Very weak solutions to the Navier-Stokes system in general unbounded domains

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We consider very weak instationary solutions u of the Navier-Stokes system in general unbounded domains $\Omega \subset \mathbb{R}^n$, $n \geq 3$, with smooth boundary, i.e., u solves the Navier-Stokes system in the sense of distributions and $u \in L^r(0,T;\tilde{L}^q(\Omega))$ where $\frac{2}{r} + \frac{n}{q} = 1$, $2 < r < \infty$. Solutions of this class have no differentiability properties and in general are not weak solutions in the sense of Leray-Hopf. However, they lie in the so-called Serrin class $L^r(0,T;\tilde{L}^q(\Omega))$ yielding uniqueness. To deal with the unboundedness of the domain we work in the spaces $\tilde{L}^q(\Omega)$ (instead of $L^q(\Omega)$) defined as $L^q \cap L^2$ when $q \geq 2$ but as $L^q + L^2$ when $1 < q < 2$. The proofs are strongly based on duality arguments and the properties of the spaces $\tilde{L}^q(\Omega)$.

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1 Introduction

We consider the instationary Navier-Stokes system

$$\begin{aligned} u_t - \Delta u + \operatorname{div}(u \otimes u) + \nabla p &= f && \text{in} \quad (0,T) \times \Omega, \\ \operatorname{div} u &= k && \text{in} \quad (0,T) \times \Omega, \\ u &= g && \text{on} \quad (0,T) \times \partial\Omega, \\ u(0) &= u_0 && \text{in} \quad \Omega, \end{aligned} \tag{1.1}$$

in a general unbounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 3$, with uniform C^2-boundary and a finite time interval $(0,T)$. Here $u = (u_1, \ldots, u_n)$ denotes the unknown velocity field, p an associated pressure, f a given external force, and u_0 denotes the initial value of u at time $t = 0$. In the most general problem the velocity

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u is not assumed to be solenoidal; rather we prescribe a function $k = \operatorname{div} u$ and also non-zero boundary values g on $\partial\Omega$. For a precise definition of uniform C^k-boundaries we refer to Definition 2.1 below. The viscosity is set to $\nu = 1$, for simplicity. In contrast to the theory of weak solutions in the Leray-Hopf class $u \in L^\infty(0,T;L^2_\sigma(\Omega)) \cap L^2(0,T;W^{1,2}_0(\Omega))$ and to strong (or regular) solutions satisfying $u_t, \Delta u \in L^r(0,T;L^q(\Omega))$, say for $r = q = 2$, our focus is put on the concept of *very weak solutions* u lying in Serrin's class $L^r(0,T;L^q(\Omega))$, $2 < r < \infty$, $n < q < \infty$, $2/r + n/q = 1$ without any differentiability properties. In general, a very weak solution does neither have a bounded kinetic energy in $L^\infty(0,T;L^2(\Omega))$ nor a finite dissipation energy in $L^2(0,T;H^1(\Omega))$. In particular, a very weak solution is not necessarily a weak solution and vice versa. However, very weak solutions lying in Serrin's uniqueness class $L^r(0,T;L^q(\Omega))$ can be shown to be unique.

This concept was mainly introduced in a series of papers by H. Amann [3, 4, 5] in the setting of Besov spaces when $k = 0$, but also used already earlier in a paper of Ch. Amrouche and V. Girault ([6]). More recently, this concept was modified by G.P. Galdi, H. Kozono, C. Simader, H. Sohr and the first author of this paper to a setting in classical L^q-spaces including the inhomogeneous data k, see [9, 10, 12] and [21]. Moreover, very weak solutions can be considered in weighted Lebesgue and Bessel potential spaces using arbitrary Muckenhoupt weights, see the work of K. Schumacher ([25, 26, 27, 28]).

It is advantageous to generalize the concept so that neither the external force nor boundary values nor initial values of a very weak solution are specified or can be defined separately from each other. This data is composed into a functional $\mathcal{F}$, the divergence k and the normal component of the trace g is composed to a functional $\mathcal{K}$,

$$\langle \mathcal{F}, \phi \rangle = (u_0, \phi(0))_\Omega + (f_1, \phi)_{T,\Omega} + (f_2, \nabla\phi)_{T,\Omega} - (g, N \cdot \nabla\phi)_{T,\partial\Omega}, \tag{1.2}$$

$$\langle \mathcal{K}, \psi \rangle = (g, \psi N)_{T,\partial\Omega} - (k, \psi)_{T,\Omega}, \tag{1.3}$$

for adequate test functions ϕ and ψ. By this setting the theory of very weak solutions is strongly related via duality arguments to the theory of strong (or regular) solutions.

A second crucial issue in our setting is the unboundedness of the underlying domain Ω. Due to counter-examples by M.E. Bogovskij and V.N. Maslennikova [7, 8] the Helmholtz decomposition of vector fields in $L^q(\Omega)$, $1 < q(\neq 2) < \infty$, on an unbounded smooth domain may fail. Hence a bounded Helmholtz projection P_q with the properties required to define the Stokes operator $A_q = -P_q\Delta$ when $q \neq 2$ may fail to exist. Therefore, in [11, 13, 14, 15] H. Kozono, H. Sohr and the first author of this article introduced the spaces

$$\tilde{L}^q(\Omega) := \begin{cases} L^q(\Omega) + L^2(\Omega), & \text{if } 1 \le q < 2, \\ L^q(\Omega) \cap L^2(\Omega), & \text{if } 2 \le q \le \infty. \end{cases} \tag{1.4}$$

For bounded domains Ω we have that $\tilde{L}^q(\Omega) = L^q(\Omega)$ with equivalent norms by Hölder's inequality. Note that functions in $\tilde{L}^q(\Omega)$ locally behave like L^q-functions, but globally like L^2-functions. Obviously, $\tilde{L}^q(\Omega)^* \cong \tilde{L}^{q'}(\Omega)$. By analogy, function spaces like $\tilde{L}^q_\sigma(\Omega)$ of solenoidal vector fields and $\tilde{W}^{k,q}(\Omega)$ of weakly differentiable functions will be defined.

As shown in [13] a Helmholtz projection $\tilde{P}_q : \tilde{L}^q(\Omega)^n \to \tilde{L}^q_\sigma(\Omega)$ is well defined, allowing to define a closed Stokes operator $\tilde{A}_q = -\tilde{P}_q\Delta$ with domain $\tilde{D}^1_q = \tilde{W}^{2,q}(\Omega)\cap\tilde{W}^{1,q}_0(\Omega)\cap\tilde{L}^q_\sigma(\Omega)$ dense in $\tilde{L}^q_\sigma(\Omega)$. The operator $\tilde{A}_q$ has similar properties as the usual Stokes operator A_q, generates an analytic semigroup $e^{-t\tilde{A}_q}$, $t \geq 0$, enjoys the property of bounded imaginary powers and maximal regularity; for details and further properties of these function spaces and operators we refer to [11, 13, 14, 15] and [24] as well as to Sect. 2.

Now we are in the position to give a precise definition of very weak solutions.

Definition 1.1. *Let $\Omega \subseteq \mathbb{R}^n$ be a uniform C^2-domain, $0 < T < \infty$, and let $2 < r < \infty$, $n < q < \infty$ and $2/r + n/q = 1$. As usual, let q' denote the conjugate exponent to q.*

(i) Let the test function space for the functional $\mathcal{F}$ be defined by

$$\mathcal{T}^{1,r',q'}(T,\Omega) := \{\phi \in L^{r'}(0,T;\tilde{D}^1_{q'}) \cap W^{1,r'}(0,T;\tilde{L}^{q'}(\Omega)) \colon \phi(T) = 0\}$$

and equipped with the norm

$$\|\phi\|_{\mathcal{T}^{1,r',q'}(T,\Omega)} := \|\phi_t\|_{L^{r'}(0,T;\tilde{L}^{q'}(\Omega))} + \|\phi\|_{L^{r'}(0,T;\tilde{D}^1_{q'})}.$$

Then the set of bounded functionals on $\mathcal{T}^{1,r',q'}(T,\Omega)$ is denoted by $\mathcal{T}^{-1,r,q}(T,\Omega)$. Moreover, we need the set of functionals

$$L^r(0,T;\tilde{G}^{-1}_q(\Omega)) = L^{r'}(0,T;\tilde{G}_{q'}(\Omega))^*$$

where $\tilde{G}^{-1}_q(\Omega)$ is the dual space to $\tilde{G}_{q'}(\Omega) = \{\nabla p : p \in L^{q'}_{\mathrm{loc}}(\Omega), \nabla p \in \tilde{L}^{q'}(\Omega)\}$. For a functional $\mathcal{K} \in L^r(0,T;\tilde{G}^{-1}_q(\Omega)$ we simply write $\langle \mathcal{K}, \psi\rangle$, $\nabla\psi \in L^{r'}(0,T;\tilde{G}_{q'}(\Omega))$, instead of $\langle \mathcal{K}, \nabla\psi\rangle$ or $\langle \mathcal{K}, [\psi]\rangle$ where two representatives in an equivalence class $[\psi]$ differ by an additive constant.

(ii) For given data $\mathcal{F} \in \mathcal{T}^{-1,r,q}(T,\Omega)$ and $\mathcal{K} \in L^r(0,T;\tilde{G}^{-1}_q(\Omega))$ we call $u \in L^r(0,T;\tilde{L}^q(\Omega))$ a very weak solution to the Navier-Stokes system with data $\mathcal{F}$, $\mathcal{K}$ if the conditions

$$\begin{aligned} -(u,\phi_t)_{T,\Omega} - (u,\Delta\phi)_{T,\Omega} - (u\otimes u,\nabla\phi)_{T,\Omega} &= \langle \mathcal{F},\phi\rangle, \\ (u,\nabla\psi)_{T,\Omega} &= \langle \mathcal{K},\psi\rangle \end{aligned}$$

hold for all test functions $\phi \in \mathcal{T}^{1,r',q'}(T;\Omega)$ and $\nabla\psi \in L^{r'}(0,T;\tilde{G}_{q'}(\Omega))$.

The large generality of the data space is of course an advantage. In fact, it is optimal in the following sense: Every vector field $u \in L^r(0,T;\tilde{L}^q(\Omega))$ is a very weak solution to some data $\mathcal{F}$, $\mathcal{K}$, namely to

$$\begin{aligned}\langle \mathcal{F}, \phi\rangle &= -(u, \phi_t)_{T,\Omega} - (u, \Delta\phi)_{T,\Omega} - (u \otimes u, \nabla\phi)_{T,\Omega},\\ \langle \mathcal{K}, \psi\rangle &= (u, \nabla\psi)_{T,\Omega}.\end{aligned}$$

For details we refer to Sect. 2.

Our first main results deals with the linearized problem, i.e., the nonstationary Stokes problem, when omitting the term $\operatorname{div}(u \otimes u)$ in (1.1) or $(u \otimes u, \nabla\phi)_{T,\Omega}$ in Definition 1.1 (ii). Note that in the linear case the condition $2/r + n/q = 1$ is not needed. The term $\tau(\Omega)$ will be explained in Sect. 2.

Theorem 1.2 (Existence and Uniqueness for the Stokes Problem). *Let $\Omega \subseteq \mathbb{R}^n$ be a uniform C^2-domain and $0 < T < \infty$. Let furthermore $1 < r, q < \infty$. Then for every $\mathcal{F} \in \mathcal{T}^{-1,r,q}(T,\Omega)$ and $\mathcal{K} \in L^r(0,T;\tilde{G}_q^{-1}(\Omega))$ there exists a unique very weak solution $u \in L^r(0,T;\tilde{L}^q(\Omega))$ to the Stokes system with data $\mathcal{F}$ and $\mathcal{K}$. This solution satisfies the* a priori *estimate*

$$\|u\|_{L^r(0,T;\tilde{L}^q(\Omega))} \le C\big(\|\mathcal{F}\|_{\mathcal{T}^{-1,r,q}(T,\Omega)} + \|\mathcal{K}\|_{L^r(0,T;\tilde{G}_q^{-1}(\Omega))}\big)$$

with a constant $C = C(\tau(\Omega), r, q, T)$.

The second main result states the existence of a very weak solution for the Navier-Stokes system (1.1).

Theorem 1.3 (Existence for the Navier-Stokes Problem). *Let $\Omega \subseteq \mathbb{R}^n$ be a uniform C^2-domain and let $0 < T < \infty$. Assume that $\mathcal{F} \in \mathcal{T}^{-1,r,q}(T,\Omega)$ and $\mathcal{K} \in L^r(0,T;\tilde{G}_q^{-1}(\Omega))$ where $2 < r < \infty$, $n < q < \infty$, and Serrin's condition $\frac{2}{r} + \frac{n}{q} = 1$ is satisfied.*

(i) There exists an $\eta = \eta(\tau(\Omega), r, q, T) > 0$ with the following property: if

$$\|\mathcal{F}\|_{\mathcal{T}^{-1,r,q}(T,\Omega)} + \|\mathcal{K}\|_{L^r(0,T;\tilde{G}_q^{-1}(\Omega))} \le \eta, \tag{1.5}$$

then there exists a very weak solution $u \in L^r(0,T;\tilde{L}^q(\Omega))$ to the Navier-Stokes system with data $\mathcal{F}$, $\mathcal{K}$ in the sense of Definition 1.1. The a priori estimate

$$\|u\|_{L^r(0,T;\tilde{L}^q(\Omega))} \le C\big(\|\mathcal{F}\|_{\mathcal{T}^{-1,r,q}(T,\Omega)} + \|\mathcal{K}\|_{L^r(0,T;\tilde{G}_q^{-1}(\Omega))}\big) \tag{1.6}$$

holds with a constant $C = C(\tau(\Omega), r, q, T)$.

(ii) There exists a $T' \in (0,T)$ such that there is a very weak solution $u \in L^r(0,T';\tilde{L}^q(\Omega))$ to the Navier-Stokes system with data $\mathcal{F}|_{[0,T']} \in \mathcal{T}^{-1,r,q}(T',\Omega)$, $\mathcal{K}|_{[0,T']} \in L^r(0,T';\tilde{G}_q^{-1}(\Omega))$.

Further results in Sect. 3 and 4 deal with questions of regularity of very weak solutions to the Stokes and Navier-Stokes system; see Theorems 3.2 and 3.3, Proposition 4.2, Theorem 4.7 and, concerning $L^4(L^4)$-integrability, Theorem 4.4. Uniqueness of very weak solutions is discussed in Theorem 4.1 and Corollary 4.3. Particular emphasis is also put on conditions on data u_0, f_1, f_2 etc. to guarantee that the functionals $\mathcal{F}, \mathcal{K}$ lie in $\mathcal{T}^{-1,r,q}(T,\Omega)$ and $L^r(0,T;\tilde{G}_q^{-1}(\Omega))$, respectively; see Propositions 2.4, 3.4 and 4.5, Corollary 4.6 and Proposition 4.8. The results of this paper will be applied in a forthcoming article to prove regularity results for weak solutions of the Navier-Stokes system [16].

2 Preliminaries

Definition 2.1. *For $k \in \mathbb{N}_0$ and $\lambda \in (0,1]$ a domain $\Omega \subset \mathbb{R}^n$ is called* uniform $C^{k,\lambda}$-domain *if there are positive constants α, β, K such that for all $x_0 \in \partial\Omega$ there exist - after an orthogonal and an affine coordinate transform - a real-valued function h of class $C^{k,\lambda}$ and a neighborhood $U_{\alpha,\beta,h}(x_0)$ of x_0 with the following properties: h is defined on the closed ball $\overline{B'_\alpha(0)} \subseteq \mathbb{R}^{n-1}$ with $\|h\|_{C^{k,\lambda}} \leq K$ and $h(0) = 0$ and, if $k \geq 1$, $h'(0) = 0$; moreover,*

$$\begin{aligned} U_{\alpha,\beta,h}(x_0) :&= \{(y', y_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \colon |y'| < \alpha, |h(y') - y_n| < \beta\}, \\ U^-_{\alpha,\beta,h}(x_0) :&= \{(y', y_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \colon |y'| < \alpha, h(y') - \beta < y_n < h(y')\} \\ &= \Omega \cap U_{\alpha,\beta,h}(x_0), \\ \partial\Omega \cap U_{\alpha,\beta,h}(x_0) &= \{(y', y_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \colon |y'| < \alpha, \ h(y') = y_n\}. \end{aligned}$$

The triple (α, β, K) is called the *type* of Ω, for short $\tau(\Omega) = (\alpha, \beta, K)$. For a constant C in some estimate we will write $C = C(\tau(\Omega))$ if it does depend only on α, β and K, but in no other way on Ω. A uniform C^k-domain is defined in an obviously analogous way. We note that bounded and exterior domains are included, as long as the boundary is smooth enough.

In addition to the spaces $\tilde{L}^q(\Omega)$, see (1.4), we define for $1 < q < \infty$, $1 \leq \rho \leq \infty$ the Lorentz spaces

$$\tilde{L}^{q,\rho}(\Omega) := \begin{cases} L^{q,\rho}(\Omega) + L^2(\Omega), & q < 2, \\ L^{q,\rho}(\Omega) \cap L^2(\Omega), & q > 2, \end{cases} \tag{2.1}$$

letting the case $q = 2$ undefined; here $L^{q,\rho}(\Omega)$ denotes a usual Lorentz space.

For spaces of Sobolev-type we proceed analogously: For $k \in \mathbb{N}$ and $1 \leq q \leq \infty$ we let

$$\tilde{W}^{k,q}(\Omega) := \begin{cases} W^{k,2}(\Omega) + W^{k,q}(\Omega), & 1 \leq q < 2, \\ W^{k,2}(\Omega) \cap W^{k,q}(\Omega), & 2 \leq q \leq \infty. \end{cases} \tag{2.2}$$

Similarly, we define the spaces $\tilde{W}_0^{1,q}(\Omega)$, $1 < q < 2$ and $2 \leq q < \infty$, based on the classical Sobolev spaces $W_0^{1,q}(\Omega)$ and $W_0^{1,2}(\Omega)$.

The $\tilde{L}^q$- and $\tilde{W}^{k,q}(\Omega)$-spaces have the following properties; for a proof see [24]:

- Let $1 \le q \le r < \infty$. Then $(\tilde{L}^q(\Omega))^* = \tilde{L}^{q'}(\Omega)$ and $\|u\|_{\tilde{L}^q} \le \|u\|_{\tilde{L}^r}$. If $1 \le r, p, q \le \infty$, $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$ and $u \in \tilde{L}^p$, $v \in \tilde{L}^q$, then $uv \in \tilde{L}^r$ and $\|uv\|_{\tilde{L}^r} \le \|u\|_{\tilde{L}^p}\|v\|_{\tilde{L}^q}$.

- Let $1 < q, r < \infty$, $0 < \theta < 1$ and s be defined by $\frac{1}{s} = \frac{1-\theta}{q} + \frac{\theta}{r}$. Then in the sense of complex interpolation spaces
$$\left[\tilde{L}^q(\Omega), \tilde{L}^r(\Omega)\right]_\theta = \tilde{L}^s(\Omega).$$

- Let $1 \le r \ne q \le \infty$, $0 < \theta < 1$, $1 \le \rho \le \infty$, and define $1 < s \ne 2 < \infty$ by $\frac{1}{s} = \frac{1-\theta}{q} + \frac{\theta}{r}$. Then
$$\left(\tilde{L}^q(\Omega), \tilde{L}^r(\Omega)\right)_{\theta,\rho} = \tilde{L}^{s,\rho}(\Omega).$$
For $s = 2$ and $\rho = 2$ we get $\left(\tilde{L}^q(\Omega), \tilde{L}^r(\Omega)\right)_{\theta,2} = L^2(\Omega)$.

- Let $m \in \mathbb{N}$, $1 \le q < \infty$ and $\Omega \subseteq \mathbb{R}^n$ be a uniform C^2-domain. Then
$$\tilde{W}^{m,q}(\Omega) \hookrightarrow \tilde{L}^r(\Omega)$$
if either $q \le r \le \infty$ and $mq > n$, or $q \le r < \infty$ and $mq = n$, or $q \le r \le \frac{nq}{n-mq}$ and $mq < n$.

Concerning the Helmholtz projection on $\tilde{L}^q(\Omega)$ for a domain $\Omega \subseteq \mathbb{R}^n$ of uniform type C^1 we have the following result, see [13]. We define

$$\tilde{L}^q_\sigma(\Omega) := \begin{cases} L^q_\sigma(\Omega) + L^2_\sigma(\Omega), & 1 < q < 2 \\ L^q_\sigma(\Omega) \cap L^2_\sigma(\Omega), & 2 \le q < \infty \end{cases}, \tag{2.3}$$

equipped with the norm of $\tilde{L}^q(\Omega)$, and gradient spaces by

$$\tilde{G}_q(\Omega) := \begin{cases} G_q(\Omega) + G_2(\Omega), & 1 < q < 2, \\ G_q(\Omega) \cap G_2(\Omega), & 2 \le q < \infty, \end{cases} \tag{2.4}$$

based on the usual gradient space $G_q(\Omega) = \{\nabla p : p \in L^q_{\text{loc}}(\Omega), \nabla p \in L^q(\Omega)\}$. The norm in $\tilde{G}_q(\Omega)$ is denoted by $\|\cdot\|_{\tilde{G}_q(\Omega)} := \|\cdot\|_{\tilde{L}^q(\Omega)}$. Then the space $\tilde{L}^q(\Omega)$ admits the direct algebraic and topological decomposition

$$\tilde{L}^q(\Omega) = \tilde{L}^q_\sigma(\Omega) \oplus \tilde{G}_q(\Omega).$$

The corresponding projection $\tilde{P}_q$ from $\tilde{L}^q(\Omega)$ onto its range $\tilde{L}^q_\sigma(\Omega)$ and with kernel $\tilde{G}_q(\Omega)$ has a norm bounded by a constant $c = c(q, \tau(\Omega))$. We have the duality relations $\left(\tilde{P}_q\right)^* = \tilde{P}_{q'}$, $\tilde{L}^q_\sigma(\Omega)^* \cong \tilde{L}^{q'}_\sigma(\Omega)$ and $\tilde{G}^{-1}_q(\Omega) := \tilde{G}_q(\Omega)^* \cong \tilde{G}_{q'}(\Omega)$.

Using the Helmholtz projection $\tilde{P}_q$ we define the Stokes operator $\tilde{A}_q$, $1 < q < \infty$, for a uniform C^2-domain $\Omega \subseteq \mathbb{R}^n$. Let

$$\mathcal{D}(\tilde{A}_q) := \begin{cases} \mathcal{D}_q + \mathcal{D}_2, & 1 < q < 2, \\ \mathcal{D}_q \cap \mathcal{D}_2, & 2 \le q < \infty, \end{cases} \tag{2.5}$$

where $\mathcal{D}_q := L^q_\sigma(\Omega) \cap W^{1,q}_0(\Omega) \cap W^{2,q}(\Omega)$. Then the Stokes operator $\tilde{A}_q : \mathcal{D}(\tilde{A}_q) \subseteq \tilde{L}^q_\sigma(\Omega) \to \tilde{L}^q_\sigma(\Omega)$ is defined by $\tilde{A}_q u := -\tilde{P}_q \Delta u$ and has the following properties, see [15]:

- $\tilde{A}_q$ is a densely defined closed operator in $\tilde{L}^q_\sigma(\Omega)$ satisfying $\big(\tilde{A}_q\big)^* = \tilde{A}_{q'}$.
- $\tilde{A}_q$ satisfies the resolvent estimate

  $$\|\lambda u; \nabla^2 u; \tilde{A}_q u\|_{\tilde{L}^q(\Omega)} \le C \|(\lambda + \tilde{A}_q) u\|_{\tilde{L}^q(\Omega)}$$

  for $\lambda \in \mathbb{C} \setminus \{0\}$ with $|\lambda| \ge \delta > 0$ and $|\arg(\lambda)| < \theta$, $\frac{\pi}{2} < \theta < \pi$, and with a constant $C = C(q, \delta, \theta, \tau(\Omega)) > 0$.
- $\tilde{A}_q$ generates an analytic semigroup $e^{-t\tilde{A}_q}$, $t \ge 0$, having the bound

  $$\|e^{-t\tilde{A}_q} f\|_{\tilde{L}^q(\Omega)} \le C e^{\delta t} \|f\|_{\tilde{L}^q(\Omega)}$$

  for all $f \in \tilde{L}^q_\sigma(\Omega)$ and $t \ge 0$ with a constant $C = C(q, \delta, \tau(\Omega))$, $\delta > 0$.

It is unknown whether the resolvent estimate holds uniformly in λ as $|\lambda| \to 0$. Therefore, the semigroup may increase exponentially fast and the maximal regularity estimate in Theorem 2.2 below is stated only for finite time intervals. For this reason, the operator $\tilde{A}_q$ has often to be replaced by $I + \tilde{A}_q$ in the following. Note that from time to time we omit the symbols Ω and T for domain and length of the time interval, respectively, when this data is known from the context.

Theorem 2.2. ([14, Theorem 1.4]) *Let $\Omega \subseteq \mathbb{R}^n$ be a uniform C^2-domain and $1 < r, q < \infty$, $0 < T < \infty$.*

(i) Let an external force $f \in L^r(0, T; \tilde{L}^q_\sigma(\Omega))$ and an initial value $u_0 \in \mathcal{D}(\tilde{A}_q)$ (for simplicity) be given. Then there exists a unique solution $u \in L^r(0, T; \mathcal{D}(\tilde{A}_q)) \cap W^{1,r}(0, T; \tilde{L}^q_\sigma(\Omega))$ of the abstract Cauchy problem

$$u_t + \tilde{A}_q u = f, \quad u(0) = u_0.$$

It satisfies the estimate

$$\|u\|_{L^r(0,T;\mathcal{D}(\tilde{A}_q))} + \|u_t\|_{L^r(0,T;\tilde{L}^q)} \le C\big(\|u_0\|_{\mathcal{D}(\tilde{A}_q)} + \|f\|_{L^r(0,T;\tilde{L}^q)}\big)$$

with a constant $C = C(q, r, T, \tau(\Omega)) > 0$. It can be represented by the variation of constants formula

$$u(t) = e^{-t\tilde{A}_q} u_0 + \int_0^t e^{-(t-\tau)\tilde{A}_q} f(\tau)\, d\tau \qquad \text{for a.a. } 0 \le t \le T.$$

(ii) If $f \in L^r(0,T;\tilde{L}^q(\Omega))$ and $u_0 \in \mathcal{D}(\tilde{A}_q)$, then there is a unique $u \in L^r(0,T;\mathcal{D}(\tilde{A}_q)) \cap W^{1,r}(0,T;\tilde{L}^q_\sigma(\Omega))$ and a unique $\nabla p \in L^r(0,T;\tilde{G}^{1,q}(\Omega))$ solving the initial value problem

$$u_t - \Delta u + \nabla p = f, \quad u(0) = u_0.$$

It satisfies the estimate

$$\|u\|_{L^r(0,T;\tilde{W}^{2,q}(\Omega))} + \|u_t;\, \nabla p\|_{L^r(0,T;\tilde{L}^q(\Omega))} \leq C\big(\|u_0\|_{\mathcal{D}(\tilde{A}_q)} + \|f\|_{L^r(0,T;\tilde{L}^q(\Omega))}\big)$$

with a constant $C = C(q,r,T,\tau(\Omega))$.

A further crucial property of the Stokes operator $1 + \tilde{A}_q$ is the fact that it admits bounded imaginary powers, see [22, 23]. Hence complex interpolation methods can be used to describe domains of fractional powers $(1+\tilde{A}_q)^\alpha$, $-1 \leq \alpha \leq 1$. To be more precise, for $0 \leq \alpha \leq 1$ let the domain of the fractional power $(1+\tilde{A}_q)^\alpha$ be denoted by

$$\tilde{D}^\alpha_q = \tilde{D}^\alpha_q(\Omega) = \mathcal{D}((1+\tilde{A}_q)^\alpha), \tag{2.6}$$

equipped with the norm $\|(1+\tilde{A}_q)^\alpha \cdot\|_{\tilde{L}^q}$. If $-1 \leq \alpha < 0$ define $\tilde{D}^\alpha_q$ as the completion of $\tilde{L}^q_\sigma(\Omega)$ in the norm $\|(1+\tilde{A}_q)^\alpha \cdot\|_{\tilde{L}^q}$. These spaces are reflexive and satisfy the duality relation $(\tilde{D}^\alpha_q)^* = \tilde{D}^{-\alpha}_{q'}$. As special cases we get that $\tilde{D}^0_q = \tilde{L}^q_\sigma(\Omega)$, $\tilde{D}^1_q = \mathcal{D}(\tilde{A}_q)$, and

$$\tilde{D}^{1/2}_q = \tilde{W}^{1,q}_0 \cap \tilde{L}^q_\sigma(\Omega) \quad \text{with norm } \|(1+\tilde{A}_q)^{1/2}\cdot\|_{\tilde{L}^q} \sim \|\cdot\|_{\tilde{W}^{1,q}(\Omega)}.$$

Moreover, for $-1 \leq \alpha \leq \beta \leq 1$ the operator $(1+\tilde{A}_q)^{\beta-\alpha}$ is an isomorphism between $\tilde{D}^\beta_q$ and $\tilde{D}^\alpha_q$. Finally, we obtain the interpolation result

$$\big[\tilde{D}^\alpha_q, \tilde{D}^\beta_q\big]_\theta = \tilde{D}^\gamma_q, \tag{2.7}$$

when $-1 \leq \alpha \leq \beta \leq 1$ and $(1-\theta)\alpha + \theta\beta = \gamma$, $\theta \in (0,1)$. This result is the basis to prove the following embedding estimate and L^r–L^q-estimates of the Stokes semigroup, cf. [24, Proposition 3, Theorem 1]:

- Let $n \geq 3$, $0 \leq \alpha \leq 1$, $1 < q \leq r < \infty$, and $\frac{1}{r} = \frac{1}{q} - \frac{2\alpha}{n}$. Then
$$\|u\|_{\tilde{L}^r(\Omega)} \leq C\|(1+\tilde{A}_q)^\alpha u\|_{\tilde{L}^q(\Omega)} \tag{2.8}$$
for all $u \in \tilde{D}^\alpha_q$ with a constant $C = C(\tau(\Omega), q, \alpha)$.

- Let $n \geq 3$, $1 < q \leq r < \infty$, and $\alpha := \frac{n}{2}\left(\frac{1}{q} - \frac{1}{r}\right) \geq 0$. Then for every $f \in \tilde{L}^q_\sigma(\Omega)$ and $t > 0$
$$\|e^{-t\tilde{A}_r} f\|_{\tilde{L}^r(\Omega)} \leq Ce^{\delta t}(1+t)^\alpha t^{-\alpha}\|f\|_{\tilde{L}^q(\Omega)}, \tag{2.9}$$
$$\|\nabla e^{-t\tilde{A}_r} f\|_{\tilde{L}^r(\Omega)} \leq Ce^{\delta t}(1+t)^{\alpha+\frac{1}{2}} t^{-\alpha-\frac{1}{2}}\|f\|_{\tilde{L}^q(\Omega)} \tag{2.10}$$
with a constant $C = C(\tau(\Omega), r, q, \delta) > 0$ and with any $\delta > 0$.

Let us have a close look at the test function space $\mathcal{T}^{1,r',q'} = \mathcal{T}^{1,r',q'}(T,\Omega)$ in Definition 1.1 and the functionals $\mathcal{F}$, $\mathcal{K}$, see (1.2), (1.3), respectively.

Lemma 2.3. *Let $1 < r, q < \infty$, $0 < T < \infty$ and $\Omega \subseteq \mathbb{R}^n$ be a C^2-domain.*

(i) For every $v \in L^{r'}(0,T;\tilde{L}^{q'}_\sigma(\Omega))$ there exists a unique solution $\phi = \phi(v) \in \mathcal{T}^{1,r',q'}$ to the backward Stokes equation

$$-\phi_t + \tilde{A}_q \phi = v \text{ on } (0,T), \quad \phi(T) = 0.$$

It is represented by the formula

$$\phi(v)(T-t) = \int_0^t e^{-(t-\tau)\tilde{A}_{q'}} v(T-\tau)\, d\tau.$$

The map $v \mapsto \phi(v)$ is linear and satisfies with a constant $C = C(q,r,T,\tau(\Omega)) > 0$ the bound

$$\|\phi(v)\|_{\mathcal{T}^{1,r',q'}(T,\Omega)} \le C\|v\|_{L^{r'}(0,T;\tilde{L}^{q'}(\Omega))}.$$

Moreover, for every $v \in L^{r'}(0,T;\tilde{L}^{q'}(\Omega))$ there exists a unique solution $\phi(v)$ and an associated pressure $\psi = \psi(v)$ with $\nabla\psi \in L^{r'}(0,T;\tilde{G}^{q'}(\Omega))$ such that

$$-\phi_t - \Delta\phi + \nabla\psi = v \text{ on } (0,T), \quad \phi(T) = 0,$$

satisfying with a constant $C = C(q,r,T,\tau(\Omega)) > 0$ the estimate

$$\|\phi(v)_t;\ \nabla^2\phi(v);\ \nabla\psi(v)\|_{L^{r'}(0,T;\tilde{L}^{q'}(\Omega))} \le C\|v\|_{L^{r'}(0,T;\tilde{L}^{q'}(\Omega))}.$$

(ii) Assume that $1 < \rho_1, \gamma_1 < \infty$ satisfy $\frac{2}{\rho_1} + \frac{n}{\gamma_1} = 2 + \left(\frac{2}{r} + \frac{n}{q}\right)$, and that $\frac{1}{q} < \frac{1}{\gamma_1} \le \frac{1}{q} + \frac{2}{n}$. Then $\mathcal{T}^{1,r',q'} \hookrightarrow L^{\rho_1'}(0,T;\tilde{L}^{\gamma_1'})$, and for each $v \in L^{r'}(0,T;\tilde{L}^{q'})$

$$\|\phi(v)\|_{L^{\rho_1'}(0,T;\tilde{L}^{\gamma_1'}(\Omega))} \le C\|v\|_{L^{r'}(0,T;\tilde{L}^{q'}(\Omega))}. \tag{2.11}$$

(iii) Assume that $1 < \rho_2, \gamma_2 < \infty$ satisfy $\frac{2}{\rho_2} + \frac{n}{\gamma_2} = 1 + \left(\frac{2}{r} + \frac{n}{q}\right)$, and that $\frac{1}{q} \le \frac{1}{\gamma_2} \le \frac{1}{q} + \frac{1}{n}$. Then $\nabla\mathcal{T}^{1,r',q'} \hookrightarrow L^{\rho_2'}(0,T;\tilde{L}^{\gamma_2'})$, and for each $v \in L^{r'}(0,T;\tilde{L}^{q'})$

$$\|\nabla\phi(v)\|_{L^{\rho_2'}(0,T;\tilde{L}^{\gamma_2'}(\Omega))} \le C\|v\|_{L^{r'}(0,T;\tilde{L}^{q'}(\Omega))}. \tag{2.12}$$

Proof. (i) follows directly from the maximal regularity of the Stokes equation, cf. Theorem 2.2, and a variable transformation $\tilde{t} := T - t$.

(ii) First let $\frac{1}{\gamma_1} = \frac{1}{q} + \frac{2}{n}$. Then the embedding $\tilde{D}^1_{q'} \hookrightarrow \tilde{L}^{\gamma_1'}$, see (2.8), implies that $\mathcal{T}^{1,r',q'} \hookrightarrow L^{r'}(0,T;\tilde{D}^1_{q'}) \hookrightarrow L^{r'}(0,T;\tilde{L}^{\gamma_1'})$ and that (2.11) holds.

Next we assume that $\frac{1}{q} \le \frac{1}{\gamma_1} < \frac{1}{q} + \frac{2}{n}$. With the embedding estimate (2.9), $\alpha_1 := \frac{n}{2}\big(\frac{1}{\gamma_1} - \frac{1}{q}\big) \in [0,1)$, and the Hardy-Littlewod-Sobolev inequality we get that

$$\begin{aligned}
\|\phi(v)\|_{L^{\rho_1'}(\tilde L^{\gamma_1'})} &\le \left(\int_0^T \left(\int_0^t \big\|e^{-(t-\tau)\tilde A_{\gamma_1'}} v(T-\tau)\big\|_{\tilde L^{\gamma_1'}}\,d\tau\right)^{\rho_1'} dt\right)^{1/\rho_1'}\\
&\le C\left(\int_0^T \left(\int_0^T |t-\tau|^{-\alpha_1}\|v(T-\tau)\|_{\tilde L^{q'}}\,d\tau\right)^{\rho_1'} dt\right)^{1/\rho_1'}\\
&\le C\|v\|_{L^{r'}(\tilde L^{q'})}.
\end{aligned}$$

(iii) Assume first that $\frac{1}{\gamma_2} = \frac{1}{q} + \frac{1}{n}$. Since $\tilde D_q^{1/2} \hookrightarrow \tilde W^{1,q}(\Omega)$, due to (2.8) with $\alpha = \frac{1}{2}$ and u replaced by $(1+\tilde A_q)^{1/2}\phi(v)$ where $v \in L^{r'}(\tilde L^{q'})$ we get the estimate

$$\begin{aligned}
\|\nabla\phi(v)\|_{L^{r'}(\tilde L^{\gamma_2'})} &\le C\|(1+\tilde A_{\gamma_2'})^{1/2}\phi(v)\|_{L^{r'}(\tilde L^{\gamma_2'})}\\
&\le C\|(1+\tilde A_{q'})\phi(v)\|_{L^{r'}(\tilde L^{q'})} \le C\|v\|_{L^{r'}(\tilde L^{q'})}.
\end{aligned}$$

Finally, we assume $\frac{1}{q} \le \frac{1}{\gamma_2} < \frac{1}{q} + \frac{1}{n}$. It suffices to show that $\nabla\phi(v) \in L^{\rho_2'}(\tilde L^{\gamma_2'})$. Indeed, with $\alpha_2 := \frac{n}{2}\big(\frac{1}{\gamma_2} - \frac{1}{q}\big) \in \big[0, \frac{1}{2}\big)$ we obtain that

$$\begin{aligned}
\|\nabla\phi(v)\|_{L^{\rho_2'}(\tilde L^{\gamma_2'})} &\le \left(\int_0^T \left(\int_0^t \|\nabla e^{-(t-\tau)\tilde A_{\gamma_2'}} v(T-\tau)\|_{\tilde L^{\gamma_2'}} d\tau\right)^{\rho_2'} dt\right)^{1/\rho_2'}\\
&\le \left(\int_0^T \left(\int_0^T |t-\tau|^{-\alpha_2-1/2}\|v(T-\tau)\|_{\tilde L^{q'}} d\tau\right)^{\rho_2'} dt\right)^{1/\rho_2'}.
\end{aligned}$$

Then the Hardy-Littlewood-Sobolev inequality implies that $\|\nabla\phi(v)\|_{L^{\rho_2'}(\tilde L^{\gamma_2'})} \le C\|v\|_{\tilde L^{r'}(\tilde L^{q'})}$. □

The functionals $\mathcal F \in \mathcal T^{-1,r,q}(T,\Omega)$ and $\mathcal K \in L^r(0,T;\tilde G_q^{-1}(\Omega))$ will have the following canonical forms (cf. (1.2), (1.3)):

$$\langle \mathcal F, \phi\rangle = (u_0, \phi(0))_\Omega + (f_1,\phi)_{T,\Omega} + (f_2, \nabla\phi)_{T,\Omega} - (g, N\cdot\nabla\phi)_{T,\partial\Omega} \tag{2.13}$$
$$\langle \mathcal K, \psi\rangle = (g, \psi N)_{T,\partial\Omega} - (k,\psi)_{T,\Omega} \tag{2.14}$$

for every $\phi \in \mathcal T^{1,r',q'}(T,\Omega)$ and $\nabla\psi \in L^{r'}(0,T;\tilde G_{q'}(\Omega))$, respectively. Related to the Navier-Stokes system a typical force term to be considered is $f_2 = u\otimes u$. Sufficient conditions on the functions u_0, f_1, f_2 and g and k will be specified in the sequel.

Proposition 2.4. *Let $\Omega \subset \mathbb R^n$ be a uniform C^2-domain, $0 < T < \infty$, and let Serrin exponents $2 < r < \infty$, $n < q < \infty$, $\frac{2}{r} + \frac{n}{q} = 1$, $n \ge 3$, be given. Then* *the following conditions on u_0, f_1, f_2 and g, k are sufficient for $\mathcal{F}$ and $\mathcal{K}$ to be contained in the data spaces $\mathcal{T}^{-1,r,q}(T,\Omega)$ and $L^r(0,T;\tilde{G}_q^{-1}(\Omega))$, respectively.*

(i) The "optimal" condition on u_0 in terms of real interpolation theory is

$$u_0 \in \big(\tilde{D}_q^{-1}, \tilde{L}_\sigma^q(\Omega)\big)_{1/r',r},$$

i.e. $u_0 \in \tilde{D}_q^{-1}$ and $\int_0^T \|e^{-t\tilde{A}_q}u_0\|_{\tilde{L}^q}^r\, dt < \infty$.

In particular, the conditions $u_0 \in \tilde{L}_\sigma^\rho(\Omega)$ and $\int_0^T \|e^{-t\tilde{A}_\rho}u_0\|_{\tilde{L}^q}^r\, dt < \infty$ for some $1<\rho<\infty$ imply that $u_0 \in \big(\tilde{D}_q^{-1}, \tilde{L}_\sigma^q(\Omega)\big)_{1/r',r}$.

Moreover, $u_0 \in \tilde{L}_\sigma^{n,r}(\Omega)$ and, if even $r \ge n \ge 3$, $u_0 \in \tilde{L}_\sigma^n(\Omega)$ are sufficient conditions.

For $n=3$ the L^2-conditions $u_0 \in L_\sigma^2(\Omega)$ together with

$$\int_0^T \big\|(1+\tilde{A}_2)^{\frac{1}{4}+\frac{1}{r}}e^{-t\tilde{A}_2}u_0\big\|_2^r\, dt < \infty$$

are sufficient. This holds e.g. when $u_0 \in \tilde{D}_2^{1/4}$.

(ii) For f_1 the condition $f_1 \in L^1(0,T;\tilde{L}^q(\Omega))$ as well as $f_1 \in L^{\rho_1}(0,T;\tilde{L}^{\gamma_1}(\Omega))$, where $1<\rho_1,\gamma_1<\infty$, $\frac{2}{\rho_1}+\frac{n}{\gamma_1}=3$ and $\frac{1}{q}<\frac{1}{\gamma_1}\le\frac{1}{q}+\frac{2}{n}$, is sufficient.

In particular, when $n=3$, this includes the condition $f_1 \in L^{4/3}(0,T;L^2(\Omega))$.

(iii) For f_2 it suffices to require the condition $f_2 \in L^{\rho_2}(0,T;\tilde{L}^{\gamma_2}(\Omega))$, where $1<\rho_2,\gamma_2<\infty$, $\frac{2}{\rho_2}+\frac{n}{\gamma_2}=2$, and $\frac{1}{q}\le\frac{1}{\gamma_2}\le\frac{1}{q}+\frac{1}{n}$.

In the L^2-context over $\mathbb{R}^3$, we require $f_2 \in L^4(0,T;L^2(\Omega))$ if $q \le 6$ (or $r \ge 4$).

(iv) In view of $\mathcal{F}$ and the boundary value g it is sufficient to require that $g \in L^r(0,T;L^\rho(\partial\Omega)\cap L^\gamma(\partial\Omega))$ where $\frac{q}{n'}\le\rho\le q$ and $\frac{2}{n'}\le\gamma\le 2$.

Concerning the functional $\mathcal{K}$ the assumptions that $g\cdot N$ has compact support in $\partial\Omega$, $g\cdot N \in L^r(0,T;L^{q/n'}(\partial\Omega))$ and $\int_{\partial\Omega} g\cdot N\, dS = 0$ are sufficient. No assumption on the tangential part of g is needed.

(v) Let $k \in L^r(0,T;L^q(\Omega))$ have compact support in $\overline{\Omega}$ and satisfy $\int_\Omega k\, dx = 0$. Then the functional $\psi \mapsto (k,\psi)_{T,\Omega}$ is contained in $L^{r'}(0,T;\tilde{G}_{q'}^{-1}(\Omega))$.

Proof. (i) We must show that the functional $(u_0,\phi(0))_\Omega$ is bounded in $\phi \in \mathcal{T}^{1,r',q'}(T,\Omega)$. The "optimal" condition is determined by the optimal space for the trace $\phi(0)$, i.e., by the real interpolation space $(\tilde{L}_\sigma^{q'}, \tilde{D}_{q'}^1)_{1/r,r'}$, cf. [2, Theorem III.4.10.2]. Hence the optimal space for u_0 is the dual space of $(\tilde{L}_\sigma^{q'}, \tilde{D}_{q'}^1)_{1/r,r'}$. By the duality theorem for the real interpolation method, cf. [30, Theorem 1.11.2], $(\tilde{L}_\sigma^q, (\tilde{D}_{q'}^1)^*)_{1/r,r} = (\tilde{D}_q^{-1}, \tilde{L}_\sigma^q)_{1/r',r}$ using the duality relation $(\tilde{D}_{q'}^1)^* = \tilde{D}_q^{-1}$. Since $\tilde{D}_q^1 = \mathcal{D}(I+\tilde{A}_q)$, $(I+\tilde{A}_q)^{-1}\tilde{D}_q^{-1} \cong \tilde{L}_\sigma^q$ and $I+\tilde{A}_q$ generates the exponentially decreasing analytic semigroup $e^{-t}e^{-t\tilde{A}_q}$, the condition $u_0 \in (\tilde{D}_q^{-1}, \tilde{L}_\sigma^q)_{1/r',r}$ or equivalently $(I+\tilde{A}_q)^{-1}u_0 \in (\tilde{L}_\sigma^q, \tilde{D}_q^1)_{1/r',r}$ is characterized by the finiteness of the norm

$$\|(I+\tilde{A}_q)^{-1}u_0\|_{\tilde{L}^q} + \left(\int_0^\infty \big\|(I+\tilde{A}_q)e^{-t(I+\tilde{A}_q)}(I+\tilde{A}_q)^{-1}u_0\big\|_{\tilde{L}^q}^r\, dt\right)^{1/r}.$$

By [30, Theorem 1.14.5] this norm is equivalent to the norm $\left(\int_0^T \left\|e^{-t\tilde{A}_q}u_0\right\|_{\tilde{L}^q}^r dt\right)^{1/r}$.

If $\int_0^T \left\|e^{-t\tilde{A}_\rho}u_0\right\|_{\tilde{L}^q}^r dt < \infty$, then obviously $u_0 \in (\tilde{D}_q^{-1}, \tilde{L}_\sigma^q)_{1/r',r}$.

The next two conditions are immediate consequences of [24, Theorem 2]. If $u_0 \in \tilde{L}_\sigma^{n,r}(\Omega)$ or $u_0 \in \tilde{L}_\sigma^n(\Omega)$ and $r \geq n \geq 3$, then $\int_0^T \left\|e^{-t\tilde{A}_n}u_0\right\|_{\tilde{L}^q(\Omega)}^r dt$ is finite.

Now let $n = 3$. For the L^2-condition we use the embedding estimate (2.8) to get that $\|e^{-t\tilde{A}_2}u_0\|_{\tilde{L}^q} \leq C\|(1+\tilde{A}_2)^{\frac{1}{4}+\frac{1}{r}}e^{-t\tilde{A}_2}u_0\|_2$ and continue as above.

Finally, if $u_0 \in \tilde{D}_2^{1/4}$, we follow the proof of [29, Lemma IV.1.5.3]. With $v_0 := (1+\tilde{A}_2)^{1/4}u_0$, we note that $r > 2$ and get from the moment inequality $\|(I+\tilde{A}_q)^\alpha v\|_{\tilde{L}^q} \leq C\|v\|_{\tilde{L}^q}^{1-\alpha}\|(I+\tilde{A}_q)v\|_{\tilde{L}^q}^\alpha$, that

$$
\begin{aligned}
\int_0^T \left\|(1+\tilde{A}_2)^{\frac{1}{r}+\frac{1}{4}}e^{-t\tilde{A}_2}u_0\right\|_2^r dt &= \int_0^T \left\|((1+\tilde{A}_2)^{1/2})^{2/r}e^{-t\tilde{A}_2}v_0\right\|_2^r dt \\
&\leq C\int_0^T \left\|(1+\tilde{A}_2)^{1/2}e^{-t\tilde{A}_2}v_0\right\|_2^2 \left\|e^{-t\tilde{A}_2}v_0\right\|_2^{r-2} dt \\
&\leq C\|v_0\|_2^{r-2}\int_0^T \left\|(1+\tilde{A}_2)^{1/2}e^{-t\tilde{A}_2}v_0\right\|_2^2 dt.
\end{aligned}
$$

The integral on the right-hand side is estimated as follows:

$$
\begin{aligned}
\int_0^T \left\|(1+\tilde{A}_2)^{1/2}e^{-t\tilde{A}_2}v_0\right\|_2^2 dt &= \int_0^T \Big(\big(e^{-2t\tilde{A}_2}v_0, v_0\big)_\Omega + \big(\tilde{A}_2 e^{-2t\tilde{A}_2}v_0, v_0\big)_\Omega\Big)\, dt \\
&= \int_0^T \|e^{-t\tilde{A}_2}v_0\|_2^2\, dt - \frac{1}{2}\int_0^T \frac{d}{dt}\big(e^{-2t\tilde{A}_2}v_0, v_0\big)_\Omega\, dt \\
&\leq \|v_0\|_2^2 + \frac{1}{2}\Big((v_0, v_0)_\Omega - (e^{-2T\tilde{A}_2}v_0, v_0)_\Omega\Big) \\
&\leq C\|v_0\|_2^2.
\end{aligned}
$$

Combining these estimates we get that $\int_0^T \left\|(1+\tilde{A}_2)^{\frac{1}{r}+\frac{1}{4}}e^{-t\tilde{A}_2}u_0\right\|_2^r dt \leq C\|u_0\|_{\tilde{D}_2^{1/4}}^r$.

(ii) Note that $\mathcal{T}^{1,r',q'} \hookrightarrow L^\infty(0,T;\tilde{L}^{q'})$ since $\phi(T) = 0$ and $\phi_t \in L^{r'}(0,T;\tilde{L}^{q'})$ for $\phi \in \mathcal{T}^{1,r',q'}$. Hence $f_1 \in L^1(0,T;\tilde{L}^q)$ defines a bounded functional on $\mathcal{T}^{1,r',q'}$.

The other assertions follow immediately from Lemma 2.3 (ii).

(iii) These assertions evidently follow from Lemma 2.3 (iii).

(iv) For $\phi \in \mathcal{T}^{1,r',q'}(T,\Omega)$ note, since $q' < 2$, that $\nabla\phi \in L^{r'}(0,T;W^{1,q'}(\Omega) + W^{1,2}(\Omega))$. Therefore, for each $\varepsilon > 0$, we find $\vartheta_1 \in L^{r'}(0,T;W^{1,q'}(\Omega))$ and $\vartheta_2 \in L^{r'}(0,T;W^{1,2}(\Omega))$ such that $\nabla\phi = \vartheta_1 + \vartheta_2$ and

$$
\|\vartheta_1\|_{L^{r'}(W^{1,q'})} + \|\vartheta_2\|_{L^{r'}(W^{1,2})} \leq \|\nabla\phi\|_{L^{r'}\tilde{W}^{1,q'})} + \varepsilon.
$$

We can now estimate:

$$
|(g, N\cdot\nabla\phi)_{T,\partial\Omega}| \leq \|g\|_{L^r(L^\rho(\partial\Omega))}\|\vartheta_1\|_{L^{r'}(L^{\rho'}(\partial\Omega))} + \|g\|_{L^r(L^\gamma(\partial\Omega))}\|\vartheta_2\|_{L^{r'}(L^{\gamma'}(\partial\Omega))}.
$$

Note that $q' \le \rho' \le \frac{(n-1)q'}{n-q'}$ and $2 \le \gamma' \le \frac{(n-1)2}{n-2}$. Therefore, using trace estimates (cf. [1, Theorem 5.36]) we get that

$$\|\vartheta_1\|_{L^{r'}(0,T;L^{\rho'}(\partial\Omega))} \le C\|\vartheta_1\|_{L^{r'}(0,T;W^{1,q'}(\Omega))}$$
$$\|\vartheta_2\|_{L^{r'}(0,T;L^{\gamma'}(\partial\Omega))} \le C\|\vartheta_2\|_{L^{r'}(0,T;W^{1,2}(\Omega))},$$

implying that

$$|(g, N\cdot\nabla\phi)_{\partial\Omega}| \le C\big(\|g\|_{L^r(L^\rho(\partial\Omega))} + \|g\|_{L^r(L^\gamma(\partial\Omega))}\big)\big(\|\nabla\phi\|_{L^{r'}(\tilde{W}^{1,q'})} + \varepsilon\big)$$

for all $\varepsilon > 0$, and thus also for $\varepsilon = 0$. This finishes the proof.

Concerning the functional $\mathcal{K}$ and corresponding assumptions on g let $S \subset \partial\Omega$ denote the compact support of $g \cdot N$ in $\partial\Omega$. Choose a bounded Lipschitz domain $\Omega_0 \subset \Omega$ with the property that $S \subset \partial\Omega_0$. Then for a.a. $t \in [0,T)$ there exists a constant $m(t) \in \mathbb{R}$ (measurable in t) such that $\|\psi(t) - m(t)\|_{L^{q'}(\Omega_0)} \le C\|\nabla\psi(t)\|_{L^{q'}(\Omega_0)}$ with some constant C depending on Ω_0 and q only. Hence, for any $\nabla\psi \in L^{r'}(0,T;\tilde{G}_{q'}(\Omega))$ and for a.a. $t \in [0,T)$,

$$\begin{aligned}|(g(t), \psi(t)N)_{\partial\Omega}| &\le |(g(t)\cdot N, \psi(t))_S| = |(g(t)\cdot N, \psi(t) - m(t))_S| \\ &\le \|g(t)\|_{L^{q/n'}(S)}\|\psi(t) - m(t)\|_{L^{(q/n')'}(S)} \\ &\le \|g(t)\|_{L^{q/n'}(\partial\Omega)}\|\psi(t) - m(t)\|_{L^{(q/n')'}(\partial\Omega_0)}.\end{aligned}$$

By the trace estimate we can continue by

$$\|\psi(t) - m(t)\|_{L^{(q/n')'}(\partial\Omega_0)} \le C\|\psi(t) - m(t)\|_{W^{1,q'}(\Omega_0)} \le C\|\nabla\psi(t)\|_{L^{q'}(\Omega_0)}.$$

Finally, since Ω_0 is a bounded subset of Ω, the norm $\|\nabla\psi(t)\|_{L^{q'}(\Omega_0)}$ can be estimated by $C\|\nabla\psi(t)\|_{\tilde{L}^{q'}(\Omega_0)} \le C\|\nabla\psi(t)\|_{\tilde{L}^{q'}(\Omega)}$. Summarizing these estimates we conclude that

$$|(g, \psi N)_{T,\partial\Omega}| \le C\|g\|_{L^r(0,T;L^{q/n'}(\partial\Omega))}\|\psi\|_{L^{r'}(0,T;\tilde{G}_{q'}(\Omega))},$$

and get the assertion.

(v) Let K denote the support of k, which is compact in $\overline{\Omega}$. Then for a.a. $t \in [0,T)$ there exists $m(t) \in \mathbb{R}$ such that $\|\psi(t) - m(t)\|_{L^{q'}(K)} \le C\|\nabla\psi(t)\|_{\tilde{L}^{q'}(\Omega)}$. We thus get that

$$|(k(t), \psi(t))_\Omega| = |(k(t), \psi(t) - m(t))_K| \le C\|k(t)\|_{L^q}\|\nabla\psi(t)\|_{\tilde{L}^{q'}}$$

which leads to the estimate $|(k,\psi)_{T,\Omega}| \le C\|k\|_{L^r(0,T;L^q)}\|\nabla\psi\|_{L^{r'}(0,T;\tilde{L}^{q'})}$. □

Remark 2.5. *Assume that $\mathcal{F} \in \mathcal{T}^{-1,r,q}(T,\Omega)$ and $\mathcal{K} \in L^r(0,T;\tilde{G}_q^{-1}(\Omega))$ are given. Then we can restrict $\mathcal{F}$ and $\mathcal{K}$ to smaller time intervals. For $\mathcal{K}$ it is clear how this is to be understood. For $\mathcal{F}$ we note that for the test function spaces it holds that $\mathcal{T}^{1,r',q'}(T',\Omega) \hookrightarrow \mathcal{T}^{1,r',q'}(T,\Omega)$ for $0 < T' \le T$ in the sense that we can just extend any $\phi \in \mathcal{T}^{1,r,q}(T',\Omega)$ by 0 to the larger time interval $[0,T]$, hereby preserving the norm. Hence we can restrict the functional $\mathcal{F}$ to those functions ϕ which vanish on $[T',T]$. We will write $\mathcal{F}|_{[0,T']}$ for this restriction.*

For later use we need two further technical lemmata.

Proposition 2.6. *Let $\Omega \subseteq \mathbb{R}^n$ be a uniform C^2-domain, $0 < T < \infty$, and $1 < r, q < \infty$. Let $\mathcal{F} \in \mathcal{T}^{-1,r,q}(T,\Omega)$ and $\mathcal{K} \in L^r(0,T;\tilde{G}_q^{-1}(\Omega))$ be given. Then for any $\varepsilon > 0$ there is a $0 < T' < T$ such that*

$$\|\mathcal{F}|_{[0,T']}\|_{\mathcal{T}^{-1,r,q}(T',\Omega)} + \|\mathcal{K}\|_{L^r(0,T';\tilde{G}_q^{-1}(\Omega))} < \varepsilon.$$

Proof. First we need to find a better representation for $\|\mathcal{F}\|_{\mathcal{T}^{-1,r,q}(T,\Omega)}$. Recall that the test function space $\mathcal{T}^{1,r',q'}(T,\Omega)$ is a closed subspace of $L^{r'}(0,T;\tilde{D}^1_{q'}) \cap W^{1,r'}(0,T;\tilde{L}^{q'}_\sigma(\Omega))$. By the Hahn-Banach theorem there is a norm preserving extension $\hat{\mathcal{F}}$ of $\mathcal{F}$ onto that space, i.e.

$$\begin{aligned}\hat{\mathcal{F}} &\in \big(L^{r'}(0,T;\tilde{D}^1_{q'}) \cap W^{1,r'}(0,T;\tilde{L}^{q'}_\sigma(\Omega))\big)^* \\ &= \big(L^{r'}(0,T;\tilde{D}^1_{q'})\big)^* + \big(W^{1,r'}(0,T;\tilde{L}^{q'}_\sigma(\Omega))\big)^*.\end{aligned}$$

Hence we find a representation

$$\langle \mathcal{F}, \phi\rangle = \int_0^T \big(\langle f(t), \phi(t)\rangle_{\tilde{D}_q^{-1},\tilde{D}^1_{q'}} + \langle f_1(t), \phi(t)\rangle_{\tilde{L}^q_\sigma,\tilde{L}^{q'}_\sigma\rangle} + \langle f_2(t), \phi_t(t)\rangle_{\tilde{L}^q_\sigma,\tilde{L}^{q'}_\sigma}\big)\, dt$$

for all $\phi \in \mathcal{T}^{1,r',q'}(T,\Omega)$ with functions $f \in L^r(0,T;\tilde{D}_q^{-1})$, f_1, $f_2 \in L^r(0,T;\tilde{L}^q_\sigma(\Omega))$. Consequently, on $(0,T')$, $0 < T' < T$, we obtain the estimate

$$\|\mathcal{F}|_{[0,T']}\|_{\mathcal{T}^{1,r,q}(T',\Omega)} \leq \|f\|_{L^r(0,T';\tilde{D}_q^{-1})} + \|f_1\|_{L^r(0,T';\tilde{L}^q)} + \|f_2\|_{L^r(0,T';\tilde{L}^q)}.$$

The right hand side tends to 0 as $T' \to 0$ by Lebesgue's theorem on dominated convergence. It is also clear that $\|\mathcal{K}\|_{L^r(0,T';\tilde{G}_q^{-1}(\Omega))} \to 0$. □

Lemma 2.7. *Let $1 < r_1, r_2, q_1, q_2 < \infty$ and $\alpha \geq 0$.*
(i) The space $L^{r_1}(0,T;\tilde{L}^{q_1}_\sigma(\Omega)) \cap L^{r_2}(0,T;\tilde{D}^{-\alpha}_{q_2})$ is dense in $L^{r_2}(0,T;\tilde{D}^{-\alpha}_{q_2})$.
(ii) The space $\mathcal{T}^{1,r_1,q_1}(T,\Omega) \cap \mathcal{T}^{1,r_2,q_2}(T,\Omega)$ is dense in $\mathcal{T}^{1,r_1,q_1}(T,\Omega)$.

Proof. (i) The set of simple functions $v = \sum_{j=1}^N \chi_{M_j} v_j$, $N \in \mathbb{N}$, with pairwise disjoint measurable subsets M_j of $[0,T)$ and $v_j \in \tilde{D}^{-\alpha}_{q_2}$ is dense in $L^{r_2}(0,T;\tilde{D}^{-\alpha}_{q_2})$ by the definition of Bochner spaces. Also, the space $C^\infty_{0,\sigma}(\Omega)$ is dense in $\tilde{D}^{-\alpha}_{q_2}$, since it is dense in $\tilde{L}^{q_2}_\sigma(\Omega)$, which again is continuously and densely embedded into $\tilde{D}^{-\alpha}_{q_2}$. Combining, we find that the simple functions $\sum_j \chi_{M_j}\phi_j$ with $\phi_j \in C^\infty_{0,\sigma}(\Omega)$ are dense in $L^{r_2}(0,T;\tilde{D}^{-\alpha}_{q_2})$. In particular the intersection space above is dense.

(ii) Let $\phi \in \mathcal{T}^{1,r_1,q_1}(T,\Omega)$ and define $v := -\phi_t + \tilde{A}_{q_1}\phi \in L^{r_1}(0,T;\tilde{L}^{q_1}_\sigma(\Omega))$. For $\varepsilon > 0$ we find by part (i) of this lemma a function $v_\varepsilon \in L^{r_1}(0,T;\tilde{L}^{q_1}_\sigma(\Omega)) \cap L^{r_2}(0,T;\tilde{L}^{q_2}_\sigma(\Omega))$ such that $\|v - v_\varepsilon\|_{L^{r_1}(0,T;\tilde{L}^{q_1})} < \varepsilon$. For this v_ε we get that $\phi(v_\varepsilon) \in \mathcal{T}^{1,r_1,q_1} \cap \mathcal{T}^{1,r_2,q_2}$. Hence $\|\phi - \phi(v_\varepsilon)\|_{\mathcal{T}^{1,r_1,q_1}} \leq C\|v - v_\varepsilon\|_{L^{r_1}(\tilde{L}^{q_1})} \leq C\varepsilon$. This implies the claimed density. □

3 Very Weak Solutions to the Stokes System

Definition 3.1. *Let $\Omega \subseteq \mathbb{R}^n$ be a uniform C^2-domain. Let $1 < r, q < \infty$ and $0 < T < \infty$. For data $\mathcal{F} \in \mathcal{T}^{-1,r,q}(T,\Omega)$ and $\mathcal{K} \in L^r(0,T;\tilde{G}_q^{-1}(\Omega))$ we call $u \in L^r(0,T;\tilde{L}^q(\Omega))$ a* very weak solution *of the Stokes system with data $\mathcal{F}$, $\mathcal{K}$ if*

$$\begin{aligned} -(u,\phi_t)_{T,\Omega} - (u,\Delta\phi)_{T,\Omega} &= \langle \mathcal{F},\phi\rangle, \\ (u,\nabla\psi)_{T,\Omega} &= \langle \mathcal{K},\psi\rangle \end{aligned}$$

hold for all $\phi \in \mathcal{T}^{1,r',q'}(T,\Omega)$ and $\nabla\psi \in L^{r'}(0,T;\tilde{G}^{q'}(\Omega))$.

Proof of Theorem 1.2 For every $v \in L^{r'}(0,T;\tilde{L}^{q'})$ there exists by Lemma 2.3 (ii) a unique function $\phi(v) \in \mathcal{T}^{1,r',q'}$ and a unique gradient $\nabla\psi(v) \in L^{r'}(0,T;\tilde{G}^{q'})$ such that

$$v = -\phi(v)_t - \Delta\phi(v) + \nabla\psi(v).$$

These functions depend linearly on v and satisfy the maximal regularity estimate

$$\|\phi(v)\|_{\mathcal{T}^{1,r',q'}} + \|\nabla\psi(v)\|_{L^{r'}(\tilde{L}^{q'})} \le C\|v\|_{L^{r'}(\tilde{L}^{q'})}$$

with a constant $C = C(\tau(\Omega), r, q, T)$. Now we define $u \in L^r(0,T;\tilde{L}^q)$ via duality as the linear functional on $L^{r'}(0,T;\tilde{L}^{q'})$ acting for every v as

$$(u,v)_{T,\Omega} = \langle \mathcal{F},\phi(v)\rangle + \langle \mathcal{K},\psi(v)\rangle.$$

This vector field u is indeed a very weak solution, since for every $\phi \in \mathcal{T}^{1,r',q'}$ and $\nabla\psi \in L^{r'}(0,T;\tilde{G}^{q'})$ we have with $w = -\phi_t - \Delta\phi + \nabla\psi$ that

$$-(u,\phi_t)_{T,\Omega} - (u,\Delta\phi)_{T,\Omega} + (u,\nabla\psi)_{T,\Omega} = (u,w)_{T,\Omega} = \langle \mathcal{F},\phi\rangle + \langle \mathcal{K},\psi\rangle.$$

The *a priori* estimate follows from the observation that for every $v \in L^{r'}(0,T;\tilde{L}^{q'})$

$$\begin{aligned} |(u,v)_{T,\Omega}| &\le \|\mathcal{F}\|_{\mathcal{T}^{-1,r,q}}\|\phi(v)\|_{\mathcal{T}^{1,r',q'}} + \|\mathcal{K}\|_{L^r(0,T;\tilde{G}_q^{-1})}\|\nabla\psi(v)\|_{L^{r'}(0,T;\tilde{L}^{q'})} \\ &\le C\big(\|\mathcal{F}\|_{\mathcal{T}^{-1,r,q}} + \|\mathcal{K}\|_{L^r(\tilde{G}_q^{-1})}\big)\|v\|_{L^{r'}(\tilde{L}^{q'})}. \end{aligned}$$

It remains to prove uniqueness of very weak solutions. To this end assume that $\mathcal{F} = 0$ and $\mathcal{K} = 0$. Then we have for every $v \in L^{r'}(0,T;\tilde{L}^{q'})$ that

$$(u,v)_{T,\Omega} = \langle \mathcal{F},\phi(v)\rangle + \langle \mathcal{K},\psi(v)\rangle = 0,$$

proving that $u = 0$ a.e. This finishes the proof. □

We now focus on the case $\mathcal{K} = 0$, i.e., the very weak solution u is contained in the solenoidal space $L^r(0,T;\tilde{L}^q_\sigma(\Omega))$.

First of all, we give optimal conditions for the data $\mathcal{F}$ such that a very weak solution $u \in L^r(0,T;\tilde{L}^q_\sigma(\Omega))$ is contained in a space of the same type, but with different exponents.

Theorem 3.2. *Let $\Omega \subseteq \mathbb{R}^n$ be a uniform C^2-domain and $0 < T < \infty$. Assume that exponents $1 < r, q < \infty$ and a data functional $\mathcal{F} \in \mathcal{T}^{-1,r,q}(T, \Omega)$ are given and that $u \in L^r(0, T; \tilde{L}^q_\sigma(\Omega))$ is the uniquely determined very weak solution to the Stokes system with data $\mathcal{F}$ (and $\mathcal{K} = 0$). Also let $1 < r_0, q_0 < \infty$.*

Then $u \in L^{r_0}(0, T; \tilde{L}^{q_0}(\Omega))$ if and only if $\mathcal{F}$ is also contained in $\mathcal{T}^{-1,r_0,q_0}(T, \Omega)$.

Proof. First assume that $\mathcal{F} \in \mathcal{T}^{-1,r_0,q_0}$. We want to prove by duality arguments that $u \in L^{r_0}(0, T; \tilde{L}^{q_0})$. For any $v \in L^{r_0'}(0, T; \tilde{L}^{q_0'}_\sigma) \cap L^{r'}(0, T; \tilde{L}^{q'}_\sigma)$ we get by Lemma 2.3 a corresponding $\phi = \phi(v) \in \mathcal{T}^{1,r',q'}$. Then it holds that

$$(u, v)_{T,\Omega} = -(u, \phi_t)_{T,\Omega} + (u, \tilde{A}_{q'}\phi)_{T,\Omega} = \langle \mathcal{F}, \phi \rangle$$

for all such v, since u is a very weak solution. We get that

$$|(u, v)_{T,\Omega}| = |\langle \mathcal{F}, \phi \rangle| \le \|\mathcal{F}\|_{\mathcal{T}^{-1,r_0,q_0}} \|\phi\|_{\mathcal{T}^{1,r_0',q_0'}}$$

where by maximal regularity $\|\phi\|_{\mathcal{T}^{1,r_0',q_0'}} \le C\|v\|_{L^{r_0'}(\tilde{L}^{q_0'})}$. Hence, for any v, we get the estimate

$$|(u, v)_{T,\Omega}| \le C\|\mathcal{F}\|_{\mathcal{T}^{-1,r_0,q_0}} \|v\|_{L^{r_0'}(0,T;\tilde{L}^{q_0'})}.$$

Since by Lemma 2.7 (ii) (with $\alpha = 0$) the space $L^{r_0'}(0, T; \tilde{L}^{q_0'}_\sigma) \cap L^{r'}(0, T; \tilde{L}^{q'}_\sigma)$ is dense in $L^{r_0'}(0, T; \tilde{L}^{q_0'}_\sigma)$, we conclude that $u \in L^{r_0}(0, T; \tilde{L}^{q_0}_\sigma)$.

Now assume that $u \in L^{r_0}(0, T; \tilde{L}^{q_0})$. We want to prove that $\mathcal{F}$ must have been an element of $\mathcal{T}^{-1,r_0,q_0}$. For any $\phi \in \mathcal{T}^{1,r_0,q_0} \cap \mathcal{T}^{1,r,q}$ we can estimate

$$\begin{aligned}
|\langle \mathcal{F}, \phi \rangle| &\le |-(u, \phi_t)_{T,\Omega}| + |(u, \tilde{A}_{q_0'}\phi)_{T,\Omega}| \\
&\le \|u\|_{L^{r_0}(\tilde{L}^{q_0})} \big(\|\phi_t\|_{L^{r_0'}(\tilde{L}^{q_0'})} + \|\tilde{A}_{q_0'}\phi\|_{L^{r_0'}\tilde{L}^{q_0'})} \big) \\
&\le \|u\|_{L^{r_0}(\tilde{L}^{q_0})} \|\phi\|_{\mathcal{T}^{1,r_0',q_0'}}.
\end{aligned}$$

By density of $\mathcal{T}^{1,r,q} \cap \mathcal{T}^{1,r_0,q_0}$ in $\mathcal{T}^{1,r_0,q_0}$, cf. Lemma 2.7 (ii), we get $\mathcal{F} \in \mathcal{T}^{-1,r_0,q_0}$, finishing the proof. □

Now we discuss higher order differentiability in space. The following theorem is an extension of the preceding one, but the proof is essentially the same.

Theorem 3.3. *Let $\Omega \subseteq \mathbb{R}^n$ be a uniform C^2-domain, $0 < T < \infty$ and $1 < r, q < \infty$. Further, for $\mathcal{F} \in \mathcal{T}^{-1,r,q}(T, \Omega)$, let $u \in L^r(0, T; \tilde{L}^q_\sigma(\Omega))$ be the uniquely determined very weak solution to the Stokes system with data $\mathcal{F}$ (and $\mathcal{K} = 0$).*

If $\mathcal{F}$ satisfies for some $0 \le \alpha \le 1$ and $1 < r_*, q_* < \infty$ a bound of the form

$$|\langle \mathcal{F}, \phi(v) \rangle| \le C\|(1 + \tilde{A}_{q'})^{-\alpha} v\|_{L^{r_*'}(0,T;\tilde{L}^{q_*'}_\sigma(\Omega))} \tag{3.1}$$

for all $v \in L^{r'}(0, T; \tilde{L}^{q'}_\sigma(\Omega)) \cap L^{r_*'}(0, T; \tilde{D}^{-\alpha}_{q_*'})$ and corresponding $\phi(v) \in \mathcal{T}^{1,r',q'}(T, \Omega)$, then $u \in L^{r_*}(0, T; \tilde{D}^{\alpha}_{q_*})$.

Proof. For any function $v \in L^{r'}(0,T;\tilde{L}^{q'}_\sigma(\Omega)) \cap L^{r'_*}(0,T;\tilde{D}^{-\alpha}_{q'_*})$ and corresponding $\phi = \phi(v)$ we get by the assumptions on $\mathcal{F}$ that

$$|(u,v)_{T,\Omega}| = |-(u,\phi_t)_{T,\Omega} + (u,\tilde{A}_{q'}\phi)_{T,\Omega}| = |\langle \mathcal{F},\phi\rangle| \le C\|v\|_{L^{r'_*}(0,T;\tilde{D}^{-\alpha}_{q'_*})}.$$

Then Lemma 2.7 and the duality $\tilde{D}^{-\alpha}_{q'_*} = (\tilde{D}^{\alpha}_{q_*})^*$ prove that $u \in L^{r_*}(0,T;\tilde{D}^{\alpha}_{q_*})$. □

The condition in the above theorem is very abstract. To be more precise, let us consider the three-dimensional case only.

Proposition 3.4. *Let Ω, T and r, q, r_*, q_*, α be as in the preceding Theorem 3.3. Assume that a data functional $\mathcal{F} \in \mathcal{T}^{-1,r,q}(T,\Omega)$ is given in the special form*

$$\langle \mathcal{F},\phi\rangle = (u_0,\phi(0))_\Omega + (f_1,\phi)_{T,\Omega} + (f_2,\nabla\phi)_{T,\Omega}$$

for all $\phi \in \mathcal{T}^{1,r',q'}(T,\Omega)$. Then the following conditions on u_0, f_1, f_2 are sufficient to guarantee the estimate (3.1) in Theorem 3.3:

(i) Let $u_0 \in \tilde{L}^\rho_\sigma(\Omega)$ for some $\rho \in (1,\infty)$ and $\int_0^T \|(1+\tilde{A}_\rho)^\alpha e^{-\tau\tilde{A}_\rho}u_0\|^{r_*}_{\tilde{L}^{q_*}}\,d\tau < \infty$. Even the condition $u_0 \in \tilde{L}^\gamma_\sigma(\Omega)$ with $0<\gamma<\infty$, $0 < \frac{3}{\gamma} < \frac{2}{r_*} + \frac{3}{q_*} - 2\alpha$ is sufficient.

(ii) Let $q_* \ge 2$, $2\alpha < \frac{3}{q_*} + \frac{1}{2}$, and let $\gamma \in (1,\infty)$ defined by $\frac{1}{\gamma} := \frac{1}{4} + \frac{1}{2}\big(\frac{2}{r_*} + \frac{3}{q_*} - 2\alpha\big)$. Then the condition $f_1 \in L^\gamma(0,T;L^2(\Omega))$ is sufficient.

(iii) The condition $f_2 \in L^\gamma(0,T;L^2(\Omega))$ is sufficient, if $q_* \ge 2$, $2\alpha < \frac{3}{q_*} - \frac{1}{2}$ and if $\frac{1}{\gamma} = -\frac{1}{4} + \frac{1}{2}\big(\frac{2}{r_*} + \frac{3}{q_*} - 2\alpha\big) \in (0,1)$.

Proof. (i) For given $v \in L^{r'}(0,T;\tilde{L}^{q'}_\sigma(\Omega)) \cap L^{r'_*}(0,T;\tilde{D}^{-\alpha}_{q'_*})$ and corresponding $\phi = \phi(v) \in \mathcal{T}^{1,r',q'}$ Lemma 2.3 yields

$$\begin{aligned}(u_0,\phi(0))_\Omega &= \int_0^T \big(u_0, e^{-(T-\tau)\tilde{A}_{q'}}v(T-\tau)\big)_\Omega\,d\tau \\ &= \int_0^T \big((1+\tilde{A}_\rho)^\alpha e^{-\tau\tilde{A}_\rho}u_0, (1+\tilde{A}_{q'_*})^{-\alpha}v(\tau)\big)_\Omega\,d\tau.\end{aligned}$$

Then we estimate

$$|(u_0,\phi(0))_\Omega| \le \left(\int_0^T \big\|(1+\tilde{A}_\rho)^\alpha e^{-\tau\tilde{A}_\rho}u_0\big\|^{r_*}_{\tilde{L}^{q_*}}\,d\tau\right)^{1/r_*} \|v\|_{L^{r'_*}(0,T;\tilde{D}^{-\alpha}_{q'_*})}.$$

This proves the first condition for u_0.

Now assume that $u_0 \in \tilde{L}^\gamma_\sigma(\Omega)$, $0 < \frac{3}{\gamma} < \frac{2}{r_*} + \frac{3}{q_*} - 2\alpha$. We estimate, using (2.9),

$$\big\|(1+\tilde{A}_\rho)^\alpha e^{-\tau\tilde{A}_\rho}u_0\big\|_{\tilde{L}^{q_*}} \le C\tau^{-\alpha-\beta}\|u_0\|_{\tilde{L}^\gamma}$$

with $\beta = \frac{3}{2}\big(\frac{1}{\gamma} - \frac{1}{q_*}\big)$ if $q^* > \gamma$ (or with $\beta = 0$ if $\gamma \ge q_*$ using the Hölder inequality $\|\cdot\|_{\tilde{L}^{q_*}} \le \|\cdot\|_{\tilde{L}^\gamma}$) and continue with

$$\int_0^T \big\|(1+\tilde{A}_\rho)^\alpha e^{-\tau\tilde{A}_\rho}u_0\big\|^{r_*}_{\tilde{L}^{q_*}}\,d\tau \le C\int_0^T \tau^{-(\alpha+\beta)r_*}\,d\tau\;\|u_0\|^{r_*}_{\tilde{L}^\gamma}.$$

The right hand side is finite if and only if $\frac{3}{\gamma} < \frac{2}{r_*} + \frac{3}{q_*} - 2\alpha$.

(ii) Since $|(f_1, \phi)_{T,\Omega}| \le \|f_1\|_{L^\gamma(L^2)}\|\phi\|_{L^{\gamma'}(L^2)}$, we only estimate

$$
\begin{aligned}
\|\phi\|_{L^{\gamma'}(L^2)}^{\gamma'} &= \int_0^T \|\phi(T-t)\|_2^{\gamma'}\,dt \\
&\le \int_0^T \left(\int_0^t \left\|e^{-(t-\tau)\tilde{A}_{q'}} v(T-\tau)\right\|_2 d\tau\right)^{\gamma'} dt \\
&= \int_0^T \left(\int_0^t \left\|(1+\tilde{A}_{q'})^{\alpha} e^{-(t-\tau)\tilde{A}_{q'}} (1+\tilde{A}_{q'})^{-\alpha} v(T-\tau)\right\|_2 d\tau\right)^{\gamma'} dt \\
&\le C\int_0^T \left(\int_0^t \left\|(1+\tilde{A}_{q'})^{\alpha+\beta} e^{-(t-\tau)\tilde{A}_{q'}} (1+\tilde{A}_{q'})^{-\alpha} v(T-\tau)\right\|_{\tilde{L}^{q'_*}} d\tau\right)^{\gamma'} dt
\end{aligned}
$$

with $\beta = \frac{3}{2}\left(\frac{1}{q'_*} - \frac{1}{2}\right)$. Then, by the analyticity of the Stokes semigroup and the Hardy-Littlewood-Sobolev inequality,

$$
\begin{aligned}
\|\phi\|_{L^{\gamma'}(L^2)}^{\gamma'} &\le C\int_0^T \left(\int_0^T |t-\tau|^{-\alpha-\beta}\|v(T-\tau)\|_{\tilde{D}^{-\alpha}_{q'_*}}\,d\tau\right)^{\gamma'} dt \\
&\le C\|v\|^{\gamma'}_{L^{r'_*}(0,T;\tilde{D}^{-\alpha}_{q'_*})}.
\end{aligned}
$$

The condition $2\alpha < \frac{3}{q_*} + \frac{1}{2}$ is needed for $\alpha + \beta < 1$. This shows the assertion.

(iii) The proof is almost the same as for f_1. In this case we use that

$$
\begin{aligned}
\|\nabla\phi\|_{L^{\gamma'}(L^2)}^{\gamma'} &\le C\int_0^T \left(\int_0^T |t-\tau|^{-\alpha-\beta-1/2}\|v(T-\tau)\|_{\tilde{D}^{-\alpha}_{q'_*}} d\tau\right)^{\gamma'} dt \\
&\le C\|v\|^{\gamma'}_{L^{r'_*}(0,T;\tilde{D}^{-\alpha}_{q'_*})}.
\end{aligned}
$$

The condition $2\alpha < \frac{3}{q_*} - \frac{1}{2}$ is needed for $\alpha + \beta + \frac{1}{2} < 1$. □

4 The Navier-Stokes System

Proof of Theorem 1.3 (i) We consider the modified data functional $\phi \mapsto \langle \mathcal{F}, \phi\rangle + (\hat{f}_2(u,u), \nabla\phi)_{T,\Omega}$, where $\hat{f}_2(u_1,u_2) = u_1 \otimes u_2$, and show that $\hat{f}_2 \in \mathcal{T}^{-1,r,q}(T,\Omega)$. Indeed, $\hat{f}_2$ has the form of the function f_2 discussed in Proposition 2.4 (iii) with $\rho_2 = \frac{r}{2}$ and $\gamma_2 = \frac{q}{2}$. Hence by (2.12)

$$
\begin{aligned}
|(\hat{f}_2(u_1,u_2), \nabla\phi)_{T,\Omega}| &\le C\|u_1\|_{L^r(0,T;\tilde{L}^q)}\|u_2\|_{L^r(0,T;\tilde{L}^q)}\|\nabla\phi\|_{L^{(r/2)'}(\tilde{L}^{(q/2)'})} \\
&\le C\|u_1\|_{L^r(0,T;\tilde{L}^q)}\|u_2\|_{L^r(0,T;\tilde{L}^q)}\|v\|_{L^{r'}(\tilde{L}^{q'})} \qquad (4.1)
\end{aligned}
$$

and $\|v\|_{L^{r'}(\tilde{L}^{q'})} \le C\|\phi\|_{\mathcal{T}^{1,r',q'}}$ where $C = C(\tau(\Omega), r, q, \delta)e^{\delta T}$.

We define a map $S\colon L^r(0,T;\tilde{L}^q(\Omega)) \to L^r(0,T;\tilde{L}^q(\Omega))$ as follows: for $u \in L^r(0,T;\tilde{L}^q(\Omega))$ let $S(u)$ be the unique solution to

$$\begin{aligned} -(S(u),\phi_t)_{T,\Omega} - (S(u),\Delta\phi)_{T,\Omega} &= \langle \mathcal{F},\phi\rangle + (\hat{f}_2(u,u),\nabla\phi)_{T,\Omega}, \\ (S(u),\nabla\psi)_{T,\Omega} &= \langle \mathcal{K},\psi\rangle \end{aligned}$$

for all $\phi \in \mathcal{T}^{1,r',q'}$ and $\nabla\psi \in L^{r'}(0,T;\tilde{G}^{q'})$. Then we get the estimate

$$\|S(u)\|_{L^r(\tilde{L}^q)} \le C_1\big(\|\mathcal{F}\|_{\mathcal{T}^{-1,r,q}} + \|\mathcal{K}\|_{L^r(\tilde{G}_q^{-1})}\big) + C_1C_2\|u\|^2_{L^r(\tilde{L}^q)} \tag{4.2}$$

for all $u \in L^r(0,T;\tilde{L}^q)$ where $C_1 = C_1'e^{\delta T}$ is the constant coming from the linear system and $C_2 = C_2'e^{\delta T}$ is the constant from the control of the nonlinearity.

Since very weak solutions of the Navier-Stokes system are fixed points of the mapping S and vice versa, all we need to show is the existence of a fixed point using Banach's theorem. We put

$$\eta := \frac{3}{16C_1^2C_2}, \quad \rho := \frac{1}{4C_1C_2}.$$

Assume $\|u\|_{L^r(\tilde{L}^q)} \le \rho$. Then from (1.5) and (4.2) we see that

$$\begin{aligned} \|S(u)\|_{L^r(\tilde{L}^q)} &\le C_1\eta + C_1C_2\rho^2 \\ &\le \frac{3}{16C_1C_2} + \frac{1}{16C_1C_2} = \frac{1}{4C_1C_2} = \rho, \end{aligned}$$

which implies that S is a selfmap of the closed ball $\overline{B}_\rho(0) \subset L^r(0,T;\tilde{L}^q)$.

Now consider $u_1,u_2 \in \overline{B}_\rho(0)$. Then

$$\begin{aligned} -(S(u_1)-S(u_2),\phi_t)_{T,\Omega} - (S(u_1)-S(u_2),\Delta\phi)_{T,\Omega} &= (u_1\otimes u_1 - u_2\otimes u_2,\nabla\phi)_{T,\Omega}, \\ (S(u_1)-S(u_2),\nabla\psi)_{T,\Omega} &= 0 \end{aligned}$$

for all $\phi \in \mathcal{T}^{1,r',q'}$ and $\nabla\psi \in L^{r'}(\tilde{G}^{q'})$. Hence, from linear theory, we get

$$\begin{aligned} \|S(u_1)-S(u_2)\|_{L^r(\tilde{L}^q)} &\le C_1C_2\big(\|u_1\|_{L^r(\tilde{L}^q)} + \|u_2\|_{L^r(\tilde{L}^q)}\big)\|u_1-u_2\|_{L^r(\tilde{L}^q)} \\ &\le 2\rho C_1C_2\|u_1-u_2\|_{L^r(\tilde{L}^q)} = \frac{1}{2}\|u_1-u_2\|_{L^r(\tilde{L}^q)}, \end{aligned}$$

implying that S is strictly contractive on $\overline{B}_\rho(0)$. By Banach's fixed point theorem, there exists a unique fixed point $u \in \overline{B}_\rho(0)$, i.e. a very weak solution in this ball. Note that this does not yet imply uniqueness of very weak solutions globally. The very weak solution u satisfies $S(u) = u$ and hence

$$\begin{aligned} \|u\|_{L^r(\tilde{L}^q)} = \|S(u)\|_{L^r(\tilde{L}^q)} &\le C_1\big(\|\mathcal{F}\|_{\mathcal{T}^{-1,r,q}} + \|\mathcal{K}\|_{L^r(\tilde{G}_q^{-1})}\big) + C_1C_2\|u\|^2_{L^r(\tilde{L}^q)} \\ &\le C_1\big(\|\mathcal{F}\|_{\mathcal{T}^{-1,r,q}} + \|\mathcal{K}\|_{L^r(\tilde{G}_q^{-1})}\big) + \frac{1}{4}\|u\|_{L^r(\tilde{L}^q)}, \end{aligned}$$

yielding the *a priori* estimate (1.6) by absorption.

(ii) By Proposition 2.6 there exists a $T' \in (0,T)$ such that

$$\|\mathcal{F}|_{[0,T']}\|_{\mathcal{T}^{-1,r,q}(T',\Omega)} + \|\mathcal{K}\|_{L^r(0,T';\tilde{G}_q^{-1}(\Omega))} \leq \eta,$$

with $\eta > 0$ as in (i). Thus there is a very weak solution $u \in L^r(0,T';\tilde{L}^q(\Omega))$ on the possibly smaller time interval $[0,T')$ to the restricted data $\mathcal{F}|_{[0,T']}$, $\mathcal{K}|_{[0,T']}$. □

Theorem 4.1 (Uniqueness)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be a uniform C^2-domain, $0 < T < \infty$, and let Serrin exponents $2 < r < \infty$, $n < q < \infty$, $\frac{2}{r} + \frac{n}{q} = 1$ be given. Assume that $\mathcal{F} \in \mathcal{T}^{-1,r,q}(T,\Omega)$ and $\mathcal{K} \in L^r(0,T;\tilde{G}_q^{-1}(\Omega))$. Then there is at most one very weak solution $u \in L^r(0,T;\tilde{L}^q(\Omega))$ to the Navier-Stokes system with data $\mathcal{F}$, $\mathcal{K}$.*

Proof. Let u and v be very weak solutions to the Navier-Stokes system with data $\mathcal{F}$, $\mathcal{K}$. Put

$$T_{max} := \sup\{T_0 \colon u = v \text{ a.e. on } [0,T_0)\}$$

including $T_{max} = 0$ if $u \neq v$ on every interval $[0,T_0)$, $0 < T_0 \leq T$. We need to show that $T_{max} = T$. To this end we assume $T_{max} < T$ and derive a contradiction.

Fixing an arbitrary $T' \in (T_{max}, T]$, the difference $u - v$ of the two solutions solves the system

$$\begin{aligned} -(u-v,\phi_t)_{T',\Omega} - (u-v,\Delta\phi)_{T',\Omega} &= (u \otimes u - v \otimes v, \nabla\phi)_{T',\Omega}, \\ (u-v,\nabla\psi)_{T',\Omega} &= 0 \end{aligned}$$

for every $\phi \in \mathcal{T}^{1,r',q'}(T',\Omega)$ and $\nabla\psi \in L^{r'}(0,T';\tilde{G}^{q'}(\Omega))$. Consequently, by linear theory, we can estimate

$$\begin{aligned} \|u-v\|_{L^r(0,T';\tilde{L}^q)} &\leq C_1C_2\big(\|u \otimes (u-v)\|_{L^{r/2}(0,T';\tilde{L}^{q/2})} + \|(u-v)\otimes v\|_{L^{r/2}(0,T';\tilde{L}^{q/2})}\big) \\ &\leq C_1C_2\big(\|u\|_{L^r(T_{max},T';\tilde{L}^q)} + \|v\|_{L^r(T_{max},T';\tilde{L}^q)}\big)\|u-v\|_{L^r(0,T';\tilde{L}^q)} \end{aligned}$$

where $C_1 = C_1' e^{\delta T}$ and $C_2 = C_2' e^{\delta T}$ are constants as in the proof of Theorem 1.2. Choosing $T' > T_{max}$ such that

$$C_1C_2\big(\|u\|_{L^r(T_{max},T';\tilde{L}^q)} + \|v\|_{L^r(T_{max},T';\tilde{L}^q)}\big) < 1,$$

we obtain that $\|u-v\|_{L^r(0,T';\tilde{L}^q)} = 0$, which contradicts the maximality of T_{max}. Hence, $u = v$ a.e. on $[0,T)$. □

In the following analysis of regularity we will assume for simplicity that $\mathcal{K} = 0$, i.e., any very weak solution belongs to the solenoidal space $L^r(0,T;\tilde{L}^q_\sigma(\Omega))$.

Proposition 4.2. *Let a uniform C^2-domain $\Omega \subseteq \mathbb{R}^n$ and a finite time interval $[0,T)$ be given. Assume that $2 < r_1, r_2 < \infty$ and $n < q_1, q_2 < \infty$ are two pairs of Serrin exponents, i.e. $\frac{2}{r_1} + \frac{n}{q_1} = 1 = \frac{2}{r_2} + \frac{n}{q_2}$. Assume that $\mathcal{F} \in \mathcal{T}^{-1,r_1,q_1}(T,\Omega) \cap \mathcal{T}^{-1,r_2,q_2}(T,\Omega)$ (and $\mathcal{K} = 0$) and that $u \in L^{r_1}(0,T;\tilde{L}^{q_1}_\sigma(\Omega))$ is a very weak solution to the Navier-Stokes system.*

Then u is also contained in the Serrin class $L^{r_2}(0,T;\tilde{L}^{q_2}_\sigma(\Omega))$.

Proof. Let $v \in L^{r_1'}(0,T;\tilde{L}_\sigma^{q_1'}) \cap L^{r_2'}(0,T;\tilde{L}_\sigma^{q_2'})$. With $\phi(v) \in \mathcal{T}^{1,r_1',q_1'} \cap \mathcal{T}^{1,r_2',q_2'}$ we then see that

$$\begin{aligned}|(u,v)_{T,\Omega} &= | - (u,\phi_t)_{T,\Omega} + (u,\tilde{A}_{q_1}\phi)_{T,\Omega}| = |\langle F,\phi\rangle + (u\otimes u,\nabla\phi)_{T,\Omega}| \\ &\le C\|\mathcal{F}\|_{\mathcal{T}^{-1,r_2,q_2}}\|v\|_{L^{r_2'}(\tilde{L}^{q_2'})} + \|u\|^2_{L^{r_1}(\tilde{L}^{q_1})}\|\nabla\phi\|_{L^{(r_1/2)'}(\tilde{L}^{(q_1/2)'})}.\end{aligned}$$

In a first step assume that q_2 satisfies the relation

$$\frac{1}{q_2} < \frac{2}{q_1} < \frac{1}{q_2} + \frac{1}{n}, \tag{4.3}$$

cf. Lemma 2.3 (iii) with $\gamma_2 = q_1/2$, $q = q_2$ and $\rho_2 = r_1/2$, $r = r_2$. From (2.12) we conclude that $\|\nabla\phi\|_{L^{(r_1/2)'}(\tilde{L}^{(q_1/2)'})} \le C\|v\|_{L^{r_2'}(\tilde{L}^{q_2'})}$. Combining the estimates we see that

$$|(u,v)_{T,\Omega}| \le C(\|\mathcal{F}\|_{\mathcal{T}^{-1,r_2,q_2}} + \|u\|^2_{L^{r_1}(\tilde{L}^{q_2})})\|v\|_{L^{r_2'}(L^{q_2'})},$$

giving by a duality and density argument that $u \in L^{r_2}(0,T;\tilde{L}_\sigma^{q_2}(\Omega))$.

Now we have to get rid of the restriction on q_2, see (4.3). To this end, first observe that, by interpolation, the data $\mathcal{F}$ are contained in any $\mathcal{T}^{-1,\tilde{r},\tilde{q}}(T,\Omega)$, as long as $\tilde{q} \in [\min(q_1,q_2),\max(q_1,q_2)]$, where of course $\tilde{r}$ and $\tilde{q}$ are Serrin exponents, i.e. $2/\tilde{r} + n/\tilde{q} = 1$. This enables us to iterate the procedure.

Note that the iteratively defined sequences

$$a_0 := b_0 := \frac{1}{q_1} \in \left(0, \frac{1}{n}\right), \qquad a_{k+1} := 2a_k - \frac{1}{n}, \quad b_{k+1} := 2b_k$$

satisfy $a_k < a_{k-1} < 1/n$ and $0 < b_{k-1} < b_k$ for all $k \ge 1$ such that even $a_k \to -\infty$ and $b_k \to \infty$ as $k \to \infty$. Therefore, for some finite k_0, depending only on $a_0 = \frac{1}{q_1}$, we have the inclusion $\frac{1}{q_2} \in \left(0, \frac{1}{n}\right) \subseteq (a_k, b_k)$ for all $k \ge k_0$. This ensures that the above procedure can be used repeatedly, leading to the property $u \in L^{r_2}(0,T;\tilde{L}^{q_2}(\Omega))$ after a finite number of steps. □

Corollary 4.3. *If $\mathcal{F} \in \mathcal{T}^{-1,r_1,q_1}(T,\Omega) \cap \mathcal{T}^{-1,r_2,q_2}(T,\Omega)$, and if very weak solutions $u_1 \in L^{r_1}(0,T;\tilde{L}^{q_1}(\Omega))$ and $u_2 \in L^{r_2}(0,T;\tilde{L}^{q_2}(\Omega))$ are given, then $u_1 = u_2$ almost everywhere on $[0,T)$.*

Proof. This follows directly from the above theorem, the uniqueness theorem, cf. Theorem 4.1, and the fact that $\mathcal{T}^{1,r_1',q_1'} \cap \mathcal{T}^{1,r_2',q_2'}$ is dense in both $\mathcal{T}^{1,r_1',q_1'}$ and $\mathcal{T}^{1,r_2',q_2'}$, see Lemma 2.7. □

In the theory of the Navier-Stokes system it is a very important question whether a given solution u satisfies the condition $u \otimes u \in L^2(0,T;L^2(\Omega))$. In this case a solution may satisfy the energy equality rather than the energy inequality and very weak solutions may be identified with a weak solution to prove regularity results for weak solutions, cf. [16]. One sufficient condition in this step is clearly $u \in L^4(0,T;\tilde{L}^4(\Omega))$, at least for finite times T. We give here a sufficient condition for a very weak solution to be contained in that space.

Theorem 4.4. *Let $\Omega \subseteq \mathbb{R}^3$ be a uniform C^2-domain, $0 < T < \infty$, and let Serrin exponents $\frac{16}{5} \le r \le 16$, $\frac{24}{7} \le q \le 8$, $\frac{2}{r} + \frac{3}{q} = 1$ be given. Assume that $u \in L^r(0,T;\tilde{L}^q_\sigma(\Omega))$ is a very weak solution to the Navier-Stokes system with data $\mathcal{F} \in \mathcal{T}^{-1,r,q}(T,\Omega)$ (and $\mathcal{K} = 0$).*

If $\mathcal{F}$ is also contained in $\mathcal{T}^{-1,4,4}(T,\Omega)$, then $u \in L^4(0,T;\tilde{L}^4(\Omega))$. In particular, $u \otimes u \in L^2(0,T;L^2(\Omega))$.

Proof. Again, the idea of the proof is similar to the one of Proposition 4.2. For any $v \in L^{4/3}(0,T;\tilde{L}^{4/3}_\sigma) \cap L^{r'}(0,T;\tilde{L}^{q'}_\sigma)$ and corresponding $\phi = \phi(v) \in \mathcal{T}^{1,r',q'} \cap \mathcal{T}^{1,4/3,4/3}$ we find that

$$|(u,v)_{T,\Omega}| = |-(u,\phi_t)_{T,\Omega} + (u,\tilde{A}_{q'}\phi)_{T,\Omega}| \le |\langle \mathcal{F},\phi\rangle| + |(u\otimes u,\nabla\phi)_{T,\Omega}|$$

where

$$|\langle \mathcal{F},\phi\rangle| \le \|\mathcal{F}\|_{\mathcal{T}^{-1,4,4}}\|\phi\|_{\mathcal{T}^{1,4/3,4/3}} \le C\|\mathcal{F}\|_{\mathcal{T}^{-1,4,4}}\|v\|_{L^{4/3}(0,T;\tilde{L}^{4/3})}.$$

The other term is treated as follows. First we consider the case $r = 16$, $q = 24/7$ and obtain the estimate

$$|(u\otimes u,\nabla\phi)_{T,\Omega}| \le \|u\|^2_{L^{16}(0,T;\tilde{L}^{24/7})}\|\nabla\phi\|_{L^{8/7}(0,T;\tilde{L}^{12/5})}.$$

Now we apply (2.8) to $\nabla\phi(t)$ and see that for a.a. t

$$\|\nabla\phi(t)\|_{\tilde{L}^{12/5}} \le C\|(1+\tilde{A}_{12/5})^{1/2}\phi(t)\|_{\tilde{L}^{12/5}} \le C\|(1+\tilde{A}_{4/3})\phi(t)\|_{\tilde{L}^{4/3}}.$$

Since $\|(1+\tilde{A}_{4/3})\phi\|_{L^{8/7}(\tilde{L}^{4/3})} \le C\|v\|_{L^{8/7}(\tilde{L}^{4/3})}$ by Lemma 2.3, we get that

$$|(u\otimes u,\nabla\phi)_{T,\Omega}| \le C\|u\|^2_{L^r(0,T;\tilde{L}^q)}\|v\|_{L^{8/7}(0,T;\tilde{L}^{4/3})}.$$

Let us now consider the case $r < 16$, $q > 24/7$. Again we first estimate

$$|(u\otimes u,\nabla\phi)_{T,\Omega}| \le \|u\|^2_{L^r(0,T;\tilde{L}^q(\Omega))}\|\nabla\phi\|_{L^{(r/2)'}(0,T;\tilde{L}^{(q/2)'}(\Omega))}.$$

Form (2.12) (with $\gamma_2 = q/2$, $\rho_2 = r/2$) we get

$$\|\nabla\phi\|_{L^{(r/2)'}(0,T;\tilde{L}^{(q/2)'})} \le C\|v\|_{L^{8/7}(0,T;\tilde{L}^{4/3})}.$$

From here on we can again treat the cases $r = 16$ and $r < 16$ together. By Hölder's inequality we find that $\|v\|_{L^{8/7}(\tilde{L}^{4/3})} \le T^{1/8}\|v\|_{L^{4/3}(\tilde{L}^{4/3})}$. Hence

$$|(u,v)_{T,\Omega}| \le C\big(\|F\|_{\mathcal{T}^{-1,4,4}} + \|u\|^2_{L^r(\tilde{L}^q)}\big)\|v\|_{L^{4/3}(\tilde{L}^{4/3})}.$$

Since $L^{4/3}(0,T;\tilde{L}^{4/3}_\sigma) \cap L^{r'}(0,T;\tilde{L}^{q'}_\sigma)$ is dense in $L^{4/3}(0,T;\tilde{L}^{4/3}_\sigma)$ we find that $u \in L^4(0,T;\tilde{L}^4_\sigma)$. This finishes the proof. □

Similarly to Proposition 2.4 we now discuss more concrete conditions on u_0, f_1, f_2 for a functional $\mathcal{F}$ to be contained in $\mathcal{T}^{-1,4,4}(T,\Omega)$.

Proposition 4.5. *Let $\Omega \subseteq \mathbb{R}^n$ be a uniform C^2-domain, $n \geq 3$, $0 < T < \infty$, and let $1 < r, q < \infty$ be Serrin exponents. Assume that $\mathcal{F} \in \mathcal{T}^{-1,r,q}(T,\Omega)$ is given by*

$$\langle \mathcal{F}, \phi \rangle := (u_0, \phi(0))_\Omega + (f_1, \phi)_{T,\Omega} + (f_2, \nabla\phi)_{T,\Omega}. \tag{4.4}$$

Then the condition $\mathcal{F} \in \mathcal{T}^{-1,4,4}(T,\Omega)$ can be ensured if
(i) concerning u_0

$$u_0 \in L^2_\sigma(\Omega), \qquad \int_0^T \|e^{-\tau\tilde{A}_2} u_0\|^4_{\tilde{L}^4(\Omega)}\, d\tau < \infty$$

or $u_0 \in \tilde{L}^\gamma_\sigma(\Omega)$, $\frac{1}{4} \leq \frac{1}{\gamma} < \frac{1}{4} + \frac{1}{2n}$, and
(ii) concerning f_1, f_2

$$f_1 \in L^{\rho_1}(0,T;\tilde{L}^{\gamma_1}(\Omega)), \quad f_2 \in L^{\rho_2}(0,T;\tilde{L}^{\gamma_2}(\Omega)),$$

where $1 < \rho_1, \gamma_1, \rho_2, \gamma_2 < \infty$,

$$\frac{2}{\rho_1} + \frac{n}{\gamma_1} = 2 + \frac{n+2}{4}, \quad \frac{2}{\rho_2} + \frac{n}{\gamma_2} = 1 + \frac{n+2}{4}$$

and $\frac{1}{4} < \frac{1}{\gamma_1} \leq \frac{1}{4} + \frac{2}{n}$ and $\frac{1}{4} \leq \frac{1}{\gamma_2} \leq \frac{1}{4} + \frac{1}{n}$. This includes the case $\rho_1 = 4$, $\frac{1}{\gamma_1} = \frac{1}{4} + \frac{2}{n}$ and $\rho_2 = 4$, $\frac{1}{\gamma_2} = \frac{1}{4} + \frac{1}{n}$, and, when $n = 3$, the case $f_1 \in L^{8/7}(0,T;L^2(\Omega))$, $f_2 \in L^{8/3}(0,T;L^2(\Omega))$.

Proof. (i) For $\phi \in \mathcal{T}^{1,4/3,4/3}(T,\Omega)$ we let $v := -\phi_t + \tilde{A}_{q'}\phi$. Using Lemma 2.3 we estimate

$$\begin{aligned}
|(u_0, \phi(0))_{T,\Omega}| &= \left| \int_0^T (u_0, e^{-\tau\tilde{A}_{q'}} v(\tau))_\Omega\, d\tau \right| = \left| \int_0^T (e^{-\tau\tilde{A}_2} u_0, v(\tau))_\Omega\, d\tau \right| \\
&\leq \left(\int_0^T \|e^{-\tau\tilde{A}_2} u_0\|^4_{\tilde{L}^4}\, d\tau \right)^{1/4} \|v\|_{L^{4/3}(0,T;\tilde{L}^{4/3})} \\
&\leq \left(\int_0^T \|e^{-\tau\tilde{A}_2} u_0\|^4_{\tilde{L}^4}\, d\tau \right)^{1/4} \|\phi\|_{\mathcal{T}^{1,4/3,4/3}}.
\end{aligned}$$

Hence the functional $\phi \mapsto (u_0, \phi(0))_\Omega$ is contained in the space $\mathcal{T}^{-1,4,4}(T,\Omega)$.

For the second condition on u_0 we use $\tilde{L}^r$-$\tilde{L}^q$-estimates, see (2.9). Since $\|e^{-\tau\tilde{A}_2} u_0\|_{\tilde{L}^4} \leq C\tau^{-\alpha}\|u_0\|_{\tilde{L}^\gamma}$ for $0 < \tau \leq T$, with $\alpha = \frac{n}{2}\left(\frac{1}{\gamma} - \frac{1}{4}\right) < \frac{1}{4}$, we get that

$$\int_0^T \|e^{-\tau\tilde{A}_2} u_0\|^4_{\tilde{L}^4}\, d\tau \leq C\|u_0\|^4_{\tilde{L}^\gamma} \int_0^T \tau^{-4\alpha}\, d\tau < \infty.$$

(ii) For the condition on f_1 we use (2.11) to prove that

$$\begin{aligned}|(f_1,\phi)_{T,\Omega}| &\le \|f_1\|_{L^{\rho_1}(\tilde L^{\gamma_1})}\|\phi\|_{L^{\rho_1'}(\tilde L^{\gamma_1'})}\\ &\le C\|f_1\|_{\tilde L^{\rho_1}(\tilde L^{\gamma_1})}\|v\|_{L^{4/3}(\tilde L^{4/3})} \le C\|f_1\|_{\tilde L^{\rho_1}(\tilde L^{\gamma_1})}\|\phi\|_{\mathcal{T}^{1,4/3,4/3}}.\end{aligned}$$

Concerning the estimates for f_2 we use Hölder's inequality for $|(f_2,\nabla\phi)_{T,\Omega}|$, then (2.12) to see that $\|\nabla\phi\|_{L^{\rho_2'}(\tilde L^{\gamma_2'})} \le C\|v\|_{L^{4/3}(0,T;\tilde L^{4/3})}$, and complete the proof as above. □

Combining the sufficient conditions of Proposition 2.4 and Proposition 4.5 as well as the existence theorems Theorem 1.3 and Theorem 4.4 we get the following result in the three-dimensional case.

Corollary 4.6. *Let $\Omega \subset \mathbb{R}^3$ be a uniform C^2-domain, $0<T<\infty$, $\frac{16}{5}\le r\le 16$, $\frac{24}{7}\le q\le 8$, $\frac{2}{r}+\frac{3}{q}=1$. Assume functions $u_0\in L^2_\sigma(\Omega)$ with*

$$e^{-\tau\tilde A_2}u_0 \in L^4(0,T;\tilde L^4(\Omega)),$$

and $f_1\in L^r(0,T;\tilde L^{\gamma_1}(\Omega))$ where $\frac{1}{\gamma_1}=\frac{1}{q}+\frac{2}{3}$ together with one of the conditions

$$f_1\in L^4(0,T;\tilde L^{12/11}(\Omega)) \quad \textit{or} \quad L^{8/7}(0,T;L^2(\Omega)),$$

and $f_2\in L^r(0,T;\tilde L^{\gamma_2}(\Omega))$ where $\frac{1}{\gamma_2}=\frac{1}{q}+\frac{1}{3}$ together with one of the conditions

$$f_2\in L^4(0,T;\tilde L^{12/7}(\Omega)) \quad \textit{or} \quad L^{8/3}(0,T;L^2(\Omega))$$

are given. Consider the data functionals $\mathcal{K}=0$ and $\mathcal{F}$ as in (4.4). Then there exists a constant $\eta=\eta(\tau(\Omega),q,T)>0$ with the following property: if

$$\int_0^T \|e^{-\tau\tilde A_2}u_0\|^r_{\tilde L^q(\Omega)}\,d\tau \le \eta$$

and

$$\|f_1\|_{L^r(0,T;\tilde L^{\gamma_1}(\Omega))}\le\eta \quad \textit{or} \quad \|f_1\|_{L^{4/3}(0,T;L^2(\Omega))}\le\eta,$$

and

$$\|f_2\|_{L^r(0,T;\tilde L^{\gamma_2}(\Omega))}\le\eta \quad \textit{or} \quad \|f_2\|_{L^4(0,T;L^2(\Omega))}\le\eta \ \textit{ if } q\le 6,$$

then there is a very weak solution $u\in L^r(0,T;\tilde L^q_\sigma(\Omega))$ to the data $\mathcal{F}$ with the additional property $u\otimes u\in L^2(0,T;L^2(\Omega))$.

Proof. By Proposition 2.4 the functional $\mathcal{F}$ is contained in $\mathcal{T}^{-1,r,q}(T,\Omega)$; moreover, $\|\mathcal{F}\|_{\mathcal{T}^{-1,r,q}}\le\eta'$ if $\eta>0$ is chosen small enough. Here η' is the constant from Theorem 1.3. Then, by Theorem 1.3, there exists a very weak solution $u\in L^r(0,T;\tilde L^q_\sigma(\Omega))$. By Proposition 4.5 it also holds that $\mathcal{F}\in\mathcal{T}^{-1,4,4}(T,\Omega)$. Now we use Theorem 4.4 and get that $u\otimes u\in L^2(0,T;L^2(\Omega))$. □

In the final results we will improve space regularity, cf. Theorem 3.3 for the linear case. After an abstract result in Theorem 4.7 we discuss more concrete conditions on the data functional in Proposition 4.8 below.

Theorem 4.7. *Let $\Omega \subseteq \mathbb{R}^n$ be a uniform C^2-domain, $0 < T < \infty$, and let Serrin exponents $2 < r < \infty$, $n < q < \infty$, $\frac{2}{r} + \frac{n}{q} = 1$ be given. Assume that a very weak solution $u \in L^r(0,T;\tilde{L}^q_\sigma(\Omega))$ to the Navier-Stokes equations is given for data $\mathcal{F} \in \mathcal{T}^{-1,r,q}(T,\Omega)$ and $\mathcal{K} = 0$.*

Furthermore, consider exponents $1 < r_*, q_* < \infty$ and $0 \le \alpha < \frac{1}{2}$ such that

$$\frac{2}{q} + \frac{2\alpha - 1}{n} < \frac{1}{q_*} \le \frac{2}{q}, \qquad \frac{2}{r_*} + \frac{n}{q_*} = 1 + 2\alpha.$$

Alternatively, we consider the limit case where $\alpha = \frac{1}{2}$ and $r_* = \frac{r}{2}$, $q_* = \frac{q}{2}$. If for $\mathcal{F}$ the additional inequality

$$|\langle \mathcal{F}, \phi(v)\rangle| \le C_{\mathcal{F}} \|(1 + \tilde{A}_{q_*'})^{-\alpha} v\|_{L^{r_*'}(0,T;\tilde{L}^{q_*'}(\Omega))} \tag{4.5}$$

holds for all $v \in L^{r'}(0,T;\tilde{L}^{q'}_\sigma(\Omega)) \cap L^{r_*'}(0,T;\tilde{D}^{-\alpha}_{q_*'})$, then u also satisfies

$$u \in L^{r_*}(0,T;\tilde{D}^{\alpha}_{q_*}).$$

Proof. Consider any $v \in L^{r'}(0,T;\tilde{L}^{q'}_\sigma) \cap L^{r_*'}(0,T;\tilde{D}^{-\alpha}_{q_*'})$ and the corresponding $\phi = \phi(v) \in \mathcal{T}^{1,r',q'} \cap \mathcal{T}^{1,r_*',q_*'}$. Then $(u,v)_{T,\Omega} = \langle \mathcal{F}, \phi\rangle + (u \otimes u, \nabla\phi)_{T,\Omega}$ admits the estimate

$$|(u,v)_{T,\Omega}| \le C_{\mathcal{F}} \|(1 + \tilde{A}_{q_*'})^{-\alpha} v\|_{L^{r_*}(\tilde{L}^{q_*'})} + \|u\|^2_{L^r(\tilde{L}^q)} \|\nabla\phi\|_{L^{(r/2)'}(\tilde{L}^{(q/2)'})}.$$

Let us estimate the term $\|\nabla\phi\|_{L^{(r/2)'}(0,T;\tilde{L}^{(q/2)'})}$ by v in $L^{r_*'}(0,T;\tilde{D}^{-\alpha}_{q_*'})$.

In the limit case $\alpha = \frac{1}{2}$, $r_* = \frac{r}{2}$, $q_* = \frac{q}{2}$ we exploit (2.10) and the maximal regularity estimate for $(1 + \tilde{A}_{q_*})^{-1/2}\phi$ and $(1 + \tilde{A}_{q_*})^{-1/2} v$ to obtain that

$$\begin{aligned}\|\nabla\phi\|_{L^{(r/2)'}(0,T;\tilde{L}^{(q/2)'})} &\le C\|(1 + \tilde{A}_{q_*})^{\alpha}\phi\|_{L^{r_*'}(0,T;\tilde{L}^{q_*'})}\\ &\le C\|(1 + \tilde{A}_{q_*})^{-\alpha} v\|_{L^{r_*'}(0,T;\tilde{L}^{q_*'})}.\end{aligned}$$

Now consider $0 \le \alpha < \frac{1}{2}$. We define $0 \le \beta \le \frac{1}{2}$ by $\beta = \frac{n}{2}\left(\frac{2}{q} - \frac{1}{q_*}\right)$ and see that the assumptions on q_* imply that $\frac{1}{2} + \alpha + \beta < 1$. This and Lemma 2.3 allows to continue as follows:

$$\begin{aligned}\|\nabla\phi(T-t)\|_{\tilde{L}^{(q/2)'}} &\le C\|(1 + \tilde{A}_{q'})^{\frac{1}{2}}\phi(T-t)\|_{\tilde{L}^{(q/2)'}}\\ &\le C\|(1 + \tilde{A}_{q_*'})^{\frac{1}{2}+\beta}\phi(T-t)\|_{\tilde{L}^{q_*'}}\\ &\le C\int_0^t \|(1 + \tilde{A}_{q_*'})^{\frac{1}{2}+\beta} e^{-(t-\tau)\tilde{A}_{q_*'}} v(T-\tau)\|_{\tilde{L}^{q_*'}}\, d\tau.\end{aligned}$$

Then we insert $1 = (1 + \tilde{A}_{q_*})^{\alpha}(1 + \tilde{A}_{q_*})^{-\alpha}$ and use the fact that the semigroup commutes with the fractional powers. Hence we find by the Hardy-Littlewood-Sobolev inequality that $\|\nabla\phi\|_{L^{(r/2)'}(0,T;\tilde{L}^{(q/2)'})}$ is bounded by

$$C\left(\int_0^T \left(\int_0^t \|(1 + \tilde{A}_{q_*'})^{\frac{1}{2}+\alpha+\beta} e^{-(t-\tau)\tilde{A}_{q_*'}} (1 + \tilde{A}_{q_*'})^{-\alpha} v(T-\tau)\|_{\tilde{L}^{q_*'}} d\tau\right)^{(r/2)'} dt\right)^{1/(r/2)'}$$

$$\le C\left(\int_0^T \left(\int_0^T |t-\tau|^{-\frac{1}{2}-\alpha-\beta} \|(1 + \tilde{A}_{q_*'})^{-\alpha} v(T-\tau)\|_{\tilde{L}^{q_*'}}\right)^{(r/2)'} dt\right)^{1/(r/2)'}$$

$$\le C\|(1 + \tilde{A}_{q_*'})^{-\alpha} v\|_{L^{r_*'}(0,T;\tilde{L}^{q_*'})}.$$

Putting together all the pieces we find that

$$|(u,v)_{T,\Omega}| \le \big(C_{\mathcal{F}} + C\|u\|^2_{L^r(0,T;\tilde{L}^q)}\big)\|v\|_{L^{r_*'}(0,T;\tilde{D}^{-\alpha}_{q_*'})}$$

for all $v \in L^{r'}(0,T;\tilde{L}^{q'}) \cap L^{r_*'}(0,T;\tilde{D}^{-\alpha}_{q_*'})$. By a density argument, see Lemma 2.7, this implies by duality that $u \in L^{r_*}(0,T;\tilde{D}^{\alpha}_{q_*})$. This finishes the proof. □

Proposition 4.8. *Let $n = 3$, Ω, T, r, q, $r_* = \frac{r}{2}$, $q_* = \frac{q}{2}$, $\alpha = \frac{1}{2}$ be as in Theorem 4.7. Let the functional $\mathcal{F} \in \mathcal{T}^{-1,r,q}(T,\Omega)$ be given as in (4.4) for some functions $u_0 \in \tilde{L}^{\gamma}_{\sigma}(\Omega)$, $3 < \gamma < \infty$, and $f_1 \in L^{r/2}(0,T;\tilde{L}^{\gamma_1}(\Omega))$, $\frac{1}{\gamma_1} = \frac{1}{3} + \frac{2}{q}$, as well as $f_2 \in L^{r/2}(0,T;\tilde{L}^{q/2}(\Omega))$.*

Then a very weak solution $u \in L^r(0,T;\tilde{L}^q(\Omega))$ to the Navier-Stokes equations with data $\mathcal{F}$ (and $\mathcal{K} = 0$) even satisfies $u \in L^{r/2}(0,T;\tilde{W}^{1,q/2}_0(\Omega))$.

Proof. Let us prove the estimate

$$|\langle \mathcal{F}, \phi(v)\rangle| \le C_{\mathcal{F}}\|(1 + \tilde{A}_{(q/2)'})^{-1/2} v\|_{L^{(r/2)'}(0,T;\tilde{L}^{(q/2)'}(\Omega))}$$

for all $v \in L^{r'}(0,T;\tilde{L}^{q'}_{\sigma}(\Omega)) \cap L^{(r/2)'}(0,T;\tilde{D}^{-1/2}_{(q/2)'})$. The condition on u_0 is justified by Proposition 3.4. Concerning f_1 it suffices to estimate ϕ as follows:

$$\begin{aligned}\|\phi\|_{L^{(r/2)'}(0,T;\tilde{L}^{(\gamma_1)'})} &\le C\|(1 + \tilde{A}_{(q/2)'})^{1/2}\phi\|_{L^{(r/2)'}(0,T;\tilde{L}^{(q/2)'})}\\ &\le C\|(1 + \tilde{A}_{(q/2)'})^{-1/2} v\|_{L^{(r/2)'}(0,T;\tilde{L}^{(q/2)'})}.\end{aligned}$$

For f_2 we simply note that

$$\|\nabla\phi\|_{L^{(r/2)'}(0,T;\tilde{L}^{(q/2)'})} \le C\|(1 + \tilde{A}_{(q/2)'})^{1/2}\phi\|_{L^{(r/2)'}(0,T;\tilde{L}^{(q/2)'})}$$

and continue as for f_1. Now we use Theorem 4.7 to see that $u \in L^{r/2}\big(0,T;\tilde{D}^{1/2}_{q/2}\big)$. Identifying the space $\tilde{D}^{1/2}_{q/2}$ with $\tilde{W}^{1,q/2}_0(\Omega) \cap \tilde{L}^{q/2}_{\sigma}(\Omega)$ we get the assertion. □

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