Local strong solutions of the nonhomogeneous Navier-Stokes system with control of the interval of existence

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Abstract

Consider a bounded domain $\Omega \subseteq \mathbb{R}^3$ with smooth boundary $\partial\Omega$, a time interval $[0,T)$, $0 < T \leq \infty$, and in $[0,T) \times \Omega$ the nonhomogeneous Navier-Stokes system $u_t - \Delta u + u \cdot \nabla u + \nabla p = f$, $u|_{t=0} = u_0$, $\operatorname{div} u = k$, $u|_{\partial\Omega} = g$, with sufficiently smooth data f, u_0, k, g. In this general case there are mainly known two classes of weak solutions, the class of global weak solutions, similar as in the well known case $k = 0$, $g = 0$ which need not be unique, see [5], and the class of local very weak solutions, see [1], [2], [3], [4], which are uniquely determined but need neither have differentiability properties nor satisfy the energy inequality. Our aim is to introduce a new class of local strong solutions for the general case $k \neq 0$, $g \neq 0$, satisfying similar regularity and uniqueness properties as in the known case $k = 0$, $g = 0$. For slightly restricted data this class coincides with the corresponding class of very weak solutions yielding new regularity results. Further, through the given data we obtain a control on the interval of existence of the strong solution.

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1 Introduction

Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain with boundary $\partial\Omega$ of class $C^{2,1}$ and let $[0,T)$, $0 < T \le \infty$, be the time interval. Then we consider in $[0,T) \times \Omega$ the general nonhomogeneous Navier-Stokes system

$$u_t - \Delta u + u \cdot \nabla u + \nabla p = f,\ u|_{t=0} = u_0,\ \operatorname{div} u = k,\ u|_{\partial\Omega} = g, \tag{1.1}$$

where ∇p means the associated pressure. We refer to [3] and [5] for very weak and weak solutions of this system, respectively; for a review on very weak solutions see [4]. However, the focus of this paper is put on the existence of local in time strong solutions.

In the following we construct u in (1.1) in the form $u = v + E$, see Definition 1.2, where $v|_{t=0} = u|_{t=0}$, $E|_{t=0} = 0$. Therefore in the following we use the notation

$$u|_{t=0} = v|_{t=0} = u_0 = v_0.$$

For simplicity we use for weak and strong solutions the same data class to exploit from the beginning both theories simultaneously; see [5] for a more general theory of weak solutions.

Next we describe the general assumptions on the data f, $u_0 = v_0$, k and g; here $N(x)$ denotes the outward normal vector at $x \in \partial\Omega$.

Assumption 1.1

$$\text{(i)} \quad f = \operatorname{div} F,\ F \in L^{s/2}(0,T; L^{q/2}(\Omega)), \tag{1.2}$$
$$\textit{with} \quad 4 \le s \le 8,\ 4 \le q \le 6,\ \frac{2}{s} + \frac{3}{q} = 1,$$

$$\text{(ii)} \quad v_0 \in L^2_\sigma(\Omega),\ \|v_0\|_{B^{q,s}_T(\Omega)} := \left(\int_0^T \left\| e^{-tA} v_0 \right\|_q^s \mathrm{d}t \right)^{1/s} < \infty, \tag{1.3}$$

$$\text{(iii)} \quad k \in L^s(0,T; L^q(\Omega)),\ g \in L^s(0,T; W^{-\frac{1}{q},q}(\partial\Omega)) \tag{1.4}$$
$$\textit{with compatibility condition} \int_\Omega k(t)\,\mathrm{d}x = \langle g(t), N \rangle_{\partial\Omega}$$
$$\textit{for a.a. } t \in [0,T).$$

Here $L^r(\Omega)$ denotes the usual Lebesgue space of functions (or vector or matrix fields) with norm $\|\cdot\|_r$ and pairing $\langle \cdot,\cdot \rangle_\Omega$ with its dual $L^{r'}(\Omega)$, $1 < r < \infty$, $r' = \frac{r}{r-1}$. Moreover, $L^r_\sigma(\Omega) = \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_r}$ where $C^\infty_{0,\sigma}(\Omega) := \{v = (v_1, v_2, v_3) \in C^\infty_0(\Omega); \operatorname{div} v = 0\}$. Usual Bochner spaces are denoted by $L^s(0,T; L^q(\Omega))$ with norm $\|\cdot\|_{q,s,T}$, $1 < q, s < \infty$, and with pairing $\langle \cdot,\cdot \rangle_{\Omega,T}$.

The nonlinear term $u\cdot\nabla u$ is defined by $u\cdot\nabla u = (u\cdot\nabla u_1, u\cdot\nabla u_2, u\cdot\nabla u_3)$ where $u\cdot\nabla = u_1D_1+u_2D_2+u_3D_3$ and $D_j = \partial/\partial x_j$, $j = 1,2,3$. When $\operatorname{div} u = 0$, we obtain that $u\cdot\nabla u = \operatorname{div}(uu) = \nabla\cdot(uu)$ where $uu = (u_iu_j)_{i,j=1,2,3}$. Further, for $F = (F_{ij})_{i,j=1,2,3}$, we have $\operatorname{div} F = (D_1F_{1j}+D_2F_{2j}+D_3F_{3j})_{j=1,2,3}$.

The initial value norm $\|v_0\|_{B_T^{q,s}(\Omega)}$ is a so-called Besov space norm, see [3], [6]–[9], and Section 3 for details. The space $W^{-\frac{1}{q},q}(\partial\Omega)$ is a Sobolev trace space of negative order $-\frac{1}{q}$, namely the dual of the trace space $W^{\frac{1}{q},q'}(\partial\Omega)$, see [13, I.3.6, (3.6.9)]. For $L^s(0,T;W^{-\frac{1}{q},q}(\partial\Omega))$ we denote the norm by $\|\cdot\|_{-\frac{1}{q},q,s,T}$ and use the pairing $\langle\cdot,\cdot\rangle_{\partial\Omega,T}$. Concerning the trace $u|_{\partial\Omega} = g$ see [3, Remarks 1 and 3(2)].

Let $P = P_q : L^q(\Omega) \to L^q_\sigma(\Omega)$ denote the Helmholtz projection and $A = A_q = -P\Delta$ in (1.3) the Stokes operator, $1 < q < \infty$.

To obtain a precise definition of weak and strong solutions u for (1.1), see [5, (1.2)–(1.6)], we use in $[0,T)\times\Omega$ a fixed solution

$$E = E_{k,g} \in L^s(0,T;L^q(\Omega)) \tag{1.5}$$

of the linear Stokes system

$$E_t - \Delta E + \nabla h = 0,\ E|_{t=0} = 0,\ \operatorname{div} E = k,\ E|_{\partial\Omega} = g \tag{1.6}$$

with associated pressure ∇h.

We know, see [3, Theorem 4], that there exists a unique solution $E = E_{k,g} \in L^s(0,T;L^q(\Omega))$ of (1.6) satisfying for each $w \in C_0^\infty([0,T);C^\infty_{0,\sigma}(\overline{\Omega}))$, $C^\infty_{0,\sigma}(\overline{\Omega}) = \{v|_\Omega\,;\, v\in C^\infty_{0,\sigma}(\mathbb{R}^3)\}$, with $w|_{\partial\Omega} = 0$

$$-\langle E, w_t\rangle_{\Omega,T} - \langle E,\Delta w\rangle_{\Omega,T} + \langle g, N\cdot\nabla w\rangle_{\partial\Omega,T} = 0, \tag{1.7}$$

$$\operatorname{div} E = k,\ N\cdot E|_{\partial\Omega} = N\cdot g \ \text{ a.e. in } (0,T), \tag{1.8}$$

as well as

$$\begin{aligned}
&(A^{-1}PE)_t \in L^s(0,T;L^q_\sigma(\Omega)),\\
&A^{-1}PE \in C([0,T);L^q_\sigma(\Omega)),\ A^{-1}PE|_{t=0} = 0,\\
&\|(A^{-1}PE)_t\|_{q,s,T} + \|E\|_{q,s,T} \le C\big(\|k\|_{q,s,T} + \|g\|_{-\frac{1}{q},q,s,T}\big)
\end{aligned} \tag{1.9}$$

with $C = C(\Omega,q) > 0$.

The condition $E|_{\partial\Omega} = g$ is well defined in the sense of boundary distributions, see [3, Remarks 3, (2)]. Moreover, the condition $E|_{t=0} = 0$ is defined by (1.9) in the generalized sense that $A^{-1}PE|_{t=0} = 0$, see [3, (1.6)]. This means that

$$\langle E(.),v\rangle_\Omega :\ t \to \langle E(t),v\rangle_\Omega,\ t\in[0,T),\ \text{is continuous} \tag{1.10}$$

for each test function $v \in C^\infty_{0,\sigma}(\Omega)$ and even for all v in the domain of $A_{q'}$. In particular, we conclude from (1.10) that $E|_{t=0} = 0$ is defined modulo gradients.

To give the system (1.1) a precise meaning we set

$$u = v + E,\ \ E = E_{k,g} \tag{1.11}$$

and choose a vector field v satisfying in $[0,T) \times \Omega$ the system

$$v_t - \Delta v + (v+E)\cdot\nabla(v+E) + \nabla p^* = f,\ v|_{t=0} = v_0, v|_{\partial\Omega} = 0, \operatorname{div} v = 0 \tag{1.12}$$

which is called the *perturbed Navier-Stokes system*, see [5], with associated pressure ∇p^*.

The following definition extends the well known special case $k = 0$, $g = 0$ to our general case $k \neq 0$, $g \neq 0$ including the additional terms $\langle (v+E)(v+E), \nabla w\rangle_{\Omega,T}$ and $-\langle k(v+E), w\rangle_{\Omega,T}$ in (1.15) below. Correspondingly, we obtain two additional terms in the energy inequality (1.16), see Section 4.1, c) concerning these terms.

Definition 1.2 (Weak and strong solutions for (1.1).) *Suppose $f, u_0 = v_0, k, g, q, s$ satisfy Assumption 1.1, and let $E = E_{k,g}$ be as in* (1.5)-(1.6).

(1) A vector field v in $[0,T)\times\Omega$ is called a weak solution of the perturbed system (1.12) *with data f, v_0, and $u := v + E$ is called a* weak solution of the general system (1.1) *with data $f, u_0 = v_0, k, g$, if the following conditions are satisfied:*

a) $v \in L^\infty_{\mathrm{loc}}([0,T); L^2_\sigma(\Omega)) \cap L^2_{\mathrm{loc}}([0,T); W^{1,2}_0(\Omega))$, (1.13)

b) $v : [0,T) \to L^2_\sigma(\Omega)$ *is weakly continuous and* $v|_{t=0} = v_0$ (1.14)

c) $$-\langle v, w_t\rangle_{\Omega,T} + \langle \nabla v, \nabla w\rangle_{\Omega,T} - \langle (v+E)(v+E), \nabla w\rangle_{\Omega,T} - \langle k(v+E), w\rangle_{\Omega,T} = \langle v_0, w(0)\rangle_\Omega - \langle F, \nabla w\rangle_{\Omega,T} \tag{1.15}$$
for each $w \in C_0^\infty([0,T); C^\infty_{0,\sigma}(\Omega))$,

d) $$\frac{1}{2}\|v(t)\|_2^2 + \int_0^t \|\nabla v\|_2^2 \,\mathrm{d}\tau \leq \frac{1}{2}\|v_0\|_2^2 - \int_0^t \langle F, \nabla v\rangle_\Omega \,\mathrm{d}\tau + \int_0^t \langle (v+E)E, \nabla v\rangle_\Omega \,\mathrm{d}\tau + \frac{1}{2}\int_0^t \langle k(v+2E), v\rangle_\Omega \,\mathrm{d}\tau \tag{1.16}$$
for each $t \in [0,T)$.

(2) Let v be a weak solution of (1.12) *with data f, v_0 and let*

$$v \in L^s(0,T; L^q(\Omega)). \tag{1.17}$$

Then v *is called a* strong solution of the perturbed system (1.12) *with data* f, v_0, *and*

$$u = v + E_{k,g} \tag{1.18}$$

is called a strong solution of the general system (1.1) *with data* $f, u_0 = v_0, k, g$.

We see that in the well known case $E_{k,g} = 0$, the weak solution v is a Hopf type weak solution, and $u = v$ is the usual strong solution.

In the following we will get that the strong solutions v and $u = v + E_{k,g}$ have similar uniqueness and regularity properties as in the known case $E_{k,g} = 0$.

2 Main results

We are mainly interested in strong solutions v and $u + v$, given in Definition 1.2, which are also weak solutions. Therefore, for simplicity, we used the same data class in Definition 1.2 for weak and strong solutions. Indeed, the data class in [5, Theorem 1.4], for (global) weak solutions is slightly more general than that in Definition 1.2.

An important aspect in the main Theorem 2.1 below is that the existence of a strong solution in a given interval $[0, T)$ can be proven if the norm $b(T)$ of the data, see (2.1), satisfies a smallness condition $b(T) \leq \varepsilon^*(\Omega, q)$. Since $b(T)$ tends to zero for $T \to 0$, we can determine some interval $[0, T^*)$, $0 < T^* \leq T$, satisfying $b(T^*) \leq \varepsilon^*(\Omega, q)$ and yielding precisely the existence interval for the local strong solution, see Corollary 2.2 below. Usually, in the well known case $k = 0$, $g = 0$, the existence of a strong solution has been shown only in some “sufficiently small” subinterval of $[0, T)$.

Theorem 2.1 (Existence of a strong solution in $[0, T)$). *Let* $f = \operatorname{div} F$, $u_0 = v_0, k, g, q, s, \frac{2}{s} + \frac{3}{q} = 1$ *be given as in Assumption 1.1, let* $E = E_{k,g}$ *be as in* (1.5)–(1.9), *and let*

$$b(T) := \|v_0\|_{B_T^{q,s}(\Omega)} + \|F\|_{\frac{q}{2},\frac{s}{2},T} + \|k\|_{q,s,T} + \|g\|_{-\frac{1}{q},q,s,T} \tag{2.1}$$

be the data norm in $[0, T)$.

There exists a constant $\varepsilon^* = \varepsilon^*(\Omega, q) > 0$ *such that if*

$$b(T) \leq \varepsilon^*, \tag{2.2}$$

then there exist in $[0, T)$ *uniquely determined strong solutions* v *of the perturbed system* (1.12) *and* $u = v + E$ *of the general system* (1.1), *respectively.*

Since $b(T) \to 0$ for $T \to 0$ we obtain well defined existence intervals $[0, T^*)$, $0 < T^* \le T$, for strong solutions. Thus we get the following result.

Corollary 2.2 (Interval of existence of local strong solutions) *Let* $f = \operatorname{div} F$, $u_0 = v_0, k, g, \varepsilon^* = \varepsilon^*(\Omega, q)$, *and* $E = E_{k,g}$ *be as in Theorem 2.1.*

Then each $[0, T^*)$, $0 < T^* \le T$, *defined by* $b(T^*) \le \varepsilon^*$, *is an interval of existence of uniquely determined strong solutions* v *of* (1.12) *and* $u = v + E$ *of* (1.1), *respectively, with* T *replaced by* T^*.

The next result yields the regularity of strong solutions.

Theorem 2.3 (Regularity result for strong solutions) *Let* $f = \operatorname{div} F$, $u_0 = v_0, k, g, q, s$ *satisfy Assumption 1.1, and let* $E = E_{k,g}$ *be as in* (1.5)–(1.9). *Assume the following additional regularity properties of the data,*

$$\begin{aligned} &F \in L^s(0,T;W^{1,q}(\Omega)),\ k \in L^s(0,T;W^{1,q}(\Omega)),\ k_t \in L^s(0,T;L^q(\Omega)), \quad (2.3)\\ &g \in L^s(0,T;W^{2-1/q,q}(\partial\Omega)),\ g_t \in L^s(0,T;W^{-\frac{1}{q},q}(\partial\Omega)),\ v_0 \in W^{2,q}(\Omega),\\ &g(0) = 0,\ k(0) = 0, \end{aligned}$$

and assume that v *and* $u = v + E$ *are strong solutions in* $[0, T)$ *as given in Theorem 2.1.*

Then v, E *satisfy, additionally to* (1.13)–(1.16) *and* (1.7)–(1.10) *respectively, the following regularity properties*

$$\begin{aligned} &v \in L^\infty_{\mathrm{loc}}([0,T);W^{1,2}_0(\Omega)) \cap L^2_{\mathrm{loc}}([0,T);W^{2,2}(\Omega)), \quad (2.4)\\ &v_t \in L^2_{\mathrm{loc}}([0,T);L^2_\sigma(\Omega)),\\ &E \in L^s(0,T;W^{2,q}(\Omega)),\ E_t \in L^s(0,T;L^q(\Omega)), \quad (2.5) \end{aligned}$$

and $u = v + E$ *satisfies corresponding additional regularity properties. In particular,*

$$u \in L^2_{\mathrm{loc}}([0,T);W^{2,2}(\Omega)),\ u_t \in L^2_{\mathrm{loc}}([0,T);L^2_\sigma(\Omega)). \quad (2.6)$$

Remarks 2.4 (1) In order to compare the class of strong solutions in Definition 1.2 with the class of very weak solutions in [3, Theorem 1] let us restrict, for simplicity, the condition for F in (1.2) and for v_0 in (1.3) as follows:

$$F \in L^s(0,T;L^q(\Omega)), v_0 = 0. \quad (2.7)$$

Then the data class in Assumption 1.1, restricted by (2.7), is contained in the data class of very weak solutions in [3, Theorem 1]. Let D_r be this restricted data class. Then there is some $0 < T^* \le T$ such that in $[0, T^*)$ the solution class VD_r of very weak solutions coincides with the solution class SD_r of strong solutions, see [3, (4.23)] for very weak and (2.2) for strong solutions. Then it holds $VD_r = SD_r$ in $[0,T^*)$: Each such strong solution is a very weak one and each such very weak solution is a strong one, because of the uniqueness of very weak solutions. However, the very weak solutions VD_r need not have any differentiability property in space and time, and need not satisfy any energy inequality. These are weaker conditions as for the usual (possibly non uniquely determined) weak solutions - this is the reason for the notion “very weak”, see [3, p. 425].

Since $VD_r = SD_r$ in $[0,T^*)$, our result shows, at least for slightly restricted data, that the very weak solution class VD_r has the same regularity properties as the class of strong solutions SD_r. Thus the notion “very weak” seems to be no longer justified.

(2) Let u be a strong solution as in Theorem 2.1. Then we can use similar arguments as in [13, V. Theorem 1.8.2] and obtain for smooth data $f,k,g,v_0 \in C^\infty$ that v and $u = v + E_{k,g}$ satisfy $v,u \in C^\infty((0,T)\times \Omega)$.

(3) Let v be a strong solution of (1.1) as in Definition 1.2. Then we can replace the energy inequality (1.16) by the corresponding energy equality as in the known case $k=0, g=0$.

3 Preliminaries

In Assumption 1.1 we already used for $0 < T \le \infty$ the Besov-type space

$$B_T^{q,s}(\Omega) := \left\{ v \in L^2_\sigma(\Omega);\, \|v\|_{B_T^{q,s}} := \left(\int_0^T \|e^{-\tau A} v\|_q^s \,\mathrm{d}\tau \right)^{1/s} < \infty \right\} \qquad (3.1)$$

with norm $\|v\|_{B_T^{q,s}} = \|v\|_{B_T^{q,s}(\Omega)}$. This normed space, which has been introduced in [6], [7], [8], is important for our results. Equipped with the norm $v \mapsto \|v\|_{B_T^{q,s}} + \|v\|_{L^2_\sigma}$ it is a Banach space.

In (3.1) $A = A_q$ denotes the Stokes operator, and $S(\tau) = e^{-\tau A}, 0 \le \tau < \infty$, the analytic semi-group generated by $-A$. Using the fractional powers A^α we will exploit for $v \in L^2_\sigma(\Omega)$ the estimates

$$\|A^{-\alpha} v\|_q \le C\|v\|_2 \qquad (3.2)$$

$$\|S(\tau)v\|_q \le C\tau^{-\alpha} e^{-\delta\tau} \|A^{-\alpha} v\|_q \le C\tau^{-\alpha} e^{-\delta\tau}\|v\|_2, \qquad (3.3)$$

with $0 < \alpha < \frac{3}{4}$, $2\alpha + \frac{3}{q} = \frac{3}{2}$, $\delta > 0$, $C = C(\Omega, q, \alpha, \delta) > 0$, see [6, (1.14)]. By (3.3) we conclude that the function $\tau \mapsto \|S(\tau)v\|_q^s$ is well defined on $(0,T)$. Therefore, $v \in B_T^{q,s}(\Omega) \subseteq L^2_\sigma(\Omega)$ simply means that this function is Lebesgue integrable on $[0,T)$.

Moreover, let $W^{1,2}_{0,\sigma}(\Omega) = \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_{W^{1,2}}}$. Then we define the Banach space

$$X := \big\{ v \in L^2_{\mathrm{loc}}([0,T); W^{1,2}_{0,\sigma}(\Omega)); \qquad (3.4)$$
$$(A^{-\frac{1}{2}}v)_t, A^{\frac{1}{2}}v \in L^{s/2}(0,T; L^{q/2}(\Omega)),\ A^{-\frac{1}{2}}v|_{t=0} = 0 \big\}$$

equipped with the norm

$$\|v\|_X := \|(A^{-\frac{1}{2}}v)_t\|_{\frac{q}{2},\frac{s}{2},T} + \|A^{\frac{1}{2}}v\|_{\frac{q}{2},\frac{s}{2},T}.$$

Additionally to the data given in Assumption 1.1, in the following propositions we need a vector field

$$f_0 \in L^{s/2}(0,T; L^{q/2}(\Omega)) \text{ with } q, s \text{ as in (1.2).} \qquad (3.5)$$

Note that $f = \operatorname{div} F$, f_0 in (1.2), (3.5), respectively, satisfy

$$F \in L^2_{\mathrm{loc}}([0,T); L^2(\Omega)),\ f_0 \in L^2_{\mathrm{loc}}([0,T); L^2(\Omega)). \qquad (3.6)$$

Next we consider several well known results on the linear nonstationary Stokes system in $[0,T) \times \Omega$ given by

$$v_t - \Delta v + \nabla h = f + f_0,\ \operatorname{div} v = 0,\ v|_{\partial\Omega} = 0,\ v|_{t=0} = v_0 \qquad (3.7)$$

with associated pressure ∇h.

Proposition 3.1 *Let $f = \operatorname{div} F, f_0, v_0$ be as in Assumption 1.1 and* (3.5)*, and let $E_{f,f_0,v_0} := v \in L^2_{\mathrm{loc}}([0,T); W^{1,2}_{0,\sigma}(\Omega))$ be a weak solution of the system* (3.7) *in the usual sense, defined by the relation*

$$-\langle v, w_t\rangle_{\Omega,T} + \langle \nabla v, \nabla w\rangle_{\Omega,T} = \langle v_0, w(0)\rangle_\Omega - \langle F, \nabla w\rangle_{\Omega,T} + \langle f_0, w\rangle_{\Omega,T} \quad (3.8)$$

for each $w \in C^\infty_0([0,T); C^\infty_{0,\sigma}(\Omega))$. Then we obtain the following properties:

(i) The function

$$v = E_{f,f_0,v_0} : [0,T) \to L^2_\sigma(\Omega) \qquad (3.9)$$

is strongly continuous, after redefinition on a null set of $[0,T)$, and it holds the energy equality

$$\frac{1}{2}\|v(t)\|_2^2 + \int_0^t \|\nabla v\|_2^2 \,\mathrm{d}\tau = \frac{1}{2}\|v_0\|_2^2 + \langle f_0, v\rangle_{\Omega,t} - \langle F, \nabla v\rangle_{\Omega,t} \qquad (3.10)$$

for each $t \in [0,T)$. Further, v is uniquely determined by f, f_0, v_0.

(ii) Let $S(t) := e^{-tA}$, $t \in [0,T)$. *Then* $v = E_{f,f_0,v_0}$ *has the representation*

$$v(t) = S(t)v_0 + \int_0^t S(t-\tau)Pf_0(\tau)\,\mathrm{d}\tau \qquad (3.11)$$
$$+ A^{\frac{1}{2}}\int_0^t S(t-\tau)A^{-\frac{1}{2}}P\,\mathrm{div}\,F(\tau)\,\mathrm{d}\tau$$

for each $t \in [0,T)$, *and it holds* $E_{f,f_0,v_0} = S(\cdot)v_0 + E_{f,f_0,0}$.

(iii) If $v_0 = 0$, *then* $\tilde{v} = E_{f,f_0,0} \in X$ *as in* (3.4), *i.e.*

$$(A^{-\frac{1}{2}}\tilde{v})_t,\ A^{\frac{1}{2}}\tilde{v} \in L^{s/2}(0,T;L^{q/2}(\Omega)),\ A^{-\frac{1}{2}}\tilde{v}|_{t=0} = 0, \qquad (3.12)$$

and

$$(A^{-\frac{1}{2}}\tilde{v})_t + A^{\frac{1}{2}}\tilde{v} = A^{-\frac{1}{2}}P\,\mathrm{div}\,F + A^{-\frac{1}{2}}Pf_0,\ t \in [0,T), \qquad (3.13)$$

$$\|\tilde{v}\|_X \le C\big(\|F\|_{\frac{q}{2},\frac{s}{2},T} + \|f_0\|_{\frac{q}{2},\frac{s}{2},T}\big) \qquad (3.14)$$

with some constant $C = C(\Omega,q) > 0$ *independent of* T.

(iv) Conversely, let $\tilde{v} \in L^2_{\mathrm{loc}}([0,T);W^{1,2}_{0,\sigma}(\Omega))$ *satisfy the properties* (3.12), (3.13), (3.14). *Then* $\tilde{v} = E_{f,f_0,0}$ *is a weak solution of the system* (3.7) *with* $v_0 = 0$, *and* $E_{f,f_0,v_0} = E_{f,f_0,0} + Sv_0$ *is a weak solution of* (3.7) *with the given data* f, f_0, v_0.

(v) There holds

$$\|E_{f,f_0,v_0}\|_{q,s,T} \le \|v_0\|_{B^{q,s}_T(\Omega)} + \|E_{f,f_0,0}\|_{q,s,T} < \infty, \qquad (3.15)$$

and there exists some constant $C = C(\Omega,q) > 0$ *independent of* T *such that*

$$\|E_{f,f_0,0}\|_{q,s,T} \le C\big(\|(A^{-\frac{1}{2}}E_{f,f_0,0})_t\|_{\frac{q}{2},\frac{s}{2},T} + \|A^{\frac{1}{2}}E_{f,f_0,0}\|_{\frac{q}{2},\frac{s}{2},T}\big). \qquad (3.16)$$

Proof (i) See [13, IV. Definition 2.1.1 and Theorem 2.3.1] concerning existence and uniqueness of v. Using (3.6) and (3.10) we obtain for $v = E_{f,f_0,v_0}$ and $0 < T' < T$ the estimate

$$\frac{1}{2}\|v\|^2_{2,\infty,T'} + \|\nabla v\|^2_{2,2,T'} \le 2\|v_0\|_2^2 + 8\|f_0\|^2_{2,1,T'} + 4\|F\|^2_{2,2,T'}. \qquad (3.17)$$

(ii) The representation (3.11) follows from [13, IV. Theorem 2.4.1], [3, (2.16)]. Note that $A^{-\frac{1}{2}}P\,\mathrm{div}$, defined by $\langle A^{-\frac{1}{2}}P\,\mathrm{div}F, \varphi\rangle = \langle -F, \nabla A^{-\frac{1}{2}}\varphi\rangle$ for $\varphi \in C^\infty_{0,\sigma}(\Omega)$, is a bounded operator satisfying

$$\|A^{-\frac{1}{2}}P\,\mathrm{div}\,F(t)\|_{\frac{q}{2}} \le C\|F(t)\|_{\frac{q}{2}} \quad \text{for a.a. } t \in [0,T) \qquad (3.18)$$

with $C = C(\Omega, q) > 0$; see [3, Examples 3), (2.14)], [13, IV (2.1.8)].

(iii) Applying $A^{-\frac{1}{2}}$ to (3.11) when $v_0 = 0$, we get for $\tilde{v} := E_{f,f_0,0}$ that $A^{-\frac{1}{2}}\tilde{v}$ is a weak solution of (3.13) in $[0,T) \times \Omega$. By the maximal regularity estimate, see, e.g., [6, (2.7)], we obtain $\tilde{v} \in X$ and the estimate (3.14).

(iv) Let $\tilde{v} \in L^2_{\text{loc}}([0,T); W^{1,2}_{0,\sigma}(\Omega))$ satisfy (3.12), (3.13), (3.14). Testing (3.13) with $A^{1/2}w$, $w \in C_0^\infty([0,T); C^\infty_{0,\sigma}(\Omega))$, we obtain the relation (3.8) for $\tilde{v} = E_{f,f_0,0}$ where $v_0 = 0$. Here we need the properties

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \tilde{v}, w\rangle_\Omega = \frac{\mathrm{d}}{\mathrm{d}t}\,\langle A^{-\frac{1}{2}}\tilde{v}, A^{\frac{1}{2}}w\rangle_\Omega \in L^1(0,T),$$

and $v_0 = 0$, $w(T) = 0$ yielding

$$-\langle \tilde{v}, w_t\rangle_{\Omega,T} = -\int_0^T \langle A^{-\frac{1}{2}}\tilde{v}, (A^{\frac{1}{2}}w)_t\rangle_\Omega \,\mathrm{d}t = \int_0^T \langle (A^{-\frac{1}{2}}\tilde{v})_t, A^{\frac{1}{2}}w\rangle_\Omega \,\mathrm{d}t.$$

Next we use that $E_{0,0,v_0} = Sv_0$ satisfies (3.8) for $f = 0$, $f_0 = 0$ as weak solution. This implies, together with (3.8) for $\tilde{v} = E_{f,f_0,0}$, that $E_{f,f_0,v_0} = E_{0,0,v_0} + E_{f,f_0,0}$ solves (3.8) and is a weak solution of (3.7).

(v) Setting $\tilde{v} = E_{f,f_0,0}$ when $v_0 = 0$, we obtain using (3.11), (3.13) the representation

$$\begin{aligned} A^{-\frac{1}{2}}\tilde{v}(t) &= \int_0^t S(t-\tau)A^{-\frac{1}{2}}Pf_0 \,\mathrm{d}\tau + \int_0^t S(t-\tau)A^{-\frac{1}{2}}P \operatorname{div} F \,\mathrm{d}\tau \\ &= \int_0^t S(t-\tau)\big((A^{-\frac{1}{2}}\tilde{v})_t + A^{\frac{1}{2}}\tilde{v}\big)\,\mathrm{d}\tau. \end{aligned} \tag{3.19}$$

By the fractional Sobolev embedding estimate $\|w\|_q \le c\|A^{\frac{3}{2q}}w\|_{q/2}$ and the Hardy-Littlewood inequality, see [15, p. 140], we obtain (3.16) from (3.19), cf. [6, (2.24)].

Next we obtain for $E_{f,f_0,v_0} = E_{f,f_0,0} + Sv_0$ that

$$\|E_{f,f_0,v_0}\|_{q,s,T} \le \|Sv_0\|_{q,s,T} + \|E_{f,f_0,0}\|_{q,s,T} = \|v_0\|_{B^{q,s}_T(\Omega)} + \|E_{f,f_0,0}\|_{q,s,T}$$

which proves (3.15).

This completes the proof of Proposition 3.1. ■

4 Proof of the main results

4.1 Proof of Theorem 2.1

Let $f = \operatorname{div} F, u_0 = v_0, k, g, E = E_{k,g}, 0 < T \le \infty$ be given as in Theorem 2.1. We will need several steps for the proof.

a) **Preliminaries:** Consider a solution u of (1.1) in the form $u = v + E_{k,g}$ with v satisfying (1.12) - (1.16). To work in the space X, see (3.4), we have to turn to $\hat{v} = v - E_{f,0,v_0}$ which is a solution of the equation

$$\hat{v}_t - \Delta\hat{v} + \nabla h = k(\hat{v} + \widehat{E}) - \operatorname{div}\big((\hat{v} + \widehat{E})(\hat{v} + \widehat{E})\big), \quad \widehat{E} := E_{f,0,v_0} + E_{k,g}\,, \tag{4.1}$$

together with $\operatorname{div}\hat{v} = 0$, $\hat{v}(0) = 0$ and $\hat{v}|_{\partial\Omega} = 0$. Here $k(\hat{v} + \widehat{E})$ plays the role of f_0 in Proposition 3.1. To reformulate the fixed point problem (4.1) we define

$$\begin{aligned} \widehat{F}(\hat{v}) &:= -(\hat{v} + \widehat{E})(\hat{v} + \widehat{E}),\\ \hat{f}(\hat{v}) &:= \operatorname{div}\widehat{F}(\hat{v}),\\ \hat{f}_0(\hat{v}) &:= k(\hat{v} + \widehat{E}). \end{aligned} \tag{4.2}$$

By Hölder's inequality and (3.15) we obtain that $\widehat{E}, E_{f,0,v_0}, E = E_{k,g} \in L^s(0,T;L^q(\Omega))$ as well as $\hat{v}, v \in L^s(0,T;L^q(\Omega))$, and that

$$\begin{aligned} \|\widehat{F}(\hat{v})\|_{\frac{q}{2},\frac{s}{2},T} &\le \|\hat{v} + \widehat{E}\|^2_{q,s,T} \le \big(\|\hat{v}\|_{q,s,T} + \|\widehat{E}\|_{q,s,T}\big)^2 < \infty,\\ \|\hat{f}_0(\hat{v})\|_{\frac{q}{2},\frac{s}{2},T} &\le \|k\|_{q,s,T}\big(\|\hat{v}\|_{q,s,T} + \|\widehat{E}\|_{q,s,T}\big) < \infty. \end{aligned} \tag{4.3}$$

Correspondingly, we set

$$\begin{aligned} F(v) &:= -(v + E)(v + E) = -(\hat{v} + \widehat{E})(\hat{v} + \widehat{E}),\\ f(v) &:= \operatorname{div}F(v) = \operatorname{div}\widehat{F}(\hat{v}) = \hat{f}(\hat{v}),\\ f_0(v) &:= k(v + E) = k(\hat{v} + \widehat{E}), \end{aligned} \tag{4.4}$$

and get estimates of $F(v)$, $f_0(v)$ in $L^{\frac{s}{2}}(0,T;L^{\frac{q}{2}}(\Omega))$ as those for $\widehat{F}(\hat{v})$, $\hat{f}_0(\hat{v})$ in (4.3).

Next we mention an estimate for $b(T)$ as given in (2.1). We obtain, using (3.15), (3.16), (3.14) with $f_0 = 0$ and (1.9), the estimate

$$\begin{aligned} \|\widehat{E}\|_{q,s,T} &\le \|E_{f,0,v_0}\|_{q,s,T} + \|E\|_{q,s,T}\\ &\le \|v_0\|_{B^{q,s}_T(\Omega)} + \|E_{f,0,0}\|_{q,s,T} + \|E\|_{q,s,T}\\ &\le \|v_0\|_{B^{q,s}_T(\Omega)} + C\big(\|F\|_{\frac{q}{2},\frac{s}{2},T} + \|k\|_{q,s,T} + \|g\|_{-\frac{1}{q},q,s,T}\big) \end{aligned}$$

with constant $C = C(\Omega,q) > 0$. We may assume that $C \ge 1$. Hence

$$\|\widehat{E}\|_{q,s,T} \le Cb(T),\ \|k\|_{q,s,T} \le b(T) \tag{4.5}$$

with $C = C(\Omega,q) \ge 1$ independent of T.

b) **Properties of $\mathcal{F}(\hat{v})$:** Let $\hat{v} \in X$ and let $\mathcal{F}(\hat{v}) := w$ be the solution of the system

$$(A^{-\frac{1}{2}}w)_t + A^{\frac{1}{2}}w = A^{-\frac{1}{2}}P \operatorname{div} \widehat{F}(\hat{v}) + A^{-\frac{1}{2}}P\hat{f}_0(\hat{v}), \;\; w \in X, \tag{4.6}$$

as in (3.13).

Using, step by step, (3.16), (3.14) and (4.3), we obtain that

$$\begin{aligned} \|\mathcal{F}(\hat{v})\|_{q,s,T} = \|w\|_{q,s,T} &\leq C_1\|w\|_X \\ &\leq C_2\big(\|\widehat{F}(\hat{v})\|_{\frac{q}{2},\frac{s}{2},T} + \|\hat{f}_0(\hat{v})\|_{\frac{q}{2},\frac{s}{2},T}\big) \\ &\leq C_3\big((\|\hat{v}\|_{q,s,T} + \|\widehat{E}\|_{q,s,T})^2 + \|k\|_{q,s,T}(\|\hat{v}\|_{q,s,T} + \|\widehat{E}\|_{q,s,T})\big). \end{aligned} \tag{4.7}$$

Moreover, by (3.16) for v and (4.5) for $\widehat{E}$, k, we get that

$$\|\mathcal{F}(\hat{v})\|_{q,s,T} \leq C_4\big(\|\hat{v}\|_X + b(T)\big)^2 \tag{4.8}$$

with constants $C_1, C_2, C_3, C_4 > 0$ depending on Ω, q. Consequently, by (4.7), (4.8)

$$\begin{aligned} \|\mathcal{F}(\hat{v})\|_{q,s,T} \leq C_1\|\mathcal{F}(\hat{v})\|_X &\leq C_4(\|\hat{v}\|_X + b)^2, \\ \|\mathcal{F}(\hat{v})\|_X &\leq a(\|\hat{v}\|_X + b)^2, \;\; a = C_4/C_1, \;\; b = b(T) \end{aligned} \tag{4.9}$$

with constants $C_1, C_4 > 0$ depending on Ω, q.

Next we estimate the expression $\mathcal{F}(\hat{v}) - \mathcal{F}(\tilde{v})$ with $\hat{v}, \tilde{v} \in X$, using the representation formula (3.11) with v, f_0, F replaced by $\mathcal{F}(\hat{v}) - \mathcal{F}(\tilde{v})$ and $\hat{f}_0(\hat{v}) - \hat{f}_0(\tilde{v})$, $\widehat{F}(\hat{v}) - \widehat{F}(\tilde{v})$, respectively. We obtain that

$$\begin{aligned} &(\mathcal{F}(\hat{v}) - \mathcal{F}(\tilde{v}))(t) \\ &= \int_0^t A^{\frac{1}{2}}S(t-\tau)A^{-\frac{1}{2}}P[\operatorname{div}(\widehat{F}(\hat{v}) - \widehat{F}(\tilde{v})) + \hat{f}_0(\hat{v}) - \hat{f}_0(\tilde{v})]\,\mathrm{d}\tau \\ &= \int_0^t A^{\frac{1}{2}}S(t-\tau)A^{-\frac{1}{2}}P[\operatorname{div}((\hat{v} + \widehat{E})(\hat{v} - \tilde{v}) + (\hat{v} - \tilde{v})(\tilde{v} + \widehat{E})) + k(\hat{v} - \tilde{v})]\,\mathrm{d}\tau. \end{aligned}$$

Then we apply the same arguments as in (4.7), (4.8), (4.9) to get for $\hat{v}, \tilde{v} \in X$ the estimate

$$\begin{aligned} \|\mathcal{F}(\hat{v}) - \mathcal{F}(\tilde{v})\|_X &\leq C_1'\big(\|\hat{v}\|_X + b + \|\tilde{v}\|_X + b + b\big)\|\hat{v} - \tilde{v}\|_X) \\ &\leq C_2'\big(\|\hat{v}\|_X + b + \|\tilde{v}\|_X + b\big)\|\hat{v} - \tilde{v}\|_X \end{aligned}$$

with $C_1', C_2' > 0$ depending on Ω, q. We may assume that $C_2' = a$ with a as in (4.9). Thus we obtain that

$$\|\mathcal{F}(\hat{v}) - \mathcal{F}(\tilde{v})\|_X \leq a\big(\|\hat{v}\|_X + b + \|\tilde{v}\|_X + b\big)\|\hat{v} - \tilde{v}\|_X. \tag{4.10}$$

c) **The condition $\mathcal{F}(\hat{v}) = \hat{v}$, $\hat{v} \in X$:** Assume that $\hat{v} \in X$ satisfies the condition $\mathcal{F}(\hat{v}) = \hat{v}$. Then we show that

$$v := \hat{v} + E_{f,0,v_0} \text{ is a strong solution of (1.12)} \tag{4.11}$$

as in Definition 1.2 (2).

For the proof we first consider (4.6), and using (3.12), (3.13), (3.14) with v, F, f_0 replaced by $w = \mathcal{F}(\hat{v}), \widehat{F}(\hat{v}), \hat{f}_0(\hat{v})$ together with the estimate (4.3), we conclude with (3.16) that

$$w \in L^s(0,T; L^q(\Omega)). \tag{4.12}$$

Then we use Proposition 3.1 (iv) to see that w is a weak solution of the system (3.7) with v, f, f_0 replaced by w, $\hat{f}(\hat{v})$, $\hat{f}_0(\hat{v})$, and with $v(0) = w(0) = 0$.

Now we consider $\widehat{E} = E_{f,0,v_0} + E$, $E = E_{k,g}$, as in (4.1) and use the relation (3.8) for $v = E_{f,0,v_0}$ with $f = \mathrm{div} F, f_0 = 0$ as well as for the weak solution $w = \mathcal{F}(\hat{v})$ of (3.8). We conclude that

$$v = w + E_{f,0,v_0} \text{ satisfies (1.15) with } E = E_{k,g}. \tag{4.13}$$

Thus it holds (3.8) with F, f_0 replaced by $F(v) + F$, $f_0(v)$, see (4.4). Using Proposition 3.1, (i) we obtain that $v = w + E_{f,0,v_0}$ satisfies the properties (1.13)–(1.16).

To prove (1.16) we use (3.10) with F, f_0 replaced by $F(v) + F$, $f_0(v)$ and the following elementary calculations:

$$\begin{aligned}
\int_0^t \langle (v+E)v, \nabla v\rangle_\Omega \, \mathrm{d}\tau &= \int_0^t \langle vv, \nabla v\rangle_\Omega \, \mathrm{d}\tau + \int_0^t \langle Ev, \nabla v\rangle_\Omega \, \mathrm{d}\tau \\
&= 0 - \frac{1}{2}\int_0^t \langle kv, v\rangle_\Omega \, \mathrm{d}\tau,
\end{aligned}$$

$$\int_0^t \langle (v+E)(v+E), \nabla v\rangle_\Omega \, \mathrm{d}\tau + \int_0^t \langle k(v+E), v\rangle_\Omega \, \mathrm{d}\tau =$$

$$\int_0^t \langle (v+E)E, \nabla v\rangle_\Omega \, \mathrm{d}\tau + \frac{1}{2}\int_0^t \langle k(v+2E), v\rangle_\Omega \, \mathrm{d}\tau.$$

By (4.11) and (3.15) $v \in L^s(0,T; L^q(\Omega))$, and, consequently, v is a strong solution as in Definition 1.2.

Conversely, assume that v is a strong solution as in Definition 1.2. Then we show that

$$\hat{v} := v - E_{f,0,v_0} \in X \text{ and } \mathcal{F}(\hat{v}) = \hat{v}. \tag{4.14}$$

Since by Proposition 3.1 (iv) v is also a weak solution as in Definition 1.2, we obtain, using (1.15), (3.11) and (4.4), the representation

$$v(t) = S(t)v_0 + \int_0^t S(t-\tau)Pf_0(v)\,\mathrm{d}\tau + \int_0^t A^{\frac{1}{2}}S(t-\tau)A^{-\frac{1}{2}}P\mathrm{div}(F+F(v))\,\mathrm{d}\tau \tag{4.15}$$

with F as in (1.2). Subtracting the integral representation of $E_{f,0,v_0}$ we arrive at the formula

$$\hat{v}(t) = \int_0^t S(t-\tau)Pf_0(v) + \int_0^t A^{\frac{1}{2}}S(t-\tau)A^{-\frac{1}{2}}P\mathrm{div}F(v)\,\mathrm{d}\tau; \tag{4.16}$$

in particular, $\hat{v}(t) \in X$. Since $f_0(v) = \hat{f}_0(\hat{v})$, $F(v) = \widehat{F}(\hat{v})$, the right-hand side of (4.16) coincides with $\mathcal{F}(\hat{v})$ and (4.14) is proved.

d) **Uniqueness of $\hat{\boldsymbol{v}} = \boldsymbol{\mathcal{F}}(\hat{\boldsymbol{v}})$:** Suppose that $\hat{v}_1, \hat{v}_2 \in X$ are fixed points of $\mathcal{F}$. Then we conclude from (4.10) with $\|\cdots\|_X$ replaced by $\|\cdots\|_{q;s,T}$ that

$$\begin{aligned}\|\hat{v}_1 - \hat{v}_2\|_{q,s,T} &= \|\mathcal{F}(\hat{v}_1) - \mathcal{F}(\hat{v}_2)\|_{q,s,T} \\ &\leq a\big(\|\hat{v}_1\|_{q,s,T} + b(T) + \|\hat{v}_2\|_{q,s,T} + b(T)\big)\|\hat{v}_1 - \hat{v}_2\|_{q,s,T}\end{aligned} \tag{4.17}$$

with $b = b(T)$ as in (4.5), and with $a = a(\Omega, q) > 0$ as in (4.9).

Consider any subinterval $[0, T')$, $0 < T' < T$. Then we obtain the same estimate (4.17) with $\|\cdots\|_{q,s,T}$, $b(T)$ replaced by $\|\cdots\|_{q,s,T'}$, $b(T')$, and choose $0 < T' < T$ such that

$$a\big(\|\hat{v}_1\|_{q,s,T'} + b(T') + \|\hat{v}_2\|_{q,s,T'} + b(T')\big) \leq \frac{1}{2};$$

thus we conclude that $\frac{1}{2}\|\hat{v}_1 - \hat{v}_2\|_{q,s,T'} \leq 0$, $\hat{v}_1 = \hat{v}_2$. Finally, we repeat this argument with $[0, T')$ replaced by $[T', 2T')$ with the same constant a, and so on. This yields $\hat{v}_1 = \hat{v}_2$ in $[0, T)$.

e) **Fixed point problem $\hat{\boldsymbol{v}} = \boldsymbol{\mathcal{F}}(\hat{\boldsymbol{v}})$:** Here we use similar arguments as in the existence proof of very weak solutions, see [3], [6], [7].

Let $\hat{v}, \tilde{v} \in X$. Then $\mathcal{F}(\hat{v}), \mathcal{F}(\tilde{v})$ satisfy the estimates (4.9), (4.10), i.e., with $a = a(\Omega, q) > 0$ and $b = b(T)$

$$\|\mathcal{F}(\hat{v})\|_X \leq a(\|\hat{v}\|_X + b)^2, \tag{4.18}$$

$$\|\mathcal{F}(\hat{v}) - \mathcal{F}(\tilde{v})\|_X \leq a\big(\|\hat{v}\|_X + b + \|\tilde{v}\|_X + b\big)\|\hat{v} - \tilde{v}\|_X. \tag{4.19}$$

For the given data $f = \operatorname{div} F$, v_0, k, g as in Assumption 1.1 and with $b = b(T)$ defined in (2.1) we suppose the smallness condition

$$4ab = 4a\big(\|v_0\|_{B_T^{q,s}} + \|F\|_{\frac{q}{2},\frac{s}{2},T} + \|k\|_{q,s,T} + \|g\|_{-\frac{1}{q},q,s,T}\big) < 1. \tag{4.20}$$

Using $4ab < 1$ we choose

$$0 < y_1 := 2b(1 + \sqrt{1 - 4ab})^{-1} < 2b, \quad y_1 = ay_1^2 + b > b$$

and the closed ball $B := \{v \in X; \|v\|_X \leq y_1 - b\}$. Then, if $\hat{v} \in B$, we obtain from the estimate

$$\|\mathcal{F}(\hat{v})\|_X \leq a(\|\hat{v}\|_X + b)^2 \leq ay_1^2 = y_1 - b$$

that $\mathcal{F}(B) \subseteq B$. Further we use (4.19) and obtain with $\hat{v}, \tilde{v} \in B$ that

$$\|\mathcal{F}(\hat{v}) - \mathcal{F}(\tilde{v})\|_X \leq 2ay_1\|\hat{v} - \tilde{v}\|_X \leq 4ab\|\hat{v} - \tilde{v}\|_X.$$

Thus $\mathcal{F} : B \to B$ is a strict contraction, and Banach's fixed point theorem yields a $\hat{v} \in B$ satisfying $\hat{v} = \mathcal{F}(\hat{v})$, see [13, V, (4.2.21)] and [14, Lemma 10.2].

Using Part c) above we conclude that $v := \hat{v} + E_{f,0,v_0}$ is a strong solution of the system (1.12), and $u = v + E_{k,g}$ is a strong solution of the general system (1.1). Moreover, by Part d), v and $u = v + E_{k,g}$ are uniquely determined.

Setting $\varepsilon^* = \varepsilon^*(\Omega, q) := \frac{1}{8a}$ we thus obtain the existence and uniqueness of a strong solution v of (1.12) and of a strong solution $u = v + E_{k,g}$ of (1.1) provided the condition $b = b(T) \leq \varepsilon^*$ is satisfied. This completes the proof of Theorem 2.1. ■

4.2 Proof of Corollary 2.2

Let $f, v_0, k, g, q, s, [0,T), E = E_{k,g}$, $b = b(T)$ and $\varepsilon^* = \varepsilon^*(\Omega, q) > 0$ be given as in Corollary 2.2.

Since $b(T) \to 0$ as $T \to 0$, we find some T^*, $0 < T^* \leq T$, satisfying $b(T^*) \leq \varepsilon^*$. Applying Theorem 2.1 we get the uniquely determined solutions $v, u = v + E$. ■

4.3 Proof of Theorem 2.3

Assume that the given data $f = \operatorname{div} F, v_0, k, g$ additionally satisfies the regularity conditions (2.3). From [3, Corollary 5] we obtain for the solution $E = E_{k,g}$ of the Stokes system (1.6), in addition to (1.9), that

$$E \in L^s(0,T; W^{2,q}(\Omega)), \;\; E_t \in L^s(0,T; L^q(\Omega)). \tag{4.21}$$

Then we have to prove the regularity properties (2.4) for the solution v of (1.12), written in the form

$$\begin{aligned} v_t - \Delta v + (v+E)\cdot\nabla v + \nabla p^* = f^* := \operatorname{div} F - (v+E)\cdot\nabla E \qquad (4.22)\\ v|_{t=0} = v_0, v|_{\partial\Omega} = 0,\ \operatorname{div} v = 0. \end{aligned}$$

For a moment let $v := \hat{v}$ be the corresponding solution for the well known case $k = 0$, $g = 0$, $E = 0$. In this case the regularity properties (2.4) for $v = \hat{v}$ have been shown in [13, V. Theorem 1.8.1, pp. 298], where the critical expression, now written in the form $\hat{v}\cdot\nabla\hat{v}$, has been treated using the Yosida operators $J_k = \left(I + \frac{1}{k}A^{1/2}\right)^{-1}$, $k \in \mathbb{N}$, see [16], and $\hat{v}\cdot\nabla J_k\hat{v}$ with $\hat{v} \in L^s(0,T;L^q(\Omega))$.

We can reduce our regularity problem for (4.22) to this known case. Since $v + E \in L^s(0,T;L^q(\Omega))$, we use the approximation $(v+E)\cdot\nabla J_k v$, $k \in \mathbb{N}$, for the critical term $(v+E)\cdot\nabla v$. Furthermore, since by (4.21) $E, \nabla E \in L^s(0,T;L^q(\Omega))$ and $v \in L^\infty_{\text{loc}}([0,T);L^2(\Omega))$, and since $q \geq 4, s \geq 4$, we obtain that

$$v\cdot\nabla E, E\cdot\nabla E \in L^2_{\text{loc}}([0,T);L^2(\Omega)). \qquad (4.23)$$

Thus we obtain from (4.23), (2.3) that $f^* \in L^2_{\text{loc}}([0,T);L^2(\Omega))$. Then we get for (4.22) - as in [13, V. Theorem 1.8.1] - the properties (2.4), (2.5), (2.6). This completes the proof of Theorem 2.3. ■

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