UNIQUENESS CRITERIA AND STRONG SOLUTIONS OF THE BOUSSINESQ EQUATIONS IN COMPLETELY GENERAL DOMAINS

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Abstract. Consider the instationary Boussinesq equations in a completely general domain $\Omega \subseteq \mathbb{R}^3$ and a time interval $[0,T[$. We deal with existence of strong solutions of the Boussinesq equations. These results will be used to prove uniqueness criteria for the Boussinesq equations which are based on the local or global identification of a weak solution with a strong solution.

1. Introduction and main results

Let $\Omega \subseteq \mathbb{R}^3$ be a *general domain*, i.e. a nonempty, open and connected subset and let $[0,T[\,, 0 < T \leq \infty$, be a time interval. We consider the Boussinesq equations

$$\begin{aligned}
u_t - \Delta u + u \cdot \nabla u + \nabla p &= \theta g + f_1 \qquad && \text{in }]0,T[\times\Omega\,, \\
\operatorname{div} u &= 0 && \text{in }]0,T[\times\Omega\,, \\
\theta_t - \Delta\theta + u \cdot \nabla\theta &= f_2 && \text{in }]0,T[\times\Omega\,, \\
u = 0\,, \qquad \theta &= 0 && \text{on }]0,T[\times\partial\Omega\,, \\
u = u_0\,, \qquad \theta &= \theta_0 && \text{at } t = 0\,,
\end{aligned} \tag{1.1}$$

where u denotes the velocity of the fluid, θ the difference of the temperature to a fixed reference temperature and p denotes the pressure. Further, $u_0\,, \theta_0$ are the initial values, f_1 the external force per unit mass and f_2 the external thermal radiation per heat capacity. For mathematical completeness we allow a time dependent gravitational force $g = g(t,x)$. However, in most applications the gravitational force is a constant vector field in time. The Boussinesq equations are a widely used model of motion of a viscous, incompressible buoyancy-driven fluid flow coupled with heat convection, see [19, 24]. The Boussinesq equations have been investigated in many paper, see e.g. [1, 2, 3, 13, 14, 15, 18, 20, 23] and papers cited there.

We need the following space of test functions:

$$C_0^\infty([0,T[;C_{0,\sigma}^\infty(\Omega)) := \{\, w|_{[0,T[\times\Omega}\ ; w \in C_0^\infty(]-1,T[\times\Omega)\ ; \operatorname{div} w = 0\,\}.$$

Motivated by the definition of a weak solution of the instationary Navier-Stokes equations in the sense of Leray-Hopf we give the following

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Definition 1.1. Let $\Omega \subseteq \mathbb{R}^3$ be a general domain, let $0 < T \leq \infty$, $g \in L^{8/5}_{\mathrm{loc}}([0,T[; L^4(\Omega))$. Assume $f_1, f_2 \in L^1_{\mathrm{loc}}([0,T[; L^2(\Omega))$ and $u_0 \in L^2_\sigma(\Omega)$, $\theta_0 \in L^2(\Omega)$. A pair

$$
\begin{aligned}
&u \in L^\infty_{\mathrm{loc}}([0,T[; L^2_\sigma(\Omega)) \cap L^2_{\mathrm{loc}}([0,T[; W^{1,2}_{0,\sigma}(\Omega)),\\
&\theta \in L^\infty_{\mathrm{loc}}([0,T[; L^2(\Omega)) \cap L^2_{\mathrm{loc}}([0,T[; H^1_0(\Omega)),
\end{aligned}
\tag{1.2}
$$

is called a weak solution of the Boussinesq system (1.1) if the following properties are fulfilled:

(i) The functions $u : [0,T[\to L^2_\sigma(\Omega)$ and $\theta : [0,T[\to L^2(\Omega)$ are weakly continuous.

(ii) We have

$$
\begin{aligned}
&-\langle u, w_t\rangle_{\Omega,T} + \langle \nabla u, \nabla w\rangle_{\Omega,T} + \langle u\cdot\nabla u, w\rangle_{\Omega,T}\\
&\quad = \langle f_1, w\rangle_{\Omega,T} + \langle \theta g, w\rangle_{\Omega,T} + \langle u_0, w\rangle_\Omega
\end{aligned}
$$

for all $w \in C^\infty_0([0,T[; C^\infty_{0,\sigma}(\Omega))$.

(iii) There holds

$$
\begin{aligned}
&-\langle \theta, \phi_t\rangle_{\Omega,T} + \langle \nabla\theta, \nabla\phi\rangle_{\Omega,T} + \langle u\cdot\nabla\theta, \phi\rangle_{\Omega,T}\\
&\quad = \langle f_2, \phi\rangle_{\Omega,T} + \langle \theta_0, \phi(0)\rangle_\Omega
\end{aligned}
$$

for all $\phi \in C^\infty_0([0,T[; C^\infty_0(\Omega))$.

In the identities above $\langle\cdot,\cdot\rangle_\Omega$, $\langle\cdot,\rangle_{\Omega,T}$ denotes the usual L^2-scalar product in Ω and in $]0,T[\times\Omega$, respectively.

Due to the weak continuity of (u,θ) in the definition above we have that $u(t) \in L^2_\sigma(\Omega)$ and $\theta(t) \in L^2(\Omega)$ are well defined for all $t \in [0,T[$. Especially $u(0) = u_0$ and $\theta(0) = \theta_0$. If $g \in L^\infty(]0,T[\times\Omega)$ we can show with the Faedo-Galerkin method analogously as in [18, Theorem 1] that there exists a weak solution of (1.1) in $[0,T[\times\Omega$. Moreover, there exists a distribution p, called an associated pressure, such that

$$
u_t - \Delta u + u\cdot\nabla u + \nabla p = \theta g + f_1
$$

holds in the sense of distributions in $]0,T[\times\Omega$, see [21, Section V.1.7].

Up to now it is not known if weak solutions (u,θ) of the three-dimensional Boussines equations are uniquely determined and regular. In [17] it is proved that uniqueness and regularity holds if additionally *Serrin's condition* $u \in L^s(0,T; L^q(\Omega))$ holds where $1 < s,q < \infty$ with $\frac{2}{s} + \frac{3}{q} = 1$. In general domains, which may have several exits to infinity or may have edges and corners, only the L^2-approach to the Stokes operator is available. We arrive at the following definition.

Definition 1.2. Consider data as in Definition 1.1. We call (u,θ) a *strong solution* of (1.1) if (u,θ) is a weak solution of (1.1) and $u \in L^8(0,T; L^4(\Omega))$.

The crucial point in the definition above is the fact that we have required no additional integrability condition for θ. Our first main result deals with existence of strong solutions of the Boussinesq equations in general domains. For the construction of strong solutions of the instationary Navier-Stokes system (see (1.10) below) in general domains we refer to [7, 9] and to [21, Section V.4.2]. We denote by $\Delta = \Delta_2$, $A = A_2$ the Laplace and Stokes operator, respectively. For further information about these operators we refer to the preliminaries.

Theorem 1.3. *Let $\Omega \subseteq \mathbb{R}^3$ be a general domain, let $0 < T \le \infty$, let $g \in L^{8/3}(0,T;L^4(\Omega))$. Assume $f_1 \in L^{4/3}(0,T;L^2(\Omega))$, $f_2 \in L^1(0,T;L^2(\Omega))$ and $u_0 \in L^2_\sigma(\Omega)$, $\theta_0 \in L^2(\Omega)$.*

(i) *There exists an absolute constant $\epsilon_* > 0$ (independent of $\Omega, T, g, f_1, f_2, u_0, \theta_0$) with the following property: If the conditions*

$$\Big(\int_0^T \|e^{-tA}u_0\|_4^8\,dt\Big)^{1/8} \le \epsilon_*\,, \tag{1.3}$$

$$\|g\|_{4,\frac{8}{3};T}\Big(\int_0^T \|e^{t\Delta}\theta_0\|_4^{8/3}\,dt\Big)^{3/8} \le \epsilon_*\,, \tag{1.4}$$

$$\|f_1\|_{2,\frac{4}{3};T} + \|g\|_{4,\frac{8}{3};T}\|f_2\|_{2,1;T} \le \epsilon_*\,, \tag{1.5}$$

are satisfied, then there exists a uniquely determined strong solution (u,θ) of the Boussinesq equations (1.1) in $[0,T[\times\Omega$.

(ii) *The condition*

$$\int_0^\infty \|e^{-tA}u_0\|_4^8\,dt < \infty \tag{1.6}$$

is necessary and sufficient for the existence of $0 < T' \le T$ and a strong solution (u,θ) of (1.1) in $[0,T'[\times\Omega$.

Remark. Combining (1.4) and (3.9) below it follows that ϵ_* can be replaced by a smaller constant (if necessary), to be denoted again by ϵ_*, such that if (1.3), (1.5) and

$$\|g\|_{4,\frac{8}{3};T}\|\theta_0\|_2 \le \epsilon_* \tag{1.7}$$

are satisfied, then there exists a strong solution (u,θ) of (1.1) in $[0,T[\times\Omega$.

For existence of strong solutions of the Boussinesq equations if $\Omega \subseteq \mathbb{R}^3$ is a smooth bounded domain we refer to [17, Theorems 1.3, 1.4]. From Theorem 1.3 (i) with $f_1 := f$, $f_2 := 0$, $g := 0$ and initial values u_0 and $\theta_0 := 0$ we obtain the following result, c.f. [9, Theorem 4.1] with a different condition on f.

Theorem 1.4. *Let $\Omega \subseteq \mathbb{R}^3$ be a general domain, let $u_0 \in L^2_\sigma(\Omega)$, let $0 < T \le \infty$, $f \in L^{4/3}(0,T;L^2(\Omega))$. Then there exists an absolute constant $\epsilon_* > 0$ (independent of Ω, T, f, u_0) with the following property: Assume that the conditions*

$$\Big(\int_0^T \|e^{-tA}u_0\|_4^8\,dt\Big)^{1/8} \le \epsilon_*\,, \tag{1.8}$$

$$\|f\|_{2,\frac{4}{3};T} \le \epsilon_* \tag{1.9}$$

are satisfied. Then there exists a strong solution $u \in L^8(0,T;L^4(\Omega))$ of the instationary Navier-Stokes equations

$$\begin{aligned} u_t - \Delta u + u\cdot\nabla u + \nabla p &= f \quad \text{in }]0,T[\times\Omega\,,\\ \operatorname{div} u &= 0 \quad \text{in }]0,T[\times\Omega\,,\\ u &= 0 \quad \text{on }]0,T[\times\partial\Omega\,,\\ u &= u_0 \quad \text{at } t=0\,, \end{aligned} \tag{1.10}$$

with initial value u_0 *and external force* f.

Before we start to present uniqueness criteria for the Boussinesq equations we need the following definition.

Definition 1.5. Consider data as in Definition 1.1, assume additionally $g \in L^{8/3}_{\mathrm{loc}}([0,T[;L^4(\Omega))$ and let (u,θ) be a weak solution of (1.1). We say that (u,θ) satisfies the *strong energy inequality* if there is a null set $N \subseteq]0,T[$ such that

$$\frac{1}{2}\|u(t)\|_2^2 + \int_s^t \|\nabla u\|_2^2\,d\tau \le \frac{1}{2}\|u(s)\|_2^2 + \int_s^t \langle \theta g, u\rangle_\Omega\,d\tau + \int_s^t \langle f_1, u\rangle_\Omega\,d\tau \quad (1.11)$$

for all $s \in (\,]0,T[\setminus N\,) \cup \{0\}$ and all $t \in [s,T[$.

The additional integrability assumption on g is needed to guarantee that $\int_s^t \langle \theta g, u\rangle_\Omega\,d\tau$ exists. We proceed with a general uniqueness theorem which will be the basis for Corollary 1.7 and Theorem 1.8 below. The central idea is the local construction of a strong solution and the identification of this solution with (u,θ) and (v,Θ). For uniqueness and regularity results for the Navier-Stokes equations which are based on the method of the identification of a strong solution with a weak solution we refer to [4, 5, 6, 8, 10].

Theorem 1.6. *Let* $\Omega \subseteq \mathbb{R}^3$ *be a general domain, let* $0 < T \le \infty$, *let* $g \in L^{8/3}(0,T;L^4(\Omega))$. *Assume* $f_1 \in L^{4/3}(0,T;L^2(\Omega))\,, f_2 \in L^1(0,T;L^2(\Omega))$ *and* $u_0 \in L^2_\sigma(\Omega)\,, \theta_0 \in L^2(\Omega)$. *Consider weak solutions* (u,θ) *and* (v,Θ) *of the Boussinesq equations (1.1). We assume that the following properties are fulfilled:*

(i) *There holds that*

$$\int_0^\infty \|e^{-\tau A}u(t_0)\|_4^8\,d\tau < \infty \quad \text{for all } t_0 \in [0,T[. \quad (1.12)$$

(ii) (u,θ) *and* (v,Θ) *satisfy the strong energy inequality (1.11).*

(iii) *At least one of the functions* $u : [0,T[\to L^2_\sigma(\Omega)$ *or* $v : [0,T[\to L^2_\sigma(\Omega)$ *is strongly continuous.*

Then $u(t) = v(t)$ *and* $\theta(t) = \Theta(t)$ *for all* $t \in [0,T[$.

From [17, Theorem 1.5] it follows that weak solutions (u,θ) and (v,Θ) of the Boussinesq equations (1.1) coincide if additionally $u\,, v \in L^8(0,T;L^4(\Omega))$. In the following corollary we will show that the assumption $v \in L^8(0,T;L^4(\Omega))$ can be replaced by the weaker assumption that (v,Θ) satisfies the strong energy inequality (1.11).

Corollary 1.7. *Let* $\Omega \subseteq \mathbb{R}^3$ *be a general domain, let* $0 < T \le \infty$, *let* $g \in L^{8/3}(0,T;L^4(\Omega))$. *Further let* $f_1 \in L^{4/3}(0,T;L^2(\Omega))\,, f_2 \in L^1(0,T;L^2(\Omega))$ *and* $u_0 \in L^2_\sigma(\Omega)\,, \theta_0 \in L^2(\Omega)$. *Assume that* $(u,\theta)\,, (v,\Theta)$ *are weak solutions of (1.1) and that the following conditions are fulfilled:*

(i) $u \in L^8(0,T;L^4(\Omega))$ *or* $u \in L^\infty(0,T;\mathcal{D}(A^{1/4}))$.

(ii) (v,Θ) *satisfies the strong energy inequality (1.11).*

Then $u(t) = v(t)$ *and* $\theta(t) = \Theta(t)$ *for all* $t \in [0,T[$. *Especially it follows that every strong solution of (1.1) and every weak solution of (1.1) which satisfies the strong energy inequality (1.11) coincide.*

We can apply the results of Theorem 1.3 and Corollary 1.7 to obtain an uniqueness and regularity result for (1.1) which is based on the smallness of $\|u\|_{L^s(0,T;L^4(\Omega))}$ where $\frac{2}{s}+\frac{3}{4}>1$ is allowed. For this theorem we need that $\Omega \subseteq \mathbb{R}^3$ is a domain such that

$$\|e^{-tA}v\|_4 \le c\|v\|_4 \tag{1.13}$$

holds for all $v \in L^4(\Omega)\cap L^2_\sigma(\Omega)$ with $c=c(\Omega)>0$. Especially the estimate above is fulfilled if $-A_4$ generates an uniformly bounded semigroup on $L^4_\sigma(\Omega)$. Therefore (see [11]) we have that (1.13) is satisfied in the following cases:

(i) $\Omega = \mathbb{R}^n$,
(ii) Ω is a bounded domain with $\partial\Omega \in C^{2,1}$,
(iii) Ω is a half space,
(iv) Ω is an exterior domain with $\partial\Omega \in C^{2,1}$.

In the following theorem we need no smallness conditions on f_1, f_2, g, c.f. Theorem 1.3.

Theorem 1.8. *Let $\Omega \subseteq \mathbb{R}^3$ be a domain such that (1.13) holds. Let $f_1 \in L^s(0,T;L^2(\Omega))\,, g \in L^\mu(0,T;L^4(\Omega))$ where $s>4/3\,,\mu>8/3$. Assume $0<T\le\infty$, let $f_2 \in L^1(0,T;L^2(\Omega))$ and $u_0 \in L^2_\sigma(\Omega)\cap L^4(\Omega)\,,\theta_0 \in L^2(\Omega)$. Further consider $1 \le s < 8$. Then there exists a constant $c_* = c_*(\Omega,s)>0$ such that if (u,θ) is a weak solution of (1.1) satisfying*

$$\|u_0\|_4^{\frac{8}{s}-1}\Big(\int_0^T \|u(\tau)\|_4^s\,d\tau\Big)^{1/s} \le c_*\,, \tag{1.14}$$

then (u,θ) is in fact a strong solution of (1.1). Furthermore, every weak solution (v,Θ) of (1.1) satisfying the strong energy inequality (1.11) coincides with (u,θ).

The present paper is organized as follows. After some preliminaries in Section 2 we deal with the proof of Theorem 1.3. The proofs of Theorem 1.6, Corollary 1.7 and Theorem 1.8 can be found in Sections 4-6.

2. Preliminaries

Given a domain $\Omega \subseteq \mathbb{R}^n\,, n\in\mathbb{N}$, and $1\le q\le\infty\,, k\in\mathbb{N}$, we need the usual Lebesgue and Sobolev spaces, $L^q(\Omega)$ and $W^{k,q}(\Omega)$ with norm $\|\cdot\|_{L^q(\Omega)}=\|\cdot\|_q$ and $\|\cdot\|_{W^{k,q}(\Omega)}=\|\cdot\|_{k,q}$, respectively. For two measurable functions $f\,,g$ with the property $f\cdot g\in L^1(\Omega)$, where $f\cdot g$ means the usual scalar product of vector or matrix fields, we set $\langle f,g\rangle_\Omega := \int_\Omega f(x)\cdot g(x)\,dx$. Note that the same symbol $L^q(\Omega)$ etc. will be used for spaces of scalar-, vector- or matrix-valued functions. By $v\otimes v=(v_iv_j)_{i,j=1}^n$ we denote the usual tensor product of $v\in\mathbb{R}^n$. Let $C^m(\Omega)\,, m=0,1,\dots,\infty$, denote the space of functions for which all partial derivatives of order $|\alpha|\le m$ ($|\alpha|<\infty$ if $m=\infty$) exist and are continuous. Further $C_0^m(\Omega)$ is the set of all functions from $C^m(\Omega)$ with compact support in Ω and $C^\infty_{0,\sigma}(\Omega) := \{\, v\in C_0^\infty(\Omega);\ \operatorname{div} v = 0\,\}$. Introduce $L^q_\sigma(\Omega) := \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_q}\,, 1<q<\infty$, and $W^{1,2}_{0,\sigma}(\Omega) := \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_{W^{1,2}}}$.

Given a Banach space X, $1\le p\le\infty$, and an interval $]0,T[$ we denote by $L^p(0,T;X)$ the space of (equivalence classes of) strongly measurable functions $f:]0,T[\to X$ such that $\|f\|_p := \Big(\int_0^T \|f(t)\|_X^p\,dt\Big)^{\frac{1}{p}} < \infty$ if $1\le p<\infty$

and $\|f\|_\infty := \operatorname{ess\,sup}_{t\in]0,T[} \|f(t)\|_X$ if $p=\infty$. Moreover, we define the set of *locally integrable* functions

$$L^p_{\mathrm{loc}}([0,T[;X) := \{\, u : [0,T[\to X \text{ strongly measurable}, \\ u \in L^p(0,T';X) \text{ for all } 0<T'<T\}.$$

If $X = L^q(\Omega)$, $1 \le q \le \infty$ we denote the norm of $L^p(0,T;L^q(\Omega))$ by $\|f\|_{q,p;T}$. In the following fix a general domain $\Omega \subseteq \mathbb{R}^3$. Let $P : L^2(\Omega) \to L^2_\sigma(\Omega)$ be the *Helmholtz projection.* We need the *Stokes operator* $A : \mathcal{D}(A) \subseteq L^2_\sigma(\Omega) \to L^2_\sigma(\Omega)$ and the *Laplace operator* $\Delta : \mathcal{D}(\Delta) \subseteq L^2(\Omega) \to L^2(\Omega)$. For a definition and further properties of these well known operators we refer to [21, Sections II.3.3 and III.2.1]. Fix $\alpha \in [-1,1]$. Introduce the fractional powers $A^\alpha, (-\Delta)^\alpha$ as in [21, Section II.3.2]. There holds that $A^\alpha : \mathcal{D}(A^\alpha) \to L^2_\sigma(\Omega)$ with dense domain $D(A^\alpha) \subseteq L^2_\sigma(\Omega)$ and dense range $\mathcal{R}(A^\alpha) \subseteq L^2_\sigma(\Omega)$ is a well defined, injective, closed operator such that

$$(A^\alpha)^{-1} = A^{-\alpha}, \quad \mathcal{R}(A^\alpha) = \mathcal{D}(A^{-\alpha}).$$

The same properties hold for $(-\Delta)^\alpha, \alpha \in [-1,1]$.

In general $\mathcal{D}(A^\alpha)$ will be equipped with the graph norm $\|u\|_2 + \|A^\alpha \phi\|_2$ which makes $\mathcal{D}(A^\alpha)$ to a Banach space since A^α is closed. Analogously $\mathcal{D}((-\Delta)^\alpha)$ becomes a Banach space when equipped with $\|\phi\|_2 + \|(-\Delta)^\alpha \phi\|_2$.

It is well known that $-A$ generates a uniformly bounded, analytic semigroup $\{e^{-tA}; t \ge 0\}$ on $L^2_\sigma(\Omega)$ and that Δ generates a uniformly bounded, analytic semigroup $\{e^{t\Delta}; t \ge 0\}$ on $L^2(\Omega)$. The decay estimates

$$\|A^\alpha e^{-tA} u\|_2 \le t^{-\alpha}\|u\|_2, \quad t>0, u \in L^2_\sigma(\Omega), \tag{2.1}$$

$$\|(-\Delta)^\alpha e^{t\Delta} \phi\|_2 \le t^{-\alpha}\|\phi\|_2, \quad t>0, \phi \in L^2(\Omega), \tag{2.2}$$

are satisfied for all $\alpha \in [0,1]$. For a proof of (2.1) we refer to [21, IV.(1.5.15)]. Analogously (2.2) holds. We need that

$$\|u\|_4 \le K\|A^{3/8} u\|_2 \quad \forall u \in \mathcal{D}(A^{3/8}), \tag{2.3}$$

$$\|\phi\|_4 \le K\|(-\Delta)^{3/8} \phi\|_2 \quad \forall \phi \in \mathcal{D}((-\Delta)^{3/8}), \tag{2.4}$$

are satisfied with an absolute constant $K>0$ (independent of Ω, u, ϕ). For a proof of (2.3) we refer to [21, Lemma III.2.4.2]. The proof of (2.4) is analogous.

Let us introduce the generalized operators $A^{-1/2}P\mathrm{div}$ and $(-\Delta)^{-1/2}\mathrm{div}$. Fix $F \in L^2(\Omega)$. By [21, Lemma III.2.6.1] there exists a unique element in $L^2_\sigma(\Omega)$ to be denoted by $A^{-1/2}P\mathrm{div}F$ such that

$$\langle A^{-1/2}P\mathrm{div}F, A^{1/2}w\rangle_\Omega = -\langle F, \nabla w\rangle_\Omega \quad \forall w \in W^{1,2}_{0,\sigma}(\Omega).$$

Further

$$\|A^{-1/2}P\mathrm{div}F\|_2 \le \|F\|_2. \tag{2.5}$$

Analogously $(-\Delta)^{-1/2}\mathrm{div}F$ is well defined by

$$\langle(-\Delta)^{-1/2}\mathrm{div}F, (-\Delta)^{1/2}\phi\rangle_\Omega = -\langle F, \nabla\phi\rangle_\Omega \quad \forall \phi \in H^1_0(\Omega).$$

In the lemma below we formulate integral equations which characterize weak solutions of the Boussinesq system (1.1).

Lemma 2.1. *Let $\Omega \subseteq \mathbb{R}^3$ be a general domain, let $0 < T \le \infty$, $g \in L^{8/5}_{loc}([0,T[; L^4(\Omega))$. Assume $f_1, f_2 \in L^1_{loc}([0,T[; L^2(\Omega))$ and $u_0 \in L^2_\sigma(\Omega)$, $\theta_0 \in L^2(\Omega)$. Then (u,θ) satisfying (1.2) is a weak solution of (1.1) if and only if the integral equations*

$$
\begin{aligned}
u(t) = e^{-tA}u_0 &+ \int_0^t e^{-(t-\tau)A}Pf_1(\tau)\,d\tau + \int_0^t e^{-(t-\tau)A}P\big(\theta(\tau)g(\tau)\big)\,d\tau \\
&- A^{1/2}\int_0^t e^{-(t-\tau)A}A^{-1/2}P\mathrm{div}\big(u(\tau)\otimes u(\tau)\big)\,d\tau\,,
\end{aligned}
\tag{2.6}
$$

$$
\begin{aligned}
\theta(t) = e^{t\Delta}\theta_0 &+ \int_0^t e^{(t-\tau)\Delta}f_2(\tau)\,d\tau \\
&- (-\Delta)^{1/2}\int_0^t e^{(t-\tau)\Delta}(-\Delta)^{-1/2}\mathrm{div}\big(\theta(\tau)u(\tau)\big)\,d\tau
\end{aligned}
\tag{2.7}
$$

are satisfied for almost all $t \in [0,T[$.

Proof. For a proof of (2.6) we refer to [21, Section IV.2.4]. To prove (2.7) we replace $-A$ by Δ and use an analogous argumentation as in the proof of (2.6). $\square$

3. Proof of Theorem 1.3

The idea of the proof of Theorem 1.3 is to construct $u \in L^8(0,T; L^4_\sigma(\Omega))$, $\theta \in L^{8/3}(0,T; L^4(\Omega))$ fulfilling (3.12) below and to show that (u,θ) is indeed a strong solution of (1.1). To solve this system with a fixed point result we need the estimates of Lemmas 3.1, 3.2. Since we allow a general domain instead of a smooth bounded domain we have to define the bilinear forms $\mathcal{F}_1, \mathcal{F}_2$ in a slightly different way than in [17, Lemma 3.2 with $q = 4, q_1 = 4, \alpha = \frac{7}{8}$].

Lemma 3.1. *Let $\Omega \subseteq \mathbb{R}^3$ be a general domain, let $0 < T \le \infty$, $g \in L^{8/3}(0,T; L^4(\Omega))$.*

(i) *Define the bilinear form*

$$
\begin{aligned}
&\mathcal{F}_1 : L^8(0,T; L^4_\sigma(\Omega)) \times L^8(0,T; L^4_\sigma(\Omega)) \to L^8(0,T; L^4_\sigma(\Omega))\,,\\
&\big(\mathcal{F}_1(u,v)\big)(t) := -A^{1/2}\int_0^t e^{-(t-\tau)A}A^{-1/2}P\mathrm{div}\big(u(\tau)\otimes v(\tau)\big)\,d\tau
\end{aligned}
$$

for a.a. $t \in [0,T[$.

Then

$$
\|\mathcal{F}_1(u,v)\|_{4,8;T} \le K\|u\otimes v\|_{2,4;T} \le K\|u\|_{4,8;T}\|v\|_{4,8;T} \tag{3.1}
$$

for all $u, v \in L^8(0,T; L^4_\sigma(\Omega))$ where $K > 0$ is an absolute constant (independent of Ω, T, g).

(ii) *Define the bilinear form*

$$
\begin{aligned}
&\mathcal{F}_2 : L^8(0,T; L^4_\sigma(\Omega)) \times L^{8/3}(0,T; L^4(\Omega)) \to L^{8/3}(0,T; L^4(\Omega))\,,\\
&\big(\mathcal{F}_2(u,\theta)\big)(t) := -(-\Delta^{1/2})\int_0^t e^{(t-\tau)\Delta}(-\Delta)^{-1/2}\mathrm{div}\big(\theta(\tau)u(\tau)\big)\,d\tau
\end{aligned}
$$

for a.a. $t \in [0,T[$.

Then

$$\|\mathcal{F}_2(u,\theta)\|_{4,\frac{8}{3};T} \le K\|\theta u\|_{2,2;T} \le K\|u\|_{4,8;T}\|\theta\|_{4,\frac{8}{3};T} \tag{3.2}$$

for all $u \in L^8(0,T;L^4_\sigma(\Omega))$, $\theta \in L^{8/3}(0,T;L^4(\Omega))$ *with an absolute constant* $K > 0$.

(iii) *Define the linear map*

$$\mathcal{L} : L^{8/3}(0,T;L^4(\Omega)) \to L^8(0,T;L^4_\sigma(\Omega))\,,$$

$$\big(\mathcal{L}\theta\big)(t) := \int_0^t e^{-(t-\tau)A} P\big(\theta(\tau)g(\tau)\big)\,d\tau \quad \text{for a.a. } t \in [0,T[.$$

Then

$$\|\mathcal{L}\theta\|_{4,8;T} \le K\|\theta g\|_{2,\frac{4}{3};T} \le K\|g\|_{4,\frac{8}{3};T}\|\theta\|_{4,\frac{8}{3};T} \tag{3.3}$$

for all $\theta \in L^{8/3}(0,T;L^4(\Omega))$ *with an absolute constant* $K > 0$.

Proof. Fix $u\,, v \in L^8(0,T;L^4_\sigma(\Omega))$ and $\theta \in L^{8/3}(0,T;L^4(\Omega))$.

Proof of (i). Estimates (2.1), (2.3), (2.5) in combination with the closedness of $A^{7/8}$ and [12, Theorem 3.7.12] imply

$$\begin{aligned}
\|(\mathcal{F}_1(u,v))(t)\|_4 &\le K\Big\|A^{7/8}\int_0^t e^{-(t-\tau)A}A^{-1/2}P\mathrm{div}\big(u(\tau)\otimes v(\tau)\big)\,d\tau\Big\|_2 \\
&= K\Big\|\int_0^t A^{7/8}e^{-(t-\tau)A}A^{-1/2}P\mathrm{div}\big(u(\tau)\otimes v(\tau)\big)\,d\tau\Big\|_2 \\
&\le K\int_0^T |t-\tau|^{-7/8}\|u(\tau)\otimes v(\tau)\|_2\,d\tau
\end{aligned} \tag{3.4}$$

for a.a. $t \in [0,T[$ with an absolute constant $K > 0$. Since $(1-\frac{7}{8})+\frac{1}{8} = \frac{1}{4}$ we can apply the Hardy-Littlewood inequality (see [22, Theorem V.1]) and Hölder's inequality to (3.4) and obtain

$$\|\mathcal{F}_1(u,v)\|_{4,8;T} \le K\|u\otimes v\|_{2,4;T} \le K\|u\|_{4,8;T}\|v\|_{4,8;T}$$

with an absolute constant $K > 0$.

Proof of (ii). As a consequence of (2.2), (2.4), an analogous estimate of (2.5) for $(-\Delta)^{-1/2}\mathrm{div}$ and [12, Theorem 3.7.12] we get

$$\begin{aligned}
\|(\mathcal{F}_2(u,\theta))(t)\|_4 &\le K\Big\|(-\Delta)^{7/8}\int_0^t e^{(t-\tau)\Delta}(-\Delta)^{-1/2}\mathrm{div}\big(\theta(\tau)u(\tau)\big)\,d\tau\Big\|_2 \\
&\le K\int_0^T |t-\tau|^{-7/8}\|\theta(\tau)u(\tau)\|_2\,d\tau
\end{aligned}$$

for a.a. $t \in [0,T[$ with an absolute constant $K > 0$. Using the Hardy-Littlewood inequality in the form $(1-\frac{7}{8})+\frac{1}{\left(\frac{8}{3}\right)} = \frac{1}{2}$ and Hölder's inequality yields

$$\|\mathcal{F}_2(u,\theta)\|_{4,\frac{8}{3};T} \le K\|\theta u\|_{2,2;T} \le K\|u\|_{4,8;T}\|\theta\|_{4,\frac{8}{3};T} \tag{3.5}$$

with an absolute constant $K > 0$.

Proof of (iii). From (2.3) it follows

$$\|(\mathcal{L}\theta)(t)\|_4 \le K\int_0^T |t-\tau|^{-3/8}\|\theta(\tau)g(\tau)\|_2\,d\tau \tag{3.6}$$

for a.a. $t \in [0,T[$ with an absolute constant $K > 0$. Applying the Hardy-Littlewood inequality to (3.6) shows that (3.3) holds. $\square$

Our next lemma reads as follows:

Lemma 3.2. *Let $\Omega \subseteq \mathbb{R}^3$ be a general domain, let $0 < T \leq \infty$. The following statements are fulfilled:*

(i) *For $f_1 \in L^{4/3}(0,T;L^2(\Omega))$ define*

$$(\mathcal{G}f_1)(t) := \int_0^t e^{-(t-\tau)A} P f_1(\tau)\, d\tau \quad \textit{for a.a. } t \in [0,T[.$$

Then

$$\|\mathcal{G}f_1\|_{4,8;T} \leq K\|f_1\|_{2,\frac{4}{3};T}. \tag{3.7}$$

with an absolute constant $K > 0$ (independent of Ω, T, f_1).

(ii) *For $f_2 \in L^1(0,T;L^2(\Omega))$ define*

$$(\mathcal{H}f_2)(t) := \int_0^t e^{(t-\tau)\Delta} f_2(\tau)\, d\tau \quad \textit{for a.a. } t \in [0,T[.$$

Then

$$\|\mathcal{H}f_2\|_{4,\frac{8}{3};T} \leq K\|f_2\|_{2,1;T} \tag{3.8}$$

with an absolute constant $K > 0$.

(iii) *We have*

$$\left(\int_0^T \|e^{t\Delta}\theta_0\|_4^{8/3}\, dt\right)^{3/8} \leq K\|\theta_0\|_2 \tag{3.9}$$

for all $\theta_0 \in L^2(\Omega)$ with an absolute constant $K > 0$.

Proof. In this proof we will make frequent use of the following interpolation inequality: For $E \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$ there holds

$$\begin{aligned} \|E\|_{4,\frac{8}{3};T} &\leq \|E\|_{2,\infty;T}^{1/4}\|E\|_{6,2;T}^{3/4} \\ &\leq K\|E\|_{2,\infty;T}^{1/4}\|(-\Delta)^{1/2}E\|_{2,2;T}^{3/4} \\ &\leq K\big(\|E\|_{2,\infty;T} + \|(-\Delta)^{1/2}E\|_{2,2;T}\big) \end{aligned} \tag{3.10}$$

with an absolute constant $K > 0$.

(i) Replace θg by f_1 in the proof of (3.3).

(ii) An analogous energy estimate as in ([21, IV.(2.3.4)]) yields

$$\|\mathcal{H}f_2\|_{2,\infty;T}^2 + \|(-\Delta)^{1/2}\mathcal{H}f_2\|_{2,2;T}^2 \leq K\|f\|_{2,1;T}^2$$

with an absolute constant $K > 0$. Therefore

$$\|\mathcal{H}f_2\|_{2,\infty;T} + \|(-\Delta)^{1/2}\mathcal{H}f_2\|_{2,2;T} \leq K\|f\|_{2,1;T}.$$

Consequently, by (3.10) it follows that (3.8) is satisfied.

(iii) Putting $\|e^{t\Delta}\theta_0\|_{2,\infty;T} \leq \|\theta_0\|_2$ and $\|(-\Delta)^{1/2}e^{t\Delta}\theta_0\|_{2,2;T} \leq \|\theta_0\|_2$ (analogous to [21, Lemma IV.1.5.3]) in (3.10) we get (3.9). $\square$

Now we have all ingredients at hand to show that the statements formulated in Theorem 1.3 are fulfilled.

Proof of Theorem 1.3. Step 1. Define $X := L^8(0,T;L^4_\sigma(\Omega))\,, Y := L^{8/3}(0,T;L^4(\Omega))$ and let $\mathcal{F}_1, \mathcal{F}_2\,, \mathcal{L}$ be defined as in Lemma 3.1. Choose an absolute constant $K > 0$ (independent on Ω, T, g) such that (3.1), (3.2), (3.3) are fulfilled. Furthermore introduce

$$\begin{aligned} E_1(t) &:= e^{-tA}u_0 + \int_0^t e^{-(t-\tau)A} Pf_1(\tau)\, d\tau\,, \\ E_2(t) &:= e^{t\Delta}\theta_0 + \int_0^t e^{(t-\tau)\Delta} f_2(\tau)\, d\tau \end{aligned}$$

for a.a. $t \in [0,T[$. From [1, Lemma 5.1] with $\eta = \frac{1}{2}$ and $\alpha_1 := K\,, \alpha_2 := K$ and $\alpha_3 := K\|g\|_{4,\frac{8}{3};T}$ we obtain the existence of a constant $c_* = c_*(K) > 0$ such that if

$$\|E_1\|_{4,8;T} + \|g\|_{4,\frac{8}{3};T}\|E_2\|_{4,\frac{8}{3};T} \leq c_* \tag{3.11}$$

then there exist $u \in L^8(0,T;L^4_\sigma(\Omega))\,, \theta \in L^{8/3}(0,T;L^4(\Omega))$ fulfilling

$$\begin{aligned} u &= E_1 + \mathcal{F}_1(u,u) + \mathcal{L}\theta\,, \\ \theta &= E_2 + \mathcal{F}_2(u,\theta). \end{aligned} \tag{3.12}$$

By construction c_* is an absolute constant. Looking at (3.7), (3.8) we get the estimate

$$\begin{aligned} &\|E_1\|_{4,8;T} + \|g\|_{4,\frac{8}{3};T}\|E_2\|_{4,\frac{8}{3};T} \\ &\leq K_*\Big[\Big(\int_0^T \|e^{-tA}u_0\|_4^8\, dt\Big)^{1/8} + \|f_1\|_{2,\frac{4}{3};T} + \|g\|_{4,\frac{8}{3};T}\Big(\int_0^T \|e^{t\Delta}\theta_0\|_4^{8/3}\, dt\Big)^{3/8} \\ &\quad + \|g\|_{4,\frac{8}{3};T}\|f_2\|_{2,1;T}\Big] \end{aligned}$$

with an absolute constant $K_* > 0$. Define $\epsilon_* := \frac{c_*}{3K_*}$. Thus, if (1.3), (1.4), (1.5) are satisfied, then (3.11) is fulfilled and consequently there is a solution $u \in L^8(0,T;L^4_\sigma(\Omega))\,, \theta \in L^{8/3}(0,T;L^4(\Omega))$ of (3.12).

Step 2. Let (u,θ) with $u \in L^8(0,T;L^4_\sigma(\Omega))$ and $\theta \in L^{8/3}(0,T;L^4(\Omega))$ be a solution of (3.12). Consider any $0 < T' \leq T$ with $T' < \infty$. It follows $u \otimes u \in L^2(0,T';L^2(\Omega))$. Consequently, from [21, Lemma IV.2.4.2] (with $F = u \otimes u$) we get

$$\mathcal{F}_1(u,u) \in \mathcal{LH}_{T'} := L^\infty(0,T';L^2_\sigma(\Omega)) \cap L^2(0,T';W^{1,2}_{0,\sigma}(\Omega)). \tag{3.13}$$

From $P(\theta g) \in L^{4/3}(0,T;L^2(\Omega))$ and [21, Lemma IV.2.4.2] (with $f = \theta g$) it follows $\mathcal{L}\theta \in \mathcal{LH}_{T'}$. Thus $u \in \mathcal{LH}_{T'}$. Using $\theta u \in L^2(0,T;L^2(\Omega))$ and an analogous version of [21, Lemma IV.2.4.2] with A replaced by $-\Delta$ we get

$$\theta \in L^\infty(0,T';L^2(\Omega)) \cap L^2(0,T';H^1_0(\Omega)).$$

Looking at Lemma 2.1 we get that (u,θ) is also a weak solution of (1.1) in $[0,T[\times\Omega$. Since $u \in L^8(0,T;L^4(\Omega))$ we have that (u,θ) is indeed a strong solution of (1.1). The uniqueness of a strong solution of the Boussinesq equations (1.1) in $[0,T[\times\Omega$ follows from Theorem 4.1 below.

Step 3. From (3.9) we get $\int_0^T \|e^{t\Delta}\theta_0\|_4^{8/3}\, dt < \infty$. The sufficiency of (1.6) for the existence of $0 < T' \leq T$ and a strong solution of (1.1) in $[0,T'[\times\Omega$

follows from Theorem 1.3 (i). Assume that (u,θ) is a strong solution of (1.1) in $[0,T'[\times\Omega$ where $0<T'\le T$. From (2.6) we get

$$e^{-tA} = u(t) - \mathcal{F}_1(u,u)(t) - \mathcal{L}\theta(t) - \int_0^t e^{-(t-\tau)A} P f_1(\tau)\,d\tau\,, \quad \text{a.a. } t\in[0,T[.$$

By Lemma 3.1 (i), (iii) and Lemma 3.2 (i) it follows $e^{-tA}u_0 \in L^8(0,T';L^4(\Omega))$. From (2.1), (2.3) we get $\int_{T'}^\infty \|e^{-tA}u_0\|_4^8\,dt \le c\int_{T'}^\infty t^{-3}\|u_0\|_2^8\,dt < \infty$. Altogether (1.6) holds. □

4. Proof of Theorem 1.6

For the proof of Theorem 1.6 we need the following result.

Theorem 4.1. *Let $\Omega\subseteq\mathbb{R}^3$ be a general domain, let $0<T\le\infty\,, g\in L^{8/3}_{loc}([0,T[;L^4(\Omega))$, let $f_1\in L^{4/3}_{loc}([0,T[;L^2(\Omega))\,, f_2\in L^1_{loc}([0,T[;L^2(\Omega))$. Assume that (u,θ) and (v,Θ) are weak solutions of (1.1) such that additionally $u\,,v\in L^8_{loc}([0,T[;L^4(\Omega))$. Then $u(t)=v(t)$ and $\theta(t)=\Theta(t)$ for almost all $t\in[0,T[$.*

Proof. The proof is analogous to the proof of [17, Theorem 1.5] with exponents $s=8\,,q=4$ and $s_1=\frac{8}{3}\,,q_1=4$. Indeed, since [17, Lemma 3.2] can be replaced by Lemma 3.1 there are no problems occurring in this proof although we consider a general domain instead of a smooth bounded domain in [17, Theorem 1.5]. □

We proceed with the following lemma.

Lemma 4.2. *Let $\Omega\subseteq\mathbb{R}^3$ be a general domain, let $0<T\le\infty$, let $g\in L^{8/3}(0,T;L^4(\Omega))$. Consider $f_1\in L^{4/3}(0,T;L^2(\Omega))\,,f_2\in L^1(0,T;L^2(\Omega))$ and $u_0\in L^2_\sigma(\Omega)\,,\theta_0\in L^2(\Omega)$. Let $(u,\theta)\,,(v,\Theta)$ be weak solutions of (1.1) in $[0,T[\times\Omega$, let $t_0\in[0,T[$ such that $u(t_0)=v(t_0)\,,\theta(t_0)=\Theta(t_0)$. Assume that (u,θ) and (v,Θ) satisfy (1.11) with $s=t_0$ and all $t\ge t_0$. Further assume*

$$\int_0^\infty \|e^{-tA}u(t_0)\|_4^8\,dt<\infty. \tag{4.1}$$

Then there exists $\delta>0$ such that $u(t)=v(t)$ and $\theta(t)=\Theta(t)$ for all $t\in[t_0,t_0+\delta[$.

Proof. Define

$$\tilde u(t):=u(t+t_0)\,,\quad \tilde\theta(t):=\theta(t+t_0)\,, \tag{4.2}$$

$$\tilde v(t):=v(t+t_0)\,,\quad \tilde\Theta(t):=\Theta(t+t_0) \tag{4.3}$$

for all $t\in[0,T-t_0[$. A standard argumentation, which uses the weak continuity of $(u,\theta)\,,(v,\Theta)$, shows that $(\tilde u,\tilde\theta)\,,(\tilde v,\tilde\Theta)$ are weak solutions of (1.1) in $[0,T-t_0[\times\Omega$ with initial values $u_0:=u(t_0)\,,\theta_0:=\theta(t_0)$ and $f_1(\cdot+t_0)$, $f_2(\cdot+t_0)\,,g(\cdot+t_0)$.

Define $f(t):=f_1(t+t_0)+\theta(t+t_0)g(t+t_0)$, a.a. $t\in[0,T-t_0[$. By Theorem 1.4 with $f\in L^{8/3}(0,T-t_0;L^4(\Omega))$ and initial value u_0 there exists $0<\delta\le T-t_0$ and a strong solution $w\in L^8(0,\delta;L^4(\Omega))$ of the Navier-Stokes equations (1.10). By construction, $\tilde u$ is a weak solution of (1.10) in $[0,T-t_0[\times\Omega$ satisfying the energy inequality, i.e. (1.11) holds with $s=0$ and all $t\in[0,T-t_0[$. Altogether, all requirements of Serrin's uniqueness theorem (see [21, Theorem V.1.5.1]) are fulfilled. We obtain $\tilde{u}(t) = w(t)$ for a.a. $t \in [0,\delta[$ and consequently $\tilde{u} \in L^8(0,\delta;L^4(\Omega))$. In the same way as above we can prove (after a possible reduction of δ) that $\tilde{v} \in L^8(0,\delta;L^4(\Omega))$.

Now the requirements of Theorem 4.1 are fulfilled and therefore $\tilde{u}(t) = \tilde{v}(t)$ and $\tilde{\theta}(t) = \tilde{\Theta}(t)$ for a.a. $t \in [0,\delta[$. This means $u(t) = v(t)\,, \theta(t) = \Theta(t)$ for a.a. $t \in [t_0, t_0+\delta[$. By weak continuity these identities are even fulfilled for all $t \in [0,\delta[$. □

Proof of Theorem 1.6. From Lemma 4.2 we get the existence of $\delta > 0$ such that $u(t) = v(t)$ and $\theta(t) = \Theta(t)$ for all $t \in [0,\delta[$. Define

$$t_* := \sup\{\, t \in]0,T[; u(\tau) = v(\tau) \text{ and } \theta(\tau) = \Theta(\tau) \quad \text{for all } \tau \in [0,t[\,\}.$$

To finish the proof we have to show $t_* = T$. Assume by contradiction that $t_* < T$. By weak continuity $u(t_*) = v(t_*)$. Consequently, looking at assumption (iii) of Theorem 1.6 we see that $u\,, v : [0,t_*] \to L^2_\sigma(\Omega)$ are both strongly continuous functions.

From (1.11) we obtain $0 < t_j < t_*\,, j \in \mathbb{N}$, with $t_j \to t_*$ as $j \to \infty$ such that

$$\frac{1}{2}\|u(t)\|_2^2 + \int_{t_j}^t \|\nabla u\|_2^2\,d\tau \le \frac{1}{2}\|u(t_j)\|_2^2 + \int_{t_j}^t \langle \theta g, u\rangle_\Omega\,d\tau + \int_{t_j}^t \langle f_1, u\rangle_\Omega\,d\tau \quad (4.4)$$

for all $t \in [t_j, T[$ and $j \in \mathbb{N}$. Due to the strong continuity of u we can pass to the limit in (4.4) and obtain

$$\frac{1}{2}\|u(t)\|_2^2 + \int_{t_*}^t \|\nabla u\|_2^2\,d\tau \le \frac{1}{2}\|u(t_*)\|_2^2 + \int_{t_*}^t \langle \theta g, u\rangle_\Omega\,d\tau + \int_{t_*}^t \langle f_1, u\rangle_\Omega\,d\tau$$

for all $t \in [t_*, T[$. Analogously we can prove that (v,Θ) satisfies (1.11) with $s = t_*$ and all $t \in [t_*,T[$. From Lemma 4.2 we obtain $\epsilon > 0$ such that $u(t) = v(t)$ and $\theta(t) = \Theta(t)$ for all $t \in [t_*, t_*+\epsilon[$. This is a contradiction to the definition of t_*. Thus $t_* = T$. □

5. Proof of Corollary 1.7

Step 1. Assume $u \in L^8(0,T;L^4(\Omega))$. Then $u \otimes u \in L^2_{\text{loc}}([0,T[;L^2(\Omega))$. By [21, Theorem IV.2.3.1] with $f = \theta g + f_1$ we get that $u : [0,T[\to L^2_\sigma(\Omega)$ is strongly continuous and that (u,θ) satisfy (1.11) (even as an equality). Considering (u,θ) as a weak solution of the Boussinesq equations (1.1) in $[t_0, T-t_0[$ we obtain from Theorem 1.3 (ii) that (1.12) holds. Theorem 1.6 yields $u(t) = v(t)\,, \theta(t) = \Theta(t)$ for all $t \in [0,T[$.

Step 2. Assume $u \in L^\infty(0,T;\mathcal{D}(A^{1/4}))$. Using the trivial imbedding $\mathcal{D}(A^{1/4}) \hookrightarrow L^2_\sigma(\Omega)$ we can prove that $u : [0,T[\to \mathcal{D}(A^{1/4})$ is weakly continuous. Especially $u(t) \in \mathcal{D}(A^{1/4})$ for all $t \in [0,T[$. Further, the continuous imbedding $\mathcal{D}(A^{1/4}) \hookrightarrow L^3(\Omega)$ implies $u \in L^\infty(0,T;L^3(\Omega))$. By interpolation and Sobolev's imbedding theorem

$$\|u \otimes u\|_{2,2;T} \le c\|u\|_{L^2(0,T;H^1(\Omega))}\|u\|_{3,\infty;T}.$$

Therefore, see [21, Theorem IV.2.3.1], it follows that $u : [0,T[\to L^2_\sigma(\Omega)$ is strongly continuous and that (u,θ) fulfils (1.11). From (2.3) and [21, IV.(1.5.24)] we get

$$\int_0^T \|e^{-tA}u_0\|_4^8\,dt \le K\int_0^T \|A^{1/8}e^{-tA}A^{1/4}u_0\|_2^8\,dt \le K\|A^{1/4}u_0\|_2 \tag{5.1}$$

with an absolute constant $K>0$. The claim follows from Theorem 1.6. □

6. Proof of Theorem 1.8

Let us introduce for $0<T'\le T\le\infty$ and $f\in L^s(0,T;L^q(\Omega))$ the notation $\|f\|_{q,s;T',T} := \left(\int_{T'}^T \|f(t)\|_q^s\,dt\right)^{\frac{1}{s}}$.

Step 1. (Preparation) Let ϵ_* be the constant constructed in Theorems 1.3, 1.4. From (1.13) we get

$$\Big(\int_0^\delta \|e^{-\tau A}u_0\|_4^8\,d\tau\Big)^{\frac{1}{8}} \le c_1\delta^{1/8}\|u_0\|_4 \tag{6.1}$$

for all $\delta>0$ with a fixed constant $c_1=c_1(\Omega,q)>0$. Define

$$c_* := \min\Big\{\frac{\epsilon_*^{8/s}}{c_1^{8/s}2^{1/s}},\epsilon_*\Big\}.$$

Let us introduce exponents $1<\tilde\mu,\tilde s<\infty$ by

$$\frac{1}{8/3}=\frac{1}{\mu}+\frac{1}{\widetilde\mu},\quad \frac{1}{4/3}=\frac{1}{s}+\frac{1}{\widetilde s}.$$

Now we define

$$\delta := \min\Big\{\Big(\frac{\epsilon_*}{c_1\|u_0\|_4}\Big)^8,\Big(\frac{\epsilon_*}{\|g\|_{4,\mu;T}\|\theta\|_{2,\infty;T}}\Big)^{\widetilde\mu},$$
$$\Big(\frac{\epsilon_*}{2\|g\|_{4,\mu;T}\|f_2\|_{2,1;T}}\Big)^{\widetilde\mu},\Big(\frac{\epsilon_*}{2\|f_1\|_{2,s;T}\|}\Big)^{\widetilde s},T\Big\}.$$

Consider any $t_0,t_1\in[0,T[$ with $|t_1-t_0|\le\delta$. Then

$$\begin{aligned}\|g\|_{4,\frac{8}{3};t_0,t_1}\|\theta(t_0)\|_2 &\le (t_1-t_0)^{1/\widetilde\mu}\|g\|_{4,\mu;t_0,t_1}\|\theta\|_{2,\infty;T}\\ &\le \delta^{1/\widetilde\mu}\|g\|_{4,\mu;T}\|\theta\|_{2,\infty;T}\\ &\le \epsilon_*.\end{aligned}\tag{6.2}$$

Analogously

$$\begin{aligned}\|f_1\|_{2,\frac{4}{3};t_0,t_1}+\|g\|_{4,\frac{8}{3};t_0,t_1}\|f_2\|_{2,1;t_0,t_1} &\le \delta^{1/\widetilde s}\|f_1\|_{2,s;T}+\delta^{1/\widetilde\mu}\|g\|_{4,\mu;T}\|f_2\|_{2,1;T}\\ &\le \epsilon_*.\end{aligned}\tag{6.3}$$

Step 2. (Regularity on $[0,\delta[$) By construction $\left(\int_0^\delta \|e^{-\tau A}u_0\|_4^8\,d\tau\right)^{1/8}\le\epsilon_*$. Inserting $t_0:=0, t_1:=\delta$ in (6.2), (6.3) we see that (1.3), (1.5), (1.7) are satisfied on $[0,\delta[$. By Theorem 1.3 (i) there exists a strong solution (v,Θ) of the Boussinesq equations (1.1) in $[0,\delta[\times\Omega$ which coincides by Corollary 1.7 with (u,θ). Thus $u\in L^8(0,\delta;L^4(\Omega))$.

Step 3. (Global regularity) Fix an arbitrary $t\in[\delta,T-\frac{\delta}{2}[$.

Assertion. There exists $t_0 \in]t-\frac{\delta}{2}, t[\setminus N$ such that

$$(t_1-t_0)^{\frac{s}{8}}\|u(t_0)\|_4^s \le \frac{2}{\delta}\int_{t-\frac{\delta}{2}}^t (t_1-\tau)^{\frac{s}{8}}\|u(\tau)\|_4^s\,d\tau \tag{6.4}$$

where $t_1 := t+\frac{\delta}{2}$ and $N \subseteq]0,T[$ denotes the null set in Definition 1.5.

Proof of (6.4). If such a t_0 would not exist we could integrate the estimate

$$(t_1-t_0)^{\frac{s}{8}}\|u(t_0)\|_4^s > \frac{2}{\delta}\int_{t-\frac{\delta}{2}}^t (t_1-\tau)^{\frac{s}{8}}\|u(\tau)\|_4^s\,d\tau\,, \quad \text{a.a. } t_0 \in]t-\frac{\delta}{2}, t[$$

over $]t-\frac{\delta}{2}, t[$ and get a contradiction to (6.4). □

Using the definition of c_*, δ and (6.1), (6.4) we obtain

$$\begin{aligned}\Big(\int_0^{t_1-t_0}\|e^{-\tau A}u(t_0)\|_4^8\,d\tau\Big)^{\frac{1}{8}} &\le c_1\big((t_1-t_0)^{\frac{s}{8}}\|u(t_0)\|_4^s\big)^{\frac{1}{s}}\\ &\le c_1\Big(\frac{2}{\delta}\int_{t-\frac{\delta}{2}}^t (t_1-\tau)^{\frac{s}{8}}\|u(\tau)\|_4^s\,d\tau\Big)^{\frac{1}{s}}\\ &\le c_1 2^{\frac{1}{s}}\delta^{\frac{1}{8}-\frac{1}{s}}\Big(\int_0^T\|u(\tau)\|_4^s\,d\tau\Big)^{1/s}\\ &\le c_1 2^{\frac{1}{s}}\Big(\frac{\epsilon_*}{c_1\|u_0\|_4}\Big)^{1-\frac{8}{s}}\frac{c_*}{\|u_0\|_4^{\frac{8}{s}-1}}\\ &\le \epsilon_*.\end{aligned} \tag{6.5}$$

From (6.2), (6.3), (6.5) in combination with Theorem 1.3 (i) we get the existence of a strong solution (v,Θ) of the Boussinesq equations (1.1) in $[0,t_1-t_0[\times\Omega$ with initial values $v(0)=u(t_0)\,,\Theta(0)=\theta(t_0)$ and $f_1(\cdot+t_0), f_2(\cdot+t_0), g(\cdot+t_0)$. As in (4.2) introduce $\tilde{u}(t) := u(t+t_0)\,,\tilde{\theta}(t) := \theta(t+t_0)$ for $t\in[0,t_1-t_0[$. From Corollary 1.7 we get $\tilde{u}=v$ and $\tilde{\theta}=\Theta$ in $[0,t_1-t_0[$ and consequently $u\in L^8(t_0,t_1;L^4(\Omega))$. Altogether $u\in L^8(t,t+\frac{\delta}{2};L^4(\Omega))$ for all $t\in[\delta,T-\frac{\delta}{2}[$. Since δ is independent of t we obtain $u\in L^8(\delta,T;L^4(\Omega))$. It follows $u\in L^8(0,T;L^4(\Omega))$ which means that (u,θ) is a strong solution of (1.1) in $[0,T[\times\Omega$. The uniqueness statement at the end of Theorem 1.8 follows from Corollary 1.7. □

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