

Finite sections of truncated Toeplitz operators

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Abstract

We describe the C^* -algebra associated with the finite sections discretization of truncated Toeplitz operators on the model space K_u^2 where u is an infinite Blaschke product. As consequences, we get a stability criterion for the finite sections discretization and results on spectral and pseudospectral approximation.

Keywords: Model spaces, truncated Toeplitz operators, Widom's identity, stability of the finite sections discretization

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1 Truncated Toeplitz operators

Let H^2 denote the standard Hardy space on the unit disk \mathbb{D} , i.e. the Hilbert space of all holomorphic functions on \mathbb{D} which have square-summable Taylor coefficients. As usual, we identify H^2 with its space of non-tangential boundary functions, which is a closed subspace of the Lebesgue space $L^2(\mathbb{T})$ with normalized Lebesgue measure m on the unit circle \mathbb{T} . The orthogonal projection from $L^2(\mathbb{T})$ onto H^2 is denoted by P .

Every function $a \in L^\infty(\mathbb{T})$ defines an operator of multiplication on $L^2(\mathbb{T})$, which we denote by aI . The *Toeplitz operator* induced by a is the operator $T(a) := PaI|_{H^2}$, acting from H^2 to H^2 . The Toeplitz operator with generating function $a(z) = z$ is the operator S of forward shift, $(Sf)(z) = zf(z)$. Its adjoint S^* , the backward shift operator, is given by $(S^*f)(z) = z^{-1}(f(z) - f(0))$.

Let u be a non-constant inner function, i.e., u is holomorphic on \mathbb{D} and $|u(t)| = 1$ for $t \in \mathbb{T}$ (the following becomes trivial when u is constant). The subspace $K_u^2 := H^2 \ominus uH^2$ is a proper nontrivial invariant subspace of S^* . Conversely, every proper nontrivial invariant subspace of S^* is of this form by a celebrated theorem of Beurling. The spaces K_u^2 are also known as *model spaces*. We denote the orthogonal projection from $L^2(\mathbb{T})$ onto K_u^2 by P_u . If M_u and $M_{\bar{u}}$ denote the operators of multiplication by u and \bar{u} on $L^2(\mathbb{T})$, then $P_u = P - M_u P M_{\bar{u}}$.

For $a \in L^\infty(\mathbb{T})$, the *truncated Toeplitz operator* (TTO for short) generated by a is the operator $T_u(a) := P_u a I|_{K_u^2}$ acting from K_u^2 to K_u^2 . Truncated Toeplitz operators share many of their properties with their relatives, the Toeplitz operators

on H^2 , to which we here sometimes refer as *classical* Toeplitz operators, but there are also some striking differences. For example, the function a is in general not uniquely determined by the operator $T_u(a)$ it generates, and the truncated shift S_u , i.e., the TTO with generating function $a(z) = z$, is the sum of a unitary and a compact operator (hence a Fredholm operator with index 0), whereas its classical counterpart S is a proper isometry (and a Fredholm operator of index -1). Moreover, whereas the spectrum of the classical shift S is the closed unit disk \mathbb{D} , the spectrum of the truncated shift S_u coincides with the so-called *spectrum*

$$\sigma(u) := \{\lambda \in \overline{\mathbb{D}} : \liminf_{z \rightarrow \lambda} |u(z)| = 0\}$$

of the inner function u ([7, Lemma 2.5]).

The following collection of positive results is taken from and proved in [2]. We denote by $\mathbb{T}_u(C)$ the smallest closed C^* -subalgebra of $L(K_u^2)$ which contains the truncated shift S_u and the identity operator (the notation will be justified by assertion (d) in the theorem below). Further, we write $\text{Comm } \mathcal{A}$ for the commutator ideal of a C^* -algebra \mathcal{A} , i.e., for the smallest closed ideal of \mathcal{A} which contains all commutators $ab - ba$ with $a, b \in \mathcal{A}$. The essential spectrum of an operator A is denoted by $\sigma_{\text{ess}}(A)$, and its essential norm by $\|A\|_{\text{ess}}$.

Theorem 1 *Let u be a non-constant inner function. Then*

- (a) *for $a, b \in C(\mathbb{T})$, $T_u(a)T_u(b) - T_u(ab)$ is compact.*
- (b) $\text{Comm}(\mathbb{T}_u(C)) = K(K_u^2)$.
- (c) $\mathbb{T}_u(C)/K(K_u^2)$ *is $*$ -isomorphic to $C(\sigma(u) \cap \mathbb{T})$.*
- (d) *for $a \in C(\mathbb{T})$, the TTO $T_u(a)$ is compact if and only if $a(\sigma(u) \cap \mathbb{T}) = \{0\}$.*
- (e) $\mathbb{T}_u(C) = \{T_u(a) + K : a \in C(\mathbb{T}), K \in K(K_u^2)\}$.
- (f) *for $a \in C(\mathbb{T})$, $\sigma_{\text{ess}}(T_u(a)) = a(\sigma_{\text{ess}}(S_u))$.*
- (g) *for $a \in C(\mathbb{T})$, $\|T_u(a)\|_{\text{ess}} = \sup\{|a(t)| : t \in \sigma(u) \cap \mathbb{T}\}$.*
- (h) *Every operator in $\mathbb{T}_u(C)$ is the sum of a normal and a compact operator.*

Moreover,

$$\{0\} \longrightarrow K(K_u^2) \xrightarrow{\text{id}} \mathbb{T}_u(C) \xrightarrow{\pi} C(\sigma(u) \cap \mathbb{T}) \longrightarrow \{0\}$$

is a short exact sequence, with the mapping π given by $T_u(a) + K \mapsto a|_{\sigma(u) \cap \mathbb{T}}$.

Here is an outline of the contents of the paper. In Section 2 we will single out a sequence (P_{u_n}) of finite rank projections which converge strongly to the identity operator on K_u^2 . The operator $P_{u_n}T_u(a)P_{u_n}$ is considered as a finite section of the truncated Toeplitz operator $T_u(a)$. In Section 4, Theorem 10, we describe the C^* -algebra $\mathcal{S}(\mathbb{T}_u(C))$ generated by all sequences of the form $(P_{u_n}T_u(a)P_{u_n})_{n \geq 1}$ with a a continuous function. This description is based of a formula of Widom-type that we will derive in Section 3. As consequences of Theorem 10, we get a

stability criterion and results on spectral approximation. The stability criterion (Theorem 11) says that a sequence (A_n) in $\mathcal{S}(\mathbb{T}_u(C))$ is stable if and only if its strong limit A is invertible. In the case when the A_n are the finite sections of a truncated Toeplitz operator, this result is due to Treil [8]. One advantage of Theorem 11 is that it implies (without any additional effort) results on spectral and pseudospectral approximation as well as on the asymptotic behavior of the small singular values of A_n ; see the end of Section 4.

2 A filtration and Widom's identity

Recall that a *filtration* on a Hilbert space H is a sequence $\mathcal{P} = (P_n)$ of orthogonal projections of finite rank on H which converges strongly to the identity operator on H . To define a filtration on the model space K_u^2 we specify u to be a Blaschke product, as follows. A single *Blaschke factor* is a function on the unit disk of the form

$$b_\lambda(z) := \begin{cases} z & \text{if } \lambda = 0, \\ \frac{\lambda - z}{1 - \bar{\lambda}z} \frac{|\lambda|}{\lambda} & \text{if } \lambda \in \mathbb{D} \setminus \{0\}. \end{cases} \quad (1)$$

A *Blaschke product* is then a function

$$u = \prod_{\lambda \in \mathbb{D} : k(\lambda) > 0} b_\lambda^{k(\lambda)} \quad (2)$$

which satisfies the *Blaschke condition*

$$\sum_{\lambda \in \mathbb{D}} k(\lambda)(1 - |\lambda|) < \infty. \quad (3)$$

If u is a finite Blaschke product, i.e., if u is of the form (2) with $k(\lambda) = 0$ for all but finitely many $\lambda \in \mathbb{D}$, then (3) is satisfied. Conversely, if (3) holds, then every disk $\{z \in \mathbb{D} : |z| \leq r\}$ with $0 < r < 1$ contains only finitely many λ with $k(\lambda) \neq 0$. Thus, if u in (2) is an (infinite) Blaschke product, the number of its non-one factors is countable. We order the λ with $k(\lambda) \neq 0$ in a sequence $(\lambda_k)_{k \geq 1}$ in such a way that $|\lambda_k| \leq |\lambda_{k+1}|$ for all k . Then (2) and (3) can be written as

$$u = \prod_{k=1}^{\infty} b_{\lambda_k} \quad \text{with} \quad \sum_{k=1}^{\infty} (1 - |\lambda_k|) < \infty. \quad (4)$$

Blaschke products are inner functions. If u is an infinite Blaschke product, we use its product representation (4) to define a filtration on the associated model space K_u^2 . For $n \geq 1$, set $u_n := \prod_{k=1}^n b_{\lambda_k}$ and let P_{u_n} denote the orthogonal projection from $L^2(\mathbb{T})$ onto $K_{u_n}^2$. The projections P_{u_n} own the following properties.

Proposition 2 (a) *The projections P_{u_n} have a finite spatial rank.*

- (b) $P_{u_n} \rightarrow P_u$ on $L^2(\mathbb{T})$ strongly as $n \rightarrow \infty$.
(c) $P_{u_m}P_{u_n} = P_{u_n}P_{u_m} = P_{u_{\min\{m,n\}}}$ for $m, n \geq 1$.
(d) $P_{u_n}P_u = P_uP_{u_n} = P_{u_n}$ and $P_{u_n}P = PP_{u_n} = P_{u_n}$ for $n \geq 1$.

Hints to the proof. Assertion (a) is the "Lemma on Finite Dimensional Subspaces" in [4, p. 33]. For assertion (b) observe that the u_n converge to u uniformly on compact subsets of \mathbb{D} by the "Lemma on Blaschke Products" in [4, p. 280]. Thus, the model spaces $K_{u_n}^2$ converge to K_u^2 by the "Theorem on Lower Limits" in [4, p. 34] (note that the limit of the u_n exists; so the set of its limit points contains only one element, which clearly coincides with the greatest common divisor in the formulation of that theorem). Thus, $P_{u_n} \rightarrow P_u$ on $L^2(\mathbb{T})$ strongly by the definition in [4, p. 34].

Finally, u_n is a divisor of u and of u_m for $m \geq n$. Corollary 8 in [4, p. 19] then implies that $K_{u_n}^2 \subseteq K_{u_m}^2 \subseteq K_u^2 \subseteq H^2$, whence assertions (c) and (d). ■

Thus, the restrictions of the projections P_{u_n} to the model space K_u^2 form a filtration on K_u^2 by the preceding proposition. We denote this filtration by \mathcal{P}_u and write $\mathcal{P}_u = (P_{u_n})_{n \geq 1}$, not distinguishing between a projection P_{u_n} on $L^2(\mathbb{T})$ and its restriction to K_u^2 .

The study of the finite sections discretization (FSD for short) for (classical) Toeplitz operators is dominated by Widom's identity

$$P_n T(ab) P_n = P_n T(a) P_n T(b) P_n + P_n H(a) H(\tilde{b}) P_n + R_n H(\tilde{a}) H(b) R_n. \quad (5)$$

To explain this identity, we need some notation. Let $J : L^2(\mathbb{T}) \rightarrow L^2(\mathbb{T})$ denote the operator $(Jf)(t) = t^{-1}f(t^{-1})$. One easily checks that, for every function $a \in L^\infty(\mathbb{T})$, JaJ is the operator of multiplication by the function $\tilde{a}(t) := a(t^{-1})$. Then $H(a) := PaIJ|_{H^2}$ is the (classical) Hankel operator, $R_n := H(t^n)$, and $P_n := R_n^2$. Note that P_n is just the orthogonal projection onto the linear span of the functions t^n with $n \in \{0, 1, \dots, n-1\}$ and that $R_n = R_n^*$.

Our goal is to achieve a comparable identity for the finite sections of TTO, where the role of the R_n in Widom's identity is played by the operators

$$R_{u_n} := P_{u_n} M_{u_n} J \quad \text{and} \quad R_{u_n}^* = J M_{\overline{u_n}} P_{u_n}. \quad (6)$$

Theorem 3 (Widom's identity for TTO) *Let $a, b \in L^\infty(\mathbb{T})$. Then*

$$\begin{aligned} P_{u_n} T_u(ab) P_{u_n} \\ = P_{u_n} T_u(a) P_{u_n} T_u(b) P_{u_n} + P_{u_n} H(a) H(\tilde{b}) P_{u_n} + R_{u_n} H(\tilde{a}) H(b) R_{u_n}^*. \end{aligned}$$

Proof. By Proposition 2 (d) and since $P_{u_n} = P - M_{u_n} P M_{\overline{u_n}}$, we find

$$\begin{aligned} P_{u_n} T_u(a) P_{u_n} T_u(b) P_{u_n} \\ = P_{u_n} P a P_{u_n} b P P_{u_n} \end{aligned}$$

$$\begin{aligned}
&= P_{u_n} P a P b P P_{u_n} - P_{u_n} a M_{u_n} P M_{\bar{u}_n} b P_{u_n} \\
&= P_{u_n} P a b P P_{u_n} - P_{u_n} P a Q b P P_{u_n} - P_{u_n} M_{u_n} a P b M_{\bar{u}_n} P_{u_n} \quad \text{with } Q := I - P \\
&= P_{u_n} a b P_{u_n} - P_{u_n} P a Q J^2 Q b P P_{u_n} \\
&\quad - P_{u_n} M_{u_n} P a P b M_{\bar{u}_n} P_{u_n} - P_{u_n} M_{u_n} Q a P b M_{\bar{u}_n} P_{u_n} \\
&= P_{u_n} T_u(ab) P_{u_n} - P_{u_n} H(a) H(\tilde{b}) P_{u_n} - P_{u_n} M_{u_n} Q a P b M_{\bar{u}_n} P_{u_n}.
\end{aligned}$$

In the last line we used that $P_{u_n} M_{u_n} P f \in P_{u_n} (u_n H^2) = 0$ for $f \in L^2(\mathbb{T})$, whence $P_{u_n} M_{u_n} P = 0$. Then $P M_{\bar{u}_n} P_{u_n} = (P_{u_n} M_{u_n} P)^* = 0$, too, and we can proceed with

$$\begin{aligned}
&P_{u_n} T_u(a) P_{u_n} T_u(b) P_{u_n} \\
&= P_{u_n} T_u(ab) P_{u_n} - P_{u_n} H(a) H(\tilde{b}) P_{u_n} \\
&\quad - P_{u_n} M_{u_n} Q a P b P M_{\bar{u}_n} P_{u_n} - P_{u_n} M_{u_n} Q a P b Q M_{\bar{u}_n} P_{u_n} \\
&= P_{u_n} T_u(ab) P_{u_n} - P_{u_n} H(a) H(\tilde{b}) P_{u_n} - P_{u_n} M_{u_n} Q a P b Q M_{\bar{u}_n} P_{u_n} \\
&= P_{u_n} T_u(ab) P_{u_n} - P_{u_n} H(a) H(\tilde{b}) P_{u_n} - P_{u_n} M_{u_n} J^2 Q a P b Q J^2 M_{\bar{u}_n} P_{u_n} \\
&= P_{u_n} T_u(ab) P_{u_n} - P_{u_n} H(a) H(\tilde{b}) P_{u_n} - R_{u_n} J Q a P b Q J R_{u_n}^*.
\end{aligned}$$

Since $J Q a P b Q J = H(\tilde{a}) H(b)$, this is the assertion ■

3 Hankel operators by Blaschke products

The operators R_n in Widom's identity (5) can be identified with the (classical) Hankel operators $H(t^n)$. Similarly,

$$\begin{aligned}
R_{u_n} &= P_{u_n} M_{u_n} J \\
&= (P - M_{u_n} P M_{\bar{u}_n}) M_{u_n} J \\
&= (P M_{u_n} - M_{u_n} P) J \\
&= (P M_{u_n} - P M_{u_n} P) J \quad (\text{since } M_{u_n} P f \in H^2 \text{ for } f \in H^2) \\
&= P M_{u_n} Q J,
\end{aligned}$$

which can be identified with the (classical) Hankel operator $H(M_{u_n})$ on H^2 . Analogously, $R_{u_n}^*$ can be identified with $H(M_{u_n})^* = H(\widetilde{M_{u_n}})$. We will see that the operators R_{u_n} in Widom's identity for TTO play a quite different role compared with the R_n in (5). We start with some general properties of Hankel operators generated by inner functions.

Proposition 4 *Let $u \in H^\infty$ and $|u(t)| = 1$ for $t \in \mathbb{T}$. Then the Hankel operator $H(u) = P u Q J$ is a partial isometry, the range and initial projection of which are given by $H(u) H(u)^* = P - u P \bar{u} I$ and $H(u)^* H(u) = P - \bar{u} P \tilde{u} I$.*

Proof. Since $u\bar{u} = 1$ and $PuP = uP$,

$$\begin{aligned} H(u)H(u)^* &= PuQJJQ\bar{u}P \\ &= PuQ\bar{u}P = Pu\bar{u}P - PuP\bar{u}P = P - uP\bar{u}I. \end{aligned} \quad (7)$$

Using this identity and $P\bar{u}Q = 0$ we obtain

$$\begin{aligned} H(u)H(u)^*H(u) &= PuQJ - uP\bar{u}PuQJ \\ &= PuQJ - uP\bar{u}uQJ + uP\bar{u}QPuQJ = PuQJ = H(u), \end{aligned}$$

i.e., $H(u)$ is an isometry. Replacing u in (7) by $\bar{\bar{u}}$ (which is also in H^∞) and taking into account that $H(\bar{\bar{u}}) = H(u)^*$, the identity for $H(u)^*H(u)$ follows. ■

Corollary 5 $R_{u_n}R_{u_n}^* = P_{u_n}$, $R_{u_n}^*R_{u_n} = P_{\bar{\bar{u}_n}}$, $P_{u_n}R_{u_n} = R_{u_n}$, $R_{u_n}^*P_{u_n} = R_{u_n}^*$.

The following convergence result for the R_{u_n} is in sharp contrast with the R_n in Widom's identity (5), which converge weakly to zero.

Theorem 6 $R_{u_n} = H(M_{u_n}) \rightarrow H(M_u)$ **-strongly*.

In the proof of this result, we will make use of the following well known assertion.

Lemma 7 *Let A_n, A be bounded linear operators on a Hilbert space H . If $A_n \rightarrow A$ weakly and $\|A_n x\| \rightarrow \|Ax\|$ for all $x \in H$, then $A_n \rightarrow A$ strongly.*

Proof. It is clearly sufficient to prove the following fact for elements x_n, x of H : if $x_n \rightarrow x$ weakly and $\|x_n\| \rightarrow \|x\|$, then $\|x_n - x\| \rightarrow 0$. This follows from

$$\begin{aligned} \|x_n - x\|^2 &= \langle x_n - x, x_n - x \rangle \\ &= \langle x_n, x_n \rangle - \langle x_n, x \rangle - \langle x, x_n \rangle + \langle x, x \rangle \\ &= \|x_n\|^2 + \|x\|^2 - \langle x_n, x \rangle - \langle x, x_n \rangle \end{aligned}$$

which goes to 0 by hypothesis. ■

Proof of Theorem 6. We first show that $R_{u_n} = H(M_{u_n}) \rightarrow H(M_u)$ weakly. Indeed, the uniform convergence of u_n to u on compact subsets of \mathbb{D} implies the convergence of the k th Taylor coefficient of u_n to the k th Taylor coefficient of u for every $k \in \mathbb{Z}^+$. Together with the uniform boundedness of the operators H_{u_n} , this fact implies the weak convergence of $H(M_{u_n})$ to $H(M_u)$.

Next we show that $\|H(M_{u_n})x\| \rightarrow \|H(M_u)x\|$ for every $x \in H^2$. Once this is done, the strong convergence of $H(M_{u_n})$ to $H(M_u)$ follows from Lemma 7.

We start with showing that

$$P_{\bar{\bar{u}_n}} \rightarrow P_{\bar{\bar{u}}} \quad \text{strongly as } n \rightarrow \infty. \quad (8)$$

Indeed, let b_λ be a single Blaschke factor as in (1). For $t \in \mathbb{T}$ and $\lambda \neq 0$ we then have

$$\overline{b_\lambda}(t) = \frac{\overline{\lambda - t^{-1}} |\lambda|}{1 - \overline{\lambda} t^{-1} \lambda} = \frac{\bar{\lambda} - t |\lambda|}{1 - \lambda t \bar{\lambda}} = b_{\bar{\lambda}}(t).$$

For $\lambda = 0$, the equality $\overline{b_\lambda} = b_{\bar{\lambda}}$ on \mathbb{T} is evident. Moreover, if (λ_n) is a sequence in \mathbb{D} satisfying the Blaschke condition, then the sequence $(\bar{\lambda}_n)$ also satisfies this condition. So we can apply the assertion of Proposition 2 (b) to the functions $\prod_{k=1}^{\infty} b_{\bar{\lambda}_k}$ and $\prod_{k=1}^n b_{\bar{\lambda}_k}$ in place of u and u_n to get the assertion (8). From (8) and Corollary 5 we then conclude that

$$H(M_{u_n})^* H(M_{u_n}) \rightarrow H(M_u)^* H(M_u) \quad \text{strongly,}$$

from which we obtain

$$\langle H(M_{u_n})^* H(M_{u_n})x, x \rangle - \langle H(M_u)^* H(M_u)x, x \rangle = \|H(M_{u_n})x\|^2 - \|H(M_u)x\|^2 \rightarrow 0$$

for every $x \in H^2$. This proves the strong convergence of $R_{u_n} = H(M_{u_n})$ to $H(M_u)$. The strong convergence of the adjoint operators follows as above, by working with the Blaschke product \tilde{u} in place of u . ■

Corollary 8 *Let L be a compact operator on H^2 . Then*

$$\|R_{u_n} L R_{u_n}^* - P_{u_n} H(M_u) L H(M_u)^* P_{u_n}\| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Indeed, since L and $H(M_u) L H(M_u)^*$ are compact, we derive from Proposition 2 (b) and Theorem 6 that both sequences $(R_{u_n} L R_{u_n}^*)$ and $(P_{u_n} H(M_u) L H(M_u)^* P_{u_n})$ converge to $H(M_u) L H(M_u)^*$ in the norm. ■

4 The algebra of the FSD for TTO

With every filtration $\mathcal{P} = (P_n)$ on a Hilbert space H , there are naturally associated some algebraic objects. By $\mathcal{F}^{\mathcal{P}}$ we denote the set of all sequences $\mathbf{A} = (A_n)$ of operators $A_n : \text{im } P_n \rightarrow \text{im } P_n$ for which the sequence $(A_n P_n)$ converges *-strongly to some operator $W(\mathbf{A})$ on H . Provided with element-wise defined operations and the supremum norm, $\mathcal{F}^{\mathcal{P}}$ becomes a C^* -algebra, the set $\mathcal{G}^{\mathcal{P}}$ of all sequences in $\mathcal{F}^{\mathcal{P}}$ which converge to 0 in the norm is a closed ideal of $\mathcal{F}^{\mathcal{P}}$, and the mapping W , also called the consistency map of the filtration \mathcal{P} , is a *-homomorphism from $\mathcal{F}^{\mathcal{P}}$ to the algebra $L(H)$ of the bounded linear operators on H .

We prepare the proof of Theorem 10 below by an assertion of independent interest.

Proposition 9 *Let $\mathcal{P} = (P_n)$ be a filtration on a Hilbert space H . Then the ideal $\mathcal{G}^{\mathcal{P}}$ is contained in the smallest closed subalgebra \mathcal{J} of $\mathcal{F}^{\mathcal{P}}$ which contains all sequences $(P_n K P_n)$ with K compact if and only if \mathcal{P} is injective.*

Proof. The "only if"-part of the assertion is evident. For the "if"-part we are going to show that, for each $n_0 \in \mathbb{N}$, there is a sequence (G_n) in \mathcal{J} with $G_{n_0} \neq 0$ and $G_n = 0$ for all $n \neq n_0$. Since the matrix algebras $\mathbb{C}^{k \times k}$ are simple, this fact already implies that each sequence (G_n) with arbitrarily prescribed $G_{n_0} \in L(\text{im } P_{n_0})$ and $G_n = 0$ for $n \neq n_0$ belongs to \mathcal{J} . Since $\mathcal{G}^{\mathcal{P}}$ is generated by sequences of this special form, the assertion follows.

For $n_0 \in \mathbb{N}$, put

$$\mathbb{N}_{<} := \{n \in \mathbb{N} : \text{im } P_n \cap \text{im } P_{n_0} \text{ is a proper subspace of } \text{im } P_{n_0}\},$$

and set $\mathbb{N}_{>} := \mathbb{N} \setminus (\{n_0\} \cup \mathbb{N}_{<})$. The set $\mathbb{N}_{<}$ is at most countable, and none of the closed linear spaces $\text{im } P_n \cap \text{im } P_{n_0}$ has interior points relative to $\text{im } P_{n_0}$. By the Baire category theorem, $\bigcup_{n \in \mathbb{N}_{<}} (\text{im } P_n \cap \text{im } P_{n_0})$ is a proper subset of $\text{im } P_{n_0}$. Choose a unit vector

$$f \in \text{im } P_{n_0} \setminus \bigcup_{n \in \mathbb{N}_{<}} (\text{im } P_n \cap \text{im } P_{n_0}).$$

Then $\|P_n f\| < 1$ for all $n \in \mathbb{N}_{<}$ by the Pythagoras theorem. (Indeed, otherwise $\|P_n f\| = 1$, and the equality $1 = \|f\|^2 = \|P_n f\|^2 + \|f - P_n f\|^2$ implies $f = P_n f$, whence $f \in \text{im } P_n$.)

Let $Q_n := I - P_n$. If $n \in \mathbb{N}_{>}$, then $\text{im } P_n \cap \text{im } P_{n_0} = \text{im } P_{n_0}$ by the definition of $\mathbb{N}_{>}$. Thus, $\text{im } P_{n_0} \subseteq \text{im } P_n$, and since no two of the projections P_n coincide, this implies that $\text{im } P_{n_0}$ is a proper subspace of $\text{im } P_n$ and $\text{im } Q_n$ is a proper subspace of $\text{im } Q_{n_0}$ for $n \in \mathbb{N}_{>}$. Again by the Baire category theorem, $\bigcup_{n \in \mathbb{N}_{>}} \text{im } Q_n$ is a proper subset of $\text{im } Q_{n_0}$. Choose a unit vector

$$g \in \text{im } Q_{n_0} \setminus \bigcup_{n \in \mathbb{N}_{>}} \text{im } Q_n.$$

As above, $\|Q_n g\| < 1$ for all $n \in \mathbb{N}_{>}$. Consider the operator $K : x \mapsto \langle x, g \rangle f$ on H . Its adjoint is the operator $x \mapsto \langle x, f \rangle g$, and

$$P_n K Q_n K^* P_n x = \langle P_n x, f \rangle \langle Q_n g, g \rangle P_n f = \langle x, P_n f \rangle \|Q_n g\|^2 P_n f.$$

If $n \in \mathbb{N}_{<}$ then $\|P_n f\| < 1$, and if $n \in \mathbb{N}_{>}$ then $\|Q_n g\| < 1$ by construction. In both cases, $\|P_n K Q_n K^* P_n\| < 1$. In case $n = n_0$, the operator $P_n K Q_n K^* P_n x = \langle x, f \rangle f$ is an orthogonal projection of norm 1, which we call P . The sequence $\mathbf{K} := (P_n K Q_n K^* P_n)$ belongs to the algebra \mathcal{J} since

$$(P_n K Q_n K^* P_n) = (P_n K K^* P_n) - (P_n K P_n) (P_n K^* P_n).$$

As $r \rightarrow \infty$, the powers \mathbf{K}^r converge in the norm of $\mathcal{F}^{\mathcal{P}}$ to the sequence (G_n) with $G_{n_0} = P \neq 0$ and $G_n = 0$ if $n \neq n_0$. Indeed, since $P_n \rightarrow I$ strongly, one has $\|Q_n g\| < 1/2$ for n large enough, whence $\|P_n K Q_n K^* P_n\| < 1/2$ for these n , whereas $\|P_n K Q_n K^* P_n\| < 1$ for the remaining (finitely many) n as seen above. Since $\mathbf{K}^r \in \mathcal{J}$ and \mathcal{J} is closed, the sequence (G_n) has the claimed properties. ■

The goal in this section is to study the FSD of TTO with respect to the filtration \mathcal{P}_u . In accordance with Theorem 1 (e), we define the corresponding (full) algebra of the FSD as the smallest closed subalgebra $\mathcal{S}(\mathbb{T}_u(C))$ of $\mathcal{F}^{\mathcal{P}_u}$ which contains all sequences $(P_{u_n}(T_u(a) + K)P_{u_n})_{n \geq 1}$ with $a \in C(\mathbb{T})$ and $K \in K(K_u^2)$.

Theorem 10 $\mathcal{S}(\mathbb{T}_u(C))$ consists of all sequences $(P_{u_n}(T_u(a) + K)P_{u_n} + G_n)$ with $a \in C(\mathbb{T})$, $K \in K(K_u^2)$ and $(G_n) \in \mathcal{G}^{\mathcal{P}_u}$.

Proof. The proof runs parallel to that of Theorem [3, 1.53]; so we address to some main steps only.

For a moment, let \mathcal{S}_1 denote the set of all sequences of the mentioned form. The sequences $(P_{u_n}(T_u(a) + K)P_{u_n})$ are contained in $\mathcal{S}(\mathbb{T}_u(C))$ by definition, and since the filtration \mathcal{P}_u is injective, we conclude from Proposition 9 that the ideal $\mathcal{G}^{\mathcal{P}_u}$ of the zero sequences is also contained in $\mathcal{S}(\mathbb{T}_u(C))$. Thus, $\mathcal{S}_1 \subseteq \mathcal{S}(\mathbb{T}_u(C))$.

For the reverse inclusion, we prove that \mathcal{S}_1 is a closed subalgebra of $\mathcal{S}(\mathbb{T}_u(C))$. If $a, b \in C(\mathbb{T})$ then, by Theorem 3 (Widom's identity) and Corollary 8,

$$\begin{aligned} & P_{u_n}T_u(a)P_{u_n} \cdot P_{u_n}T_u(b)P_{u_n} \\ &= P_{u_n}T_u(ab)P_{u_n} - P_{u_n}H(a)H(\tilde{b})P_{u_n} - R_{u_n}H(\tilde{a})H(b)R_{u_n}^* \\ &= P_{u_n}T_u(ab)P_{u_n} - P_{u_n}(H(a)H(\tilde{b}) + H(M_u)H(\tilde{a})H(b)H(M_u)^*)P_{u_n} + G_n \\ &= P_{u_n}T_u(ab)P_{u_n} - P_{u_n}KP_{u_n} + G_n \end{aligned}$$

with a compact operator K and a sequence $(G_n) \in \mathcal{G}^{\mathcal{P}_u}$. Thus,

$$(P_{u_n}T_u(a)P_{u_n})(P_{u_n}T_u(b)P_{u_n}) \in \mathcal{S}_1.$$

It follows now in a standard way that \mathcal{S}_1 is an algebra. To prove that \mathcal{S}_1 is closed, let $((P_{u_n}(T_u(a_m) + K_m)P_{u_n} + G_n^m)_{n \geq 1})_{m \geq 1}$ be a sequence in \mathcal{S}_1 which converges in the norm of $\mathcal{F}^{\mathcal{P}_u}$. Let W denote the consistency map of the filtration \mathcal{P}_u , i.e., $W((A_n)) = \text{s-lim } A_n P_{u_n}$. Then

$$(W((P_{u_n}(T_u(a_m) + K_m)P_{u_n} + G_n^m)_{n \geq 1}))_{m \geq 1} = (T_u(a_m) + K_m)_{m \geq 1}$$

is a Cauchy sequence in $\mathbb{T}_u(C)$. Since $\mathbb{T}_u(C)$ is a closed algebra, this sequence converges in $\mathbb{T}_u(C)$. The limit of this sequence is of the form $T_u(a) + K$ with $a \in C(\mathbb{T})$ and K compact by Theorem 1 (e). (But note that the representation of the limit in that form is not unique by Theorem 1 (d).) It is now easy to see that

$$(P_{u_n}(T_u(a_m) + K_m)P_{u_n})_{n \geq 1} \rightarrow (P_{u_n}(T_u(a) + K)P_{u_n})_{n \geq 1}$$

in the norm of $\mathcal{F}^{\mathcal{P}_u}$ as $m \rightarrow \infty$. Then, finally, the sequence $((G_n^m)_{n \geq 1})_{m \geq 1}$ converges; its limit is in $\mathcal{G}^{\mathcal{P}_u}$. \blacksquare

Theorem 11 A sequence $\mathbf{A} = (A_n) \in \mathcal{S}(\mathbb{T}_u(C))$ is stable if and only if the operator $W(\mathbf{A}) = \text{s-lim } A_n P_{u_n}$ is invertible.

Proof. By Theorem 10, we have to show that the sequence $\mathbf{A} := (P_{u_n}(T_u(a) + K)P_{u_n})$ (with $a \in C(\mathbb{T})$ and $K \in K(K_u^2)$) is stable if and only if the operator $A := T_u(a) + K$ is invertible. Since the stability of \mathbf{A} implies the invertibility of A by Polski's theorem, we are left with the reverse implication.

So let A be invertible. By inverse closedness of C^* -algebras, $A^{-1} \in \mathbb{T}_u(C)$; hence, $A^{-1} = T_u(b) + L$ with a certain function $b \in C(\mathbb{T})$ and a compact operator L by Theorem 1 (e). Using Widom's identity as in the proof of the previous theorem and employing assertion (a) of Theorem 1, we conclude that there are compact operators R_1, R_2 and a sequence $(G_n) \in \mathcal{G}^{\mathcal{P}_u}$ such that

$$\begin{aligned} & P_{u_n}T_u(a)P_{u_n} \cdot P_{u_n}T_u(b)P_{u_n} - P_{u_n} \\ &= P_{u_n}(T_u(ab) - I)P_{u_n} - P_{u_n}R_1P_{u_n} + G_n \\ &= P_{u_n}R_2P_{u_n} - P_{u_n}R_1P_{u_n} + G_n. \end{aligned}$$

Thus, the sequence $(P_{u_n}T_u(b)P_{u_n})$ is a right inverse of the sequence $(P_{u_n}T_u(a)P_{u_n})$ modulo the ideal

$$\mathcal{J} := \{(P_{u_n}KP_{u_n} + G_n) : K \in K(K_u^2), (G_n) \in \mathcal{G}^{\mathcal{P}_u}\}$$

of $\mathcal{S}(\mathbb{T}_u(C))$. A similar computation shows that it is a left inverse modulo \mathcal{J} , too. Then $(P_{u_n}T_u(b)P_{u_n})$ is also an inverse of $(P_{u_n}AP_{u_n})$ modulo \mathcal{J} . Now the assertion follows from the Lifting Theorem [3, 5.37] in its simplest form, i.e., with $\mathcal{J}/\mathcal{G}^{\mathcal{P}_u}$ consisting of one elementary ideal only. \blacksquare

The following is certainly the most important consequence of Theorem 11. The definition of a fractal algebra is in [3].

Corollary 12 *The algebra $\mathcal{S}(\mathbb{T}_u(C))$ is fractal.*

Note that the consistency map of \mathcal{P}_u is fractal; so the assertion of the corollary follows from Theorems [3, 1.69] and 11. \blacksquare

Sequences in fractal algebras are distinguished by their excellent convergence properties. To mention only a few of them, let $\sigma(a)$ denote the spectrum of an element a of a C^* -algebra with identity element e , write $\sigma_2(a)$ for the set of the singular values of a , i.e., $\sigma_2(a)$ is the set of all non-negative square roots of elements in the spectrum of a^*a and finally, for $\varepsilon > 0$, let $\sigma^{(\varepsilon)}(a)$ refer to the ε -pseudospectrum of a , i.e. to the set of all $\lambda \in \mathbb{C}$ for which $a - \lambda e$ is not invertible or $\|(a - \lambda e)^{-1}\| \geq 1/\varepsilon$. Let further

$$d_H(M, N) := \max \left\{ \max_{m \in M} \min_{n \in N} |m - n|, \max_{n \in N} \min_{m \in M} |m - n| \right\}$$

denote the Hausdorff distance between the non-empty compact subsets M and N of the complex plane.

Theorem 13 *Let (A_n) be a sequence in $\mathcal{S}(\mathbb{T}_u(C))$ with strong limit A . Then the following set-sequences converge with respect to the Hausdorff distance as $n \rightarrow \infty$:*

- (a) $\sigma(A_n) \rightarrow \sigma(A)$ if (A_n) is self-adjoint;
- (b) $\sigma_2(A_n) \rightarrow \sigma_2(A)$;
- (c) $\sigma^{(\varepsilon)}(A_n) \rightarrow \sigma^{(\varepsilon)}(A)$.

The proof follows immediately from the stability criterion in Theorem 11 and from Theorems 3.20, 3.23 and 3.33 in [3]. Note that in general one cannot remove the assumption $(A_n) = (A_n)^*$ in assertion (a), whereas (c) holds without this assumption. ■

The notion of a Fredholm sequence was introduced in [6]; see also [3, Chapter 6]. In the present setting, the Fredholm property of a sequence $(A_n) \in \mathcal{S}(\mathbb{T}_u(C))$ means nothing but the invertibility of the coset $(A_n) + \mathcal{J}$ in the quotient algebra $\mathcal{S}(\mathbb{T}_u(C))/\mathcal{J}$, and the results of [6] specify as follows. Let $\sigma_1(a) \leq \dots \leq \sigma_n(A) = \|A\|$ denote the singular values of the $n \times n$ -matrix A .

Theorem 14 *Let (A_n) be a sequence in $\mathcal{S}(\mathbb{T}_u(C))$ with strong limit A . Then*

- (a) (A_n) is a Fredholm sequence if and only if A is a Fredholm operator.
- (b) If A is a Fredholm operator and $\dim \ker A = k$, then

$$\lim_{n \rightarrow \infty} \sigma_k(A_n) = 0 \quad \text{and} \quad \liminf_{n \rightarrow \infty} \sigma_{k+1}(A_n) > 0.$$

Assertion (b) allows the numerical determination of the kernel dimension of a Fredholm operator $A \in \mathbb{T}_u(C)$.

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