$

$$\|m_{t,s}^1 \cdot m_{t,s}^3\|_{\mathcal{M}} \le Ce^{-t/(16\omega)}\Big(1+\Big(\frac{t}{\omega}\Big)^6\Big). \tag{3.19}$$

Proof Obviously, $m_{t,s}^1 = \eta_1(\xi)e^{-|\zeta|^2 t_*}$ satisfies the Hörmander-Mikhlin estimate $\|m_{t,s}^1\|_{\mathcal{M}} \le c\exp\big(-\frac{t}{16\omega}\big)\Big(1+\big(\frac{t}{\omega}\big)^2\Big)$ since supp $\eta_1' \subset B_4 \setminus B_{1/4}$. Now (3.19) is an immediate consequence.

Concerning the pointwise estimate (3.2) of $m_{t,s}^2(\xi)$ we note that

$$\frac{1}{h_s(\zeta)t_*}\Big(e^{h_s t_*} - 1 - \frac{h_s^2 t_*^2}{2} - \frac{h_s^4 t_*^4}{4!}\Big)$$

can be written as a convergent power series in the complex variable $h_s(\zeta)t_*$ starting with the terms $1 + \frac{1}{6}h_s^2 t_*^2 + \frac{1}{5!}h_s^4 t_*^4$. Hence $m_{t,s}^2$ satisfies the estimate

$$\big||\xi|^\alpha D^\alpha m_{t,s}^2(\xi)\big| \le c(1+t_*^5) \quad \text{for } \xi \in \mathbb{R}^3 \text{ with } |h_s(\zeta)t_*| \le 1$$

and all $\alpha \in \mathbb{N}_0^3$, $|\alpha| \le 2$. Now let $\xi \in \mathbb{R}^3$ and $|h_s(\zeta)t_*| > 1$ but $|\zeta| < 1$ so that $h_s(\zeta)$ is purely imaginary and hence $|e^{h_s t_*}| \le 1$. Then we get the estimate

$$\big||\xi|^\alpha D^\alpha m_{t,s}^2(\xi)\big| \le c(1+t_*^5) \quad \text{for } \ |\zeta| < 1 < |h_s(\zeta)t_*|.$$

Finally we consider $\xi \in$ supp η_1' satisfying $|\zeta| \ge 1$ and $|h_s(\zeta)t_*| > 1$. Depending on $s \in [0,1]$ we have $h_s(\zeta) \ge 0$ or $h_s(\zeta) < 0$; hence a reasonable estimate of $m_{t,s}^2(\xi)$ is not available due to the exponential term $\exp(h_s t_*)$. However, the product $m_{t,s}^1 \cdot m_{t,s}^2$ involves an exponential term with exponent

$$\begin{aligned}\big(-|\zeta|^2 + (2s-1)|\zeta|\sqrt{|\zeta|^2-1}\big)t_* &\le -|\zeta|\big(|\zeta| - \sqrt{|\zeta|^2-1}\big)t_* \\ &= -\frac{|\zeta|t_*}{|\zeta| + \sqrt{|\zeta|^2-1}} \le -\frac{t_*}{2}.\end{aligned}$$

This argument yields the estimate $|m_{t,s}^1 \cdot m_{t,s}^2(\xi)| \le ce^{-t_*/2}(1+t_*^5)$ for these $\xi \in$ supp η_1'; it also holds for derivatives $|\xi|^\alpha D^\alpha$, $\alpha \in \mathbb{N}_0^3$, $|\alpha| \le 2$. Summarizing the previous discussion we get (3.18). □

It remains to discuss the operators defined by the multiplier functions M_1 and M_2. We write

$$\begin{aligned}M_1(\xi) &= \eta_1'(\xi)\sqrt{1-|\zeta|^2}\ \chi_{B_1}(\zeta) \\ &= \sqrt{1-|\zeta|^2}\ \chi_{B_1}(\zeta) + \big(\eta_1'(\xi)-1\big)\sqrt{1-|\zeta|^2}\ \chi_{B_1}(\zeta)\end{aligned}$$

where $\xi \mapsto \big(\eta_1'(\xi)-1\big)\sqrt{1-|\zeta|^2}\,\chi_{B_1}(\zeta)$ defines a smooth multiplier satisfying (3.2) with an absolute bound $\|\cdot\|_{\mathcal{M}}$. Moreover, the first term is the classical Bochner-Riesz multiplier which defines a bounded operator on $L^p(\mathbb{R}^3)$ provided $\frac{6}{5} < p < 6$, cf. [9, Theorem 10.4.6]. Hence we proved the following result.

Lemma 3.4. *The multiplier operator defined by the multiplier function M_1 is bounded on $L^p(\mathbb{R}^3)$ if and only if $\frac{6}{5} < p < 6$. The operator norm on $L^p(\mathbb{R}^3)$ is independent of μ, ν, k and ω.*

To prove the corresponding result for M_2 we consider an even more general multiplier function with the singular term $(|\zeta|^2-1)^\lambda$ where $\lambda \in \mathbb{C}$. The first result concerns properties of inverse Fourier transforms to be used as kernels in convolution integrals.

Lemma 3.5. *On $\mathbb{R}^3$ and for $\operatorname{Re}\lambda > -1$ let*

$$K_\lambda(x) := \mathcal{F}^{-1}\big((|\zeta|^2-1)_+^\lambda g(|\zeta|)\big)$$

where $g \in C^\infty\big([1,\infty)\big)$ satisfies $g(\tau) = 1$ for $1 \le \tau \le 2$ but $g(\tau) = 0$ for $\tau \ge 4$. Then there exists a constant $C = C(\operatorname{Re}\lambda) > 0$ such that

$$|K_\lambda(x)| \le \frac{C}{(1+|x|)^{\operatorname{Re}\lambda+2}}\big(1+|\operatorname{Im}\lambda|\big)^{[\operatorname{Re}\lambda]+2}\ \big(\times\ \log(2+|x|) \quad \textit{when } \operatorname{Re}\lambda \in \mathbb{N}_0\big). \tag{3.20}$$

Proof Evidently, $(|\zeta|^2-1)_+^\lambda g(|\zeta|) \in L^1(\mathbb{R}^3)$ with L^1-norm bounded by $\frac{c}{\operatorname{Re}\lambda+1}$. This proves (3.20) for small $|x|$. It remains to consider (3.20) for large $|x|$.

Since the Fourier transform of K_λ is radially symmetric, the same holds for K_λ itself, and, using the classical Bessel function $\mathcal{J}_{1/2}(r) = \left(\frac{2}{\pi r}\right)^{1/2} \sin r$,

$$\begin{aligned} K_\lambda(x) &= \frac{2}{|x|} \int_0^\infty \sin(2\pi|x|\tau) g(\tau)(\tau^2-1)_+^\lambda \, \tau \, \mathrm{d}\tau \\ &= \frac{2}{|x|} \int_1^4 \sin(2\pi|x|\tau)(\tau^2-1)^{\operatorname{Re}\lambda} e^{i \operatorname{Im}\lambda \ln(\tau^2-1)} \, \tau g(\tau) \, \mathrm{d}\tau. \end{aligned}$$

In the following we will omit the constant 2π in the integrand (or replace x by $2\pi x$) and omit the constant 2 in front of the integral. It will suffice to compute the real part of $K_\lambda(x)$, the imaginary part being just a minor modification.

Assume for a moment that $\operatorname{Re}\lambda > 0$. Then, after the above-mentioned simplification, we get with an integration by parts for $k_\lambda(x) \triangleq \operatorname{Re} K_\lambda(x)$ that

$$\begin{aligned} k_\lambda(x) &= \frac{1}{|x|} \int_1^4 \frac{\mathrm{d}}{\mathrm{d}\tau}\Big(-\frac{1}{|x|}\cos(|x|\tau)\Big)(\tau^2-1)^{\operatorname{Re}\lambda} \cos\big(\operatorname{Im}\lambda \ln(\tau^2-1)\big) \, \tau g(\tau) \, \mathrm{d}\tau \\ &= \frac{2}{|x|^2} \int_1^4 \cos(|x|\tau)(\tau^2-1)^{\operatorname{Re}\lambda-1} \big[\operatorname{Re}\lambda \cos\big(\operatorname{Im}\lambda\ln(\tau^2-1)\big) \\ &\qquad -\operatorname{Im}\lambda \sin\big(\operatorname{Im}\lambda\ln(\tau^2-1)\big)\big] \tau^2 g(\tau)\, \mathrm{d}\tau \\ &\quad + \frac{1}{|x|^2} \int_1^4 \cos(|x|\tau)(\tau^2-1)^{\operatorname{Re}\lambda} \cos\big(\operatorname{Im}\lambda\ln(\tau^2-1)\big) \frac{\mathrm{d}}{\mathrm{d}\tau}\big(\tau g(\tau)\big) \, \mathrm{d}\tau. \end{aligned} \tag{3.21}$$

Note that the boundary terms at $\tau = 1$ and $\tau = 4$ vanish. In the last integral with term $(\tau^2-1)^{\operatorname{Re}\lambda}$ a further integration by parts immediately yields another power $|x|^{-1}$; in this sense this integral will have better decay properties than the first integral. To perform a further integration by parts in the other integral and to gain another term $|x|^{-1}$ we have to assume that $\operatorname{Re}\lambda > 1$. Then we find a polynomial $p_n = p_n(\operatorname{Re}\lambda, \operatorname{Im}\lambda)$ of degree at most n (for the moment $n = 2$) such that

$$\begin{aligned} k_\lambda(x) = &\frac{p_2}{|x|^3} \int_1^4 \sin(|x|\tau)(\tau^2-1)^{\operatorname{Re}\lambda-2} \cos\big(\operatorname{Im}\lambda\ln(\tau^2-1)\big) g_1(\tau) \, \mathrm{d}\tau \\ &+ \text{ similar terms where sine and cosine may be replaced by} \\ &\qquad \text{cosine and sine or terms where the power of } (\tau^2-1) \\ &\qquad \text{is larger than } \operatorname{Re}\lambda - 2; \end{aligned} \tag{3.22}$$

here $g_1 \in C^\infty\big([1,\infty)\big)$, $\operatorname{supp} g_1 \subset [1,4]$.

We may repeat this procedure and arrive after ($[\operatorname{Re}\lambda]+1$) steps at the formula

$$\begin{aligned} k_\lambda(x) = \frac{p_{[\operatorname{Re}\lambda]+1}}{|x|^{[\operatorname{Re}\lambda]+2}} \int\limits_1^4 & \cos(|x|\tau)(\tau^2-1)^{\operatorname{Re}\lambda-[\operatorname{Re}\lambda]-1} \\ & \cdot \cos\big(\operatorname{Im}\lambda \ln(\tau^2-1)\big) g_2(\tau)\,\mathrm{d}\tau + \dots \end{aligned} \tag{3.23}$$

provided that $\operatorname{Re}\lambda > [\operatorname{Re}\lambda] \ge 0$. Now we use the change of variables $s = |x|\tau$ to see that a typical leading term in $k_\lambda(x)$ (concerning decay as $|x| \to \infty$) has the form

$$\begin{aligned} k_\lambda(x) = \frac{p_{[\operatorname{Re}\lambda]+1}}{|x|^{2\operatorname{Re}\lambda-[\operatorname{Re}\lambda]+1}} \int\limits_{|x|}^{4|x|} & \cos s\big((s-|x|)(s+|x|)\big)^{\operatorname{Re}\lambda-[\operatorname{Re}\lambda]-1} \\ & \cdot \cos\Big(\operatorname{Im}\lambda \ln\Big(\frac{s^2}{|x|^2}-1\Big)\Big) g_2\Big(\frac{s}{|x|}\Big)\,\mathrm{d}s + \dots \end{aligned} \tag{3.24}$$

This integral will be considered separately on $[|x|, |x|+1]$ and $[|x|+1, 4|x|]$. On $[|x|, |x|+1]$ we estimate as follows: there exist $c_\lambda = C(\operatorname{Re}\lambda) > 0$ such that

$$\begin{aligned} \Bigg|\int\limits_{|x|}^{|x|+1} (\dots)\Bigg| & \le c_\lambda |x|^{\operatorname{Re}\lambda-[\operatorname{Re}\lambda]-1} \int\limits_{|x|}^{|x|+1} (s-|x|)^{\operatorname{Re}\lambda-[\operatorname{Re}\lambda]-1}\,\mathrm{d}s \\ & \le c_\lambda |x|^{\operatorname{Re}\lambda-[\operatorname{Re}\lambda]-1}, \end{aligned} \tag{3.25}$$

since $\operatorname{Re}\lambda - [\operatorname{Re}\lambda] - 1 \in (-1,0)$. On $[|x|+1, 4|x|]$ we replace the variable s by $t = s - |x|$ to get from (3.24) the integral

$$\int\limits_1^{3|x|} \cos(t+|x|)\big(t(t+2|x|)\big)^{\operatorname{Re}\lambda-[\operatorname{Re}\lambda]-1} \cos\Big(\operatorname{Im}\lambda \ln\Big(\frac{(t+|x|)^2}{|x|^2}-1\Big)\Big) g_2\Big(\frac{t+|x|}{|x|}\Big)\,\mathrm{d}t. \tag{3.26}$$

With an integration by parts we arrive at the boundary terms

$$\begin{aligned} & \sin(4|x|)(3|x|\,5|x|)^{\operatorname{Re}\lambda-[\operatorname{Re}\lambda]-1} \cos(\dots) g_2(\dots) \\ & \quad - \sin(1+|x|)(1+2|x|)^{\operatorname{Re}\lambda-[\operatorname{Re}\lambda]-1} \cos(\dots) g_2(\dots), \end{aligned} \tag{3.27}$$

which can be estimated for $|x| > 1$ by $C|x|^{\mathrm{Re}\,\lambda-[\mathrm{Re}\,\lambda]-1}$, and the new integrals

$$
\begin{aligned}
-\int_1^{3|x|} &\sin(t+|x|)(t+2|x|)^{\mathrm{Re}\,\lambda-[\mathrm{Re}\,\lambda]-1}\\
&\cdot \frac{\mathrm{d}}{\mathrm{d}t}\Big[t^{\mathrm{Re}\,\lambda-[\mathrm{Re}\,\lambda]-1}\cos\Big(\mathrm{Im}\,\lambda\ln\Big(\frac{s^2}{|x|^2}-1\Big)\Big)g_2\Big(\frac{s}{|x|}\Big)\Big]\,\mathrm{d}t \qquad (3.28)\\
-\int_1^{3|x|} &\sin\,(t+|x|)\;c_\lambda\;\big(t(t+2|x|)\big)^{\mathrm{Re}\,\lambda-[\mathrm{Re}\,\lambda]-2}\,t\,\cos(\dots)g_2(\dots)\,\mathrm{d}t. \qquad (3.29)
\end{aligned}
$$

The integral in (3.29) is immediately estimated by

$$
c_\lambda\int_1^{3|x|}(t+2|x|)^{\mathrm{Re}\,\lambda-[\mathrm{Re}\,\lambda]-2}\,\mathrm{d}t \le c_\lambda|x|^{\mathrm{Re}\,\lambda-[\mathrm{Re}\,\lambda]-1}. \qquad (3.30)
$$

In (3.28) we estimate the first two terms of the integrand by $C|x|^{\mathrm{Re}\,\lambda-[\mathrm{Re}\,\lambda]-1}$. Hence it remains to show that

$$
\begin{aligned}
\int_1^{3|x|}\;&\Big|\,\frac{\mathrm{d}}{\mathrm{d}t}\Big[t^{\mathrm{Re}\,\lambda-[\mathrm{Re}\,\lambda]-1}\cos\Big(\mathrm{Im}\,\lambda\ln\Big(\frac{t(t+2|x|)}{|x|^2}\Big)\Big)g_2\Big(1+\frac{t}{|x|}\Big)\Big]\,\Big|\,\mathrm{d}t \qquad (3.31)\\
&\le c_\lambda(1+|\mathrm{Im}\,\lambda|)
\end{aligned}
$$

for $|x| > 1$. A first term due to the differentiation of $t^{\mathrm{Re}\,\lambda-[\mathrm{Re}\,\lambda]-1}$ yields the bound in (3.31) since $\mathrm{Re}\,\lambda-[\mathrm{Re}\,\lambda]-2 < -1$. The second term comes from the differentiation of the cosine function, yields the factor $\mathrm{Im}\,\lambda$ and exploits the estimate

$$
\Big|\frac{\mathrm{d}}{\mathrm{d}t}\ln\Big(\frac{t(t+2|x|)}{|x|^2}\Big)\Big| = \frac{2(t+|x|)}{t(t+2|x|)} \le \frac{2}{t}; \qquad (3.32)
$$

hence we are led to the integration of $t^{\mathrm{Re}\,\lambda-[\mathrm{Re}\,\lambda]-2}$ and again to (3.31). Finally $\big|\frac{\mathrm{d}}{\mathrm{d}t}g_2\big(1+\frac{t}{|x|}\big)\big| \le \frac{c}{|x|} \le \frac{c}{t}$ for $t \in [1, 3|x|]$. Now (3.31) is proved. Summarizing the ideas from (3.24)–(3.31) we arrive at the estimate

$$
|k_\lambda(x)| \le C\frac{(1+|\mathrm{Im}\,\lambda|)^{[\mathrm{Re}\,\lambda]+2}}{|x|^{\mathrm{Re}\,\lambda+2}} \qquad (3.33)
$$

and hence (3.20) provided $\mathrm{Re}\,\lambda \notin \mathbb{N}_0$.

In the case when $\mathrm{Re}\,\lambda \in \mathbb{N}_0$ we start as for $\mathrm{Re}\,\lambda \notin \mathbb{N}_0$ with integration by parts but stop before reaching (3.23) with the term $(\tau^2-1)^{\mathrm{Re}\,\lambda-[\mathrm{Re}\,\lambda]-1} = (\tau^2-1)^{-1}$,

i. e., we stop with the identities

$$
\begin{aligned}
k_\lambda(x) &= \frac{p_{\operatorname{Re}\lambda}}{|x|^{\operatorname{Re}\lambda+1}} \int\limits_1^4 \cos(|x|\tau)\, \cos\big(\operatorname{Im}\lambda \ln(\tau^2-1)\big) g_2(\tau)\,\mathrm{d}\tau + \dots \\
&= \frac{p_{\operatorname{Re}\lambda}}{|x|^{\operatorname{Re}\lambda+2}} \int\limits_{|x|}^{4|x|} \cos s\, \cos\Big(\operatorname{Im}\lambda \ln\Big(\frac{s^2}{|x|^2}-1\Big)\Big) g_2\Big(\frac{s}{|x|}\Big)\,\mathrm{d}s + \dots,
\end{aligned}
$$

where $s = |x|\tau$, cf. (3.23)–(3.24). Since the integral $\int_{|x|}^{|x|+1}(\dots)\,\mathrm{d}s$ is bounded uniformly in $|x|$, cf. (3.25), it suffices to estimate the integral

$$
\int\limits_1^{3|x|} \cos(t+|x|) \cos\Big(\operatorname{Im}\lambda \ln\Big(\frac{s^2}{|x|^2}-1\Big)\Big) g_1\Big(\frac{s}{|x|}\Big)\,\mathrm{d}t,
$$

cf. (3.26). Using integration by parts in $t = s - |x|$ we get in $|x|$ uniformly bounded boundary terms, cf. (3.27), and the new integral

$$
+\int\limits_1^{3|x|} \sin(t+|x|)(\operatorname{Im}\lambda) \frac{\mathrm{d}}{\mathrm{d}t}\Big[\ln\Big(\frac{s^2}{|x|^2}-1\Big)\Big] \sin(\dots) g_2(\dots)\,\mathrm{d}t \tag{3.34}
$$

plus a further integral involving $\frac{\mathrm{d}}{\mathrm{d}t} g_2(\frac{s}{|x|})$, cf. (3.28). The integral corresponding to (3.29) vanishes since $c_\lambda = 0$. On $[1, 3|x|]$ we use (3.32) to estimate the derivative of the logarithm; moreover, $g_2(\frac{s}{|x|})$ is bounded from below by a positive constant. Hence (3.34) is bounded by $C(1 + |\operatorname{Im}\lambda|)\log|x|$. □

Remark 3.6. Note that the estimate (3.20) for the Bochner-Riesz kernel $K_\lambda(x)$ is - up to the additional term $\log(2+|x|)$ when $\operatorname{Re}\lambda \in \mathbb{N}_0$ - the same as for the kernel with Fourier transform $(1-|\zeta|^2)_+^\lambda$, cf. [8, Appendix B5]. Therefore, the asymptotic structure of K_λ is more or less not a consequence of special integral identities of Bessel functions, used for the usual Bochner-Riesz multipliers, see [8, Appendix B3], but only of their asymptotic and oscillatory behavior. Similarly, we could proceed in the n-dimensional case where, however, the computations would become much more complicated.

Together with Lemma 3.5 we are in a position to modify the proof of [8, Theorem 10.4.6] to get mapping properties of the convolution operator $K_\lambda *$ on L^p-spaces.

Lemma 3.7. *The convolution operator $K_\lambda *$ has the properties:*

$$
\|K_\lambda * f\|_2 \le C\|f\|_2 \quad \textit{for } \operatorname{Re}\lambda \ge 0, \tag{3.35}
$$

$$
\|K_\lambda * f\|_p \le C(\operatorname{Re}\lambda)(1+|\operatorname{Im}\lambda|)^q\, \|f\|_p \quad \textit{for } \operatorname{Re}\lambda > \frac{1}{4} \tag{3.36}
$$

when $1 \le p \le \frac{4}{3}$ and $p > \frac{3}{2+\operatorname{Re}\lambda}$. Here $q \in \mathbb{N}$ depends on $[\operatorname{Re}\lambda]$.

Proof Since $\hat{K}_\lambda$ is bounded when $\mathrm{Re}\,\lambda \geq 0$, Plancherel's Theorem shows (3.35) for $p = 2$. By (3.20) $K_\lambda \in L^1(\mathbb{R}^3)$ for $\mathrm{Re}\,\lambda > 1$; this proves (3.36) when $p = 1$ and $\mathrm{Re}\,\lambda > 1$.

To prove (3.36) for $p > 1$ we choose a radially symmetric function $\varphi \in C_0^\infty(B_2)$ with $\varphi(x) = 1$ for $x \in \overline{B}_1$, and define

$$\psi_j(x) = \psi(2^{-j}x) = \varphi(2^{-j}x) - \varphi(2^{-j+1}x), \quad j \in \mathbb{N}.$$

Note that $\mathrm{supp}\,\psi_j \subset \overline{B}_{2^{j+1}} \setminus B_{2^{j-1}}$ and $\hat{\psi}_j(\zeta) = 2^{3j}\hat{\psi}(2^j\zeta)$. Moreover, we define

$$T_0^\lambda(x) = (\varphi K_\lambda)(x), \quad T_j^\lambda(x) = (\psi_j K_\lambda)(x), \quad j \in \mathbb{N}.$$

Then formally

$$K_\lambda * f = T_0^\lambda * f + \sum_{j=1}^\infty T_j^\lambda * f. \tag{3.37}$$

Since K_λ is a bounded function, T_0^λ is bounded with compact support. Hence $T_0^\lambda *$ is a bounded convolution operator on each space $L^p(\mathbb{R}^3)$, $1 \leq p \leq \infty$, with norm growing polynomially in $|\mathrm{Im}\,\lambda|$.

Concerning T_j^λ our aim is to show the following assertion:

Claim: For $p \geq 1$ satisfying $\frac{3}{4} \leq \frac{1}{p} < \frac{1}{3}(2 + \mathrm{Re}\,\lambda)$ (hence $\mathrm{Re}\,\lambda > \frac{1}{4}$) the estimate

$$\|T_j^\lambda * f\|_p \leq C(\mathrm{Re}\,\lambda)(1 + |\mathrm{Im}\,\lambda|)^q j\, 2^{-j\delta}\|f\|_p, \quad j \in \mathbb{N}, \tag{3.38}$$

holds; here $q = q([\mathrm{Re}\,\lambda]) \in \mathbb{N}$, $\delta = \frac{1}{2}(1 + 2\mathrm{Re}\,\lambda) - 3(\frac{1}{p} - \frac{1}{2}) > 0$ and C is independent of $j \in \mathbb{N}$.

We note that (3.38) makes the formal identity (3.37) rigorous and proves estimate (3.36) for p, λ satisfying $\frac{3}{4} \leq \frac{1}{p} < \frac{1}{3}(2 + \mathrm{Re}\,\lambda)$; in particular, (3.36) holds when $p = \frac{4}{3}, \mathrm{Re}\,\lambda > \frac{1}{4}$. In the following a constant $C > 0$ always depends on $\mathrm{Re}\,\lambda$ and $\mathrm{Im}\,\lambda$, but only in a polynomial way on $|\mathrm{Im}\,\lambda|$, and is independent of $j \in \mathbb{N}$.

Proof of the Claim. We decompose the function f into pieces $f\chi_{Q_k}$, $k \in \mathbb{N}$, where the Q_k's are pairwise disjoint cubes of side length $R = 2^{j+1}$. Since T_j^λ is supported in a cube of side length $2R = 2^{j+2}$, for each $x \in \mathbb{R}^3$ at most 5^3 pieces $T_j^\lambda * f\chi_{Q_k}$, $k \in \mathbb{N}$, overlap. Hence an estimate like $\|T_j^\lambda * f\|_p \leq C\|f\|_p$ valid for all f with support in a cube of side length R suffices to get the general estimate $\|T_j^\lambda * f\|_p \leq 5^3 C\|f\|_p$ for all $f \in L^2(\mathbb{R}^3)$, cf. [9, Exercise 10.4.4]. Using this idea let us consider f with support in a cube of side length 2^{j+1} so that $\mathrm{supp}\,T_j^\lambda * f$ is contained in a cube of side length $3 \cdot 2^{j+1}$. Then for $1 \leq p \leq 2$ we have

$$\|T_j^\lambda * f\|_p^2 \leq c\, 2^{6j(\frac{1}{p} - \frac{1}{2})}\|T_j^\lambda * f\|_2^2 = c\, 2^{6j(\frac{1}{p} - \frac{1}{2})}\|\hat{T}_j^\lambda \hat{f}\|_2^2. \tag{3.39}$$

To proceed with the term $\|\hat{T}_j^\lambda \hat{f}\|_2^2$ we use polar coordinates in ζ-space. Since T_j^λ and $\hat{T}_j^\lambda$ are radial, we get

$$\|\hat{T}_j^\lambda \hat{f}\|_2^2 = \int_0^\infty |\hat{T}_j^\lambda(r,0,0)|^2 \Big(\int_{\mathbb{S}^2} |\hat{f}(r\theta)|^2 \,\mathrm{d}\theta \Big) r^2 \,\mathrm{d}r.$$

By the Restriction Theorem 3.8 below applied to $r^{-3} f(\frac{x}{r})$ with Fourier transform $\hat{f}(r\zeta)$, $\zeta \in \mathbb{R}^3$, we see that

$$\int_{\mathbb{S}^2} |\hat{f}(r\theta)|^2 \,\mathrm{d}\theta \le C_p^2 \Big(\int_{\mathbb{R}^3} r^{-3p} \big| f\big(\frac{x}{r}\big)\big|^p \,\mathrm{d}x \Big)^{2/p} = C_p^2 \Big(r^{-3(p-1)} \int_{\mathbb{R}^3} |f(y)|^p \,\mathrm{d}y \Big)^{2/p}$$

which holds for $1 \le p \le \frac{4}{3}$. Therefore

$$\begin{aligned} \|\hat{T}_j^\lambda \hat{f}\|_2^2 &\le C_p^2 \, \|f\|_p^2 \int_0^\infty |\hat{T}_j^\lambda(r,0,0)|^2 \, r^{2-6(p-1)/p} \,\mathrm{d}r \\ &= C_p^2 \, \|f\|_p^2 \int_{\mathbb{R}^3} |\hat{T}_j^\lambda(\zeta)|^2 \, |\zeta|^{-6/p'} \,\mathrm{d}\zeta. \end{aligned} \tag{3.40}$$

Next we decompose the integral on the right-hand side of (3.40) into integrals over $B_{1/2}$ and its complement.

To estimate the integral over $B_{1/2}$ we claim that for all $M > 4$ and $\beta < 3$ there exists a constant $C_{M,\beta} > 0$ such that

$$\int_{B_{1/2}} |\hat{T}_j^\lambda(\zeta)|^2 \, |\zeta|^{-\beta} \,\mathrm{d}\zeta \le C_{M,\beta} \, 2^{-2j(M-3)}. \tag{3.41}$$

Indeed, we may assume that $\operatorname{supp} \hat{K}_\lambda \subset \overline{B}_2 \setminus B_1$ by modifying the function g in the definition of K_λ in Lemma 3.5. Hence

$$|\hat{T}_j^\lambda(\zeta)| = |\hat{K}_\lambda * \hat{\psi}_j(\zeta)| \le 2^{3j} \int_{1 \le |\zeta - z| \le 2} (|\zeta - z|^2 - 1)^{\mathrm{Re}\,\lambda} |\hat{\psi}(2^j z)| \,\mathrm{d}z$$

so that for $\zeta \in B_{1/2}$ we can restrict z to $|z| \ge \frac{1}{2}$. Since ψ and $\hat{\psi}$ are Schwartz functions, we get for $|z| \ge \frac{1}{2}$ that

$$|\hat{\psi}(2^j z)| \le C_M |2^j z|^{-M} \le C_M 2^{-jM}.$$

Thus $|\hat{T}_j^\lambda(\zeta)| \le C_M 2^{-j(M-3)}$ for $|\zeta| \le \frac{1}{2}$. Finally, $|\zeta|^{-\beta}$ is integrable on $B_{1/2}$ as $\beta < 3$. Thus (3.41) follows.

Concerning the integral in (3.40) over $\mathbb{R}^3 \setminus B_{1/2}$ recall from (3.20) that

$$|T_j^\lambda(x)| \le C(1+|x|)^{-2-\mathrm{Re}\,\lambda} \log(2+|x|)\, \psi_j(x) \le Cj\, 2^{-(2+\mathrm{Re}\,\lambda)j}$$

since ψ_j is supported in $2^{j-1} \le |x| \le 2^{j+1}$; note that the term $\log(2+|x|)$ and the factor j are needed only when $\lambda \in \mathbb{N}$. Hence

$$\|\mathcal{F}(T_j^\lambda)\|_2^2 = \|T_j^\lambda\|_2^2 \le C\, 2^{-2j(2+\mathrm{Re}\,\lambda)} j^2\, 2^{3j} = Cj^2\, 2^{-(1+2\mathrm{Re}\,\lambda)j}. \tag{3.42}$$

Summarizing (3.40) with (3.42) and (3.41) (with $\beta = 3/p' < 3$ which holds for $p < 2$) we arrive at the estimate

$$\|\hat{T}_j^\lambda \hat{f}\|_2^2 \le Cj^2\, 2^{-(1+2\mathrm{Re}\,\lambda)j}\, \|f\|_p^2,$$

and, returning to $\|T_j^\lambda f\|_p$, by (3.39)

$$\|T_j^\lambda * f\|_p \le Cj\, 2^{\{3(\frac{1}{p}-\frac{1}{2})-\frac{1}{2}(1+2\mathrm{Re}\,\lambda)\}j}\, \|f\|_p. \tag{3.43}$$

The power of 2 is negative iff $p > \frac{3}{2+\mathrm{Re}\,\lambda}$.

Now the inequality (3.38) is proved when $\frac{3}{2+\mathrm{Re}\,\lambda} < p \le \frac{4}{3}$; this also implies the restriction $\mathrm{Re}\,\lambda > \frac{1}{4}$. □

The proof of Lemma 3.7 is a minor modification of the proof of the L^p-boundedness of the Bochner-Riesz multiplier $(1-|\zeta|^2)_+^\lambda$. The log-term in Lemma 3.5 is responsible for the additional factor j in (3.38), but of no consequence for further convergence properties. A crucial step in the previous proof was the Restriction Theorem for the Fourier transform, see [9, Theorem 10.4.5]:

Theorem 3.8 (Restriction Theorem for $\mathbb{S}^2 \subset \mathbb{R}^3$)**.** *Let $1 \le p \le \frac{4}{3}$ and $q = \frac{p'}{2}$. Then the operator*

$$R_{p\to q} : L^p(\mathbb{R}^3) \to L^q(\mathbb{S}^2), \quad f \mapsto \hat{f}\big|_{\mathbb{S}^2},$$

is bounded. In particular, $R_{p\to 2}$ is bounded for $1 \le p \le \frac{4}{3}$.

Theorem 3.9. *For $\mathrm{Re}\,\lambda > \frac{1}{4}$ we have the estimate*

$$\|K_\lambda * f\|_p \le c_p \|f\|_p, \quad f \in L^p(\mathbb{R}^3), \tag{3.44}$$

provided $\left|\frac{1}{p} - \frac{1}{2}\right| < \frac{1}{6}(1+2\mathrm{Re}\,\lambda)$. For $\lambda = \frac{1}{2}$ this condition reduces to the inclusion $\frac{6}{5} < p < 6$.

Proof By Lemma 3.7 the estimate (3.44) is proved for $\mathrm{Re}\,\lambda > \frac{1}{4}$ when $\frac{3}{2+\mathrm{Re}\,\lambda} < p \le \frac{4}{3}$ or $p = 2$. Hence, by complex interpolation, we get the assertion for all p satisfying $\frac{3}{2+\mathrm{Re}\,\lambda} < p \le 2$, i.e., when $0 \le \frac{1}{p} - \frac{1}{2} < \frac{1}{6}(1+2\mathrm{Re}\,\lambda)$. Then a duality argument proves the Theorem. □

Proof of Theorem 1.1 We only have to summarize several estimates of Sect. 2 and 3. The solution g of (1.4) $\stackrel{\wedge}{=}$ (2.9) has been decomposed into terms g_0, g_1, g_2 where by (3.4), (3.5) $\|g_2\|_q \leq c\|A\|_q$. For g_0 a similar estimate holds; however, by (3.8) c is multiplied by the term $\big(1+\frac{1}{\omega(2\mu+\nu)}\big)^4$. Concerning the crucial term g_1 we exploit that $\frac{6}{5} < q < 6$ and estimate using the representation of the multipliers $m_{t,s}$ in (3.17) together with (3.10), (3.11), (3.12) that

$$\begin{aligned}
\|g_1\|_q &\leq c\int_0^\infty \Big(\int_0^1 \big(t\|m^1_{t,s}m^2_{t,s}\|_{\mathcal{M}}(\|M_1\|_{\mathcal{M}} + \|M_2\|_{\mathcal{M}}) + \|m^1_{t,s}m^3_{t,s}\|_{\mathcal{M}}\big)\|\hat{A}_t\|_q \,\mathrm{d}s\Big)\,\mathrm{d}t \\
&\leq c\int_0^\infty t\, e^{-t/\omega}\Big(1+\frac{t}{\omega}\Big)^7 \|\hat{A}\|_q \,\mathrm{d}t \\
&\leq c(\omega(2\mu+\nu))^2.
\end{aligned}$$

Adding the estimates of g_0, g_2 and exploiting (2.20) we arrive at (1.5). □

References

[1] Desjardins, B., and Esteban, M.J., On weak solutions for fluid-rigid structure interaction: compressible and incompressible models. *Comm. Partial Differential Equations* 25, 1399–1413 (2000)

[2] Farwig, R., An L^q-analysis of viscous fluid flow past a rotating obstacle. *Tôhoku Math. J.* 58, 129–147 (2005)

[3] Farwig, R., Estimates of lower order derivatives of viscous fluid flow past a rotating obstacle. *Banach Center Publications,* Warszawa 70, 73–84 (2005)

[4] Farwig, R., Hishida, T., and Müller, D., L^q-Theory of a singular "winding" integral operator arising from fluid dynamics. *Pacific J. Math.* 215, 297–312 (2004)

[5] Farwig, R., Guenther, R.B., Thomann, E.A., and Nečasová, Š., The fundamental solution of linearized nonstationary Navier-Stokes equations of motion around a rotating and translating body. *Discrete Cont. Dyn. Syst. Ser. A* 34, 511-529 (2014)

[6] Feireisl, E., On the motion of rigid bodies in a viscous compressible fluid. *Arch. Ration. Mech. Anal.* 167, 281-308 (2003).

[7] Galdi, G.P., and Kyed, M., A simple proof of L^q-estimates for the steady-state Oseen and Stokes equations in a rotating frame. Part I: Strong solutions. *Proc. Amer. Math. Soc.* 141, 573583 (2013)

[8] Grafakos, L., *Classical Fourier Analysis. Graduate Texts in Mathematics.* Vol. 249, Springer Berlin, 2nd ed., 2008

[9] Grafakos, L., *Modern Fourier Analysis. Graduate Texts in Mathematics.* Vol. 250, Springer Berlin, 2nd ed., 2011

[10] Hishida, T., The Stokes operator with rotation effect in exterior domains. *Analysis* 19, 51–67 (1999)

[11] Hishida, T., An existence theorem for the Navier-Stokes flow in the exterior of a rotating obstacle. *Arch. Ration. Mech. Anal.* 150, 307–348 (1999)

[12] Hishida, T., On the Navier-Stokes flow around a rigid body with a prescribed rotation. *Nonlinear Analysis* 47, 4217–4231 (2001)

[13] Kračmar, S., Nečasová, Š., and Novotný, A., The motion of a compressible viscous fluid around rotating body. Preprint of the Institute of Mathematics, Academy of Sciences of the Czech Republic IM–2013–34.